A GENERAL NULLSTELLENSATZ FOR GENERALIZED SPACES

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ABSTRACT. We give a general Nullstellensatz for the generic model of a geometric theory, useful as a source of nongeometric sequents validated by the generic model, and characterize the first-order formulas validated by the genetric model. XXX

1. INTRODUCTION

Generic models. Let $\mathbb{T}$ be a geometric theory, such as the theory of rings, local rings or intervals. We follow Caramello's terminology [4] to mean by *geometric theory* a system given by a set of sorts, a set of finitary function symbols, a set of finitary relation symbols and a set of axioms, consisting of geometric sequents (sequents of the form $\varphi \vdash_{\vec{x}} \psi$ where φ and ψ are geometric formulas, that is formulas built from equality and the relation symbols by the logical connectives $\top \bot \wedge \vee \exists$ and by arbitrary set-indexed disjunctions $\bigvee$).

A fundamental result is that there is a *generic model* $U_{\mathbb{T}}$ of $\mathbb{T}$, a model such that for any geometric sequent σ, the following notions coincide:

(1) The sequent σ is provable modulo $\mathbb{T}$.
(2) The sequent σ holds for any $\mathbb{T}$-model in any Grothendieck topos.
(3) The sequent σ holds for $U_{\mathbb{T}}$.

One could argue that it is this model which mathematicians implicitly refer to when they utter the phrase "Let M be a $\mathbb{T}$-model.". It can typically not be realized as a set-theoretic model, consisting of a set for each sort, a function for each function symbol and so on; instead it is a model in a custom-tailored syntactically constructed Grothendieck topos, the *classifying topos* $\mathrm{Set}[\mathbb{T}]$ of $\mathbb{T}$, hence consists of an object of $\mathrm{Set}[\mathbb{T}]$ for each sort, a morphism for each function symbol and so on.

To state what it means for a $\mathbb{T}$-structure in a topos $\mathcal{E}$ to verify the axioms of $\mathbb{T}$, rendering it a model, the *internal language* of $\mathcal{E}$ is used, roughly reviewed in Section **??** below. We write "$\mathcal{E} \models \alpha$" to mean that the statement α holds from the internal point of view of $\mathcal{E}$. Since this language is a form of a higher-order dependently-typed intuitionistic logic, the classifying topos $\mathrm{Set}[\mathbb{T}]$ can be regarded as a higher-order completion of the geometric theory $\mathbb{T}$. The generic model enjoys the universal property that any $\mathbb{T}$-model in any (Grothendieck) topos $\mathcal{E}$ is the pullback of $U_{\mathbb{T}}$ along an essentially unique geometric morphism $\mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

Nongeometric sequents. Crucially, the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) relating provability and truth in $\mathrm{Set}[\mathbb{T}]$ only pertains to geometric sequents. The generic model may validate additional nongeometric sequents which are not provable from the axioms of $\mathbb{T}$ in first-order or even higher-order logic, and these nongeometric sequents may be quite surprising and have useful consequences.

One of the most celebrated such sequents arises in the case that $\mathbb{T}$ is the theory of local rings. In this case, the classifying topos $\mathrm{Set}[\mathbb{T}]$ is also known as the *big Zariski*

topos of $\mathrm{Spec}(\mathbb{Z})$ from algebraic geometry, the topos of sheaves over the site of schemes locally of finite presentation, and the generic model is the functor $\underline{\mathbb{A}}^1$ of points of the affine line, the functor which maps any (l.o.f.p.) scheme X to $\mathrm{Hom}(X, \mathbb{A}^1) = \mathcal{O}_X(X)$.

From the point of view of the topos, the ring object $\underline{\mathbb{A}}^1$ is not only a local ring, but even a field in the sense that any nonzero element is invertible. As this condition is of nongeometric form, it is not inherited by arbitrary local rings, which are indeed typically not fields. However, any intuitionistic consequence of this condition which is of geometric form is inherited by any local ring in any topos. Hence we may, when verifying a general fact about local rings, suppose without loss of generality that the given ring is a field. This observation is due to Kock [5], who exploited it to develop projective geometry over local rings, and was further used by Reyes to prove a Jacobian criterion for étale morphisms [6].

