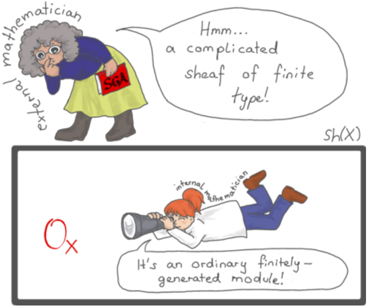

Using the internal language of toposes in algebraic geometry

am Tag, an dem Marc im Curry Club vorträgt

Outline

What is a topos?

Formal definition

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier.

Motto

A topos is a category which is sufficiently rich to support an **internal language**.

Examples

- Set: category of sets
- $\mathrm{Sh}(X)$: category of set-valued sheaves on a space X

What is the internal language?

The internal language of a topos $\mathcal{E}$ allows to

1. construct objects and morphisms of the topos,
2. formulate statements about them and
3. prove such statements

in a **naive element-based** language:

externally	internally to $\mathcal{E}$
object of $\mathcal{E}$	set/type
morphism in $\mathcal{E}$	map of sets
monomorphism	injective map
epimorphism	surjective map
ring object	ring
module object	module

The internal language of $\mathrm{Sh}(X)$

Let X be a topological space. Then we recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ.

The internal language of Sh(X)

Let X be a topological space. Then we recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ.

$$\begin{aligned}
U \models f = g : \mathcal{F} \quad &:\Longleftrightarrow f|_U = g|_U \in \Gamma(U, \mathcal{F}) \\
U \models \varphi \wedge \psi \quad &:\Longleftrightarrow U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi \quad &:\Longleftrightarrow \cancel{U \models \varphi \text{ or } U \models \psi} \\
&\qquad \text{there exists a covering } U = \textstyle\bigcup_i U_i \text{ s.th. for all } i\text{:} \\
&\qquad\qquad U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi \quad &:\Longleftrightarrow \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall f \colon \mathcal{F}.\, \varphi(f) &:\Longleftrightarrow \text{for all sections } f \in \Gamma(V, \mathcal{F}), V \subseteq U\text{: } V \models \varphi(f) \\
U \models \exists f \colon \mathcal{F}.\, \varphi(f) &:\Longleftrightarrow \text{there exists a covering } U = \textstyle\bigcup_i U_i \text{ s.th. for all } i\text{:} \\
&\qquad\qquad \text{there exists } f_i \in \Gamma(U_i, \mathcal{F}) \text{ s.th. } U_i \models \varphi(f_i)
\end{aligned}$$

The internal language of $\mathrm{Sh}(X)$

Crucial property: Locality

If $U = \bigcup_i U_i$, then $U \models \varphi$ iff $U_i \models \varphi$ for all i.

Crucial property: Soundness

If $U \models \varphi$ and φ implies ψ constructively, then $U \models \psi$.

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A simple look at the constructive nature

- $U \models f = 0$ iff $f|_U = 0 \in \Gamma(U, \mathcal{F})$.
- $U \models \neg\neg(f = 0)$ iff $f = 0$ on a dense open subset of U.

The little Zariski topos

Definition

The **little Zariski topos** of a scheme X is the category $\mathrm{Sh}(X)$ of set-valued sheaves on X.

Basic look and feel

- Internally, the structure sheaf $\mathcal{O}_X$ looks like

  an ordinary ring.

- Internally, a sheaf of $\mathcal{O}_X$-modules looks like

  an ordinary module on that ring.

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set/type
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
tensor product of sheaves	tensor product of modules
rank function	minimal number of generators
sheaf of Kähler differentials	module of Kähler differentials
dimension of X	Krull dimension of $\mathcal{O}_X$

Using the dictionary

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$$\Downarrow$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Using the dictionary

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

Any sheaf of modules of finite type on a reduced scheme is locally free *on a dense open subset.*

Ravi Vakil: “Important hard exercise” (13.7.K).

