Homework 8

(1)

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad m = E[X] = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$C = Cov(X) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$Y = AX + b = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

a) $P(X_2 > 0) = ?$

$$X_2 \sim N(\mu, \sigma^2) \implies X_2 \sim (2, 1)$$

$$P(X_2 > 0) = 1 - P(X_2 \leq 0)$$

$$= 1 - \Phi\left(\frac{0-2}{\sqrt{1}}\right)$$

$$= 1 - \Phi(-2) = \Phi(2)$$

b) $E[Y] = ?$

$$E[Y] = A\,E[X] + b$$

$$A\,E[X] = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times1 + 1\times2 \\ -1\times1 + 1\times2 \\ 1\times1 + 3\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$$

$$E[Y] = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 8 \end{pmatrix}$$

c) $\mathrm{Cov}(Y) = A \cdot \mathrm{Cov}(X) \cdot A^T = ACA^T$

$$C = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ -3 & 0 \\ 7 & 4 \end{bmatrix}$$

$$C_Y = (AC)A^T = \begin{bmatrix} 9 & 3 \\ -3 & 0 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & -6 & 18 \\ -6 & 3 & -3 \\ 18 & -3 & 19 \end{bmatrix}$$

d) $P(Y_2 \leq 2) = ?$

$$Y_2 \sim N(1, 3)$$

$$P(Y_2 \leq 2) = \Phi\left(\frac{2-1}{\sqrt{3}}\right)$$

$$= \Phi\left(\frac{1}{\sqrt{3}}\right)$$

2 Given $C_X = QDQ^T$

$Y = Q^T X$

$$\mathrm{cov}(Y) = E[YY^T]$$

$$= E[Q^T X X^T Q]$$

$$= Q^T C_X Q$$

$$= Q^T Q D Q^T Q$$

$$= D$$

$\mathrm{cov}(Y_i, Y_j) = 0$ for $i \neq j$

$\boxed{\mathrm{cov}(Y) = D}$ components of Y are uncorrelated.