A general Nullstellensatz for generalised spaces

– an invitation –

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This set of slides will be updated soon with more annotations. Right now important references are not properly mentioned, including:

- Marc Bezem, Ulrik Buchholtz and Thierry Coquand. *Syntactic forcing models for coherent logic.*
- Olivia Caramello. *Universal models and definability.*
- Carsten Butz and Peter Johnstone. *Classifying toposes for first-order theories.*

Fig.: The Event Horizon Telescope picture of the central black hole in the galaxy M87

At this meeting we had lots of discussions on the notion of space. Here is a picture of actual space, or rather the lack thereof, released only yesterday.

The mystery of nongeometric sequents

Let $\mathbb{T}$ be a geometric theory, for instance the theory of rings.

sorts, function symbols, relation symbols, geometric sequents as axioms

sorts: R
fun. symb.: $0, 1, -, +, \cdot$
axioms: $(\top \vdash_{x,y:R} xy = yx), \ldots$

$\mathbb{Z}$ $\quad$ $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$ $\quad$ $\mathcal{O}_X$ $\quad$ $U_{\mathbb{T}}$

A *geometric sequent* is a syntactical expression of the form $(\varphi \vdash_{x_1 : X_1, \ldots, x_n : X_n} \psi)$, where $x_1 : X_1, \ldots, x_n : X_n$ is a list of variable declarations, the X_i ranging over the available sorts, and φ and ψ are *geometric formulas*. Often the variable context is abbreviated to $\vec{x} : \vec{X}$ or even just $\vec{x}$. Such a sequent is read as "in the context of variables $\vec{x}$, φ entails ψ".

Geometric formulas are built from atomic propositions (using equality or the relation symbols) using the connectives $\top$, $\bot$, $\wedge$, $\bigvee$ (set-indexed disjunction) and $\exists$. Geometric formulas may not contain $\neg$, $\Rightarrow$, $\forall$.

There is a notion of a *model* of a geometric theory in a given topos. For instance, a ring in the usual sense is a model of the theory of rings in the topos Set. The structure sheaf of a scheme X is a model in the topos $\mathrm{Sh}(X)$ of set-valued sheaves on X.

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Theorem. There is a **generic model** $U_{\mathbb{T}}$. It is **conservative** in that for any geometric sequent σ the following notions coincide:

1. The sequent σ holds for $U_{\mathbb{T}}$.
2. The sequent σ holds for any $\mathbb{T}$-model in any topos.
3. The sequent σ is provable modulo $\mathbb{T}$.

Among all models in any topos, the *universal* or *generic* one is special. It enjoys the universal property that any model in any topos can be obtained from it by pullback along an essentially unique geometric morphism. It is intriguing from a logical point of view because it has exactly those properties which are shared by any model in any topos.

One could argue, with a certain amount of success, that the generic model of the theory of rings is what a mathematician implicitly refers to when she utters the phrase "Let R be a ring". This point of view is fundamental to the slogan *continuity is geometricity*, as expounded for instance in Continuity and geometric logic by Steve Vickers.

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Observation (Kock). The generic local ring is a field:

$$(x = 0 \Rightarrow \bot) \vdash_{x:R} (\exists y : R.\, xy = 1)$$

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Crucially, the conservativity statement only pertains to properties which can be put as geometric sequents. Generic models may have additional nongeometric properties. Because conservativity does not apply to them, they are not shared by all models in all toposes – but any consequences which can be put as geometric sequents are.

For instance, if we want to verify a geometric sequent for all local rings, we may freely use the displayed field axiom. Hence one reason why these nongeometric sequents are interesting is because they provide us with new reduction strategies ("without loss of generality").

Construction of the generic model

The generic model is **not** the same as …

- the **initial model** (think $\mathbb{Z}$) or
- the **free model on one generator** (think $\mathbb{Z}[X]$).

Set-based models are **too inflexible**.

Definition. The **syntactic site** $\mathcal{C}_{\mathbb{T}}$ has …

1. objects: $\{x_1 : X_1, \ldots, x_n : X_n.\, \varphi\}$ (shorter: $\{\vec{x}.\, \varphi\}$)
2. morphisms: eqv. classes of provably functional formulas
3. coverings: provably jointly surjective families

The topos of sheaves over $\mathcal{C}_{\mathbb{T}}$ is the **classifying topos** $\mathrm{Set}[\mathbb{T}]$.
The generic model interprets a sort X by よ$\{x : X.\, \top\}$.

In case the theory $\mathbb{T}$ is a Horn theory (for instance if it is an equational theory), the *term algebra* (the set of terms in the empty context modulo provable equality) is a model of $\mathbb{T}$. While such models do enjoy some nice categorical properties, they are in general *not* the generic model.

