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An extended TODIM approach with intuitionistic linguistic numbers

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Abstract

In many practical problems, multiple alternatives are ranked with respect to related criteria, and the criteria's weights are usually unknown. In order to solve this kind of multicriteria decision making (MCDM) problem, this paper proposes an interactive MCDM approach based on an acronym in Portuguese of interactive and multicriteria decision-making (TODIM) method and nonlinear programming (NLP) with intuitionistic linguistic numbers (ILNs). First, by comparing the existing operations and comparison methods for ILNs, new operations and a new comparison method are defined based on linguistic scale functions to obtain rational results. Second, considering their linguistic terms, membership degrees, and nonmembership degrees as a whole, the generalized distance between ILNs is defined with an adjustable parameter. Third, the total ranking of alternatives is obtained using the proposed NLP-based TODIM approach based on the generalized ILN distance. Finally, an example of selecting hotels from a tourism website is presented to verify the validity and feasibility of the proposed approach. A comparison with existing methods is also conducted and analyzed.

Keywords: intuitionistic linguistic numbers; TODIM; nonlinear programming; generalized distance function

1. Introduction

As society rapidly develops, the factors that people should consider in decision-making problems continue to increase. As a result, multicriteria decision-making (MCDM) problems have become an important focus of research in recent years (Ilgin et al., 2015; Gul et al., 2016; Peng et al., 2016). In situations with incomplete information in uncertain environments, crisp numbers cannot meet the requirements of describing alternatives with respect to different criteria. In order to evaluate alternatives accurately, Zadeh (1965) proposed fuzzy sets (FSs), which are characterized by a membership degree; subsequently, numerous methods have been proposed based on FSs to deal with various MCDM problems (Yager, 1977; Chen, 2000; Wang et al., 2016b). However, some researchers hold the view that the single membership degree of an FS is inadequate when considering

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nonmembership degrees would contribute valuable information. Based on that idea, Atanassov (1986, 1989) proposed intuitionistic FSs (IFSs), which are characterized by both membership degrees and nonmembership degrees; IFSs are a generalization of the FS concept. Since then, interval-valued IFSs (Atanassov and Gargov, 1989; Wu et al., 2013; Cao et al., 2016), intuitionistic triangular fuzzy numbers (Shu et al., 2006; Wang and Zhang, 2009), and triangular intuitionistic fuzzy numbers (Zhang and Liu, 2010; Ye, 2012) have been defined and researched as extensions of IFSs in order to address MCDM problems (Borana et al., 2009; Zhang and Liu, 2011; Tian et al., 2015; Nan et al., 2016). Furthermore, Wang and Li (2010) proposed intuitionistic linguistic sets (ILSs) that consider the convenience of linguistic variables and the comprehensiveness of IFSs in assessing alternatives.

Since Wang and Li (2010) proposed ILSs, a number of research studies (Liu, 2013; Liu and Wang, 2014; Su et al., 2014; Wang et al., 2014b, 2015) have been conducted based on the concept. Because intuitionistic linguistic numbers (ILNs) are the predominant carriers of information in ILSs, many applications of ILSs are based on ILNs. Liu (2013) defined an intuitionistic linguistic power generalized weighted average operator and an intuitionistic linguistic power generalized ordered weighted average operator, using the operators to propose two corresponding MCDM methods. Su et al. (2014) proposed a new method using the quasi-arithmetic Atanassov's intuitionistic linguistic ordered weighted averaging distance (Quasi-AILOWAD) operator. Wang et al. (2014b, 2015) defined the intuitionistic linguistic ordered weighted averaging operator, intuitionistic linguistic hybrid aggregation operator, intuitionistic linguistic ordered weighted geometric operator, and intuitionistic linguistic hybrid geometric operator; they then proposed MCDM methods based on the different operators in which criteria values are ILNs and the criteria weight information is completely known. As methods based on ILNs have proliferated, researchers have explored many potential applications of ILNs. However, all of the methods described above are based on the aggregation operator, such that the differences of alternatives under different criteria are lost. Furthermore, the operations and comparison methods for ILNs are unreasonable because their linguistic terms, membership degrees, and nonmembership degrees are separated to calculate results. Therefore, given the validity of ILNs in expressing information, additional research should explore rational operations, comparison methods, and MCDM methods with ILNs.

In general, when it comes to considering the behavior of decision makers, an acronym in Portuguese of interactive and multicriteria decision-making (TODIM) method (Gomes and Lima, 1992), which was proposed by Gomes and Lima based on the prospect theory, would be the preferable method comparing with another methods (Gomes and González, 2012; Gomes et al., 2013; Tan et al., 2015; Zhou et al., 2016). Later, Gomes and Rangel (2009) defined a reference value for the rents of these properties using the TODIM method of multicriteria decision aiding. Zhang and Xu (2014) extended the TODIM method in order to consider the decision maker's psychological behavior in hesitant fuzzy environments. Krohling et al. (2013) developed an intuitionistic fuzzy TODIM method, which accounts for uncertainty modeled by intuitionistic trapezoidal fuzzy numbers in the decision matrix. Lourenzutti and Krohling (2013) proposed a generalization of the TODIM method that considers both intuitionistic fuzzy information and an underlying random vector that affects the performance of the alternatives. Ji et al. (2016) proposed a projection-based TODIM method for multivalued neutrosophic environments and applied it to personnel selection. Wang et al. (2016a) proposed a likelihood function for multihesitant fuzzy linguistic term elements, which was then embedded into TODIM in order to address decision-making problems in which decision makers exhibit bounded rationality, while hesitance and repetitiveness exist in the linguistic evaluation information. In order to solve hybrid MCDM problem, Fan et al. (2013) extended the TODIM method to three formats of attribute values (crisp numbers, interval numbers, and fuzzy numbers). Based on these existing studies, TODIM's main advantages are easily identifiable: (1) The TODIM method takes the decision makers' behavior into consideration based on prospect theory; and (2) the potential value of gains and losses, which can be adjusted by the factor of the losses, can be used to reflect risk preferences.

Nevertheless, in both ILN methods based on various kinds of aggregation operators and extended TODIM methods with different forms of fuzzy numbers, the criteria weights are completely determined by decision makers; furthermore, the TODIM method has not been extended to ILNs. Therefore, this paper addresses a nonlinear programming (NLP) based TODIM approach with ILNs. The main motivation and contributions of this study are summarized as follows.

1. In order to overcome problems resulting from the separation of linguistic terms, membership degrees, and nonmembership degrees in computing, this paper defines new operations for ILNs based on linguistic scale functions and operations defined in previous papers. The rationality and properties of the new operations are discussed and proven. Similarly, the previous comparison methods of ILNs are discussed, and a new comparison method is defined that can improve the accuracy of calculation to some extent.
2. Based on linguistic scale functions, this paper proposes a generalized distance between any two ILNs. When the parameter λ is assigned different values, Hamming distance and Euclidean distance can be obtained. The linguistic scale functions and the parameter can improve the flexibility of this generalized distance.
3. Considering both facts that criteria weights are usually unknown and advantage of the TODIM method in reflecting decision makers' risk preferences, this paper proposes a novel NLP-based TODIM approach with ILNs. The proposed approach is capable of improving the adaptability of ILNs in practice, in addition to effectively solving MCDM problems that know only alternatives and criteria.

The remainder of this paper is organized as follows. Section 2 briefly reviews the concepts of linguistic term sets, linguistic scale functions, and ILNs. Section 3 defines the new operations and comparison method for ILNs and discusses related properties of these operations. Section 4 defines the generalized distance between ILNs and proposes a modified NLP-based TODIM approach based on the distance. Section 5 demonstrates the feasibility and applicability of the proposed method through a case study involving selecting hotels on a tourism website; the final results are then compared with those produced by other existing methods. Finally, Section 6 presents conclusions.

2. Preliminaries

This section reviews and discusses some related basic concepts, including linguistic term sets and their extension, linguistic scale functions, and intuitionistic linguistic sets.

2.1. Linguistic term sets and their extension

In actuality, during the process of solving MCDM problems, many aspects of different activities can only be assessed in a qualitative form rather than in a quantitative form; therefore, in situations with vague or imprecise knowledge, it is highly convenient to use linguistic variables to assess alternatives with respect to concerned criteria (Wei et al., 2011).

Definition 1 (Rodríguez et al., 2012). *Let $S = \{s_i | i = 0, 1, \ldots, 2t\}$ be a finite and totally ordered discrete linguistic term set accompanied by s_i, which represents a possible value for a linguistic variable; then the following characteristics are true (Delgado et al., 1992).*

1. The set is ordered: if $i > j$, then $s_i > s_j$.
2. There exists a negation operator: $neg(s_i) = s_{2t-i}$.

Aggregated results cannot usually match the elements in the language assessment scale. In order to preserve those results, Xu (2004) extended the discrete term set S to a continuous term set $\bar{S} = \{s_\alpha | s_0 \leq s_\alpha \leq s_t, \alpha \in [0, l]\}$, in which $s_i > s_j$ if $i > j$, and $l(l > 2t)$ is a sufficiently large positive integer; furthermore, the elements in the set $\bar{S}$ meet all of the characteristics described above. If $s_\alpha \in S$, then s_α is called an original linguistic term; otherwise, s_α is called a virtual linguistic term, which does not have any practical meaning, and its main role is to rank the alternatives (Martinez et al., 2010).

