Probability & Probability Distributions

BU609-2

Objectives

- Understand basic probability terms & concepts
- Be able to work with the most common probability distributions
  - Use Excel functions or tables

Probability Concepts

"The only certainty is uncertainty, the only constant is change"

Probability

- Likelihood of an event; Chance of something happening; Relative rate of occurrence of events
- $0 \leq p \leq 1$
- Commonly stated as %
- Common measure of risk (uncertainty) in finance and decision analysis

Risk Management

- Options
  - Avoid/prevent, manage/control/mitigate, accept/embrace
- Basic Process
  - Identify potential risks, assign likelihoods (probabilities), determine expected impact, prioritize and develop plans
- Applications: Insurance, Construction, Finance, Healthcare, …

Notation & Terminology

- Random Experiment: Process with well defined, but uncertain, outcomes
- Trial: Conducting a random experiment once
- Outcome/Sample Point: Possible result of a random experiment
- Sample Space (S): Set of all possible outcomes of a random experiment

Events

- Subsets of *S* (possible outcomes)
- Qualitative (category or type) or Quantitative (number)
- Simple (one basic outcome) or Compound (combination of outcomes)
- Probabilities are normally "assigned" to simple events & calculated for other events
- P(E) = probability of event E

Example: TSX Composite Index

- TSX index over 2 consecutive days

Simple Event	Day 1	Day 2
E_1	Up	Up
E_2	Up	Down
E_3	Down	Up
E_4	Down	Down

- Compound Events
  - Index is up on first day (E_1 and E_2)
- Conditional Events
  - Index is Up on second day **given** it was Down on first day (E_3)

Example: Bradley Investments

Bradley has invested in two stocks, Markley Oil and Collins Mining. Bradley has determined that the possible outcomes of these investments three months from now are as follows.

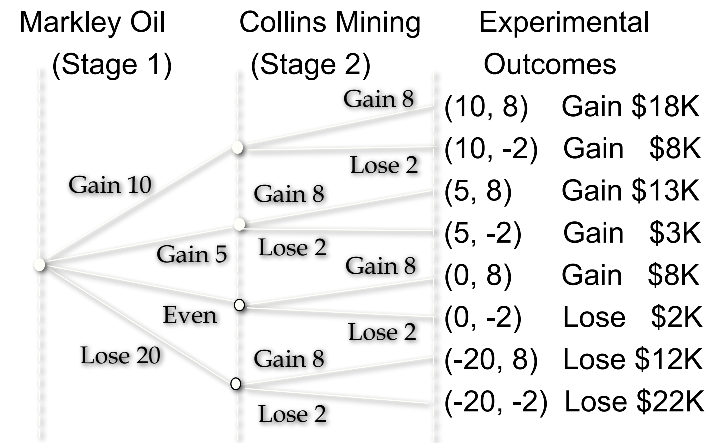
Investment Gain or Loss in 3 Months (in $000)

Markley Oil	Collins Mining
10	8
5	-2
0	
-20	

Identify: *Experiment, Sample Space, Sample Point, Event*

Example: Bradley Investments

- Probability Tree



Determining Probabilities (1)

- Classical
  - Make logical assumptions (such as equally likely, independence) about the design of the random experiment
- Example:
  - Roll a fair die and record the face up
  - S={1,2,3,4,5,6}
  - Term "fair" means each face is equally likely to show up, so P(i) = 1/6, for i=1,2,3,4,5,6

Determining Probabilities (2)

- Relative Frequency
  - Conduct large # of trials of the random experiment, then use the relative frequency of occurrence of an event as an estimate of its probability
  - Let A be any event in s
  - P(A) ≈ a / n, where a is number of times event A was observed in n trials
  - Most common approach for business (CRM, revenue management, …)

Relative Frequency Example

- Lucas Tool Rental

  LTL wants to assign probabilities to the # of floor polishers it rents per day. Office records show the following frequencies of daily rentals for the last 40 days.

