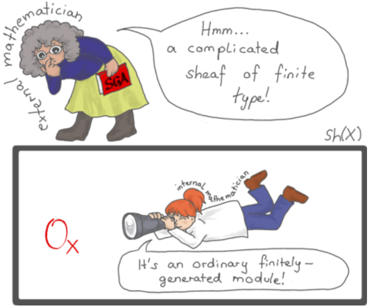


Using the internal language of toposes in algebraic geometry

Ingo Blechschmidt
University of Augsburg

Topos à l'IHES
November 27th, 2015

Outline

1. Basic applications of the internal language
2. The $\Diamond$-translation
3. Quasicoherence of sheaves of modules
4. The relative and internal spectrum

Abstract

We describe how the internal language of certain toposes, the associated petit and gros Zariski toposes of a scheme, can be used to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry.

The starting point is that, from the internal point of view, sheaves of rings and sheaves of modules look just like plain rings and plain modules. In this way, some concepts and statements of scheme theory can be reduced to concepts and statements of intuitionistic linear algebra.

Furthermore, modal operators can be used to model phrases such as "on a dense open subset it holds that" or "on an open neighbourhood of a given point it holds that". These operators define certain subtoposes; a generalization of the double-negation translation is useful in order to understand the internal universe of those subtoposes from the internal point of view of the ambient topos.

A particularly interesting task is to internalize the construction of the relative spectrum, which, given a quasicoherent sheaf of algebras on a scheme X, yields a scheme over X. From the internal point of view, this construction should simply reduce to an intuitionistically sensible variant of the ordinary construction of the spectrum of a ring, but it turns out that this expectation is too naive and that a refined approach is necessary.

Several Topos theory questions

29

20

Hey. I have a few off the wall questions about topos theory and algebraic geometry.

1. Do the following few sentences make sense?

Every scheme X is pinned down by its Hom functor Hom(-,X) by the yoneda lemma, but since schemes are locally affine varieties, it is actually just enough to look at the case where "-" is an affine scheme. So you could define schemes as particular functors from CommRing^op to Sets. In this setting schemes are thought of as sheaves on the "big zariski site".

If that doesn't make sense my next questions probably do not either.

2 The category of sheaves on the big zariski site forms a topos T, the category of schemes being a subcategory. It is convenient to reason about toposes in their own "internal logic". Has there been much thought done about the internal logic of T, or would the logic of T require too much commutative algebra to feel like logic? Along these lines, have there been attempts to write down an elementary list of axioms which capture the essense of this topos? I am thinking of how Anders Kock has some really nice ways to think of differential geometry with his SDG.

ct.category-theory topos-theory lo.logic ag.algebraic-geometry

share cite improve this question

edited Oct 24 '09 at 17:53
Ben Webster ♦
25.3k ● 4 ● 73 ● 167

asked Oct 24 '09 at 16:49
Steven Gubkin
6,003 ● 2 ● 47 ● 87

Exploiting the internal language

A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to the **spectrum of a commutative ring**:

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

Exploiting the internal language

A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to the **spectrum of a commutative ring**:

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

The topos $\mathrm{Sh}(X)$ is the **petit Zariski topos** of X.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set/type
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set/type
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
coherent sheaf	coherent module
tensor product of sheaves	tensor product of modules
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf o	
morph	
monon	
epimoi	
sheaf o	
sheaf o	
sheaf o	
finite l	
coherei	
tensor	
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