2.2. Linguistic scale functions

In different situations, linguistic scale functions assign different semantic values to linguistic terms. Therefore, these functions have the capacity to provide more deterministic results based on semantics (Wang et al., 2014a).

Definition 2 (Wang et al., 2014a). *If $\theta_i \in [0, 1]$ is a numeric value, then the linguistic scale function f conducts the mapping from s_i to θ_i $(i = 0, 1, 2 \ldots, 2t)$, and it can be represented in the following form:*

$$f : s_i \to \theta_i \, (i = 0, 1, 2, \ldots, 2t) \, ,$$

where $0 \leq \theta_0 < \theta_1 < \cdots < \theta_{2t}$.

The symbols θ_i $(i = 0, 1, 2, \ldots, 2t)$ are used to express the linguistic term $s_i \in S (i = 0, 1, 2, \ldots, 2t)$ according to this function, and the semantics of the linguistic terms are denoted by the function/value.

Three common linguistic scale functions are provided to address different problems.

1. The first linguistic scale function is defined as follows:

$$f_1 \left(s_x\right) = \theta_x = \frac{x}{2t} \quad (x = 0, 1, 2, \ldots, 2t) \, .$$

This function is simple and similar to the subscript function $I(s_i) = i$; therefore, it is commonly used.

2. Decision makers' mental stimulation caused by decision criteria includes both good and bad aspects. Therefore, a composite assessment scale function is proposed and expressed as follows:

$$f_2(s_y) = \theta_y = \begin{cases} \dfrac{a^t - a^{t-y}}{2a^t - 2} & (y = 0, 1, 2, \ldots, t) \\ \dfrac{a^t + a^{y-t} - 2}{2a^t - 2} & (y = t+1, t+2, \ldots, 2t) \end{cases}$$

The value of parameter a can be used to adjust the absolute deviation between any two adjacent linguistic subscripts, and it can be obtained through a subjective approach (Bao et al., 2010). Suppose there are two indicators expressed by I_A and I_B, where I_A is more significant than I_B, with an importance ratio of m. Let k represent the scale level and $a = \sqrt[k]{m}$. Generally, most researchers believe that the upper limit of the importance ratio is $m = 9$. If the scale level is 7, then $a = \sqrt[7]{9} \approx 1.4$.

3. In prospect theory, the decision makers' sensitivity regarding the gap between "good" and "slightly good" is greater than their sensitivity regarding the gap between "good" and "very good." Based on these gaps, a linguistic scale function that relates to the concept of prospect theory is defined as follows:

$$f_3(s_z) = \theta_z = \begin{cases} \dfrac{t^\alpha - (t-z)^\alpha}{2t^\alpha} & (z = 0, 1, 2, \ldots, t) \\ \dfrac{t^\beta + (z-t)^\beta}{2t^\beta} & (z = t+1, t+2, \ldots, 2t) \end{cases}.$$

In this function, $\alpha, \beta \in (0, 1]$, and if $\alpha = \beta = 1$, then $\theta_z = \frac{z}{2t}$.

Each of the above functions can be expanded to $f^* : \bar{S} \to R^+ (R^+ = \{r | r \geq 0, r \in R\})$, which is a strictly monotonically increasing and continuous function. Therefore, the mapping from $\bar{S}$ to R^+ is one-to-one due to its monotonicity, and the inverse function of f^* exists and is denoted by f^{*-1}.

2.3. Intuitionistic linguistic set

Definition 3 (Wang and Li, 2010). *Let X be a universe of discourse and $s_{\theta(x)} \in S$; then, an ILN set A in X is an object having the following form:*

$$A = \left\{ \left(x, \langle s_{\theta(x)}, \mu_A(x), \nu_A(x) \rangle \right) : x \in X \right\},$$

which is characterized by a linguistic term $s_{\theta(x)}$, a membership degree $\mu_A(x)$, and a nonmembership degree $\nu_A(x)$ of the element x to $s_{\theta(x)}$, where $\mu_A(x) : X \to [0, 1]$, $\nu_A(x) : X \to [0, 1]$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Let $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$, then $\pi_A(x)$ is called the degree of hesitancy of x to $s_{\theta(x)}$.

When $\mu_A(x) = 1$ and $\nu_A(x) = 0$, the ILN set is reduced to the linguistic term set. In particular, when X has only one element, the ILN set A is reduced to $\langle s_{\theta(x)}, \mu_A(x), \nu_A(x) \rangle$, which we call it an ILN.

Example 1. Let the linguistic set $S = \{s_i | i = 0, 1, \ldots, 2t\} = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$; then, $h = \langle s_{\theta(h)}, \mu(h), \nu(h)\rangle = \langle s_2, 0.6, 0.3\rangle$ is an ILN in X.

3. New operations and comparison method for ILNs

Previous researchers have defined several operations and comparison methods, using them to manage ILNs; however, each of these operations and comparison methods has its limitations. This section discusses in detail the limitations of various operations and comparison methods. In order to overcome these limitations, new operations and a new comparison method for ILNs are defined in different subsections.

3.1. Operations of ILNs

Definition 4 (Wang and Li, 2010). *Let $h_1 = \langle s_{\theta(h_1)}, \mu(h_1), \nu(h_1)\rangle$ and $h_2 = \langle s_{\theta(h_2)}, \mu(h_2), \nu(h_2)\rangle$ be any two ILNs; then, some operations of h_1 and h_2 are defined as follows:*

(1) $h_1 \oplus h_2 = \langle s_{\theta(h_1)+\theta(h_2)}, \frac{\theta(h_1)\mu(h_1)+\theta(h_2)\mu(h_2)}{\theta(h_1)+\theta(h_2)}, \frac{\theta(h_1)\nu(h_1)+\theta(h_2)\nu(h_2)}{\theta(h_1)+\theta(h_2)}\rangle$;
(2) $h_1 \otimes h_2 = \langle s_{\theta(h_1)\theta(h_2)}, \mu(h_1)\mu(h_2), \nu(h_1) + \nu(h_2)\rangle$;
(3) $\lambda h_1 = \langle s_{\lambda\theta(h_1)}, \mu(h_1), \nu(h_1)\rangle$, $\lambda \geq 0$;
(4) $h_1^\lambda = \langle s_{(\theta(h_1))^\lambda}, (\mu(h_1))^\lambda, 1 - (1 - (\nu(h_1))^\lambda)\rangle$, $\lambda \geq 0$.

However, some obvious limitations exist for these operations:

(1) Linguistic terms have the advantage of flexibly expressing the uncertainty and ambiguity of practical problems; however, linguistic terms can lose this advantage if all operations are carried out based directly on the subscripts of linguistic terms.
(2) The result of the operation $h_1 \otimes h_2$ may be not an ILN. It is obvious that the nonmembership degrees of linguistic terms are simply added in the operation $h_1 \otimes h_2$. Suppose $h_1 = \langle s_3, 0.3, 0.6\rangle$ and $h_2 = \langle s_2, 0.4, 0.5\rangle$; then, $h_1 \otimes h_2 = \langle s_6, 0.12, 1.1\rangle$. Although this result has no practical meaning, it is unreasonable and counterintuitive.

Moreover, the operations in Definition 4 have a common limitation that they lack the negation operator. For the purpose of overcoming the limitations listed above, new operations of ILNs based on linguistic scale functions are defined as follows.

