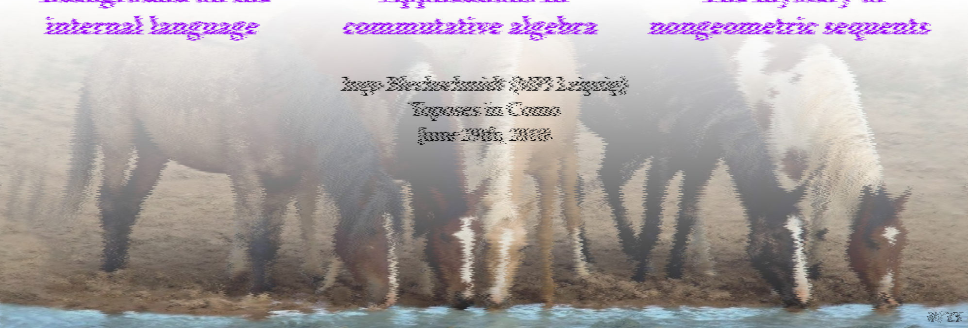
How topos theory can help algebra

– *an invitation* –

1 Background on the internal language

2 Applications in commutative algebra

3 The mystery of nongeometric sequents



Ingo Blechschmidt (MPI Leipzig)

Toposes in Como

June 28th, 2018

How topos theory can help algebra

– an invitation –

1. **Background on the internal language**
2. **Applications in commutative algebra**
3. **The mystery of nongeometric sequents**

Ingo Blechschmidt (MPI Leipzig)
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June 29th, 2018

Motivating testcases

Let A be a ring (commutative, with unit, $1 = 0$ allowed).
Assume that A is reduced: If $x^n = 0$, then $x = 0$.

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \qquad \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

A baby application

Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

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Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

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Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

Proof. Assume not. Then there is a maximal ideal $\mathfrak{m}$. The matrix is surjective over the field $A/\mathfrak{m}$. This is a contradiction to basic linear algebra.

A child application

Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

Proof. Assume not. Then there is a minimal prime ideal $\mathfrak{p}$. The matrix is injective over the field $A_\mathfrak{p} = A[(A \setminus \mathfrak{p})^{-1}]$. This is a contradiction to basic linear algebra.

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Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

Proof. Assume not. Then there is a prime ideal $\mathfrak{p}$. The matrix is surjective over the field $\mathrm{Quot}(A/\mathfrak{p})$. This is a contradiction to basic linear algebra.

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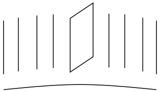

Generic freeness

Let B be an A-algebra of finite type ($\cong A[X_1, \ldots, X_n]/\mathfrak{a}$).
Let M be a finitely generated B-module ($\cong B^m/U$).
If $f = 0$ is the only element of A such that

1. $B[f^{-1}]$ and $M[f^{-1}]$ are free modules over $A[f^{-1}]$,
2. $A[f^{-1}] \to B[f^{-1}]$ is of finite presentation and
3. $M[f^{-1}]$ is finitely presented as a module over $B[f^{-1}]$,

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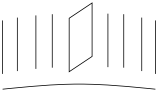

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Proof. See [Stacks Project, Tag 051Q].

The internal language of a topos

For any topos $\mathcal{E}$ and any formula φ, we define the meaning of

"$\mathcal{E} \models \varphi$" ("$\varphi$ holds in the internal universe of $\mathcal{E}$")

using (Shulman's extension of) the **Kripke–Joyal semantics**.

$\mathrm{Set} \models \varphi$ "φ holds in the usual sense."	$\mathrm{Sh}(X) \models \varphi$ "φ holds continuously."	$\mathrm{Eff} \models \varphi$ "φ holds computably."

Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if φ entails ψ intuitionistically, then $\mathcal{E} \models \psi$.

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Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if φ entails ψ intuitionistically, then $\mathcal{E} \models \psi$.

no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

The internal language of $\mathrm{Sh}(X)$

Let X be a topological space. We recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ. Write “$\mathrm{Sh}(X) \models \varphi$” to mean $X \models \varphi$.