A related nongeometric sequent is valid in the little Zariski topos of the spectrum of a ring A, the classifying topos of local localizations of A. If A is reduced, the generic model validates the dual condition that any noninvertible element is zero. This property has been used to give a short and even constructive proof of Grothendieck's generic freeness lemma, substantially improving on previously published proofs [2].

In time, further nongeometric sequents holding in the big Zariski topos of an arbitrary base scheme have been found [3, Section 18.4]. These include:

- $\underline{\mathbb{A}}^1$ is *anonymously algebraically closed* in the sense that any monic polynomial $p : \underline{\mathbb{A}}^1[T]$ of degree at least one does *not not* have a zero.

- The Nullstellensatz holds: Let $f_1, \ldots, f_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ be polynomials without a common zero in $(\underline{\mathbb{A}}^1)^n$. Then there are polynomials $g_1, \ldots, g_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ such that $\sum_i g_i f_i = 1$.

- Any function $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is given by a unique polynomial.

- $\underline{\mathbb{A}}^1$ is microaffine: Let $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \,|\, \varepsilon^2 = 0\}$. Let $f : \Delta \to \underline{\mathbb{A}}^1$ be an arbitrary function. Then there are unique elements $a, b : \underline{\mathbb{A}}^1$ such that $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon : \Delta$.

- $\underline{\mathbb{A}}^1$ is *synthetically quasicoherent*: For any finitely presented $\underline{\mathbb{A}}^1$-algebra A, the canonical homomorphism $A \to (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(A)}$, where $\mathrm{Spec}(A)$ is defined as the set of $\underline{\mathbb{A}}^1$-algebra-homomorphisms $A \to \underline{\mathbb{A}}^1$, is bijective.

All of these nongeometric sequents are useful for the purposes of synthetic algebraic geometry, the desire to carry out algebraic geometry in a language close to the simple language on the 19th and the beginning of the 20th century while still being fully rigorous and fully general, working over arbitrary base schemes instead of restricting to the field of complex numbers.

Characterizing nongeometric sequents. Referring to one of the previous examples, Tierney remarked around the time that those sequents were first studied that "[it] is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas" [7, p. 209]. In view of their importance, is there a way to discover nongeometric sequents in a systematic fashion? To characterize the nongeometric sequents holding in classifying toposes? To this end, Wraith put forward a specific conjecture [8, p. 336]:

> *The problem of characterising all the non-geometric properties of*
> *a generic model appears to be difficult. If the generic model of*

> *a geometric theory $\mathbb{T}$ satisfies a sentence α then any geometric consequence of $\mathbb{T} + \alpha$ has to be a consequence of $\mathbb{T}$. We might call α $\mathbb{T}$-redundant. Does the generic $\mathbb{T}$-model satisfy all $\mathbb{T}$-redundant sentences?*

This question was recently answered in the negative by Bezem, Buchholtz and Coquand [1]; hence the characterization we propose is necessarily more nuanced.

Our starting point was the empirical observation [3, p. 164] that in the case of the big Zariski topos, every true known nongeometric sequent followed from just a single such, namely the synthetic quasicoherence of the generic model, and in earlier work we surmised that one could formulate an appropriate metatheorem explaining this observation and generalizing it to arbitrary classifying toposes [3, Speculation 22.1]. This hope turned out to be true, in the sense we will now make precise.

A general Nullstellensatz. To explain the relevant background, the somewhat vague question "to which extent does the classifying topos $\mathrm{Set}[\mathbb{T}]$ realize that it is the classifying topos for $\mathbb{T}$?" is a useful guiding principle. This is easiest to visualize with a concrete example for $\mathbb{T}$, such as the theory of rings.