Working internally to toposes

Let $\mathcal{C}$ be a site. We recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for objects $U \in \mathcal{C}$ and formulas φ. Write “$\mathrm{Sh}(\mathcal{C}) \models \varphi$” for $1 \models \varphi$.

$$
\begin{array}{ll}
U \models \top & \text{iff true} \\
U \models \bot & \text{iff } \cancel{\text{false}} \text{ the empty family is a covering of } U \\
U \models s = t : F & \text{iff } s|_U = t|_U \in F(U) \\
U \models \varphi \wedge \psi & \text{iff } U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi & \text{iff } \cancel{U \models \varphi \text{ or } U \models \psi} \text{ there exists a covering } (U_i \to U)_i \\
 & \qquad \text{such that for all } i\text{: } U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi & \text{iff for all } V \to U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall s : F.\, \varphi(s) & \text{iff for all } V \to U \text{ and sections } s_0 \in F(V)\text{: } V \models \varphi(s_0) \\
U \models \exists s : F.\, \varphi(s) & \text{iff } \cancel{\text{there exists } s_0 \in F(U) \text{ such that } U \models \varphi(s_0)} \\
 & \text{there exists a covering } (U_i \to U)_i \text{ such that for all } i\text{:} \\
 & \qquad \text{there exists } s_0 \in F(U_i) \text{ such that } U_i \models \varphi(s_0)
\end{array}
$$

A selection of nongeometric properties

The **generic object** validates:

1. $\forall x, y : U_{\mathbb{T}}.\ \neg\neg(x = y)$.
2. $\forall x_1, \ldots, x_n : U_{\mathbb{T}}.\ \neg\forall y : U_{\mathbb{T}}.\ \bigvee_{i=1}^{n} y = x_i$.
3. $(U_{\mathbb{T}})^{U_{\mathbb{T}}} \cong 1 \amalg U_{\mathbb{T}}$.

The **generic ring** validates:

1. $\forall x : U_{\mathbb{T}}.\ \neg\neg(x = 0)$.
2. $\forall x : U_{\mathbb{T}}.\ (x = 0 \Rightarrow 1 = 0) \Rightarrow (\exists y : U_{\mathbb{T}}.\ xy = 1)$.

The **generic local ring** validates:

1. $\neg\forall x : U_{\mathbb{T}}.\ \neg\neg(x = 0)$.
2. $\forall a_0, \ldots, a_{n-1} : U_{\mathbb{T}}.\ \neg\neg\exists x : U_{\mathbb{T}}.\ x^n + a_{n-1}x^{n-1} + \cdots + a_0x^0 = 0$.
3. Let $\Delta = \{\varepsilon : U_{\mathbb{T}} \mid \varepsilon^2 = 0\}$. For any map $f : \Delta \to U_{\mathbb{T}}$, there are unique elements $a, b : U_{\mathbb{T}}$ s. th. $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon : \Delta$.

Affine schemes

Let A be a ring. Is there a **free local ring** $A \to A'$ over A?

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] & & \underset{\text{local}}{R} \\
& \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

For a fixed ring R, the localization $A' := A[S^{-1}]$ with $S := f^{-1}[R^\times]$ would do the job. (S is a *filter*.)

Hence we need the **generic filter**.

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If A is reduced ($x^n = 0 \Rightarrow x = 0$):

$A^\sim$ is a **field**: $\forall x : A^\sim. \, (\neg(\exists y : A^\sim. \, xy = 1) \Rightarrow x = 0)$.
$A^\sim$ has **¬¬-stable equality**: $\forall x, y : A^\sim. \, \neg\neg(x = y) \Rightarrow x = y$.
$A^\sim$ is **anonymously Noetherian**.

A systematic source of nongeometricity?

Empirical fact. In **synthetic algebraic geometry**, every known property of $\underline{\mathbb{A}}^1$ followed from its **synthetic quasicoherence**:

For any finitely presented $\underline{\mathbb{A}}^1$-algebra A, the canonical map

$$A \longrightarrow (\underline{\mathbb{A}}^1)^{\mathrm{Hom}_{\underline{\mathbb{A}}^1}(A,\underline{\mathbb{A}}^1)}, \quad s \longmapsto (x \mapsto x(s))$$

is an isomorphism of $\underline{\mathbb{A}}^1$-algebras.

1. Does a general metatheorem explain this observation?
2. Is there a systematic source in any classifying topos?
3. Is there even an exhaustive source?

$$\begin{tikzcd} \mathbb{T}\text{ proves }\alpha \arrow[r, Rightarrow, bend right] & \alpha\text{ holds for }U_{\mathbb{T}} \arrow[l, Rightarrow, bend right, "\times"'] \arrow[r, Rightarrow, bend right] & \alpha\text{ is }\mathbb{T}\text{-redundant} \arrow[l, Rightarrow, bend right, "\times"'] \end{tikzcd}$$

A systematic source of nongeometricity?

