EXPLORING MATHEMATICAL OBJECTS FROM CUSTOM-TAILORED MATHEMATICAL UNIVERSES

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Abstract. foo

Toposes can be pictured as mathematical universes in which we can do mathematics in. Most mathematicians spend all their professional life in just a single topos, the so-called *standard topos*. However, besides the standard topos, there is a colorful host of alternate toposes which are just as worthy of mathematical study and in which mathematics plays out slightly differently.

For instance, there are toposes in which the axiom of choice and the intermediate value theorem from undergraduate calculus fail, toposes in which any function $\mathbb{R} \to \mathbb{R}$ is continuous, toposes in which infinitesimal numbers exist and toposes whose properties depend on certain circumstances of our physical world.

The purpose of this contribution is twofold.

(1) We give a glimpse of the toposophic landscape, presenting several specific toposes and exploring their peculiar properties.
(2) We explicate how each topos provides a distinct lens through which the usual mathematical objects of the standard topos can be viewed.

Viewed through such a lens, a given mathematical object can have different properties than it has when considered normally. In particular, it can have better properties for the purposes of specific applications, especially if the topos is custom-tailored to the object in question. This change of perspective has been used in mathematical practice and demonstrates that toposes go much beyond being logicians' testbeds. To give just a taste of what is possible, through the lens provided by an appropriate topos, any given ring can look like a field.

We argue that toposes and specifically the change in perspective provided by toposes are ripe for philosophical analysis. In particular, there are the following connections with topics in the philosophy of mathematics:

(1) Toposes enrich the realism/anti-realism debate in that they point the picture that the platonic heaven of mathematical objects is not unique: besides the standard heaven of the standard topos, we could fathom the alternate heavens of all other toposes, all embedded in a secord-order heaven.
(2) Mathematics is not only about studying mathematical objects, but also about studying the relations between mathematical objects. The distinct view on mathematical objects provided by any topos uncovers relations which otherwise remain hidden.
(3) In some cases, a mathematical relation can be expressed quite succinctly using the language of a specific topos and not so succinctly using the language of the standard topos. This phenomenon showcases the importance of *appropriate language*.

(4) Toposes provide new impetus to study constructive mathematics and intuitionistic logic, in particular also to restrict to intuitionistic logic on the meta level and to consider the idea that the platonic heaven might be governed by intuitionistic logic.

This note only touches on these topics and invites further research.

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1. Toposes as alternate mathematical universes

Formally, a topos is a certain kind of *category*, containing objects and morphisms between those objects. For instance, the standard topos contains all sets as its objects and all maps between sets as morphisms. A careful definition of the notion of a topos requires some amount of category theory, but, as will be outlined in the following sections, exploring the mathematical universe of a given topos does not.

Put briefly, the axioms state that a topos should share several categorical properties with the category of sets (namely having finite limits, being cartesian closed and having a subobject classifier); they ensure that each topos contains its own versions of familiar mathematical objects such as natural numbers, real numbers, groups and manifolds, and is closed under the usual constructions such as cartesian products or quotients.

Given a topos $\mathcal{E}$, we write "$\mathcal{E} \models \varphi$" to denote that a mathematical statement φ *holds in* $\mathcal{E}$. The meaning of "$\mathcal{E} \models \varphi$" is defined by recursion on the structure of φ following the so-called *Kripke–Joyal translation rules*. For instance, the rule for translating conjunction reads

$$\mathcal{E} \models (\alpha \wedge \beta) \qquad \text{iff} \qquad \mathcal{E} \models \alpha \quad \text{and} \quad \mathcal{E} \models \beta.$$

The remaining translation rules are more involved; we do not list them here for the case of a general topos $\mathcal{E}$, but we will state them in the next sections for several specific toposes.

In the definition of $\mathcal{E} \models \varphi$, the statement φ can be any statement in the language of a general version of higher-order predicate calculus with dependent types. In practice almost any mathematical statement can be interpreted in a given topos.[1]

It is by the Kripke–Joyal translation rules that we can access the alternate universe of a topos. In the special case of the standard topos Set, the definition of "Set $\models \varphi$" unfolds to φ for any statement φ. Hence a statement holds in the standard topos if and only if it holds in the usual mathematical sense.

1.1. The logic of toposes.

By their definition as special kinds of categories, toposes are merely algebraic structures not unlike groups and vector spaces are. Hence we need to argue why we picture toposes as mathematical universes while we do not elevate other kinds of algebraic structures in the same way. For us, the reason is given by the following metatheorem:

[1] The main exceptions are statements from set theory, which typically make substantial use of a global membership predicate "$\in$". Toposes only support a typed *local* membership predicate, where we may write "$x \in A$" only in the context of some fixed sort type B such that x is of type B and A is of type $P(B)$, the power type of B.