MISCONCEPTIONS ABOUT K_X

by Steven L. KLEIMAN

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

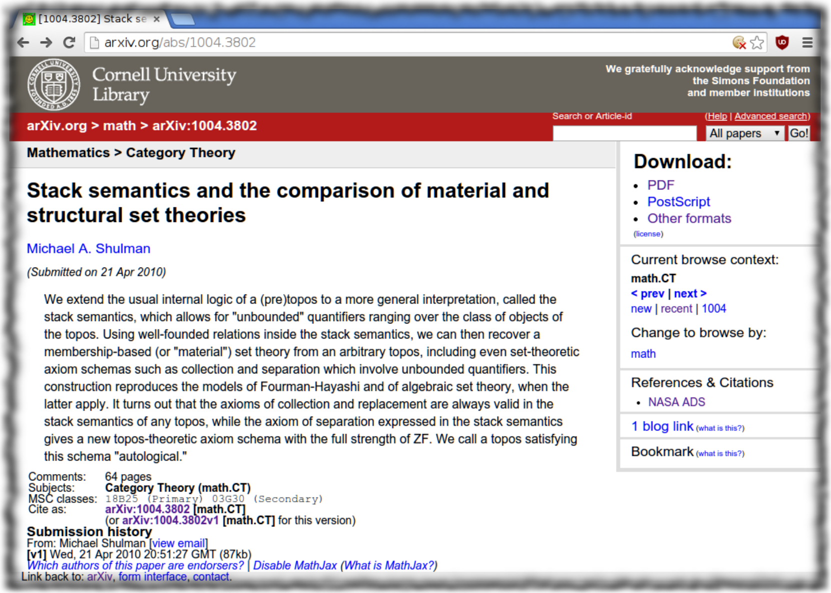
$$U \mapsto \Gamma(U, O_X)_{tot}, \qquad (*)$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \mathrm{Spec}\,(A)$ is

See the notes for more dictionary entries.

The simple definition of $\mathcal{K}_X$ allows to give an internal account of the basics of the theory of Cartier divisors, for instance giving an easy description of the line bundle associated to a Cartier divisor.

Praise for Mike Shulman



[1004.3802] Stack se

arxiv.org/abs/1004.3802

Cornell University Library

We gratefully acknowledge support from the Simons Foundation and member institutions

arXiv.org > math > arXiv:1004.3802

Search or Article-id (Help | Advanced search) All papers Go!

Mathematics > Category Theory

Stack semantics and the comparison of material and structural set theories

Michael A. Shulman

(Submitted on 21 Apr 2010)

We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for "unbounded" quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or "material") set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema "autological."

Comments: 64 pages
Subjects: **Category Theory (math.CT)**
MSC classes: 18B25 (Primary) 03G30 (Secondary)
Cite as: **arXiv:1004.3802 [math.CT]**
(or **arXiv:1004.3802v1 [math.CT]** for this version)

Submission history
From: Michael Shulman [view email]
[v1] Wed, 21 Apr 2010 20:51:27 GMT (87kb)
Which authors of this paper are endorsers? | Disable MathJax (What is MathJax?)
Link back to: arXiv, form interface, contact.

Download:

- PDF
- PostScript
- Other formats

(license)

Current browse context:
math.CT
< prev | next >
new | recent | 1004

Change to browse by:
math

References & Citations
- NASA ADS

1 blog link (what is this?)

Bookmark (what is this?)

The internal language of a topos supports

- first-order logic,
- higher-order logic (for instance quantification over subsets),
- dependent types, and
- unbounded quantification.

The first three items are standard. The fourth is due to Mike Shulman. Combined, it's possible to interpret "essentially all of constructive mathematics" internal to a topos.

Restrictions persist for operations with a "set-theoretical flavor" like building an infinite union of iterated powersets, for example $\bigcup_{n\in\mathbb{N}} P^n(\mathbb{N})$.

Using the dictionary

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$$\Downarrow$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Using the dictionary

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

Any sheaf of modules of finite type on a reduced scheme is locally free *on a dense open subset*.

Ravi Vakil: “Important hard exercise” (13.7.K).

A curious property

Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.