Definition 5. *Let $h_1 = \langle s_{\theta(h_1)}, \mu(h_1), \nu(h_1)\rangle$ and $h_2 = \langle s_{\theta(h_2)}, \mu(h_2), \nu(h_2)\rangle$ be any two ILNs, and let f^* and f^{*-1} be a linguistic scale function and its inverse function respectively. For convenience, let $f_1^* = f^*(s_{\theta(h_1)})$ and $f_2^* = f^*(s_{\theta(h_2)})$. Then, the following new operations can be defined for h_1 and h_2:*

(1) $neg(h_1) = \langle f^{*-1}(f^*(s_{2t}) - f_1^*), \nu(h_1), \mu(h_1)\rangle$;
(2) $h_1 \oplus h_2 = \langle f^{*-1}(f_1^* + f_2^*), \frac{f_1^*\mu(h_1)+f_2^*\mu(h_2)}{f_1^*+f_2^*}, \frac{f_1^*\nu(h_1)+f_2^*\nu(h_2)}{f_1^*+f_2^*}\rangle$;

(3) $h_1 \Theta h_2 = \langle f^{*-1}(f_1^* - f_2^*), \frac{f_1^* \mu(h_1) - f_2^* \mu(h_2)}{f_1^* - f_2^*}, \frac{f_1^* \nu(h_1) - f_2^* \nu(h_2)}{f_1^* - f_2^*} \rangle$;
(4) $h_1 \otimes h_2 = \langle f^{*-1}(f_1^* f_2^*), \mu(h_1)\mu(h_2), \nu(h_1)\nu(h_2) \rangle$;
(5) $\frac{h_1}{h_2} = \langle f^{*-1}(\frac{f_1^*}{f_2^*}), \frac{\mu(h_1)}{\mu(h_2)}, \frac{\nu(h_1)}{\nu(h_2)} \rangle$;
(6) $\lambda h_1 = \langle f^{*-1}(\lambda f_1^*), \mu(h_1), \nu(h_1) \rangle, \ \lambda \geq 0$;
(7) $h_1^{\lambda} = \langle f^{*-1}((f_1^*)^{\lambda}), (\mu(h_1))^{\lambda}, (\nu(h_1))^{\lambda} \rangle, \ \lambda \geq 0$.

Definition 5 makes it quite clear that f^* is a mapping from the linguistic term s_i to the numeric value θ_i; in contrast, f^{*-1} is a mapping from θ_i to s_i. Moreover, different results can be obtained by adopting different linguistic scale functions, which can reflect actual semantic situations. Therefore, decision makers can select linguistic scale functions as required. In the process of aggregation, the results of $h_1 \oplus h_2$, $h_1 \Theta h_2$, $h_1 \otimes h_2$, $\frac{h_1}{h_2}$, λh_1, and h_1^{λ} have no practical meaning.

Example 2. Recalling the linguistic term set defined in Example 1, let $h_1 = \langle s_3, 0.3, 0.6 \rangle$, $h_2 = \langle s_2, 0.4, 0.5 \rangle$, $h_3 = \langle s_{5.5316}, 0.3435, 0.5564 \rangle$, $h_4 = \langle s_{0.8345}, 0.12, 0.3 \rangle$, $\lambda = 2$ and $t = 3$. Then the results can be calculated as follows:

$$\text{If } a = 1.4,\ f_2(s_y) = \theta_y = \begin{cases} \dfrac{a^t - a^{t-y}}{2a^t - 2} & (y = 0, 1, 2, \ldots, t) \\ \dfrac{a^t + a^{y-t} - 2}{2a^t - 2} & (y = t+1, t+2, \ldots, 2t) \end{cases}, \text{ then}$$

(1) $neg(h_1) = \langle s_3, 0.6, 0.3 \rangle$;
(2) $h_1 \oplus h_2 = \langle s_{5.5316}, 0.3435, 0.5564 \rangle$;
(3) $h_3 \Theta h_2 = \langle s_3, 0.3, 0.6 \rangle$;;
(4) $h_1 \otimes h_2 = \langle s_{0.8345}, 0.12, 0.3 \rangle$;
(5) $\frac{h_4}{h_2} = \langle s_3, 0.3, 0.6 \rangle$;
(6) $2h_1 = \langle s_6, 0.3, 0.6 \rangle$;
(7) $h_1^2 = \langle s_{1.1365}, 0.09, 0.36 \rangle$.

Through Example 2, it is obvious that Definition 5 contains the negation operator, which is very useful in the process of normalizing a decision matrix. Then, linguistic scale functions are used to adjust the values of the linguistic terms to different situations, and the different values are integrated into the membership and nonmembership degrees in the calculation. Furthermore, it is quite clear that the results above are ILNs, and the following theorem about ILNs can be proven easily.

Theorem 1. *Let h_1 and h_2 be any two ILNs; then, the following properties are true*:

(1) $h_1 \oplus h_2 = h_2 \oplus h_1$;
(2) $h_1 \otimes h_2 = h_2 \otimes h_1$;
(3) $h_1 \oplus (h_2 \oplus h_3) = (h_1 \oplus h_2) \oplus h_3$;
(4) $(h_1 \Theta h_2) \oplus h_3 = h_1$;
(5) $h_1 \otimes (h_2 \otimes h_3) = (h_1 \otimes h_2) \otimes h_3$;
(6) $\frac{h_1 \otimes h_2}{h_2} = h_1$;
(7) $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$;

(8) $\lambda_1 h_1 \oplus \lambda_2 h_1 = (\lambda_1 + \lambda_2) h_1,\ \lambda_1, \lambda_2 \geq 0$;
(9) $(h_1 \otimes h_2)^{\lambda} = h_1^{\lambda} \otimes h_2^{\lambda},\ \lambda \geq 0$;
(10) $h_1^{\lambda_1+\lambda_2} = h_1^{\lambda_1} \otimes h_1^{\lambda_2},\ \lambda_1, \lambda_2 \geq 0$.

3.2. Comparison method for ILNs

A method for comparing fuzzy numbers represents an important part of the calculation process for fuzzy numbers. Previous studies have proposed several comparison methods for ILNs and used them to obtain the overall order of alternatives. However, the effect of linguistic terms seems to be insignificant in these comparison methods. This section discusses five comparison methods in detail; then, based on the discussion, a new comparison method is proposed.

Definition 6 (Wang and Li, 2010). *For an ILN $h = \langle s_{\theta(h)}, \mu(h), \nu(h) \rangle$, the expected value $E_1(h)$, score function $S_1(h)$, and accuracy function $H_1(h)$ of h are defined as follows:*

$$E_1(h) = s_{\theta(h).(\mu(h)+1-\nu(h)/2)},$$

$$S_1(h) = \frac{\theta(h) \cdot (\mu(h) + 1 - \nu(h))}{2} \cdot (\mu(h) - \nu(h)),$$

$$H_1(h) = \frac{\theta(h) \cdot (\mu(h) + 1 - \nu(h))}{2} \cdot (\mu(h) + \nu(h)).$$

The order relationship for any two ILNs h_1 and h_2 can be defined as follows:

(1) if $S_1(h_1) > S_1(h_2)$, then $h_1 > h_2$;
(2) if $S_1(h_1) = S_1(h_2)$, then

$$\text{if } H_1(h_1) > H_1(h_2), \quad h_1 > h_2;$$

$$\text{if } H_1(h_1) = H_1(h_2), \quad h_1 = h_2.$$

Definition 7 (Liu, 2013). *For an ILN $h = \langle s_{\theta(h)}, \mu(h), \nu(h) \rangle$, the score function $S_2(h)$ and accuracy function $H_2(h)$ of h are defined as follows:*

$$S_2(h) = \frac{\theta(h)}{2t} \cdot \frac{\mu(h) + 1 - \nu(h)}{2},$$

$$H_2(h) = \frac{\theta(h)}{2t} \cdot (\mu(h) + \nu(h)).$$

The order relationship for any two ILNs h_1 and h_2 can be defined as follows:

(1) if $S_2(h_1) > S_2(h_2)$, then $h_1 > h_2$;
(2) if $S_2(h_1) = S_2(h_2)$, then

$$\text{if } H_2(h_1) > H_2(h_2), \quad h_1 > h_2;$$

$$\text{if } H_2(h_1) = H_2(h_2), \quad h_1 = h_2.$$

Definition 8 (Liu and Wang 2014). *For an ILN $h = \langle s_{\theta(h)}, \mu(h), \nu(h)\rangle$, the expected value $E_3(h)$, score function $S_3(h)$, and accuracy function $H_3(h)$ of h are defined as follows:*

$$E_3(h) = s_{(\theta(h)\times(\mu(h)+1-\nu(h)))/2},$$
$$S_3(h) = E_3(h)(\mu(h) - \nu(h)),$$
$$H_3(h) = E_3(h)\cdot(\mu(h) + \nu(h)).$$

The order relationship for any two ILNs h_1 and h_2 can be defined as follows:

(1) if $E_3(h_1) > E_3(h_2)$, then $h_1 > h_2$;
(2) if $E_3(h_1) = E_3(h_2)$ and $S_3(h_1) > S_3(h_2)$, then $h_1 > h_2$;
(3) if $E_3(h_1) = E_3(h_2)$ and $S_3(h_1) = S_3(h_2)$, then

$$\text{if } H_3(h_1) > H_3(h_2), \quad h_1 > h_2;$$
$$\text{if } H_3(h_1) = H_3(h_2), \quad h_1 = h_2.$$

Definition 9 (Wang et al., 2014b). *For an ILN $h = \langle s_{\theta(h)}, \mu(h), \nu(h)\rangle$, the score function $S_4(h)$ and accuracy function $H_4(h)$ of h are defined as follows:*

$$S_4(h) = \theta(h)\cdot(\mu(h) - \nu(h)),$$
$$H_4(h) = \theta(h)\cdot(\mu(h) + \nu(h)).$$