Let A be a ring. A simple version of the classical Nullstellensatz states: *For any polynomials f and g over A, if any zero of f is also a zero of g, then there is a polynomial h such that $g = hf$.* The polynomial h can be regarded as an "algebraic certificate" of the hypothesis. This principle holds for instance in the case that A is an algebraically closed field and that g is the unit polynomial. It is also true, without any restriction on g, for the generic ring.

We could try to generalize the Nullstellensatz to arbitrary geometric theories $\mathbb{T}$ as follows. *For any geometric sequent σ, if σ holds for a given $\mathbb{T}$-model M then σ is provable modulo $\mathbb{T}$.* In place of the algebraic certificate we now have a logical certificate, a *proof* of σ. However, this generalized statement is typically false, even for the generic model $U_{\mathbb{T}}$: The statement

$$\mathrm{Set}[\mathbb{T}] \models \ulcorner \text{for any geometric sequent } \sigma,$$
$$\text{if } \sigma \text{ holds for } U_{\mathbb{T}} \text{ then } \underline{\mathbb{T}} \text{ proves } \sigma \urcorner$$

does not hold.[1]

For instance, let $\mathbb{T}$ be the theory of rings and let σ be the sequent $(\top \vdash 1 + 1 = 0)$. Since there is an intuitionistic proof that $\mathbb{T}$ does not prove σ and toposes are sound with respect to intuitionistic logic, the statement $\ulcorner \underline{\mathbb{T}} \text{ proves } \sigma \urcorner$ is false from the internal point of view of $\mathrm{Set}[\mathbb{T}]$. However, it is not the case that the statement $\ulcorner 1 + 1 = 0 \text{ in } U_{\mathbb{T}} \urcorner$ is false from the internal point of view. In fact, this statement holds in a nontrivial slice of $\mathrm{Set}[\mathbb{T}]$, the open subtopos coinciding with the classifying topos of the theory of rings of characteristic two.

Intuitively, the problem is that while the meaning of $\ulcorner \underline{\mathbb{T}} \text{ proves } \sigma \urcorner$ is fixed, the meaning of $\ulcorner \sigma \text{ holds for } U_{\mathbb{T}} \urcorner$ can vary with the slice. This problem can be solved by passing from $\underline{\mathbb{T}}$ to a varying theory, the internal theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ defined in Section **??**. If $\mathbb{T}$ is the theory of rings, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is the $\mathrm{Set}[\mathbb{T}]$-theory of $U_{\mathbb{T}}$-algebras. Unlike $\underline{\mathbb{T}}$,

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[1]Here $\underline{\mathbb{T}}$ is the internal geometric theory induced by $\mathbb{T}$, obtained by pulling back the set of sorts, the set of function symbols and so on along the geometric morphism $\mathrm{Set}[\mathbb{T}] \to \mathrm{Set}$. For instance, if $\mathbb{T}$ is the theory of rings, then from the internal point of view of $\mathrm{Set}[\mathbb{T}]$ the theory $\underline{\mathbb{T}}$ will again be the theory of rings. More details will be given in Section **??**. The corner quotes indicate that for sake of readability, the translation into formal language is to be carried out by the reader.

this theory is not the pullback of an external geometric theory. We then have, subject to some qualifications made precise in Section **??**:

Theorem 1.1. *Let* $\mathbb{T}$ *be a geometric theory. Then, internally to* $\mathrm{Set}[\mathbb{T}]$:

$$A \text{ geometric}^* \text{ sequent } \sigma \text{ holds for } U_{\mathbb{T}} \text{ if and only if } \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves}^* \sigma. \qquad (\ddagger)$$

To illustrate Theorem 1.1, let $\mathbb{T}$ be the theory of rings and let σ be of the form $(f(x) = 0 \vdash_x g(x) = 0)$ for some polynomials f and g. To say that σ holds for $U_{\mathbb{T}}$ amounts to saying that any zero $x : U_{\mathbb{T}}$ of f is also a zero of g, and to say that $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves σ amounts to saying that in $U_{\mathbb{T}}[X]/(f(X))$, the free $U_{\mathbb{T}}$-algebra on one generator X subject to the relation $f(X) = 0$, the relation $g([X]) = 0$ holds. Hence we obtain

$$\mathrm{Set}[\mathbb{T}] \models \forall f, g : U_{\mathbb{T}}[X]. \left((\forall x : U_{\mathbb{T}}.\ f(x) = 0 \Rightarrow g(x) = 0) \Longleftrightarrow \exists h \in U_{\mathbb{T}}[X].\ g = hf \right).$$