ON THE SPECTRUM OF A RINGED TOPOS 209

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec(**E**, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator $\Diamond$. Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond,$$

$\Diamond : \Omega_\mathcal{E} \to \Omega_\mathcal{E}$

where the translation $\varphi \mapsto \varphi^\Diamond$ is given by:

$$\begin{aligned}
(s = t)^\Diamond &:\equiv \Diamond(s = t) \\
(\varphi \wedge \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \wedge \psi^\Diamond) \\
(\varphi \vee \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \vee \psi^\Diamond) \\
(\varphi \Rightarrow \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \Rightarrow \psi^\Diamond) \\
(\forall x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\forall x : X.\, \varphi^\Diamond(x)) \\
(\exists x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\exists x : X.\, \varphi^\Diamond(x))
\end{aligned}$$

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator $\Diamond$.
Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond.$$

$\Diamond : \Omega_\mathcal{E} \to \Omega_\mathcal{E}$

Let X be a scheme. Depending on $\Diamond$, $\mathrm{Sh}(X) \models \Diamond\varphi$ means that φ holds on …

- … a dense open subset.
- … a schematically dense open subset.
- … a given open subset U.
- … an open subset containing a given closed subset A.
- … an open neighbourhood of a given point $x \in X$.

Can tackle the question "$\varphi^\Diamond \stackrel{?}{\Rightarrow} \Diamond\varphi$" logically.

The $\Diamond$-translation is a generalization of the *double negation translation*, which is well-known in logic. The double negation translation has the following curious property: A formula φ admits a classical proof if and only if the translated formula $\varphi^{\neg\neg}$ admits an intuitionistic proof.

The $\Diamond$-translation has been studied before (see for instance Aczel: *The Russell–Prawitz modality*, and Escardó, Oliva: *The Peirce translation and the double negation shift*), but to the best of my knowledge, this application – expressing the internal language of subtoposes in the internal language of the ambient topos – is new.

For ease of exposition, assume that X is irreducible with generic point ξ. Let $\Diamond :\equiv \neg\neg$.

Then $\mathrm{Sh}(X) \models \Diamond\varphi$ means that φ holds on a dense open subset of X, while $\mathrm{Sh}(X) \models \varphi^\Diamond$ means that φ holds at the generic point (taking stalks of all involved sheaves).

The question "does $\varphi^\Diamond$ imply $\Diamond\varphi$?" therefore means: Does φ spread from the generic point to a dense open subset?

For the special case of the double negation translation, a general answer to this purely logical question has long been known: This holds if φ is a *geometric formula* (doesn't contain $\Rightarrow$ and $\forall$).

Let $\mathcal{F}$ be a sheaf of modules on a locally ringed space X. Assume that the stalk $\mathcal{F}_x$ at some point $x \in X$ vanishes. Then in general it does *not* follow that $\mathcal{F}$ vanishes on some open neighbourhood of x.

This can be understood in logical terms: The statement that $\mathcal{F}$ vanishes,

$$\forall s : \mathcal{F}.\ s = 0,$$

is not a geometric formula.

However, if $\mathcal{F}$ is additionally supposed to be of finite type, then it *does* follow that $\mathcal{F}$ vanishes on an open neighbourhood. This too can be understood in logical terms: If $\mathcal{F}$ is of finite type, then internally there are generators $s_1, \ldots, s_n$ of $\mathcal{F}$. Thus the vanishing of $\mathcal{F}$ can be reformulated as

$$s_1 = 0 \wedge \cdots \wedge s_n = 0,$$

and this condition is manifestly geometric.

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Diamond_f\text{-sheaf for any } f : \mathcal{O}_X,$$
$$\text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Diamond_f\text{-sheaf for any } f : \mathcal{O}_X, \\ \text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

In particular: If $\mathcal{E}$ is quasicoherent, then internally

$$(f \text{ invertible} \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0$$

for any $f : \mathcal{O}_X$ and $s : \mathcal{E}$.

The sheaf condition and the sheafification functor can be described purely internally. An object M is *separated* with respect to $\Diamond$ if and only if, from the internal point of view,

$$\forall x, y : M.\ \Diamond(x = y) \Rightarrow x = y.$$

It is a *sheaf* with respect to $\Diamond$, if furthermore

$$\forall K \subseteq M.\ \Diamond(\exists x : M.\ K = \{x\}) \Longrightarrow \exists x : M.\ \Diamond(x \in K).$$

The second condition displayed on the previous slide is equivalent to the separatedness condition. In the special case $\mathcal{E} = \mathcal{O}_X$, $s = 1$ it reduces to Mulvey's "somewhat obscure formula". We now understand this condition in its proper context.