# Polishers Rented	# of Days	Probability
0	4	.10 = 4/40
1	6	.15 = 6/40
2	18	.45 etc.
3	10	.25
4	2	.05
	40	1.00

Determining Probabilities (3)

- Subjective Assessment
  - Use past experience or personal preference to assign probability
- Why:
  - If economic conditions & company circumstances change rapidly, historical data often inappropriate
  - Can use available data as well as our experience and intuition
  - Best probability estimates often a combination of classical or relative frequency with subjective estimates

Example: Bradley Investments

Applying the subjective method an **analyst** made the following probability assignments.

Outcome	Net Gain/Loss	Prob.
(10, 8)	$18,000 Gain	.20
(10, -2)	8,000 Gain	.08
(5, 8)	13,000 Gain	.16
(5, -2)	3,000 Gain	.26
(0, 8)	8,000 Gain	.10
(0, -2)	2,000 Loss	.12
(-20, 8)	12,000 Loss	.02
(-20, -2)	22,000 Loss	.06

Determining Probabilities (4)

- Theoretical
  - Assumptions about experiment tell us how to assign probabilities, using known probability distributions
  - Examples: Binomial, Normal, Uniform, Exponential, Poisson, ...
  - Apply to quantitative data only (but can use proportions with qualitative data)
  - Will use this approach. Methods (2) & (3) are more commonly used in business

Probability Essentials

- Probability of event A in sample space *S* is denoted P(A)
- P(A) defined for all events of interest:
  1. $0 \le P(A) \le 1$
  2. $P(S) = 1$
  3. $P(\Phi) = 0$, where Φ is the **Null** set

Give me a marketing example where
P(sale) = 1, where P(sale) = 0

Events & Probabilities

- An **event** is a collection of one or more sample points
- Probability of any event, p(E), is equal to sum of probabilities of sample points in event
- Example:

Event *M* = Markley Oil Profitable

$M = \{(10, 8), (10, -2), (5, 8), (5, -2)\}$

$P(M) = P(10, 8)+P(10, -2)+P(5, 8)+P(5, -2)$

$= .2 + .08 + .16 + .26 = .70$

Event *C* = Collins Mining Profitable

$P(C) = .48$ (found using same logic)

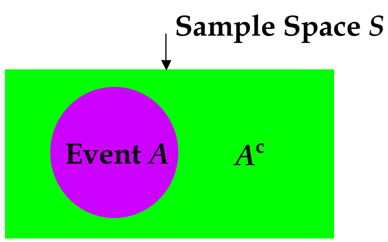
This is **enumeration** approach

Basic Probability Relationships

- Complement of an Event
- Union of Two Events
- Intersection of Two Events
- Mutually Exclusive Events
- 2 Laws (Addition, Multiplication)

Complement of an Event

- **Complement** of event *A* is defined to be event consisting of all sample points that are **not** in *A*
- Complement of *A* is denoted by A^c or $\bar{A}$
- **Venn diagram** illustrates the concept of a complement
- $P(A) + P(A^c) = 1$

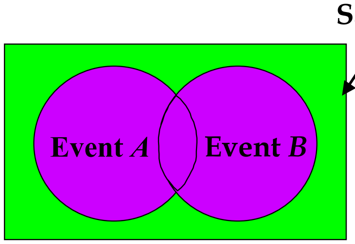


Venn Diagrams

- Many probability concepts can be easily understood with Venn Diagrams
- Rectangle represents the sample space *S* (with total area of 1). Events such as A & B (subsets of *S*) are normally shown as circles, ellipses, etc.
- Prob. of an event can be interpreted as area of the circle or ellipse for the event
- Many probability rules can be explained using Venn Diagrams

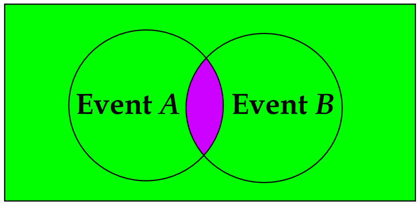
Union of Two Events

- **Union** of events *A* and *B* is event containing all sample points **in *A* or *B* or both**
- Denoted by $A \cup B$ or 'A or B'



Intersection of Two Events

- **Intersection** of events *A* and *B* is set of all sample points **in both *A* and *B***
- Denoted by $A \cap B$ or AB or 'A and B'



Union Example (Enumeration)