Theorem 1.1. *Let $\mathcal{E}$ be a topos and φ be a statement such that $\mathcal{E} \models \varphi$. If φ intuitionistically entails a further statement ψ, then $\mathcal{E} \models \psi$.*

XXX explain intuitionistically

XXX explain that this metatheorem allows us to *reason* in a topos

XXX include something like this: A natural question is this: *Which of the familiar mathematical facts of the standard topos carry over to arbitrary toposes $\mathcal{E}$?* For any given mathematical statement φ and topos $\mathcal{E}$, we can unroll the definition of $\mathcal{E} \models \varphi$ to try to check whether φ holds in $\mathcal{E}$ on an individual case-by-case basis; but it turns out that there is a general metatheorem ...

XXX Mathematicians are familiar with the fact that the usual objects of mathematical study are governed by the laws of ordinary *classical reasoning*. ...

1.2. **Models of set theory.** Toposes can be pictured as "universes in which we can do mathematics in" in much the same way as models of set theory can be viewed in this way. In fact, to any model M of a set theory such as ZF or ZFC, there is a topos Set_M such that a statement holds in the internal language of Set_M if and only if it holds in M.[2] This includes V, the trivial inner model consisting of all sets there are, in case such a thing exists in one's chosen ontology; in this case the associated topos Set_V is the category of all sets, and a statement holds in Set_V if and only if it's true in the ordinary mathematical sense.

Carrying the analogy further, just as using forcing and other techniques we can construct new models of set theory from given ones, thereby exploring the set-theoretic multiverse, we can construct new toposes from given toposes. However, there are two important differences between the notion of mathematical universe as provided by toposes and as provided by models of set theory, both regarding the subject matter and the reasons for why we are interested in them.

Firstly, toposes are more general than models of set theory. By definition, a model of **ZFC** will always satisfy the axioms of **ZFC**; in contrast, most toposes do not even validate the law of excluded middle, much less so the axiom of choice (which implies the law of excluded middle in presence of the remaining axioms of intuitionistic Zermelo–Fraenkel set theory [**XXX**]). This failure of classicality is often for interesting geometric or algebraic reasons instead of logical issues, as will be detailed in Section **??**.

Secondly, there is a difference in motivation. The main philosophical reason for studying models of set theory is to study which notions of sets are coherent: Does the cardinality of the reals need to be the cardinal directly succeeding $\aleph_0$, the cardinality of the naturals? No, there are models of set theory in which the continuum hypothesis fails. Do non-measurable sets of reals need to exist? No, in models of $\mathbf{ZF} + \mathbf{AD}$, Zermelo–Fraenkel set theory plus the axiom of determinacy, it's a theorem that every subset of $\mathbb{R}^n$ is Lebesgue-measurable. Can the axiom of choice be added to the axioms of **ZF** without causing inconsistency? Yes, if M is a model of **ZF** then L^M, the set of definable sets of M, is a model of **ZFC**. [**XXX, maybe SEP?**]

While toposes can be used for similar such purposes, and indeed have been, especially to explore the various intuitionistic notions of sets, an important aspect of

[2]The topos Set_M can be described as follows: Its objects are the elements of M, that is the things which M believes to be sets, and its morphisms are those things which M believes to be maps. The topos Set_M validates the axioms of **ETCS** [**XXX**], and for models which are not elementarily equivalent, their associated toposes will not be equivalent.

topos theory is that toposes are used to explore the standard mathematical universe: Truth in the effective topos tells us what is computable; truth in sheaf toposes tells us what's true locally, in a fashion which depends continuously on parameters; the little Zariski topos of a general commutative ring can be used to pretend that the given ring is a field; toposes adapted to synthetic differential geometry can be used to rigorously work with infinitesimals. All of these examples will be presented in more detail in Section **??**.

In a certain precise way, toposes allow us to study the common objects of mathematics from a different point of view – one such view for every topos – and it is a beautiful and intriguing fact that with the sole exception of the law of excluded middle, the laws of logic apply to mathematical objects also when viewed through the lens of a specific topos.

