I. ADMM Scheme

The original problem we want to solve is

$$\min_x \|\mathcal{T}(\mathbf{x})\mathbf{D}\|_F^2 + \lambda\|P_\Gamma \mathbf{x} - \mathbf{b}\|.$$

or equivalently

$$\min_x \sum_{i=1}^N \|\mathcal{T}(\mathbf{x})\mathbf{d}_i\|_F^2 + \lambda\|P_\Gamma \mathbf{x} - \mathbf{b}\|.$$

where $\mathbf{D} = [\mathbf{d}_1, \ldots, \mathbf{d}_N]$. Here we may write

$$\mathcal{T}(\mathbf{x})\mathbf{d} = \begin{bmatrix} \mathbf{PC}_i\mathbf{M}_x \\ \mathbf{PC}_i\mathbf{M}_y \end{bmatrix} \mathbf{x}$$

or more compactly $\mathcal{T}(\mathbf{x})\mathbf{d} = \mathbf{PC}_i\mathbf{Mx}$, with $\mathbf{M} = [\mathbf{M}_x^*, \mathbf{M}_y^*]^*$ and where $\mathbf{P}$ and $\mathbf{C}_i$ are understood to be applied to each block of $\mathbf{Mx}$.

We introduce the splitting $\mathbf{Y} = \mathbf{Mx}$, which gives the augmented cost:

$$\min_{\mathbf{x},\mathbf{Y}} \sum_{i=1}^N \|\mathbf{PC}_i\mathbf{Y}\|_F^2 + \lambda\|P_\Gamma \mathbf{x} - \mathbf{b}\|_2^2 + \gamma\|\mathbf{Y} - \mathbf{Mx} + \mathbf{L}\|_F^2.$$

where $\gamma > 0$ is a fixed parameter, $\mathbf{L}$ represents a collection of Lagrange multipliers. This results in the ADMM scheme:

$$\mathbf{Y}^{(n+1)} = \arg\min_{\mathbf{Y}} \sum_{i=1}^N \|\mathbf{PC}_i\mathbf{Y}\|_F^2 + \gamma\|\mathbf{Y} - \mathbf{Mx}^{(n)} + \mathbf{L}^{(n)}\|_F^2 \tag{1}$$

$$\mathbf{x}^{(n+1)} = \arg\min_{\mathbf{x}} \lambda\|P_\Gamma \mathbf{x} - \mathbf{b}\|_2^2 + \gamma\|\mathbf{Y}^{(n+1)} - \mathbf{Mx} + \mathbf{L}^{(n)}\|_F^2 \tag{2}$$

$$\mathbf{L}^{(n+1)} = \mathbf{L}^{(n)} + \mathbf{Y}^{(n+1)} - \mathbf{Mx}^{(n+1)} \tag{3}$$

To optimize (1), we set its gradient equal to zero, and obtain

$$\underbrace{\left[\sum_{i=1}^N (\mathbf{C}_i^*\mathbf{P}^*\mathbf{PC}_i) + \gamma\mathbf{I}\right]}_{\mathbf{R}} \mathbf{Y} = \gamma(\mathbf{Mx}^{(n)} - \mathbf{L}^{(n)})$$

This we may solve by PCG. A natural preconditioner is to form a circulant approximation of $\mathbf{R}$ by replacing the projection $\mathbf{P}^*\mathbf{P}$ with identity, i.e.

$$\mathbf{S} = \sum_{i=1}^N (\mathbf{C}_i^*\mathbf{C}_i) + \gamma\mathbf{I} = \mathbf{F}(\mathbf{\Sigma} + \gamma\mathbf{I})\mathbf{F}^*$$

where $\mathbf{\Sigma}$ is a diagonal matrix representing multiplication by spatially gridded samples of the sum-of-squares polynomial, and so $\mathbf{\Sigma}+\gamma\mathbf{I}$ can be interpreted as an annihilation mask with improved conditioning. Alternatively, we could use the preconditioner $\mathbf{S}$ as a surrogate for $\mathbf{R}$, which then could be solved exactly with two FFTs. This would be super-fast, since the whole least-squares problem would only require maybe $\sim 20-50$ FFTs, and not the current ~ 2000.

