Masterthesis: The power of two paths in grid computing networks

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Abstract

This thesis researched the behavior of forwarding algorithms on ring structured distributed systems. We first describe how the different algorithms actually behave and what their basic properties are. Next, we will simulate the algorithms, where we will find left/right forward to be the best performing algorithm when nodes can only see their neighbors. When nodes can see all other nodes, random unvisited has the best performance, but only slightly better than the coprime offset family. An upper bound is found for the simulations using multiple CPUs per server.

When validating those results numerically, it seems the algorithms behave alike. This means the simulation does not contain serious faults. We have also found that the numerical model can be optimized by lumping redundant states. While this increases the performance of the validation, the actual benefit is disappointing, due to the non-polynomial nature of the algorithms. We have also proven the equivalence between multiple algorithms under certain conditions.

Acknowledgments

This thesis is not just the work of one person, I could never accomplish this without the people around me. I would like to take this opportunity to thank some of them in particular.

First and foremost, my supervisor Professor dr. Benny Van Houdt. First of all, he taught me a whole new field in the area of computer science, the part that interested me the most during the course of my studies. Secondly, for being my supervisor: he helped me out when he could and suggested ideas when I got stuck. I was always welcome to drop by and reflect thoughts.

I would also like to thank my family, especially my parents. It is because of their financial support and trust I could complete my education. They gave me the freedom of making my own choices and motivated me when I needed it. My sisters, Anneleen and Evelien have been a great help and source of moral support these five years. Anneleen deserves a special note for proofreading this thesis and other assignments I had to write in English.

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Contents

Introduction

This thesis researches the behavior of different forwarding algorithms in a ring structured distributed system.

Section 1 precisely describes the setup of the system. It specifies the general assumptions made in this document and gives a short overview of the different algorithms we have reviewed. We specify how each algorithms works and their properties are summed up.

Next, section 2 provides a short introduction about the simulator we wrote and how to use it. Further, it contains the results of the simulations we have performed. Curiosities encountered in the results are simulated using other parameters. The section finishes with the attempt to transform the results of single CPU servers into the results of multiple CPU servers.

The following part, section 3, is used to validate the results of the simulation using numerical algorithms. Some of the algorithms are modeled into Markov Chains to get numerically exact results. Afterwards, the structure of the Markov Chains are reviewed and optimized. At the end of this section we will try to find equivalencies between the algorithms under certain circumstances.

Finally, in section 4, we will give our thoughts on the results obtained in this thesis.

1 Setup

We are using a ring-structured distributed system of N nodes. Each node is connected to two neighbors, its left and right. The purpose of these nodes is to process incoming jobs. When a node is busy while a job arrives, it must forward the incoming job to another node. When a job has visited all nodes and none of them was found idle, the job is dropped.

Jobs have an arrival time, a length, the ID of the first node and optional metadata. They arrive at each node independently as a Poisson process at rate λ. Their length is exponentially distributed with mean μ (unless otherwise noted, assume $\mu = 1$). Although each job has a length, this length may not be known in advance. Finally, the metadata is optional and may be used by the nodes to pass information among the job (e.g. a list of visited nodes).

Nodes can use different algorithms to determine whereto a job will be forwarded, some algorithms require the node to keep state information. The performance of the algorithms is the main focus of this thesis. Different techniques will be discussed and simulated. Afterwards, some results of the simulation will be validated using numerical techniques. Note that the cost of forwarding a job is ignored. Together with the presumption a job must visit each node before being dropped, this means a job arriving at any node will be processed if and only if at least one server is idle.

The performance of the forwarding algorithms is measured by the average number of nodes a job must visit before being executed. The goal of the algorithms is to minimize this number by spreading the load evenly along the ring.

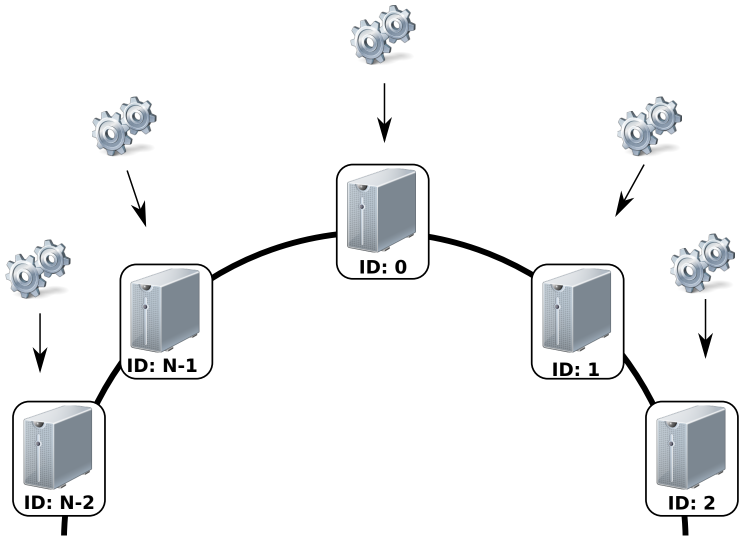


Figure 1: A ring structured network

1.1 Forwarding algorithms

Nodes that must forward a job will choose another node of the ring. Nodes have no information about other nodes, so they have no idea which nodes are idle or busy. The algorithms are grouped into two categories: forward to neighbor and forward anywhere. The first group allows a busy node to forward an incoming job to either its left or its right neighbor, where the latter may forward these jobs to any node in the ring. Because the amount of dropped jobs is equal for each forwarding algorithm, these jobs will be ignored when computing the average number of forwards. One should note that the loss rate of jobs in the system is the same as the Erlang-b loss rate.

1.2 Forward to neighbor

Forward right

A busy node using this technique will forward a job to its right neighbor. The job will keep traveling in this direction until an idle node is found, where it will be processed, or until it has visited all nodes when it will be dropped. Unless otherwise noted, this algorithm is used as a baseline in all further tests. This algorithm does not use any space in the job's metadata, nor must nodes save states.

Left/right forward

A variant to the previous algorithm is the left/right forward technique. Instead of forwarding each job to its right neighbor, a busy node will alternate the direction after forwarding an incoming job. To avoid a job coming back, this initial direction is saved in the job's metadata. Busy nodes receiving a job from another neighbor must forward it in the same direction as specified in the job's metadata. This algorithm requires the job to keep 1 bit representing the direction in the job's metadata, it also needs to save 1 bit state information in each node to keep track of the last direction a new job was forwarded to.

Random left/right forward with parameter p

This technique is a variant of the left/right forward algorithm. However, instead of alternating the direction for each new job, a node will forward a job to its right with probability p and to its left with probability $1 - p$. Like the previous technique, the direction is saved in the job's metadata and subsequent nodes must maintain this direction when forwarding. Unlike left/right forward, nodes do not need to save state information.

Position dependent forwarding

As shown in figure 1, each node in the ring has a unique ID. The IDs are ordered clockwise on the ring. Nodes using this algorithm will always forward an incoming job into the same direction: to the right when the node's id is even, to the left otherwise. Just like the previous algorithm, the direction is saved in the job's metadata and this direction must be used when other nodes have to forward the job. The initial direction of incoming jobs can be derived from the node's ID, therefore the node needs no state information.

1.3 Forward anywhere

The ring structure can be used in real networks, however in many cases the ring is just a virtual overlay over another structure (e.g. the internet). In these networks each node is able to connect each other node directly and more sophisticated forwarding algorithms can be used.

Random unvisited

The random unvisited algorithm is the most basic algorithm in this category. Every time a job must be forwarded, a list of unvisited nodes is generated and a random node is chosen from that list. The current node is added to the list of visited nodes, which is found in the job's metadata. This list can be saved using $N - 1$ bits (-1 because the initial node is saved separately). Each unvisited node must have an equal probability of being picked.

Coprime offset

Another algorithm is coprime offset. This algorithm generates a list of all numbers smaller than N, and coprime to N. The first time a job is forwarded, the next number of this list is selected. This is the job's forwarding offset and it is saved in the job's metadata. When a job is forwarded, it is sent exactly this many nodes farther. Because this number and N are coprime, it will visit all nodes exactly once before being returned to its originating node.

Example: Consider a ring size of $N = 10$ in which every node is busy. The list of coprimes is: $[1, 3, 7, 9]$. Let's assume a job arrives at node 3 and the last time node 3 forwarded a job, the job was given offset 1. Because this node is busy, the next number on the list (3) is selected and saved in the job's metadata. All nodes are busy so the job visits all these nodes before being dropped: 3 (arrival), $6, 9, 2, 5, 8, 1, 4, 7, 0$. Node 0 will drop the job because the next node would be 3, which is the node on which the job arrived.

Because the list of coprimes can be regenerated each time a job arrives, it must not be saved. Both the job and the node must keep an index to a coprime. The size of this list $\varphi(N)$, ergo an index to an element in this list requires $\lceil log_2 \varphi(N) \rceil$ bits (see section 1.4 for the definition of $\varphi(N)$).

Random coprime offset

The random coprime offset algorithm is almost equal to coprime offset. The difference between them is the decision of the offset value. Where it is the next number on the list in coprime offset, a random value is taken from the list when using random coprime offset. As coprime offset, the offset must be saved in the job's metadata. Since the offset for incoming jobs is chosen at random, the nodes do not need to keep state information.

1.4 Properties of Algorithms

We have reviewed some of the basic properties for each algorithm. They are summarized in table 1. The function $\varphi(N)$ is Euler's totient function and counts the coprimes of N up to N [1].

$$\begin{aligned} \varphi(N) &= \sum_{0<p<N, p \perp N} 1 \\ &= N \cdot \prod_{p|N, p \text{ is prime}} (1 - \frac{1}{p}) \end{aligned}$$

Note that this value is upper bound by $N-1$ and equals $N-1$ when N is prime.

	Size known	First forward	Possible paths	Space requirement (job)	Space requirement (node)
Forward right	N	1	1	0	0
Left/right forward	N	2	2	1	1
Random left/right forward (p)	N	2	2 [(1)]	1	0
Position dependent forward	N	1	1	1	0
Random unvisited	Y	$N-1$	$(N-1)!$	$N-1$	0
Coprime offset	Y	$\varphi(N)$	$\varphi(N)$	$\lceil log_2 \varphi(N) \rceil$	$\lceil log_2 \varphi(N) \rceil$
Random coprime offset	Y	$\varphi(N)$	$\varphi(N)$	$\lceil log_2 \varphi(N) \rceil$	0

Table 1: Properties of forwarding algorithms

Size known Represents whether the size of the ring must be known to the nodes in order for the algorithm to function. This implies that a change in the ring (node joins or leaves) must be propagated over the ring.

[1]For random left/right forward where $p \neq 0.5$, the probability of each path is different.

First forward When the node on which the incoming job arrives is busy, this number represents the number of candidates to which the node can forward the job.

Possible paths Assuming the system is saturated, this number represents the number of possible paths a job can travel until being dropped. The probability of each path is the same (unless otherwise noted).

Space requirement (job) This value represents the number of bits required in the job's metadata.

Space requirement (node) This value represents the number of bits required to save the node's state information.

2 Simulation

To evaluate the different algorithms discussed in the previous section, 2 methods will be used. Our first method is using a simulator. The second method is the numerical evaluation of the simulation results using MATLAB. The validation method is further discussed in section 3.

The simulation is accomplished using a custom simulator. A continuous time simulator is written in C++, using no external requirements but the STL and OpenMP [3]. The source code of the simulator can be found in appendix A or at `http://code.google.com/p/powerofpaths/`.

The simulator can be controlled using a command line interface of which the usage is described below:

```
Usage: -r -s long -j double -a double -n long -p long -l long -t
    long -h type
        -r      Random seed
        -s      Set seed                        (default: 0)
        -j      Job length                      (default: 1.0)
        -a      Load                            (default: 1.0)
        -n      Ring size                       (default: 100)
        -c      Processing units per node       (default: 1)
        -p      Print progress interval         (default: -1 -
            disabled)
        -l      Simulation length               (default: 3600)
        -t      Repetition                      (default: 1)
        -h      Print this help
         type   right | switch | randswitch | evenswitch | prime |
             randprime | randunvisited | totop
```

Listing 1: Simulator usage description

2.1 Measure

The goal of the algorithms is to distribute the jobs evenly along the ring. This implies that the number of nodes a job must visit should be low. As a measure for our experiments, we will be using the average number of times a job is forwarded before it is executed. Since the number of forwards of a job that could not be executed is the same for each algorithm (i.e. $N-1$: traversing each link but one), and the loss rate of each algorithm is the same (because forwards are instantaneous), we will not take these dropped jobs into account when computing the average.

