AN ELEMENTARY AND CONSTRUCTIVE PROOF OF GROTHENDIECK'S GENERIC FREENESS LEMMA

INGO BLECHSCHMIDT

Abstract. We present a new and direct proof of Grothendieck's generic freeness lemma in its general form. Unlike the previously published proofs, it doesn't proceed in a series of reduction steps and is fully constructive, not involving the axiom of choice or even the law of excluded middle. It was found by unwinding the result of a general topos-theoretic technique.

We prove Grothendieck's generic freeness lemma for rings and modules in the following form.

Theorem 1. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. If $f = 0$ is the only element of A such that*

(1) *the $A[f^{-1}]$-modules $B[f^{-1}]$ and $M[f^{-1}]$ are free,*
(2) *the $A[f^{-1}]$-algebra $B[f^{-1}]$ is of finite presentation and*
(3) *the $B[f^{-1}]$-module $M[f^{-1}]$ is finitely presented,*

then $1 = 0$ in A.

Previously known proofs either only cover the case where A is a Noetherian integral domain, where one can argue by *dévissage* (see for instance [4, Lemme 6.9.2], [6, Thm. 24.1] or [5, Thm. 14.4]), or proceed in a series of intermediate steps, reducing to that case (see for instance [7] or [8, Tag 051Q]); but in fact, a direct proof is possible and shorter.

Grothendieck's generic freeness lemma is often presented in contrapositive form or in the following geometric variant:

Theorem 2. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. Then the space* $\mathrm{Spec}(A)$ *contains a dense open U such that over U,*

(a) *$B^\sim$ and $M^\sim$ are free as sheaves of $A^\sim$-modules,*
(b) *$B^\sim$ is of finite presentation as a sheaf of $A^\sim$-algebras and*
(c) *$M^\sim$ is finitely presented as a sheaf of $B^\sim$-modules.*

Theorem 2 immediately follows from Theorem 1 by defining U as the union of all those basic opens $D(f)$ such that (1), (2) and (3) hold. It's clear that (a), (b) and (c) hold over U, and U is dense for if V is an arbitrary open such that $U \cap V = \emptyset$, the open V is itself empty: Let $h \in A$ be such that $D(h) \subseteq V$. The hypothesis implies the assumptions of Theorem 1 for the datum $(A[h^{-1}], B[h^{-1}], M[h^{-1}])$. Thus $1 = 0 \in A[h^{-1}]$, so h is nilpotent and $D(h) = \emptyset$.

The new proof was found using a general topos-theoretical technique which we believe to be useful in other situations as well. This technique allows to view reduced rings and their modules from a different point of view, one from which reduced rings look like fields. Since Grothendieck's generic freeness is trivial for fields, this technique yields a trivial proof for reduced rings. The proof presented here was obtained by unwinding the topos-theoretic proof, yielding a self-contained argument without any references to topos theory. We refer readers who want to learn about this technique to a forthcoming companion paper [1].

Acknowledgments. The proof presented here was prompted by user HeinrichD on MathOverflow [3] and greatly benefited from discussions with Martin Brandenburg, who employed the constructive version in a paper of his [2]. I'm grateful to Marc Nieper-Wißkirchen for carefully guiding my PhD studies at the University of Augsburg, where most of the work for this paper was carried out.

1. The proof of the finitely-generated case

The following proposition is just a special instance of Grothendieck's generic freeness lemma. Its proof is easier and shorter than the proof of the general case, which is why we present it here. The general proof will not refer to this one.

Proposition 3. *Let A be a reduced ring. Let M be a finitely generated A-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is a finite free $A[f^{-1}]$-module, then $1 = 0$ in A.*

Proof. We proceed by induction on the length of a given generating family of M. Let M be generated by $(v_1, \ldots, v_m)$.

We show that the family $(v_1, \ldots, v_m)$ is linearly independent. Let $\sum_i a_i v_i = 0$. Over $A[a_i^{-1}]$, the vector $v_i \in M[a_i^{-1}]$ is a linear combination of the other generators. Thus $M[a_i^{-1}]$ can be generated as an $A[a_i^{-1}]$-module by fewer than m generators. The induction hypothesis, applied to this module, yields that $1 = 0$ in $A[a_i^{-1}]$. Since A is reduced, this amounts to $a_i = 0$.

We finish by using the assumption for $f = 1$. □

We remark that proof takes a somewhat curious course: Our goal is to verify $1 = 0$, but as an intermediate step we verify that M is free, which after the fact will be a trivial statement.

2. The proof of the general case

Proof of Theorem 1. Let B be generated by $(x_1, \ldots, x_n)$ as an A-algebra and let M be generated by $(v_1, \ldots, v_m)$ as a B-module. We endow the sets

$$\mathcal{I} := \{(i_1, \ldots, i_n) \,|\, i_1, \ldots, i_n \geq 0\} \quad \text{and}$$
$$\mathcal{J} := \{(\ell, i_1, \ldots, i_n) \,|\, 1 \leq \ell \leq m, i_1, \ldots, i_n \geq 0\}$$

with the lexicographic order. The family $(w_J)_{J \in \mathcal{J}} := (x_1^{i_1} \cdots x_n^{i_n} v_\ell)_{(\ell, i_1, \ldots, i_n) \in \mathcal{J}}$ generates M as an A-module, and we'll call a subfamily $(w_J)_{J \in \mathcal{J}' \subseteq \mathcal{J}}$ *good* if and only if for all $J \in \mathcal{J}$, the vector w_J is a linear combination of the vectors $(w_{J'})_{J' \in \mathcal{J}', J' \preceq J}$, and if $(\ell, i_1, \ldots, i_n) \notin \mathcal{J}'$ implies $(\ell, k_1, \ldots, k_n) \notin \mathcal{J}'$ for all $k_1 \geq i_1, \ldots, k_n \geq i_n$. Figure 1 shows how a good generating family can look like. Similarly, we define when a subfamily of the canonical generating family $(x_1^{i_1} \cdots x_n^{i_n})_{(i_1, \ldots, i_n) \in \mathcal{I}}$ of B is good (which is just the special case $m = 1$).

