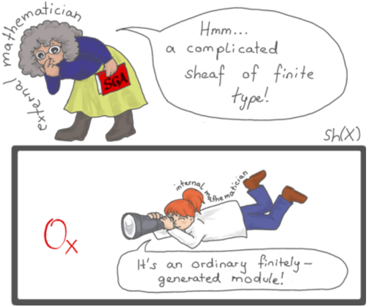

Using the internal language of toposes in algebraic geometry

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Topos à l'IHES
November 27th, 2015

Outline

Several Topos theory questions

29

20

Hey. I have a few off the wall questions about topos theory and algebraic geometry.

1. Do the following few sentences make sense?

Every scheme X is pinned down by its Hom functor Hom(-,X) by the yoneda lemma, but since schemes are locally affine varieties, it is actually just enough to look at the case where "-" is an affine scheme. So you could define schemes as particular functors from CommRing^op to Sets. In this setting schemes are thought of as sheaves on the "big zariski site".

If that doesn't make sense my next questions probably do not either.

2 The category of sheaves on the big zariski site forms a topos T, the category of schemes being a subcategory. It is convenient to reason about toposes in their own "internal logic". Has there been much thought done about the internal logic of T, or would the logic of T require too much commutative algebra to feel like logic? Along these lines, have there been attempts to write down an elementary list of axioms which capture the essense of this topos? I am thinking of how Anders Kock has some really nice ways to think of differential geometry with his SDG.

ct.category-theory topos-theory lo.logic ag.algebraic-geometry

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edited Oct 24 '09 at 17:53
Ben Webster ♦
25.3k ● 4 ● 73 ● 167

asked Oct 24 '09 at 16:49
Steven Gubkin
6,003 ● 2 ● 47 ● 87

Exploiting the internal language

A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to the **spectrum of a commutative ring**:

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

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The topos $\mathrm{Sh}(X)$ is the **petit Zariski topos** of X.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set/type
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

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monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
coherent sheaf	coherent module
tensor product of sheaves	tensor product of modules
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

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epimo	
sheaf o	
sheaf o	
sheaf o	
finite l	
cohere	
tensor	
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

MISCONCEPTIONS ABOUT K_X

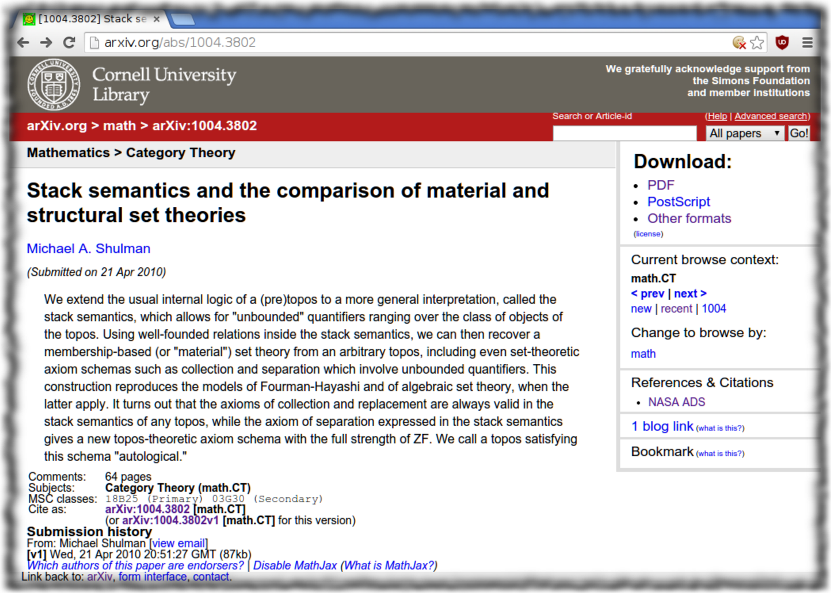
by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

$$(*) \qquad U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \mathrm{Spec}\,(A)$ is

Praise for Mike Shulman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Mathematics > Category Theory

Stack semantics and the comparison of material and structural set theories

Michael A. Shulman

(Submitted on 21 Apr 2010)

We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for "unbounded" quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or "material") set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema "autological."