Event *M* = Markley Oil Profitable

Event *C* = Collins Mining Profitable

$M \cup C$ = Markley Oil Profitable <u>or</u> Collins Mining Profitable

$M \cup C$ = {(10,8), (10,-2), (5,8), (5,-2), (0,8), (-20,8)}

$P(M \cup C)$ = P(10,8) + P(10,-2) + P(5,8) + P(5,-2) + P(0,8) + P(-20,8)

= .20 + .08 + .16 + .26 + .10 + .02

= .82

Used **enumeration** approach

Intersection Example (Enumerate)

- Example:

Event M = Markley Oil Profitable

Event C = Collins Mining Profitable

$M \cap C$ = **Markley Oil Profitable**

<u>and</u> Collins Mining Profitable

$M \cap C = \{(10, 8), (5, 8)\}$

$$P(M \cap C) = P(10, 8) + P(5, 8)$$
$$= .20 + .16$$
$$= .36$$

Addition Law Example

- **Addition law** provides way to compute prob. of event A or B **or** both A and B occurring.
- Written as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Example: Markley Oil **or** Collins Mining Profitable

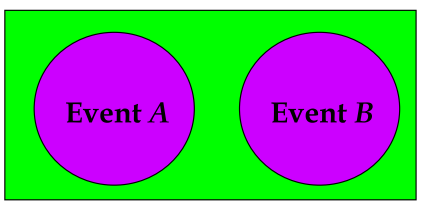
We know: $P(M) = .70$, $P(C) = .48$, $P(M \cap C) = .36$

Thus: $P(M \cup C) = P(M) + P(C) - P(M \cap C)$
$$= .70 + .48 - .36$$
$$= .82$$

Same result as obtained earlier using enumeration

Mutually Exclusive Events

- Two events are said to be **mutually exclusive** if the events have no sample points in common. That is, when one event occurs, the other cannot occur.



- Addition Law for Mutually Exclusive Events:

$$P(A \cup B) = P(A) + P(B)$$

Conditional Probability

- The prob. of an event **given that another event has occurred** is called a **conditional probability**
- The conditional probability of A **given** B is denoted by $P(A \mid B)$
- Additional info can be used in revising probability
- A conditional probability is computed as follows:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

- Example: Collins Mining Profitable **given** Markley Oil Profitable

$$P(C \mid M) = \frac{P(C \cap M)}{P(M)} = \frac{.36}{.70} = .51$$

Life Insurance Mortality Table

Age	Remaining Years of Life: Both Sexes	Male	Female
0	76.1	73.0	79.0
1	75.6	72.6	78.6
5	71.7	68.7	74.7
10	66.8	63.8	69.7
15	61.9	58.9	64.8
20	57.1	54.2	59.9
25	52.4	49.6	55.1
30	47.7	44.9	50.2
35	43.0	40.4	45.4
40	38.4	35.9	40.7
45	33.9	31.5	36.0
50	29.4	27.1	31.5
55	25.2	23.0	27.1
60	21.2	19.2	22.9
65	17.5	15.7	18.9
70	14.1	12.5	15.3
75	11.1	9.8	11.9
80	8.3	7.3	8.9
85	6.1	5.4	6.4

All Races, 1996

These tables were issued in 1997 by the National Center for Health Statistics, a unit of the U.S. Department of Health and Human Services.

http://www.budgetrates.com/mortality.htm

Expected Values

$P(x \geq 38.4 \mid \text{age } 40) = 0.5$

$P(x \geq 35.9 \mid \text{male } 40) = 0.5$

Conditional Probability (cont.)

- Can easily calculate from tabular data:
  - Example re Annual Salary ($)

JOB	<40K	40-60K	>60K	Total
MNGMNT	2	10	8	20
SUPRVSR	5	6	4	15
WORKER	48	15	2	65
TOTAL	55	31	14	100

- P($<40K | Mngr) = 2/20 = .1
- P(Mngr | $<40K) = 2/55 = .036

Multiplication Law

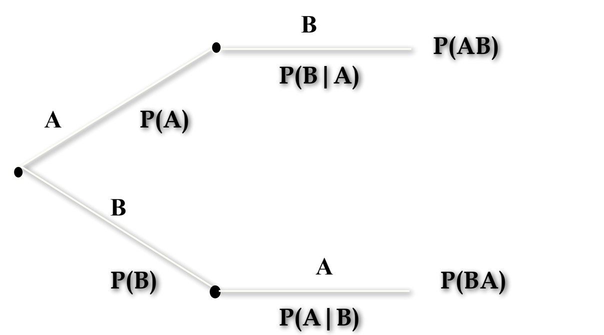
- Computes probability of intersection of two events
- $P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$
- Example: Markley Oil and Collins Mining Profitable

