HIGHER DIRECT IMAGES FOR DUMMIES

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Abstract. Your abstract.

1. Introduction

- Give definitions of presheaves, sheaves, stalks.

2. Locally constant sheaves

- Give definition.
- Explain sources of examples (and their usefulness): constant sheaves (duh), direct images (sometimes), sheaves of solutions of differential equations; and of course examples induced by covering spaces, representations of the fundamental groupoid, and vector bundles with flat connection.
- Explain the equivalence of locally constant sheaves and covering spaces; also discuss how f^* and f_* look in this picture.
- Explain the equivalence of locally constant sheaves and representations of the fundamental groupoid. Calculate examples of the induced representation, for instance with the sheaf $\{f \mid f' = \frac{1}{2z} f\}$ on $\mathbb{C}^\times$ (which has monodromy $f \mapsto -f$). Also explain how to restrict on representations of the fundamental group at some fixed base point (find out the technical hypotheses needed for this).
- Explain the equivalence of locally constant sheaves and vector bundles with flat connection. Warn that a visual representation of a vector bundles necessarily lives in $\mathbb{R}^3$ and thus, somewhat unfortunately, comes with an induced connection.

3. Direct images

- Give the definition.
- Explain how to easily calculate the stalks of the direct image in case the map in question was closed. Explain how this is related to "proper base change". Explain why closedness is intuitively necessary. Of course, give `https://www2.math.uni-paderborn.de/fileadmin/Mathematik/People/wedhorn/Lehre/SkriptMannigfaltigkeiten.pdf` as a reference.
- Give description in terms of generators and relations (in the case that this should turn out to be helpful).

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4. Sheaf cohomology and higher direct images

4.1. Basic definition and properties.

Definition 4.1. Let $\mathcal{E}$ be a sheaf of abelian groups on a topological space X. Then the *n-th cohomology of X with values in $\mathcal{E}$* is the abelian group $H^n(X, \mathcal{E}) := R^n\Gamma_X(\mathcal{E})$, i.e. the value of the n-th right derived functor of the global sections functor $\Gamma_X : \mathrm{AbSh}(X) \to \mathrm{Ab}$.

Unrolling the definitions, the sheaf cohomology $H^n(X, \mathcal{E})$ is the cohomology of the cochain complex $\Gamma_X(\mathcal{I}^\bullet)$, where $0 \to \mathcal{E} \to I^0 \to I^1 \to \cdots$ is a resolution of $\mathcal{E}$ by injective objects. However, this description is rarely useful in practice. We will discuss techniques for calculating sheaf cohomology below.

Example 4.2. $H^0(X, \mathcal{E}) \cong \mathcal{E}(X)$.

With singular or simplicial cohomology, the choice of a coefficient ring does not matter much in many applications; there is always the universal coefficient theorem relating $\mathbb{Z}$-valued cohomology with cohomology with values in other rings. *This is decidely not so with sheaf cohomology.* Sheaf cohomology is as much about the topological space as it is about the sheaf in question. The sheaf cohomology groups depend hugely on the used sheaf.

Unfortunately, there is no useful intuition about sheaf cohomology along the ways of "closed cochains modulo exact cochains". Instead, some intuition on what the elements of the sheaf cohomology groups look like can be gained by Čech methods (see below).

The basic proposition about sheaf cohomology is that short exact sequences of sheaves induce long exact sequences in cohomology.

Proposition 4.3. *Let X be a topological space. Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be a short exact sequence of sheaves of abelian groups on X. (This means that the induced sequences $0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$ on stalks are exact for all points $x \in X$.) Then there is a natural long exact sequence*

$$0 \to \mathcal{E}(X) \to \mathcal{F}(X) \to \mathcal{G}(X) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^2(X, \mathcal{E}) \to \cdots .$$

4.2. Relation to ordinary cohomology.

Proposition 4.4. *Let X be a locally contractible space. Let R be a ring and $\underline{R}$ be the constant sheaf with stalks R. Then $H^\bullet_{sing}(X, R) \cong H^\bullet(X, \underline{R})$.*

Proof. See [1]. □

XXX: Include comparison theorem with cohomology with values in local systems.

4.3. Why sheaf cohomology?

Sheaf cohomology can be motived from many different angles.