The order relationship for any two ILNs h_1 and h_2 can be defined as follows:

(1) If $S_4(h_1) > S_4(h_2)$, then $h_1 > h_2$;
(2) If $S_4(h_1) = S_4(h_2)$, then

$$\text{if } H_4(h_1) > H_4(h_2), \ h_1 > h_2;$$
$$\text{if } H_4(h_1) = H_4(h_2), \ h_1 = h_2.$$

Definition 10 (Wang et al., 2015). *For an ILN $h = \langle s_{\theta(h)}, \mu(h), \nu(h)\rangle$, the score function $S_5(h)$ and accuracy function $H_5(h)$ of h are defined as follows:*

$$S_5(h) = \theta(h)\cdot(1 + \mu(h) - \nu(h)),$$
$$H_5(h) = \theta(h)\cdot(1 - \mu(h) - \nu(h)).$$

The order relationship for any two ILNs h_1 and h_2 can be defined as follows:

(1) if $S_5(h_1) > S_5(h_2)$, then $h_1 > h_2$;
(2) if $S_5(h_1) = S_5(h_2)$, then

$$\text{if } H_5(h_1) > H_5(h_2), \ h_1 > h_2;$$
$$\text{if } H_5(h_1) = H_5(h_2), \ h_1 = h_2.$$

The five comparison methods described above are widely applied in MCDM problems, and they allow most ILNs to be calculated and compared in order to obtain their order relationship. However, these comparison methods have both common restrictions and individual restrictions.

(1) The subscripts of linguistic terms are used instead of the linguistic terms in all of the comparison methods above. The practicability of MCDM strategies based on these comparison methods is an issue that is worthwhile to be investigated, similarly, the expansibility of linguistic terms is not good.
(2) The influence of membership degree together with non-membership degree exceeds the impact of linguistic terms. Suppose $h_1 = \langle s_1, 0.8, 0.1\rangle$ and $h_2 = \langle s_6, 0.1, 0.9\rangle$; then, $h_1 > h_2$ according to the Definitions 6 through 10. However, when decision makers evaluate the alternatives with respect to different criteria, the linguistic terms are the main evaluation values. Therefore, it seems unreasonable that the scores of ILNs are largely influenced by membership degrees and nonmembership degrees.
(3) Several ILNs would be equivalent when the membership and nonmembership degrees of ILNs meet specific conditions. For example, let $\langle s_\alpha, 0, 1\rangle$ be any one ILN; then, the score and accurate value of the ILN would be equal to zero when applying the comparison methods in Definitions 6, 8, and 10. In this case, it is obvious that $\langle s_0, 0, 1\rangle = \langle s_2, 0, 1\rangle = \langle s_4, 0, 1\rangle = \langle s_6, 0, 1\rangle$, and this is illogical.
(4) There are only two differences between Definitions 6 and 8. First, it is not difficult to determine that the expected ILN values can be calculated based on the same formula according to Definitions 6 and 8; however, there is no need to compare the expected ILN values in Definition 6. Second, the scores and accurate values of ILNs are calculated based on the subscripts of the expected values in Definition 6, but in Definition 8, the expected values are directly involved in arithmetic. Moreover, according to the operations of linguistic terms (Xu, 2005), the results show no difference.
(5) The differences between the ILNs' membership and nonmembership degrees have too much influence on the results of applying the comparison methods in Definitions 6, 8, and 9. Assume $h_1 = \langle s_6, 0.4, 0.5\rangle$ and $h_2 = \langle s_1, 0.5, 0.4\rangle$, then $h_1 < h_2$ according to the Definitions 6, 8, and 9. Because the membership degree of h_1 is lower than the nonmembership degree of h_1, the score of h_1 is negative. In contrast, the score of h_2 is positive, and as a result, $h_1 < h_2$. This represents an unacceptable result.

In order to overcome these problems and increase the influence of linguistic terms, the following new comparison method is proposed based on the new score function and accuracy function.

Definition 11. *Let $h_i = \langle s_{\theta(h_i)}, \mu(h_i), \nu(h_i)\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of ILNs. When $d = \max\limits_{\substack{i=1,2,\ldots,n\\ j=1,2,\ldots,n}} \{|\theta(h_i) - \theta(h_j)|\}$, score function $S(h_i)$ and accuracy function $H(h_i)$ of h_i are defined as follows:*

$$S(h_i) = \left(f_i^*\right)^d \cdot \left(1 + \frac{1 + \mu(h_i) - \nu(h_i)}{2}\right),$$

$$H(h_i) = \left(f_i^*\right)^d \cdot \left(\mu(h_i) + \nu(h_i)\right),$$

where $f_i^* = f^*(s_{\theta(h_i)})$.

The order relationship for h_1 and h_2 can be defined as follows:

(1) if $S(h_1) > S(h_2)$, then $h_1 > h_2$;
(2) if $S(h_1) = S(h_2)$, then

$$\text{if } H\left(h_1\right) > H\left(h_2\right),\ h_1 > h_2;$$

$$\text{if } H\left(h_1\right) = H\left(h_2\right),\ h_1 = h_2.$$

Example 3. Recall the linguistic term set defined in Example 1. Using Definition 11 and the examples in the discussion above, the results of two ILNs defined with different values are as follows:

$$\text{If } a = 1.4,\ f_2\left(s_y\right) = \theta_y = \begin{cases} \dfrac{a^t - a^{t-y}}{2a^t - 2} & (y = 0, 1, 2, \ldots, t) \\ \dfrac{a^t + a^{y-t} - 2}{2a^t - 2} & (y = t+1, t+2, \ldots, 2t) \end{cases}.$$

(1) Suppose $h_1 = \langle s_1, 0.8, 0.1\rangle$ and $h_2 = \langle s_6, 0.1, 0.9\rangle$, then $d = 5$, $S(h_1) = 0.0011$, and $S(h_2) = 1.1$. Therefore, $h_1 < h_2$.
(2) Suppose $h_1 = \langle s_0, 0, 1\rangle$, $h_2 = \langle s_2, 0, 1\rangle$, $h_3 = \langle s_4, 0, 1\rangle$, and $h_4 = \langle s_6, 0, 1\rangle$, then $d = 6$, $S(h_1) = 0$, $S(h_2) = 0.0033$, $S(h_3) = 0.0539$, and $S(h_4) = 1$. Therefore, $h_1 < h_2 < h_3 < h_4$.
(3) Suppose $h_1 = \langle s_6, 0.4, 0.5\rangle$ and $h_2 = \langle s_1, 0.5, 0.4\rangle$, then $d = 5$, $S(h_1) = 1.45$, and $S(h_2) = 0.0008$ such that $h_1 > h_2$.

Obviously, instead of subscripts, linguistic scale functions are used to deal with linguistic terms. At the same time, the impact of linguistic terms is improved. According to the logical results in Example 3, the new comparison method is more reasonable than the method described in Definitions 6 through 10.

4. NLP-based TODIM approach with ILNs

In the practical applications, many problems can be resolved using the MCDM method. However, due to the complexity of real-world problems, the weights in the problems often cannot be precisely determined. NLP is one common method used to calculate the weights. The TODIM method measures the dominance degree of each alternative over the others by establishing a multicriteria value function based on prospect theory (Kahneman and Tversky, 1979). Based on the obtained dominance degrees, the ranking of alternatives can be determined. The main advantage of TODIM is its ability to capture the DM's behavior. This section defines the distance measure between two ILNs; then, based on the distance measure, an extended TODIM approach is proposed based on the NLP for uncertain MCDM problems with ILNs.

4.1. Distance between two ILNs

Definition 12. *Let $h_1 = \langle s_{\theta(h_1)}, \mu(h_1), \nu(h_1)\rangle$ and $h_2 = \langle s_{\theta(h_2)}, \mu(h_2), \nu(h_2)\rangle$ be any two ILNs, and let f^* be a linguistic scale function. Then the generalized distance between h_1 and h_2 can be defined as follows:*

$$d_g(h_1, h_2) = \left(\frac{1}{2}\left(\left|f^*\left(s_{\theta(h_1)}\right) \cdot \mu(h_1) - f^*\left(s_{\theta(h_2)}\right) \cdot \mu(h_2)\right|^\lambda + \left|f^*\left(s_{\theta(h_1)}\right) \cdot (1-\nu(h_1)) - f^*\left(s_{\theta(h_2)}\right) \cdot (1-\nu(h_2))\right|^\lambda\right)\right)^{1/\lambda}. \quad (1)$$

When $\lambda = 1$ or $\lambda = 2$, Equation (1) is reduced to the Hamming distance or Euclidean distance, respectively.