It is clear that when the system load approaches 0, the probability that a node is busy will also approach 0 and the average number of forwards will therefore also approach 0. On the other hand, when the load approaches ∞, each node's probability of being busy will approach 1 and therefore the number of forwards will be $N - 1$ and the job will fail. A system with load > 1 is called an overloaded system. For the purpose of this thesis, only loads up to 1 are discussed.

We will compare each algorithm to a baseline result. The baseline used in this thesis is the forward right algorithm. This means that each graph will show its result relative to the forward right results. The results given by the simulator were obtained using a ring size of 100 and using a random seed for each run.

The absolute performance of the baseline algorithm is shown in figure 2.

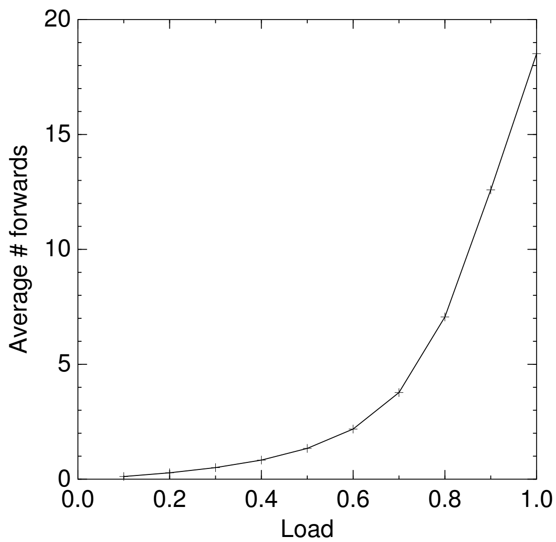


Figure 2: The forward right baseline result

2.2 Results

Left/right forward

It is intuitively clear that alternating the forwarding direction of incoming jobs should distribute the load better than keeping the same direction, certainly under temporary local heavy load. Figure 3 shows the improvement made by the left/right forward algorithm over the forward right method. The performance gain is at least 1% and up to over 4% under medium load.

Random left/right forward with parameter p

For $p = 0.5$, one would expect the results of this algorithm being similar to those obtained in the previous simulation. However, it seems the small change in the algorithm worsened the results significantly.

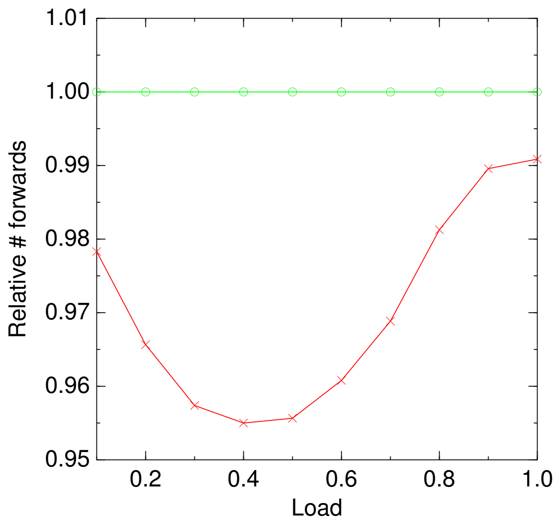


Figure 3: Left/right forward

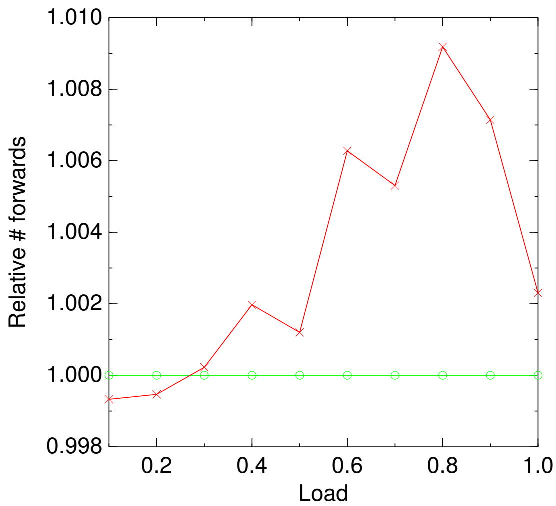


Figure 4: Random left/right forward with parameter 0.5

Figure 4 shows the results of this algorithm for $p = 0.5$. For $p = 0$, the algorithm is equal to the forward right algorithm. Hence, we should investigate how different values of p influence the final results. Figure 5 shows the results obtained for $0 \leq p \leq 0.5$. We find no value of p for which this algorithm performs better than forward right.

Position dependent forwarding

This technique groups every 2 nodes into small virtual clusters. When a job arrives at a busy node, the job will be forwarded to the other node in this cluster. Jobs leaving a cluster will seem to do this in a random direction ($p = 0.5$). Since

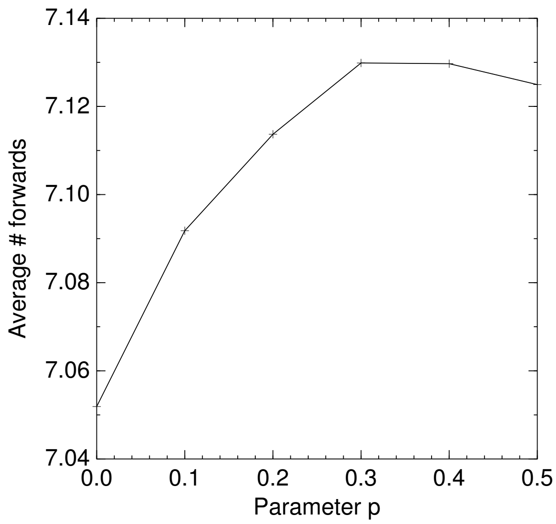


Figure 5: Random left/right forward with load 0.8

the load is concentrated per cluster instead of being distributed over the whole system, this technique performs worse than other techniques. The results are shown in figure 6.

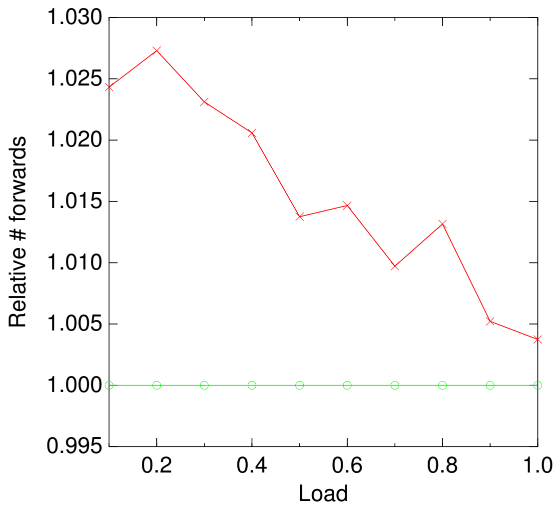


Figure 6: Position dependent forwarding

Random unvisited

Unlike the previous algorithms, this algorithm is not bound to its neighbors when forwarding a job. Lifting this constraint allows a serious performance boost.

Figure 7 shows the results of this algorithm. The performance benefit is at least 10% in the range $0.20 - 1.00$, and even up to 30% under medium load.

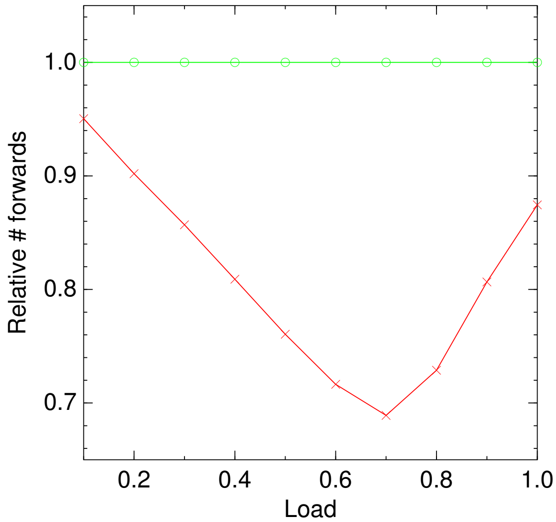
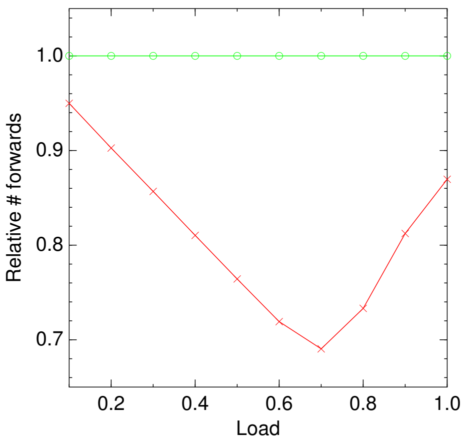


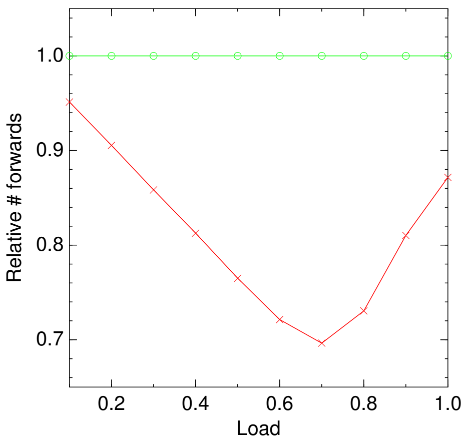
Figure 7: Random unvisited forwarding

(Random) Coprime offset

Two other algorithms that are not restricted to forwarding to a neighbor are coprime offset and random coprime offset. Figure 8 shows the results of both these algorithms. The difference of these algorithms with themselves and the random unvisited algorithms is not clearly visible when comparing both to the forward right algorithm. To put these results in perspective, we included figure 9 where coprime offset and random coprime offset are depicted relative to random unvisited. Although the difference is very small, it seems the random unvisited algorithm shows a better performance than the other two. However, coprime offset and random coprime offset require $\lceil log_2\varphi(N)\rceil$ bits in the job's metadata instead of N for random unvisited.



(a) Coprime offset



(b) Random coprime offset

Figure 8: (Random) Coprime offset

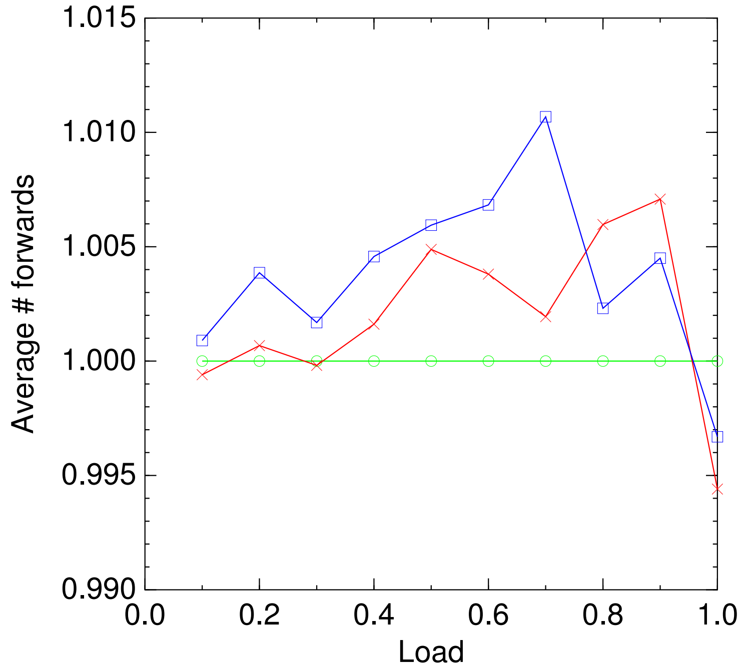


Figure 9: Random unvisited (green), coprime offset (blue), random coprime offset (Red)

2.3 Multiple execution units

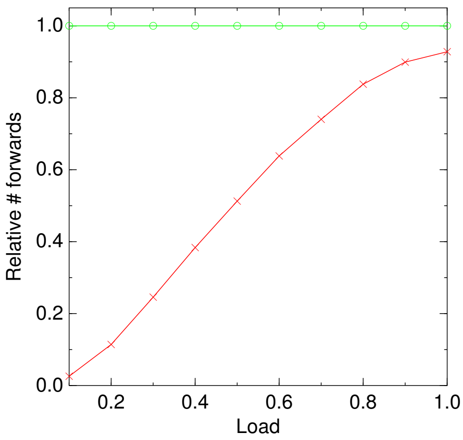
So far, all simulations were executed using 1 CPU per server. This is not a realistic assumption for most distributed systems. This section is intended to research the behavior of the algorithms when using multiple CPUs, and comparing the results to a system with 1 CPU per server, and to each other.

Per algorithm, two simulations were performed and compared. One using 1 CPU per server and one using 4 CPUs per server. See table 2 for more information about the tests.