The Leray spectral sequence. If $f : X \to Y$ is any continuous map, the cohomology of X, Y, and the fibers of f are related. The Leray spectral sequence expresses this relationship in a precise way, describing how the cohomology of X is composed by the cohomology of Y and the cohomology of the fibers. However, even if one is only interested in the cohomology of X with values in $\mathbb{Z}$ or in a field, the cohomology of non-constant sheaves appears in the spectral sequence. Only in special cases, such that f is a fibration, does cohomology with values in local systems suffice.

Global sections. In complex and algebraic geometry, one is often interested in the space of global sections of a sheaf $\mathcal{E}$. For instance, in the case that $\mathcal{E} = \mathcal{O}_X(D)$ is the line bundle associated to a divisor D on a complex manifold or scheme X, $H^0(X, \mathcal{E})$ is the space of meromorphic functions on X with zeroes and poles behaviour governed by D. However, this space, or even its dimension, is hard to compute. Therefore one turns to the *Euler characteristic* of $\mathcal{E}$, the alternating sum

$$\chi(X, \mathcal{E}) := \dim H^0(X, \mathcal{E}) - \dim H^1(X, \mathcal{E}) + \dim H^2(X, \mathcal{E}) \pm \cdots$$

in which the global sections dimension appears as the first summand. This quantity is in general easier to compute, since it is additive in short exact sequences (if $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is exact, then $\chi(X, \mathcal{F}) = \chi(X, \mathcal{E}) + \chi(X, \mathcal{G})$) and constant in certain families. Obviously, sheaf cohomology is needed to define the Euler characteristic.

Classifying geometric objects. For certain choices of the sheaf, sheaf cohomology classifies geometric objects. For instance, if X is a complex manifold, $H^1(X, \mathcal{O}_X^\times)$ classifies complex line bundles on X up to isomorphism. Here, $\mathcal{O}_X^\times$ is the sheaf of nowhere vanishing holomorphic functions. More generally, $H^1(X, \mathcal{A}ut(\mathcal{E}))$ classifies sheaves which are locally isomorphic to a fixed sheaf $\mathcal{E}$ and $H^1(X, \mathcal{G})$ classifies $\mathcal{G}$-torsors on X.

Irreducible spaces. The singular cohomology of irreducible topological spaces – spaces which cannot be written as the nontrivial union of two proper closed subsets – is always zero (in positive degrees). This is not a problem when working with manifolds, which are irreducible only in the one-point case. But it is a problem in algebraic geometry, where schemes with their Zariski topology are often irreducible. For such cases, one is forced to look at cohomology with values in non-constant sheaves.

4.4. **Čech methods.**

- Give the definition (of the ordered Čech complex).
- Give lemma that Čech cohomology = sheaf cohomology. Sketch the very basic idea of the proof.
- Also give lemma that sheaf cohomology = singular cohomology. Again, sketch the first step of the proof. Link to the very detailed reference `http://www3.nd.edu/~lnicolae/sheaves_coh.pdf`. Also check out `http://www.math.ru.nl/~mgroth/teaching/algtopII14/Section08.pdf` and `http://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomolog`
- Explain how Mayer–Vietoris is a special case of the Čech-to-sheaf spectral sequence.
- Give the definition of higher direct images.

Locally constant sheaves

Example 4.5. Consider S^1 together with the locally constant sheaf of typical stalk $\mathbb{Z}^2$ and monodromy representation $[\gamma].(a, b) = (b, a)$ where $[\gamma]$ is a generator of $\pi_1 S^1$. We want to calculate its sheaf cohomology via cohomology with local coefficient system. The bundle of coefficients E is given as

$$E := [0, 1] \times \mathbb{Z}^2 / \sim$$

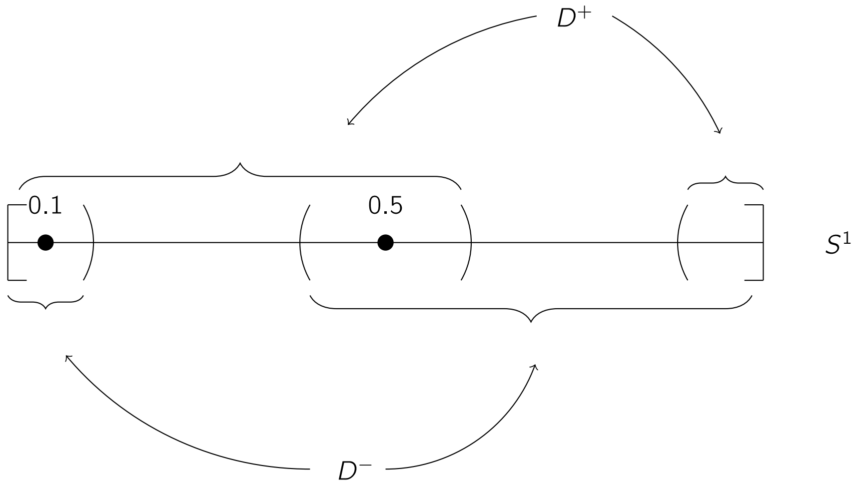
with the identification $(0,(a,b)) \sim (1,[\gamma].(a,b)) = (1,(b,a))$ and the obvious projection $E \to S^1$. We want to apply the Mayer-Vietoris sequence to the standard decomposition $S^1 = D^+ \cup D^-$ of S^1 into two open intervals yielding the exactness of