  We know: $P(M) = .70$, $P(C \mid M) = .51$

  Thus: $P(M \cap C) = P(M)P(C \mid M) = (.70)(.51) = .36$

  Result is same as obtained earlier using enumeration

Mult. Law – Tree Diagram



P(BA) = P(AB)

Mult. Law (Independent Events)

- Events A & B are **independent** if $P(A \mid B) = P(A)$
- Multiplication Law for Independent Events:

  $P(A \cap B) = P(A)P(B)$
- Multiplication law can also be used to test if two events are independent
- Example: Are M and C independent?

Does $P(M \cap C) = P(M)P(C)$?

We know: $P(M \cap C)=.36$, $P(M)=.70$, $P(C)=.48$

But: $P(M)P(C) = (.70)(.48) = .34$

Since $.34 \neq .36$, M and C are not independent.

Six Sigma Payoff

# Operations, Components	Yield % (99%)	Yield % (99.9996%)	Improvement %
1	99.00	100.00	1
10	90.44	100.00	10.56
100	36.60	99.96	173
500	0.66	99.80	15089
1000	0.00	99.60	2,307,009
5000	0.00	98.02	$6.5*10^{23}$

DPMO, CTQ, First time yield vs Throughput yield

Summary re Events

- S = Sample space (all events)
- Φ = Null (impossible) event
- $E_1 \cup E_2 = E_1$ union E_2; either E_1 or E_2 or both
- $E_1 \cap E_2 = E_1$ intersection E_2; both E_1 & E_2 occur
- E_1^C (complement) = E_1 does not occur
- Mutually Exclusive = E_1 & E_2 cannot occur at the same time; $E_1 \cap E_2 = \Phi$
- Collectively exhaustive = set of all events in S; at least one must occur in a trial
- Independent = knowledge of one event does not help determine probability of another event

Summary of Prob. Approaches

- Enumeration
- Venn diagram
- Probability tree diagram
  - for conditional probabilities
- Table
  - particularly helpful for two events (A, B) and working with conditional probabilities
- Probability Laws

Sometimes one approach is easier

Probability Distributions

Random Variables & Prob. Dist.

- **Random Variable** (R.V.)
  - Outcome on an **experiment**
  - Will consider only **numerical** outcomes here
- Can be **discrete** (distinct point values) or **continuous** (an interval)
- **Probability distribution** (model) defines values a specific random variable takes on; can be **discrete** (e.g. Binomial, Poisson), or **continuous** (e.g. Normal, Uniform, Exponential)
  - $p(x)$ for discrete (individual probabilities)
  - $f(x)$ for continuous (defines a curve, with area = 1)

Discrete Prob. Distributions

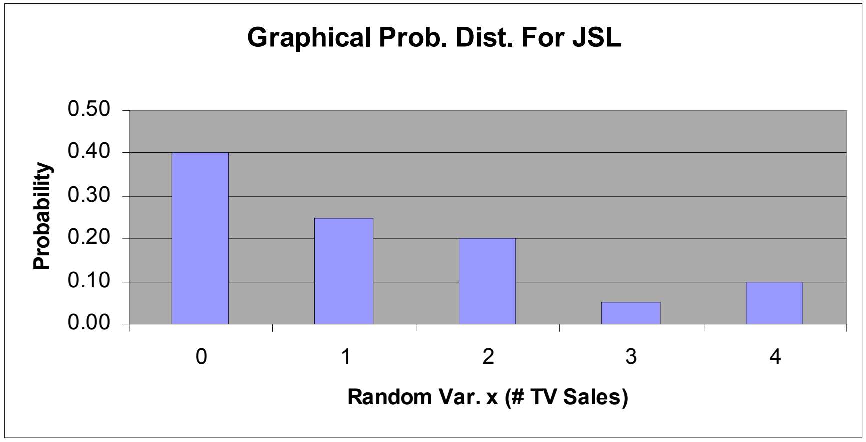
- Describes how probabilities are distributed over values of random variable
- Can describe a discrete probability distribution with a **table**, **graph**, or **equation**
  - Option: **cumulative prob. dist.**
- Mathematical equation called a **probability function** ($p(x)$), which provides probability for each value of random variable
- Required conditions for a discrete prob. function are: $p(x) \geq 0$ and $\Sigma p(x) = 1$
- **Watch end points (include or not)!**
  - Number line diagram helps