The absolute spectrum

Let A be a commutative ring (in Set).

Is there a **free local ring** $A \to A'$ over A?

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

No, if we restrict to Set.

Yes, if we allow a change of topos: Then $A \to \mathcal{O}_{\operatorname{Spec} A}$ is the universal localization.

Details on this point of view can be found in one of Peter Arndt's very nice answers on MathOverflow:

`http://mathoverflow.net/a/14334/31233`

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \text{topological space of the prime ideals of } A$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \cancel{\text{topological space of the prime ideals of } A}$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned} \operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \text{topological space of the prime filters of } A \end{aligned}$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned} \operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \cancel{\text{topological space of the prime filters of } A} \end{aligned}$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned}\operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \cancel{\text{topological space of the prime filters of } A} \\ &:= \text{locale of the prime filters of } A\end{aligned}$$

Define the frame of opens of $\operatorname{Spec} A$ to be the frame of radical ideals in A.

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned} \operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \cancel{\text{topological space of the prime filters of } A} \\ &:= \text{locale of the prime filters of } A \end{aligned}$$

Define the frame of opens of $\operatorname{Spec} A$ to be the frame of radical ideals in A.

This gives an internal description of Monique Hakim's spectrum functor $\mathrm{RT} \to \mathrm{LRT}$.

Monique Hakim constructed in her thesis a very general spectrum functor, taking a ringed topos to a locally ringed one, using explicit calculations with sites.

Using the internal language allows to reduce these calculations to a minimum. One constructs the spectrum as the sheaf topos over an internal locale and then uses the general theorem that toposes over the base $\mathcal{E}$ are the same as toposes internal to $\mathcal{E}$.

As a byproduct one obtains that Hakim's spectrum is *localic* over the base.

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe $\boldsymbol{\underline{\mathrm{Spec}}_X\, \mathcal{A}}$, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\, \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe $\underline{\mathbf{Spec}}_X\, \boldsymbol{\mathcal{A}}$, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\, \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X\, \mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe $\underline{\mathbf{Spec}}_X\,\boldsymbol{\mathcal{A}}$, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\,\mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X\,\mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

Its **points** are those prime filters G of $\mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \varphi(f) \in G \Longrightarrow f \text{ invertible in } \mathcal{O}_X.$$

The stated condition on I is, under the assumption that $\mathcal{A}$ is quasicoherent, equivalent to the condition that I is quasicoherent (as an $\mathcal{O}_X$-module).

The relative spectrum is thus constructed as a certain sublocale of the absolute one. The two constructions coincide if and only if the dimension of the base scheme is ≤ 0.

If X is not a scheme or $\mathcal{A}$ is not quasicoherent, the construction still gives rise to a locally ringed locale over X which satisfies the universal property

$$\mathrm{Hom}_{\mathrm{LRL}/X}(T, \underline{\mathrm{Spec}}_X \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all locally ringed locales $T \xrightarrow{\mu} X$ over X.

The relative spectrum, reformulated

Let $B \to A$ be an algebra in topos.

Is there a **free local and local-over-*B* ring** $A \to A'$ over A?

$$\begin{tikzcd}
B \arrow[r] \arrow[rrr, bend left, "\text{local}"] \arrow[drr, bend right, "\text{local}"'] & A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& & \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

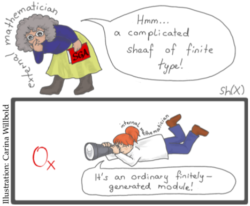
Form limits in the category of **locally ringed locales** by **relocalizing** the corresponding limit in ringed locales.

