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Semantifying formal concept analysis using description logics☆

Yuncheng Jiang

School of Computer Science, South China Normal University, Guangzhou 510631, China

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ABSTRACT

Formal Concept Analysis (FCA) is a field of applied mathematics with its roots in order theory, in particular the theory of complete lattices. Over the past 20 years, FCA has been widely studied. Description Logics (DLs) are a family of knowledge representation languages which can be used to represent the terminological knowledge of an application domain in a structured and formally well-understood way. Nowadays, properties and semantics of ontology constructs mainly are determined by DLs. The current research progress and the existing problems of FCA are analyzed. In this paper, we semantify FCA with DLs, in other words, we present an extended FCA (i.e., semantic FCA) by using the concepts of DLs to act as the attributes of formal contexts. Furthermore, we semantify the three components (i.e., formal concepts, attribute implications, and concept lattices) of traditional FCA. In addition, we also study the attribute reduction of formal contexts, formal concepts, and concept lattices from a semantics point of view.

1. Introduction

Formal Concept Analysis (FCA) [1] is a field of applied mathematics with its roots in order theory, in particular the theory of complete lattices. FCA emerged in the 1980s from efforts to restructure lattice theory with the purpose of providing a lattice-theoretic formalization of the notions of a concept and a conceptual hierarchy [2]. Over the past 20 years, FCA has been widely studied [3–8]. As an effective mathematical tool for conceptual data analysis and knowledge processing, FCA has extensively been applied to various fields such as information retrieval [9–11], data mining [12–16], ontology engineering [17–19], services computing [20], Semantic Web [21,22], software engineering [23–26], and description logics [27–30]. In essence, FCA is based on a formalization of the philosophical understanding of a concept as a unit of thought constituted by its extent and intent. The extent of a concept is understood as the collection of all objects belonging to the concept and the intent as the multitude of all attributes common to all those objects [31].

In the basic setting of FCA, the input data consist of a table describing objects and attributes, and their relationship. Concretely, in FCA data is represented in a binary matrix called a formal context. A formal context is a simple way of specifying which objects have which attributes. Formally, a formal context is defined as a triple (G, M, J) consisting of a set G of objects, a set M of attributes, and a binary relation $J \subseteq G \times M$ indicating that each object of G has what attributes in M [32]. A formal context in FCA corresponds to an information system in rough set theory [33,34] and soft set theory [35–37]. Obviously, a background knowledge that a user may have regarding the input data is not taken into account in the basic setting. As pointed out by Belohlavek and Vychodil [38], the background knowledge may reflect additional information regarding the data or a particular type of knowledge that a user is looking for in the data. An absence of background knowledge may result in extraction of a large number of formal concepts including those that may seem artificial, and hence, not interesting to the user because they are not congruent with his background knowledge. On the other hand, an appropriate treatment of background knowledge may result in a focused knowledge extraction. Therefore, it is necessary to add some background knowledge to formal contexts.

One particular type of background knowledge that Belohlavek and Vychodil deal with in [38] is relative importance of attributes. Such type of background knowledge is commonly used in human categorization. Roughly speaking, they present an approach to modeling background knowledge that represents user's priorities regarding attributes and their relative importance. Such priorities serve as a constraint, only those formal concepts that are compatible with user's priorities are considered relevant, extracted from data, and presented to the user. In this paper we will consider another particular type of background knowledge.

It is well-known that ontologies provide a formal specification of a shared conceptualization [39]. Being machine readable and constructed from the consensus of a community of users or

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E-mail address: ycjiang@scnu.edu.cn.

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domain experts, they represent a very reliable and structured knowledge source. Due to this reason, and thanks to initiatives such as the Semantic Web, which brought the creation of thousands of domain ontologies [40], ontologies have been extensively exploited in knowledge-based systems [41]. The Web Ontology Language (OWL [42]) which comprises three sublanguages of increasing expressive power: OWL Lite, OWL DL and OWL Full, and its revision OWL 2 [43], are well-known languages for ontology modeling. Nowadays, properties and semantics of ontology constructs mainly are determined by Description Logics (DLs) [44], a class of knowledge representation formalisms in the tradition of semantic networks and frames, which can be used to represent the terminological knowledge of an application domain in a structured and formally well-understood way. DL systems provide their users with inference services (like computing the subsumption hierarchy) that deduce implicit knowledge from the explicitly represented knowledge. DLs have proved to be very useful as ontology languages. For instance, OWL Lite, OWL DL and OWL 2 are close equivalents to $\mathcal{SHIF}(\mathbf{D})$, $\mathcal{SHOIN}(\mathbf{D})$ and $\mathcal{SROIQ}(\mathbf{D})$, respectively [44,45].

Obviously, the attributes in formal contexts are very simple. In other words, each attribute is only a word or a term, and expressive (or complex) attributes are not considered in formal contexts. From a semantic point of view, the attributes of formal contexts have not semantics. In order to extend the expressive power of FCA, we will consider the semantics of attributes, that is, we will semantify FCA. Concretely, we will extend formal concept analysis with DLs, in other words, we use the concepts of DLs to act as the attributes of formal contexts and present a kind of semantic Formal Concept Analysis (semantic FCA). On the other hand, since the ontology languages such as OWL Lite, OWL DL, and OWL 2 are equivalent to DLs [44,45], therefore, from the ontology point of view, the semantic formal concept analysis presented in this paper can also be called as ontology-based formal concept analysis. It should be noted that both the type of the background knowledge and the aims in [46] are different from those that are discussed in this paper. Ganter uses attribute implications for the purpose of attribute exploration, where attribute exploration can help users to explore the implicational logic of FCA attributes (see especially [46] for further details). Concretely, in [46] Ganter proposes an exploration procedure in which this procedure allow users to provide arbitrary (propositional) "background knowledge". Clearly, FCA attributes in [46] are still not defined, that is, the attributes of formal contexts considered in [46] have still not semantics. Technically, in this paper we semantify FCA using DLs. In fact, there are also some research on applying FCA to DLs. Baader and Sertkaya [47] describe two cases where attribute exploration from FCA can be used to compute an extended subsumption hierarchy in DL (the hierarchy of conjunctions of concepts defined in a terminology, and the hierarchy of all least common subsumers of a finite set of concept descriptions). Baader et al. [28] extend the attribute exploration algorithm from FCA to partial contexts, and show how the extended algorithm can be used to complete DL knowledge bases, using both DL reasoning and answers given by a domain expert. That is, the extended DL knowledge base is complete in a certain, well-defined sense.

The rest of this paper is organized as follows. The following section briefly reviews some background on ontologies, DLs and FCA. Next, in Section 3, we provide some motivating examples. In Section 4, we extend FCA with DLs, i.e., present a kind of semantic FCA and discuss its properties. Section 5 discusses attribute reduction of semantic FCA. Finally, in Section 6, we provide some discussion and draw the conclusion.

2. Preliminaries

For completeness of presentation and convenience of subsequent discussions, in the current section we will briefly recall some basic notions of ontologies, DLs and FCA. See especially [1,44] for further details and properties.

2.1. Ontologies and description logics

Ontologies, defined as "formal, explicit specifications of a shared conceptualization" [48], encode machine-interpretable descriptions of the concepts and the relations in a domain using abstractions as class, role or instance, which are qualified using logical axioms. Nowadays, properties and semantics of ontology constructs mainly are determined by Description Logics (DLs) [33,44], a family of logics for representing structured knowledge which have proved to be very useful as ontology languages [49]. Formally, an ontology is a triple $O = \langle \mathcal{RB}, \mathcal{TB}, \mathcal{AB} \rangle$, where $\mathcal{RB}$ (the Role Box or RBox) and $\mathcal{TB}$ (the Terminological Box or TBox) comprise the intensional knowledge, i.e., general knowledge about the world to be described (statements about roles and concepts, respectively), and $\mathcal{AB}$ (the Assertional Box or ABox) the extensional knowledge, i.e., particular knowledge about a specific instantiation of this world (statements about individuals in terms of concepts and roles) [49].

In the following, we introduce the DL $\mathcal{ALC}$ [50], which is a significant representative of DLs. It should be noted that the ontology-based FCA (see Section 4) is not restricted to $\mathcal{ALC}$. It applies to arbitrary (decidable) DLs.

We assume three alphabets of symbols, called atomic concepts (denoted by A), atomic roles (denoted by R) and individuals (denoted by a and b).

A concept (denoted by C and D) of the language $\mathcal{ALC}$ is built out of atomic concepts according to the following syntax rules:

$$
\begin{aligned}
C, D \rightarrow \; & \top \mid \text{(top concept)} \\
& \bot \mid \text{(bottom concept)} \\
& A \mid \text{(atomic concept)} \\
& \neg C \mid \text{(concept negation)} \\
& C \sqcap D \mid \text{(concept conjunction)} \\
& C \sqcup D \mid \text{(concept disjunction)} \\
& \exists R.C \mid \text{(existential quantification)} \\
& \forall R.C \mid \text{(universal quantification).}
\end{aligned}
$$

From a semantic point of view, concepts are interpreted as subsets of an abstract domain, while roles are interpreted as binary relations over such a domain. More precisely, an interpretation $I = (\Delta^I, \bullet^I)$ consists of a domain of interpretation Δ^I, and an interpretation function $\bullet^I$ mapping every atomic concept A to a subset of Δ^I and every atomic role R to a subset of $\Delta^I \times \Delta^I$.

The interpretation function $\bullet^I$ is extended to complex concepts of $\mathcal{ALC}$ (note that in $\mathcal{ALC}$ roles are always atomic) as follows:

- $A^I \subseteq \Delta^I$;
- $R^I \subseteq \Delta^I \times \Delta^I$;
- $\top^I = \Delta^I$;
- $\bot^I = \phi$;
- $(\neg C)^I = \Delta^I / C^I$;
- $(C \sqcap D)^I = C^I \cap D^I$;
- $(C \sqcup D)^I = C^I \cup D^I$;
- $(\exists R.C)^I = \{x \in \Delta^I | \exists y \in \Delta^I, \langle x, y \rangle \in R^I \wedge y \in C^I\}$;
- $(\forall R.C)^I = \{x \in \Delta^I | \forall y \in \Delta^I, \langle x, y \rangle \in R^I \rightarrow y \in C^I\}$.

An $\mathcal{ALC}$ TBox $\mathcal{TB}$ consists of a finite set of General Concept Inclusion (GCI) axioms of the form $C \sqsubseteq D$, which means that concept C is more specific than D, i.e., D subsumes C. A concept definition $C \equiv D$ (C and D are equivalent) is an abbreviation of the pair of axioms $C \sqsubseteq D$ and $D \sqsubseteq C$. In $\mathcal{TB}$ the interpretations of concepts can be restricted to the models of $\mathcal{TB}$. Based on this model-theoretic semantics, concepts can be checked for unsatisfiability: whether they are necessarily interpreted as the empty set. Another useful semantic implication is subsumption of two concepts C and D (a subset relation C^I and D^I w.r.t. all models I of $\mathcal{TB}$) denoted by $\mathcal{TB} \models C \sqsubseteq D$.

An $\mathcal{ALC}$ ABox $\mathcal{AB}$ consists of a finite set of assertions about individuals. An assertion is either a concept assertion $C(a)$ (meaning that a is an instance of C) or a role assertion $R(a, b)$ (meaning (a, b) is an instance of R). The semantics is a straightforward extension of the previous definition: an interpretation I is a model for an assertion $C(a)$ and $R(a, b)$ if and only if $a^I \in C^I$ and $(a^I, b^I) \in R^I$.

In $\mathcal{ALC}$ there is no RBox $\mathcal{RB}$, since no axioms involving roles are allowed. In more expressive DLs such as $\mathcal{SROIQ}$ [45], $\mathcal{RB}$ consists of a finite set of role axioms stating restrictions as subsumption, transitivity, cardinality, etc [49].

A DL ontology not only stores axioms and assertions, but also offers some reasoning services, such as KB satisfiability (or consistency), concept satisfiability, subsumption or instance checking.

Regarding more expressive DLs, the interested reader is referred to the handbook [44].

2.2. Formal concept analysis

Formal concept analysis (FCA), invented by R. Wille, is not only a method for data analysis and knowledge representation, but also a formal formulation for concept formation and learning [51,52]. Practically, FCA is a conceptual framework and starts with a formal context containing values of 0 and 1 in an information system. In the following, we introduce basic definitions and ideas of FCA [1,53].