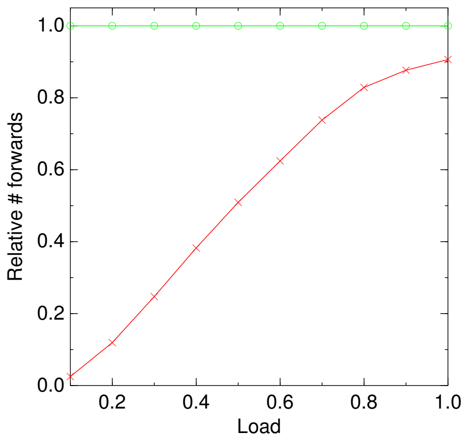
Algorithms	Forward right, Left/right forward, Random left/right (0.5), Random unvisited, Random coprime offset	
Ring size	100	25
CPUs per node	1	4
Total CPUs in system	100	
Load	0.1 – 1.0	

Table 2: Comparison of algorithms using multiple CPUs per server

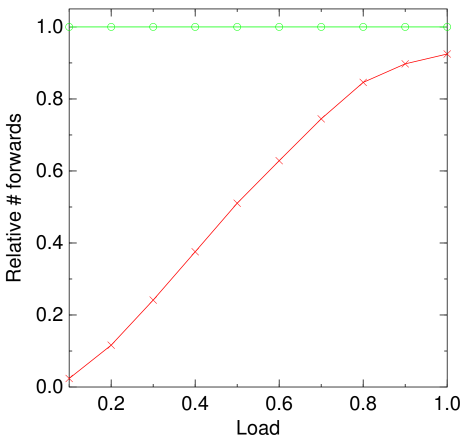
Since the ring size for the second test is 25 instead of 100, the number of forwards of the second test will be multiplied by 4 to get meaningful results. This actually makes sense because a job that is forwarded once has encountered 4 busy CPUs. The baseline for the tests is the result of the algorithm using 1 CPU (found in section 2.2), where the results of the simulation using multiple CPUs is drawn relative to the baseline result.



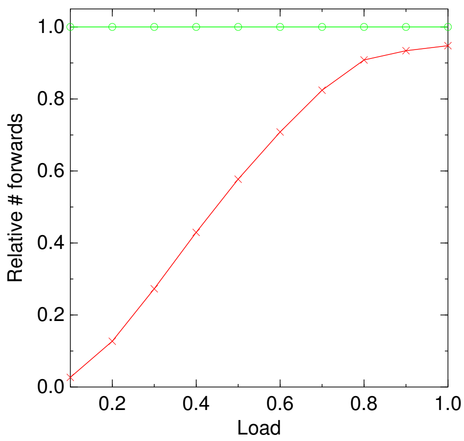
(a) Forward right



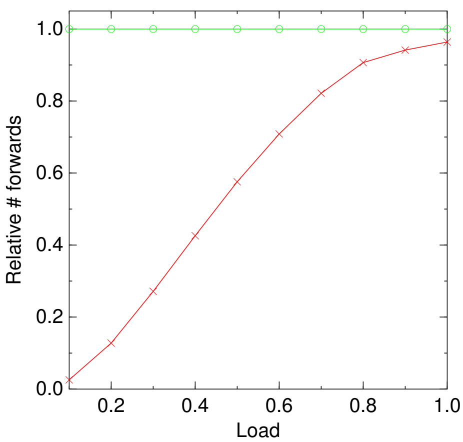
(b) Left/right forward



(c) Random left/right forward (0.5)



(d) Random unvisited



(e) Random coprime offset

Figure 10: 4 CPUs per server versus 1

# Forwards	Distribution	M	Result
0	0.5092	0	0
1	0.2118	0	0
2	0.1082	1	0.1082
3	0.0609	1	0.0609
4	0.0363	2	0.0726
5	0.0224	2	0.0449
6	0.0142	3	0.0426
7	0.0091	3	0.0272
8	0.0058	4	0.0233
9	0.0037	4	0.0147
Total	0.9816		0.3944
Weighted total	0.3944/0.9816 = 0.4018		
Simulation result	0.4171		

Table 3: Comparison when load=0.5

Figure 10 shows the performance of 4 CPUs versus 1 CPU for different algorithms. For each tested algorithm, we obtain the same result. This means the results of each algorithm are influenced in the same way when using multiple CPUs per server. We can use this knowledge to generalize the results of section 2.2.

Since we assume each algorithm is influenced the same way, we will investigate one of them in depth. We will try to transform the results of an algorithm with ring size N and 1 CPU per server into the results of the same algorithm with ring size N/c and c CPUs per server. Every other parameter should be unchanged.

Let d be the distribution of the number of forwards, stored as a row vector. We will build a transformation vector M in the form $[\lfloor 0/c \rfloor, \lfloor 1/c \rfloor, \ldots, \lfloor (N-1)/c \rfloor]'$. This vector represents the number of times a job would be forwarded if there would be c CPUs per server. $d * M$ equals the average number of forwards when using such a system. To ignore the dropped jobs, the results must be weighted for only the completed jobs. In our example, we will transform the results of the forward right algorithm using ring size $N = 10$ and 1 CPU into the results of a ring with size $N = 5$ and $c = 2$ CPUs per server, see table 3 for a worked out example and figure 11 for a comparison of the transformation and a simulation result.

While deriving the transformation, we did not take into account that for each choice the algorithm makes, c CPUs are bound to this choice. The real simulation chooses a node and must first test all those CPUs where the transformation only counts the $n \cdot c$ forwards using a better choice for each forward. This makes the transformation nothing more than an approximation and an upper bound of the real results.

3 Numerical Validation

To validate the results obtained in the previous section, we modeled the scheduling algorithms into Markov Chains. Using the steady state distribution

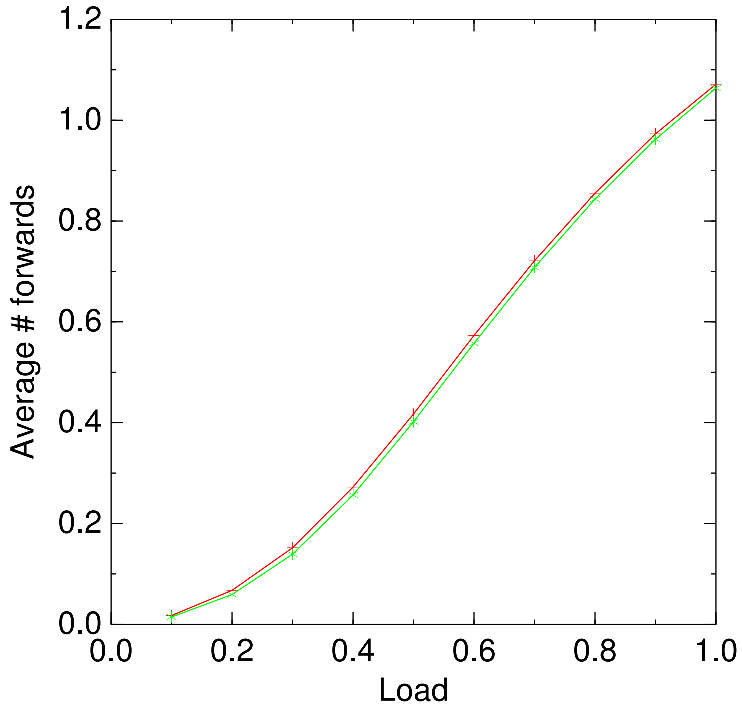


Figure 11: Multiple CPUs result derived of the result for 1 CPU (green) versus an actual simulation result using 4 CPUs (red)

of these chains, we can derive the average number of forwards and the average loss. For N nodes in a ring, the Markov chain consists of 2^N states, where the n-th bit represents whether the n-th server is busy (1) or idle (0). To optimize the computation time and memory requirements, we used sparse matrices for the validation. The validation code is written in MATLAB, it can be found in appendix B or on `http://code.google.com/p/powerofpaths/`.

The validation of the results happens in a different environment than the simulation. Because of the non-polynomial execution time of the algorithm, the size of the ring is reduced to 10. Therefore, the results of this validation are smaller but the relative results are still relevant.

Not every algorithm can be modeled efficiently into a Markov Chain. Computing the results of algorithms requiring state information in the nodes is infeasible for a decent ring size, therefore we did not model left/right forward and the coprime offset algorithms. We also ignored the position dependent forwarding algorithm since its results seemed logical and further investigation would not reveal useful information.

3.1 Comparison

Forward right

Modeling a technique into a Markov Chain is an easy operation for most algorithms. The example given below is for a ring of 3 nodes. For convenience, the states are represented in their binary form.

$$Q = \begin{array}{cc} & \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\ \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} & \begin{bmatrix} -3\lambda & \lambda & \lambda & 0 & \lambda & 0 & 0 & 0 \\ \mu & -3\lambda-\mu & 0 & \lambda & 0 & 2\lambda & 0 & 0 \\ \mu & 0 & -3\lambda-\mu & 2\lambda & 0 & 0 & \lambda & 0 \\ 0 & \mu & \mu & -3\lambda-2\mu & 0 & 0 & 0 & 3\lambda \\ \mu & 0 & 0 & 0 & -3\lambda-\mu & \lambda & 2\lambda & 0 \\ 0 & \mu & 0 & 0 & \mu & -3\lambda-2\mu & 0 & 3\lambda \\ 0 & 0 & \mu & 0 & \mu & 0 & -3\lambda-2\mu & 3\lambda \\ 0 & 0 & 0 & \mu & 0 & \mu & \mu & -3\mu \end{bmatrix} \end{array}$$

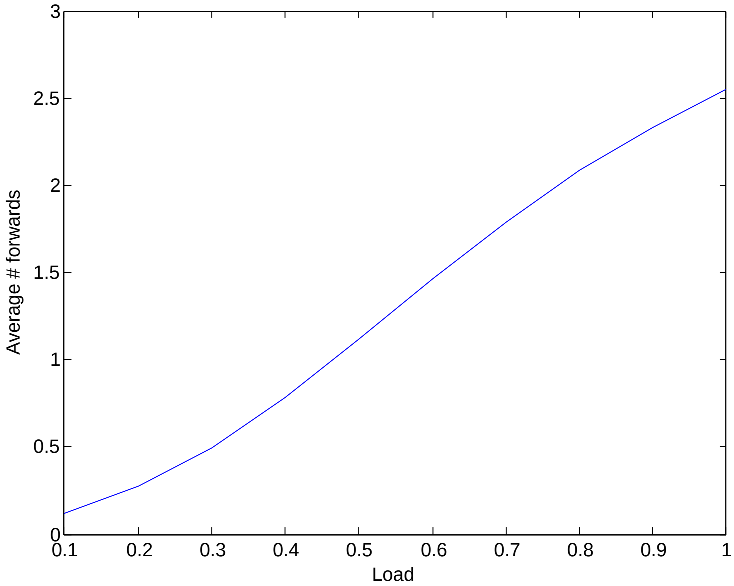


Figure 12: Validation of forward right

Like to the simulation section, this method will be the baseline result in our other results.

Random left/right forward with parameter p

This matrix is very similar to the one above. But we need to take into account the parameters p and $1 - p$ instead of 1 and 0.

$$
Q = \begin{array}{cc}
 & \begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \end{array} \\
\begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} &
\begin{bmatrix}
-3\lambda & \lambda & \lambda & 0 & \lambda & 0 & 0 & 0 \\
\mu & -3\lambda-\mu & 0 & (2-p)\lambda & 0 & (1+p)\lambda & 0 & 0 \\
\mu & 0 & -3\lambda-\mu & (1+p)\lambda & 0 & 0 & (2-p)\lambda & 0 \\
0 & \mu & \mu & -3\lambda-2\mu & 0 & 0 & 0 & 3\lambda \\
\mu & 0 & 0 & 0 & -3\lambda-\mu & (2-p)\lambda & (1+p)\lambda & 0 \\
0 & \mu & 0 & 0 & \mu & -3\lambda-2\mu & 0 & 3\lambda \\
0 & 0 & \mu & 0 & \mu & 0 & -3\lambda-2\mu & 3\lambda \\
0 & 0 & 0 & \mu & 0 & \mu & \mu & -3\mu
\end{bmatrix}
\end{array}
$$

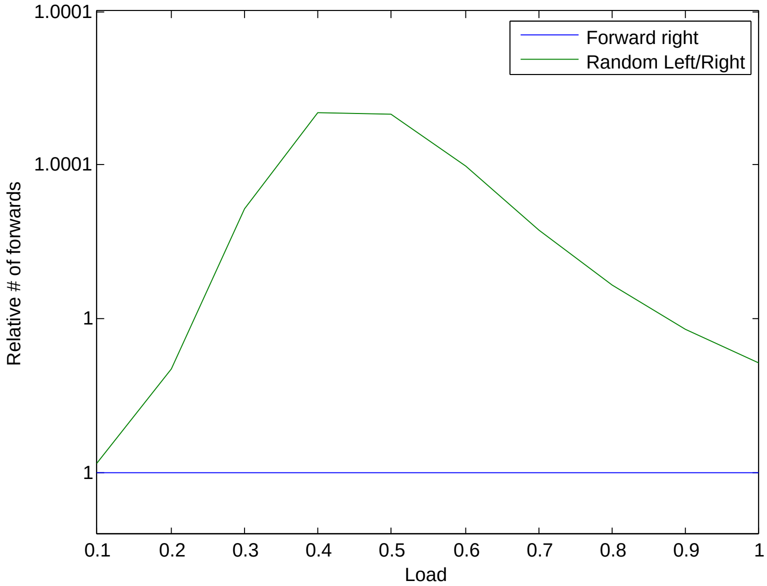


Figure 13: Validation of Random left/right with $p = 0.5$

It seems the results of the simulation fit. Although the results in 13 are much smoother than those in 4, we must take into account that these results are computed exact and using a smaller ring size.