$$H^{k-1}(D^+,E) \oplus H^{k-1}(D^-,E) \longrightarrow H^{k-1}(D^+ \cap D^-,E) \longrightarrow H^k(S^1,E) \longrightarrow H^k(D^+,E) \oplus H^k(D^-,E)$$

On a simply connected space a local system of coefficients is globally constant and this shows that $H^k(S^1,E) = 0$ for $k > 1$. For $k = 1$ the sequence becomes

$$\begin{tikzcd}
H^0(D^+,E) \oplus H^0(D^-,E) \arrow[r] \arrow[d] & H^0(D^+ \cap D^-,E) \arrow[r] \arrow[d] & H^1(S^1,E) \arrow[r] & 0 \\
\mathbb{Z}^2 \oplus \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \oplus \mathbb{Z}^2 & &
\end{tikzcd} \tag{1}$$

We have to identify the lower arrow.



For an arbitrary space X a cochain representing a class in $H^0(X,E)$ assigns to every point $x \in X$ a value in the corresponding fibre E_x. The cochain condition means that this value is constant on connected components of X. There are no nontrivial coboundaries in this dimension. Therefore we have a very clear understanding of how the groups in the diagram 1 are isomorphic to $\mathbb{Z}^k$ namely by evaluation at some fixed points:

$$\begin{tikzcd}
H^0(D^+,E) \oplus H^0(D^-,E) \arrow[r] \arrow[d, "\mathrm{ev}_{0.5}"', "\mathrm{ev}_{0.5}"] & H^0(D^+ \cap D^-,E) \arrow[r] \arrow[d, "\mathrm{ev}_{0.1} \oplus \mathrm{ev}_{0.5}"] & H^1(S^1,E) \arrow[r] & 0 \\
\mathbb{Z}^2 \oplus \mathbb{Z}^2 \arrow[r] & \mathbb{Z}^2 \oplus \mathbb{Z}^2 & &
\end{tikzcd} \tag{2}$$

Consider the cohomology class in $H^0(D^+,E)$ represented by the cocycle which maps $0.5 \mapsto (a,b) \in \mathbb{Z}^2$. This cocycle restricted to D^+ also maps 0.1 to (a,b) because both of these points lie in the same connected component of D^+. Thus under isomorphisms 2 the restriction homomorphism $H^0(D^+,E) \to H^0(D^+ \cap D^-,E)$ maps $(a,b) \mapsto ((a,b),(a,b))$. On the other hand the cohomology class in $H^0(D^-,E)$ represented by the cocycle which maps $0.5 \mapsto (c,d) \in \mathbb{Z}^2$ maps $0.1 \mapsto (d,c)$ since we have to let the monodromy act first. Therefore the restriction homomorphism $H^0(D^-,E) \to$

$H^0(D^+ \cap D^-, E)$ maps $(c, d) \mapsto ((d, c), (c, d))$. Mere calculation shows that the cokernel of $H^0(D^+, E) \oplus H^0(D^-, E) \to H^0(D^+ \cap D^-, E)$ is isomorphic to $\mathbb{Z}$ yielding

$$H^1(S^1, E) \cong \mathbb{Z}.$$

Direct Images

Example 4.6. Consider the 2-sheeted covering $p : S^1 \to S^1, z \to z^2$ viewing $S^1 \subset \mathbb{C}$. We want to describe the direct image $p_*\underline{\mathbb{Z}}$ of the constant sheaf $\underline{\mathbb{Z}}$. The sheaf $p_*\underline{\mathbb{Z}}$ is locally constant hence it is determined by its monodromy representation $\pi_1 S^1 = \mathbb{Z} \curvearrowright \mathbb{Z}^2$. The curve $\gamma : [0, 2\pi] \to S^1, t \mapsto \exp(it)$ is a generator of $\pi_1 S^1$ and we want to find a lift of this curve $I \to p_*\underline{\mathbb{Z}}$ to the étalé space of $p_*\underline{Z}$ with any given initial condition $\tilde{\gamma}(0) \in p_*\underline{\mathbb{Z}}_0$:

