Using the internal language of toposes in algebraic geometry

Ingo Blechschmidt, University of Augsburg

Summary

With the internal language of toposes, we can:

- express sheaf-theoretic concepts in a simple, element-based language and thus understand them in a more conceptual way,
- mechanically obtain corresponding sheaf-theoretic theorems for any (intuitionistic) theorem of linear or commutative algebra, and
- understand which properties spread from points to neighbourhoods.

What is a topos?

A *topos* is a category which has finite limits, is cartesian closed and has a subobject classifier. More simply, a topos is a category which has similar properties as the category of sets.

Important examples of toposes are the category of sets and the category of set-valued sheaves on a topological space.

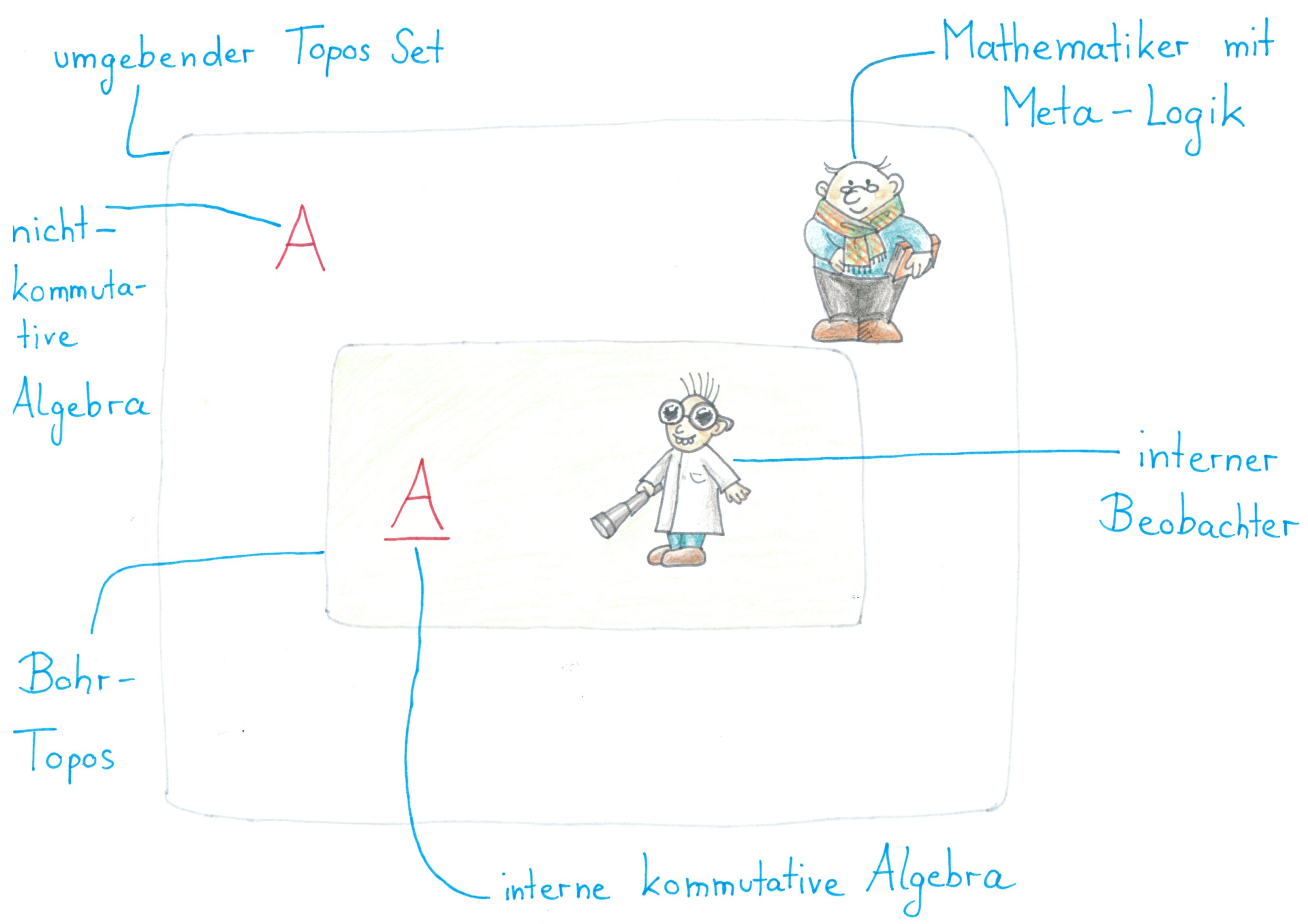
What is the internal language?

The internal language of a topos $\mathcal{E}$ allows us to construct objects and morphisms of the topos, formulate statements about them, and prove such statements in a *naive element-based* language. *Special case:* The language of the topos of sets is the usual formal mathematical language.

external point of view	internal point of view
objects of $\mathcal{E}$	sets
morphisms of $\mathcal{E}$	maps of sets

The small Zariski topos

Let X be a scheme. Let $\mathrm{Sh}(X)$ be the small Zariski topos, i. e. the topos of set-valued sheaves on X. From the point of view of $\mathrm{Sh}(X)$, the structure sheaf $\mathcal{O}_X$ looks like an *ordinary ring* (instead of a sheaf of rings), and sheaves of $\mathcal{O}_X$-modules look like *ordinary modules* on that ring.



Basic example

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. Assume that $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type. Then it is well-known that $\mathcal{F}$ is of finite type as well.

A sheaf is of finite type if and only if, internally, it is a finitely generated module. Therefore the proposition follows *at once* from interpreting the analogous statement of intuitionistic linear algebra in the little Zariski topos: Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. Assume that M and M'' are finitely generated. Then M is as well.

Caveat: Proofs by contradiction can not be interpreted with the internal language.

Locally free sheaves

Let X be a reduced scheme. The structure sheaf $\mathcal{O}_X$ looks like a *field* from the internal point of view. This is notable; recall that neither the rings of local sections nor the stalks are in general fields.

Let $\mathcal{F}$ be a finite type sheaf of $\mathcal{O}_X$-modules. Then it is well-known that $\mathcal{F}$ is locally free on a dense open subset of X.

This follows *at once* from the following statement of intuitionistic linear algebra: Let M be a finitely generated vector space. Then M is *not not* finite free.

Rational functions

The sheaf $\mathcal{K}_X$ of rational functions can internally simply be defined as the total quotient ring of $\mathcal{O}_X$.

Spreading of properties

The following meta-theorem covers a wide range of cases: Let φ be a property which can be formulated without using "$\Rightarrow$", "$\neg$", and "$\forall$". Then φ holds at a point if and only if it holds on some open neighbourhood of the point.

For instance, a sheaf of modules $\mathcal{F}$ is zero if and only if, from the internal perspective, "$\forall x \in \mathcal{F}\colon x = 0$". Because of the "$\forall$", a stalk may be zero without the sheaf being zero on a neighbourhood.

But if $\mathcal{F}$ is of finite type, the condition can be reformulated using generators as "$x_1 = 0 \wedge \cdots \wedge x_n = 0$". The meta-theorem is applicable to this statement, thus a stalk is zero if and only if $\mathcal{F}$ is zero on a neighbourhood.

Dictionary external vs. internal notions

Detailed notes are available at `http://tiny.cc/topos` (work in progress).

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