Definition 1. A formal context $K = (G, M, J)$ consists of two sets G and M and a binary relation J between G and M. The elements of G are called the objects and the elements of M are called the attributes of the context. gJm or $(g, m) \in J$ can be used to express that an object g is in a relation J with an attribute m and is read as "the object g has the attribute m".

A small context can be easily represented by a cross table, i.e., by a rectangular table the rows of which are headed by the object names and the columns headed by the attribute names. A cross in row g and column m means that the object g has the attribute m.

Definition 2. For a set $A \subseteq G$ of objects we define $A' = \{m \in M | \, gJm$ for all $g \in A\}$ (the set of attributes common to the objects in A). Correspondingly, for a set B of attributes we define $B' = \{g \in G | \, gJm$ for all $m \in B\}$ (the set of objects which have all attributes in B).

Definition 3. A formal concept of the formal context (G, M, J) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$ and $B' = A$. We call A the extent and B the intent of the formal concept (A, B). $\mathcal{B}(G, M, J)$ denotes the set of all formal concepts of the formal context (G, M, J).

Definition 4. If (A_1, B_1) and (A_2, B_2) are formal concepts of the formal context $K = (G, M, J)$, (A_1, B_1) is called a subconcept of (A_2, B_2) (or (A_2, B_2) is a superconcept of (A_1, B_1)), provided that $A_1 \subseteq A_2$ (or $B_2 \subseteq B_1$) and is denoted by $(A_1, B_1) \leq (A_2, B_2)$. The relation $\leq$ is called the hierarchical order (or simply order) of the formal concepts. The set of all formal concepts of (G, M, J) ordered in this way is denoted by $\mathcal{B}(G, M, J)$ and is called the concept lattice of the formal context (G, M, J).

Table 1
The Research Papers context.

	FCA	*Ontology*	*DLs*	*OWL*	*KR*	*DM*
P_1	×	×	×			×
P_2		×		×	×	
P_3	×	×	×		×	
P_4	×		×	×	×	×
P_5	×	×		×		×
P_6			×	×	×	

The extent A and the intent B of a formal concept (A, B) are closely connected by the relation J. Each of the two parts determines the other and thereby the concept, since $A' = B$ and $B' = A$, respectively. We have the following properties:

Theorem 1. *If (G, M, J) is a context, $A, A_1, A_2 \subseteq G$ are sets of objects and $B, B_1, B_2 \subseteq M$ are sets of attributes, then*
(1) $A_1 \subseteq A_2 \Rightarrow A_2{'} \subseteq A_1{'}$, $B_1 \subseteq B_2 \Rightarrow B_2{'} \subseteq B_1{'}$;
(2) $A \subseteq A''$, $B \subseteq B''$;
(3) $A' = A'''$, $B' = B'''$;
(4) $A \subseteq B' \Leftrightarrow B \subseteq A' \Leftrightarrow A \times B \subseteq J$.
The basic theorem on concept lattices is as follows (see more details in [1]):

Theorem 2. *Let T be an index set and, for every $t \in T$. The concept lattice $\mathcal{B}(G, M, J)$ is a complete lattice in which infimum and supermum are given by:*

$$\wedge_{t\in T} (A_t, B_t) = (\cap_{t\in T} A_t, (\cup_{t\in T} B_t)''),$$
$$\vee_{t\in T} (A_t, B_t) = ((\cup_{t\in T} A_t)'', \cap_{t\in T} B_t).$$

3. Motivating examples

In order to illustrate semantic operations for formal concept analysis using domain ontologies (i.e., description logic knowledge bases), we provide some motivating examples.

Example 1. Consider a formal context (G, M, J) named Research Papers. Suppose that the set G is defined by the following six objects representing six different papers:

$$G = \{P_1, P_2, \ldots, P_6\}$$

and the set M is defined by six possible attributes (i.e., keywords) of these objects:

$$M = \{FCA, Ontology, DLs, OWL, KR, DM\}$$

where *FCA* stands for Formal Concept Analysis, *DLs* stands for Description Logics, *OWL* stands for Web Ontology Language, *KR* stands for Knowledge Representation, and *DM* stands for Data Mining.

Furthermore, suppose the papers are related to the above attributes according to the binary relation J defined by Table 1.

According to Table 1, we say that, for example, the paper P_3 has, or is described by, four attributes (or keywords), namely, *FCA*, *Ontology*, *DLs*, and *KR*, and vice versa, these attributes apply to the object (i.e., paper) P_3. A formal concept of the Research Papers context is, for instance:

$$(\{P_3, P_4\}, \{FCA, DLs, KR\})$$

since both P_3 and P_4 have the attributes *FCA*, *DLs*, and *KR*, and vice versa, all these attributes apply to both the objects P_3, P_4. Furthermore, we can construct the Concept Lattice from the formal context of Table 1.

Obviously, the formal context Research Papers is correct from a syntax point of view. However, it is not correct from a semantics point of view since there are some semantic relations

Table 2
The correct Research Papers context under ontology.

	FCA	*Ontology*	*DLs*	*OWL*	*KR*	*DM*
P_1	×	×	×		×	×
P_2		×	×	×	×	
P_3	×	×	×		×	
P_4	×	×	×	×	×	×
P_5	×	×	×	×	×	×
P_6		×	×	×	×	

Table 3
The Purchase Houses context.

	expensive	*beautiful*	*large*	*cheap*	*wooden*
H_1	×	×	×	×	×
H_2		×	×	×	
H_3	×		×		×
H_4		×		×	×
H_5	×	×	×	×	

Table 4
The correct Purchase Houses context under ontology.

	expensive	*beautiful*	*large*	*wooden*
H_1	×	×	×	×
H_2		×	×	
H_3	×		×	×
H_4		×		×
H_5	×	×	×	

between the attributes of the formal context Research Papers under domain ontology. For example, some semantic relations are as follows: $OWL \sqsubseteq DLs$, $OWL \sqsubseteq Ontology$, $DLs \sqsubseteq KR$. Under these semantic relations, it is easy to know that there are some incorrect (or inappropriate) relations in J. For instance, $(P_4, OWL) \in J$ and $(P_4, Ontology) \notin J$ are two incorrect relations, since $OWL \sqsubseteq Ontology$ and $(P_4, OWL) \in J$, we must have $(P_4, Ontology) \in J$. According to the above semantic relations, we can obtain the following correct Research Papers context (see Table 2, we also can get another correct context, i.e., delete some relations).

We have shown that $(\{P_3, P_4\}, \{FCA, DLs, KR\})$ is a formal concept under the formal context of Table 1, however, it is not a formal concept under the formal context of Table 2. It is easy to know that $(\{P_1, P_3, P_4, P_5\}, \{FCA, Ontology, DLs, KR\})$ is a formal concept under the formal context of Table 2.

Furthermore, if we consider the following semantic relations: $FCA \equiv CL$, $DLs \equiv TLs$, where CL stands for Concept Lattice, TLs stands for Terminological Logics, then we can look $(\{P_1, P_3, P_4, P_5\}, \{CL, Ontology, DLs, KR\})$, $(\{P_1, P_3, P_4, P_5\}, \{FCA, Ontology, TLs, KR\})$, and $(\{P_1, P_3, P_4, P_5\}, \{CL, Ontology, TLs, KR\})$ as formal concepts under the formal context of Table 2 although both CL and TLs are not attributes of the formal context of Table 2.

Example 2. Let us consider another formal context named Purchase Houses (see Table 3).

Assume that there is an ontology as follows: $expensive \equiv \neg cheap$. Clearly, the Purchase Houses context is not correct from a semantics point of view under this ontology since there exist some conflicts (i.e., $(H_1, expensive) \in J$, $(H_1, cheap) \in J$ and $(H_5, expensive) \in J$, $(H_5, cheap) \in J$) in J. One of the correct Purchase Houses contexts as follows (see Table 4, we also can get another correct context, i.e., we may reserve the attribute *cheap* and remove the attribute *expensive*):

Similarly to Example 1, we can obtain more appropriate (or accurate) formal concepts and concept lattice under the formal context of Table 4 than under the formal context of Table 3.

To cope with the situations mentioned in Examples 1 and 2, we have to provide some novel approaches to deal with FCA (including formal contexts, formal concepts, and concept lattices) from a semantics point of view. The purpose of the present paper is to provide some semantic operations for FCA by using ontology (i.e., DL) reasoning.

4. Semantifying FCA using DLs

In this section we will extend FCA with DLs, i.e., we use the concepts of DLs to act as the attributes of formal contexts. We first define some notions of semantic FCA from a semantics point of view. We then provide some properties for semantic FCA.

In the rest of this paper, we assume that $\mathcal{DL}$ is an arbitrary (decidable) description logic such as $\mathcal{ALC}$ [50], $\mathcal{SHOIQ}$ [54], or $\mathcal{SROIQ}(\mathbf{D})$ [45].

A distinguishing feature of FCA is an inherent integration of three components: discovery of clusters (so-called formal concepts) in data, discovery of data dependencies (so-called attribute implications) in data, and visualization of formal concepts and attribute implications by a single hierarchical diagram (so-called concept lattice) [38]. In what follows, we will extend the three components to semantic case, respectively.

4.1. Semantic formal concepts

Firstly, we extend the notion of formal contexts, i.e., introduce the definition of semantic formal contexts before give the definition of semantic formal concepts.

Definition 5. Let DL be an arbitrary (decidable) $\mathcal{DL}$, $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology) expressed in $\mathcal{DL}$, Σ be a set of $\mathcal{DL}$-concepts of $\mathcal{DO}$, Γ be a set of individuals of $\mathcal{DO}$, and $I = (\Delta^I, \bullet^I)$ be a model of $\mathcal{DL}$-ontology (i.e., $\mathcal{DL}$-knowledge base). A semantic formal context (semantic context for short) $K = (G, M, J)$ w.r.t. $\mathcal{DO}$ consists of two sets G and M and a binary relation J between G and M, where $G \subseteq \Gamma$, $M \subseteq \Sigma$, and $J = I|_{\{M,G\}}$ (i.e., the restriction of I on $\{M, G\}$).

Analogously to Definition 1, the elements of G are called the objects (or individuals) and the elements of M are called the attributes (or $\mathcal{DL}$-concepts) of the context. aJC or $(a, C) \in J$ (where $a \in G$, $C \in M$) can be used to express that an object a is in a relation J with an attribute C and is read as "the object a has the attribute C", in other words, $(a, C) \in J$ if and only if $a^J \in C^J$.

Obviously, if the domain ontology (i.e., DL-ontology) $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ is consistent, we can obtain a correct semantic formal context. For example, the contexts of Tables 1 and 3 cannot appear under definition of semantic formal context. Conversely, a semantic formal context K can be treated as a DL-ontology $\mathcal{DO}' = \langle \mathcal{AB}', \mathcal{TB}', \mathcal{RB}' \rangle$, where $\mathcal{TB}' = \phi$ and $\mathcal{RB}' = \phi$, that is, $\mathcal{DO}' = \mathcal{AB}'$. However, when we utilize the semantic formal context K, we can exploit the DL-ontology $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ (especially the $\mathcal{TB}$ and $\mathcal{RB}$) to implement semantic operations for K, where $\mathcal{DO}$ is the ontology behind the semantic formal context K (see Example 3).

To illustrate this idea, let us consider the following example.

Example 3. Let us continue to consider Example 1. Assume that $G = \{P_1, P_2, \ldots, P_6\}$ is a set of objects (or individuals), and $M = \{FCA, Ontology, DLs, OWL, KR, DM\}$ is a set of attributes (or $\mathcal{DL}$-concepts). Furthermore, $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ is a domain ontology

Table 5
The semantic Research Papers context w.r.t. ontology.