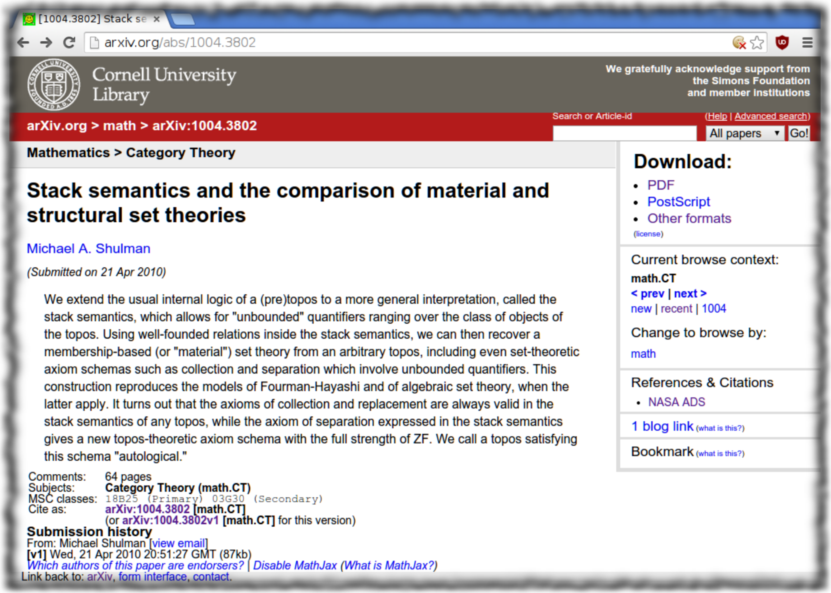
The objective

Understand notions and statements of **algebraic geometry** as notions and statements of (intuitionistic) **commutative algebra** internal to suitable **toposes**.

Further topics in the little Zariski topos:

- The sheaf $\mathcal{K}_X$ of rational functions
- Transfer principles $M \leftrightarrow M^\sim$
- The curious role of affine open subsets
- Quasicoherence
- Spreading from points to neighbourhoods
- The relative spectrum

Praise for Mike Shulman



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Mathematics > Category Theory

Stack semantics and the comparison of material and structural set theories

Michael A. Shulman

(Submitted on 21 Apr 2010)

We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for "unbounded" quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or "material") set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema "autological."

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Subjects: **Category Theory (math.CT)**
MSC classes: 18B25 (Primary) 03G30 (Secondary)
Cite as: **arXiv:1004.3802 [math.CT]**
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The sheaf of rational functions

Classical definition

The sheaf K_X of **rational functions** on a scheme X is the sheafification of the presheaf

$$U \subseteq X \longmapsto \Gamma(U, \mathcal{O}_X)[S(U)^{-1}],$$

where

$$S(U) = \{ s \in \Gamma(U, \mathcal{O}_X) \mid s \in \mathcal{O}_{X,x} \text{ is regular for all } x \in U \}.$$

Internal definition

K_X is the total quotient ring of $\mathcal{O}_X$.

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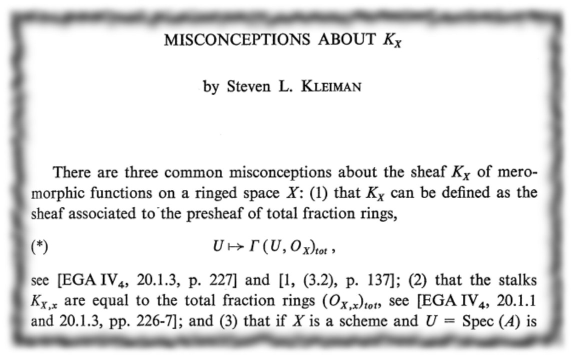

MISCONCEPTIONS ABOUT K_X

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

(*) $$U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \text{Spec}(A)$ is

Transfer principles

Question: How do the properties of

- an A-module M in Set and
- the $\mathcal{O}_X$-module $M^\sim$ in $\mathrm{Sh}(X)$, where $X = \operatorname{Spec} A$, relate?

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Observation: $M^\sim = \underline{M}[\mathcal{F}^{-1}]$, where

- $\underline{M}$ is the constant sheaf with stalks M on X and
- $\mathcal{F} \hookrightarrow \underline{A}$ is the **generic filter**.

Note: M and $\underline{M}$ share all first-order properties.

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Note: M and $\underline{M}$ share all first-order properties.

Answer: $M^\sim$ inherits those properties of M which are **stable under localization**.

The curious role of affine open subsets

Question: Why do the following identities hold, for quasicoherent sheaves and affine open subsets U?

$$\begin{aligned}
\Gamma(U, \mathcal{E}/\mathcal{F}) &= \Gamma(U, \mathcal{E})/\Gamma(U, \mathcal{F}) \\
\Gamma(U, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) &= \Gamma(U, \mathcal{E}) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, \mathcal{F}) \\
\Gamma(U, \mathcal{E}_{\mathrm{tors}}) &= \Gamma(U, \mathcal{E})_{\mathrm{tors}} \quad \text{(sometimes)} \\
\Gamma(U, \mathcal{K}_X) &= \operatorname{Quot} \mathcal{O}_X(U) \quad \text{(sometimes)} \\
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\end{aligned}$$

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&\vdots
\end{aligned}$$

Answer: Because localization commutes with quotients, tensor products, torsion submodules (sometimes), …

A curious property

Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.

ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in $\mathbf{E}$ the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in $\mathrm{Spec}(\mathbf{E}, A)$ the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n (x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Diamond_f\text{-sheaf for any } f : \mathcal{O}_X,$$
$$\text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

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In particular: If $\mathcal{E}$ is quasicoherent, then internally

$$(f \text{ invertible} \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0$$

for any $f : \mathcal{O}_X$ and $s : \mathcal{E}$.

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator. Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond,$$

where the translation $\varphi \mapsto \varphi^\Diamond$ is given by:

$$\begin{aligned}
(s = t)^\Diamond &:\equiv \Diamond(s = t) \\
(\varphi \wedge \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \wedge \psi^\Diamond) \\
(\varphi \vee \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \vee \psi^\Diamond) \\
(\varphi \Rightarrow \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \Rightarrow \psi^\Diamond) \\
(\forall x \colon X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\forall x \colon X.\, \varphi^\Diamond(x)) \\
(\exists x \colon X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\exists x \colon X.\, \varphi^\Diamond(x))
\end{aligned}$$

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Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator. Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond.$$

Let X be a scheme. Depending on $\Diamond$, $\mathrm{Sh}(X) \models \Diamond\varphi$ means that φ holds on …

- … a dense open subset.
- … a schematically dense open subset.
- … a given open subset U.
- … an open subset containing a given closed subset A.
- … an open neighbourhood of a given point $x \in X$.

Can tackle the question "$\varphi^\Diamond \overset{?}{\Rightarrow} \Diamond\varphi$" logically.

The absolute spectrum

Let A be a commutative ring (in Set).

Is there a **free local ring** $A \to A'$ over A?

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

No, if we restrict to Set.

Yes, if we allow a change of topos: Then $A \to \mathcal{O}_{\operatorname{Spec} A}$ is the universal localization.

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \text{topological space of the prime ideals of } A$$

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The absolute spectrum, internalized

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To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned}\operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \cancel{\text{topological space of the prime filters of } A} \\ &:= \text{locale of the prime filters of } A\end{aligned}$$

Define the frame of opens of $\operatorname{Spec} A$ to be the frame of radical ideals in A. Universal property:

$$\operatorname{Hom}_{\mathrm{LRT}/\mathcal{E}}(T, \operatorname{Spec} A) \cong \operatorname{Hom}_{\mathrm{Ring}(\mathcal{E})}(A, \mu_* \mathcal{O}_T)$$

for all locally ringed toposes T equipped with a geometric morphism $T \xrightarrow{\mu} \mathcal{E}$.

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be an algebra. Can we describe $\underline{\mathbf{Spec}}_X\, \boldsymbol{\mathcal{A}}$ internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{LRS}/X}(T, \underline{\mathrm{Spec}}_X\, \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all locally ringed spaces over X.

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for all locally ringed spaces over X.

Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X \mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f \colon \mathcal{O}_X.\, \forall s \colon \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

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$$\forall f\colon \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

Its **points** are those prime filters G of $\mathcal{A}$ such that

$$\forall f\colon \mathcal{O}_X.\, \varphi(f) \in G \Longrightarrow f \text{ invertible in } \mathcal{O}_X.$$

The relative spectrum, reformulated

Let $B \to A$ be an algebra in a topos.

Is there a **free local and local-over-*B* ring** $A \to A'$ over A?

$$\begin{tikzcd}
B \arrow[r] \arrow[rrr, bend left, "\text{local}"] \arrow[drr, bend right, "\text{local}"'] & A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& & \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

Form limits in the category of **locally ringed locales** by **relocalizing** the corresponding limit in ringed locales.

The big Zariski topos

Definition

The **big Zariski topos** $\mathrm{Zar}(S)$ of a scheme S is the category $\mathrm{Sh}(\mathrm{Sch}/S)$. It consists of certain functors $(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$.