As in the simulation section, we have validated the results for different values of p. These results are shown in figure 14.

The random left/right forward algorithm does not seem a viable choice for real implementations. It has a small performance drawback compared to forward right and has no other benefits.

Random unvisited

This problem can be modeled much more efficiently than the techniques. Since the next node is chosen at random, it is not necessary to store the position of the nodes. The information we need to save consists only of the number of servers that are currently busy. This problem is similar to modeling an Erlang-B loss

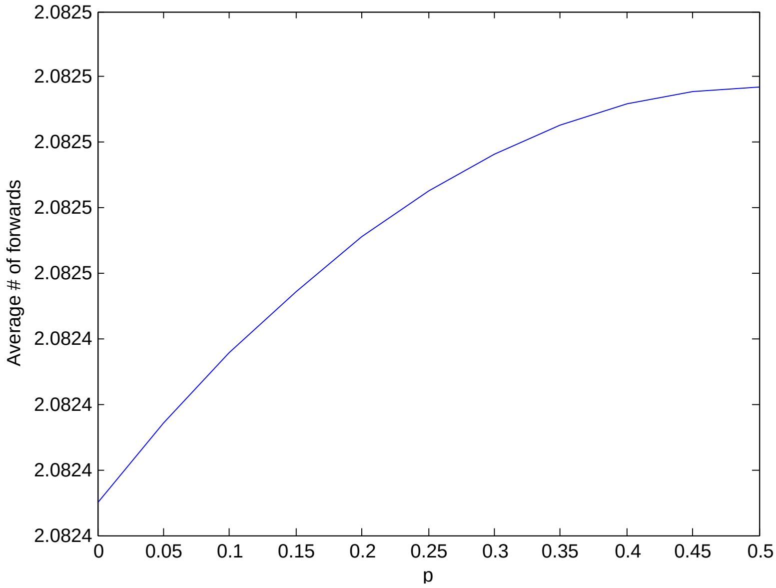


Figure 14: Performance of random left/right with load= 0.8

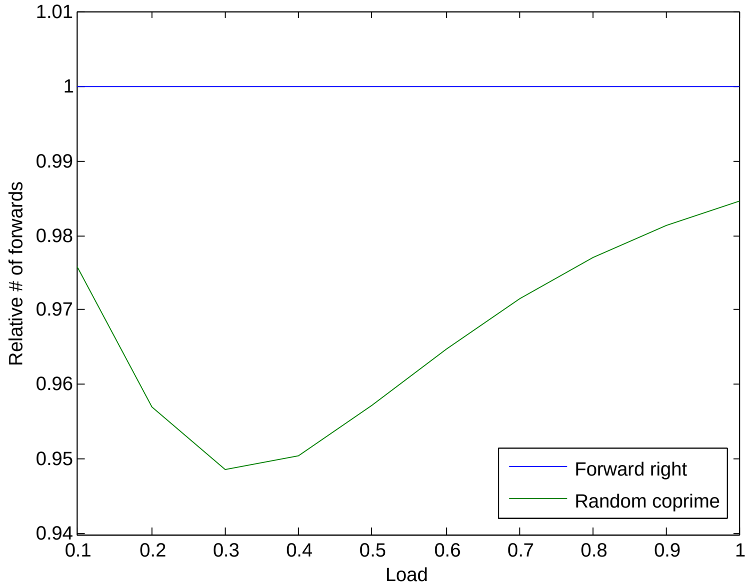
system. The number of states in this Markov Chain is linear to N, which is much more dense than the previous given models. For $N = 3$, the matrix is given below.

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -3\lambda & 3\lambda & 0 & 0 \\ \mu & -3\lambda-\mu & 3\lambda & 0 \\ 0 & \mu & -3\lambda-\mu & 3\lambda \\ 0 & 0 & \mu & -\mu \end{bmatrix} \end{array}$$

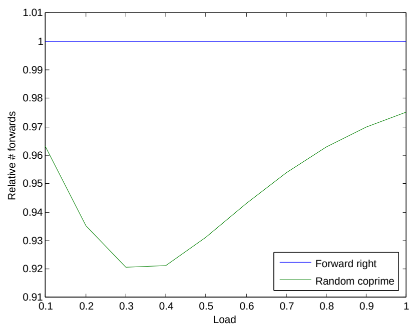
Random coprime offset

Modeling this technique yields different results for various ring sizes. The performance of this algorithm depends on $\varphi(N)$. The results in figure 15 are very similar to those found in the simulation (figure 8b).

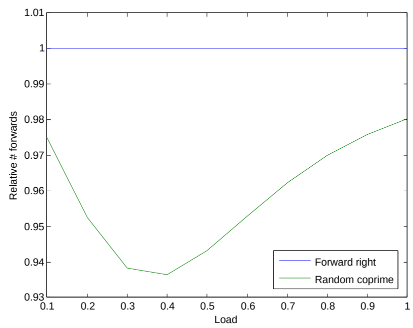
The results show a performance gain up to 5% for a ring size of $N = 10$. The list of coprimes in that scenario is $1, 3, 7, 9$, so 4 possible choices. When increasing the ring size to 11, a prime number, the list of coprimes expands to 1..10 (because 11 is prime), so 10 possible choices. This increases the relative performance gain up to 8% (figure 15b). To make clear the increased performance is not due to the increase of N, figure 15c shows the performance gain for $N = 12$. For $N = 10$ and $N = 12$, a job can follow 4 possible routes, for $N = 11$, 10 different routes can be chosen.



(a) $N = 10$



(b) $N = 11$



(c) $N = 12$

Figure 15: Validation of the random coprime offset algorithm

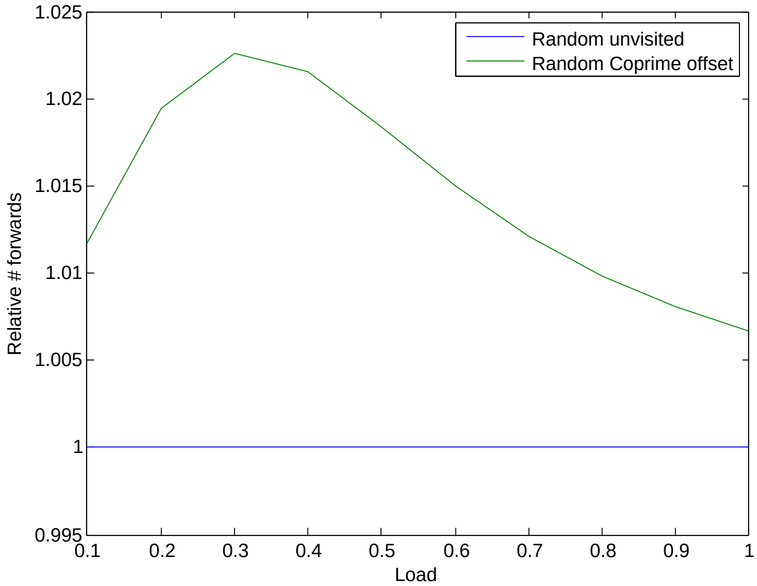


Figure 16: Performance of random coprime offset relative to random unvisited

3.2 Lumped states

Except for random unvisited, each described technique is modeled into a Markov Chain with N^2 states. However, many of these states are redundant: for example, for $N = 3$ the states 001, 010 and 100 all represent a state where one of the nodes is busy. For states representing multiple busy nodes, the space between these servers is critical information. Redundant states can be generated from one state: bitrotate the state by 1 until you get the original state. Each encountered state is redundant. Example: the states below are redundant and can therefore be lumped into one state:

```
001101 = 011010 = 110100 = 101001 = 010011 = 100110
```

The example model of the forward right algorithm can be lumped into the following Markov Chain:

$$Q = \begin{array}{c} \\ 000 \\ 001 \\ 011 \\ 111 \end{array} \begin{array}{c} \begin{array}{cccc} 000 & 001 & 011 & 111 \end{array} \\ \begin{bmatrix} -3\lambda & 3\lambda & 0 & 0 \\ \mu & 3\lambda - \mu & 3\lambda & 0 \\ 0 & 2\mu & 3\lambda - 2\mu & 3\lambda \\ 0 & 0 & 3\mu & -3\mu \end{bmatrix} \end{array}$$

Computing the steady state distribution of a Markov Chain is subject to both time and memory constraints. Using sparse matrices for our algorithms reduces the memory constraint so much it is no longer a bottleneck. Two factors are important when working with matrices: the number of elements and the number of nonzero elements. We will show that both factors are reduced significantly by lumping the matrices.

For unlumped Markov Chains modeling the Random forward algorithm, a matrix consists of 2^N states, an exponential growth. Lumping these matrices results in a number of states equal to $\frac{1}{N}\sum_{d|N}(2^{N/d} \cdot \varphi(d))$ [2]. Although this result greatly reduces the number of states, its complexity is still non-polynomial.

The number of nonzero elements for unlumped Markov Chains is $(N+1)2^N$. For lumped matrices, we were not able to derive an exact formula, however, figure 18 shows a clear reduction as well. Yet, this result does not seem polynomial either.

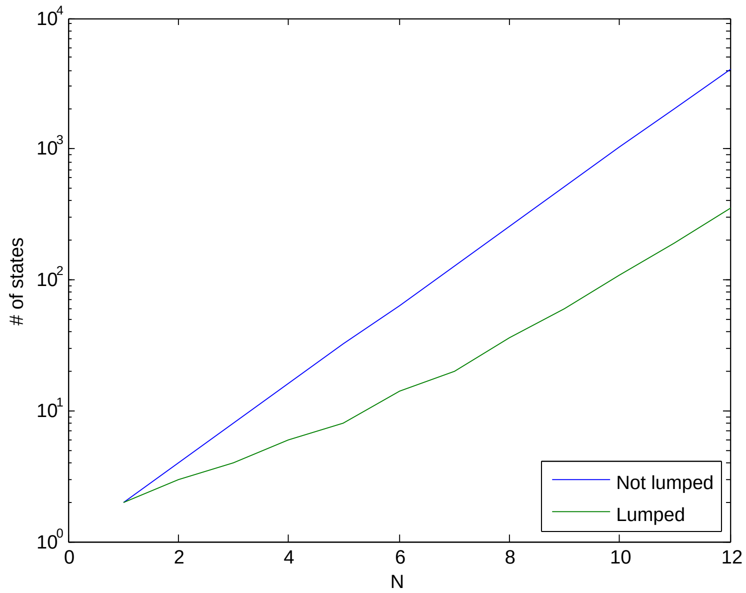


Figure 17: Number of states

It seems lumping is a good technique to push the boundaries of the validation by reducing two important factors of the computing time. However, it is no silver bullet: both the number of states and the number of nonzero elements are non-polynomial after lumping the matrices.

3.3 Equivalent algorithms

Various algorithms are equivalent or very similar under special conditions. We will discuss some of these equivalencies.

Low N

For $N \leq 3$ all algorithms are equivalent. This is obvious for $N = 2$. For $N = 3$, there are 4 possible states with the same behavior for each algorithm:

All nodes idle All 3 nodes are idle, an incoming job will be executed by first node.