	FCA	*Ontology*	*DLs*	*OWL*	*KR*	*DM*
P_1	×	×	×	×	×	
P_2	×	×				×
P_3		×	×	×	×	
P_4	×				×	×
P_5		×	×	×	×	×
P_6	×				×	×

(i.e., DL-ontology) expressed in $\mathcal{DL}$ as follows:

$$\begin{aligned}
\mathcal{RB} = {} & \phi,\\
\mathcal{TB} = {} & \{FCA \equiv AppliedMathematics \sqcap OrderTheory,\\
& CLattice \equiv AppliedMathematics \sqcap OrderTheory,\\
& Ontology \equiv KnowledgeEngineering \sqcap OntologyLanguages,\\
& DLs \equiv KR \sqcap OntologyLanguages,\\
& TLs \equiv KR \sqcap OntologyLanguages,\\
& CLanguages \equiv DLs,\\
& OWL \equiv DLs \sqcap KnowledgeEngineering,\\
& KE \equiv KnowledgeEngineering,\\
& DM \equiv MachineLearning \sqcap Database,\\
& KDD \equiv DM\}, \text{ and}\\
\mathcal{AB} = {} & \{AppliedMathematics(P_1), AppliedMathematics(P_2),\\
& AppliedMathematics(P_4),\\
& AppliedMathematics(P_6), OrderTheory(P_1), OrderTheory(P_2),\\
& OrderTheory(P_4),\\
& OrderTheory(P_6), KnowledgeEngineering(P_1),\\
& KnowledgeEngineering(P_2),\\
& KnowledgeEngineering(P_3), KnowledgeEngineering(P_5),\\
& OntologyLanguages(P_1),\\
& OntologyLanguages(P_2), OntologyLanguages(P_3),\\
& OntologyLanguages(P_5), KR(P_1),\\
& KR(P_3), KR(P_4), KR(P_5), KR(P_6), MachineLearning(P_2),\\
& MachineLearning(P_4),\\
& MachineLearning(P_5), MachineLearning(P_6), Database(P_2),\\
& Database(P_4),\\
& Database(P_5), Database(P_6)\}.
\end{aligned}$$

Suppose that we have an interpretation I such that $I \vDash \mathcal{TB}$ and $I \vDash \mathcal{AB}$. For example,

$$\begin{aligned}
& AppliedMathematics^I = \{P_1, P_2, P_4, P_6\},\\
& OrderTheory^I = \{P_1, P_2, P_4, P_6\},\\
& KnowledgeEngineering^I = \{P_1, P_2, P_3, P_5\},\\
& OntologyLanguages^I = \{P_1, P_2, P_3, P_5\},\\
& KR^I = \{P_1, P_3, P_4, P_5, P_6\},\\
& MachineLearning^I = \{P_2, P_4, P_5, P_6\}, \text{ and}\\
& Database^I = \{P_2, P_4, P_5, P_6\}.
\end{aligned}$$

According to the above interpretation I, we can get the following binary relation J (i.e., $I|_{\{M,G\}}$) between G and M by exploiting the model-theoretic semantics of DL (see Section 2.1):

$$\begin{aligned}
FCA^J &= \{P_1, P_2, P_4, P_6\}, Ontology^J = \{P_1, P_2, P_3, P_5\},\\
DLs^J &= \{P_1, P_3, P_5\},\\
OWL^J &= \{P_1, P_3, P_5\}, KR^J = \{P_1, P_3, P_4, P_5, P_6\},\\
DM^J &= \{P_2, P_4, P_5, P_6\}.
\end{aligned}$$

Thus, we can obtain the semantic Research Papers context (G, M, J) with its tabular representation as in Table 5.

From Example 3 we know that we can obtain a semantic formal context through a description logic ontology (or knowledge base) $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$, in other words, we can transform a description logic ontology into a semantic formal context. There are two points we have to point out here. Firstly, DL knowledge bases use an open-world semantics, i.e., absence of information is interpreted as lack of knowledge, not as negation of information. On the other hand, from a DL viewpoint formal contexts in FCA have a closed-world semantics. In the rest of this paper, we require closed-world knowledge about individuals (i.e., objects) in semantic FCA, therefore, we use models as a closed-world representation of individuals in a DL setting (see Example 3). Secondly, both DLs and FCA use Unique Name Assumption (UNA), that is, different names (i.e., objects or individuals, attributes or concepts) always refer to different entities in the world. It should be noted that UNA is a concept from ontology languages and logics. In logics such as DLs with the unique name assumption, different names always refer to different entities. However, OWL does not make this assumption (it provides explicit constructs to express whether two names denote the same or distinct entities).

Now we show the approach to construct a semantic formal context using DL ontology (or DL knowledge base).

Assume that we will construct the semantic formal context $K = (G, M, J)$, where $G = \{g_1, g_2, \ldots, g_n\}$ is a set of objects (or individuals), and $M = \{C_1, C_2, \ldots, C_m\}$ is a set of attributes (or DL-concepts). Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), where $\mathcal{TB}$ consists of a finite set of equivalent axioms of the form $C \equiv D$, $\mathcal{RB}$ consists of a finite set of role axioms such as role equivalence axioms, transitive role axioms, disjoint role axioms, reflexive role axioms, and symmetric role axioms (but $\mathcal{RB}$ does not include role inclusion axioms), $\mathcal{AB} \neq \phi$, and $G \subseteq ind(\mathcal{AB})$, $M \subseteq con(\mathcal{TB})$ and $ac(\mathcal{DO}) \subseteq con(\mathcal{AB})$, where $ind(\mathcal{AB})$ stands for the set of all individuals of $\mathcal{AB}$, $con(\mathcal{TB})$ stands for the set of all concepts of $\mathcal{TB}$, $con(\mathcal{AB})$ stands for the set of all concepts of $\mathcal{AB}$, and $ac(\mathcal{DO})$ stands for the set of all atomic concepts of $\mathcal{DO}$. The approach to construct a semantic formal context using DL ontology is as Algorithm 1.

It should be noted that $G \subseteq ind(\mathcal{AB})$, $M \subseteq con(\mathcal{TB})$ and $ac(\mathcal{DO}) \subseteq con(\mathcal{AB})$ in order to ensure the semantic formal context can be generated correctly. In other words, we only can extract the set of objects G and the set of attributes M of the context from the ontology $\mathcal{DO}$. Clearly, if some object or attribute of semantic formal context which we want to construct is not in the ontology $\mathcal{DO}$, we have to update the ontology in advance.

Obviously, a semantic formal context can be easily represented by a cross table. Hence, we can define formal concepts of semantic formal context using Definition 2 from a syntax point of view. However, compare to formal context (see Definition 1), the differences between formal context and semantic formal context (see Definition 5) are: there is an ontology behind the semantic formal context, and the attributes of the semantic formal context are DL concepts and these attributes (i.e., DL concepts) are defined in the ontology. On the other hand, it is well-known that DL systems [44,55,56] provide their users with inference services (like computing the subsumption hierarchy and the instance checking) that deduce implicit knowledge from explicitly represented knowledge, that is, DLs have reasoning power. For instance, given an arbitrary DL knowledge base (or DL ontology) $\mathcal{DO}$, we have $I \vDash C \sqcap D \equiv \neg(\neg C \sqcup \neg D)$ and $I \vDash \forall R.C \equiv \neg\exists R.\neg C$ for any $\mathcal{DL}$-concepts C, D and $\mathcal{DL}$-role R in $\mathcal{DO}$, and any model I of $\mathcal{DO}$. Namely, we have that $\mathcal{DO} \vDash C \sqcap D \equiv \neg(\neg C \sqcup \neg D)$ and $\mathcal{DO} \vDash \forall R.C \equiv \neg\exists R.\neg C$. Therefore, we need to redefine the notion of formal concepts in semantic FCA. In the following we will introduce a preliminary definition.

Definition 6. Given two sets $M_1 = \{C_1, C_2, \ldots, C_m\}$ and $M_2 = \{D_1, D_2, \ldots, D_n\}$ of attributes of a semantic formal context, where $C_i (1 \leq i \leq m)$ and $D_j (1 \leq j \leq n)$ are concepts of a domain ontology (i.e., DL ontology) $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$. If $\forall C_i \in M_1$, there exists $D_j \in M_2$ satisfies $\mathcal{DO} \vDash C_i \equiv D_j$ (denoted by $C_i \equiv_{\mathcal{DO}} D_j$), then we say that M_1 is a logical subset of M_2 with respect to $\mathcal{DO}$ (denoted by $M_1 \subseteq_{\mathcal{DO}} M_2$).

Algorithm 1. Construct the semantic formal context K=(G, M, J)

Input: a set of objects G={g_1, g_2, …, g_n}, a set of attributes (or $\mathcal{DL}$-concepts) M={C_1, C_2, …, C_m}, a domain ontology (i.e., DL-ontology) $\mathcal{DO}$=⟨$\mathcal{AB}$, $\mathcal{TB}$, $\mathcal{RB}$⟩, where $\mathcal{TB}$ consists of a finite set of equivalent axioms, $\mathcal{RB}$ does not include role inclusion axioms, $\mathcal{AB}\neq\phi$, $G\subseteq ind(\mathcal{AB})$, $M\subseteq con(\mathcal{TB})$, and $ac(\mathcal{DO})\subseteq con(\mathcal{AB})$

Output: the semantic formal context K=(G, M, J) w.r.t. $\mathcal{DO}$

(1) $J=\phi$;
(2) obtain a primitive interpretation P in terms of $\mathcal{AB}$ using the following method: if $A(a)\in\mathcal{AB}$, then we have that $a^P\in A^P$;
(3) according to the model-theoretic semantics of DLs extend the primitive interpretation P to a model I of $\mathcal{DO}$, that is, $I\vDash\mathcal{RB}$, $I\vDash\mathcal{TB}$, and $I\vDash\mathcal{AB}$;
(4) compute the sets C_i^I ($1\le i\le m$);
(5) $C_i^H=C_i^I\cap\{g_1, g_2, \ldots, g_n\}$ ($1\le i\le m$);
(6) obtain the binary relation J between G and M using the approach as follows: $(g_i, C_j)\in J$ iff $g_i\in C_j^H$ ($1\le i\le n$, $1\le j\le m$), in other words, $J=\{(g_i, C_j)|\, g_i\in C_j^H, 1\le i\le n, 1\le j\le m\}$;
(7) output (G, M, J);
(8) end.

If $M_1 \subseteq_{\mathcal{DO}} M_2$ and $M_2 \subseteq_{\mathcal{DO}} M_1$, then we say that M_1 and M_2 are logical equivalent with respect to $\mathcal{DO}$ (denoted by $M_1 =_{\mathcal{DO}} M_2$).

The logical intersection of M_1 and M_2 with respect to $\mathcal{DO}$ (denoted by $M_1 \cap_{\mathcal{DO}} M_2$) is defined as follows: $M_1 \cap_{\mathcal{DO}} M_2 = \{E \in M_1 | \exists D_j \in M_2, \mathcal{DO} \vDash E \equiv D_j\}$. Obviously, we also may defined the logical intersection as follows: $M_1 \cap_{\mathcal{DO}} M_2 = \{E \in M_2 | \exists C_i \in M_1, \mathcal{DO} \vDash E \equiv C_i\}$. It is easy to know that these two definitions are equivalent (i.e., $\{E \in M_1 | \exists D_j \in M_2, \mathcal{DO} \vDash E \equiv D_j\} =_{\mathcal{DO}} \{E \in M_2 | \exists C_i \in M_1, \mathcal{DO} \vDash E \equiv C_i\}$). In this paper, we use the first definition.

The logical union of M_1 and M_2 with respect to DO (denoted by $M_1 \cup_{\mathcal{DO}} M_2$) is defined as follows: $M_1 \cup_{\mathcal{DO}} M_2 = M_1 \cup M_2 - (M_1 \cap_{\mathcal{DO}} M_2)$.

The logical difference of M_1 and M_2 with respect to DO (denoted by $M_1 -_{\mathcal{DO}} M_2$) is defined as follows: $M_1 -_{\mathcal{DO}} M_2 = M_1 - (M_1 \cap_{\mathcal{DO}} M_2)$.