Basic look and feel

- For an S-scheme X, its functor of points

$$\underline{X} = \mathrm{Hom}_S(\cdot, X)$$

  is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- Internally, $\underline{\mathbb{A}}^1_S$ (given by $\underline{\mathbb{A}}^1_S(X) = \Gamma(X, \mathcal{O}_X)$) looks like a field:

$$\mathrm{Zar}(S) \models \forall x : \underline{\mathbb{A}}^1_S.\ x \neq 0 \Longrightarrow \ulcorner x \text{ inv.} \urcorner$$

Group schemes

Motto: Internal to $\mathrm{Zar}(S)$, group schemes look like ordinary groups.

group scheme	internal definition	functor of points: $X \mapsto \ldots$
$\mathbb{G}_{\mathrm{a}}$	$\underline{\mathbb{A}}^1_S$ (as additive group)	$\Gamma(X, \mathcal{O}_X)$
$\mathbb{G}_{\mathrm{m}}$	$\{x : \underline{\mathbb{A}}^1_S \mid \ulcorner x \text{ inv.} \urcorner\}$	$\Gamma(X, \mathcal{O}_X)^\times$
μ_n	$\{x : \underline{\mathbb{A}}^1_S \mid x^n = 1\}$	$\{f \in \Gamma(X, \mathcal{O}_X) \mid f^n = 1\}$
GL_n	$\{M : {\underline{\mathbb{A}}^1_S}^{n \times n} \mid \ulcorner M \text{ inv.} \urcorner\}$	$\mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$

Some internal constructions

- The functor of points of $\mathbb{P}^n_S$ has the internal description

$$\{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1_S)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/\text{scaling}.$$

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- Let $\mathcal{A}$ be an $\mathcal{O}_S$-algebra. This induces an $\underline{\mathbb{A}}^1_S$-algebra $\mathcal{A}^\sim$ internal to $\mathrm{Zar}(S)$. The functor of points of $\underline{\mathrm{Spec}}_S\,\mathcal{A}$ has the internal description

$$\mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(\mathcal{A}^\sim, \underline{\mathbb{A}}^1_S).$$

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$$\mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(\mathcal{A}^\sim, \underline{\mathbb{A}}^1_S).$$

- Let X be an S-scheme. The functor of points of $\underline{\mathrm{Spec}}_X\,\Omega^1_{X/S} \to X \to S$ has the internal description

$$\mathrm{Hom}(\Delta, \underline{X}),$$

where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1_S \mid \varepsilon^2 = 0\}$.

The étale subtopos

Recall that the **Kummer sequence** is not exact in $\mathrm{Zar}(S)$ at the third term:

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_\mathrm{m} \xrightarrow{(\cdot)^n} \mathbb{G}_\mathrm{m} \longrightarrow 1$$

But we have:

$$\mathrm{Zar}(S) \models \forall f\colon (\underline{\mathbb{A}}^1_S)^\times .\, \Diamond_{\text{ét}}(\exists g : (\underline{\mathbb{A}}^1_S)^\times .\, f = g^n),$$

where $\Diamond_{\text{ét}}$ is such that $\mathrm{Zar}(S)_{\Diamond_{\text{ét}}} \hookrightarrow \mathrm{Zar}(S)$ is the **big étale topos**. It's the largest subtopos of $\mathrm{Zar}(S)$ where

$$\ulcorner \underline{\mathbb{A}}^1_S \text{ is separably closed} \urcorner$$

holds (Wraith, Felix).

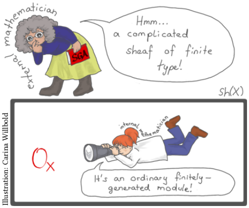
Comparing the little and the big toposes

- From the point of view of $\mathrm{Sh}(S)$, the big Zariski topos is $\mathrm{Zar}(\mathcal{O}_S|\mathcal{O}_S)$, the classifying topos of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.
- From the point of view of $\mathrm{Zar}(S)$, the little Zariski topos is the largest subtopos where $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ is bijective.

$$(\flat\underline{\mathbb{A}}^1_S)(X \xrightarrow{\mu} S) = \Gamma(X, \mu^{-1}\mathcal{O}_S)$$
$$\underline{\mathbb{A}}^1_S(X \xrightarrow{\mu} S) = \Gamma(X, \mathcal{O}_X)$$

Semi-open and open tasks

- Characterize quasicoherence in the big Zariski topos.
- Understand how to work with $\flat \dashv \sharp$.
- Do cohomology in the little Zariski topos; exploit that higher direct images look like ordinary sheaf cohomology from the internal point of view.
- Do cohomology in the big Zariski topos.
- Understand more subtoposes of the big Zariski topos.
- Derive suitable axioms for synthetic algebraic geometry.