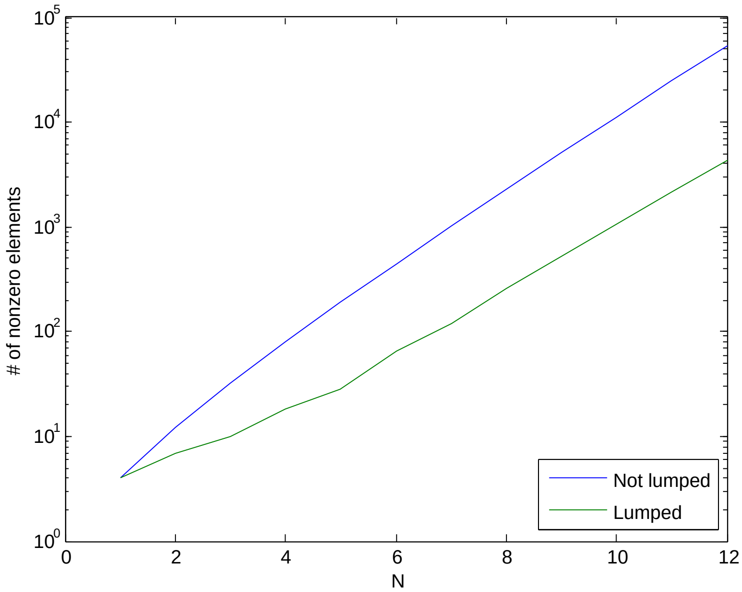


Figure 18: Number of nonzero elements

1 node busy Each node is as likely to be the next node receiving a job. For an idle node, the job is executed by that node. For the busy node, the job will need to make exactly 1 hop.

2 node busy Since each node is as likely to be the next node receiving a job, the job will be forwarded 0,1 or 2 times, all with the same probability.

Saturated The job will be dropped.

For $N > 3$, other equivalencies exist. These are described below.

High load

When the load becomes sufficiently large, all algorithms will yield the same results. In a ring under heavy load, each node will be busy nearly all the time. As long all nodes are busy, incoming jobs will be dropped. When a node becomes idle, the first event will be an incoming job. The first incoming job will claim this node. Each node has the same probability of receiving the next incoming job, so the expected value for the number of forwards is:

$$\begin{aligned} \frac{1}{N} \cdot \sum_{i=0}^{N-1} i &= \frac{1}{N} \cdot \frac{N(N-1)}{2} \\ &= \frac{N-1}{2} \end{aligned}$$

Random left/right with parameter p versus forward right

This algorithm is always equivalent to itself using parameter $1 - p$. This is intuitively clear because the probabilities of right and left are swapped, like looking to the ring in a mirror. It is clear that for $p = 0$ (and so for $p = 1$), this algorithm is equivalent to the forward right algorithm. Furthermore, for arbitrary values of p, this algorithm is still equivalent to forward right as long as the ring size $N \leq 5$. This can be proven by lumping the Markov Chains for both techniques.

For $N = 4$, define the lumped Markov Chain of the forward right algorithm.

$$Q_1 = \begin{array}{c} \\ 0000 \\ 0001 \\ 0011 \\ 0101 \\ 0111 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccc} 0000 & 0001 & 0011 & 0101 & 0111 & 1111 \end{array} \\ \begin{bmatrix} -4\lambda & 4\lambda & 0 & 0 & 0 & 0 \\ \mu & -4\lambda-\mu & 3\lambda & \lambda & 0 & 0 \\ 0 & 2\mu & -4\lambda-2\mu & 0 & 4\lambda & 0 \\ 0 & 2\mu & 0 & -4\lambda-2\mu & 4\lambda & 0 \\ 0 & 0 & 2\mu & \mu & -4\lambda-3\mu & 4\lambda \\ 0 & 0 & 0 & 0 & 4\mu & -4\mu \end{bmatrix} \end{array}$$

Now, define the lumped Markov Chain of the FR algorithm with parameter p and simplify.

$$Q_2 = \begin{array}{c} \\ 0000 \\ 0001 \\ 0011 \\ 0101 \\ 0111 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccc} 0000 & 0001 & 0011 & 0101 & 0111 & 1111 \end{array} \\ \begin{bmatrix} -4\lambda & 4\lambda & 0 & 0 & 0 & 0 \\ \mu & -\mu & \lambda+(1-p)\lambda+p\lambda+\lambda & \lambda & 0 & 0 \\ 0 & 2\mu & -2\mu & 0 & \lambda+2\lambda\cdot((1-p)+p)+\lambda & 0 \\ 0 & 2\mu & 0 & -2\mu & \lambda+p\lambda+(1-p)\lambda & 0 \\ 0 & 0 & 2\mu & \mu & -x & 4\lambda \\ 0 & 0 & 0 & 0 & 4\mu & -4\mu \end{bmatrix} \end{array}$$

$$Q_2 = \begin{array}{c} \\ 0000 \\ 0001 \\ 0011 \\ 0101 \\ 0111 \\ 1111 \end{array} \begin{array}{c} \begin{array}{cccccc} 0000 & 0001 & 0011 & 0101 & 0111 & 1111 \end{array} \\ \begin{bmatrix} -4\lambda & 4\lambda & 0 & 0 & 0 & 0 \\ \mu & -4\lambda-\mu & 3\lambda & \lambda & 0 & 0 \\ 0 & 2\mu & -4\lambda-2\mu & 0 & 4\lambda & 0 \\ 0 & 2\mu & 0 & -4\lambda-2\mu & 4\lambda & 0 \\ 0 & 0 & 2\mu & \mu & -4\lambda-3\mu & 4\lambda \\ 0 & 0 & 0 & 0 & 4\mu & -4\mu \end{bmatrix} \end{array}$$

We find $Q_1 = Q_2$ for $N = 4$. This also works for $N = 5$, but for $N \geq 6$, the matrices are different.

Random coprime offset versus forward right

For $N = 4$, this algorithm is equivalent to the forward right algorithm as well. The coprimes of 4 are 1 and 3. Meaning a jump of 1 (right neighbor) or a jump of 3 (left neighbor). It is clear this is the same as random left/right forward using parameter $p = 0.5$, which is equivalent to the forward right algorithm.

Random coprime offset versus random unvisited

The performance of the random coprime offset algorithm depends on $\varphi(N)$, which is $N - 1$ when N is prime. Figure 19 shows the average ring traversal of this algorithm and the random unvisited algorithm. It seems for prime values of N, the graphs touch. However, when looking at the actual data we see this is only true for $N = 5$.

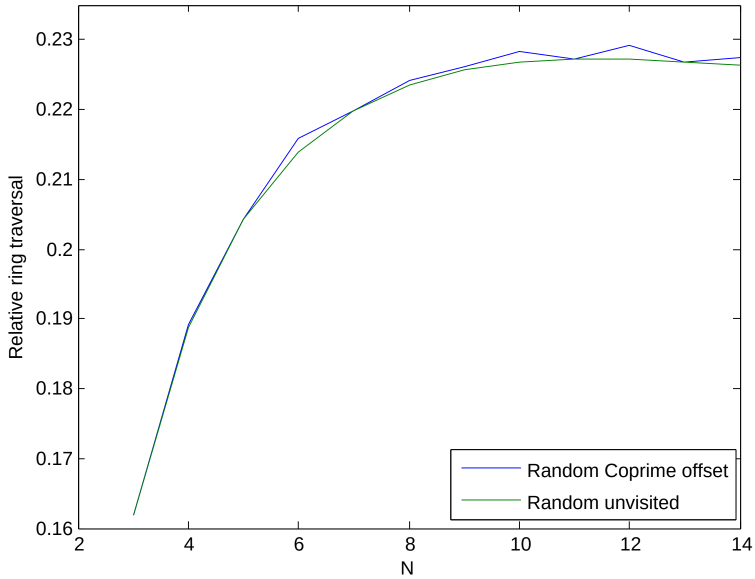


Figure 19: Ring traversal versus N for load=1.0

N	Random unvisited	Random coprime offset
3	0.161764705882353	0.161764705882353
5	0.204376460590610	0.204376460590610
7	0.219792926622134	0.219788935482327
11	0.227265522566042	0.227249844727623
13	0.226908433098761	0.226887197455899

Table 4: Relative ring traversal for load=1.0

4 Conclusion

We have simulated all our algorithms and validated the results of some of them. The ideal algorithm is depends on the specific requirements of the distributed system. However, it is safe to say some algorithms are not ideal for any situation. One should also note that in some configurations, different algorithms can yield the same results. In those cases, only memory requirements or the complexity to implement the algorithms should be decisive.

Forward to neighbors

When nodes are only directly connected to their neighbors, the choice depend on the memory requirements. When at least 1 bit can be saved in both the job's metadata as the node's state, left/right forward is an ideal choice. This algorithm provides up to 4% better result than forward right under medium load ($N = 100$).

When no memory is available, the forward right algorithm is the only possible choice while maintaining sufficient results. Additionally, it is the easiest algorithm to implement.

Both these algorithms allow nodes to come and go. The random left/right and position dependent forwarding techniques perform worse than forward right while having more requirements, therefore they should not be chosen.

Forward anywhere

If each node can reach each other node directly, the forward anywhere algorithms improve the results significantly. The best results can be achieved using the random unvisited algorithm. This algorithm performs slightly better than coprime offset and random coprime offset, but at a higher cost: the random unvisited algorithm requires $N - 1$ bits in the job's metadata.

The coprime offset and random coprime offset algorithms require less memory but with worse performance. However, when the ring size is prime, its results are only slightly less than those of random unvisited. The memory required in the job's metadata is lowered to $\lceil log_2 \varphi(N) \rceil$ bits ($\lceil log_2(N-1) \rceil$ bits when N is prime). The coprime offset algorithm yields about the same results as the random variant. In addition, it needs to save $\lceil log_2 \varphi(N) \rceil$ bits in the state of each node. Random coprime offset and random unvisited seem two very good candidates, it depends whether the small performance benefit is worth the memory and implementation cost.

Other findings

The results of using multiple CPUs per server can be approximated using the the results of 1 CPU per server and a ring size $N_1 = N_c \cdot c$. This approximation is an upper bound to the real outcome since the derivation ignores the fact that the choice for a node is no longer a choice for a CPU, it is a choice for a group of c CPUs.

For the random left/right forward, we have found out its results are worst for $p = 0.5$. We did not come up with an explanation why this algorithm is worse than forward right. This might be a start for further research.

We have also found a way to model some of these algorithms into Markov Chains. This allowed us to compute the actual result of a simulation. The drawback of our model is the limited ring size. We optimized it by lumping states and using sparse matrices, but because of the non-polynomial nature of the matrix size, these optimizations allow us to increase N only by 2 or 3.

Another finding is the equivalency between multiple algorithms, under certain conditions some algorithms model the exact same problem.

References

[1] *Euler's totient function*. URL: `http://en.wikipedia.org/wiki/Euler%27s_totient_function`.

[2] The Online Encyclopedia of Integer Sequences. *A000031*. June 2009. URL: `https://oeis.org/A000031`.

[3] *The OpenMP® API specification for parallel programming.* URL: `http://openmp.org/wp/`.

A Simulator source code

```
#include <iostream>
#include <math.h>
#include <time.h>
#include <stdlib.h>

#include "ring/ring.h"
#include "ring/job.h"
#include "ring/arriveevent.h"
#include "nodes.h"
#include "configuration.h"
using namespace pop;
using namespace std;

double exp_distr(double lambda){
    double r = (double)rand() / (double)RAND_MAX;
    return -lambda * log(r);
}

void preload(Configuration c, Ring* r){
    double rnd;
    double l;
    double load = c.length/c.arrival;
    for (unsigned int i = 0; i < r->getSize(); ++i){
        rnd = (double)rand() / (double)RAND_MAX;
        if (rand() < load){
            l=exp_distr(c.joblength);
            r->getSimulator()->addEvent(new ArriveEvent(0.0, new Job(c.makeInfoFunction(l)), r->getNode(i)));
        }
    }
}

void fillEvents(Configuration c, Ring* r){
    double t;
    double l;
    Node* n;
    for (unsigned int i = 0; i < r->getSize(); ++i){
        n=r->getNode(i);
        t = 0.0;
        while (t < c.length){
            t+=exp_distr(c.arrival);
            l=exp_distr(c.joblength);
            r->getSimulator()->addEvent(new ArriveEvent(t, new Job(c.makeInfoFunction(l)), n));
        }
    }
}


int main(int argc, char** argv) {
    Configuration c(argc, argv);

    double success = 0.0;
    double avghops = 0.0;

    cout.setf(ios::fixed,ios::floatfield);
    cout.precision(12);

#pragma omp parallel for
    for (unsigned int i = 0; i < c.repeat; ++i){
        Ring r(c.nodes, c.nodeSize, c.makeNodeFunction);

        //preload(c, &r); //disabled to compare output to more early results

        fillEvents(c, &r);

        if (c.progressinterval <= 0){
            r.getSimulator()->run();
        }else{
            r.getSimulator()->run(c.progressinterval);
        }

