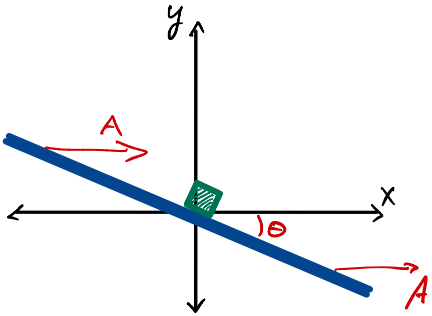
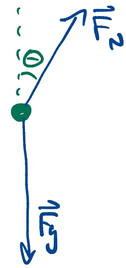
PS F2 Problem 3 - Solution



Ⓐ Equations of motion

a)



b) $\vec{F}_{net} = \vec{F}_g + \vec{F}_N$

$\vec{F}_g = -mg\hat{y}$

$\vec{F}_N = N\sin\theta\,\hat{x} + N\cos\theta\,\hat{y}$

$\Rightarrow \vec{F}_{Net} = N\sin\theta\,\hat{x} + (N\cos\theta - mg)\hat{y}$

$$\vec{a} = \frac{1}{m}\vec{F}_{net} \Rightarrow \left\{ \boxed{\begin{aligned} a_x &= \frac{N}{m}\sin\theta \\ a_y &= \frac{N}{m}\cos\theta - g \end{aligned}} \right.$$

e1

Ⓑ Equilibrium

c) To remain stationary relative to the plane:

* $y = 0\ \forall t \Rightarrow v_y = 0\ \forall t \Rightarrow \boxed{a_y = 0}$
* x moves with the plane $\Rightarrow \boxed{a_x = A}$

d) Sub into e1 $\Rightarrow A = \frac{N}{m}\sin\theta$ & $0 = \frac{N}{m}\cos\theta - g \Rightarrow g = \frac{N}{m}\cos\theta$

Divide the first eq by the second to eliminate N $\Rightarrow \frac{A}{g} = \tan\theta \Rightarrow \boxed{A = g\tan\theta}$

e) Checks: $A \propto g$ ✓

$\lim_{\theta\to 0} A = 0$ ✓ $\quad \lim_{\theta\to 90^\circ} A = \infty$ ✓ :)

Ⓒ General solution for $\vec{r}(t)$

f) $a_x = \frac{N}{m}\sin\theta;\ v_x = v_{x,0} + \int_0^t a_x\,dt' \Rightarrow \boxed{v_x(t) = \frac{N}{m}\sin\theta\ t}$

$a_y = \frac{N}{m}\cos\theta - g;\ v_y = v_{y,0} + \int_0^t a_y\,dt' \Rightarrow \boxed{v_y(t) = \left(\frac{N}{m}\cos\theta - g\right)t}$

g) $x(t) = x_0 + \int_0^t v_x(t')\,dt' \Rightarrow \boxed{x(t) = \frac{N}{2m}\sin\theta\ t^2}$

$y(t) = y_0 + \int_0^t v_y(t')\,dt' \Rightarrow \boxed{y(t) = \frac{1}{2}\left(\frac{N}{m}\cos\theta - g\right)t^2}$

e2

D) Motion of the plane's midpoint

$$\vec{a}_p(t) = A\hat{x} \Rightarrow \vec{v}_p(t) = \vec{v}_p(0) + \int_0^t \vec{a}_p(t') \Rightarrow \boxed{\vec{v}_p(t) = At\,\hat{x}}$$

$$\vec{r}_p(t) = \vec{r}_p(0) + \int_0^t \vec{v}_p(t')\,dt' \Rightarrow \boxed{\vec{r}_p(t) = \tfrac{1}{2}At^2\,\hat{x}}$$

I didn't ask you to work this out, but I'm showing you how I did 'cuz you might want to learn how!

The constraint equation

A generic line equation has the form $y = \alpha x + \beta$, where α = slope and β = y-intercept.

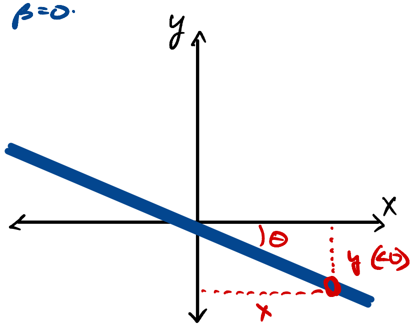
At $t=0$, the plane passes through the origin $\Rightarrow \beta = 0$.