Theorem 2. *Let $h_1 = \langle s_{\theta(h_1)}, \mu(h_1), \nu(h_1)\rangle$, $h_2 = \langle s_{\theta(h_2)}, \mu(h_2), \nu(h_2)\rangle$, and $h_3 = \langle s_{\theta(h_3)}, \mu(h_3), \nu(h_3)\rangle$ be any three ILNs; additionally, let f^* be a linguistic scale function. Then, the generalized distance $d_g(h_i, h_j)$ satisfies the following properties:*

(1) $d_g(h_1, h_2) \geq 0$;
(2) $d_g(h_1, h_2)=d_g(h_2, h_1)$;
(3) If $s_{\theta(h_1)} \leq s_{\theta(h_2)} \leq s_{\theta(h_3)}$, $\mu(h_1) \leq \mu(h_2) \leq \mu(h_3)$, and $\nu(h_1) \geq \nu(h_2) \geq \nu(h_3)$, then $d_g(h_1, h_2) \leq d_g(h_1, h_3)$ and $d_g(h_2, h_3) \leq d_g(h_1, h_3)$.

Proof. Obviously, Properties (1) and (2) are correct, and the proof of Property (3) is presented next.

Since $s_{\theta(h_1)} \leq s_{\theta(h_2)} \leq s_{\theta(h_3)}$, $\mu(h_1) \leq \mu(h_2) \leq \mu(h_3)$, $\nu(h_1) \geq \nu(h_2) \geq \nu(h_3)$, and f^* is a strictly monotonically increasing and continuous function, $f^*(s_{\theta(h_1)}) \leq f^*(s_{\theta(h_2)}) \leq f^*(s_{\theta(h_3)})$, then

$$f^*\left(s_{\theta(h_1)}\right) \cdot \mu\left(h_1\right) \leq f^*\left(s_{\theta(h_2)}\right) \cdot \mu\left(h_2\right) \leq f^*\left(s_{\theta(h_3)}\right) \cdot \mu\left(h_3\right),$$

$$\left|f^*\left(s_{\theta(h_1)}\right) \cdot \mu\left(h_1\right) - f^*\left(s_{\theta(h_2)}\right) \cdot \mu\left(h_2\right)\right|^\lambda \leq \left|f^*\left(s_{\theta(h_1)}\right) \cdot \mu\left(h_1\right) - f^*\left(s_{\theta(h_3)}\right) \cdot \mu\left(h_3\right)\right|^\lambda,$$

$$f^*\left(s_{\theta(h_1)}\right) \cdot \left(1-\nu\left(h_1\right)\right) \leq f^*\left(s_{\theta(h_2)}\right) \cdot \left(1-\nu\left(h_2\right)\right) \leq f^*\left(s_{\theta(h_3)}\right) \cdot \left(1-\nu\left(h_3\right)\right),$$

$$\left|f^*\left(s_{\theta(h_1)}\right) \cdot \left(1-\nu\left(h_1\right)\right) - f^*\left(s_{\theta(h_2)}\right) \cdot \left(1-\nu\left(h_2\right)\right)\right|^\lambda \leq \left|f^*\left(s_{\theta(h_1)}\right) \cdot \left(1-\nu\left(h_1\right)\right) - f^*\left(s_{\theta(h_3)}\right) \cdot \left(1-\nu\left(h_3\right)\right)\right|^\lambda.$$

Then

$$\left(\frac{1}{2}\left(\left|f_1^* \cdot \mu\left(h_1\right) - f_2^* \cdot \mu\left(h_2\right)\right|^\lambda + \left|f_1^* \cdot \left(1 - \nu\left(h_1\right)\right) - f_2^* \cdot \left(1 - \nu\left(h_2\right)\right)\right|^\lambda\right)\right)^{1/\lambda}$$
$$\leq \left(\frac{1}{2}\left(\left|f_1^* \cdot \mu\left(h_1\right) - f_2^* \cdot \mu\left(h_3\right)\right|^\lambda + \left|f_1^* \cdot \left(1 - \nu\left(h_1\right)\right) - f_2^* \cdot \left(1 - \nu\left(h_3\right)\right)\right|^\lambda\right)\right)^{1/\lambda}.$$

Therefore, $d_g(h_1, h_2) \leq d_g(h_1, h_3)$ and $d_g(h_2, h_3) \leq d_g(h_1, h_3)$ can be proven in a similar way. This completes the proof of Theorem 2.

In order to illustrate the application condition of Property (3), an example against Property (3) is presented next.

Example 4. Let $h_1 = \langle s_4, 0.2, 0.4\rangle$, $h_2 = \langle s_5, 0.5, 0.6\rangle$, $h_3 = \langle s_4, 0.3, 0.7\rangle$, and $\lambda = 2$. Then $d_g(h_1, h_2) = 0.1917$, $d_g(h_1, h_3) = 0.1375$, and $d_g(h_2, h_3) = 0.1689$. Therefore, $d_g(h_1, h_2) > d_g(h_1, h_3)$ and $d_g(h_2, h_3) > d_g(h_1, h_3)$.

4.2. NLP-based TODIM approach

In some cases, like if decision makers must select an appropriate hotel from a website, they need to take multiple criteria into consideration, including price, service, and location. According to the common methods, the evaluation values of the different criteria would be aggregated directly to obtain the ranking result. However, these aggregations ignore the decision makers' behavioral characteristics, such as reference dependence and loss aversion. Therefore, the TODIM method (Gomes and Lima, 1992) is often used to solve MCDM problems while considering the decision makers' behavior. In these cases, the weights of criteria are not completely certain, and decision makers cannot determine accurate weights for each criterion because of limited knowledge.

Suppose an MCDM problem has m alternatives denoted by $A = \{a_1, a_2, \ldots, a_m\}$; decision makers need to evaluate these alternatives and sort out the preferable alternative. Meanwhile, n criteria, denoted by $C = \{c_1, c_2, \ldots, c_n\}$, can be used to calculate comprehensive assessments. A number of decision makers provide their assessed values $r_{ij} (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ for alternative a_i with respect to criterion c_j. The weight of criterion c_j is w_j, and weights are uncertain and satisfy a set of constraints. Certain weights can be computed based on the NLP model. Based on the analysis above, this paper proposes a NLP-based TODIM approach. The entire steps of the approach are illustrated in Fig. 1, and each step is detailed as follows.

Step 1. Obtain the normative decision matrix D.

For the purpose of considering the assessments holistically and obtaining reasonable results to the greatest extent possible, the assessed values should be processed into ILNs before calculation. The normative decision matrix D can be obtained according to the negation operator in Definition 5.

Step 2. Calculate the relative weight.

The relative weight w_{lj} of criterion c_j to reference criterion c_l can be calculated according to the following expression:

$$w_{lj} = \frac{w_j}{w_l},$$

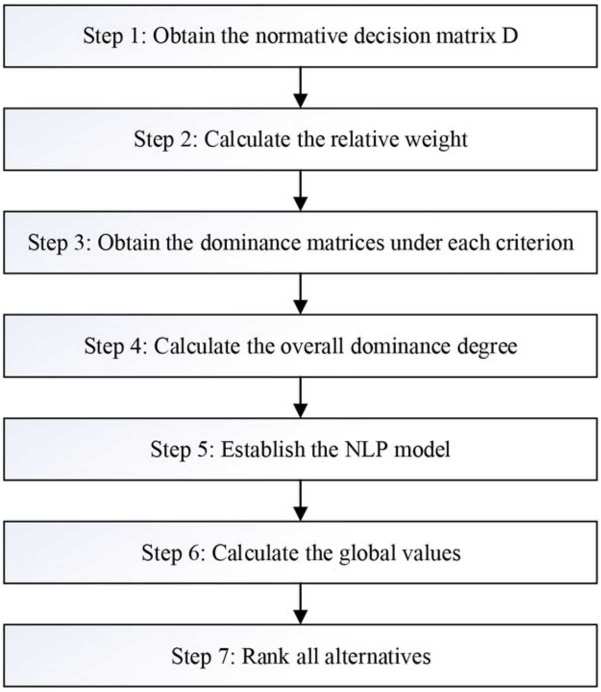

Fig. 1. Flowchart of the proposed approach.

where w_j is the weight value of criterion c_j and w_l is the maximum weight value of the criteria. In addition, note that the criterion weight is an unknown variable rather than a certain value. As a result, the dominance matrices and overall dominance degrees contain unknown variables in the first calculation, and the certain values can be obtained using certain weights obtained by the NLP model.

Step 3. Obtain the dominance matrices under each criterion.

