Optimization: Linear Programming

Objectives

- Understand linear programming and a few of its applications
- Be capable of formulating and solving basic problems, using s/w
- Be able to interpret sensitivity output provided by s/w

Mathematical Models

- Models are representations of situations
  - draw conclusions about real situation by studying & analyzing model
- Models, compared to experimenting with an actual situation, are:
  - less expensive, less time-consuming, less risky, more feasible
- However
  - models are simplifications of reality
  - more closely a model approximates & acts like real situation, better it is
  - model that seeks "right answer to wrong question" is of no value

Math Programming

- Deals with resource allocation to **maximize** or **minimize** an objective subject to certain constraints
- Types:
  - Linear, Integer, Mixed, Nonlinear, Goal (multiple objectives)
- Relatively easy to solve using modern computing technology

Why MP?

- Rooted in WWII (operational research); complex - need computer
- Applied in '60s to petroleum refinery product mix problem
- Widespread application now to many problems now, in all functional areas
  - Product mix, Project selection, Portfolio selection, Workforce balancing, Media mix, Financial planning, …

Our Focus

- Recognize when math programming is appropriate
- Develop basic linear models
- Computer solution
  - Excel, Solver
  - Many other s/w pkgs available
- Interpreting results

Lego Enterprises Exercise

- Table profit is $16; Chair profit is $10
- Table design
  - 2 large blocks (side by side)
  - 2 small blocks (stacked under, centered)
- Chair design
  - 1 large block (seat)
  - 2 small blocks (back, bottom)
- Objective: select product mix to maximize profits, using available resources

Understanding Lego Problem

- Formulate as LP
  - Decision Variables, Objective Function, Constraints
- Graph
  - Constraints, Objective function
- Find solution
- Simplex approach

LP Formulation

- Identify/define Decision Variables
  - T = # of tables
  - C = # of chairs
- Objective (as equation)
  - Maximize profit =
- Constraints (as equations)
  - For large blocks:
  - For small blocks:

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Graphing Lego Example

- Draw quadrant & axes
  - use **T** on x-axis and **C** on y-axis
- Add constraint lines
  - Find intercepts: set T to zero and solve for C, set C to zero and solve for T
- Add profit equation
  - Select reasonable value
- Move profit equation outwards, as far as feasible

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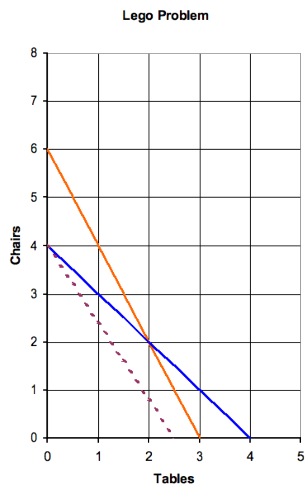
Graph



- Blue Line = Small block constraint

  2T + 2C <= 8
- Red Line = Large block constraint

  2T + C <= 6
- Dashed Line = Profit

  z = 16T + 10C

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Simplex Approach

- #1. Start at origin (all activity levels zero)
- #2. Find activity with highest marginal value
  - If no improvement possible, stop
  - Else, go with this activity
- #3. Find tightest constraint
  - Use it all
  - Substitute one activity for another, where necessary
- #4. List new soln; go back to #2

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Characteristics Of LPs

- Objective function and constraints are **linear functions**
- Constraint types are $\leq$, $=$, or $\geq$
- Variables can assume any fractional value
  - not as great a limitation as it may seem
- Decision variables are non-negative
- Maximize or Minimize single objective

Key Definitions

- **Feasible solution**: one that satisfies all constraints
  - can have many feasible solutions
- **Feasible region**: set of all feasible solutions
- **Optimal solution**: any feasible solution that optimizes the objective function
  - can have "ties"

Standard LP Form

- All constraints expressed as equalities
  - use **slack** (**≤**) or **surplus** (**≥**) variables
- All variables are nonnegative
- All variables appear on the left side of the constraint functions
- All constants appear on the right side of the constraint function

COMPUTER S/W DOES THIS FOR US!