```

```
#pragma omp critical
        {
            cout << "--------------------------------" << endl;
            cout << "Run: " << i << endl;
            cout << "Total jobs:\t\t" << r.getTotalJobs() << endl;
            cout << "Finished jobs:\t\t" << r.getFinishedJobs() << endl;
            cout << "Discarded jobs:\t\t" << r.getDiscardedJobs() << endl;
            cout << "Total hops (finished):\t" << r.getFinishedJobTotalHops() << endl;
            long totalhops = r.getFinishedJobTotalHops() + (c.nodes-1) * r.getDiscardedJobs();
            cout << "Total hops (all):\t" << totalhops << endl;
            cout << "Hops/job (finished):\t" << (double)r.getFinishedJobTotalHops()/r.getFinishedJobs() << endl;
            cout << "Hops/job (total):\t" << (double)totalhops/r.getTotalJobs() << endl;
            cout << "Success ratio:\t\t" << (100.0 * r.getFinishedJobs()) / r.getTotalJobs() << "%" << endl;
            success+=(double)(r.getFinishedJobs()) / r.getTotalJobs();
            avghops+=(double)r.getFinishedJobTotalHops()/r.getFinishedJobs();
        }
    }

    if (c.repeat > 1){
        cout << "--------------------------------" << endl;
        cout << "Avg. hops/job (finished):\t" << avghops / c.repeat << endl;
        cout << "Avg. success ratio:\t\t" << 100.0 * success / c.repeat << "%" << endl;
    }

    return 0;
}
```

Listing 2: Main.cpp

```
/*
 * configuration.h
 *
 *  Created on: Oct 6, 2011
 *      Author: ibensw
 */

#ifndef CONFIGURATION_H_
#define CONFIGURATION_H_

#include "ring/node.h"
#include "ring/job.h"

void help();

typedef pop::Node* (*makeNodeType)(unsigned int i, pop::Ring* r, unsigned int size);
typedef pop::JobInfo* (*makeInfoType)(double length);
struct Configuration{
    unsigned int seed;
    double joblength;
    double arrival;
    long nodes;
    unsigned int nodeSize;
    long progressinterval;
    long length;
    long repeat;
    makeNodeType makeNodeFunction;
    makeInfoType makeInfoFunction;

    Configuration(int argc, char** argv);
};

#endif /* CONFIGURATION_H_ */
```

Listing 3: configuration.h

```
#include <iostream>
#include <stdlib.h>
#include "configuration.h"
#include "nodes.h"

using namespace std;
```

```
void help(){
    cout << "Usage: -r -s long -j double -a double -n long -p long -l long -t long -h type" << endl;
    cout << "\t-r\tRandom seed" << endl;
    cout << "\t-s\tSet seed\t\t\t(default: 0)" << endl;
    cout << "\t-j\tJob length\t\t\t(default: 1.0)" << endl;
    cout << "\t-a\tLoad\t\t\t\t(default: 1.0)" << endl;
    cout << "\t-n\tRing size\t\t\t(default: 100)" << endl;
    cout << "\t-c\tProcessing units per node\t(default: 1)" << endl;
    cout << "\t-p\tPrint progress interval\t\t(default: -1 - disabled)" << endl;
    cout << "\t-l\tSimulation length\t\t(default: 3600)" << endl;
    cout << "\t-t\tRepetition\t\t\t(default: 1)" << endl;
    cout << "\t-h\tPrint this help" << endl;
    cout << "\t type\tright | switch | randswitch | evenswitch | prime | randprime | randunvisited | totop" << endl;
}

template <typename T>
pop::JobInfo* createJI(double len){
    return new T(len);
}

template <typename T>
pop::Node* createN(unsigned int id, pop::Ring* ring, unsigned int size){
    return new T(id, ring, size);
}

Configuration::Configuration(int argc, char** argv):
    seed(0), joblength(1.0),
    arrival(1.0), nodes(100), nodeSize(1), progressinterval(-1), length(3600),
    repeat(1), makeNodeFunction(0), makeInfoFunction(0)
{
    int c;
    int index;
    double load = 1.0;
    while ((c = getopt (argc, argv, "rs:j:a:n:c:p:l:hv:t:")) != -1){
        switch (c){
        case 'r':
            seed = time(0);
            break;
        case 's':
            seed = atol(optarg);
            break;
        case 'j':
            joblength = atof(optarg);
            break;
        case 'a':
            load = atof(optarg);
            break;
        case 'n':
            nodes = atol(optarg);
            break;
        case 'c':
            nodeSize = atol(optarg);
            break;
        case 'p':
            progressinterval = atoi(optarg);
            break;
        case 'l':
            length = atol(optarg);
            break;
        case 't':
            repeat = atol(optarg);
            break;
        case 'h':
            help();
            exit(0);
            break;
        case 'v':
            RandSwitchNode::setValue(atof(optarg));
            break;
        default:
            cout << "Unknown option: " << optopt << endl;
```

```
                break;
            }
        }

        arrival = joblength/load/nodeSize;

        srand(seed);
        cout << "Seed: " << seed << endl
                 << "Interarrival time: " << arrival << endl;

        for (index = optind; index < argc; index++){
            string arg = argv[index];
            if (arg == "right"){
                makeNodeFunction = createN<RightNode>;
                makeInfoFunction = createJI<RightNode::info_type>;
            }
            if (arg == "switch"){
                makeNodeFunction = createN<SwitchNode>;
                makeInfoFunction = createJI<SwitchNode::info_type>;
            }
            if (arg == "randswitch"){
                makeNodeFunction = createN<RandSwitchNode>;
                makeInfoFunction = createJI<RandSwitchNode::info_type>;
            }
            if (arg == "evenswitch"){
                makeNodeFunction = createN<EvenSwitchNode>;
                makeInfoFunction = createJI<EvenSwitchNode::info_type>;
            }
            if (arg == "prime"){
                makeNodeFunction = createN<PrimeNode>;
                makeInfoFunction = createJI<PrimeNode::info_type>;
            }
            if (arg == "randprime"){
                makeNodeFunction = createN<RandPrimeNode>;
                makeInfoFunction = createJI<RandPrimeNode::info_type>;
            }
            if (arg == "randunvisited"){
                makeNodeFunction = createN<RandUnvisited>;
                makeInfoFunction = createJI<RandUnvisited::info_type>;
            }
            if (arg == "totop"){
                makeNodeFunction = createN<ToTopNode>;
                makeInfoFunction = createJI<ToTopNode::info_type>;
            }
            if (arg == "rrunvisited"){
                makeNodeFunction = createN<RRUnvisited>;
                makeInfoFunction = createJI<RRUnvisited::info_type>;
            }
        }

        if (!makeNodeFunction){
            cerr << "No type given" << endl;
            exit(1);
        }
}
```

Listing 4: configuration.cpp

```
/*
 * nodes.h
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#ifndef NODES_H_
#define NODES_H_

#include "ring/node.h"
#include "ring/ring.h"
#include "ring/job.h"
#include <set>

struct DirectionInfo: public pop::JobInfo{
```

```
    inline DirectionInfo(double length):
        fLength(length), fDirection(0), fFirst(0)
    {}

    double fLength;
    short fDirection;
    pop::Node* fFirst;
};

struct VisitedInfo: public DirectionInfo{
    inline VisitedInfo(double length):
        DirectionInfo(length)
    {}

    std::set<unsigned int> visited;
};

class RightNode: public pop::Node {
public:
    typedef DirectionInfo info_type;

    RightNode(unsigned int id, pop::Ring* ring, unsigned int size);

    virtual ~RightNode(){}

    bool pushJob(pop::Job* j);
    void clearJob(pop::Job* j);

    bool wasHereFirst(pop::Job* j);
    bool accept(pop::Job* j);
};

class SwitchNode: public RightNode {
public:
    typedef DirectionInfo info_type;

    inline SwitchNode(unsigned int id, pop::Ring* ring, unsigned int size):
            RightNode(id, ring, size), last(1)
    {}

    bool pushJob(pop::Job* j);

protected:
    int last;
};

class RandSwitchNode: public RightNode {
public:
    typedef DirectionInfo info_type;

    inline static void setValue(double nv){
        v = nv;
    }

    inline RandSwitchNode(unsigned int id, pop::Ring* ring, unsigned int size):
            RightNode(id, ring, size)
    {}

    bool pushJob(pop::Job* j);

private:
    static double v;
};

class EvenSwitchNode: public RightNode{
public:
    typedef DirectionInfo info_type;

    inline EvenSwitchNode(unsigned int id, pop::Ring* ring, unsigned int size):
            RightNode(id, ring, size)
    {}

    bool pushJob(pop::Job* j);
};
```

```cpp

class PrimeNode: public SwitchNode {
public:
    typedef DirectionInfo info_type;

    static void makePrimes(unsigned int size);

    inline PrimeNode(unsigned int id, pop::Ring* ring, unsigned int size):
        SwitchNode(id, ring, size)
    {
        if (fPrimes == 0){
            makePrimes(ring->getSize());
        }
    }

    bool pushJob(pop::Job* j);

protected:
    static int* fPrimes;
    static int fPrimesLen;
};

class RandPrimeNode: public PrimeNode{
public:
    typedef DirectionInfo info_type;

    RandPrimeNode(unsigned int id, pop::Ring* ring, unsigned int size):
        PrimeNode(id, ring, size)
    {
        if (fPrimes == 0){
            makePrimes(ring->getSize());
        }
    }

    bool pushJob(pop::Job* j);
};

class RandUnvisited: public RightNode{
public:
    typedef VisitedInfo info_type;

    RandUnvisited(unsigned int id, pop::Ring* ring, unsigned int size):
        RightNode(id, ring, size)
    {}

    bool pushJob(pop::Job* j);
};

class ToTopNode: public RightNode{
public:
    typedef DirectionInfo info_type;

    ToTopNode(unsigned int id, pop::Ring* ring, unsigned int size):
        RightNode(id, ring, size)
    {}

    bool pushJob(pop::Job* j);
};

class RRUnvisited: public RightNode{
public:
    typedef VisitedInfo info_type;

    RRUnvisited(unsigned int id, pop::Ring* ring, unsigned int size):
        RightNode(id, ring, size), offset(0)
    {}

    bool pushJob(pop::Job* j);

private:
    unsigned int offset;
};

#endif /* NODES_H_ */
```

Listing 5: nodes.h

```
/*
 * nodes.cpp
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#include "nodes.h"
#include "ring/job.h"
#include "ring/ring.h"
#include "ring/finishevent.h"
#include <stdlib.h>
#include <iostream>
#include <vector>
#include <string.h>

using namespace pop;
using namespace std;

RightNode::RightNode(unsigned int id, Ring* ring, unsigned int size):
    Node(id, ring, size)
{}

bool RightNode::wasHereFirst(Job* j){
    info_type* ji = dynamic_cast<info_type*>(j->getInfo());

    if (ji->fFirst == 0){
        ji->fFirst = this;
        return false;
    }

    return (ji->fFirst == this);
}

bool RightNode::accept(Job* j){
    if (!isBusy()){
        info_type* ji = dynamic_cast<info_type*>(j->getInfo());
        fCurrents.insert(j);
        double len = ji->fLength;
        fRing->getSimulator()->addEvent(new FinishEvent(fRing->getSimulator()->getTime()+
            len, j));
        return true;
    }
    return false;
}

bool RightNode::pushJob(Job* j){
    if (wasHereFirst(j)){
        return false;
    }

    if (!accept(j)){
        j->forward(fRing->getNode(this->fId+1));
    }

    return true;
}

void RightNode::clearJob(Job* j){
    fCurrents.erase(j);
}

bool SwitchNode::pushJob(Job* j){
    if (wasHereFirst(j)){
        return false;
    }

    if (!accept(j)){
        info_type* ji = dynamic_cast<info_type*>(j->getInfo());
        if (ji->fDirection == 0){
            ji->fDirection = last;
```

```
            last*=-1;
        }

        j->forward(fRing->getNode(this->fId + ji->fDirection));
    }
    return true;
}

double RandSwitchNode::v = 0.5;

bool RandSwitchNode::pushJob(Job* j){
    if (wasHereFirst(j)){
        return false;
    }

    if (!accept(j)){
        info_type* ji = dynamic_cast<info_type*>(j->getInfo());
        if (ji->fDirection == 0){
            double rnd = (double)rand() / (double)RAND_MAX;
            ji->fDirection = (rnd < v ? 1 : -1);
        }

        j->forward(fRing->getNode(this->fId + ji->fDirection));
    }
    return true;
}

bool EvenSwitchNode::pushJob(Job* j){
    if (wasHereFirst(j)){
        return false;
    }

    if (!accept(j)){
        info_type* ji = dynamic_cast<info_type*>(j->getInfo());
        if (ji->fDirection == 0){
            ji->fDirection = ((this->getId() % 2 == 1) ? 1 : -1);
        }

        j->forward(fRing->getNode(this->fId + ji->fDirection));
    }
    return true;
}

int* PrimeNode::fPrimes = 0;
int PrimeNode::fPrimesLen = 0;

unsigned int gcd(unsigned int a, unsigned b){
    unsigned int t;
    while(b){
        t=b;
        b=a%b;
        a=t;
    }
    return a;
}

void PrimeNode::makePrimes(unsigned int size){
    vector<unsigned int> primes;
    for (unsigned int i = 1; i < size; ++i){
        if (gcd(size, i) == 1){
            cout << "RelPrime: " << i << endl;
            primes.push_back(i);
        }
    }
    fPrimesLen = primes.size();
    fPrimes = new int[fPrimesLen];
    memcpy(fPrimes, primes.data(), fPrimesLen * sizeof(unsigned int));
}

bool PrimeNode::pushJob(Job* j){
    if (wasHereFirst(j)){
        return false;
    }