2. Exploring the effective topos

Statement	in Set	in Eff
Any natural number is prime or not prime.	✓ (trivially so)	✓
There are infinitely many primes.	✓	✓
Any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not.	✓ (trivially so)	✗
Any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine.	✗	✓ (trivially so)
Any function $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓

2.1. **"Any natural number is prime or not."** Even without knowing what a prime number is, one can safely judge this statement to be true in the standard topos, since it is just an instance of the law of excluded middle.

By the Kripke–Joyal semantics, saying that it's true in the effective topos amounts to saying that there is a Turing machine which, given a natural number n as input, terminates with a correct judgment whether n is prime or not. Such a Turing machine indeed exists – writing such a program is often a first exercise in programming courses. Hence the statement is also true in the effective topos, but for the nontrivial reason that such a machine exists.

2.2. **"There are infinitely many primes."** A first-order formalization of this statement is "for any natural number n, there is a prime number p which is greater than n", and is known to be true in the standard topos by any of the many proofs of this fact.

Its external meaning when interpreted in the effective topos is that there exists a Turing machine which, given a natural number n as input, terminates with a prime number $p > n$ as output. Such a Turing machine exists, hence the statement is true in the effective topos.

2.3. **"Any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not."** More formally, the statement is

$$\forall f : \mathbb{N}^{\mathbb{N}}.\ \big((\forall n : \mathbb{N}.\ f(n) = 0) \vee \neg(\forall n : \mathbb{N}.\ f(n) = 0)\big).$$

By the law of excluded middle, this statement is trivially true in the standard topos.

Its meaning when interpreted in the effective topos is that there exists a Turing machine M which, given the description of a Turing machine F which computes a function $f : \mathbb{N} \to \mathbb{N}$ as input, terminates with a correct judgment of whether f is the zero function or not. Such a machine M does not exist, hence the statement is false in the effective topos.

A formal proof that such a machine M does not exist will reduce its assumed existence to the undecidability of the halting problem. Intuitively, the issue is the following. Turing machines are able to simulate other Turing machines. Hence M could simulate F on various inputs to search the list of function values $f(0), f(1), \ldots$ for a nonzero number. In case that after a certain number of steps a nonzero function value is found, the machine M can correctly output the judgment that f is not the zero function. But if the search only turned up zero values, it cannot come to any verdict – it cannot rule out that a nonzero function value will show up in the as yet unexplored part of the function.

2.4. **"Any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine."** The preceding examples could give the impression that what is true in the effective topos is simply a subset of what is true in the standard topos. This statement shows that the relation between these two toposes is more nuanced.

The fundamental observation of computability theory is that, in the standard topos, there are functions $\mathbb{N} \to \mathbb{N}$ which are not computable by a Turing machine. Explicit examples include the *halting function*, which maps a number n to zero or one depending on whether the n-th Turing machine (in some fixed enumeration of all Turing machines) terminates or not, and the *busy beaver function*. Cardinality arguments even show that most functions $\mathbb{N} \to \mathbb{N}$ are not computable: There are $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ functions $\mathbb{N} \to \mathbb{N}$, but only $\aleph_0$ Turing machines and hence only $\aleph_0$ functions which are computable by a Turing machine.

In contrast, in the effective topos, any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine: The external meaning of this internal statement is that there exists a Turing machine M which, given a description of a Turing machine F computing a function $f : \mathbb{N} \to \mathbb{N}$, outputs a description of a Turing machine computing f. It is trivial to program such a machine M; the machine M simply has to echo its input back to the caller.

To avert a paradox, we should point out where the proof of the fundamental observation of computability theory employs nonconstructive reasoning, for if it would admit a constructive proof, it would also hold internally to the effective topos, in contradiction to the fact that it does not. The halting function $h : \mathbb{N} \to \mathbb{N}$, defined using the case distinction

$$h : n \mapsto \begin{cases} 1, & \text{if the } n\text{-th Turing machine terminates,} \\ 0, & \text{if the } n\text{-th Turing machine does not terminate,} \end{cases}$$

cannot be given as a counterexample in the effective topos since, in the effective topos, it is not actually a total function from $\mathbb{N}$ to $\mathbb{N}$. It is only defined on those numbers n for which the n-th Turing machine terminates or does not terminate. Assuming the law of excluded middle, this is a trivial condition; but intuitionistically, it is not. The definition of the busy beaver function requires a similar case distinction and therefore also does not give rise to a well-defined counterexample within the effective topos.

2.5. **"Any function $\mathbb{R} \to \mathbb{R}$ is continuous."** In the standard topos, this statement is plainly false, with the signum and Heaviside functions being prominent

$\mathrm{Eff} \models \varphi$	iff there is a natural number e such that $e \Vdash \varphi$.

