AN ELEMENTARY AND CONSTRUCTIVE PROOF OF GROTHENDIECK'S GENERIC FREENESS LEMMA

INGO BLECHSCHMIDT

Abstract. We present a new and direct proof of Grothendieck's generic freeness lemma in its general form. Unlike the previously published proofs, it doesn't proceed in a series of reduction steps and is fully constructive, not involving the axiom of choice or even the law of excluded middle. It was found by unwinding the result of a general topos-theoretic technique.

We prove Grothendieck's generic freeness lemma for rings and modules in the following form.

Theorem 1. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. If $f = 0$ is the only element of A such that*

1. *the $A[f^{-1}]$-modules $B[f^{-1}]$ and $M[f^{-1}]$ are free,*
2. *the $A[f^{-1}]$-algebra $B[f^{-1}]$ is of finite presentation, and*
3. *the $B[f^{-1}]$-module $M[f^{-1}]$ is finitely presented,*

then $1 = 0$ in A.

Previously known proofs proceed in a series of intermediate steps, reducing to the case that A is a Noetherian integral domain where one can argue by *dévissage*; but in fact, a direct proof is possible and shorter.

Grothendieck's generic freeness lemma is often presented in contrapositive form or in the following geometric variant:

Theorem 2. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. Then the space* $\mathrm{Spec}(A)$ *contains a dense open U such that over U,*

(a) *$B^\sim$ and $M^\sim$ are free as sheaves of $A^\sim$-modules,*
(b) *$B^\sim$ is of finite presentation as a sheaf of $A^\sim$-algebras, and*
(c) *$M^\sim$ is finitely presented as a sheaf of $B^\sim$-modules.*

Theorem 2 immediately follows from Theorem 1 by defining U as the union of all the basic opens $D(f)$ such that (1), (2), and (3) hold. It's clear that (a), (b), and (c) hold over U, and U is dense for if V is an arbitrary open such that $U \cap V = \emptyset$, the open V is itself empty: Let $h \in A$ such that $D(h) \subseteq V$. The hypothesis implies the assumptions of Theorem 1 for the datum $(A[h^{-1}], B[h^{-1}], M[h^{-1}])$. Thus $1 = 0 \in A[h^{-1}]$, so h is nilpotent and $D(h) = \emptyset$.

The new proof was found using a general topos-theoretical technique which we believe to be useful in other situations as well. This technique allows to view reduced rings and their modules from a different point of view, one from which reduced rings look like fields. Since Grothendieck's generic freeness is trivial for fields, this technique yields a trivial proof for reduced rings. The proof presented here was obtained by unwinding the topos-theoretic proof, yielding a self-contained argument without any references to topos theory. We refer readers who want to learn about this technique to an accompanying note [**?**].

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XXX: References to prior proofs: EGA, Stacks Project, Charles Staats. Check this MO question

1. The proof of the general case

Lemma 3. *Let A be a ring. Let M be an A-module with generating family $(x_i)_{i \in I}$ where I is a totally ordered set. Assume that the only element $g \in A$ such that one of the x_i is an $A[g^{-1}]$-linear combination in $M[g^{-1}]$ of other generators with smaller index is $g = 0$. Then M is free with $(x_i)_{i \in I}$ as a basis.*

Proof. Let $\sum_i a_i x_i = 0$. Starting with the greatest index i which appears in that sum, we see that in $M[a_i^{-1}]$, the element x_i is an $A[g^{-1}]$-linear combination of other generators with smaller index. Thus $a_i = 0$ by assumption. □

Theorem 4. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is free as an $A[f^{-1}]$-module, then $1 = 0$ in A.*

Proof. Let B be generated by $(x_1, \ldots, x_n)$ as an A-algebra and let M be generated by $(v_1, \ldots, v_m)$ as an B-module. We endow the set

$$I := \{(\ell, j_1, \ldots, j_n) \mid \ell \in \{1, \ldots, m\}, j_1, \ldots, j_n \in \{0, 1, \ldots\}\}$$

with the lexicographic order. The family $(w_i)_{i \in I} := (x_1^{j_1} \ldots x_n^{j_n} v_\ell)_{(\ell, j_1, \ldots, j_n) \in I}$ thus generates M as an A-module, and we'll call a subfamily $(w_i)_{i \in I' \subseteq I}$ *good* if and only if for all $i \in I$, the vector w_i is a linear combination of the vectors $(w_{i'})_{i' \in I, i' \preceq i}$, and if $(\ell, j_1, \ldots, j_n) \notin I'$ implies $(\ell, k_1, \ldots, k_n) \notin I'$ for all $k_1 \geq j_1, \ldots, k_n \geq j_n$. Figure 1 shows how a good generating family can look like.

We induct on the shape of a good generating family, starting with $(w_i)_{i \in I}$.

We show that $(w_i)_{i \in I}$ is a basis of M by verifying the assumptions of Lemma 3. Thus let $g \in A$ be given such that one of the w_i is an $A[g^{-1}]$-linear combination of generators with smaller index in $M[g^{-1}]$. Removing $w_i = x_1^{j_1} \ldots x_n^{j_n} v_\ell$ and also all vectors $x_1^{k_1} \ldots x_n^{k_n} v_\ell$ where $k_1 \geq j_1, \ldots, k_n \geq j_n$, we obtain a subfamily which is still good. By induction, applied to $A[g^{-1}]$ and its module $M[g^{-1}]$, it therefore follows that $A[g^{-1}] = 0$. This implies that $g = 0$ since A is reduced.