We then proceed by induction on the shapes of a given good generating family $(w_J)_{J \in \mathcal{J}'}$ for M and a given good generating family $(s_I)_{I \in \mathcal{I}'}$ for B, starting with the canonical ones. It's reasonably obvious that this induction is well-founded; the formal statement that it is so is known as *Dickson's Lemma* in mathematical logic.

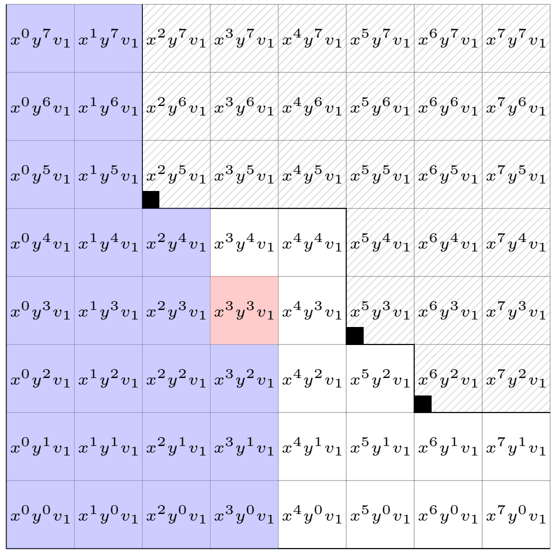

FIGURE 1. A graphical depiction of a good generating family (the non-hatched cells) in the special case $n = 2, m = 1$, writing "x" and "y" for x_1 and x_2. The hatched cells indicate vectors which have already been removed from the family. The small black squares indicate *corners*. If the vector in the red cell will be found to be expressible as a linear combination of vectors with smaller index (blue cells), it will be removed, along with the vectors in all cells to the top and to the right of the red cell.

We show that $(w_J)_{J\in\mathcal{J}'}$ is a basis of M by verifying linear independence. Thus let $\sum_J a_J w_J = 0$ in M. We show that all coefficients in this sum are zero, starting with the largest appearing index J: In the module $M[a_J^{-1}]$ over the localized ring $A[a_J^{-1}]$, the vector w_J is a linear combination of generators with smaller index. Removing $w_J = x_1^{i_1}\cdots x_n^{i_n} v_\ell$ and also all vectors $x_1^{k_1}\cdots x_n^{k_n} v_\ell$ where $k_1 \geq i_1, \ldots, k_n \geq i_n$, we obtain a subfamily which is still good for the localized module. The induction hypothesis, applied to $A[a_J^{-1}]$ and its module $M[a_J^{-1}]$, therefore implies that $A[a_J^{-1}] = 0$. Thus $a_J = 0$ since A is reduced.

Similarly, we show that the given good generating family $(s_I)_{I\in\mathcal{I}'}$ is a basis. Thus M and B are free over A. We fix for any corner J of $\mathcal{J}'$, as indicated in Figure 1, a way of expressing $w_J = \sum_K a_{JK} w_K$ as a linear combination of generators with strictly smaller index. Let $\widehat{w}_{(\ell,i_1,\ldots,i_n)} := x_1^{i_1}\cdots x_n^{i_n} V_\ell$ in the free B-module $B\langle V_1,\ldots,V_m\rangle$. The canonical map

$$\widehat{M} := B\langle V_1,\ldots,V_m\rangle/(\widehat{w}_J - \textstyle\sum_K a_{JK}\widehat{w}_K)_{J \text{ corner of } \mathcal{J}'} \longrightarrow M$$

is trivially well-defined and surjective. It is also injective, since any element of $\widehat{M}$ can be written as an A-linear combination of the vectors $(\widehat{w}_J)_{J\in\mathcal{J}'}$ by employing the corner relations a finite number of times. Therefore M is finitely presented as a B-module.

In a similar vein, a quotient algebra of $A[X_1,\ldots,X_n]$, where we mod out by a suitable ideal with as many generators as corners of $\mathcal{I}'$, is isomorphic to B. Thus B is finitely presented as an A-algebra.

We finish by using the assumption for $f = 1$. $\square$

References

[1] I. Blechschmidt. "Without loss of generality, any reduced ring is a field". 2018, in preparation.

[2] M. Brandenburg. "Operations on categories of modules are given by Schur functors". 2016. URL: `https://arxiv.org/abs/1610.02180`.

[3] MathOverflow contributors. *Constructive proof that a kernel consists of nilpotent elements (thread started by HeinrichD)*. 2016. URL: `https://mathoverflow.net/questions/250040`.

[4] J. Dieudonné and A. Grothendieck. "Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie". In: *Publ. Math. Inst. Hautes Études Sci.* 24 (1965).

[5] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*. Vol. 150. Grad. Texts in Math. Springer, 1995.

[6] H. Matsumura. *Commutative ring theory*. Vol. 8. Cambridge Stud. Adv. Math. Cambridge University Press, 1987.

[7] C. Staats. *Elementary proof of generic freeness*. 2011. URL: `https://math.uchicago.edu/~cstaats/Charles_Staats_III/Notes_and_papers_files/generic_freeness.pdf`.

[8] The Stacks Project Authors. *Stacks Project*. URL: `https://stacks.math.columbia.edu/`.