In the following, we write "$e \cdot n \downarrow$" to mean that calling the e-th Turing machine on input n terminates, and in this case denote the result by "$e \cdot n$".

$e \Vdash s = t$	iff $s = t$.
$e \Vdash \top$	is true for any number e.
$e \Vdash \bot$	is false for any number e.
$e \Vdash (\varphi \wedge \psi)$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ and $e \cdot 0 \Vdash \varphi$ and $e \cdot 1 \Vdash \psi$.
$e \Vdash (\varphi \vee \psi)$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ and if $e \cdot 0 = 0$ then $e \cdot 1 \Vdash \varphi$, and if $e \cdot 0 \neq 0$ then $e \cdot 1 \Vdash \psi$.
$e \Vdash (\varphi \Rightarrow \psi)$	iff for any number r such that $r \Vdash \varphi$, $e \cdot r \downarrow$ and $e \cdot r \Vdash \psi$.
$e \Vdash (\forall n : \mathbb{N}.\ \varphi(n))$	iff for any natural number $n_0 \in \mathbb{N}$, $e \cdot n_0 \downarrow$ and $e \cdot n_0 \Vdash \varphi(n_0)$.
$e \Vdash (\exists n : \mathbb{N}.\ \varphi(n))$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ terminate and $e \cdot 1 \Vdash \varphi(e \cdot 0)$.
$e \Vdash (\forall f : \mathbb{N}^{\mathbb{N}}.\ \varphi(f))$	iff for any function $f_0 : \mathbb{N} \to \mathbb{N}$ and any number r_0 such that f_0 is computed by the r_0-th Turing machine, $e \cdot r_0 \downarrow$ and $e \cdot r_0 \Vdash \varphi(f_0)$.
$e \Vdash (\exists f : \mathbb{N}.\ \varphi(f))$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ such that the $(e \cdot 0)$-th Turing machine computes a function $f_0 : \mathbb{N} \to \mathbb{N}$ and $e \cdot 1 \Vdash \varphi(f_0)$.

Table 1. A (fragment of) the translation rules defining the meaning of statements internal to the effective topos.

counterexamples. In the effective topos, this statement is true. A formal proof is not entirely straightforward [**XXX**], but an intuitive explanation is as follows.

What the effective topos believes to be a real number is, from the external point of view, a Turing machine X which outputs, when called with a natural number n as input, a rational approximation $X(n)$. These approximations are required to be *consistent* in the sense that $|X(n) - X(m)| \leq 1/(n+1) + 1/(m+1)$. Intuitively, such a machine X denotes the real number $\lim_{n\to\infty} X(n)$, and the approximations $X(n)$ must be within $1/(n+1)$ of the limit.

A function $f : \mathbb{R} \to \mathbb{R}$ in the effective topos is therefore given by a Turing machine M which, given the description of such a Turing machine X as input, outputs the description of a similar such Turing machine Y as output. To compute a rational approximation $Y(n)$, the machine Y may simulate X and can therefore determine arbitrarily many rational approximations $X(m)$. However, within finite time, the machine Y can only acquire finitely many such approximations. Hence a function such as the signum function, for which even rough rational approximations of $\mathrm{sgn}(x)$ require infinite precision in the input x, do not exist in the effective topos.

2.6. **The formal translation rules.** The previous examples were picked to convey an intuitive understanding of what statements in the effective topos externally mean, and to showcase a variety of different situations. The formal translation rules are given in Table 1.

2.7. **Variants of the effective topos.** The effective topos belongs to a wider class of *realizability toposes*. These can be obtained by repeating the construction of the effective topos with any other reasonable model of computation in place of

Turing machines. The resulting toposes will in general not be equivalent and reflect higher-order properties of the employed models. Two of these further toposes are of special philosphical interest.

Hypercomputation. Firstly, in place of ordinary Turing machines, one can employ the *infinite-time Turing machines* pioneered by Hamkins and Lewis [**hamkins-lewis:ittm**]. These machines model *hypercomputation* in that they can run for "longer than infinity"; more precisely, the computational steps are indexed by the ordinal numbers instead of the natural numbers. For instance, an infinite-time Turing machine can trivially decide the twin prime conjecture, by simply walking along the natural number line and recording any twin primes it finds. Then, on day ω, it can observe whether it has found finitely many twins or not.