Thus M is free. We finish by using the assumption for $f = 1$. □

Recall that a module is Noetherian if and only if all of its submodules are finitely generated. We call a module M over an A-algebra B *weakly Noetherian* (over A) if and only if, for all B-submodules $U \subseteq M$ and all $f \in A$ the condition

for all $g \in A$, if $U[(fg)^{-1}]$ is finitely generated as an $B[(fg)^{-1}]$-module, then fg is nilpotent in A

implies that f is nilpotent. Geometrically, this expresses that all quasicoherent sheaves of submodules of $M^\sim$ are, over a dense open of $\mathrm{Spec}(A)$, of finite type over $B^\sim$. An A-algebra B is *weakly Noetherian* if and only if it is weakly Noetherian as a module over itself.

This notion is interesting for the following reason.

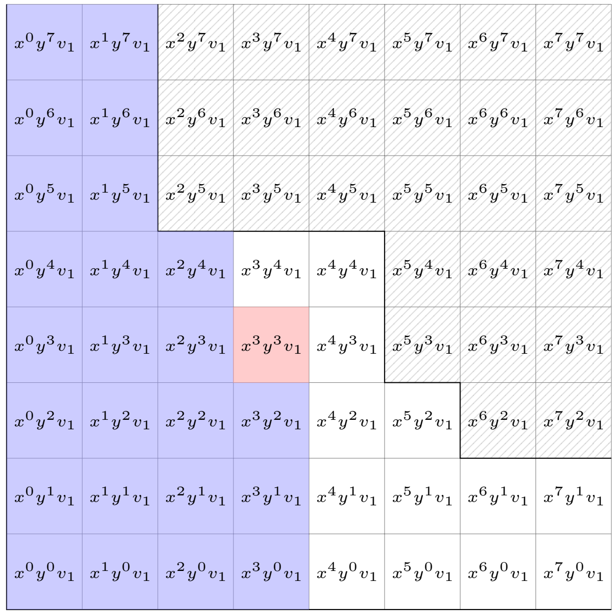

FIGURE 1. A graphical depiction of a good generating family in the special case $n = 2, m = 1$. The hatched cells indicate vectors which have already been removed from the family. If the vector in the red cell will be found to be expressible as a linear combination of vectors with smaller index (blue cells), it will be removed, along with the vectors in all cells to the top and to the right of the red cell.

Lemma 5. *Let A be a reduced ring. Then A, regarded as an algebra over itself, is weakly Noetherian.*

Proof. Let $\mathfrak{a} \subseteq A$ be an ideal. Let $f \in A$ such that, for any $g \in A$, if $\mathfrak{a}[(fg)^{-1}]$ is finitely generated, then $fg = 0$. We verify that $f = 0$.

We first show that $\mathfrak{a}[f^{-1}] = (0)$. Let $g \in \mathfrak{a}$. Then $\mathfrak{a}[(fg)^{-1}] = (1)$ is finitely generated. Thus $fg = 0$, hence $g = 0$ in $A[f^{-1}]$.

Since $\mathfrak{a}[f^{-1}]$ is finitely generated, we finish by using the assumption for $g = 1$. $\square$

The weak Noetherian condition enjoys the same basic stability properties as the usual Noetherian condition:

Lemma 6. *Let A be a ring. Let B be an A-algebra.*

(1) *If B is weakly Noetherian, then so is $B[X]$.*
(2) *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of B-modules. Then M is weakly Noetherian if and only if M' and M'' are.*

We can use these observations to finish the proof of Grothendieck's generic freeness lemma.

Proposition 7. *Let A be a reduced ring. Let B be an A-algebra of finite type. If $f = 0$ is the only element of A such that $B[f^{-1}]$ is of finite presentation as an $A[f^{-1}]$-algebra, then $1 = 0$ in A.*

Proof. We write $B = A[X_1, \ldots, X_n]/\mathfrak{a}$. By Lemma 5 and Lemma 6, the A-algebra $A[X_1, \ldots, X_n]$ is weakly Noetherian. We can therefore conclude if we can show, for any $g \in A$, that if the ideal $\mathfrak{a}[g^{-1}]$ is finitely generated, then $g = 0$. This follows immediately from the assumption (for $f = g$). □

Proposition 8. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is finitely presented as an $B[f^{-1}]$-module, then $1 = 0$ in A.*

Proof. We write $M = B^n/U$ for an B-submodule U. By Lemma 5 and Lemma 6, the B-module B^n is weakly Noetherian. We can therefore conclude if we can show, for any $g \in A$, that if the submodule $U[g^{-1}]$ is finitely generated, then $g = 0$. This follows immediately from the assumption (for $f = g$). □

2. The proof of the finitely-generated case

The following proposition is just an instance of Grothendieck's generic freeness lemma. We prove it here because it admits an easier proof.

Proposition 9. *Let A be a reduced ring. Let M be a finitely generated A-module. If $f = 0$ is the only element of A such that $M[f^{-1}]$ is a finite free $A[f^{-1}]$-module, then $1 = 0$ in A.*

Proof. We proceed by induction on the length of a given generating family of M. Let M be generated by $(v_1, \ldots, v_m)$.

We show that the family $(v_1, \ldots, v_m)$ is linear independent. Let $\sum_i a_i v_i = 0$. Over $A[a_i^{-1}]$, the vector $v_i \in M[a_i^{-1}]$ is a linear combination of the other generators. Thus $M[a_i^{-1}]$ can be generated as an $A[a_i^{-1}]$-module by fewer than m generators. The induction hypothesis, applied to this module, yields that $1 = 0$ in $A[a_i^{-1}]$. Since A is reduced, this amounts to $a_i = 0$.

We finish by using the assumption for $f = 1$. □