Example: JSL Appliances

- Using past data on TV sales, a tabular representation of the probability distribution for TV sales was developed.
-

Units Sold	# of Days	x	$p(x)$	Cum.
0	80	0	.40	.40
1	50	1	.25	.65
2	40	2	.20	.85
3	10	3	.05	.90
4	20	4	.10	1.00
	200		1.00	

(cont.)



Expected Value & Variance

- **Expected value**, or **mean**, of a random variable is a measure of its central location
  - Expected value of a discrete random variable: $E(x) = \mu = \Sigma\, x\, p(x)$
- **Variance** summarizes variability in values of a random variable
  - Variance of a discrete random variable:

$$\mathrm{Var}(x) = \sigma^2 = \Sigma(x - \mu)^2\, p(x)$$

- **Standard deviation**, s, is defined as positive square root of the variance

Binomial Prob. Distribution

- **Discrete** prob. dist.; 2 **parameters** (n, p)
- Binomial experiment: N2PINo
  - **n** trials
  - only **2** outcomes possible (success, failure)
  - probability of success (**p**) constant for all trials
  - each trial is **i**ndependent of all others
  - typically interested in total **number** of successes in n trials (k)

*Binomial (cont.)**

- Binomial Probability Function

$$f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{(n-x)}$$

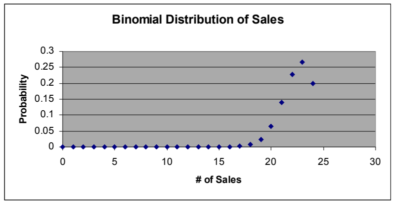
where

$f(x)$ = prob. of x successes in n trials

n = # of trials

p = prob. of success on any one trial

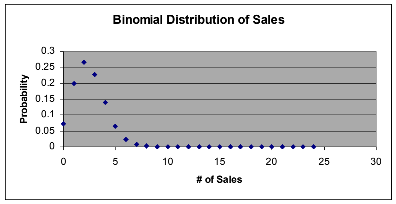
Shape Depends on p

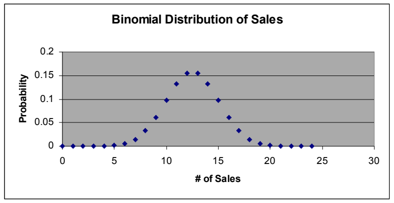


p=0.9

n=25

p =0 .1





p =0.5

Expected Value & Variance

- Expected Value: $E(x) = \mu = np$
  - Mean (average over long run)
- Variance: $Var(x) = \sigma^2 = np(1 - p) = npq$
  - Where $1 - p = q$
- Standard Deviation: σ

Excel Binomial Functions

- Alternative to Traditional Tables
- BINOMDIST(*k, n, p, cum*)
  - **cum** is a logical (0/1 or false/true) value; 1 gives **cumulative** value $p(x \le k)$; 0 gives point probability, $p(x = k)$
  - works for very large n, small p, etc.
- CRITBINOM(*n, p,* α): finds value of k such that $P(x \le k) \ge \alpha$ (cumulative prob.)
  - limited by n & p; watch for "#NUM!" msg

Q: How do you find $p(5<x<10)$? $P(x \ge 12)$?

(be careful about which end points are included)

Uniform Discrete Prob. Dist.

