USING THE INTERNAL LANGUAGE OF TOPOSES IN ALGEBRAIC GEOMETRY

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Abstract. There are several important topoi associated to a scheme, for instance the petit and gros Zariski topoi. These come with an internal mathematical language which closely resembles the usual formal language of mathematics, but is "local on the base scheme":

For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a certain condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

These expository notes give an introduction to this topic and show how the internal point of view can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry.

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1. Introduction

2. Kripke–Joyal semantics

Let X be a topological space. Later, X will be the underlying space of a scheme.

Definition 2.1 (Kripke–Joyal semantics of a sheaf topos)**.** The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ is given by the following rules, recursively in the structure of φ:

$$\begin{array}{lll}
U \models f = g : \mathcal{F} & :\Longleftrightarrow & f|_U = g|_U \in \Gamma(U, \mathcal{F}) \\
U \models \varphi \wedge \psi & :\Longleftrightarrow & U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi & :\Longleftrightarrow & \cancel{U \models \varphi \text{ or } U \models \psi} \\
 & & \text{there exists a covering } U = \bigcup_i U_i \text{ s. th. for all } i\text{:} \\
 & & \qquad U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi & :\Longleftrightarrow & \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall f : \mathcal{F}.\ \varphi(f) & :\Longleftrightarrow & \text{for all sections } f \in \Gamma(V, \mathcal{F}), V \subseteq U\text{: } V \models \varphi(f) \\
U \models \exists f : \mathcal{F}.\ \varphi(f) & :\Longleftrightarrow & \cancel{\text{there exists a section } f \in \Gamma(U, \mathcal{F}) \text{ s. th. } U \models \varphi(f)} \\
 & & \text{there exists a covering } U = \bigcup_i U_i \text{ s. th. for all } i\text{:} \\
 & & \qquad \text{there exists } f_i \in \Gamma(U_i, \mathcal{F}) \text{ s. th. } U_i \models \varphi(f_i) \\
U \models \forall \mathcal{F}.\ \varphi(\mathcal{F}) & :\Longleftrightarrow & \text{for all sheaves } \mathcal{F} \text{ on } V,\ V \subseteq U\text{: } V \models \varphi(\mathcal{F}) \\
U \models \exists \mathcal{F}.\ \varphi(\mathcal{F}) & :\Longleftrightarrow & \text{there exists a covering } U = \bigcup_i U_i \text{ s. th. for all } i\text{:} \\
 & & \qquad \text{there exists a sheaf } \mathcal{F}_i \text{ on } U_i \text{ s. th. } U_i \models \varphi(\mathcal{F}_i)
\end{array}$$

Remark 2.2. The last two rules, concerning *unbounded quantification*, are not part of the classical Kripke–Joyal semantics, but instead of Mike Shulman's stack semantics [**?**], a slight extension. They are needed so that we can use formulate universal properties in the internal language.

The rules are not all arbitrary. They are finely concerted to make the following propositions true, which are crucial for a proper appreciation of the internal language.

Proposition 2.3 (Locality of the internal language)**.** *Let $U = \bigcup_i U_i$ be covered by open subsets. Let φ be a formula. Then*

$$U \models \varphi \qquad \textit{iff} \qquad U_i \models \varphi \textit{ for each } i.$$

Proof. Induction on the structure of φ. Note that the canceled rules would make this proposition false. ☐

Proposition 2.4 (Soundness of the internal language)**.** *If a formula φ implies a further formula ψ in intuitionistic logic, then*

$$U \models \varphi \qquad \textit{implies} \qquad U \models \psi.$$

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that on each U_i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in [**?**, D1.3.1]. ☐

- fundamental properties

- geometric formulas
- geometric constructions

3. Sheaves of rings

- reducedness
- field property
- discreteness

4. Sheaves of modules

- of finite type, of finite presentation, coherent
- basic lemmas
- flatness
- important hard exercise

5. Rational functions and Cartier divisors

- internal definition of K_X
- internal definition of Cartier divisors
- correspondence between Cartier divisors and sub-O_X-modules of K_X

6. Relative spectrum

- ...

7. Modalities

- negneg
- spreading of properties from stalk to neighbourhood
- internal sheafification

8. Unsorted

- Kähler differentials
- completion of the natural numbers, rank function
- meta properties
- locally small categories
- big Zariski topos
- open/closed immersions
- morphisms of schemes...
- proper maps...

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