A GENERAL NULLSTELLENSATZ FOR GENERALIZED SPACES

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Abstract. We give a general Nullstellensatz for the generic model of a geometric theory, useful as a source of nongeometric sequents validated by the generic model, and characterize the first-order formulas validated by the genetric model. XXX

1. Introduction

Generic models. Let $\mathbb{T}$ be a geometric theory, such as the theory of rings, local rings or intervals. We follow Caramello's terminology [4] to mean by *geometric theory* a system given by a set of sorts, a set of finitary function symbols, a set of finitary relation symbols and a set of axioms, consisting of geometric sequents (sequents of the form $\varphi \vdash_{\vec{x}} \psi$ where φ and ψ are geometric formulas, that is formulas built from equality and the relation symbols by the logical connectives $\top \perp \wedge \vee \exists$ and by arbitrary set-indexed disjunctions $\bigvee$).

A fundamental result is that there is a *generic model* $U_{\mathbb{T}}$ of $\mathbb{T}$, a model such that for any geometric sequent σ, the following notions coincide:

(1) The sequent σ is provable modulo $\mathbb{T}$.
(2) The sequent σ holds for any $\mathbb{T}$-model in any Grothendieck topos.
(3) The sequent σ holds for $U_{\mathbb{T}}$.

One could argue that it is this model which mathematicians implicitly refer to when they utter "Let M be a $\mathbb{T}$-model.". It can typically not be realized as a set-theoretic model, consisting of a set for each sort, a function for each function symbol and so on; instead it is a model in a custom-tailored syntactically constructed Grothendieck topos, the *classifying topos* $\mathrm{Set}[\mathbb{T}]$ of $\mathbb{T}$, hence consists of an object of $\mathrm{Set}[\mathbb{T}]$ for each sort, a morphism for each function symbol and so on.

To state what it means for a $\mathbb{T}$-structure in a topos $\mathcal{E}$ to verify the axioms of $\mathbb{T}$, rendering it a model, the *internal language* of $\mathcal{E}$ is used, roughly reviewed in Section **??** below. We write "$\mathcal{E} \models \alpha$" to mean that the statement α holds from the internal point of view of $\mathcal{E}$. Since this language is a form of a higher-order dependently-typed intuitionistic logic, the classifying topos $\mathrm{Set}[\mathbb{T}]$ can be regarded as a higher-order completion of the geometric theory $\mathbb{T}$. The generic model enjoys the universal property that any $\mathbb{T}$-model in any (Grothendieck) topos $\mathcal{E}$ is the pullback of $U_{\mathbb{T}}$ along an essentially unique geometric morphism $\mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

Nongeometric sequents. Crucially, the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) relating provability and truth in $\mathrm{Set}[\mathbb{T}]$ only pertains to geometric sequents. The generic model may validate additional nongeometric sequents which are not provable from the axioms of $\mathbb{T}$ in first-order or even higher-order logic, and these nongeometric sequents may be quite surprising and have useful consequences.

One of the most celebrated such sequents arises in the case that $\mathbb{T}$ is the theory of local rings. In this case, the classifying topos $\mathrm{Set}[\mathbb{T}]$ is also known as the *big Zariski* *topos* of $\mathrm{Spec}(\mathbb{Z})$ from algebraic geometry, the topos of sheaves over the site of schemes locally of finite presentation, and the generic model is the functor $\underline{\mathbb{A}}^1$ of points of the affine line, the functor which maps any (l.o.f.p.) scheme X to $\mathrm{Hom}(X, \mathbb{A}^1) = \mathcal{O}_X(X)$.

From the point of view of the topos, the ring object $\underline{\mathbb{A}}^1$ is not only a local ring, but even a field in the sense that

any nonzero element is invertible.

As this condition is of nongeometric form, it is not inherited by arbitrary local rings, which are indeed typically not fields. However, any intuitionistic consequence of this condition which is of geometric form is inherited by any local ring in any topos. Hence we may, when verifying a general fact about local rings, suppose without loss of generality that the given ring is a field. This observation is due to Kock [5], who exploited it to develop projective geometry over local rings, and was further used by Reyes to prove a Jacobian criterion for étale morphisms [6].

A related nongeometric sequent is valid in the little Zariski topos of the spectrum of a ring A, the classifying topos of local localizations of A. If A is reduced, the generic model validates the dual condition that any noninvertible element is zero. This property has been used to give a short and even constructive proof of Grothendieck's generic freeness lemma, substantially improving on previously published proofs [2].