```

```
    if (!accept(j)){
        info_type* ji = dynamic_cast<info_type*>(j->getInfo());
        if (ji->fDirection == 0){
            ji->fDirection = fPrimes[last];
            ++last;
            last%=fPrimesLen;
        }

        j->forward(fRing->getNode(this->fId + ji->fDirection));
    }
    return true;
}

bool RandPrimeNode::pushJob(Job* j){
    if (wasHereFirst(j)){
        return false;
    }

    if (!accept(j)){
        info_type* ji = dynamic_cast<info_type*>(j->getInfo());
        if (ji->fDirection == 0){
            ji->fDirection = fPrimes[rand() % fPrimesLen];
        }

        j->forward(fRing->getNode(this->fId + ji->fDirection));
    }
    return true;
}

bool RandUnvisited::pushJob(Job* j){
    info_type* ji = dynamic_cast<info_type*>(j->getInfo());
    if (ji->visited.count(this->getId())){
        return false;
    }

    if (!accept(j)){
        ji->visited.insert(this->getId());

        if (fRing->getSize() == ji->visited.size()){
            return false;
        }

        unsigned int next;
        if (5*ji->visited.size() > 4*fRing->getSize()){
            unsigned int x = rand() % (fRing->getSize() - ji->visited.size());
            next = x;

            for (set<unsigned int>::iterator it = ji->visited.begin(); it != ji->visited.
                end(); it++){
                if (*it <= next){
                    ++next;
                }
            }
        }else{
            do{
                next = rand() % fRing->getSize();
            }while(ji->visited.count(next));
        }

        j->forward(fRing->getNode(next));
    }
    return true;
}

bool ToTopNode::pushJob(Job* j){
    if (wasHereFirst(j)){
        return false;
    }

    if (!accept(j)){
        info_type* ji = dynamic_cast<info_type*>(j->getInfo());
        if (ji->fDirection == 0){
            if (this->getId() > fRing->getSize()/2){
                ji->fDirection = 1;
```

```
            }else{
                ji->fDirection = -1;
            }
        }

        j->forward(fRing->getNode(this->fId + ji->fDirection));
    }
    return true;
}

bool RRUnvisited::pushJob(Job* j){
    info_type* ji = dynamic_cast<info_type*>(j->getInfo());
    if (ji->visited.count(this->getId())){
        return false;
    }

    if (!accept(j)){
        ji->visited.insert(this->getId());

        if (fRing->getSize() == ji->visited.size()){
            return false;
        }

        unsigned int next = this->getId() + offset + 1;
        ++offset;
        offset%=(fRing->getSize()-1);

        next%=fRing->getSize();
        while (ji->visited.count(next)){
            next++;
            next%=fRing->getSize();
        }

        j->forward(fRing->getNode(next));
    }
    return true;
}
```

Listing 6: nodes.cpp

```
/*
 * servernode.h
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#ifndef SERVERNODE_H_
#define SERVERNODE_H_

#include "ring/node.h"
#include "ring/ring.h"

namespace pop {

class ServerNode: public Node {
public:
    ServerNode(unsigned int id, Ring* ring);
    virtual ~ServerNode();
};

} /* namespace pop */
#endif /* SERVERNODE_H_ */
```

Listing 7: servernode.h

```
/*
 * servernode.cpp
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#include "servernode.h"
```

```cpp

namespace pop {

ServerNode::ServerNode(unsigned int id, Ring* ring):
    Node(id, ring)
    {}

ServerNode::~ServerNode() {
    // TODO Auto-generated destructor stub
}

} /* namespace pop */
```

Listing 8: servernode.cpp

```cpp
/*
 * events.h
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#ifndef EVENTS_H_
#define EVENTS_H_

#include "../simulator/event.h"
#include "job.h"

namespace pop {

class ArriveEvent : public Event {
public:
    inline ArriveEvent(double scheduled, Job* job, Node* n):
        Event(scheduled), j(job), first(n)
    {}

    inline void run(Simulator* simulator){
        j->forward(first);
        delete this;
    }

private:
    Job* j;
    Node* first;
};

} /* namespace pop */
#endif /* EVENTS_H_ */
```

Listing 9: ring/arriveevent.h

```cpp
/*
 * finishevent.h
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#ifndef FINISHEVENT_H_
#define FINISHEVENT_H_

#include "../simulator/simulator.h"
#include "../simulator/event.h"
#include "job.h"

namespace pop {

class FinishEvent: public Event {
public:
    inline FinishEvent(double scheduled, Job* job):
        Event(scheduled), j(job)
    {}

    inline void run(Simulator* simulator){
```

```
        j->finish(simulator->getTime());
        delete this;
    }

private:
    Job* j;
};

} /* namespace pop */
#endif /* FINISHEVENT_H_ */
```

Listing 10: ring/finishevent.h

```
/*
 * job.h
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#ifndef JOB_H_
#define JOB_H_

#include "node.h"

namespace pop {

class JobInfo {
public:
    inline JobInfo(){}

    virtual ~JobInfo(){}
};

class Job {
public:
    Job(JobInfo* ji);
    virtual ~Job();

    inline Node* getCurrentNode(){
        return fCurrent;
    }

    inline JobInfo* getInfo(){
        return fJobInfo;
    }

    void forward(Node* n);
    void finish(double time);
    void discard();

private:
    double fStart;
    double fFinish;
    Node* fCurrent;
    unsigned int fHops;
    JobInfo* fJobInfo;
};

} /* namespace pop */
#endif /* JOB_H_ */
```

Listing 11: ring/job.h

```
/*
 * job.cpp
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#include "job.h"
#include "ring.h"
#include <iostream>
```

```
using namespace std;

namespace pop {

Job::Job(JobInfo* ji):
	fStart(-1.0), fFinish(-1.0), fCurrent(0), fHops(-1), fJobInfo(ji)
	{}

Job::~Job() {
	if (fJobInfo){
		delete fJobInfo;
	}
}

void Job::discard(){
	fCurrent->getRing()->jobDiscarded();
	//cout << fCurrent->getId() << "\tJob discarded\t(arrival time: " << fStart << " / #hops: " << fHops << ")" << endl;
	delete this;
}

void Job::finish(double time){
	fCurrent->getRing()->jobFinished(fHops);
	fFinish = time;
	//cout << fCurrent->getId() << "\tJob finished\t(arrival time: " << fStart << " / finish time: " << fFinish << " / #hops: " << fHops << ")" << endl;
	fCurrent->clearJob(this);
	delete this;
}

void Job::forward(Node* n){
	if (!fCurrent){
		n->getRing()->jobCreated();
		fStart = n->getRing()->getSimulator()->getTime();
	}
	++fHops;
	fCurrent = n;
	if (!n->pushJob(this)){
		discard();
	}
}

} /* namespace pop */
```

Listing 12: ring/job.cpp

```
/*
 * node.h
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#ifndef NODE_H_
#define NODE_H_

#include <set>

namespace pop {
class Ring;
class Job;

class Node {
public:
	Node(unsigned int id, Ring* ring, unsigned int size = 1);
	virtual ~Node();

	inline unsigned int getId() const{
		return fId;
	}

	inline Ring* getRing() const {
		return fRing;
	}
```

```

    inline unsigned int getTotalSize() const{
        return fSize;
    }

    inline unsigned int getSize() const{
        return fCurrents.size();
    }

    inline bool isBusy() const{
        return fCurrents.size() == fSize;
    }

    virtual bool pushJob(Job* j) = 0;
    virtual void clearJob(Job* j) = 0;

protected:
    unsigned int fId;
    Ring* fRing;
    std::set<Job*> fCurrents;
    unsigned int fSize;
};

} /* namespace pop */
#endif /* NODE_H_ */
```

Listing 13: ring/node.h

```
/*
 * node.cpp
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#include "node.h"

namespace pop {

Node::Node(unsigned int id, Ring* ring, unsigned int size):
    fId(id), fRing(ring), fSize(size)
    {}

Node::~Node() {
    // TODO Auto-generated destructor stub
}

} /* namespace pop */
```

Listing 14: ring/node.cpp

```
/*
 * ring.h
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#ifndef RING_H_
#define RING_H_

#include "node.h"
#include "../simulator/simulator.h"

//#include <iostream>

namespace pop {

class Ring {
public:
    Ring(unsigned int size, unsigned int nodesize, Node* (*mkNode)(unsigned int i, Ring*
        r, unsigned int ns));
    virtual ~Ring();

```

```cpp
    inline unsigned int getSize(){
        return fSize;
    }

    inline Node* getNode(int id){
        //std::cout << "id: " << id << " = " << (id + fSize) % fSize << std::endl;
        return fRing[(id + fSize) % fSize];
    }

    inline Simulator* getSimulator(){
        return &fSimulator;
    }

    inline unsigned int getTotalJobs(){
        return jobsTotal;
    }
    inline unsigned int getDiscardedJobs(){
        return jobsDiscarded;
    }
    inline unsigned int getFinishedJobs(){
        return jobsFinished;
    }
    inline unsigned int getFinishedJobTotalHops(){
        return jobsFinishedTotalHops;
    }

    inline void jobCreated(){
        ++jobsTotal;
    }

    inline void jobFinished(unsigned int hops){
        ++jobsFinished;
        jobsFinishedTotalHops+=hops;
    }

    inline void jobDiscarded(){
        ++jobsDiscarded;
    }

private:
    unsigned int fSize;
    Node** fRing;
    Simulator fSimulator;

    unsigned int jobsTotal;
    unsigned int jobsFinished;
    unsigned int jobsDiscarded;
    unsigned int jobsFinishedTotalHops;
};

} /* namespace pop */
#endif /* RING_H_ */
```

Listing 15: ring/ring.h

```cpp
/*
 * ring.cpp
 *
 *  Created on: Sep 27, 2011
 *      Author: ibensw
 */

#include "ring.h"

namespace pop {

Ring::Ring(unsigned int size, unsigned int nodesize, Node* (*mkNode)(unsigned int i, Ring
    * r, unsigned int ns)):
    fSize(size),
    fRing(new Node*[size]),
    jobsTotal(0), jobsFinished(0), jobsDiscarded(0), jobsFinishedTotalHops(0)
{
    for (unsigned int i = 0; i < size; ++i){
        fRing[i] = mkNode(i, this, nodesize);
```

```
    }
}

Ring::~Ring() {
    for (unsigned int i = 0; i < fSize; ++i){
        delete fRing[i];
    }
    delete[] fRing;
}

} /* namespace pop */
```

Listing 16: ring/ring.cpp

```
/*
 * event.h
 *
 *  Created on: Sep 26, 2011
 *      Author: ibensw
 */

#ifndef EVENT_H_
#define EVENT_H_

//#include "simulator.h"

namespace pop {
class Simulator;

class Event {
public:
    inline Event(double scheduled):
        fScheduled(scheduled)
        {}

    inline virtual ~Event(){

    }

    inline double getScheduleTime(){
        return fScheduled;
    }

    virtual void run(Simulator* simulator) = 0;

protected:
    double fScheduled;
};

} /* namespace pop */
#endif /* EVENT_H_ */
```

Listing 17: simulator/event.h

```
/*
 * schedule.h
 *
 *  Created on: Sep 26, 2011
 *      Author: ibensw
 */

#ifndef SCHEDULE_H_
#define SCHEDULE_H_

#include "event.h"

namespace pop {

class Schedule {
public:
    inline Schedule(Event* e):
        fE(e)
    {}

```

```
    inline bool operator<(const Schedule& s) const{
        return fE->getScheduleTime() > s.fE->getScheduleTime();
    }

    inline Event* getEvent(){
        return fE;
    }

private:
    Event* fE;
};

} /* namespace pop */
#endif /* SCHEDULE_H_ */
```

Listing 18: simulator/schedule.h

```
/*
 * schedule.cpp
 *
 *  Created on: Sep 26, 2011
 *      Author: ibensw
 */

#include "schedule.h"

namespace pop {


} /* namespace pop */
```