The dominance matrices consist of the dominance degrees of one alternative over other alternative under different criteria. The dominance degree of alternative a_i over alternative a_k concerning criterion c_j can be calculated as follows:

$$\phi_j\left(a_i, a_k\right) = \begin{cases} \sqrt[q]{\left(d_g\left(h_{ij}, h_{kj}\right)\right)^q . w_{lj} / \sum_{j=1}^{n} w_{lj}} & h_{ij} > h_{kj} \\ 0 & h_i = h_j \\ -\dfrac{1}{\theta}\sqrt[q]{\left(d_g\left(h_{ij}, h_{kj}\right)\right)^q . \sum_{j=1}^{n} w_{lj} / w_{lj}} & h_{ij} < h_{kj} \end{cases},$$

where $q \geq 1$ is the regulating variable that can be determined according to the decision maker's preference, $d_g(h_{ij}, h_{kj})$ denotes the distance between h_{ij} and h_{kj} as defined in Definition 12, h_{ij} and h_{kj} can be compared using the ILN comparison method in Definition 11, and parameter θ represents the attenuation factor of the losses. Different values of this parameter lead to different shapes for the prospect value function in the negative quadrant, as illustrated in Figure 2. If $h_{ij} > h_{kj}$, $\phi_j(a_i, a_k)$ represents a gain. If $h_{ij} < h_{kj}$, $\phi_j(a_i, a_k)$ represents a loss. If $h_{ij} = h_{kj}$, $\phi_j(a_i, a_k)$ is nil.

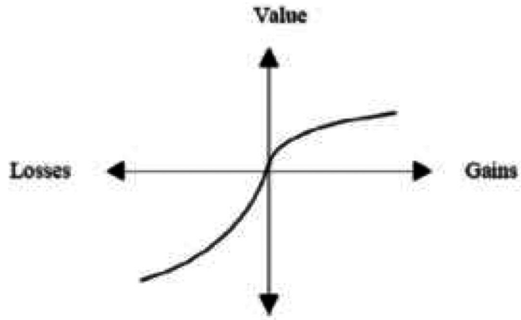

Fig. 2. Prospect value function.

Step 4. Calculate the overall dominance degree.

The overall dominance degree of alternative a_i over alternative a_k concerning criterion c_j can be calculated as follows:

$$\delta\left(a_i, a_k\right) = \sum_{j=1}^{n} \phi_j\left(a_i, a_k\right),$$

where $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, n$.

Step 5. Establish the NLP model.

Because the relative weights are uncertain variables and the overall dominance degrees can be used to order the alternatives, the nonlinear programming model can be constructed by referring to the maximizing deviations principle. This paper allows the difference values between the overall dominance degrees to be as large as possible; in this way, the ranking of alternatives can be obtained more easily. The criteria weights can be obtained using the following programming model. The set of the incomplete certain information on the criteria weights is Ω.

$$\max \sum\nolimits_{i=1}^{n} \sum\nolimits_{k=1}^{n} \delta\left(a_i, a_k\right)$$

$$\text{s.t.} \begin{cases} w \in \Omega \\ \sum\limits_{j=1}^{n} w_j = 1 \\ w_j \geq 0. \end{cases}$$

Step 6. Calculate the global values.

According to the weights calculated in Step 4, the dominance matrices that consist of certain values can be obtained. Then, the global value can be identified as follows:

$$\xi\left(a_i\right) = \frac{\sum\nolimits_{k=1}^{n} \delta\left(a_i, a_k\right) - \min\limits_{i \in n}\left\{\sum\nolimits_{k=1}^{n} \delta\left(a_i, a_k\right)\right\}}{\max\limits_{i \in n}\left\{\sum\nolimits_{k=1}^{n} \delta\left(a_i, a_k\right)\right\} - \min\limits_{i \in n}\left\{\sum\nolimits_{k=1}^{n} \delta\left(a_i, a_k\right)\right\}},$$

where $i = 1, 2, \ldots, n$.

Step 7. Rank all the alternatives.

According to the global value, the ranking result can be obtained, and the best alternative can be selected.

5. Case study

TripAdvisor.com is a famous tourism website that contains more than 300 million travel reviews submitted by real travelers. These reviews represent the honest, unbiased opinions of travelers who have had experiences with various destinations, hotels, scenic spots, and restaurants. If a traveler wants to go on a journey, he or she can choose a destination or hotel based on the latest TripAdvisor reviews. In trying to analyze these reviews, it is easy to discover that although there are more restaurants than hotels, the hotels have more reviews. Statistically, this indicates that people have greater interest in hotels; as a result, hotels are the research subjects in this paper.

Hotel pages on TripAdvisor contain the grades, the types of guests, and the criteria (such as service and cost performance) used to rate the hotel; however, the evaluation values of these criteria cannot be obtained directly through the website, and guests often consider other criteria in the process of selecting a hotel, such as location and convenience. These other criteria are evident in the detailed text comments. Therefore, several criteria can be selected to evaluate these hotels synthetically, and the weights of the criteria cannot be identified from the website. It is important to note that the grades on TripAdvisor hotel pages consist of nine scales. Based on the nine scales and the convenience of linguistic variables, nine-point linguistic terms can be used to assess the alternatives with respect to the different criteria. This approach can be used to identify a suitable hotel based on existing data from TripAdvisor.

According to TripAdvisor, there are five hotels in the city of Chiang Mai: Shangri-La Hotel, Viang Thapae Resort, the Dhara Dhevi, Napatra Hotel, and the Park Hotel. For convenience, the names of these hotels will be replaced hereafter by a_1, a_2, a_3, a_4, and a_5. All of these hotels can be assessed according to the same six criteria: location c_1, sleep quality c_2, comfort level c_3, service c_4, cost performance c_5, and cleanliness c_6. With respect to criterion c_j, text comments can be transformed into suitable linguistic terms for each alternative a_i, with the linguistic terms belonging to the linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$. Suppose the weights of the reviewers are equal, and the weight for cleanliness is greater than the weight for location, while the weight for cost performance is greater than the weight for cleanliness; in other words, $w_1 \leq w_6$ and $w_6 \leq w_5$. The specific conditions of the weights are as follows: $0.05 \leq w_1 \leq 0.1$, $0.1 \leq w_2 \leq 0.2$, $0.15 \leq w_3 \leq 0.2$, $0.1 \leq w_4 \leq 0.15$, $0.25 \leq w_5 \leq 0.3$, $0.2 \leq w_6 \leq 0.25$, and $w_1 + w_2 + ... + w_5 = 1$. Count the numbers of linguistic terms under criterion c_j for each alternative a_i, and choose the most suitable linguistic term to describe the alternative; then, the membership degree and non-membership degree of the linguistic term can be calculated according to the total text comments. The final evaluation values of the alternatives are shown in Table 1.

5.1. Illustration of the NLP-based TODIM approach

In this section, the approach proposed in Section 4 will be used to identify the most desirable alternative.

Table 1
Final evaluation values

	c_1	c_2	c_3	c_4	c_5	c_6
a_1	$\langle s_7, 0.5, 0.4\rangle$	$\langle s_8, 0.6, 0.2\rangle$	$\langle s_7, 0.9, 0.1\rangle$	$\langle s_7, 0.8, 0.2\rangle$	$\langle s_6, 0.5, 0.2\rangle$	$\langle s_7, 0.6, 0.1\rangle$
a_2	$\langle s_8, 0.9, 0.1\rangle$	$\langle s_7, 0.5, 0.4\rangle$	$\langle s_7, 0.6, 0.3\rangle$	$\langle s_6, 0.6, 0.2\rangle$	$\langle s_5, 0.8, 0.2\rangle$	$\langle s_7, 0.7, 0.2\rangle$
a_3	$\langle s_7, 0.6, 0.3\rangle$	$\langle s_8, 0.8, 0.1\rangle$	$\langle s_8, 0.6, 0.2\rangle$	$\langle s_8, 0.7, 0.1\rangle$	$\langle s_7, 0.5, 0.2\rangle$	$\langle s_8, 0.6, 0.3\rangle$
a_4	$\langle s_5, 0.8, 0.2\rangle$	$\langle s_7, 0.6, 0.1\rangle$	$\langle s_7, 0.5, 0.4\rangle$	$\langle s_6, 0.6, 0.3\rangle$	$\langle s_5, 0.6, 0.4\rangle$	$\langle s_6, 0.6, 0.3\rangle$
a_5	$\langle s_5, 0.9, 0.1\rangle$	$\langle s_6, 0.8, 0.2\rangle$	$\langle s_6, 0.6, 0.4\rangle$	$\langle s_6, 0.9, 0.1\rangle$	$\langle s_5, 0.7, 0.2\rangle$	$\langle s_6, 0.6, 0.3\rangle$

Step 1. Obtain the normative decision matrix D.