$U \models f = g : F$ iff $f|_U = g|_U \in F(U)$

$U \models \varphi \wedge \psi$ iff $U \models \varphi$ and $U \models \psi$

$U \models \varphi \vee \psi$ iff $\cancel{U \models \varphi \text{ or } U \models \psi}$ there exists a covering $U = \bigcup_i U_i$ s. th. for all i: $U_i \models \varphi$ or $U_i \models \psi$

$U \models \varphi \Rightarrow \psi$ iff for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$

$U \models \forall f : F.\, \varphi(f)$ iff for all open $V \subseteq U$ and sections $f_0 \in F(V)$: $V \models \varphi(f_0)$

$U \models \forall F.\, \varphi(F)$ iff for all open $V \subseteq U$ and sheaves F_0 over V: $V \models \varphi(F_0)$

$U \models \exists f : F.\, \varphi(f)$ iff there exists a covering $U = \bigcup_i U_i$ s. th. for all i: there exists $f_0 \in F(U_i)$ s. th. $U_i \models \varphi(f_0)$

$U \models \exists F.\, \varphi(F)$ iff there exists a covering $U = \bigcup_i U_i$ s. th. for all i: there exists a sheaf F_0 on U_i s. th. $U_i \models \varphi(F_0)$

Internalizing parameter-dependence

Let X be a space. A continuous family $(f_x)_{x \in X}$ of continuous functions (that is, a continuous function $f : X \times \mathbb{R} \to \mathbb{R}$; $f_x(a) = f(x, a)$) induces an endomorphism of the sheaf $\mathcal{C}$ of continuous functions:

$$\bar{f} : \mathcal{C} \longrightarrow \mathcal{C}, \text{ on } U\text{: } s \longmapsto (x \mapsto f_x(s(x))).$$

- $\mathrm{Sh}(X) \models$ ⌜The set $\mathcal{C}$ is the set of (Dedekind) reals⌝.
- $\mathrm{Sh}(X) \models$ ⌜The function $\bar{f} : \mathbb{R} \to \mathbb{R}$ is continuous⌝.

- Iff $f_x(-1) < 0$ for all x, then $\mathrm{Sh}(X) \models \bar{f}(-1) < 0$.
- Iff $f_x(+1) > 0$ for all x, then $\mathrm{Sh}(X) \models \bar{f}(+1) > 0$.
- Iff all f_x are increasing, then $\mathrm{Sh}(X) \models$ ⌜$\bar{f}$ is increasing⌝.

- Iff there is an open cover $X = \bigcup_i U_i$ such that for each i there is a continuous function $s : U_i \to \mathbb{R}$ with $f_x(s(x)) = 0$ for all $x \in U_i$, then $\mathrm{Sh}(X) \models \exists s : \mathbb{R}.\, \bar{f}(s) = 0$.

The little Zariski topos

Let A be a ring. Its **little Zariski topos** is equivalently

1. the classifying topos of **local localizations** of A,
2. the classifying locale of **prime filters** of A,
3. the locale given by the frame of **radical ideals** of A,
4. the topos of sheaves over the poset A with $f \preceq g$ iff $f \in \sqrt{(g)}$ and with $(f_i \to f)_i$ deemed covering iff $f \in \sqrt{(f_i)_i}$ or
5. the topos of sheaves over $\mathrm{Spec}(A)$.

Its associated topological space of points is the **classical spectrum**

$$\{\mathfrak{f} \subseteq A \mid \mathfrak{f} \text{ prime filter}\} + \text{Zariski topology}.$$

It has **enough points** if the Boolean Prime Ideal Theorem holds.