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MSC classes: 18B25 (Primary) 03G30 (Secondary)
Cite as: arXiv:1004.3802 [math.CT]
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Using the dictionary

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$$\Downarrow$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Using the dictionary

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

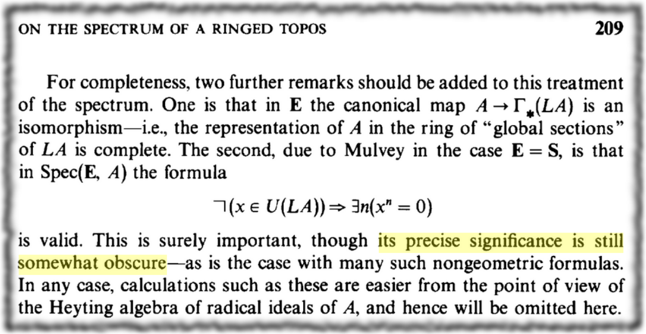
Any sheaf of modules of finite type on a reduced scheme is locally free *on a dense open subset*.

Ravi Vakil: “Important hard exercise” (13.7.K).

A curious property

Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.


ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec(**E**, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator $\Diamond$. Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond,$$

$\Diamond : \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$

where the translation $\varphi \mapsto \varphi^\Diamond$ is given by:

$$\begin{aligned}
(s = t)^\Diamond &:\equiv \Diamond(s = t) \\
(\varphi \wedge \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \wedge \psi^\Diamond) \\
(\varphi \vee \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \vee \psi^\Diamond) \\
(\varphi \Rightarrow \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \Rightarrow \psi^\Diamond) \\
(\forall x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\forall x : X.\, \varphi^\Diamond(x)) \\
(\exists x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\exists x : X.\, \varphi^\Diamond(x))
\end{aligned}$$

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$\Diamond : \Omega_\mathcal{E} \to \Omega_\mathcal{E}$

Let X be a scheme. Depending on $\Diamond$, $\mathrm{Sh}(X) \models \Diamond\varphi$ means that φ holds on …

- … a dense open subset.
- … a schematically dense open subset.
- … a given open subset U.
- … an open subset containing a given closed subset A.
- … an open neighbourhood of a given point $x \in X$.

Can tackle the question "$\varphi^\Diamond \overset{?}{\Rightarrow} \Diamond\varphi$" logically.

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Diamond_f\text{-sheaf for any } f : \mathcal{O}_X,$$
$$\text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

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In particular: If $\mathcal{E}$ is quasicoherent, then internally

$$(f \text{ invertible} \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0$$

for any $f : \mathcal{O}_X$ and $s : \mathcal{E}$.

The absolute spectrum

Let A be a commutative ring (in Set).

Is there a **free local ring** $A \to A'$ over A?

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & R \\ & A' \arrow[ur, dashed] & \end{tikzcd}$$

No, if we restrict to Set.

Yes, if we allow a change of topos: Then $A \to \mathcal{O}_{\operatorname{Spec} A}$ is the universal localization.

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \text{topological space of the prime ideals of } A$$

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Define the frame of opens of $\operatorname{Spec} A$ to be the frame of radical ideals in A.

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This gives an internal description of Monique Hakim's spectrum functor $\mathrm{RT} \to \mathrm{LRT}$.

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe $\underline{\mathbf{Spec}}_X\, \boldsymbol{\mathcal{A}}$, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\, \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

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Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X\, \mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

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Its **points** are those prime filters G of $\mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \varphi(f) \in G \Rightarrow f \text{ invertible in } \mathcal{O}_X.$$

The relative spectrum, reformulated

Let $B \to A$ be an algebra in topos.

Is there a **free local and local-over-B ring** $A \to A'$ over A?

$$\begin{tikzcd}
B \arrow[r] \arrow[rrr, bend left, "\text{local}"] \arrow[drr, bend right, "\text{local}"'] & A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& & \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

Form limits in the category of **locally ringed locales** by **relocalizing** the corresponding limit in ringed locales.

Understand notions and statements of algebraic geometry as notions and statements of algebra internal to appropriate toposes.