In fact, Definition 6 is similar to Definition 2 in [37] and Definition 2 in [57] (see [57] for some examples). Clearly, we have the following properties:

(1) $M_1 \subseteq M_2 \Rightarrow M_1 \subseteq_{\mathcal{DO}} M_2$;
(2) $M_1 = M_2 \Rightarrow M_1 =_{\mathcal{DO}} M_2$;
(3) $M_1 = M_2, M_2 \subseteq_{\mathcal{DO}} M_3 \Rightarrow M_1 \subseteq_{\mathcal{DO}} M_3$;
(4) $M_1 =_{\mathcal{DO}} M_2, M_2 \subseteq_{\mathcal{DO}} M_3 \Rightarrow M_1 \subseteq_{\mathcal{DO}} M_3$;
(5) $M_1 =_{\mathcal{DO}} M_2, M_2 \subseteq M_3 \Rightarrow M_1 \subseteq_{\mathcal{DO}} M_3$;
(6) $M_1 \subseteq_{\mathcal{DO}} M_2, M_2 \subseteq_{\mathcal{DO}} M_3 \Rightarrow M_1 \subseteq_{\mathcal{DO}} M_3$;
(7) $M_1 \cap_{\mathcal{DO}} M_2 =_{\mathcal{DO}} M_2 \cap_{\mathcal{DO}} M_1$;
(8) $M_1 \cup_{\mathcal{DO}} M_2 =_{\mathcal{DO}} M_2 \cup_{\mathcal{DO}} M_1$.

Now we give the definition of semantic formal concepts.

Definition 7. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology) and $K = (G, M, J)$ be a context w.r.t. $\mathcal{DO}$. For a set $A \subseteq G$ of objects we define $A' = \{m \in M | gJm$ for all $g \in A\}$ (the set of attributes common to the objects in A). For a set $B \subseteq_{\mathcal{DO}} M$ of attributes, we can find a set $E \subseteq M$ such that $B =_{\mathcal{DO}} E$ (by Definition 6), we define $B' = E' = \{g \in G | gJm$ for all $m \in E\}$ (the set of objects which have all attributes in E, where B and E are logical equivalent with respect to $\mathcal{DO}$).

A formal concept of the semantic formal context (G, M, J) w.r.t. a domain ontology $\mathcal{DO}$ is a pair (A, B) with $A \subseteq G$, $B \subseteq_{\mathcal{DO}} M$, $A' =_{\mathcal{DO}} B$ and $B' = A$. We call A the extent and B the intent of the formal concept (A, B). $\mathcal{B}_{\mathcal{DO}}(G, M, J)$ denotes the set of all formal concepts of the semantic formal context (G, M, J) w.r.t. $\mathcal{DO}$.

Example 4 (*Example 3 Cont'd*)**.** Let us continue to consider the semantic Research Papers context w.r.t. ontology $\mathcal{DO}$ in Example 3 (see Table 5). Suppose that $A = \{P_1, P_3, P_5\} \subseteq G$ and $B = \{Ontology, DLs, OWL, KR\} \subseteq_{\mathcal{DO}} M$, by Definition 2 we have that $A' = \{Ontology, DLs, OWL, KR\}$ and $B' = \{P_1, P_3, P_5\}$. Clearly, $A' =_{\mathcal{DO}} B$ and $B' = A$, thus, $(\{P_1, P_3, P_5\}, \{Ontology, DLs, OWL, KR\})$ is a formal concept.

According to Definition 6, since $DLs =_{\mathcal{DO}} TLs =_{\mathcal{DO}} CLanguages$, then we have that $\{Ontology, DLs, OWL, KR\} =_{\mathcal{DO}} \{Ontology, TLs, OWL, KR\} =_{\mathcal{DO}} \{Ontology, CLanguages, OWL, KR\}$. Thus, both $(\{P_1, P_3, P_5\}, \{Ontology, TLs, OWL, KR\})$ and $(\{P_1, P_3, P_5\}, \{Ontology, CLanguages, OWL, KR\})$ are also formal concepts.

There are two points we have to point out here. Regarding the set of objects $\{P_1, P_3, P_5\}$, if we only consider the formal concept from a syntax point of view, then we have only one formal concept $(\{P_1, P_3, P_5\}, \{Ontology, DLs, OWL, KR\})$. However, if we consider it from a semantics point of view, then we obtain three formal concepts $(\{P_1, P_3, P_5\}, \{Ontology, DLs, OWL, KR\})$, $(\{P_1, P_3, P_5\}, \{Ontology, TLs, OWL, KR\})$ and $(\{P_1, P_3, P_5\}, \{Ontology, CLanguages, OWL, KR\})$. Intuitively, these three formal concepts are logically equivalent (or semantically equivalent), that is, they are indistinguishable from a semantics point of view. Secondly, regarding the set of attributes $\{Ontology, TLs, OWL, KR\}$, if we only consider the formal concept at syntax level, it is easy to know that the corresponding formal concept of $\{Ontology, TLs, OWL, KR\}$ is the empty set. Clearly, it is not correct since $(\{P_1, P_3, P_5\}, \{Ontology, TLs, OWL, KR\})$ or $(\{P_1, P_3, P_5\}, \{Ontology, CLanguages, OWL, KR\})$ is a formal concept at semantic level.

Remark 1. It is obvious to see that we can realize more accurately semantic search using the formal concept of semantic FCA. For example, in semantic Web search based on FCA [22], consider the following user's query: $Q = \{Ontology, TLs, OWL, KR\}$. If we consider this query in syntax level, in Table 5 there are no concepts whose intents correspond to the given set of attributes since the attribute TLs is not in the set of attributes of formal context. However, if we consider this query in semantic level, the user can get the answer $\{P_1, P_3, P_5\}$ since $(\{P_1, P_3, P_5\}, \{Ontology, TLs, OWL, KR\})$ is a formal concept in the semantic Research Papers context w.r.t. ontology $\mathcal{DO}$. Regarding semantic Web search based on semantic FCA, we will study it deeply in other paper.

Now we give the following properties of semantic FCA.

Theorem 3. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology). If (G, M, J) is a semantic context w.r.t. $\mathcal{DO}$, $A, A_1, A_2 \subseteq G$ are sets of objects and $B, B_1, B_2 \subseteq_{\mathcal{DO}} M$ are sets of attributes, then*

(1) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq_{\mathcal{DO}} A_1'$, $B_1 \subseteq_{\mathcal{DO}} B_2 \Rightarrow B_2' \subseteq B_1'$;

(2) $A_1 = A_2 \Rightarrow A_1' =_{\mathcal{DO}} A_2'$, $B_1 =_{\mathcal{DO}} B_2 \Rightarrow B_1' = B_2'$;

(3) $A \subseteq A''$, $B \subseteq_{\mathcal{DO}} B''$;

(4) $A' =_{\mathcal{DO}} A'''$, $B' = B'''$;

(5) $A \subseteq B' \Leftrightarrow B \subseteq_{\mathcal{DO}} A' \Leftrightarrow A \times att(B) \subseteq J$, where $att(B) \subseteq M$ and $att(B) =_{\mathcal{DO}} B$.

Proof. (1) Since $A_1 \subseteq A_2$, by (1) of Theorem 1 we have that $A_2' \subseteq A_1'$. Thus, we obtain $A_2' \subseteq_{\mathcal{DO}} A_1'$. Since $B_1, B_2 \subseteq_{\mathcal{DO}} M$, then by Definition 6 we have that there are two sets $E_1, E_2 \subseteq M$ such that $B_1 =_{\mathcal{DO}} E_1$ and $B_2 =_{\mathcal{DO}} E_2$. Since $B_1 \subseteq_{\mathcal{DO}} B_2$, we get $E_1 \subseteq E_2$. By (1) of Theorem 1 we have that $E_2' \subseteq E_1'$. Since $E_1' = B_1'$ and $E_2' = B_2'$, so $B_2' \subseteq B_1'$.

(2) Since $A_1 = A_2$, we obtain $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$. According to (1), we have that $A_2' \subseteq_{\mathcal{DO}} A_1'$ and $A_1' \subseteq_{\mathcal{DO}} A_2'$. Therefore, $A_1' =_{\mathcal{DO}} A_2'$. Since $B_1, B_2 \subseteq_{\mathcal{DO}} M$, then by Definition 6 we have that there are two sets $E_1, E_2 \subseteq M$ such that $B_1 =_{\mathcal{DO}} E_1$ and $B_2 =_{\mathcal{DO}} E_2$. Since $B_1 =_{\mathcal{DO}} B_2$, then we have that $E_1 = E_2$, thus, $E_1' = E_2'$. By Definition 7, $B_1' = E_1'$ and $B_2' = E_2'$. Hence, $B_1' = B_2'$.

(3) The proof of $A \subseteq A''$ is the same as that of the first part of (2) of Theorem 1. Since $B \subseteq_{\mathcal{DO}} M$, then there is a set $E \subseteq M$ such that $B =_{\mathcal{DO}} E$. By (2) of Theorem 1, we have that $E \subseteq E''$, thus $B \subseteq_{\mathcal{DO}} E''$. Since $B =_{\mathcal{DO}} E$, then we obtain $E'' =_{\mathcal{DO}} B''$. Thus, $B \subseteq_{\mathcal{DO}} B''$.

(4) According to (3) we have that $A' \subseteq_{\mathcal{DO}} A'''$. Also by (3), we get $A \subseteq A''$, hence, in terms of (1) we yield $A''' \subseteq_{\mathcal{DO}} A'$. Therefore, $A' =_{\mathcal{DO}} A'''$. Similarly, we can obtain $B' = B'''$.

(5) Since $B \subseteq_{\mathcal{DO}} M$, then there is a set $E \subseteq M$ such that $B =_{\mathcal{DO}} E$ and $B' = E'$. Thus, we have the following: $A \subseteq B' \Rightarrow A \subseteq E' \Rightarrow E \subseteq A'$ (by (4) of Theorem 1) $\Rightarrow E \subseteq_{\mathcal{DO}} A' \Rightarrow B \subseteq_{\mathcal{DO}} A'$, and

$$B \subseteq_{\mathcal{DO}} A' \Rightarrow E \subseteq_{\mathcal{DO}} A' \Rightarrow E \subseteq A' \text{ (since } E, A' \subseteq M)$$
$$\Rightarrow A \subseteq E' \Rightarrow A \subseteq B'.$$

Thus, $A \subseteq B' \Leftrightarrow B \subseteq_{\mathcal{DO}} A'$.

By Definition 7, we have that $A \subseteq B' \Leftrightarrow A \subseteq E'$, where $E \subseteq M$ and $E =_{\mathcal{DO}} B$. By (4) of Theorem 1, we can get $A \subseteq E' \Leftrightarrow A \times E \subseteq J$. Thus, $A \subseteq B' \Leftrightarrow A \times E \subseteq J$. Let $E = att(B)$, we can obtain that $A \subseteq E' \Leftrightarrow A \times att(B) \subseteq J$. □

Example 5 (*Example 3 Cont'd*)**.** Let $A_1 = \{P_1, P_3, P_5\}$ and $A_2 = \{P_1, P_3, P_5, P_6\}$. Clearly, $A_1 \subseteq A_2$.

By Example 4, we know that $A_1' = \{Ontology, DLs, OWL, KR\}$ or $\{Ontology, TLs, OWL, KR\}$ or $\{Ontology, CLanguages, OWL, KR\}$.

According to Definition 7, we have that $A_2' = \{KR\}$. Obviously, $A_2' \subseteq_{\mathcal{DO}} A_1'$.

Let $B_1 = \{Ontology, TLs, OWL, KR\}$ and $B_2 = \{Ontology, DLs, OWL, KR, DM\}$. Then we have that $B_1 \subseteq_{\mathcal{DO}} B_2$.

In Example 4, we have shown that $B_1' = \{P_1, P_3, P_5\}$.

By Definition 7, we obtain that $B_2' = \{P_5\}$. Obviously, $B_2' \subseteq B_1'$.

Let $B = \{KR, KDD\}$. According to Definition 7, we have that $B' = \{P_4, P_5, P_6\}$. Furthermore, we have that $B'' = \{KR, KDD\}$ and $B''' = \{P_4, P_5, P_6\}$. Hence, $B \subseteq_{\mathcal{DO}} B''$ and $B' = B'''$.

4.2. Attribute implications

Now we give some notions and properties about attribute implications in semantic FCA by extending those of FCA [1,31]. Implications between attributes are statements of the form "All objects that have all attributes from the set A also have all attributes from the set B". Formally,

Definition 8. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology) and $K = (G, M, J)$ be a semantic context w.r.t. DO. An implication in K is a pair (A, B) where $A, B \subseteq_{\mathcal{DO}} M$. Usually, we denote the implication by $A \rightarrow_{\mathcal{DO}} B$. We say an implication $A \rightarrow_{\mathcal{DO}} B$ holds in the semantic context K w.r.t. $\mathcal{DO}$ if $A' \subseteq B'$.