One might wonder whether the absolute spectrum or the relative one is “more fundamental”. The absolute spectrum can be expressed using the relative one, since

$$\operatorname{Spec} A = \underline{\operatorname{Spec}}_{\operatorname{Spec} \mathbb{Z}} A^{\sim},$$

but the other way is not in general possible: The absolute spectrum is always (quasi-)compact, while the relative one is not in general.

Understand notions and statements of algebraic geometry as notions and statements of algebra internal to appropriate toposes.



- Simplify proofs and gain conceptual understanding.
- Understand relative geometry as absolute geometry.
- Develop a synthetic account of scheme theory.
- Contribute to constructive algebra.

http://tiny.cc/topos-notes

spreading of properties, general transfer principles, applications to constructive algebra, quasicoherence, internal Cartier divisors, pullback along immersions = internal sheafification, scheme dimension = internal Krull dimension of $\mathcal{O}_X$, dense = not not, modal operators, relative spectrum, other toposes, étale topology, group schemes = groups, ...

You should totally look up:

The Adventures of Sheafification Man

Spreading from points to neighbourhoods

All of the following lemmas have a short, sometimes trivial proof. Let $\mathcal{F}$ be a sheaf of finite type on a ringed space X. Let $x \in X$. Let $A \subseteq X$ be a closed subset. Then:

1. $\mathcal{F}_x = 0$ iff $\mathcal{F}|_U = 0$ for some open neighbourhood of x.
2. $\mathcal{F}|_A = 0$ iff $\mathcal{F}|_U = 0$ for some open set containing A.
3. $\mathcal{F}_x$ can be generated by n elements iff this is true on some open neighbourhood of x.
4. $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.
5. $\mathcal{F}$ is torsion iff $\mathcal{F}_\xi$ vanishes (assume X integral and $\mathcal{F}$ quasicoherent).
6. $\mathcal{F}$ is torsion iff $\mathcal{F}|_{\mathrm{Ass}(\mathcal{O}_X)}$ vanishes (assume X locally Noetherian and $\mathcal{F}$ quasicoherent).

Statements 1 and 2 follow from *one* proof in the internal language, applied to two different modal operators.

Similarly with statements 5 and 6.

The smallest dense sublocale

Let X be a reduced scheme satisfying a technical condition. Let $i : X_{\neg\neg} \to X$ be the inclusion of the smallest dense sublocale of X.

Then $i_* i^{-1} \mathcal{O}_X \cong \mathcal{K}_X$.

- This is a highbrow way of saying "rational functions are regular functions which are defined on a dense open subset".
- Another reformulation is that $\mathcal{K}_X$ is the sheafification of $\mathcal{O}_X$ with respect to the $\neg\neg$-modality.
- There is a generalization to nonreduced schemes.

Transfer principles

Let M be an A-module. How do M and the sheaf $M^\sim$ on $\operatorname{Spec} A$ relate?

Observe that $M^\sim \cong \underline{M}[\mathcal{F}^{-1}]$ is the localization of M at the **generic prime filter** and that M shares all first-order properties with the constant sheaf of modules $\underline{M}$. Therefore:

$M^\sim$ inherits all those properties of M which are **stable under localization**.

Examples: finitely generated, free, flat, …

A converse holds as well, suitably formulated.

Applications in algebra

Let A be a commutative ring. The internal language of $\mathrm{Sh}(\operatorname{Spec} A)$ allows you to say "without loss of generality, we may assume that A is local", even constructively.

The kernel of any matrix over a principial ideal domain is finitely generated.

$$\Downarrow$$

The kernel of any matrix over a Prüfer domain is finitely generated.

Hilbert's program in algebra

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides, for instance exhibiting explicit witnesses. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all prime filters") by "for the generic prime filter".

More specifically, simply interpret a given proof using prime filters in $\mathrm{Sh}(\operatorname{Spec} A)$ and let it refer to $\mathcal{F} \hookrightarrow \underline{A}$.

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

This is related (in a few cases equivalent) to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others. Their approach applies more generally.