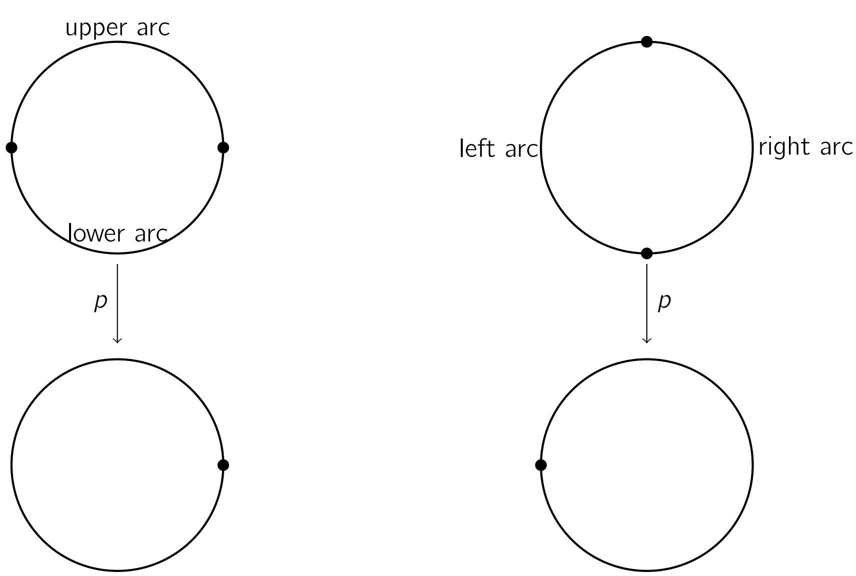
$$\begin{tikzcd} & p_*\underline{\mathbb{Z}} \arrow[d, "\pi"] \\ I \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1 \end{tikzcd}$$

We choose two trivialisations

$$p_*\underline{\mathbb{Z}}S^1 \setminus \{+1\} = \underline{\mathbb{Z}}p^{-1}S^1 \setminus \{+1\} = \underline{\mathbb{Z}}S^1 \setminus \{\pm 1\} \xrightarrow{\phi} \mathbb{Z}_{upper} \oplus \mathbb{Z}_{lower}$$

$$p_*\underline{\mathbb{Z}}S^1 \setminus \{-1\} = \underline{\mathbb{Z}}p^{-1}S^1 \setminus \{-1\} = \underline{\mathbb{Z}}S^1 \setminus \{\pm i\} \xrightarrow{\psi} \mathbb{Z}_{left} \oplus \mathbb{Z}_{right}$$

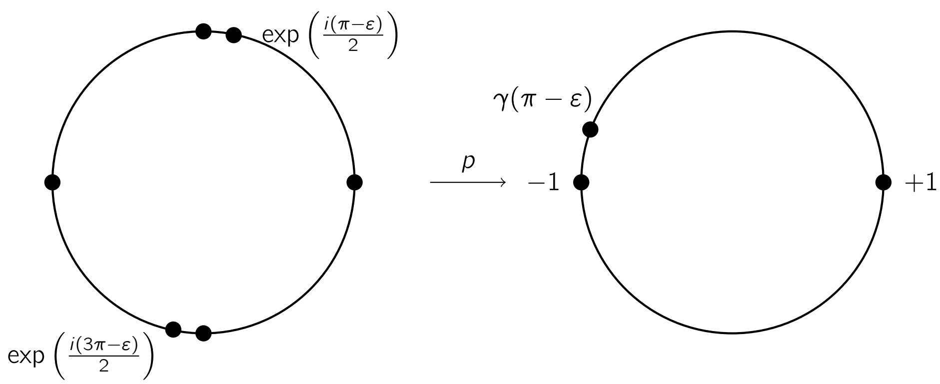
where for instance one generator of $\mathbb{Z}_{upper}$ corresponds to the locally constant real-valued function taking the value $+1$ on the upper connected arc of $S^1 \setminus \{\pm 1\}$ and vanishing on the lower one.