As an example, let us have a look at the semantic context (see Table 5) and the domain ontology from Example 3.

Example 6 (*Example 3 Cont'd*)**.** There are three objects P_1, P_3, and P_5 that have the attribute *DLs*, and all of them also have the attribute *Ontology*. Therefore, $\{DLs\} \rightarrow_{\mathcal{DO}} \{Ontology\}$ is an implication that holds in the semantic Research Papers context w.r.t. DO. Similarly, $\{DLs\} \rightarrow_{\mathcal{DO}} \{OWL\}$, $\{DLs\} \rightarrow_{\mathcal{DO}} \{KR\}$, $\{OWL\} \rightarrow_{\mathcal{DO}} \{KR\}$, and $\{DLs, OWL\} \rightarrow_{\mathcal{DO}} \{KR\}$ are also some implications that hold in the semantic Research Papers context w.r.t. $\mathcal{DO}$.

Since $DLs =_{\mathcal{DO}} TLs =_{\mathcal{DO}} CLanguages$, then we have that both $\{TLs, OWL\} \rightarrow_{\mathcal{DO}} \{KR\}$ and $\{CLanguages, OWL\} \rightarrow_{\mathcal{DO}} \{KR\}$ are also implications that hold in the semantic Research Papers context w.r.t. $\mathcal{DO}$ although both *TLs* and *CLanguages* are not the attributes of the semantic Research Papers context. Obviously, both ($\{TLs, OWL\}, \{KR\}$) and ($\{CLanguages, OWL\}, \{KR\}$) are not implications in Research Papers context at syntax level (but they are implications at semantic level).

From the logistic view, $A \rightarrow_{\mathcal{DO}} B$ in $K = (G, M, J)$ is to represent the statement "A implies B w.r.t. K" or "if A then B w.r.t. K". From a DL (or ontology) point of view, $A \rightarrow_{\mathcal{DO}} B$ in $K = (G, M, J)$ means that $(A_1 \sqcap A_2 \sqcap \cdots \sqcap A_m)^J \subseteq (B_1 \sqcap B_2 \sqcap \cdots \sqcap B_n)^J$, where $A = \{A_1, A_2, \ldots, A_m\}$, $B = \{B_1, B_2, \ldots, B_n\}$.

Definition 9. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$, $T \subseteq_{\mathcal{DO}} M$, and $A \rightarrow_{\mathcal{DO}} B$ be an implication of the context K. T is called a model of the implication $A \rightarrow_{\mathcal{DO}} B$ or is said to respect the implication $A \rightarrow_{\mathcal{DO}} B$, denoted by $T \vDash (A \rightarrow_{\mathcal{DO}} B)$, if the set T satisfies: $A \not\subseteq_{\mathcal{DO}} T$ or $B \subseteq_{\mathcal{DO}} T$, that is, if $A \subseteq_{\mathcal{DO}} T$ implies $B \subseteq_{\mathcal{DO}} T$.

Let Θ be a set of implications. We call T a model of Θ, denoted by $T \vDash \Theta$ iff $T \vDash (A \rightarrow_{\mathcal{DO}} B)$ holds for each $(A \rightarrow_{\mathcal{DO}} B) \in \Theta$.

Definition 10. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology) and $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$. An implication $A \rightarrow_{\mathcal{DO}} B$ of K follows from a set of implications Θ, denoted by $\Theta \vdash (A \rightarrow_{\mathcal{DO}} B)$, if every set $T \subseteq_{\mathcal{DO}} M$ that respects all implications from Θ also respects $A \rightarrow_{\mathcal{DO}} B$, that is, if for each set $T \subseteq_{\mathcal{DO}} M$ respecting $T \vDash \Theta$, $T \vDash (A \rightarrow_{\mathcal{DO}} B)$ holds.

Definition 11. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology) and $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$. We say that the set of implications Θ is an implicational base of K if

- Θ is sound for K, i.e., every implication from Θ holds in K, and
- Θ is complete for K, i.e., every implication that holds in K follows from Θ. That is, $\Theta \vdash (A \rightarrow_{\mathcal{DO}} B)$ holds for each implication $A \rightarrow_{\mathcal{DO}} B$ of K.

Now we give the following properties of implications in semantic FCA.

Theorem 4. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$, and $A \rightarrow_{\mathcal{DO}} B$ be an implication that holds in K. Then:*

(1) $A \rightarrow_{\mathcal{DO}} B$ follows from $\{A \rightarrow_{\mathcal{DO}} A''\}$, i.e., $\{A \rightarrow_{\mathcal{DO}} A''\} \vdash (A \rightarrow_{\mathcal{DO}} B)$;

(2) the set $\{A \rightarrow_{\mathcal{DO}} A'' | A \subseteq_{\mathcal{DO}} M\}$ is an implicational base of K.

Proof. (1) We will show that for each set $T \subseteq_{\mathcal{DO}} M$ respecting $T \vDash \{A \rightarrow_{\mathcal{DO}} A''\}$, $T \vDash (A \rightarrow_{\mathcal{DO}} B)$ holds.

Since $T \vDash \{A \rightarrow_{\mathcal{DO}} A''\}$, i.e., $T \vDash (A \rightarrow_{\mathcal{DO}} A'')$, by Definition 9 we have that $A \nsubseteq_{\mathcal{DO}} T$ or $A'' \subseteq_{\mathcal{DO}} T$. Then we obtain that $T \vDash (A \rightarrow_{\mathcal{DO}} B)$. Thus, $\{A \rightarrow_{\mathcal{DO}} A''\} \vdash (A \rightarrow_{\mathcal{DO}} B)$.

(2) To show $\{A \rightarrow_{\mathcal{DO}} A'' | A \subseteq_{\mathcal{DO}} M\}$ is an implicational base, we have to prove $\{A \rightarrow_{\mathcal{DO}} A'' | A \subseteq_{\mathcal{DO}} M\}$ is sound and complete.

By Theorem 3 we have that $A' = A'''$ for any $A \subseteq_{\mathcal{DO}} M$, then we get that $A' \subseteq A'''$. According to Definition 8 we know that $A \rightarrow_{\mathcal{DO}} A''$ holds in K. Thus, $\{A \rightarrow_{\mathcal{DO}} A'' | A \subseteq_{\mathcal{DO}} M\}$ is sound.

We have proven that $\{A \rightarrow_{\mathcal{DO}} A''\} \vdash (A \rightarrow_{\mathcal{DO}} B)$ for each implication $A \rightarrow_{\mathcal{DO}} B$, that is, $\{A \rightarrow_{\mathcal{DO}} A'' | A \subseteq_{\mathcal{DO}} M\}$ is complete. □

Theorem 5. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology) and $K = (G, M, J)$ be a context w.r.t. $\mathcal{DO}$. An implication $A \rightarrow_{\mathcal{DO}} B$ holds in (G, M, J) w.r.t. $\mathcal{DO}$ iff $B \subseteq_{\mathcal{DO}} A''$.*

Proof. "⇒". Since $A \rightarrow_{\mathcal{DO}} B$ holds in (G, M, J), then we have that $A' \subseteq B'$. Thus by Theorem 3 we have that $B'' \subseteq_{\mathcal{DO}} A''$. Since $B \subseteq_{\mathcal{DO}} B''$ (see Theorem 3), then $B \subseteq_{\mathcal{DO}} A''$.

"⇐". Since $B \subseteq_{\mathcal{DO}} A''$, then we have that $A''' \subseteq B'$ (by Theorem 3). Since $A' = A'''$ (see Theorem 3), then we obtain that $A' \subseteq B'$. Hence, $A \rightarrow_{\mathcal{DO}} B$ holds in (G, M, J) w.r.t. $\mathcal{DO}$ (by Definition 8). □

Theorem 6. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$, and $\{A\}$, $\{B\} \subseteq_{\mathcal{DO}} M$. Then the following conditions are equivalent:*

(1) $A \rightarrow_{\mathcal{DO}} B$ is an implication of K w.r.t. $\mathcal{DO}$;
(2) $B \subseteq_{\mathcal{DO}} A''$;
(3) $\forall H \in \mathcal{B}_{\mathcal{DO}}(G, M, J)$, $in(H) \vDash (A \rightarrow_{\mathcal{DO}} B)$, where $in(H)$ is the intent of H.

Proof. By Theorem 5 we have (1) ⇔ (2).

(2) ⇒ (3). To prove $in(H) \vDash (A \rightarrow_{\mathcal{DO}} B)$, we need to show that $A \nsubseteq_{\mathcal{DO}} in(H)$ or $B \subseteq_{\mathcal{DO}} in(H)$.

$\forall H \in \mathcal{B}_{\mathcal{DO}}(G, M, J)$, there are two cases: $A \nsubseteq_{\mathcal{DO}} in(H)$ or $A \subseteq_{\mathcal{DO}} in(H)$. If $A \nsubseteq_{\mathcal{DO}} in(H)$, by Definition 9 we have that $in(H) \vDash (A \rightarrow_{\mathcal{DO}} B)$. If $A \subseteq_{\mathcal{DO}} in(H)$, by Theorem 3 we obtain that $in(H)' \subseteq A'$, and furthermore, we have that $A'' \subseteq_{\mathcal{DO}} in(H)''$. Since $in(H)$ is the intent of H, so $in(H) =_{\mathcal{DO}} in(H)''$. Thus, we get $A'' \subseteq_{\mathcal{DO}} in(H)$. Since $B \subseteq_{\mathcal{DO}} A''$, then $B \subseteq_{\mathcal{DO}} in(H)$. By Definition 9 we also have that $in(H) \vDash (A \rightarrow_{\mathcal{DO}} B)$.

(3) ⇒ (2). Since $\forall H \in B_{\mathcal{DO}}(G, M, J)$, $in(H) \vDash (A \rightarrow_{\mathcal{DO}} B)$. Let $H = (A', A'')$, then we have that $A'' \vDash (A \rightarrow_{\mathcal{DO}} B)$. By Definition 9 we obtain that $A \nsubseteq_{\mathcal{DO}} A''$ or $B \subseteq_{\mathcal{DO}} A''$. According to Theorem 1, we have $A \subseteq A''$, then $A \subseteq_{\mathcal{DO}} A''$. Therefore, $B \subseteq_{\mathcal{DO}} A''$. □

In fact, Theorem 5 is a semantic extension of Proposition 19 in [1], and Theorem 6 is a semantic extension of Theorem 2 in [31].

Remark 2. In traditional FCA, there are lots of notions and properties such as pseudo-intent and Duquenne–Guigues Basis about attribute implications (see [1,31,46] for more details). Of course, we can generalize these notions and properties into semantic case. For example, we can introduce the notion of pseudo-intent in semantic FCA as follows:

Definition 12. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$. A set $P \subseteq_{\mathcal{DO}} M$ is called a pseudo-intent of K if

- P is not an intent, and
- for all pseudo-intents $Q \subset_{\mathcal{DO}} P$ it holds that $Q'' \subseteq_{\mathcal{DO}} P$.

In other words, P is called the pseudo-intent iff $P \neq_{\mathcal{DO}} P''$ and $Q'' \subseteq_{\mathcal{DO}} P$ holds for every pseudo-intent $Q \subseteq_{\mathcal{DO}} P$, $Q \neq_{\mathcal{DO}} P$.

4.3. Semantic concept lattice

In Section 4.1, we have pointed out that given a fixed set of objects A, if we only consider the formal concept at syntax level, then we have only one corresponding formal concept (A, B). However, if we consider it at semantics level, then we can obtain several formal concepts (A, B_1), (A, B_2), …, (A, B_n). Obviously, we need to redefine the notion of concept lattice for semantic FCA, that is, we will give the definition of semantic concept lattice. Intuitively, in the concept lattice for traditional context the notion of order is defined between formal concepts, however, in the semantic concept lattice for semantic context, the notion of order is defined between two sets of semantic formal concepts. Firstly, we give a property of two sets of semantic formal concepts.

Definition 13. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$. $\{(A, B_1), (A, B_2), \ldots, (A, B_m)\}$ is called a set of equivalently semantic formal concepts of K, where $A \subseteq G$, $B_1, B_2, \ldots, B_m \subseteq_{\mathcal{DO}} M$, $B_1 =_{\mathcal{DO}} B_2 =_{\mathcal{DO}} \ldots =_{\mathcal{DO}} B_m$.