Slope = $\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}$. Looking at the diagram, for the point marked:

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|} = \frac{-y}{x}$$

$$\Rightarrow \text{slope} = \frac{\Delta y}{\Delta x} = -\tan\theta$$

$\Rightarrow$ line equation is $\boxed{y = -\tan\theta\; x}$



Replacing x with $(x - x_p) \Rightarrow y_p(t) = -\tan\theta\,(x - x_p)$

From D2, $x_p = \frac{1}{2}At^2 \Rightarrow \boxed{y = -\tan\theta\left(x - \tfrac{1}{2}At^2\right)}$ e3

E) Normal force

h) Substituting the expressions for x and y from e2 into e3:

$$\Rightarrow \tfrac{1}{2}\left(\tfrac{N}{m}\cos\theta - g\right)t^2 = -\tan\theta\left[\tfrac{N}{2m}\sin\theta\; t^2 - \tfrac{1}{2}At^2\right]$$

Cancelling & simplifying & solving for N:

$$\Rightarrow \frac{N}{m}\cos\theta - 2g = \tan\theta\left[A - \frac{N}{m}\sin\theta\right] \Rightarrow \frac{N}{m}\left[\cos\theta + \tan\theta\sin\theta\right] = g + A\tan\theta$$

$$\cos\theta + \tan\theta\sin\theta = \cos\theta + \frac{\sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta}\left[\cos^2\theta + \sin^2\theta\right] = \frac{1}{\cos\theta}$$ ☺

$$\Rightarrow \frac{N}{m\cos\theta} = g + A\tan\theta \Rightarrow N = m\cos\theta\left[g + A\tan\theta\right] \Rightarrow \boxed{N = \underbrace{mg\cos\theta}_{\text{same as for a stationary ramp}} + \underbrace{mA\sin\theta}_{\text{effect of accelerating ramp}}}$$ e4

i) Units: $mg \sim$ force, $mA \sim$ force ✓

j) $\lim_{A\to 0} N \to mg\cos\theta$ ✓ The usual "block on a plane" value, when N "balances" the ⊥ component of the gravitational force.

k) $\lim_{\theta \to 0} N \to mg$ ✓ The block sits at rest on a horizontal table!

Ⓕ The actual position vs. time solution!

l) Using e4 to replace N in e2:

$$\Rightarrow \quad x(t) = \frac{\sin\theta}{2}\left(g\cos\theta + A\sin\theta\right)t^2$$

$$y(t) = \frac{1}{2}\left[\cos\theta\left(g\cos\theta + A\sin\theta\right) - g\right]t^2 = \frac{1}{2}\left[\left(\cos^2\theta - 1\right)g + \sin\theta\cos\theta A\right]t^2$$

$$= \frac{1}{2}\left[\sin\theta\cos\theta A - \sin^2 g\right]t^2 = \frac{\sin\theta}{2}\left(A\cos\theta - g\sin\theta\right)t^2$$

$$\Rightarrow \quad \boxed{\begin{aligned} x(t) &= \frac{\sin\theta}{2}\left(A\sin\theta + g\cos\theta\right)t^2 \\ y(t) &= \frac{\sin\theta}{2}\left(A\cos\theta - g\sin\theta\right)t^2 \end{aligned}} \quad e5$$

m) $y(t) = 0 \; \forall t \Rightarrow A\cos\theta = g\sin\theta \Rightarrow A = g\tan\theta$ ✓ as in B2.

Putting this value of A back into the $x(t)$ expression in e5:

$$\Rightarrow x(t) = \frac{\sin\theta}{2}\left(g\tan\theta\sin\theta + g\cos\theta\right)t^2 = \frac{1}{2}\frac{\sin\theta}{\cos\theta}\left(\sin^2\theta + \cos^2\theta\right)g t^2$$

$$= \frac{1}{2}\tan\theta\, g t^2 = \frac{1}{2}A t^2$$ ✓ agrees w/ D2.

(whew!)