Formulate Lego problem in standard form

Max $z = 16T + 10C$

s.t.

$2T + 2C <= 8$ small block constraint

$2T + C <= 6$ large block constraint

$T, C >= 0$

4 Possible LP Outcomes

- Unique optimal solution
- Alternate optimal solutions
- Unbounded problem
- Infeasible problem

Example: Unique Optimal Soln

- Solve graphically for the optimal solution:

Max $z = 6x_1 + 2x_2$

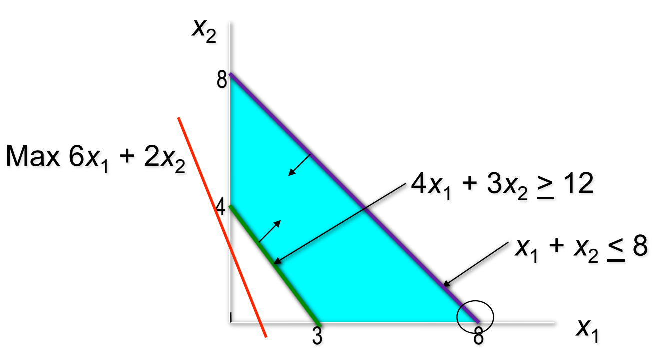
s.t. $4x_1 + 3x_2 \geq 12$

$x_1 + x_2 \leq 8$

$x_1, x_2 \geq 0$

Example: Unique Optimal

- There is only one point that satisfies both constraints



Example: Alternate Solutions

- Solve graphically for the optimal solution:

Max $z = 6x_1 + 3x_2$

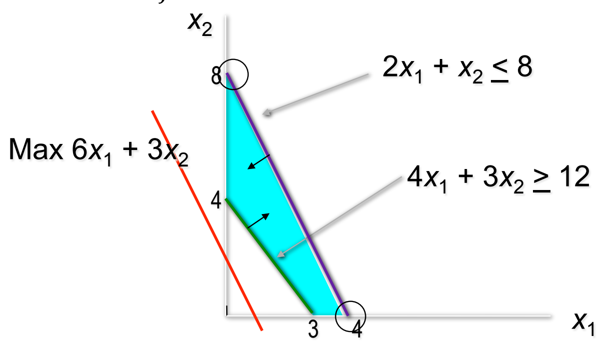
s.t. $4x_1 + 3x_2 \geq 12$

$2x_1 + x_2 \leq 8$

$x_1, x_2 \geq 0$

Example: Alternate Solutions

- There are infinite points satisfying both constraints
  - objective function falls on a constraint line



Example: Infeasible Problem

- Solve graphically for the optimal solution:

Max $z = 2x_1 + 6x_2$

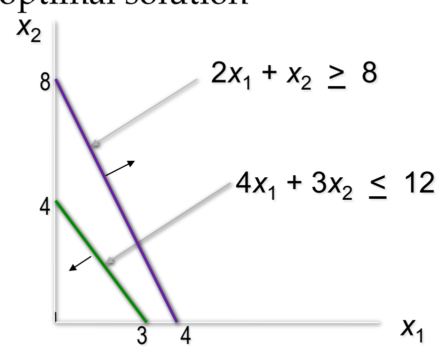
s.t. $4x_1 + 3x_2 \leq 12$

$2x_1 + x_2 \geq 8$

$x_1, x_2 \geq 0$

Example: Infeasible Problem

- No points satisfy both constraints
  - no feasible region
  - no optimal solution



Example: Unbounded Problem

- Solve graphically for the optimal solution:

Max $z = 3x_1 + 4x_2$

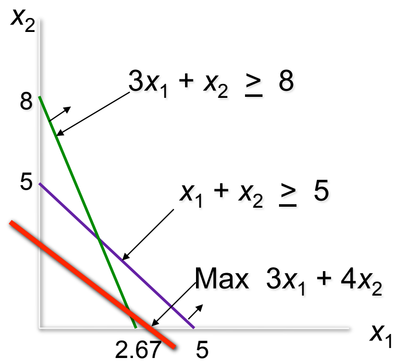
s.t. $x_1 + x_2 \geq 5$

$3x_1 + x_2 \geq 8$

$x_1, x_2 \geq 0$

Example: Unbounded Problem

- Feasible region unbounded
  - objective function can be moved outward without limit; *z* can be increased infinitely



Building LP Models: Key Questions

- What am I trying to decide?
- What is the objective? Is it to be **minimized** or **maximized**?
- What are the constraints? Are they **limitations** or **requirements**? Are they **explicit** or **implicit**?