Theorem 7. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. DO. If $\{(A_1, B_1), (A_1, B_2), \ldots, (A_1, B_m)\}$ and $\{(A_2, C_1), (A_2, C_2), \ldots, (A_2, C_n)\}$ are two sets of equivalently semantic formal concepts of K, i.e., $A_1, A_2 \subseteq G$, $B_1, B_2, \ldots, B_m, C_1, C_2, \ldots, C_n \subseteq_{\mathcal{DO}} M$, $B_1 =_{\mathcal{DO}} B_2 =_{\mathcal{DO}} \ldots =_{\mathcal{DO}} B_m$, $C_1 =_{\mathcal{DO}} C_2 =_{\mathcal{DO}} \ldots =_{\mathcal{DO}} C_n$. Then we have the following*

$A_1 \subseteq A_2$ *iff* $\forall 1 \leq i \leq n$, $1 \leq j \leq m$, $C_i \subseteq_{\mathcal{DO}} B_j$.

Proof. Since $\{(A_1, B_1), (A_1, B_2), \ldots, (A_1, B_m)\}$ and $\{(A_2, C_1), (A_2, C_2), \ldots, (A_2, C_n)\}$ are two sets of equivalently semantic formal concepts, by Definition 7 we have that ${A_1}' =_{\mathcal{DO}} B_j$, ${B_j}' = A_1$, and ${A_2}' =_{\mathcal{DO}} C_i$, ${C_i}' = A_2$, where $1 \leq i \leq n$, $1 \leq j \leq m$.

"⇒". Since $A_1 \subseteq A_2$, by Theorem 3 we have that ${A_2}' \subseteq_{\mathcal{DO}} {A_1}'$. According to ${A_1}' =_{\mathcal{DO}} B_j$ and ${A_2}' =_{\mathcal{DO}} C_i$, then we obtain that $C_i \subseteq_{\mathcal{DO}} B_j$.

"⇐". Since $C_i \subseteq_{\mathcal{DO}} B_j$, by Theorem 3 we have that ${B_j}' \subseteq {C_i}'$. In terms of ${B_j}' = A_1$ and ${C_i}' = A_2$, then we get that $A_1 \subseteq A_2$. □

Definition 14. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$. $\{(A_1, B_1), (A_1, B_2), \ldots, (A_1, B_m)\}$ and $\{(A_2, C_1), (A_2, C_2), \ldots, (A_2, C_n)\}$ are two sets of equivalently semantic formal concepts of K.

$\{(A_1, B_1), (A_1, B_2), \ldots, (A_1, B_m)\}$ is called a subconcept set of $\{(A_2, C_1), (A_2, C_2), \ldots, (A_2, C_n)\}$ (or $\{(A_2, C_1), (A_2, C_2), \ldots, (A_2, C_n)\}$ is a superconcept set of $\{(A_1, B_1), (A_1, B_2), \ldots, (A_1, B_m)\}$), provided that $A_1 \subseteq A_2$ (or $C_i \subseteq_{\mathcal{DO}} B_j$ for any $1 \leq i \leq n$, $1 \leq j \leq m$) and is denoted by $\{(A_1, B_1), (A_1, B_2), \ldots, (A_1, B_m)\} \leq \{(A_2, C_1), (A_2, C_2), \ldots, (A_2, C_n)\}$.

The relation $\leq$ is called the hierarchical order (or simply order) of the sets of equivalently semantic formal concepts. The set of all equivalently semantic formal concept sets of (G, M, J) ordered in this way is denoted by $\underline{\mathcal{SB}}(G, M, J)$ and is called the semantic concept lattice of the semantic formal context (G, M, J) w.r.t. $\mathcal{DO}$.

Example 7 (*Example 3 Cont'd*). By Example 4 we know that

$$X = \{(\{P_1, P_3, P_5\}, \{Ontology, DLs, OWL, KR\}), (\{P_1, P_3, P_5\}, \{Ontology, TLs, OWL, KR\}), (\{P_1, P_3, P_5\}, \{Ontology, CLanguages, OWL, KR\})\}$$

is a set of equivalently semantic formal concepts.

By Definition 7, we know that

$$\begin{aligned} Y &= \{(\{P_5\}, \{Ontology, DLs, OWL, KR, DM\}), (\{P_5\}, \{Ontology, TLs, OWL, KR, DM\}), (\{P_5\}, \{Ontology, CLanguages, OWL, KR, DM\})\},\\ Z &= \{(\{P_1\}, \{FCA, Ontology, DLs, OWL, KR\}), (\{P_1\}, \{FCA, Ontology, TLs, OWL, KR\}), (\{P_1\}, \{FCA, Ontology, CLanguages, OWL, KR\})\}, \text{ and}\\ U &= \{(\{P_1, P_3, P_4, P_5, P_6\}, \{KR\})\}\end{aligned}$$

are also three sets of equivalently semantic formal concepts.

According to Definition 14, it is easy to know that $U \leq X \leq Y$ and $U \leq X \leq Z$.

The line diagram in Fig. 1 represents the hierarchical order among X, Y, Z, and U.

In what follows, we give some properties of semantic concept lattices.

Theorem 8. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB}\rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. DO. If IN is an index set and, for every $t \in IN$, $A_t \subseteq G$, and $B_t \subseteq_{\mathcal{DO}} M$, then we have the following*

$$(\cup_{t\in IN} A_t)' =_{\mathcal{DO}} \cap_{DO_{t\in IN}} A_t' \text{ and } \left(\cup_{DO_{t\in IN}} B_t\right)' = \cap_{t\in IN} B_t'.$$

Proof. The proof of $(\cup_{t\in IN} A_t)' =_{\mathcal{DO}} \cap_{DO_{t\in IN}} A_t'$ is similar to that of Proposition 11 in [1], that is, according to Proposition 11 in [1], we have $(\cup_{t\in IN} A_t)' = \cap_{t\in IN} A_t'$. Since $A_t{}' \subseteq M$ for any $t \in IN$ (by Definition 7), then $\cap_{t\in IN} A_t' = \cap_{DO_{t\in IN}} A_t'$. Thus, $(\cup_{t\in IN} A_t)' =_{\mathcal{DO}} \cap_{DO_{t\in IN}} A_t'$.

By Definition 7 we have that

$$\begin{aligned}\left(\cup_{DO_{t\in IN}} B_t\right)' &= (\cup_{t\in IN} E_t)' \text{ where } E_t \subseteq M, B_t =_{\mathcal{DO}} E_t\\ &= \{g \in G |\, gJm \text{ for all } m \in \cup_{t\in IN} E_t\}\\ &= \{g \in G |\, gJm \text{ for all } m \in E_t \text{ for all } t \in IN\}, \text{ and}\end{aligned}$$

$$\begin{aligned}\cap_{t\in IN} B_t' &= \cap_{t\in IN} E_t'\\ &= \cap_{t\in IN}\{g \in G |\, gJm \text{ for all } m \in E_t\}\\ &= \{g \in G |\, gJm \text{ for all } m \in E_t \text{ for all } t \in IN\}.\end{aligned}$$

Therefore, $\left(\cup_{DO_{t\in IN}} B_t\right)' = \cap_{t\in IN} B_t'$. □

Theorem 9. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB}\rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$, and IN be an index set. The semantic concept lattice $\underline{\mathcal{SB}}$ is a complete lattice in which infimum and supremum are given by:*

$$\begin{aligned}&\wedge_{t_i,\dots,t_j\in IN}\left(\left\{\left(A_{t_i}, B_{t_i1}\right), \left(A_{t_i}, B_{t_i2}\right), \dots, \left(A_{t_i}, B_{t_in}\right)\right\}, \dots,\right.\\ &\quad \left.\left\{\left(A_{t_j}, C_{t_j1}\right), \left(A_{t_j}, C_{t_j2}\right), \dots, \left(A_{t_j}, C_{t_jm}\right)\right\}\right)\\ &= \left\{\left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1}\right)''\right),\right.\\ &\quad \left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j2}\right)''\right), \dots,\\ &\quad \left. A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_in} \cup_{DO} \dots \cup_{DO} C_{t_jm}\right)''\right\},\end{aligned}$$

$$\begin{aligned}&\vee_{t_i,\dots,t_j\in IN}\left(\left\{\left(A_{t_i}, B_{t_i1}\right), \left(A_{t_i}, B_{t_i2}\right), \dots, \left(A_{t_i}, B_{t_in}\right)\right\}, \dots,\right.\\ &\quad \left.\left\{\left(A_{t_j}, C_{t_j1}\right), \left(A_{t_j}, C_{t_j2}\right), \dots, \left(A_{t_j}, C_{t_jm}\right)\right\}\right)\\ &= \left\{\left((A_{t_i} \cup \dots \cup A_{t_j})'', B_{t_i1} \cap_{DO} \dots \cap_{DO} C_{t_j1}\right),\right.\\ &\quad \left((A_{t_i} \cup \dots \cup A_{t_j})'', B_{t_i1} \cap_{DO} \dots \cap_{DO} C_{t_j2}\right), \dots,\\ &\quad \left.\left((A_{t_i} \cup \dots \cup A_{t_j})'', B_{t_in} \cap_{DO} \dots \cap_{DO} C_{t_jm}\right)\right\}.\end{aligned}$$

Proof. Firstly, we show the formula for the infimum. Since

$$\begin{aligned}&\wedge_{t_i,\dots,t_j\in IN}\left(\left\{\left(A_{t_i}, B_{t_i1}\right), \left(A_{t_i}, B_{t_i2}\right), \dots, \left(A_{t_i}, B_{t_in}\right)\right\}, \dots,\right.\\ &\quad \left.\left\{\left(A_{t_j}, C_{t_j1}\right), \left(A_{t_j}, C_{t_j2}\right), \dots, \left(A_{t_j}, C_{t_jm}\right)\right\}\right)\\ &\quad \left\{\left(A_{t_i}, B_{t_i1}\right) \wedge \dots \wedge \left(A_{t_j}, C_{t_j1}\right), \left(A_{t_i}, B_{t_i1}\right) \wedge \dots \wedge \left(A_{t_j}, C_{t_j2}\right),\right.\\ &\quad \left.\dots, \left(A_{t_i}, B_{t_in}\right) \wedge \left(A_{t_j}, C_{t_jm}\right)\right\},\end{aligned}$$

then we only need to prove the following:

$$\begin{aligned}&\left(A_{t_i}, B_{t_i1}\right) \wedge \dots \wedge \left(A_{t_j}, C_{t_j1}\right)\\ &\quad = \left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1}\right)''\right),\\ &\left(A_{t_i}, B_{t_i1}\right) \wedge \dots \wedge \left(A_{t_j}, C_{t_j2}\right)\\ &\quad = \left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j2}\right)''\right),\\ &\dots\\ &\left(A_{t_i}, B_{t_in}\right) \wedge \dots \wedge \left(A_{t_j}, C_{t_jm}\right)\\ &\quad = \left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_in} \cup_{DO} \dots \cup_{DO} C_{t_jm}\right)''\right).\end{aligned}$$

In what follows, we only show the proof of the first equation.

Since $\left(A_{t_i}, B_{t_i1}\right)$ is a semantic formal concept, by Definition 7 we have that $A_{t_i}{}' =_{\mathcal{DO}} B_{t_i1}$ and $B_{t_i1}{}' = A_{t_i}$. Thus, $A_{t_i} \cap \dots \cap A_{t_j} = B_{t_i1}' \cap \dots \cap C_{t_j1}'$.

By Theorem 8, we obtain that $A_{t_i} \cap \dots \cap A_{t_j} = B_{t_i1}{}' \cap \dots \cap C_{t_j1}{}' = (B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1})'$. Hence, we have the following

$$\begin{aligned}&\left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1}\right)''\right)\\ &\quad = \left((B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1})', \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1}\right)''\right).\end{aligned}$$

Therefore, $\left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1}\right)''\right)$ is a semantic formal concept. Clearly, $\left(A_{t_i} \cap \dots \cap A_{t_j}, \left(B_{t_i1} \cup_{DO} \dots \cup_{DO} C_{t_j1}\right)''\right)$ is the infimum of $\left(A_{t_i}, B_{t_i1}\right)$, …, and $\left(A_{t_j}, C_{t_j1}\right)$ (since the extent of this semantic formal concept is exactly the intersection of the extents of $\left(A_{t_i}, B_{t_i1}\right)$, …, and $\left(A_{t_j}, C_{t_j1}\right)$).