Prime ideal: $0 \in \mathfrak{p}$; $x \in \mathfrak{p} \wedge y \in \mathfrak{p} \Rightarrow x + y \in \mathfrak{p}$; $1 \notin \mathfrak{p}$; $xy \in \mathfrak{p} \Leftrightarrow x \in \mathfrak{p} \vee y \in \mathfrak{p}$

Prime filter: $0 \notin \mathfrak{f}$; $x + y \in \mathfrak{f} \Rightarrow x \in \mathfrak{f} \vee y \in \mathfrak{f}$; $1 \in \mathfrak{f}$; $xy \in \mathfrak{f} \Leftrightarrow x \in \mathfrak{f} \wedge y \in \mathfrak{f}$

First steps in the little Zariski topos

Let A be a ring. Let $\mathfrak{f}_0$ be the **generic prime filter** of A; it is a subobject of the constant sheaf $\underline{A}$ of the little Zariski topos.

- The ring $A^\sim := \underline{A}[\mathfrak{f}_0^{-1}]$ is the generic local localization of A.
- Given an A-module M, we have the $A^\sim$-module $M^\sim := \underline{M}[\mathfrak{f}_0^{-1}]$.

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Definition. Let A be a ring and let M be an A-module. We define the *sheaf associated* to M on Spec A, denoted by $\tilde{M}$, as follows. For each prime ideal $\mathfrak{p} \subseteq A$, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. For any open set $U \subseteq \operatorname{Spec} A$ we define the group $\tilde{M}(U)$ to be the set of functions $s: U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in M_\mathfrak{p}$, and such that s is locally a fraction m/f with $m \in M$ and $f \in A$. To be precise, we require that for each $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U, and there are elements $m \in M$ and $f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = m/f$ in $M_\mathfrak{q}$. We make $\tilde{M}$ into a sheaf by using the obvious restriction maps.

Robin Hartshorne. Algebraic Geometry. 1977.

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Assuming the Boolean prime ideal theorem, a geometric sequent "$\forall \ldots \forall. (\cdots \Longrightarrow \cdots)$", where the two subformulas may not contain "$\Rightarrow$" and "$\forall$", holds for $M^\sim$ iff it holds for all stalks $M_\mathfrak{p}$.

$M^\sim$ inherits any property of M which is **localization-stable**.

If A is reduced ($x^n = 0 \Rightarrow x = 0$):

$A^\sim$ is a **field** (nonunits are zero).
$A^\sim$ has **$\neg\neg$-stable equality**.
$A^\sim$ is **anonymously Noetherian**.

First steps in the little Zariski topos

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ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in $\mathbf{E}$ the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec($\mathbf{E}$, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

First steps in the little Zariski topos

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Complexity reduction

The external meaning of

$$\mathrm{Spec}(A) \models \ulcorner A^\sim[X_1, \ldots, X_n] \text{ is anonymously Noetherian}\urcorner$$

is:

For any element $f \in A$ and any (not necessarily quasicoherent) sheaf of ideals $\mathcal{J} \hookrightarrow A^\sim[X_1, \ldots, X_n]|_{D(f)}$: If

for any element $g \in A$ the condition that

the sheaf $\mathcal{J}$ is of finite type on $D(g)$

implies that $g = 0$,

then $f = 0$.

which is **localization-stable**.

Revisiting the testcases

Let A be a reduced ring.

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A baby application

Let M be a surjective matrix over A with more rows than columns. Then $1 = 0$ in A.

Proof. The matrix is surjective over the field $A^\sim$. This is a contradiction to basic linear algebra. Hence $\mathrm{Spec}(A) \models \bot$, thus $1 = 0$ in A.

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Let M be an injective matrix over A with more columns than rows. Then $1 = 0$ in A.

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$$\begin{tikzcd} & M \arrow[d, dash, "\text{finitely generated}"] \\ A \arrow[r, "\text{of finite type}"'] & B \end{tikzcd}$$

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Proof. In the little Zariski topos it's **not not** the case that

1. $B^\sim$ and $M^\sim$ are free modules over $A^\sim$,
2. $A^\sim \to B^\sim$ is of finite presentation and
3. $M^\sim$ is finitely presented as a module over $B^\sim$,

by basic linear algebra over the field $A^\sim$. The claim is precisely the external translation of this fact.