Example: Production Planning

A Toronto chemical company manufactures and sells a product in 40-lb and 80-lb bags on a common production line. To meet forecasted orders, next week's production should be at least 16,000 lbs. Profit contributions are $2 per 40-lb bag, and $4 per 80-lb bag. The packaging line operates 1500 min. per week. 40-lb bags require 1.2 min of packaging time; 80-lb bags require 3 min. The company has 6000 square feet of packaging material available. Each 40-lb bag uses 6 square feet, and each 80-lb bag uses 10 square feet. How many bags of each type should be produced?

Model Development

- What do we need to decide?
- How many decision variables?

x1 =

x2 =

Model Development

- What is the objective?

Maximize total
profit

z =

Note: (\$/bag)(no. of bags) = \$. Always perform a dimensional analysis!

Model Development

- What are the constraints?

Aggregate production:

Packaging time:

Packaging materials:

Nonnegativity:

Check the dimensions!

LP Formulation (Steps)

- Define decision variables: x1, x2, …
- Objective Function (max, min)
- s.t., with constraints listed
  - Variables on left side
  - Constants on right side
  - All decision variables nonnegative
- NB: "Standard Form" requires constraints stated as equalities
  - add slack/surplus variables (computer does this automatically)

Complete Model

x1 = no. of 40-lb bags to produce
x2 = no. of 80-lb bags to produce

Maximize z = 2x1 + 4x2
subject to

$$40x_1 + 80x_2 \geq 16{,}000$$
$$1.2x_1 + 3x_2 \leq 1{,}500$$
$$6x_1 + 10x_2 \leq 6{,}000$$
$$x_1, x_2 \geq 0$$

Optimal Solution; Definitions

- Three parts: **decision variables**, **values** of decision variables, objective function **value (OFV)**
- Decision variables: **basic** (non-zero value), **non-basic** (zero)
- Basic variables are "in the solution"; non-basic are not

Sensitivity Analysis

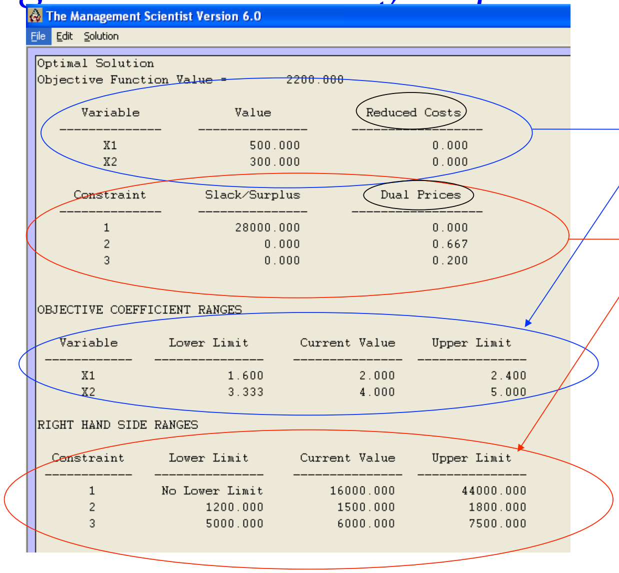
Why Sensitivity Analysis?