In case that $\mathbb{T}$ is the quotient of a Horn theory XXX

Theorem 1.1 is a source of nongeometric sequents. Indeed, it is the universal such source in the sense that any first-order formula holding for $U_{\mathbb{T}}$ can be deduced from $(\ddagger)$:

Theorem 1.2. *Let* $\mathbb{T}$ *be a geometric theory. Let* α *be a first-order formula over the signature of* $\mathbb{T}$. *Then the following statements are equivalent.*

(1) *The formula* α *holds for* $U_{\mathbb{T}}$.
(2) *The formula* α *is provable in first-order intuitionistic logic modulo the axioms of* $\mathbb{T}$ *and the additional axiom scheme* $(\ddagger)$.

Acknowledgments. XXX

2. Background on classifying toposes

3. Proofs of the main theorems

The following lemma shows that for coherent theories, there is no difference between provability and provability*.

Lemma 3.1. *Let* $\mathcal{E}$ *be a Grothendieck topos. Let* $\mathbb{T}$ *be a coherent theory internal to* $\mathcal{E}$. *Let* σ *be a coherent sequent over the signature of* $\mathbb{T}$. *Then the following statements are equivalent.*

(1) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of externally finite shape$\urcorner$.
(2) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of arbitrary external shape$\urcorner$.
(3) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of arbitrary internal shape$\urcorner$.
(4) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of internally finite shape$\urcorner$.

Proof. It is obvious that (1) implies (2) implies (3).

To verify that (3) implies (4), we can mimick the usual proof of this fact in the internal language of $\mathcal{E}$: One can construct a variant of the syntactic site of $\mathbb{T}$ which is built using only coherent sequents and finitary derivability [4, Section 1.4]. The sheaf topos over this site is another model for the classifying topos of $\mathbb{T}$. If σ is $\mathbb{T}$-derivable by a proof tree of arbitrary shape, then σ holds in this model of the classifying topos. By the analogue of Proposition **??** for this model, we obtain that σ is $\mathbb{T}$-derivable by a proof tree of finite shape.

That (4) implies (1) is a routine exercise exploiting that

$$\mathcal{E} \models \forall X. \ulcorner X \text{ is Kuratowski-finite} \urcorner \Leftrightarrow \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : X. \ \forall x : X. \ \bigvee_{i=1}^{n} x = x_i.$$

$\square$

4. THE SPECIAL CASE OF HORN THEORIES

Throughout this section, let $\mathbb{T}$ be a Horn theory.

Lemma 4.1. *Let X be a set equipped with a morphism $X \to S$ to the set of sorts of the signature Σ of $\mathbb{T}$. Let R be a set of atomic propositions in which the elements of X may appear as new constants of the respective sorts. Then there is $\mathbb{T}\langle X|R \rangle$, the free $\mathbb{T}$-model on the generators X modulo the relations R.*

Proof. The desired model can be constructed as a term algebra. As a set, it consists of the terms (in the empty context) of the signature $\Sigma + X$ modulo the equivalence relation identifying two terms if and only if $\mathbb{T} + R$ proves them to be equal. The function symbols f of Σ are interpreted by declaring $[\![f]\!]([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ and the relation symbols R are interpreted by declaring $([t_1], \ldots, [t_n]) \in [\![S]\!] \Leftrightarrow (\mathbb{T} + R \vdash S(t_1, \ldots, t_n))$.