Understanding algebraic geometry

Understand **notions of algebraic geometry** over a scheme X as **notions of algebra** internal to $\mathrm{Sh}(X)$.

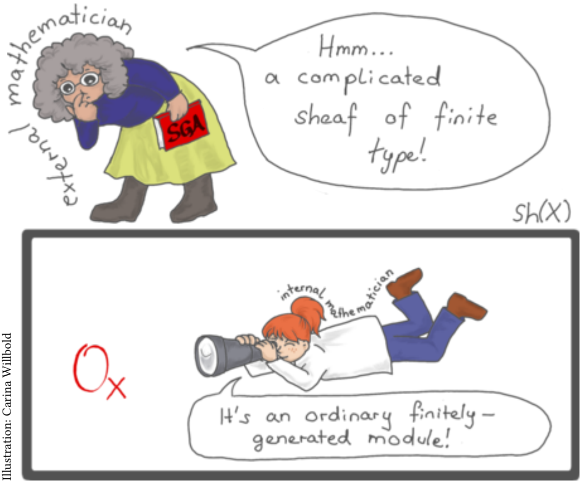

Illustration: Carina Willbold

Understanding algebraic geometry

Understand **notions of algebraic geometry** over a scheme X as **notions of algebra** internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set
sheaf of modules	module
sheaf of finite type	finitely generated module
tensor product of sheaves	tensor product of modules
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$
dimension of X	Krull dimension of $\mathcal{O}_X$
spectrum of a sheaf of $\mathcal{O}_X$-algebras	ordinary spectrum [with a twist]
big Zariski topos of X	big Zariski topos of the ring $\mathcal{O}_X$ [with a twist]
higher direct images	sheaf cohomology

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

$\Longleftarrow$

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  set of prime ideals of $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$ +
  Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

$$A \longmapsto \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\}$$

Synthetic approach: model schemes **directly as sets** in the internal universe of the **big Zariski topos** of a base scheme.

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

The big Zariski topos

Let S be a fixed base scheme.

Definition

The **big Zariski topos** $\mathrm{Zar}(S)$ of a scheme S is equivalently

1. the topos of sheaves over $(\mathrm{Aff}/S)_{\mathrm{lofp}}$,
2. the classifying topos of local rings over S or
3. the classifying $\mathrm{Sh}(S)$-topos of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.

- For an S-scheme X, its functor of points $\underline{X} = \mathrm{Hom}_S(\cdot, X)$ is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- In particular, there is the ring object $\underline{\mathbb{A}}^1$ with $\underline{\mathbb{A}}^1(T) = \mathcal{O}_T(T)$.
- This ring object is a **field**: nonzero implies invertible. [Kock 1976]

Synthetic constructions

$\mathbb{A}^n = (\underline{\mathbb{A}}^1)^n = \underline{\mathbb{A}}^1 \times \cdots \times \underline{\mathbb{A}}^1$

$\mathbb{P}^n = \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\underline{\mathbb{A}}^1)^\times$
$\cong$ set of one-dimensional subspaces of $(\underline{\mathbb{A}}^1)^{n+1}$
(with $\mathcal{O}(-1) = (\ell)_{\ell : \mathbb{P}^n}$, $\mathcal{O}(1) = (\ell^\vee)_{\ell : \mathbb{P}^n}$)

$\mathbf{Spec}(\boldsymbol{R}) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ = set of $\underline{\mathbb{A}}^1$-valued points of R

$\boldsymbol{TX} = X^\Delta$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$

A subset $U \subseteq X$ is **qc-open** if and only if for any $x : X$ there exist $f_1, \ldots, f_n : \underline{\mathbb{A}}^1$ such that $x \in U \iff \exists i.\, f_i \neq 0$.

A **synthetic affine scheme** is a set which is in bijection with $\mathrm{Spec}(R)$ for some synthetically quasicoherent $\underline{\mathbb{A}}^1$-algebra R.