Subproblem (2) we can show has an analytical solution. Setting its gradient to zero, we have

$$\left(\mathbf{P}_\Gamma^*\mathbf{P}_\Gamma + \frac{\gamma}{\lambda}\mathbf{M}^*\mathbf{M}\right)\mathbf{x} = \mathbf{P}_\Gamma^*\mathbf{b} + \frac{\gamma}{\lambda}\mathbf{M}^*(\mathbf{Y}^{(n+1)} + \mathbf{L}^{(n)}).$$

The matrix on the left is diagonal, so its inverse acts simply by an element-wise multiplication.

II. Multiple Priors

Consider the case of multiple low-rank priors

$$\min_{\mathbf{x}} \sum_j \|\mathcal{T}_j(\mathbf{x})\|_* + \lambda\|P_\Gamma\mathbf{x} - \mathbf{b}\|_2^2$$

We may adapt the IRLS algorithm to this setting by solving iterates of

$$\min_{\mathbf{x}} \sum_j \|\mathcal{T}_j(\mathbf{x})\mathbf{H}_1^{1/2}\|_F^2 + \lambda\|P_\Gamma\mathbf{x} - \mathbf{b}\|_2^2$$

which may be recast as

$$\min_{\mathbf{x},\mathbf{Y}_j} \sum_j \sum_{i=1}^N \|\mathbf{P}\mathbf{C}_{i,j}\mathbf{Y}_j\|_F^2 + \lambda\|P_\Gamma\mathbf{x} - \mathbf{b}\|_2^2 + \sum_j \gamma_j\|\mathbf{Y}_j - \mathbf{M}_j\mathbf{x} + \mathbf{L}_j\|_F^2.$$

which is fully separable in terms of the Y_j. This gives

$$\mathbf{Y}_j^{(n+1)} = \arg\min_{\mathbf{Y}_j} \sum_{i=1}^N \|\mathbf{P}\mathbf{C}_{i,j}\mathbf{Y}_j\|_F^2 + \gamma_j\|\mathbf{Y}_j - \mathbf{M}\mathbf{x}^{(n)} + \mathbf{L}^{(n)}\|_F^2 \quad (4)$$

$$\mathbf{x}^{(n+1)} = \arg\min_{\mathbf{x}} \lambda\|P_\Gamma\mathbf{x} - \mathbf{b}\|_2^2 + \sum_j \gamma_j\|\mathbf{Y}_j^{(n+1)} - \mathbf{M}\mathbf{x} + \mathbf{L}^{(n)}\|_F^2 \quad (5)$$

$$\mathbf{L}_j^{(n+1)} = \mathbf{L}_j^{(n)} + \mathbf{Y}_j^{(n+1)} - \mathbf{M}_j\mathbf{x}^{(n+1)} \quad (6)$$

and problems (4), (5) have solutions

$$\left(\mathbf{P}_\Gamma^*\mathbf{P}_\Gamma + \frac{1}{\lambda}\sum_j \gamma_j\mathbf{M}_j^*\mathbf{M}_j\right)\mathbf{x} = \mathbf{P}_\Gamma^*\mathbf{b} + \frac{1}{\lambda}\sum_j \gamma_j\mathbf{M}_j^*(\mathbf{Y}_j^{(n+1)} + \mathbf{L}_j^{(n)}).$$

and

$$\left[\sum_{i=1}^N(\mathbf{C}_{i,j}^*\mathbf{C}_{i,j}) + \gamma\mathbf{I}\right]\mathbf{Y}_j = \gamma(\mathbf{M}_j\mathbf{x}^{(n)} - \mathbf{L}_j^{(n)})$$