Listing 19: simulator/schedule.cpp

```
/*
 * simulator.h
 *
 *  Created on: Sep 26, 2011
 *      Author: ibensw
 */

#ifndef SIMULATOR_H_
#define SIMULATOR_H_

#include <queue>
#include "event.h"
#include "schedule.h"

namespace pop {

class Simulator {
public:
    Simulator();
    virtual ~Simulator();

    void run();
    void run(int infointerval);

    inline double getTime(){
        return fNow;
    }

    inline unsigned int getPendingEvents(){
        return fPending.size();
    }

    void addEvent(Event* e);

private:
    std::priority_queue<Schedule> fPending;
    double fNow;
};

} /* namespace pop */
#endif /* SIMULATOR_H_ */
```

Listing 20: simulator/simulator.h

```
/*
 * simulator.cpp
 *
 *  Created on: Sep 26, 2011
 *      Author: ibensw
 */

#include "simulator.h"
#include <iostream>
using namespace std;

namespace pop {

Simulator::Simulator():
    fNow(0.0)
        {}

Simulator::~Simulator() {
}

void Simulator::run(){
    while(!fPending.empty()){
        Schedule x = fPending.top();
        fPending.pop();
        fNow = x.getEvent()->getScheduleTime();
        x.getEvent()->run(this);
    }
}

void Simulator::run(int interval){
    int next = interval;
    while(!fPending.empty()){
        Schedule x = fPending.top();
        fPending.pop();
        fNow = x.getEvent()->getScheduleTime();
        if (fNow > next){
            cout << "Time: " << fNow << "\tPending events: " << fPending.size() << endl;
            next+=interval;
        }
        x.getEvent()->run(this);
    }
}

void Simulator::addEvent(Event* e){
    fPending.push(Schedule(e));
}

}
```

Listing 21: simulator/simulator.cpp

B MATLAB Numerical evaluation code

```
function [Q] = rightchain(size, rate)
%RIGHTCHAIN Generate a Markov Chain that always forwards right
%Parameters:
%       size    The size of the ring
%       rate    The rate of arrivals

    totalsize = 2^size;
    Q = sparse(totalsize, totalsize);

    BITS = zeros(1, size);

    for i=1:size
        BITS(i) = 2^(i-1);
    end

    for i=0:(totalsize-1)
        t=0;
        for b=1:size
            j=bitxor(i, BITS(b));
            if bitand(i, BITS(b))
                Q(i+1, j+1)=1;
            else
                r=rate;
                bt=b+1;
                while bitand(i, BITS(mod(bt-1, size)+1)) & (bt ~= (b))
                    bt=bt+1;
                    r = r + rate;
                end
                Q(i+1, j+1)=r;
            end
            t=t + Q(i+1, j+1);
        end
        Q(i+1, i+1) = -t;
    end

end
```

Listing 22: rightchain.m

```
function [Q] = randswitchchain(size, rate, p)
%RANDSWITCHCHAIN Generates a Markov Chain that randomly forward left or right
%Parameters:
%       size    The size of the Markov Chain
%       rate    The rate of arrivals
%       p               The probability a job is forwarded right
%                       (Default: 0.5)

        if nargin < 3
                p=0.5;
        end

    totalsize = 2^size;
    Q = sparse(totalsize, totalsize);

    BITS = 2.^[0:size-1];

    for i=0:(totalsize-1)
        t=0;
        for b=1:size
            j=bitxor(i, BITS(b));
            if bitand(i, BITS(b))
                Q(i+1, j+1)=1;
            else
                r=rate;
                bt=b+1;
                while bitand(i, BITS(mod(bt-1, size)+1)) & (bt ~= (b))
                    bt=bt+1;
                    r = r + rate*p;
                end
                bt=b-1;
                while bitand(i, BITS(mod(bt-1, size)+1)) & (bt ~= (b))
```

```
                    bt=bt-1;
                    r = r + rate*(1-p);
                end
                Q(i+1, j+1)=r;
            end
            t=t + Q(i+1, j+1);
        end
        Q(i+1, i+1) = -t;
    end

end
```

Listing 23: randswitchchain.m

```
function [ Q ] = rprimechain( size , rate )
%RPRIMECHAIN Generate a Markov Chain that chooses a random coprime and uses this as
    forwarding offset
%Parameters:
%        size     The size of the ring
%        rate     The arrival rate

        totalsize=2^size;
        rprimes=[];

        for i=1:(size-1)
                if gcd(size , i) == 1
                        rprimes=[rprimes i];
                end
        end

        rpcount = length(rprimes);

        %Q=zeros(totalsize);
        Q=sparse(totalsize , totalsize);

        for i=0:totalsize-1
                tot=0;
                for j=0:size-1
                        k=2^j;
                        if bitand(i,k)
                                Q(i+1, i-k+1) = 1.0;
                                tot=tot+1.0;
                        else
                                c=0;
                                for p=rprimes
                                        current=mod(j-p, size);
                                        while (bitand(i,2^current))
                                                current=mod(current-p, size);
                                                c=c+1;
                                        end
                                end
                                Q(i+1, i+k+1) = rate + c*rate/rpcount;
                                tot=tot+Q(i+1, i+k+1);
                        end
                end
                Q(i+1, i+1) = -tot;
        end

end
```

Listing 24: rprimechain.m

```
function [ Q ] = runvisitedchain( size , rate )
%RUNVISITEDCHAIN Generate a Markov Chain that forwards to an unvisited node
%Parameters:
%        size     The size of the ring
%        rate     The arrival rate

        rate = rate*size;

        Q = sparse(size+1, size+1);

        Q(1,2) = rate;
        Q(1,1) = -rate;
```

```
        Q(size+1, size) = size;
        Q(size+1, size+1) = -size;

        for i=2:size
                Q(i,i-1) = i-1;
                Q(i,i+1) = rate;
                Q(i,i) = -Q(i,i+1)-Q(i,i-1);
        end

end
```

Listing 25: runvisitedchain.m

```
function [ avg,distribution ] = avghops(Q, d)
%AVGHOPS Calculate average number of times a job is forwarded
%Parameters:
%       Q       The matrix representing a markov chain
%Optional:
%       d       Debug mode, default=1, disable debug output=0
%Return:
%       avg     The average number of forwards
%       distribution    The distribution for each possible outcome

        if nargin < 2
                d=1;
        end

        steady=full(ctmcsteadystate(Q));
        distribution=zeros(1,d);

        len=length(Q);
        states=log2(len);
        avg=0;
        total=0;

        for i=0:(states-1)
                c=0;
                prefix=((2^i)-1) * 2^(states-i);
                for j=0:(2^(states-i-1))-1
                        c=c+steady(prefix + j + 1);
                end
                total=total+c;
                if d
                        fprintf('%d hops:\t%f\n', i, c);
                end
                distribution(i+1)=c;
                avg=avg+(c*i);
        end

        loss=steady(len);
        avg=avg/(1-loss);
        if d
                fprintf('Loss:\t%f\nTotal:\t%f\nAverage #hops:\t%f\n', loss, total + loss, avg);
        end
end
```

Listing 26: avghops.m

```
function [ avg,avgp ] = ruavghops( Q, d )
%RUAVGHOPS Calculate average number of times a job is forwarded for the random unvisited chain
%Parameters:
%       Q       The matrix representing a markov chain using the random unvisited forwarding algorithm
%Optional:
%       d       Debug mode, default=1, disable debug output=0
%Return:
%       avg             The average number of times a job is forwarded
%       avgp    The distribution

        if nargin < 2
                d=1;
```

```
	end

	steady=ctmcsteadystate(Q);

	len=length(Q);

	avg = 0;
	avgp = zeros(len,1);

	for i=0:len-2
		tmpavg = 0;
		for h=0:i
			c = prod(i-h+1:i) * (len-1-i) / prod(len-1-h:len-1);
			tmpavg = tmpavg + (c * h);
			avgp(h+1) = avgp(h+1) + (c*steady(i+1));
		end
		avg=avg + steady(i+1) * tmpavg;
	end

	avgp(len) = steady(len);

	loss=steady(len);
	avg=avg/(1-loss);
	if d
		avgp
		fprintf('Loss:\t%f\nAverage #hops:\t%f\n', loss, avg);
	end

end
```

Listing 27: ruavghops.m

```
function [ pi ] = ctmcsteadystate( Q )
%CTMCSTEADYSTATE Steady state distribution of a continious time markov chain
%Parameters:
%	Q	Matrix representing a Markov Chain
%Source: http://speed.cis.nctu.edu.tw/~ydlin/course/cn/nsd2009/Markov-chain.pdf (slide 10)

	T=Q;
	len=length(Q);
	T(:,len)=ones(len, 1);
	e=zeros(1, len);
	e(len)=1;
	pi=e*inv(T);
end
```

Listing 28: ctmcsteadystate.m

```
function [ avg ] = lumpavghops(Q)
%LUMPAVGHOPS Get the average number of times a job is forwarded when the state matrix is lumped
%Parameters:
%	Q	A matrix representation of a markov Chain

	fullsize=length(Q);
	[Q S]=lump(Q);
	lumpsize=length(S);
	nodes=log2(fullsize);
	hops=zeros(1,nodes+1);

	steady=ctmcsteadystate(Q);

	hops(1)=steady(1); %zero hops
	hops(nodes+1)=steady(lumpsize); %loss
	for i=2:lumpsize-1;
		bits=ceil(log2(S(i)+1));
		hops(1)=hops(1)+(nodes-bits)/nodes*steady(i);

		for j=bits:-1:1
			c=0;
			while c<j && bitand(S(i), 2^(j-c-1))
				c=c+1;
			end
```

```
                        hops(c+1)=hops(c+1) + steady(i)/nodes;
                end
        end

        %fprintf('Sum:\t%f\n',sum(hops));

        avg=(hops(1:nodes)*[0:nodes-1]')/(1-steady(lumpsize));

end
```

Listing 29: lumpavghops.m

```
function [Ql S] = lump(Q)
%LUMP Lump a matrix representing a Markov Chain
%Parameters:
%       Q       The matrix that should be lumped
%Return:
%       Ql      The lumped matrix representation
%       S       The states that are used in the lumped matrix
%The states of the matrix Q must represent the availability of the the servers

        [S R C] = makestates(log2(length(Q)));

        Ql=sparse(length(S), length(S));

        [i j s] = find(Q);

        for x=1:length(i)
                Ql(R(i(x)),R(j(x)))=Ql(R(i(x)),R(j(x)))+s(x);
        end

        for x=1:length(S)
                Ql(x,:)=Ql(x,:)/C(x);
        end

end
```

Listing 30: lump.m

```
function [r, refindex, coverage] = makestates(rsize)
%Generate lumped states
%Parameters:
%       rsize           Size of the ring (or log2 of the number of states of the matrix)
%Return:
%       r               Vector of the remaining states, ordered
%       refindex        Reference index, each old state points to the new lumped state
%       coverage        How many states the lumped state with the same index represents

        powers = 2.^[0:rsize-1];

        function [v] = rotate(a, size)
                p=2^(size-1);
                v = a*2 + floor(a/p) - 2*p*floor(a/p);
        end

        function [r] = makesmallest(a)
                r=a;
                for i=1:(rsize-1)
                        a=rotate(a,rsize);
                        if a<r
                                r=a;
                        end
                end
        end

        refindex = [];
        for i=0:(2^rsize)-1
                refindex = [refindex makesmallest(i)];
        end

        function [c] = cover(a, size)
                c=1;
                a=makesmallest(a);
                b=rotate(a, size);
```

```
			while a ~= b
				b=rotate(b, size);
				c=c+1;
			end
	end

	function [r] = smallest(a, size)
			r=a;
			for i=1:(size-1)
				a=rotate(a,size);
				if a<r
					r=a;
				end
			end
	end

	function [v] = f(word, bits, place, size)
			if place > size
				v = word;
			elseif bits == 0
				v = f(word, bits, place+1, size);
			elseif place + bits > size
				v = f(word + powers(place), bits-1, place+1, size);
			else
				v = [f(word + powers(place), bits-1, place+1, size) f(word, bits,
					place+1, size)];
			end
	end


	function [r] = makecombs(k, n)
			leadzeros = ceil(n/k)-1;
			fullsize = n - leadzeros - 1;
			r = f(0,k-1,1,fullsize)*2 + 1;
	end

	r=[0 2^(rsize)-1];
	for i=1:rsize-1
			r = [r makecombs(i, rsize)];
	end

	s=[];
	for i=r
			s=[s refindex(i+1)];
	end

	r=unique(s);
	refindex = arrayfun(@(x) find(r == x), refindex);

	coverage=[];
	for w=r
			coverage=[coverage cover(w, rsize)];
	end

end
```

Listing 31: makestates.m