Arbeitstagung Bern–München–Verona

Modal operators for a constructive account of well quasi-orders

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Well quasi-orders

Def. Let $(X, \leq)$ be a quasi-order.

- A sequence $\alpha : \mathbb{N} \to X$ is **good** iff there exist $i < j$ with $\alpha\, i \leq \alpha\, j$.
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Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. $\square$

offensive?

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Key stability results

Assuming LEM and DC, …

Dickson: If X and Y are well, so is $X \times Y$.

Higman: If X is well, so is $X^\star$.

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Is there a procedure for reinterpreting **classical proofs** regarding well_∞ as **blueprints for constructive proofs** regarding well_{ind}?

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Constructively, well$_{\text{ind}}$ $\Rightarrow$ well$_\infty$. Moreover, if X is well$_{\text{ind}}$, then …

- for every *partial* function α, if $\forall n.\ \neg\neg(\alpha\, n\!\downarrow)$, then $\neg\neg\exists i < j.\ \alpha\, i\!\downarrow \wedge\, \alpha\, j\!\downarrow \wedge\, \alpha\, i \leq \alpha\, j$.
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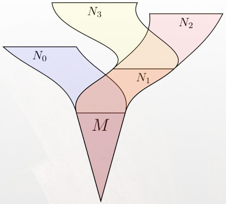
Central insight: A quasi-order X is well$_{\text{ind}}$ iff $\Box\, \forall \alpha : \mathbb{N} \to X.\ \exists i < j.\ \alpha\, i \leq \alpha\, j$.

The modal multiverse of constructive forcing

Def. A statement φ holds …

- **everywhere** ($\Box\varphi$) iff it holds in every topos (over the current base).
- **somewhere** ($\Diamond\varphi$) iff it holds in some positive topos.
- **proximally** ($\diamonddiamond\varphi$) iff it holds in some positive overt topos.



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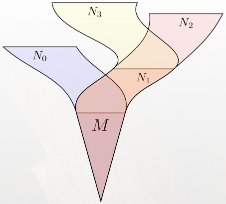

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1 A quasiorder is well$_{\text{ind}}$ iff *everywhere*, every sequence is good.

2 A relation is well-founded iff *everywhere*, there is no infinite descending chain.

3 A ring element is nilpotent iff all prime ideals *everywhere* contain it.

4 For every inhabited set X, *proximally* there is an enumeration $\mathbb{N} \twoheadrightarrow X$.

5 For every ring, *proximally* there is a maximal ideal.

6 *Somewhere*, the law of excluded middle holds.

Answering a question by Berardi–Buriola–Schuster

Def. A quasi-order X is **well$_{\mathbf{impl}}$** iff (approximately) for every monotone predicate B,

$$\text{if } B \mid_{\text{incr}} [\,], \text{ then } B \mid [\,].$$

"Assume that no matter how the empty list evolves to an *increasing* list σ, eventually $B\,\sigma$. Then no matter how the empty list evolves to an *arbitrary* list τ, eventually $B\,\tau$."

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Prop. $\text{well}_{\text{impl}} \Rightarrow \text{well}_{\text{ind}}$.

Proof. Trivially $\text{Good} \mid_{\text{incr}} [\,]$, hence $\text{Good} \mid [\,]$ by assumption. □

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Thm. $\text{well}_{\text{impl}} \Leftarrow \text{well}_{\text{ind}}$.

Proof. Let $x_0, x_1, \ldots$ be an infinite sequence (in an arbitrary topos). *Somewhere,* LEM holds and implies that there is an increasing infinite subsequence $x_{i_0}, x_{i_1}, \ldots$. *There* we have, by assumption, a finite prefix with $B\, x_{i_0} \ldots x_{i_n}$. As B is monotone, *there* we also have $B\, x_0 x_1 \ldots x_{i_n}$. So *somewhere* there is a finite prefix validating B. Hence there actually is a finite prefix validating B. □

```
data _∣_ {X : Set} (P : List X → Set) : List X → Set where
  now   : {σ : List X} → P σ → P ∣ σ
  later : {σ : List X} → ((x : X) → P ∣ (x ∷ σ)) → P ∣ σ

module _ (X : Set) (_≤_ : X → X → Set) where
  Good : List X → Set
  Good σ = ∃[ i ] ∃[ j ] (i < j × lookup σ i ≤ lookup σ j)

  Well-∞ : Set
  Well-∞ = (α : ℕ → X) → ∃[ n ] Good (take n α)

  Well-ind : Set
  Well-ind = Good ∣ []
```

```
U:**-  wqo.agda      Bot   L29   <N>   (Agda:Checked +5)

U:%*-  *All Done*    All   L1    <M>   (AgdaInfo)
```

Agda formalization in progress.