In the realizability topos made using infinite-time Turing machines, the full law of excluded middle still fails, but some instances which are wrong in the effective topos do hold in this topos. For instance, the statement "any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not" does: Its external meaning is that there is an infinite-time Turing machine M which, given the description of an infinite-time Turing machine F computing a function $f : \mathbb{N} \to \mathbb{N}$ as input, terminates (at some ordinal time step) with a correct judgment of whether f is the zero function or not. Such a machine M indeed exists: It simply has to simulate F on all inputs $0, 1, \ldots$ in order and check whether one of the resulting function values is not zero. This will require a transfinite amount of time (not least because simulating F on just one input might require a transfinite amount of time), but as an infinite-time Turing machine, M is capable of carrying out this procedure.

This realizability topos provides an intriguing environment challenging many mathematical intuitions shaped by classical logic. For instance, while from the point of view of this topos the reals are still uncountable in the sense that there is no surjection $\mathbb{N} \to \mathbb{R}$, there is an injection $\mathbb{R} \to \mathbb{N}$ [**bauer:injection**].

Machines of the real physical world. A second variant of the effective topos is obtained by using machines of the real physical world instead of abstract Turing machines in its construction. In doing so, we of course leave the realm of mathematics, as real-world machines are not objects of mathematical studies, but still it is interesting to see which commitments about the natural of the physical world imply which internal statements of the resulting topos.

For instance, Andrej Bauer proved that inside this topos any function $\mathbb{R} \to \mathbb{R}$ is continuous if, in the physical world, only finitely many computational steps can be carried out in finite time and if it is possible to form tamper-free private communication channels [**XXX**].

3. Exploring toposes of sheaves

Associated to any topological space X (such as Euclidean space), there is the *topos of sheaves over* X, $\mathrm{Sh}(X)$. To a first approximation, a statement is true in $\mathrm{Sh}(X)$ if and only if it "holds locally on X"; what $\mathrm{Sh}(X)$ believes to be a set is a "continuous family of sets, one set for each point of X". The precise rules of the Kripke–Joyal semantics of $\mathrm{Sh}(X)$ are listed in Table 2.

3.1. A geometric interpretation of double negation.

In intuitionistic logic, the double negation $\neg\neg\varphi$ of a statement φ is a slight weakening of φ; while $(\varphi \Rightarrow \neg\neg\varphi)$ is an intuitionistic tautology, the converse can only be shown for some specific statements. The internal language of $\mathrm{Sh}(X)$ gives geometric meaning to this logical peculiarity.

Namely, one can show that $\mathrm{Sh}(X) \models \neg\neg\varphi$ is equivalent to the existence of a *dense open* U of X such that $U \models \varphi$. If $\mathrm{Sh}(X) \models \varphi$, that is if $X \models \varphi$, then there obviously exists such a dense open, namely X itself; however the converse usually fails.

The only case that the law of excluded middle does hold internally to $\mathrm{Sh}(X)$ is when the only dense open of X is X itself; assuming classical logic in the metatheory, this holds if and only if every open is also closed. This is for instance satisfied if X is discrete.

An important special case is when X is the one-point space. In this case $\mathrm{Sh}(X)$ is equivalent (as categories and hence toposes) to the standard topos. If mathematics within $\mathrm{Sh}(X)$ can be described as "mathematics over X", then this observation justifies saying that "ordinary mathematics is mathematics over the point".

3.2. **Real numbers.** As detailled in Section 2.5, what the effective topos believes to be a real number is actually a Turing machine computing arbitrarily-good consistent rational approximations. A similarly drastic shift in meaning, though in an orthogonal direction, occurs with $\mathrm{Sh}(X)$. What $\mathrm{Sh}(X)$ believes to be a (Dedekind) real number a is actually a continuous family of real numbers on X, that is, a function $a : X \to \mathbb{R}$.

Such a function is everywhere positive on X if and only if, from the internal point of view $\mathrm{Sh}(X)$, the number a is positive; it is everywhere zero if and only if, internally, the number a is zero; and it is everywhere negative if and only if, internally, the number a is negative.

The law of trichotomy, stating that any real number is either negative, zero or positive, generally fails in $\mathrm{Sh}(X)$. By the Kripke–Joyal semantics, the external meaning of this internal statement is that for any continuous function $a : U \to \mathbb{R}$ defined on any open U of X, there is an open covering $U = \bigcup_i U_i$ such that on each member U_i of this covering, the function a is either everywhere negative on U_i, everywhere zero on U_i or everywhere positive on U_i. But this statement is, for most base spaces X, false. Figure 1(c) shows a counterexample.