We omit the required verifications and only remark that while the same construction can be carried out if $\mathbb{T}$ is a general geometric theory, the resulting object would in general not be a model of $\mathbb{T}$. $\square$

Lemma 4.2. *The category of $\mathbb{T}$-models is complete and cocomplete.*

Proof. Limits are computed as the limits of the underlying sets, colimits are computed by using the construction of Lemma 4.1. For instance, the coproduct of $\mathbb{T}\langle X|R \rangle$ and $\mathbb{T}\langle X'|R' \rangle$ is $\mathbb{T}\langle X \amalg X' \mid R, R' \rangle$. $\square$

Having the special case of the theory of rings in mind, we write the coproduct in the category of $\mathbb{T}$-models as "$\otimes$".

Lemma 4.3. *Let $\sigma = (\varphi_1 \wedge \cdots \wedge \varphi_n \vdash_{x_1, \ldots, x_k} \psi_1 \wedge \cdots \wedge \psi_m)$ be a Horn sequent over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) *The theory $\mathbb{T}$ proves σ.*
(2) *In $\mathbb{T}\langle x_1, \ldots, x_k|\varphi_1, \ldots, \varphi_n \rangle$, the propositions $\psi_1, \ldots, \psi_m$ hold for the k-tuple $([x_1], \ldots, [x_k])$.*

Proof. By construction of the term algebra. $\square$

Lemma 4.4. *A $\mathbb{T}$-model is a finitely presented object of the category of $\mathbb{T}$-models if and only if it is isomorphic to a model of the form $\mathbb{T}\langle X|R \rangle$ where X is Bishop-finite and R is Kuratowski-finite.*

Proof. It is an instructive exercise to verify that models of the stated form are compact. Conversely, let a $\mathbb{T}$-model M be given. Then $\mathbb{T}$ is the filtered colimit of all models over M which are of the stated form. If M is compact, the identity on M factors over such a model. Hence M is a retract of such a model and hence itself isomorphic to a model of this form. $\square$

Any $\mathbb{T}$-model A has a mirror image in the topos $\mathrm{Set}[\mathbb{T}]$, namely the functor $A^\sim$: $\mathbb{T}\text{-mod}_{\mathrm{fp}} \to \mathrm{Set}$ given by $T \mapsto A \otimes T$. This object is in a canonical way a $\mathbb{T}$-model over $U_\mathbb{T}$, hence from the point of view of $\mathrm{Set}[\mathbb{T}]$ a $\underline{\mathbb{T}}/U_\mathbb{T}$-model.

Lemma 4.5. *The functor* $(\cdot)^\sim$ *from* $\mathbb{T}$-*models to* $\underline{\mathbb{T}}/U_\mathbb{T}$-*models in* $\mathrm{Set}[\mathbb{T}]$ *is left adjoint to functor* $\Gamma = \mathrm{Hom}(1, \cdot)$ *computing global elements.*

Proof. An $U_\mathbb{T}$-algebra-homomorphism $\alpha : A^\sim \to M$ yields the $\mathbb{T}$-model-homo-morphism $\alpha_0 : A \to M(0) = \Gamma(M)$, where 0 is the initial $\mathbb{T}$-model. Conversely, a $\mathbb{T}$-model-homomorphism $\beta : A \to \Gamma(M)$ yields an $U_\mathbb{T}$-algebra-homomorphism by summing $A \to M(0) \to M(T)$ with the structure morphism $T = U_\mathbb{T}(T) \to M(T)$. $\qquad\square$

Definition 4.6. The *spectrum* $\mathrm{Spec}(M)$ of a $U_\mathbb{T}$-algebra M in $\mathrm{Set}[\mathbb{T}]$ is the result of constructing, internally to $\mathrm{Set}[\mathbb{T}]$, the set of $U_\mathbb{T}$-algebra-homomorphisms $M \to U_\mathbb{T}$.

Externally, the spectrum of M is the functor mapping a finitely presented $\mathbb{T}$-model T to $\mathrm{Hom}_T(M(T), T)$, the set of $\mathbb{T}$-homomorphisms $M(T) \to T$ compatible with the structure morphisms $T \to M(T)$ and $T \to T$.

5. Applications

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