The normative decision matrix D can be obtained according to the negation operator in Definition 5.

$$D = \begin{bmatrix} \langle s_7, 0.5, 0.4\rangle & \langle s_8, 0.6, 0.2\rangle & \langle s_7, 0.9, 0.1\rangle & \langle s_7, 0.8, 0.2\rangle & \langle s_6, 0.5, 0.2\rangle & \langle s_7, 0.6, 0.1\rangle \\ \langle s_8, 0.9, 0.1\rangle & \langle s_7, 0.5, 0.4\rangle & \langle s_7, 0.6, 0.3\rangle & \langle s_6, 0.6, 0.2\rangle & \langle s_5, 0.8, 0.2\rangle & \langle s_7, 0.7, 0.2\rangle \\ \langle s_7, 0.6, 0.3\rangle & \langle s_8, 0.8, 0.1\rangle & \langle s_8, 0.6, 0.2\rangle & \langle s_8, 0.7, 0.1\rangle & \langle s_7, 0.5, 0.2\rangle & \langle s_8, 0.6, 0.3\rangle \\ \langle s_5, 0.8, 0.2\rangle & \langle s_7, 0.6, 0.1\rangle & \langle s_7, 0.5, 0.4\rangle & \langle s_6, 0.6, 0.3\rangle & \langle s_5, 0.6, 0.4\rangle & \langle s_6, 0.6, 0.3\rangle \\ \langle s_5, 0.9, 0.1\rangle & \langle s_6, 0.8, 0.2\rangle & \langle s_6, 0.6, 0.4\rangle & \langle s_6, 0.9, 0.1\rangle & \langle s_5, 0.7, 0.2\rangle & \langle s_6, 0.6, 0.3\rangle \end{bmatrix}.$$

Step 2. Calculate the relative weight.

The relative weight w_{lj} can be calculated as follows:

$$w_l = w_5, \quad w_{l1} = \frac{w_1}{w_5}, \quad w_{l2} = \frac{w_2}{w_5}, \quad w_{l3} = \frac{w_3}{w_5}, \quad w_{l4} = \frac{w_4}{w_5}, \quad w_{l5} = 1, \quad \text{and } w_{l6} = \frac{w_6}{w_5}.$$

Step 3. Obtain the dominance matrices under each criterion.

Let $\theta = 2$, $\lambda = 2$, $q = 2$, and $f^* = f_1(s_x)$, then the dominance degree of alternative a_i over alternative a_k concerning criterion c_1 can be calculated as follows:

$$\phi_1(a_i, a_k) = \begin{bmatrix} 0 & -0.1859\sqrt{1/w_1} & -0.0438\sqrt{1/w_1} & 0.1651\sqrt{w_1} & 0.2217\sqrt{w_1} \\ 0.3718\sqrt{w_1} & 0 & 0.289\sqrt{w_1} & 0.2834\sqrt{w_1} & 0.2401\sqrt{w_1} \\ 0.0875\sqrt{w_1} & -0.1445\sqrt{1/w_1} & 0 & 0.0988\sqrt{w_1} & 0.1439\sqrt{w_1} \\ -0.0826\sqrt{1/w_1} & -0.1417\sqrt{1/w_1} & -0.0494\sqrt{1/w_1} & 0 & -0.0313\sqrt{1/w_1} \\ -0.1108\sqrt{1/w_1} & -0.1201\sqrt{1/w_1} & -0.0719\sqrt{1/w_1} & 0.0625\sqrt{w_1} & 0 \end{bmatrix},$$

$$\phi_2(a_i, a_k) = \begin{bmatrix} 0 & 0.1564\sqrt{w_2} & -0.0791\sqrt{1/w_2} & 0.0956\sqrt{w_2} & 0.0354\sqrt{w_2} \\ -0.0782\sqrt{1/w_2} & 0 & -0.1557\sqrt{1/w_2} & -0.0978\sqrt{1/w_2} & 0.1822\sqrt{w_2} \\ 0.1581\sqrt{w_2} & 0.3114\sqrt{w_2} & 0 & 0.1947\sqrt{w_2} & 0.1458\sqrt{w_2} \\ -0.0478\sqrt{1/w_2} & 0.1957\sqrt{w_2} & -0.0973\sqrt{1/w_2} & 0 & 0.069\sqrt{w_2} \\ -0.0177\sqrt{1/w_2} & -0.0911\sqrt{1/w_2} & -0.0729\sqrt{1/w_2} & -0.0345\sqrt{1/w_2} & 0 \end{bmatrix},$$

$$\phi_3(a_i,a_k)=\begin{bmatrix} 0 & 0.2231\sqrt{w_3} & -0.0773\sqrt{1/w_3} & 0.3094\sqrt{w_3} & 0.282\sqrt{w_3} \\ -0.1115\sqrt{1/w_3} & 0 & -0.0345\sqrt{1/w_3} & 0.0875\sqrt{w_3} & 0.0593\sqrt{w_3} \\ 0.1546\sqrt{w_3} & 0.069\sqrt{w_3} & 0 & 0.1564\sqrt{w_3} & 0.1275\sqrt{w_3} \\ -0.1547\sqrt{1/w_3} & -0.0438\sqrt{1/w_3} & -0.0782\sqrt{1/w_3} & 0 & 0.0364\sqrt{w_3} \\ -0.141\sqrt{1/w_3} & -0.0296\sqrt{1/w_3} & -0.0637\sqrt{1/w_3} & -0.0182\sqrt{1/w_3} & 0 \end{bmatrix},$$

$$\phi_4(a_i,a_k)=\begin{bmatrix} 0 & 0.1777\sqrt{w_4} & -0.0265\sqrt{1/w_4} & 0.1803\sqrt{w_4} & 0.0729\sqrt{w_4} \\ -0.0888\sqrt{1/w_4} & 0 & -0.0901\sqrt{1/w_4} & 0.053\sqrt{w_4} & -0.0839\sqrt{1/w_4} \\ 0.053\sqrt{w_4} & 0.1803\sqrt{w_4} & 0 & 0.1976\sqrt{w_4} & 0.025\sqrt{w_4} \\ -0.0901\sqrt{1/w_4} & -0.0265\sqrt{1/w_4} & -0.0988\sqrt{1/w_4} & 0 & -0.0956\sqrt{1/w_4} \\ -0.0364\sqrt{1/w_4} & 0.1677\sqrt{w_4} & -0.0125\sqrt{1/w_4} & 0.1912\sqrt{w_4} & 0 \end{bmatrix},$$

$$\phi_5(a_i,a_k)=\begin{bmatrix} 0 & 0.0901\sqrt{w_5} & -0.0238\sqrt{1/w_5} & 0.0707\sqrt{w_5} & 0.0476\sqrt{w_5} \\ -0.0451\sqrt{1/w_5} & 0 & -0.0283\sqrt{1/w_5} & 0.125\sqrt{w_5} & 0.0442\sqrt{w_5} \\ 0.0476\sqrt{w_5} & 0.0566\sqrt{w_5} & 0 & 0.069\sqrt{w_5} & 0.0354\sqrt{w_5} \\ -0.0354\sqrt{1/w_5} & -0.0625\sqrt{1/w_5} & -0.0345\sqrt{1/w_5} & 0 & -0.0494\sqrt{1/w_5} \\ -0.0238\sqrt{1/w_5} & -0.0221\sqrt{1/w_5} & -0.0177\sqrt{1/w_5} & 0.0988\sqrt{w_5} & 0 \end{bmatrix},$$

$$\phi_6(a_i,a_k)=\begin{bmatrix} 0 & -0.0437\sqrt{1/w_6} & -0.0797\sqrt{1/w_6} & 0.1108\sqrt{w_6} & 0.1108\sqrt{w_6} \\ 0.0875\sqrt{w_6} & 0 & -0.0444\sqrt{1/w_6} & 0.1202\sqrt{w_6} & 0.1202\sqrt{w_6} \\ 0.1593\sqrt{w_6} & 0.0888\sqrt{w_6} & 0 & 0.1186\sqrt{w_6} & 0.1186\sqrt{w_6} \\ -0.0554\sqrt{1/w_6} & -0.0601\sqrt{1/w_6} & -0.0593\sqrt{1/w_6} & 0 & 0 \\ -0.0554\sqrt{1/w_6} & -0.0601\sqrt{1/w_6} & -0.0593\sqrt{1/w_6} & 0 & 0 \end{bmatrix}.$$

Step 4. Calculate the overall dominance degree.
The overall dominance degree of alternative a_1 over alternative a_2 can be calculated as follows:

$$\delta\left(a_1,a_2\right)=-0.1859\sqrt{1/w_1}+0.1564\sqrt{w_2}+0.2231\sqrt{w_3}+0.1777\sqrt{w_4}+0.0901\sqrt{w_5}\\ -0.0437\sqrt{1/w_6}.$$

The other overall dominance degrees can be obtained in a similar way.
Step 5. Establish the NLP model.

$$\max F=1.9638\sqrt{w_1}-0.9819\sqrt{1/w_1}+1.5442\sqrt{w_2}-0.7721\sqrt{1/w_2}+1.5052\sqrt{w_3}\\ -0.7526\sqrt{1/w_3}+1.2987\sqrt{w_4}-0.6494\sqrt{1/w_4}+0.685\sqrt{w_5}-0.3425\sqrt{1/w_5}\\ +1.0348\sqrt{w_6}-0.5174\sqrt{1/w_6};$$

$$\text{s.t.}\begin{cases} 0.05 \le w_1 \le 0.1, \\ 0.1 \le w_2 \le 0.2, \\ 0.15 \le w_3 \le 0.2, \\ 0.1 \le w_4 \le 0.15, \\ 0.25 \le w_5 \le 0.3, \\ 0.2 \le w_6 \le 0.1, \\ w_1 \le w_6, \\ w_6 \le w_5, \\ w_1 + w_2 + \cdots + w_n = 1. \end{cases}$$

Thus, the weight vector of the criteria is $W = (w_1, w_2, w_3, w_4, w_5, w_6) = (0.1, 0.1577, 0.1546, 0.1377, 0.25, 0.2)$.