The weaker statement that for any real number a it's *not not* the case that $a < 0$ or $a = 0$ or $a > 0$ does hold in $\mathrm{Sh}(X)$, for this statement is an intuitionistic tautology. Its meaning is that there exists a dense open U such that U can be covered by opens on which a is either everywhere negative, everywhere zero or everywhere positive. In the example given in Figure 1(c), this open U could be taken as X with the unique zero of a removed.

3.3. **Real functions.** Let $(f_x)_{x \in X}$ be a continuous family of continuous real-valued functions; that is, each of the individual functions $f_x : \mathbb{R} \to \mathbb{R}$ should be continuous and moreover the map $\mathbb{R} \times X \to \mathbb{R}, (a, x) \mapsto f_x(a)$ should be continuous. From the point of view of $\mathrm{Sh}(X)$, this family looks like a single continuous function $f : \mathbb{R} \to \mathbb{R}$.

The internal statement that $f(-1) < 0$ means that $f_x(-1) < 0$ for all $x \in X$, and similarly so for being positive. More generally, if a and b are continuous functions $X \to \mathbb{R}$ (hence real numbers from the internal point of view), the internal statement $f(a) < b$ means that $f_x(a(x)) < b(x)$ for all $x \in X$.

The internal statement that f possesses a zero, that is, that there exists a number a such that $f(a) = 0$, means that all the functions f_x each possess a zero

$\mathrm{Sh}(X) \models \varphi$	iff $X \models \varphi$.
$U \models a = b$	iff $a = b$ on U.
$U \models \top$	is true for any open U.
$U \models \bot$	iff U is the empty open.
$U \models (\varphi \wedge \psi)$	iff $U \models \varphi$ and $U \models \psi$.
$U \models (\varphi \vee \psi)$	iff there is an open covering $U = \bigcup_i U_i$ such that, for each i, $U_i \models \varphi$ or $U_i \models \psi$.
$U \models (\varphi \Rightarrow \psi)$	iff for every open $V \subseteq U$, $V \models \varphi$ implies $V \models \psi$.
$U \models (\forall a\!:\!\mathbb{R}.\ \varphi(x))$	iff for every open $V \subseteq U$ and any continuous function $a_0 : V \to \mathbb{R}$, $V \models \varphi(a_0)$.
$U \models (\exists a\!:\!\mathbb{R}.\ \varphi(x))$	iff there is an open covering $U = \bigcup_i U_i$ such that, for each i, there exists a continuous function $a_0 : U_i \to \mathbb{R}$ with $U_i \models \varphi(a_0)$.

Table 2. A (fragment of) the translation rules defining the meaning of statements internal to $\mathrm{Sh}(X)$, the topos of sheaves over a topological space X.

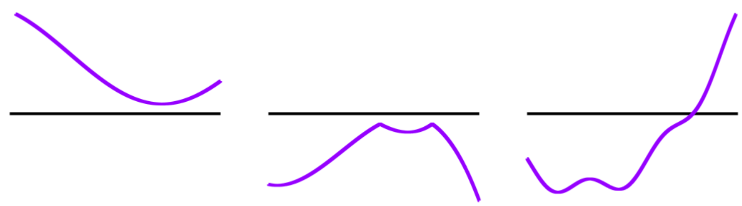

Figure 1. Three examples for what the topos $\mathrm{Sh}(X)$ each believes to be a single real number, where the base space X is the unit interval. (a) A positive real number. (b) A negative real number. (c) A number which is neither negative nor zero nor positive. Externally speaking, there is no covering of the unit interval by open subsets on which the depicted function a is either everywhere negative, everywhere zero or everywhere positive.

and that moreover, these zeros can locally be picked in a continuous fashion. More precisely, this statement means that there is an open covering $X = \bigcup_i U_i$ such that, for each i, there is a continuous function $a : U_i \to \mathbb{R}$ such that $f_x(a(x)) = 0$ for all $x \in U_i$. (On overlaps $U_i \cap U_j$, the zero-picking functions a need not agree.)

From these observations we can deduce that the intermediate value theorem of basic analysis does in general not hold in $\mathrm{Sh}(X)$ and hence does not allow for an intuitionistic proof. This theorem states: "If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(-1) < 0$ and $f(1) > 0$, there exists a number a such that $f(a) = 0$." The external meaning of this statement is that in any continuous family $(f_x)_x$ of continuous functions with $f_x(-1) < 0$ and $f_x(1) > 0$ for all $x \in X$, it's locally possible to pick zeros of the family in a continuous fashion. Figure **??** shows a counterexample to this claim.

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