The gros Zariski topos

Let X be a scheme. The **gros Zariski topos** is the topos of sheaves on Sch/X with respect to the Zariski topology. From its point of view, …

- … X-schemes look just like sets,
- … $\mathbb{P}^n_X$ is given by the naive expression

$$\{(x_0, \ldots, x_n) \mid x_1 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\text{rescaling}),$$

- … the cotangent "bundle" of an X-scheme T is

$$\text{the set of maps } \Delta \to \underline{T},$$

  where $\Delta = \{\varepsilon \in \underline{\mathbb{A}}^1_X \mid \varepsilon^2 = 0\}$.
- … affinity is a "double dual condition", and
- … the étale topology is the coarsest topology $\Diamond$ s. th.

$$\forall f : \underline{\mathbb{A}}^1_X[T].\, f \text{ is monic separable} \Rightarrow \Diamond(\exists t : \underline{\mathbb{A}}^1.\, f(t) = 0).$$

- The functor of points of $\mathbb{A}^1_X$, that is

$$\underline{\mathbb{A}}^1_X : (T/X) \longmapsto \mathcal{O}_T(T),$$

looks like a local ring and indeed like a field from the internal point of view, in the sense that

$$\forall f : \underline{\mathbb{A}}^1_X.\, \neg(f = 0) \Rightarrow f \text{ invertible}.$$

- Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Let $\mathcal{E}$ be the induced $\underline{\mathbb{A}}^1_X$-algebra given by $\mathcal{E}(T \xrightarrow{\mu} X) := (\mu^* \mathcal{A})(T)$. Then the internal Hom set $[\mathcal{E}, \underline{\mathbb{A}}^1_X]_{\underline{\mathbb{A}}^1_X}$ of $\underline{\mathbb{A}}^1_X$-algebra morphisms is the functor of points of $\underline{\mathrm{Spec}}_X(\mathcal{A})$.

- Let $\mu : T \to X$ be quasicompact and quasiseparated. Then μ is affine iff, from the internal point of view, the map

$$\underline{T} \longrightarrow [[\underline{T}, \underline{\mathbb{A}}^1_X]^\flat, \underline{\mathbb{A}}^1_X]_{\underline{\mathbb{A}}^1_X}, \; x \longmapsto \_(x)$$

into the "double dual" is bijective.

- Describing the functor of points of the projective space was suggested by Zhen Lin Low.
- The statement on the étale topology follows from Gavin Wraith's article *Generic Galois theory of local rings*.

The internal language of a topos

Let $\mathcal{E}$ be a topos. Then we can define the meaning of

$$\mathcal{E} \models \varphi \qquad (\text{“}\varphi \text{ holds in } \mathcal{E}\text{”})$$

for formulas φ over $\mathcal{E}$ using the **Kripke–Joyal semantics**.

externally	internally
object	set/type
morphism	map of sets
monomorphism	injective map
epimorphism	surjective map

The internal language of a topos

Let $\mathcal{E}$ be a topos. Then we can define the meaning of

$$\mathcal{E} \models \varphi \qquad (\text{“}\varphi \text{ holds in } \mathcal{E}\text{”})$$

for formulas φ over $\mathcal{E}$ using the **Kripke–Joyal semantics**.

externally	internally
object	set/type
morphism	map of sets
monomorphism	injective map
epimorphism	surjective map

If φ implies ψ **intuitionistically**, then $\mathcal{E} \models \varphi$ implies $\mathcal{E} \models \psi$.

More generally, for an object U of a topos $\mathcal{E}$, we define the meaning of

$$U \models \varphi \qquad (\varphi \text{ holds on } U).$$

Writing "$\mathcal{E} \models \varphi$" is then an abbreviation for "$1 \models \varphi$", where "1" denotes the terminal object of $\mathcal{E}$.

In addition to soundness with respect to intuitionistic logic, the internal language has the following two important properties:

- **Monotonicity:** If $p : V \to U$ is an arbitrary morphism and $U \models \varphi$, then also $V \models \varphi$.
- **Locality:** If $p : V \to U$ is an epimorphism and $V \models \varphi$, then also $U \models \varphi$.