A **finitely presented synthetic scheme** is a set which can be covered by finitely many qc-open f.p. synthetic affine schemes U_i such that the intersections $U_i \cap U_j$ can be covered by finitely many qc-open f.p. synthetic affine schemes.

Relations between the Zariski toposes

The big Zariski topos is a topos over the small Zariski topos:

$$\begin{array}{rrcl} \pi : & \mathrm{Zar}(A) & \longrightarrow & \mathrm{Spec}(A) \\ & \text{local } A\text{-algebra } (A \xrightarrow{\alpha} B) & \longmapsto & (A \to A[(\alpha^{-1}[B^\times])^{-1}]) \end{array}$$

This morphism is **connected** (π^{-1} is fully faithful) and **local**, so there is a preinverse

$$\begin{array}{rcl} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Zar}(A) \\ \text{local localization } (A \to B) & \longmapsto & (A \to B) \end{array}$$

which is a subtopos inclusion inducing an idempotent monad $\sharp$ and an idempotent comonad $\flat$ on $\mathrm{Zar}(S)$.

- Internally to $\mathrm{Zar}(S)$, $\mathrm{Spec}(S)$ can be constructed as the **largest subtopos** where $\flat\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is bijective.
- Internally to $\mathrm{Spec}(S)$, $\mathrm{Zar}(S)$ can be constructed as the **classifying topos** of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.
- $\mathrm{Zar}(A)$ is the **lax pullback** $(\mathrm{Set} \Rightarrow_{\mathrm{Set}[\mathrm{Ring}]} \mathrm{Set}[\mathrm{LocRing}])$.

Properties of the affine line

- $\underline{\mathbb{A}}^1$ is a local ring:

$$1 \neq 0 \qquad\qquad x + y \text{ inv.} \Longrightarrow x \text{ inv.} \vee y \text{ inv.}$$

- $\underline{\mathbb{A}}^1$ is a field:

$$\begin{aligned} \neg(x = 0) &\Longleftrightarrow x \text{ invertible} \qquad \text{[Kock 1976]} \\ \neg(x \text{ invertible}) &\Longleftrightarrow x \text{ nilpotent} \end{aligned}$$

- $\underline{\mathbb{A}}^1$ satisfies the axiom of microaffinity: Any map $f : \Delta \to \underline{\mathbb{A}}^1$ is of the form $f(\varepsilon) = a + b\varepsilon$ for unique values $a, b : \underline{\mathbb{A}}^1$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$.
- Any map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial.
- $\underline{\mathbb{A}}^1$ is anonymously algebraically closed: Any monic polynomial does *not not* have a zero.

Synthetic quasicoherence

Recall $\mathrm{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ and consider the statement

$$\text{“the canonical map} \quad \begin{array}{ccc} R & \longrightarrow & (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(R)} \\ f & \longmapsto & (\alpha \mapsto \alpha(f)) \end{array} \quad \text{is bijective”.}$$

- True for $R = \underline{\mathbb{A}}^1[X]/(X^2)$ (microaffinity).
- True for $R = \underline{\mathbb{A}}^1[X]$ (every function is a polynomial).
- True for **any** finitely presented $\underline{\mathbb{A}}^1$-algebra R.

Any known property of $\underline{\mathbb{A}}^1$ follows from this **synthetic quasicoherence**.

the mystery of nongeometric sequents

Classifying toposes in algebraic geometry

(Big) topos	classified theory
Zariski	local rings [Hakim 1972]
étale	separably closed local rings [Hakim 1972, Wraith 1979]
fppf	fppf-local rings (conjecturally: algebraically closed local rings)
ph	?? (conjecturally: algebraically closed valuation rings validating the projective Nullstellensatz)
surjective	algebraically closed geometric fields
$\neg\neg$	?? (conjecturally: algebraically closed geometric fields are integral over the base)
infinitesimal	local algebras together with a nilpotent ideal [Hutzler 2018]
crystalline	??