- Random variable can take on any one of n discrete values (x_1, x_2, x_3, …., x_n)
- All are equally likely
- $p(x = x_i) = 1/n$
- Common examples
  - Flipping fair coin
  - Tossing fair die
  - Any "equally likely" situation (often a fallback assumption)

Continuous Prob. Distributions

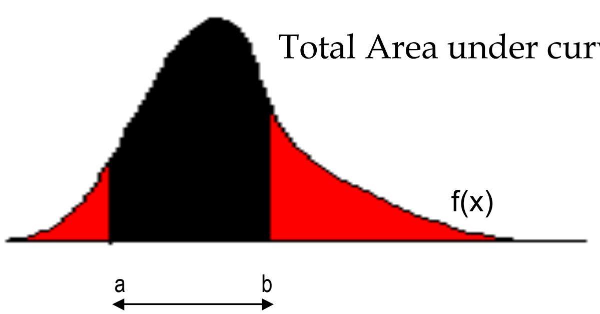
- **Continuous random variable** can assume **any** value in an interval or in a collection of intervals
- Not possible to talk about probability of random variable assuming a particular value (always zero!). **Why?**
- Instead, talk about probability of random variable assuming a value **within a given interval**
  - so need to deal with cumulative probabilities
  - requires calculus, or tables, or Excel

(cont.)

- Define a curve (probability density function, *f(x)*), such that area under it = 1.0
- Probability of random variable assuming a value within some given interval from x_1 to x_2 is defined to be area under the graph of the probability density function between x_1 and x_2 (see next slide for illustration)

Calculating Probabilities

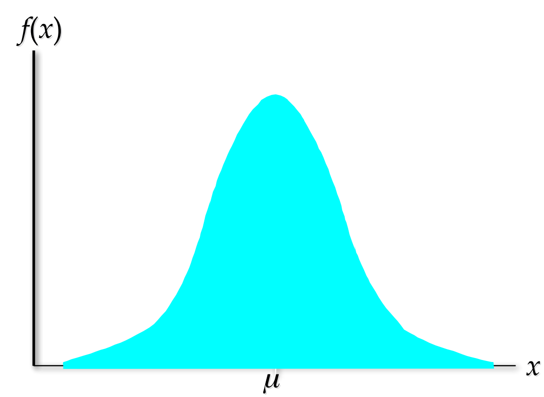
Total Area under curve = 1



$P(a<x<b)$ = (Area between a & b) / (Total area)

Normal Prob. Dist.

- Graph of Normal Probability Density Function

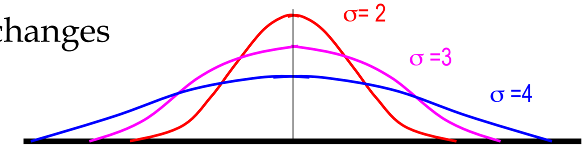


Normal Curve

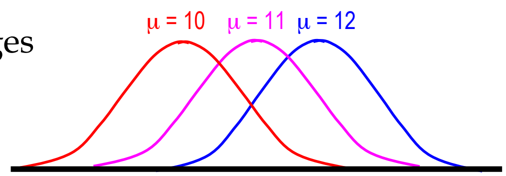
- "Bell-shaped" curve
- Highest point on normal curve is at mean (also median & mode)
- Normal curve is symmetric about mean
- Std dev determines width of curve
- Total area under curve is 1
- Probabilities for normal random variable are given by areas under curve
- 2 parameters: mean (μ), std dev. (σ)

Changing μ or σ

σ changes



μ changes



Normal Prob. Dist. (cont.)*

- Normal Probability Density Function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty \leq x \leq \infty$$

where

μ = mean

σ = standard deviation

π = 3.14159

e = 2.71828

Excel Functions

- Alternative to Traditional Tables
- NORMDIST(x, μ, σ, cum)
  - set cum = 1; why?
  - Gives "left tail" probabilities
  - Not usually interested in area from -∞ to x How do we overcome this?
- NORMINV(p, μ, σ)
  - Returns inverse of normal cumulative distribution for specified μ and σ
  - Finds x_o for $p(-\infty < x < x_o) = p$

No need to worry about "end points"

Six Sigma Payoff

σ	In Spec	Outside
1	68.26894805%	31.73105195%
3	99.73000656%	0.26999344%
6	99.99999980%	0.00000020%

$\pm 3\sigma$ = 2700 ppm outside spec limits

Many firms in 3 – 3.5σ range

$\pm 6\sigma$ = 2 ppb outside spec limits