> A. Kock has pointed out [5 (ii)] that the generic local ring satisfies the nongeometric sentence
>
> $$\forall x_1 \ldots \forall x_n . (\neg (\bigwedge_i (x_i = 0)) \to \bigvee_i (\exists y .\, x_i y = 1))$$
>
> which in classical logic defines a field! The problem of characterising all the nongeometric properties of a generic model appears to be difficult. If the generic model of a geometric theory T satisfies a sentence α then any geometric consequence of $T + (\alpha)$ has to be a consequence of T. We might call α T-redundant. Does the generic T-model satisfy all T-redundant sentences?

Gavin Wraith. Some recent developments in topos theory.
In: Proc. of the ICM (Helsinki, 1978).

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A topos-theoretic Nullstellensatz

Theorem. Internally to $\mathrm{Set}[\mathbb{T}]$:

For any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.

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The algebraic Nullstellensatz. Let A be a ring. Let $f, g \in A[X]$ be polynomials. Then, subject to some conditions:

$$\underbrace{\big(\forall x \in A.\, (f(x) = 0 \Rightarrow g(x) = 0)\big)}_{\text{algebraic truth}} \Longrightarrow \underbrace{\big(\exists h \in A[X].\, g = hf\big)}_{\text{algebraic certificate}}$$

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A naive version. "Internally to $\mathrm{Set}[\mathbb{T}]$, for any geometric sequent σ over the signature of $\underline{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}$ proves σ." **False,** for instance with the theory of rings we have

$$\begin{array}{rl} & \mathrm{Set}[\mathbb{T}] \models \neg(\ulcorner \underline{\mathbb{T}} \text{ proves } (\top \vdash 1 + 1 = 0)\urcorner) \\ \text{but} & \mathrm{Set}[\mathbb{T}] \not\models \neg(1 + 1 = 0). \end{array}$$

A varying internal theory

Theorem. Internally to $\mathrm{Set}[\mathbb{T}]$:

> *For any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.*

Definition. The theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is the internal geometric theory of **$U_{\mathbb{T}}$-algebras**, the theory which arises from $\underline{\mathbb{T}}$ by adding:

1. for each element $x : U_{\mathbb{T}}$ a constant symbol e_x,
2. for each function symbol f and n-tuple $(x_1, \ldots, x_n) \in (U_{\mathbb{T}})^n$ the axiom $(\top \vdash f(e_{x_1}, \ldots, e_{x_n}) = e_{f(x_1,\ldots,x_n)})$,
3. for each relation symbol R and n-tuple $(x_1, \ldots, x_n) \in (U_{\mathbb{T}})^n$ such that $R(x_1, \ldots, x_n)$ the axiom $(\top \vdash R(e_{x_1}, \ldots, e_{x_n}))$.

Remark. Externalising the internal classifying topos $\mathrm{Set}[\mathbb{T}][\underline{\mathbb{T}}/U_{\mathbb{T}}]$ yields the classifying topos of $\mathbb{T}$-homomorphisms.

Revisiting the test cases

Theorem. Internally to $\mathrm{Set}[\mathbb{T}]$:

> *For any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, if σ holds for $U_{\mathbb{T}}$, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.*

In the object classifier. Let $x, y : U_{\mathbb{T}}$. Assume that $\neg(x = y)$. By the Nullstellensatz $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves $(e_x = e_y \vdash \bot)$. But this is false in the $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model $U_{\mathbb{T}}/(x \sim y)$.

In the ring classifier. Let $f, g : U_{\mathbb{T}}[X]$ such that any zero of f is a zero of g. By the Nullstellensatz $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves this fact. Hence it holds in the $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model $U_{\mathbb{T}}[X]/(f)$. In this model f has the zero $[X]$. Hence also $g([X]) = 0$ in $U_{\mathbb{T}}[X]/(f)$, that is $g = hf$ for some $h : U_{\mathbb{T}}[X]$.

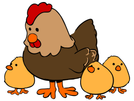
Exhaustion and extensions

Theorem. A first-order formula holds for $U_{\mathbb{T}}$ iff it is intuitionistically provable from the axioms of $\mathbb{T}$ and the scheme

$$\ulcorner \sigma \text{ holds} \urcorner \implies \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves}^{\star\star}\ \sigma \urcorner. \qquad \text{(Nullstellensatz)}$$

Theorem. Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$. Assume that $U_{\mathbb{T}}$ is contained in the subtopos $\mathrm{Set}[\mathbb{T}']$. Then internally to $\mathrm{Set}[\mathbb{T}']$:

> *A geometric* sequent σ with Horn consequent holds for $U_{\mathbb{T}'}$ iff $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ.*



Theorem. The morphism ev is an isomorphism.

$$\mathrm{ev} : \mathrm{FunctFormulas}^\star(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash) \longrightarrow P(U_{\mathbb{T}})$$

Theorem. A higher-order formula holds for $U_{\mathbb{T}}$ iff it is provable in intuitionistic higher-order logic from the axioms of $\mathbb{T}$ and the