In the special case that $\mathcal{E} = \mathrm{Sh}(X)$ is the topos of sheaves on a topological space (or locale) X, the rules of the Kripke–Joyal semantics look as follows. We tersely write "$U \models \varphi$" instead of "$\mathrm{Hom}(\_, U) \models \varphi$ for open subsets $U \subseteq X$.

$$
\begin{array}{lcl}
U \models f = g : \mathcal{F} & :\Longleftrightarrow & f|_U = g|_U \in \mathcal{F}(U) \\
U \models \varphi \wedge \psi & :\Longleftrightarrow & U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi & :\Longleftrightarrow & \cancel{U \models \varphi \text{ or } U \models \psi} \\
 & & \text{there exists a covering } U = \bigcup_i U_i \text{ s. th. for all } i\text{:} \\
 & & \qquad U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi & :\Longleftrightarrow & \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall f : \mathcal{F}.\, \varphi(f) & :\Longleftrightarrow & \text{for all sections } f \in \mathcal{F}(V), V \subseteq U\text{: } V \models \varphi(f) \\
U \models \exists f : \mathcal{F}.\, \varphi(f) & :\Longleftrightarrow & \text{there exists a covering } U = \bigcup_i U_i \text{ s. th. for all } i\text{:} \\
 & & \qquad \text{there exists } f_i \in \mathcal{F}(U_i) \text{ s. th. } U_i \models \varphi(f_i)
\end{array}
$$

Translating internal statements I

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$
\begin{aligned}
& \mathrm{Sh}(X) \models \ulcorner \alpha \text{ is injective} \urcorner \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \forall s \colon \mathcal{F}.\, \forall t \colon \mathcal{F}.\, \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s \in \mathcal{F}(U): \\
& \qquad \text{for all open } V \subseteq U, \text{ sections } t \in \mathcal{F}(V): \\
& \qquad\qquad \text{for all open } W \subseteq V: \\
& \qquad\qquad\qquad \alpha_W(s|_W) = \alpha_W(t|_W) \text{ implies } s|_W = t|_W \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s, t \in \mathcal{F}(U): \\
& \qquad \alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U \\
\Longleftrightarrow\ & \alpha \text{ is a monomorphism of sheaves}
\end{aligned}
$$

Translating internal statements II

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\begin{aligned}
& \mathrm{Sh}(X) \models \ulcorner \alpha \text{ is surjective} \urcorner \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \forall t : \mathcal{G}.\, \exists s : \mathcal{F}.\, \alpha(s) = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } t \in \mathcal{G}(U): \\
& \qquad \text{there exists an open covering } U = \textstyle\bigcup_i U_i \text{ and} \\
& \qquad \text{sections } s_i \in \mathcal{F}(U_i) \text{ such that:} \\
& \qquad\qquad \alpha|_{U_i}(s_i) = t|_{U_i} \\
\Longleftrightarrow\ & \alpha \text{ is an epimorphism of sheaves}
\end{aligned}$$

Translating internal statements III

Let X be a topological space (or locale) and let $s, t \in \mathcal{F}(X)$ be global sections of a sheaf $\mathcal{F}$ on X. Then:

$$
\begin{aligned}
& \mathrm{Sh}(X) \models \neg\neg(s = t) \\
\iff & \mathrm{Sh}(X) \models ((s = t) \Rightarrow \bot) \Rightarrow \bot \\
\iff & \text{for all open } U \subseteq X \text{ such that} \\
& \qquad \text{for all open } V \subseteq U \text{ such that} \\
& \qquad\qquad s|_V = t|_V, \\
& \qquad \text{it holds that } V = \emptyset, \\
& \text{it holds that } U = \emptyset \\
\iff & \text{there exists a dense open set } W \subseteq X \text{ such that } s|_W = t|_W
\end{aligned}
$$