- Used to determine how **optimal solution** is affected by changes, within specified ranges
  - either objective function or RHS coefficients **(only 1 at a time)**
- Important to managers who must operate in a dynamic environment with imprecise estimates of coefficients
- Sensitivity analysis allows us to ask certain what-if questions

Impact of Possible Changes

- Change objective function coefficient
  - may change optimal solution
- Change existing constraint RHS
  - changes slope; may change size of feasible region
- Add new constraint
  - may decrease feasible region (if binding)
- Remove constraint
  - may increase feasible region (if binding)

Mng. Sct. Sensitivity Report


The Management Scientist Version 6.0

File Edit Solution

Optimal Solution

Objective Function Value = 2200.000

Variable	Value	Reduced Costs
X1	500.000	0.000
X2	300.000	0.000

Constraint	Slack/Surplus	Dual Prices
1	28000.000	0.000
2	0.000	0.667
3	0.000	0.200

OBJECTIVE COEFFICIENT RANGES

Variable	Lower Limit	Current Value	Upper Limit
X1	1.600	2.000	2.400
X2	3.333	4.000	5.000

RIGHT HAND SIDE RANGES

Constraint	Lower Limit	Current Value	Upper Limit
1	No Lower Limit	16000.000	44000.000
2	1200.000	1500.000	1800.000
3	5000.000	6000.000	7500.000

Reduced Costs (Obj. Func. Coeff.)

- Reduced cost for decision variable not in solution (current value = 0) is amount variable's obj. func. coeff. would have to **improve** (increase for max, decrease for min) before variable could enter solution
- Reduced cost for **basic** decision variable **in solution** is 0
- "Could enter" = tie (non-basic, reduced cost = 0)

Obj. Func. Coeff. Ranges

- Lower limit / Upper limit
- Interval within which **original solution** remains optimal (same decision variables in solution; usually not same OFV) while keeping all other data constant
- Within range, associated **reduced cost** is valid
- **Value** of objective function **might** change in this range of optimality

Dual Price (RHS Coeff.)

- Amount objective function will **improve** per unit **increase** in constraint RHS value
  - watch sign; max vs min problems
- Reflects value of an additional unit of resource (if resource cost is sunk); reflects extra value over normal cost of resource (when resource cost is relevant)
- Always 0 for nonbinding constraint (positive slack or surplus at optimal solution)

RHS Ranges

- Lower Limit / Upper Limit
- As long as constraint RHS coefficient stays within range, associated dual price is valid
- For changes outside range, must resolve

Change Outside Range

- If change in objective function coefficient (OFC) or RHS of a constraint is outside allowable increase/decrease, the problem needs to be re-solved on the computer
- Cannot say anything about objective function value (OFV) or optimal solution (OS)

Tightening or Relaxing Constraints

- Tightening a constraint means to make it more restrictive; i.e. **decreasing** the RHS of a ≤ constraint, or **increasing** the RHS of a ≥ constraint
  - compresses feasible region
  - **may** make solution worse
- Relaxing a constraint means to make it less restrictive
  - expands feasible region
  - **may** make solution better

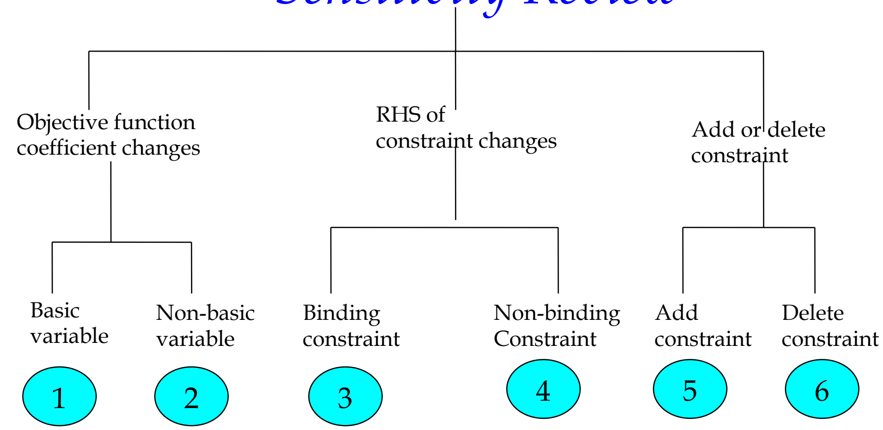
Sunk vs Relevant Costs

- When completing sensitivity analysis re constraint resources (RHS changes)
- Sunk cost: no charge for additional resources; already available
  - i.e., so much mtl available, lbr hours, etc.
- Relevant cost: charge for additional resources (overtime, raw mtl, …)
  - Need to subtract relevant cost per unit from shadow price for net profit increase