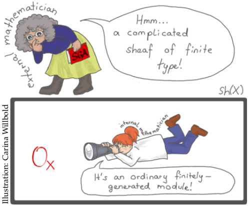

Illustration: Carina Willbold

- Simplify proofs and gain conceptual understanding.
- Understand relative geometry as absolute geometry.
- Develop a synthetic account of scheme theory.
- Contribute to constructive algebra.

http://tiny.cc/topos-notes

spreading of properties, general transfer principles, applications to constructive algebra, quasicoherence, internal Cartier divisors, pullback along immersions = internal sheafification, scheme dimension = internal Krull dimension of $\mathcal{O}_X$, dense = not not, modal operators, relative spectrum, other toposes, étale topology, group schemes = groups, ...

You should totally look up:

The Adventures of Sheafification Man

Spreading from points to neighbourhoods

All of the following lemmas have a short, sometimes trivial proof. Let $\mathcal{F}$ be a sheaf of finite type on a ringed space X. Let $x \in X$. Let $A \subseteq X$ be a closed subset. Then:

- $\mathcal{F}_x = 0$ iff $\mathcal{F}|_U = 0$ for some open neighbourhood of x.
- $\mathcal{F}|_A = 0$ iff $\mathcal{F}|_U = 0$ for some open set containing A.
- $\mathcal{F}_x$ can be generated by n elements iff this is true on some open neighbourhood of x.
- $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.
- $\mathcal{F}$ is torsion iff $\mathcal{F}_\xi$ vanishes (assume X integral and $\mathcal{F}$ quasicoherent).
- $\mathcal{F}$ is torsion iff $\mathcal{F}|_{\mathrm{Ass}(\mathcal{O}_X)}$ vanishes (assume X locally Noetherian and $\mathcal{F}$ quasicoherent).

The smallest dense sublocale

Let X be a reduced scheme satisfying a technical condition. Let $i : X_{\neg\neg} \to X$ be the inclusion of the smallest dense sublocale of X.

Then $i_* i^{-1} \mathcal{O}_X \cong \mathcal{K}_X$.

Transfer principles

Let M be an A-module. How do M and the sheaf $M^\sim$ on $\operatorname{Spec} A$ relate?

Observe that $M^\sim \cong \underline{M}[F^{-1}]$ is the localization of M at the **generic filter**. Therefore:

$M^\sim$ inherits all those properties of M which are **stable under localization**.

Examples: finitely generated, free, flat, …

A converse holds as well, suitably formulated.

Applications in algebra

Let A be a commutative ring. The internal language of $\mathrm{Sh}(\operatorname{Spec} A)$ allows you to say "without loss of generality, we may assume that A is local", even constructively.

The kernel of any matrix over a principial ideal domain is finitely generated.

$$\Downarrow$$

The kernel of any matrix over a Prüfer domain is finitely generated.

The gros Zariski topos

Let X be a scheme. The **gros Zariski topos** is the topos of sheaves on Sch/X with respect to the Zariski topology. From its point of view, …

- … X-schemes look just like sets,
- … $\mathbb{P}^n_X$ is given by the naive expression

$$\{(x_0,\ldots,x_n) \,|\, x_1 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\text{rescaling}),$$

- … affinity is a "double dual condition", and
- … the étale topology is the coarsest topology $\Diamond$ s. th.

$$\forall f : \mathbb{A}^1_X[T].\ f \text{ is monic separable} \Rightarrow \Diamond(\exists t : \mathbb{A}^1. f(t) = 0).$$

Translating internal statements

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\mathrm{Sh}(X) \models \ulcorner \alpha \text{ is injective} \urcorner$$

$$\iff \mathrm{Sh}(X) \models \forall s : \mathcal{F}.\, \forall t : \mathcal{F}.\, \alpha(s) = \alpha(t) \Rightarrow s = t$$

$\iff$ for all open $U \subseteq X$, sections $s \in \mathcal{F}(U)$:
for all open $V \subseteq U$, sections $t \in \mathcal{F}(V)$:
for all open $W \subseteq V$:
$\alpha_W(s|_W) = \alpha_W(t|_W)$ implies $s|_W = t|_W$

$\iff$ for all open $U \subseteq X$, sections $s, t \in \mathcal{F}(U)$:
$\alpha_U(s|_U) = \alpha_U(t|_U)$ implies $s|_U = t|_U$

$\iff$ α is a monomorphism of sheaves