Step 6. Calculate the global values.

The global values are $\xi(a_1) = 0.6984$, $\xi(a_2) = 0.5044$, $\xi(a_3) = 1$, $\xi(a_4) = 0$, and $\xi(a_5) = 0.2197$.

Step 7. Rank all the alternatives.

The ranking result is $a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$, which means that the best hotel is the Dhara Dhevi.

5.2. The influence of parameters and linguistic functions

This subsection defines some different values of the parameters θ, λ, and q in order to illustrate their influence on decision making according to the proposed approach. Let $f^* = f_1(s_x)$; then, the ranking results are shown in Table 2.

In order to illustrate the influence of linguistic functions, let $\theta = 2$, $\lambda = 2$, and $q = 2$, then the ranking results are listed in Table 3.

Tables 2 and 3 show that the ranking results are stable, and the values of θ, λ, q, and the linguistic functions seem to make no difference. According to the NLP model, the criteria weights are obtained by maximizing the difference between the overall dominance degrees. However, the global values are calculated based on the overall dominance degrees, such that the differences between alternatives are large, and the impact of the parameters and linguistic functions turns out to be small as a result. Even so, losses and gains change as the parameters change, making the parameters and linguistic functions meaningful to the proposed approach.

5.3. A comparison among different methods

In this subsection, different methods are used to solve the same MCDM problem in order to verify the feasibility and superiority of the approach proposed in this paper. Then, a comparison analysis is conducted based on the same case.

(1) In Liu (2013), the intuitionistic linguistic generalized weighted average (ILGWA) operator is defined to aggregate the values of the alternatives; the weights of the criteria are same as the weights in Section 5.1. The comprehensive value of each alternative is calculated as

Table 2
Ranking results based on different parameters

θ	λ	q	Ranking results
0.5	1	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
	2	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
1	1	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
	2	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
2	1	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
	2	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
10	1	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
	2	1	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		2	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
		10	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$

Table 3
Ranking results based on different linguistic functions

Linguistic functions	Ranking results
$f_1(s_x)$	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
$f_2(s_y)$	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
$f_3(s_z)$	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$

follows: $z_1 = \langle s_{6.9049}, 0.6835, 0.1676\rangle$, $z_2 = \langle s_{6.4602}, 0.7138, 0.2213\rangle$, $z_3 = \langle s_{7.65}, 0.6351, 0.1841\rangle$, $z_4 = \langle s_{5.9602}, 0.6136, 0.2731\rangle$, and $z_5 = \langle s_{5.65}, 0.7602, 0.2046\rangle$.

Then the score function can be obtained as follows:

$$S\left(z_1\right) = 0.6542,\ S\left(z_2\right) = 0.6026,\ S\left(z_3\right) = 0.6938,\ S\left(z_4\right) = 0.4994,\ \text{and}\ S\left(z_5\right) = 0.5493.$$

Thus, the ranking result of these hotels is $a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$.

(2) Su et al. (2014) defined the normalized Hamming distance between two ILNs. Based on this distance measure, AILOWAD operator is defined, and the weights of the criteria are same

Table 4
Evaluation values of ideal hotel

	c_1	c_2	c_3	c_4	c_5	c_6
a	$\langle s_8, 1, 0\rangle$	$\langle s_8, 1, 0\rangle$	$\langle s_8, 1, 0\rangle$	$\langle s_8, 1, 0\rangle$	$\langle s_8, 1, 0\rangle$	$\langle s_8, 1, 0\rangle$

Table 5
Individual distances

	c_1	c_2	c_3	c_4	c_5	c_6
a_1	0.51875	0.3	0.2125	0.3	0.5125	0.34375
a_2	0.1	0.51875	0.43125	0.475	0.5	0.34375
a_3	0.43125	0.15	0.3	0.2	0.43125	0.35
a_4	0.5	0.34375	0.51875	0.5125	0.625	0.5125
a_5	0.4375	0.4	0.55	0.325	0.53125	0.5125

Table 6
Ranking results based on different methods

Method	Ranking results
ILGWA operator in Liu (2013)	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
AILOWAD operator in Su et al. (2014)	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$
NLP-based TODIM approach	$a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$

as the weights in Section 5.1. According to the final evaluation values of the alternatives in Table 1, the evaluation values for the ideal hotel are shown in Table 4. Then the individual distances are calculated and shown in Table 5.

According to the AILOWAD operator, each aggregate value of these alternatives is listed as follows.

$$AD_1 = 0.3702,\ AD_2 = 0.4175,\ AD_3 = 0.3187,\ AD_4 = 0.5142,\ \text{and}\, AD_5 = 0.4719.$$

Therefore, the ranking result is $a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$.

The ranking results are shown together in Table 6. Based on Table 6, it is obvious that the ranking result is always $a_3 \succ a_1 \succ a_2 \succ a_5 \succ a_4$. This result conforms to the ranking of these hotels on TripAdvisor, proving the feasibility of the NLP-based TODIM approach. According the computational process of the method in Liu (2013), the ILGWA operator directly aggregates the comprehensive values, such that the differences under different criteria seem to be the same. The method in Su et al. (2014) is superior to the method in Liu (2013) for considering the distances between these hotels and the ideal hotel with respect to the six criteria. The distance between any two ILNs should satisfy the triangle inequality; however, the proof of this significant property in Su et al. (2014) is missing, such that the result based on the method in Su et al. (2014) seems unreliable.

Based on the above analysis, the advantages of the approach proposed in this paper can be summarized as follows:

(1) It is convenient to use linguistic terms to evaluate hotels along different criteria. According to the statistics, the ILNs can contain the major evaluation values from reviewers on TripAdvisor, making the evaluation values close to reality. This method utilizes linguistic scale functions; furthermore, the linguistic terms can be turned into different values under different semantic environments, which can improve the flexibility of the proposed approach. This paper defines new operations and a new comparison method, and the necessary proofs are provided in Section 3. Since the proposed approach is calculated based on the new operations and the new comparison method, the proposed approach's computation results are more reliable.
(2) The NLP model was established to obtain the weights of criteria in practical problems based on the maximizing deviations principle. The ranking result not only shows that the NLP model can overcome the subjectivity of traditional methods in determining weight coefficients, which makes evaluation results more reasonable and scientific, but also results in the proposed approach being better able to distinguish among alternatives. Compared with existing methods that use aggregation operators, the proposed approach pays attention to the differences between each alternative under each criterion. As a result, the NLP-based TODIM approach is more meaningful in addressing practical matters.
(3) As a classic method, TODIM has been used to solve various MCDM problems with different FSs. This paper extends the TODIM method into ILNs to solve MCDM problems. The proposed approach considers risk preferences of decision makers, distance measurements, and semantics involved in complex practical problems. The risk preference can be reflected by the attenuation factor of the losses θ in the process of obtaining the dominance matrices under each criterion. If the decision maker is quite sensitive to losses, then θ can be assigned a small value. However, it may be difficult for decision makers to select appropriate parameters due to their limited knowledge; in this situation, the linguistic scale function $f_1(s_x)$ can be selected for simplicity. Similarly, a Hamming distance measurement of $\lambda = 1$, risk neutral of $\theta = 1$, and $q = 1$ can be employed by default.

6. Conclusions

This paper proposes a novel TODIM approach based on the NLP in order to assess alternatives with respect to various criteria in MCDM problems. Usually, linguistic values can more conveniently express the preferences of decision makers, but the calculation becomes complex when the number of linguistic values is large. Based on statistics, many linguistic values can be put together and transformed into ILNs in order to retain the fuzziness of the original evaluation information. For the purpose of obtaining logical results, new operations and a new comparison method for ILNs are introduced before defining generalized distance between ILNs. Then the NLP-based TODIM approach is proposed using the generalized distance.

The proposed approach not only can be used to deal with situations with uncertain criteria weights but also considers the decision makers' risk preferences, as demonstrated in the illustrative example and comparative analysis. However, this approach still has some limitations. For one, the proposed approach based on the TODIM method cannot handle problems in which the evaluation values are not preprocessed before calculation. Furthermore, existing parameters in computing can cause undiscovered problems. In future research, we will study these problems and propose corresponding approaches with ILNs.

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