Constructive mathematics for mathematical phantoms: **synthetic algebraic geometry**

– an invitation –

CM:FP 2023 in Niš
June 27th, 2023

Ingo Blechschmidt

Not in this talk

“Can we salvage the result if we require the function to be uniformly continuous?”

“Can we weaken dependent choice to countable choice?”

“Can we weaken the decidability assumption?”

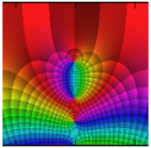
“Can pointfree topology help?”

Mathematical phantoms

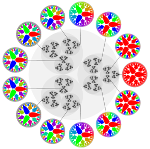
Gavin Wraith

One of the recurring themes of mathematics, and one that I have always found seductive, is that of

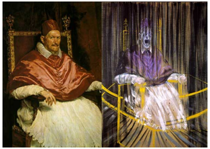
- *the nonexistent entity which ought to be there but apparently is not;*
- *which nevertheless obtrudes its effects so convincingly that one is forced to concede a broader notion of existence.*



$\mathbb{C}$



$\mathbb{Q}_p$

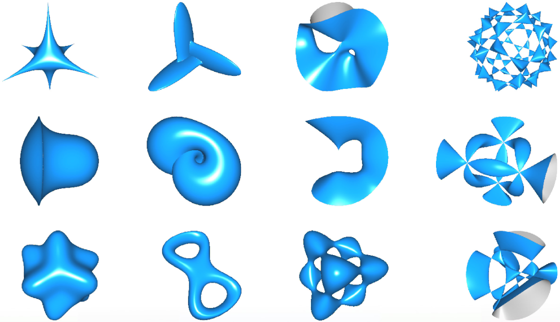


$\mathbb{F}_1$

∞

A glimpse of algebraic geometry

Algebraic geometry studies **solution sets** of polynomial systems of equations, and spaces obtained by **gluing** such sets:



C. Stussak, P. Schenzel. Interactive visualisation of algebraic surfaces as a tool for shape creation. Int. J. Arts Technol. 4:2 (2011), pp. 216–218

Concrete results such as Fermat's Last Theorem: For $n \geq 3$, no positive integers satisfy

$$a^n + b^n = c^n.$$

Mathematics vs. algebraic geometry

Let k be a base field, e.g. $\mathbb{Q}, \mathbb{F}_p, \ldots$

Functions

Which functions $k^2 \to k$ are there?

$$(x, y) \mapsto x^3 + xy^2 - y^4$$

✓ polynomial

$$(x, y) \mapsto \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{else.} \end{cases}$$

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✗ no algebraic certificate:

$$x = \ldots x^2 \ldots ?!$$

Transfinite methods?

The standard road to algebraic geometry:

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2. Put the Zariski topology on k^n.
3. Add non-maximal prime ideals to soberify the space.
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despite:

- concrete subject matter
- practical computer algebra systems for computations
- high-level proofs often constructive

Synthetic algebraic geometry

Postulate. We have a k-algebra R such that

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2. R is **quasicoherent**,

and such that we have

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Prop. ("Nullstellensatz") If $\mathrm{Spec}(A) = \emptyset$, then $A = 0$.

Cor. For all $x \in R$, if $x \neq 0$, then x is invertible.

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Prop. It is *not the case* that for every $\varepsilon \in R$, if $\varepsilon^2 = 0$ then $\varepsilon = 0$.

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Prop. ("Nullstellensatz") If $\mathrm{Spec}(A) = \emptyset$, then $A = 0$.

Cor. For all $x \in R$, if $x \neq 0$, then x is invertible.

Prop. It is *not the case* that for every $\varepsilon \in R$, if $\varepsilon^2 = 0$ then $\varepsilon = 0$.

Cor. It is *not the case* that for every $x \in R$, either $x = 0$ or $x \neq 0$.