HW 8: ECE 601 Machine Learning for Engineers

Important Notes:

(a) When a HW question asks for writing a code, you would need to include the entire code as well as the output of the program as well as any other analysis requested in the question.

(b) Don't panic about the length of the HW assignment. HW assignments are treated as opportunities for improving learning and understanding, so I might include some extra text to help you better understand the concepts or learn about a point that was not covered during the class. The actual work needed from you is indeed manageable.

(c) Note that there are no programming assignments this week, so you can submit your work as a single PDF.

Remember from your undergrad probability that:

If X is a normal random variable with mean μ and variance σ^2, i.e, $X \sim N(\mu, \sigma^2)$, then

$$\begin{aligned} f_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \\ F_X(x) &= P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \\ P(a < X \leq b) &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right). \end{aligned}$$

In answering the following questions you can leave your answers in terms of the Φ function.

1. Let $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a normal random vector with the following mean and covariance matrices

$$\mathbf{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let also

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \mathbf{AX} + \mathbf{b}.$$

(a) Find $P(X_2 > 0)$.

(b) Find expected value vector of $\mathbf{Y}$, $\mathbf{m_Y} = E\mathbf{Y}$.

(c) Find the covariance matrix of $\mathbf{Y}$, $\mathbf{C_Y}$.

(d) Find $P(Y_2 \leq 2)$.

2. (Whitening/decorrelating transformation) Let $\mathbf{X}$ be an n-dimensional zero-mean random vector. Since $\mathbf{C_X}$ is a real symmetric matrix, we conclude that it can be diagonalized. That is, there exists an n by n matrix $\mathbf{Q}$ such that

$$\mathbf{QQ}^T = \mathbf{I} \qquad (\mathbf{I} \text{ is the identity matrix})$$
$$\mathbf{C_X} = \mathbf{QDQ}^T,$$

where $\mathbf{D}$ is a diagonal matrix

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & \ldots & 0 \\ 0 & d_{22} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & d_{nn} \end{bmatrix}.$$

Now suppose we define a new random vector $\mathbf{Y}$ as $\mathbf{Y} = \mathbf{Q}^T\mathbf{X}$, thus

$$\mathbf{X} = \mathbf{QY}.$$

Show that $\mathbf{Y}$ has a diagonal covariance matrix, and conclude that components of $\mathbf{Y}$ are uncorrelated, i.e., $\mathrm{Cov}(Y_i, Y_j) = 0$ if $i \neq j$.