Let us describe the lift $\tilde{\gamma}$ with the initial condition $\tilde{\gamma}(0) = [\psi^{-1}(1,0)]_1$. At time $t = \pi - \varepsilon$ this lift still has the value $[\psi^{-1}(1,0)]_{\gamma(\pi-\varepsilon)}$ and we claim:

$$\begin{aligned} \tilde{\gamma}(\pi-\varepsilon) &= [\psi^{-1}(1,0)]_{\gamma(\pi-\varepsilon)} && (3)\\ &= [\phi^{-1}(0,1)]_{\gamma(\pi-\varepsilon)} && (4) \end{aligned}$$

The point $\tilde{\gamma}(\pi-\varepsilon)$ is obtained as follows: consider the locally constant real-valued function $f \in p_*\underline{\mathbb{Z}}(S^1 \setminus \{-1\}) = \underline{\mathbb{Z}}(S^1 \setminus \{\pm i\})$ having the value 1 on the left arc and 0 on the right one. This is a section of $p_*\underline{\mathbb{Z}}$ over the open set $S^1 \setminus \{-1\}$. The preimage of $\gamma(\pi-\varepsilon)$ consists of the two points $\exp\left(\frac{i(\pi-\varepsilon)}{2}\right)$ and $\exp\left(\frac{i(3\pi-\varepsilon)}{2}\right)$:



On the other hand the real-valued function $g := \phi^{-1}(0,1) \in p_*\underline{\mathbb{Z}}(S^1 \setminus \{+1\}) = \underline{\mathbb{Z}}(S^1 \setminus \{\pm 1\})$ having the value 0 on the upper arc and 1 on the lower one represents exactly the same stalk of $p_*\underline{\mathbb{Z}}$ at the point $\gamma(\pi-\varepsilon)$. This proves claim (4).

A similar argument shows

$$\begin{aligned} \tilde{\gamma}(2\pi-\varepsilon) &= [\phi^{-1}(0,1)]_{\gamma(2\pi-\varepsilon)}\\ &= [\psi^{-1}(0,1)]_{\gamma(2\pi-\varepsilon)} \end{aligned}$$

and then $\tilde{\gamma}(2\pi) = [\psi^{-1}(1,0)]_1$. Together with a similar calculation on the other generator of the stalk at 1 we conclude that in the monodromy representation $\pi_1 S^1 \curvearrowright \mathbb{Z}^2$ the generator of $\pi_1 S^1$ interchanges the two summands of $\mathbb{Z}^2$ – the stalk of $p_*\underline{\mathbb{Z}}$ at the point $1 \in S^1$.

The Serre Spectral Sequence

Example 4.7. Every covering is a fibration and hence yields a Serre spectral sequence converging to the cohomology of the total space. Revisiting Example 4.6 we observe that every preimage of a point consists of two discrete points and hence alle higher direct images $R^i p_*\underline{\mathbb{Z}}, i > 0$ vanish. Therefore the Serre spetral sequence looks as follows:

1			
0	$H^0(S^1, p_*\underline{\mathbb{Z}})$	$H^1(S^1, p_*\underline{\mathbb{Z}})$	$H^2(S^1, p_*\underline{\mathbb{Z}})$
	0	1	2

Thus the E_2 page equals the limiting page E_∞ and we have an isomorphism

$$H^k(S^1, p_*\underline{\mathbb{Z}}) \cong H^k(S^1, \mathbb{Z})$$

for any $k \geq 0$. The left hand side expression is sheaf cohomology and the right hand one may be interpreted as singular cohomology with coefficients in $\mathbb{Z}$.

For $k = 0$ this is obvious since $H^0(S^1, p_*\underline{\mathbb{Z}}) = \Gamma p_*\underline{\mathbb{Z}} = \mathbb{Z}$. For $k > 0$ this is some statement about the singular cohomology of S^1 with local coefficient system $p_*\underline{\mathbb{Z}}$ of typical fibre $\mathbb{Z}^2$ (cf. 4.5).

Example 4.8. Let $M = S^n$ and

$$S^{n-1} \hookrightarrow SM \to S^n$$

its unit sphere bundle. We want to calculate the integral homology of SM. The E^2 entries are

n	$\mathbb{Z}$	$\mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}$
	0	$n-1$

- definition
- many examples, also with differentials and ring structure
- Klein bottle
- Unit sphere tangent bundle of spheres

References

[1] D. Cibotaru. *Sheaf cohomology*. 2005. url: `http://www3.nd.edu/~lnicolae/sheaves_coh.pdf`.