Translating internal statements I

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\begin{aligned}
&\mathrm{Sh}(X) \models \ulcorner \alpha \text{ is injective} \urcorner \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \forall s\,{:}\,\mathcal{F}.\,\forall t\,{:}\,\mathcal{F}.\,\alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\ &\text{for all open } U \subseteq X, \text{ sections } s \in \mathcal{F}(U): \\
&\qquad \text{for all open } V \subseteq U, \text{ sections } t \in \mathcal{F}(V): \\
&\qquad\qquad \text{for all open } W \subseteq V: \\
&\qquad\qquad\qquad \alpha_W(s|_W) = \alpha_W(t|_W) \text{ implies } s|_W = t|_W \\
\Longleftrightarrow\ &\text{for all open } U \subseteq X, \text{ sections } s, t \in \mathcal{F}(U): \\
&\qquad \alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U \\
\Longleftrightarrow\ &\alpha \text{ is a monomorphism of sheaves}
\end{aligned}$$

Translating internal statements II

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$
\begin{aligned}
& \mathrm{Sh}(X) \models \ulcorner \alpha \text{ is surjective} \urcorner \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \forall t : \mathcal{G}.\, \exists s : \mathcal{F}.\, \alpha(s) = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } t \in \mathcal{G}(U): \\
& \qquad \text{there exists an open covering } U = \textstyle\bigcup_i U_i \text{ and} \\
& \qquad \text{sections } s_i \in \mathcal{F}(U_i) \text{ such that:} \\
& \qquad\qquad \alpha|_{U_i}(s_i) = t|_{U_i} \\
\Longleftrightarrow\ & \alpha \text{ is an epimorphism of sheaves}
\end{aligned}
$$

Translating internal statements III

Let X be a topological space (or locale) and let $s, t \in \mathcal{F}(X)$ be global sections of a sheaf $\mathcal{F}$ on X. Then:

$$
\begin{aligned}
& \mathrm{Sh}(X) \models \neg\neg(s = t) \\
\iff & \mathrm{Sh}(X) \models ((s = t) \Rightarrow \bot) \Rightarrow \bot \\
\iff & \text{for all open } U \subseteq X \text{ such that} \\
& \qquad \text{for all open } V \subseteq U \text{ such that} \\
& \qquad\qquad s|_V = t|_V, \\
& \qquad \text{it holds that } V = \emptyset, \\
& \text{it holds that } U = \emptyset \\
\iff & \text{there exists a dense open set } W \subseteq X \text{ such that } s|_W = t|_W
\end{aligned}
$$

Spreading from points to neighbourhoods

All of the following lemmas have a short, sometimes trivial proof. Let $\mathcal{F}$ be a sheaf of finite type on a ringed space X. Let $x \in X$. Let $A \subseteq X$ be a closed subset. Then:

1. $\mathcal{F}_x = 0$ iff $\mathcal{F}|_U = 0$ for some open neighbourhood of x.
2. $\mathcal{F}|_A = 0$ iff $\mathcal{F}|_U = 0$ for some open set containing A.
3. $\mathcal{F}_x$ can be generated by n elements iff this is true on some open neighbourhood of x.
4. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.
5. $\mathcal{F}$ is torsion iff $\mathcal{F}_\xi$ vanishes (assume X integral and $\mathcal{F}$ quasicoherent).
6. $\mathcal{F}$ is torsion iff $\mathcal{F}|_{\mathrm{Ass}(\mathcal{O}_X)}$ vanishes (assume X locally Noetherian and $\mathcal{F}$ quasicoherent).

The smallest dense sublocale

Let X be a reduced scheme satisfying a technical condition. Let $i : X_{\neg\neg} \to X$ be the inclusion of the smallest dense sublocale of X.

Then $i_* i^{-1} \mathcal{O}_X \cong \mathcal{K}_X$.

- This is a highbrow way of saying "rational functions are regular functions which are defined on a dense open subset".
- Another reformulation is that $\mathcal{K}_X$ is the sheafification of $\mathcal{O}_X$ with respect to the $\neg\neg$-modality.
- There is a generalization to nonreduced schemes.

Applications in algebra

Let A be a commutative ring. The internal language of $\mathrm{Sh}(\operatorname{Spec} A)$ allows you to say “without loss of generality, we may assume that A is local”, even constructively.

The kernel of any matrix over a principial ideal domain is finitely generated.

$$\Downarrow$$

The kernel of any matrix over a Prüfer domain is finitely generated.

Hilbert's program in algebra

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides, for instance exhibiting explicit witnesses. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all prime filters") by "for the generic prime filter".

More specifically, simply interpret a given proof using prime filters in $\mathrm{Sh}(\operatorname{Spec} A)$ and let it refer to $\mathcal{F} \hookrightarrow \underline{A}$.

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

This is related (in a few cases equivalent) to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others. Their approach is more versatile.