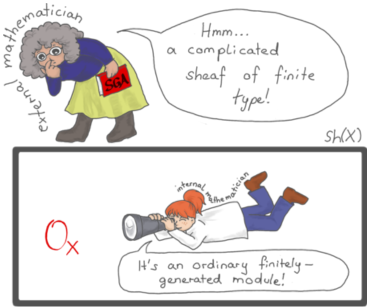

Synthetic algebraic geometry

a case study in applied topos theory

&

the phenomenon of nongeometric sequents

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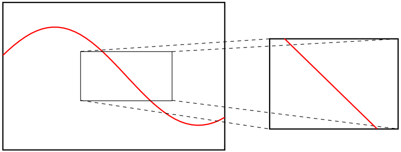
Synthetic differential geometry

The axiom of microaffinity

Let $\Delta = \{\varepsilon \in \mathbb{R} \mid \varepsilon^2 = 0\}$. For any function $f : \Delta \to \mathbb{R}$, there are unique numbers $a, b \in \mathbb{R}$ such that $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon \in \Delta$.



- The **derivative** of f as above at zero is b.
- Manifolds are **just sets**.
- A **tangent vector** to M is a map $\Delta \to M$.

Toposes provide models for this theory.

The internal universe of a topos

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier, for instance

- **Set**, the category of sets;
- $\mathbf{Sh}(X)$, the category of set-valued sheaves over a space X;
- **Eff**, the effective topos (roughly: a category of data types).

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no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

Curious universes

- Eff $\models$ “There are infinitely many prime numbers.” ✓
  External meaning: There is a Turing **machine** producing arbitrarily many prime numbers.

- Eff $\not\models$ “Any Turing machine halts or doesn’t halt.” ✗
  External meaning: There is a **halting oracle** which determines whether any given machine halts or doesn’t halt.

- $\mathrm{Sh}(X) \not\models$ “Any cont. function with opposite signs has a zero.” ✗
  External meaning: Zeros can be picked **locally continuously** in continuous families of continuous functions.

Approaches to algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  set of prime ideals of $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$ +
  Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

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Synthetic approach: model schemes **directly as sets** in a certain nonclassical set theory.

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

The big Zariski topos

Let S be a fixed base scheme.

Definition

The **big Zariski topos** $\mathrm{Zar}(S)$ is the category $\mathrm{Sh}(\mathrm{Sch}/S)$. It consists of functors $(\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition that

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

is a limit diagram for any scheme $U = \bigcup_i U_i$ over S.

- For an S-scheme X, its functor of points $\underline{X} = \mathrm{Hom}_S(\cdot, X)$ is an object of $\mathrm{Zar}(S)$. It feels like **the set of points** of X.
- In particular, there is the ring object $\underline{\mathbb{A}}^1$.
- $\mathrm{Zar}(S)$ classifies local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$.

Properties of the affine line

- $\underline{\mathbb{A}}^1$ is a local ring:

$$1 \neq 0 \qquad\qquad x + y \text{ inv.} \Longrightarrow x \text{ inv.} \vee y \text{ inv.}$$

- $\underline{\mathbb{A}}^1$ is a field:

$$\neg(x = 0) \Longleftrightarrow x \text{ inv.}$$
$$\neg(x \text{ inv.}) \Longleftrightarrow x \text{ nilpotent}$$

- $\underline{\mathbb{A}}^1$ satisfies the axiom of microaffinity.
- $\underline{\mathbb{A}}^1$ is anonymously algebraically closed: Any monic polynomial does *not not* have a zero.
- $\underline{\mathbb{A}}^1$ is of unbounded Krull dimension.
- Any function $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial.

Synthetic constructions

$\mathbb{A}^n = (\underline{\mathbb{A}}^1)^n = \underline{\mathbb{A}}^1 \times \cdots \times \underline{\mathbb{A}}^1$

$\mathbb{P}^n = \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\underline{\mathbb{A}}^1)^\times$
$\cong$ set of one-dimensional subspaces of $(\underline{\mathbb{A}}^1)^{n+1}$
(with $\mathcal{O}(-1) = (\ell)_{\ell : \mathbb{P}^n}$, $\mathcal{O}(1) = (\ell^\vee)_{\ell : \mathbb{P}^n}$)

Spec(R) $= \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1) =$ set of $\underline{\mathbb{A}}^1$-valued points of R

TX $= \mathrm{Hom}(\Delta, X)$, where $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$

A subset $U \subseteq X$ is **qc-open** if and only if for any $x : X$ there exist $f_1, \ldots, f_n : \underline{\mathbb{A}}^1$ such that $x \in U \iff \exists i.\, f_i \neq 0$.

A **synthetic affine scheme** is a set which is in bijection with Spec(R) for some finitely presented algebra R.

A **synthetic scheme** is a set which can be covered by finitely many qc-open synthetic affine schemes U_i such that the intersections $U_i \cap U_j$ can be covered by finitely many qc-open synthetic affine schemes.

Synthetic quasicoherence

Recall $\mathrm{Spec}(R) = \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1)}(R, \underline{\mathbb{A}}^1)$ and consider the statement

"the canonical map $\begin{array}{ccc} R & \longrightarrow & \mathrm{Hom}(\mathrm{Spec}(R), \underline{\mathbb{A}}^1) \\ f & \longmapsto & (\alpha \mapsto \alpha(f)) \end{array}$ is bijective".

- True for $R = \underline{\mathbb{A}}^1[X]/(X^2)$ (microaffinity).
- True for $R = \underline{\mathbb{A}}^1[X]$ (every function is a polynomial).
- True for **any** finitely presented $\underline{\mathbb{A}}^1$-algebra R.

Any known property of $\underline{\mathbb{A}}^1$ follows from this **synthetic quasicoherence**.

Example. Let $f : \underline{\mathbb{A}}^1$ such that $f \neq 0$. Set $R = \underline{\mathbb{A}}^1/(f)$.
Then $\mathrm{Spec}(R) = \emptyset$. Thus $\mathrm{Hom}(\mathrm{Spec}(R), \underline{\mathbb{A}}^1)$ is a singleton.
Hence $R = 0$. Therefore f is invertible.

Nongeometric sequents

Let $\mathbb{T}$ be a **geometric theory** (rings, intervals, …).

For a **geometric sequent** $\forall \vec{x}.\,(\varphi \Rightarrow \psi)$, the following are equivalent:

1. It is **provable** by $\mathbb{T}$.
2. It holds **for all models** of $\mathbb{T}$ in all toposes.
3. It holds for the **generic model** of $\mathbb{T}$ in its **classifying topos**.

- Additional **nongeometric sequents** may hold in a classifying topos, for instance "$\underline{\mathbb{A}}^1$ is synthetically quasicoherent" in $\mathrm{Zar}(S)$.
- These are **$\mathbb{T}$-redundant**, but the converse is false.
- Are they precisely the consequences of synthetic quasicoherence?
- Applications: synthetic algebraic geometry, generic freeness, …