The proof of the formula for the supremum is similar. □

Obviously, Theorems 8 and 9 are semantic extensions of Proposition 11 and Theorem 3 in [1], respectively. The main difference between Theorems 3 and 9 in [1] is: in Theorem 3 in [1] the operations are defined between formal concepts, however, in Theorem 9 the operations are defined between two sets of semantic formal concepts (i.e., two sets of equivalently semantic formal concepts, see Definition 13).

5. Attribute reduction

Attribute reduction in FCA is an important step in reducing computational complexity in obtaining concept lattices. Ganter and Wille demonstrated several results about attribute and object reduction in [1], considering the irreducible elements in the concept lattice. In recent years, much attention has been paid to knowledge (or attribute) reduction in FCA [32,58–60]. In fact, attribute reduction in this section is related to feature selection based on FCA [61], since feature selection methods can eliminate unfavorable features such as noisy, redundant and irrelevant features. Thus, feature selection contributes to reduce the high dimensionality of data and to restrict the input which can contain missing values data into a single or a subset of features. There is a variety of approaches for feature selection and feature selection has been used for a wide variety of applications [62–67].

In order to obtain the concept lattices with relatively less attributes, in this section we will study the attribute reduction of the (semantic) formal contexts, formal concepts, and concept lattices from a semantics point of view.

Firstly, we present the conditions for justifying whether a semantic formal context, formal concept, or concept lattice is redundant.

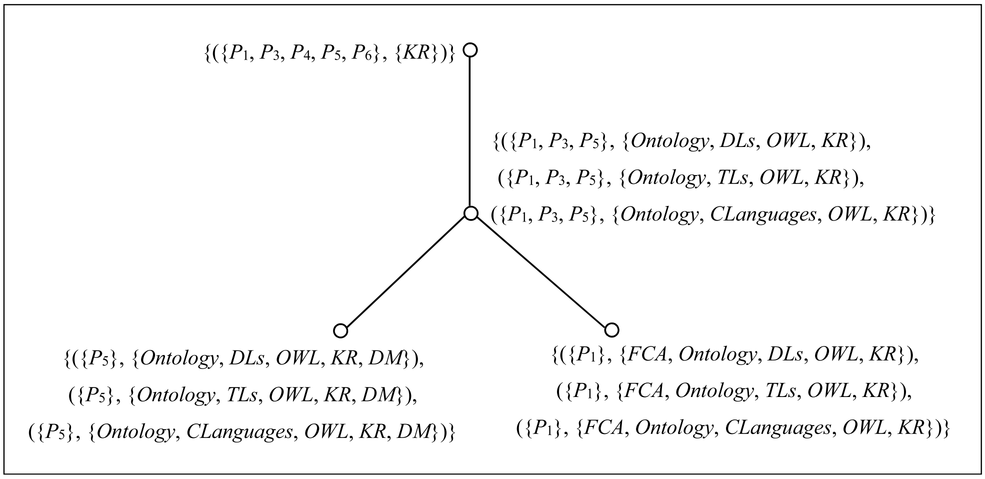

Fig. 1. Hierarchical order of the sets of equivalently semantic formal concepts.

Definition 15. Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. DO. If there are attributes $C_1, C_2 \in M$ such that $C_1 \equiv_{\mathcal{DO}} C_2$, then we say that the semantic formal context K has redundancy (or K is redundant), and the attribute C_1 (or C_2) is called redundant attribute in K. The set of all redundant attributes in K is denoted by $red(K)$.

Let (A, B) be a semantic formal concept of K w.r.t. $\mathcal{DO}$. If there are attributes $D_1, D_2 \in B$ such that $D_1 \equiv_{\mathcal{DO}} D_2$, then we say that the semantic formal concept (A, B) has redundancy (or (A, B) is redundant), and the attribute D_1 (or D_2) is called redundant attribute in (A, B). The set of all redundant attributes in (A, B) is denoted by $red(A, B)$.

Let $\underline{\mathcal{SB}}(G, M, J)$ be the semantic concept lattice of $K = (G, M, J)$ w.r.t. $\mathcal{DO}$, (A, B) be a semantic formal concept, that is, $(A, B) \in \underline{\mathcal{SB}}(G, M, J)$. If (A, B) has redundancy, then we call $\underline{\mathcal{SB}}(G, M, J)$ has redundancy (or $\underline{\mathcal{SB}}(G, M, J)$ is redundant), and the formal concept (A, B) is called redundant concept in $\underline{\mathcal{SB}}(G, M, J)$.

Remark 3. In Definition 15, we justify whether a semantic formal context (or formal concept, or concept lattice) is redundant by using logical equivalence w.r.t. DL-ontology (see Definition 6) reasoning of DLs. Intuitively, it is a reasonable condition from a semantics point of view. Clearly, if there are two logical equivalent attributes C and D (i.e., for any object a and for any model I, we have $a \in C^I \Leftrightarrow a \in D^I$), we only need one of them (i.e., C or D), in other words, one of them is redundant. However, logical equivalence is a strong condition. Certainly, we can also define redundancy by making use of the following weak condition, i.e., use the model J of $\mathcal{DO}$ to define the notion of redundancy:

Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$. If there are attributes $C_1, C_2 \in M$ such that $C_1^J = C_2^J$, then we say that the semantic formal context K has redundancy (or K is redundant), and the attribute C_1 (or C_2) is called redundant attribute in K. Correspondingly, we also can give the notions of redundancy of formal concepts and concept lattices. In this paper, we use Definition 15 to define the notions of redundancy.

On the other hand, a formal context may have a great many attributes in the practical applications. To reduce computational complexity in obtaining concept lattices, we also may relax the condition (i.e., logical equivalence) in Definition 15 by using logical subsumption reasoning of DLs to define redundancy of formal contexts (or formal concepts, or concept lattices). For example, if there are attributes $C_1, C_2 \in M$ such that $C_1 \sqsubseteq_{\mathcal{DO}} C_2$, then we call the semantic formal context $K = (G, M, J)$ has redundancy, and the attribute C_1 (or C_2) is called redundant attribute in K. Regarding the approach and properties of this attribute reduction, we need to study them deeply in other place.

Table 6
A semantic Research Papers context w.r.t. ontology.

	FCA	*Ontology*	*DLs*	*TLs*	*OWL*	*DM*	*KDD*
P_1	×	×	×	×	×		
P_2	×	×				×	×
P_3		×	×	×	×		
P_4	×					×	×
P_5		×	×	×	×	×	×
P_6	×					×	×

There are two points that we have to point out here. Firstly, it is possible that there exist several redundant attributes (resp., redundant concepts) in a given semantic formal context or formal concept (resp., concept lattice) (see Example 8). Secondly, to determine whether a redundancy exists in semantic formal contexts, formal concepts, or concept lattices, from Definition 15 we know that we must rely on DL reasoning systems such as Pellet and HermiT [44,55,56] to determine logical equivalence w.r.t. DL-ontology such as $\mathcal{DO} \models C_1 \equiv C_2$.

Example 8. Let us continue to consider Example 1. Assume that $G = \{P_1, P_2, \ldots, P_6\}$ is a set of objects (or individuals), and $M = \{FCA, Ontology, DLs, TLs, OWL, DM, KDD\}$ is a set of attributes (or $\mathcal{DL}$-concepts). Furthermore, $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB} \rangle$ is a domain ontology (i.e., DL-ontology) presented in Example 3. $K = (G, M, J)$ is a semantic Research Papers context with its tabular representation as in Table 6.

According to Definition 7 we know that $(\{P_1, P_3, P_5\}, \{Ontology, DLs, TLs, OWL\})$ and $(\{P_2, P_4, P_6\}, \{FCA, DM, KDD\})$ are two formal concepts.

Since $DLs \equiv_{\mathcal{DO}} TLs$, $DM \equiv_{\mathcal{DO}} KDD$, by Definition 15 we know that the above formal context (see Table 6) has redundancy. DLs (or TLs) and DM (or KDD) are redundant attributes.

Likewise, both $(\{P_1, P_3, P_5\}, \{Ontology, DLs, TLs, OWL\})$ and $(\{P_2, P_4, P_6\}, \{FCA, DM, KDD\})$ have redundancy.

Clearly, let $\underline{\mathcal{SB}}(G, M, J)$ be the semantic concept lattice of the semantic formal context (G, M, J) shown as Table 6. If $(\{P_1, P_3, P_5\}, \{Ontology, DLs, TLs, OWL\}) \in \underline{\mathcal{SB}}(G, M, J)$, then by Definition 15 we know that the $\underline{\mathcal{SB}}(G, M, J)$ is redundant.

To reduce attributes (i.e., remove the redundant attributes) in semantic formal context, we have to find the redundant attributes firstly. The approach is as Algorithm 2.

Algorithm 2. Find the redundant attributes in semantic formal context

Input: a domain ontology (i.e., DL-ontology) $\mathcal{DO}$, and a semantic formal context $K=(G, M, J)$ where $M=\{C_1, C_2, \ldots, C_m\}$

Output: the set of redundant attributes $red(K)$

```
red(K):=ϕ
M′:=M
forall attributes in M′ do
   if there exist C_i and C_j (1≤i, j≤m, i≠j) such that C_i≡_DO C_j, then
      red(K):=red(K)+{C_i} (or red(K):=red(K)+{C_j})
      M′:=M′−{C_i} (or M′:=M′−{C_j})
   endif
endfor
output red(K)
```

Table 7
A reduced semantic Research Papers context w.r.t. ontology.

	FCA	*Ontology*	*TLs*	*OWL*	*KDD*
P_1	×	×	×	×	
P_2	×	×			×
P_3		×	×	×	
P_4	×				×
P_5		×	×	×	×
P_6	×				×

Clearly, we also can find the redundant attributes in semantic formal concepts by using Algorithm 2.

Example 9 (*Example 8 Cont'd*)**.** Consider the semantic formal context given in Table 6. By Algorithm 2 we have that $red(K) = \{DLs, DM\}$ (or $\{DLs, KDD\}$, $\{TLs, DM\}$, or $\{TLs, KDD\}$).

Similarly, we have that $red(\{P_1, P_3, P_5\}, \{Ontology, DLs, TLs, OWL\}) = \{DLs\}$ (or $\{TLs\}$), and $red(\{P_2, P_4, P_6\}, \{FCA, DM, KDD\}) = \{DM\}$ (or $\{KDD\}$).

Intuitively, if there exist redundant attributes $red(K)$ (resp., $red(A, B)$) in a semantic formal context $K = (G, M, J)$ (resp., formal concept (A, B)), we should remove these attributes $red(K)$ (resp., $red(A, B)$) from the formal context K (resp., formal concept (A, B)). Let us continue to consider Example 8.

Example 10 (*Example 8 Cont'd*)**.** By Example 9 we know that $red(K) = \{DLs, DM\}$, thus, we can remove the attributes DLs and DM from Table 6. In other words, we can obtain the following formal context (see Table 7).

By Example 9 we also know that $red(\{P_1, P_3, P_5\}, \{Ontology, DLs, TLs, OWL\}) = \{DLs\}$ and $red(\{P_2, P_4, P_6\}, \{FCA, DM, KDD\}) = \{DM\}$, then we may remove the attributes DLs and DM from the corresponding formal concept, respectively. That is, we get the following:

$(\{P_1, P_3, P_5\}, \{Ontology, TLs, OWL\})$ and $(\{P_2, P_4, P_6\}, \{FCA, KDD\})$.

It is easy to verify that both $(\{P_1, P_3, P_5\}, \{Ontology, TLs, OWL\})$ and $(\{P_2, P_4, P_6\}, \{FCA, KDD\})$ are two formal concepts of the semantic formal context $(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$ (i.e., Table 7).

In fact, the construction of Example 10 can be adapted to other cases, leading to a general result as follows.