Sensitivity Workshop

- What happens if OFC changes?
  - 40 lb bag profit is $2.20, $1.70, $1.50, $2.50
  - 80 lb bag profit is $4.50, $3.50, $3.00, $5.00
- What if constraint RHS changes?
  - Aggregate demand > 8000, >40,000, >50,000
  - Pkg Line time increases by 200 min, 400 min
  - Pkg Line time decreases by 200 min, 400 min
  - Pkg mtl increases by 1000, 2000 ft^2
  - Pkg mtl decreases by 1000, 2000 ft^2

Sensitivity Review



1. Obj Func, Basic Variable

- If OF (objective function) coefficient of basic variable changes within allowable limits, OF slope changes:
  - Optimal solution (basic vars) unchanged
  - Objective function value changes
    - » New OFV can be determined by substituting OS into new objective function.
- Reduced cost is always zero for a basic variable

2. Obj Func, Non-basic Variable

- If objective function coefficient of basic variable changes within allowable limits,
  - Optimal solution is unchanged (DVs & values)
  - Objective function value is unchanged
- Reduced cost is usually non-zero
  - If reduced cost is zero, indicates existence of alternate optimal solution**
- For maximization problem
  - Reduced cost >= 0
  - Upper Limit = Current Value + Reduced Cost
  - Lower Limit = Negative Infinity (No Lower Limit)
- For minimization problem
  - Reduced cost <= 0
  - Upper Limit = Infinity (No Upper Limit)
  - Lower Limit = Current Value + Reduced Cost

3. RHS, Binding Constraint

- If RHS of binding constraint changes within allowable limits
  - Optimal solution changes (some/all DV values)
  - OFV changes
  - New OFV = old OFV + change in RHS*(dual price)
- Dual price for maximization problem
  - Positive for <= type constraint
  - Negative for >= type constraint
- Dual price for minimization problem
  - Positive for <= type constraint
  - Negative for >= type constraint

4. RHS, Non-binding Constraint

- If RHS of non-binding constraint changes within allowable limits
  - Optimal solution is unchanged & OFV is unchanged
- Dual price for non-binding constraint is always zero
- Non-binding constraint of <= type
  - Upper Limit = Infinity (No Upper Limit)
  - Lower Limit = Final Value - Slack
- Non-binding constraint of >= type
  - Upper Limit = Final Value - Surplus
  - Lower Limit = Negative infinity (No Lower Limit)

5. Add a Constraint

- If current optimal solution satisfies constraint,
  - Optimal solution is unchanged
  - OFV is unchanged
- If current optimal solution violates new constraint,
  - Need to re-solve problem using computer
  - OFV and optimal solution changes

6. Deleting a Constraint

- If deleted constraint is binding
  - Both optimal solution & OFV change
  - Need to re-solve problem on computer to obtain new solution values
- If deleted constraint is non-binding
  - Both optimal solution & OFV unchanged
  - May effect feasible region

Other Types of Changes

- Add a variable
  - No sensitivity analysis can be done; re-solve
- Change coefficients on LHS of constraints
  - No sensitivity analysis can be done using solver; re-solve

Shadow vs Dual Prices

- Shadow Price: Amount objective function will **change** per unit **increase** in RHS value of constraint. Excel Solver gives this result.
  - An increase relaxes a ≤ constraint, and tightens a ≥ constraint
- Some s/w pkgs (Management Scientist) calculate a dual price
  - Amount objective function will **improve** per unit increase in constraint RHS value
  - so same for max, but opposite sign for min

Common LP Applications

- Operations
  - Diet, Production Planning, Blending, Transportation, Product Mix
- Marketing
  - Media Selection, Marketing Research
- Finance
  - Portfolio Selection, Financial Planning
- HR
  - Workforce assignment
- Other
  - Revenue management, Data Envelope Analysis (compare efficiencies of multiple units)