Theorem 10. *Let $\mathcal{DO} = \langle \mathcal{AB}, \mathcal{TB}, \mathcal{RB}\rangle$ be a domain ontology (i.e., DL-ontology), $K = (G, M, J)$ be a semantic context w.r.t. $\mathcal{DO}$. Then we have the following:*

(1) $(A, B) \in \mathcal{B}_{\mathcal{DO}}(G, M, J)$ iff $(A, B\backslash red(A, B)) \in \mathcal{B}_{\mathcal{DO}}(G, M\backslash red(K), J \cap (G\times (M\backslash red(K))))$;

(2) $\{(A_1, B_1), \ldots, (A_1, B_m)\} \leq \{(A_2, C_1), \ldots, (A_2, C_n)\}$ in $\underline{\mathcal{SB}}(G, M, J)$ iff

$$\{(A_1, B_1\backslash red(A_1, B_1)), \ldots, (A_1, B_m\backslash red(A_1, B_m))\} \leq \{(A_2, C_1\backslash red(A_2, C_1)), \ldots, (A_2, C_n\backslash red(A_2, C_n))\} \text{ in } \underline{\mathcal{SB}}(G, M\backslash red(K), J \cap (G \times (M\backslash red(K)))).$$

Proof. (1) Without loss of generality, we assume that $M = \{D_1, D_2, \ldots, D_s, \ldots, D_i, D_{i+1}, \ldots, D_k, D_{k+1}, \ldots, D_{l-1}, D_l, \ldots, D_{j-1}, D_j, \ldots, D_h\}$, $B = \{D_s, \ldots, D_{l-1}\}$, $red(K) = \{D_{i+1}, \ldots, D_{j-1}\}$, and $red(A, B) = \{D_{i+1}, \ldots, D_{l-1}\}$. Then we have the following

$B\backslash red(A, B) = \{D_s, \ldots, D_i\}$, $M\backslash red(K) = \{D_1, D_2, \ldots, D_s, \ldots, D_i, D_j, \ldots, D_h\}$. Hence, $B\backslash red(A, B) \subseteq M\backslash red(K)$.

"⇒". Since $(A, B) \in \mathcal{B}_{\mathcal{DO}}(G, M, J)$, that is, $(A, \{D_s, \ldots, D_{l-1}\}) \in \mathcal{B}_{\mathcal{DO}}(G, M, J)$, then $(A, \{D_s, \ldots, D_{l-1}\})$ is a formal concept. By Definition 7, we have $A' =_{\mathcal{DO}} B =_{\mathcal{DO}} \{D_s, \ldots, D_{l-1}\}$ and $B' = \{D_s, \ldots, D_{l-1}\}' = A$ in (G, M, J). That is, $\{m \in M|\ gJm$ for all $g \in A\} =_{\mathcal{DO}} \{D_s, \ldots, D_{l-1}\}$ and $\{g \in G|\ gJm$ for all $m \in \{D_s, \ldots, D_{l-1}\}\} = A$.

Since $red(K) = \{D_{i+1}, \ldots, D_{j-1}\}$ and $red(A, B) = \{D_{i+1}, \ldots, D_{l-1}\}$, then for any $D_u \in B\backslash red(A, B) = \{D_s, \ldots, D_i\}$, there is a attribute $D_v \in red(A, B) = \{D_{i+1}, \ldots, D_{l-1}\}$ such that $D_u \equiv_{\mathcal{DO}} D_v$. Thus, for any model $I = (\Delta^I, \bullet^I)$ of $\mathcal{DO}$, we have $D_u^I = D_v^I$, that is, for any $g \in A$ in G, gJD_u iff gJD_v in (G, M, J), where $J = I|_{\{M,G\}}$ (see Definition 5).

Let $J' = I|_{\{M\backslash red(K),G\}}$. Since $D_u \in B\backslash red(A, B) \subseteq M\backslash red(K)$, then we have that gJD_u in (G, M, J) iff $gJ'D_u$ in $(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$.

Therefore, gJD_u for all $g \in A$ in (G, M, J) iff $gJ'D_u$ for all $g \in A$ in $(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$. Then we have that $A' =_{\mathcal{DO}} B\backslash red(A, B)$ in $(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$.

Since $B' = \{D_s, \ldots, D_{l-1}\}' = A$ in (G, M, J), that is, $\{g \in G|\ gJm$ for all $m \in \{D_s, \ldots, D_{l-1}\}\} = A$ in (G, M, J). Since for any $g \in A$ in G, gJD_u iff gJD_v in (G, M, J), where $J = I|_{\{M,G\}}$, then

$$\{g \in G|\ gJm \text{ for all } m \in \{D_s, \ldots, D_{l-1}\}\} = \{g \in G| gJ'm \text{ for all } m \in \{D_s, \ldots, D_i\}\}.$$

Therefore, $(B\backslash red(A, B))' = \{D_s, \ldots, D_i\}' = A$ in $(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$.

According to Definition 7, we know that

$$(A, B\backslash red(A, B)) \in B_{\mathcal{DO}}(G, M\backslash red(K), J \cap (G \times (M\backslash red(K)))).$$

"⇐". The proof is similar to that of "⇒".

(2) By (1) we know that $(A_1, B_u), (A_2, C_v) \in \mathcal{B}_{\mathcal{DO}}(G, M, J)$ iff $(A_1, B_u\backslash red(A_1, B_u)), (A_2, C_v\backslash red(A_2, C_v)) \in \mathcal{B}_{\mathcal{DO}}(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$ $(1 \leq u \leq m, 1 \leq v \leq n)$. Then we have that

$\{(A_1, B_1), \ldots, (A_1, B_m)\}, \{(A_2, C_1), \ldots, (A_2, C_n)\} \in \underline{\mathcal{SB}}(G, M, J)$ iff

$\{(A_1, B_1\backslash red(A_1, B_1)), \ldots, (A_1, B_m\backslash red(A_1, B_m))\}, \{(A_2, C_1\backslash red(A_2, C_1)), \ldots, (A_2, C_n\backslash red(A_2,$

$C_n))\} \in \underline{\mathcal{SB}}(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$.

"⇒". Since $\{(A_1, B_1), \ldots, (A_1, B_m)\} \leq \{(A_2, C_1), \ldots, (A_2, C_n)\}$ in $\underline{\mathcal{SB}}(G, M, J)$, by Definition 14 we have that $A_1 \subseteq A_2$ (or $C_v \sqsubseteq_{\mathcal{DO}} B_u$ for any $1 \leq v \leq n$, $1 \leq u \leq m$). Thus,

$\{(A_1, B_1\backslash red(A_1, B_1)), \ldots, (A_1, B_m\backslash red(A_1, B_m))\} \leq \{(A_2, C_1\backslash red(A_2, C_1)), \ldots, (A_2, C_n\backslash red(A_2, C_n))\}$ in $\underline{\mathrm{SB}}(G, M\backslash red(K), J \cap (G \times (M\backslash red(K))))$ since $A_1 \subseteq A_2$ (or $C_v\backslash red(A_2, C_v) \sqsubseteq_{\mathcal{DO}} B_u\backslash red(A_1, B_u)$ for any $1 \leq v \leq n$, $1 \leq u \leq m$).

"⇐". The proof is similar to that of "⇒". □

6. Discussion and conclusion

In Sections 3 and 4, the definitions and properties of semantic FCA are provided. Now we briefly discuss the technical implementations and applications of semantic FCA.

Regarding semantic FCA, the first question to consider is how to construct a semantic formal context. In fact, given a set of objects, a set of attributes (or $\mathcal{DL}$-concepts), and a domain ontology (i.e., DL-ontology), a semantic formal context can be generated automatically by using Algorithm 1. It should be noted that semantic formal context construction in Algorithm 1 need to use DL reasoners such as HermiT [55] and Pellet [56] (see the list of DL reasoners[1] for more details). The next question to consider is how to build a semantic concept lattice using a semantic formal context. Since a semantic formal context is also a formal context, thus, on the basis of semantic formal context, we can build a concept lattice (denoted by CL) automatically by exploiting the algorithms of traditional concept lattice construction [1,68,69]. However, this concept lattice CL is not a semantic concept lattice. We must generate the corresponding semantic concept lattice (denoted by SCL) using the concept lattice CL.

From Definition 14 we know that a semantic concept lattice consists of sets of equivalently semantic formal concepts. From Definition 13 we know that a set of equivalently semantic formal concepts is a set $\{(A, B_1), (A, B_2), \ldots, (A, B_m)\}$, where A is a set of objects of semantic formal context, B_i $(1 \leq i \leq m)$ is a set of attributes of semantic formal context, and $B_1, B_2, \ldots$, and B_m are logical equivalent (see Definition 6). Obviously, we can obtain the corresponding semantic concept lattice SCL by adding sets of equivalently semantic formal concepts to the known concept lattice CL. The approach is shown as Algorithm 3.

Algorithm 3. Generate the semantic concept lattice
Input: a concept lattice CL w.r.t. domain ontology $\mathcal{DO}$
Output: the corresponding semantic concept lattice SCL of CL w.r.t. $\mathcal{DO}$

```
SCL:=ϕ
forall formal concepts (A, B) in CL do
  SCL:=SCL∪{(A, B)}
endfor
forall formal concepts (A1, B1) and (A2, B2) in CL do
  if (A1, B1)≤(A2, B2) then
    {(A1, B1)}≤{(A2, B2)} in SCL
  endif
endfor
forall set {(A, B)} of equivalently semantic formal concepts in SCL do
  if there exist B1, B2, …, and Bm such that B1=DO B2=DO…=DO Bm=DO B then
    add {(A, B1), (A, B2), …, (A, Bm)} to {(A, B)}, i.e.,
    {(A, B)}:={(A, B)}∪{(A, B1), (A, B2), …, (A, Bm)}
  endif
endfor
output SCL
```

[1] http://owl.cs.manchester.ac.uk/tools/list-of-reasoners/.

It should be noted that the generation of semantic concept lattice in Algorithm 3 also need to use DL reasoners (i.e., DL reasoners are needed to decide $B_1 =_{\mathcal{DO}} B_2 =_{\mathcal{DO}} \ldots =_{\mathcal{DO}} B_m =_{\mathcal{DO}} B$).

The construction of semantic formal contexts and semantic concept lattices is discussed above. Clearly, we can find the redundant attributes in semantic formal contexts and semantic formal concepts using Algorithm 2. When we find the redundant attributes of semantic formal contexts or semantic formal concepts, from Definition 15 we know that the attribute reductions of semantic formal contexts, semantic formal concepts, and semantic concept lattices can be implemented directly. For example, assume that $(\{g_1, g_2, \ldots, g_m\}, \{C_1, C_2, \ldots, C_n\})$ is a semantic formal concept and $C_1 =_{\mathcal{DO}} C_i =_{\mathcal{DO}} C_j$ $(i < j)$, from Definition 15 we know that $(\{g_1, g_2, \ldots, g_m\}, \{C_1, C_2, \ldots, C_n\})$ can be reduced to $(\{g_1, g_2, \ldots, g_m\}, \{C_1, C_2, \ldots, C_{i-1}, C_{i+1}, \ldots, C_{j-1}, C_{j+1}, \ldots, C_n\})$, $(\{g_1, g_2, \ldots, g_m\}, \{C_2, \ldots, C_{i-1}, C_i, C_{i+1}, \ldots, C_{j-1}, C_{j+1}, \ldots, C_n\})$, or $(\{g_1, g_2, \ldots, g_m\}, \{C_2, \ldots, C_{i-1}, C_{i+1}, \ldots, C_{j-1}, C_j, C_{j+1}, \ldots, C_n\})$.

Regarding the applications of semantic FCA, since traditional FCA has potential applications in many different fields such as information retrieval, data mining, ontology engineering, services computing, Semantic Web, and software engineering, naturally, semantic FCA has also potential applications in these fields. For example, Remark 1 gives an application of semantic search using semantic FCA.

In this paper, we semantify FCA with DLs, in other words, we present an extended FCA (i.e., semantic FCA) by using the concepts of DLs to act as the attributes of formal contexts. More concretely, we semantify the three components of traditional FCA, that is, we extend the formal concepts, attribute implications, and concept lattices in traditional FCA to semantic case. In addition, we also study the attribute reduction of the semantic formal contexts, formal concepts, and concept lattices from a semantics point of view. As far as future directions are concerned, these will include extending fuzzy FCA (resp., temporal FCA) with fuzzy DLs (resp., temporal DLs). At the same time, how to apply semantic FCA presented in this paper in different fields will also be pursued, especially, we will study semantic Web search (semantic search) based on semantic FCA.

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