In time, further nongeometric sequents holding in the big Zariski topos of an arbitrary base scheme have been found [3, Section 18.4]. These include:

- $\underline{\mathbb{A}}^1$ is *anonymously algebraically closed* in the sense that any monic polynomial $p : \underline{\mathbb{A}}^1[T]$ of degree at least one does *not not* have a zero.
- $\underline{\mathbb{A}}^1$ fulfills the following version of the Nullstellensatz: Let $f_1, \ldots, f_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ be polynomials without a common zero in $(\underline{\mathbb{A}}^1)^n$. Then there are polynomials $g_1, \ldots, g_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ such that $\sum_i g_i f_i = 1$.
- Any function $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is given by a unique polynomial.
- $\underline{\mathbb{A}}^1$ fulfills the axiom of microaffinity: Let $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$. Let $f : \Delta \to \underline{\mathbb{A}}^1$ be an arbitrary function. Then there are unique elements $a, b : \underline{\mathbb{A}}^1$ such that $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon : \Delta$.
- $\underline{\mathbb{A}}^1$ is *synthetically quasicoherent* in the sense that for any finitely presented $\underline{\mathbb{A}}^1$-algebra A, the canonical homomorphism $A \to (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(A)}$, where $\mathrm{Spec}(A)$ is defined to be the set of $\underline{\mathbb{A}}^1$-algebra-homomorphisms $A \to \underline{\mathbb{A}}^1$, is bijective.

All of these nongeometric sequents are useful for the purposes of synthetic algebraic geometry, the desire to carry out algebraic geometry in a language close to the simple language on the 19th and the beginning of the 20th century while still being fully rigorous and fully general, working over arbitrary base schemes instead of restricting to the field of complex numbers.

Characterizing nongeometric sequents. Referring to one of the previous examples, Tierney remarked around the time that those sequents were first studied that "[it] is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas" [7, p. 209]. In view of their importance, is there a way to discover nongeometric sequents in a systematic fashion? To characterize the nongeometric sequents holding in classifying toposes? To this end, Wraith put forward a specific conjecture [8, p. 336]:

> *The problem of characterising all the non-geometric properties of a generic model appears to be difficult. If the generic model of a geometric theory $\mathbb{T}$ satisfies a sentence α then any geometric consequence of $\mathbb{T} + \alpha$ has to be a consequence of $\mathbb{T}$. We might call α $\mathbb{T}$-redundant. Does the generic $\mathbb{T}$-model satisfy all $\mathbb{T}$-redundant sentences?*

This question was recently answered in the negative by Bezem, Buchholtz and Coquand [1]; hence the characterization we propose is necessarily more nuanced.

Our starting point was the empirical observation [3, p. 164] that in the case of the big Zariski topos, every true known nongeometric sequent followed from just a single such, namely the synthetic quasicoherence of the generic model, and in earlier work we surmised that one could formulate an appropriate metatheorem explaining this observation and generalizing it to arbitrary classifying toposes [3, Speculation 22.1]. This hope turned out to be true, in the sense we will now make precise.

A general Nullstellensatz. To explain the relevant background, the somewhat vague question "to which extent does the classifying topos Set[$\mathbb{T}$] realize that it is the classifying topos for $\mathbb{T}$?" is a useful guiding principle.

From the point of view of Set[$\mathbb{T}$], the model $U_\mathbb{T}$ is an ordinary set-theoretic model. While in general properties of a specific set-theoretic model do not follow solely from the axioms of the theory, it is on first sight conceivable that this could be true for $U_\mathbb{T}$. That it, is is reasonable to wonder whether

$$\mathrm{Set}[\mathbb{T}] \models \ulcorner \text{a geometric sequent } \sigma \text{ holds for } U_\mathbb{T} \text{ iff } \underline{\mathbb{T}} \text{ proves } \sigma \urcorner$$

holds.[1] While the "if" direction certainly holds (observing that the generic model $U_\mathbb{T}$ is, from the point of view of Set[$\mathbb{T}$], a model of $\underline{\mathbb{T}}$), the "only if" direction typically fails.

For instance, let $\mathbb{T}$ be the theory of rings and let σ be the sequent $(\top \vdash 1 + 1 = 0)$. Since there is an intuitionistic proof that $\mathbb{T}$ does not prove σ and toposes are sound with respect to intuitionistic logic, the statement $\ulcorner \underline{\mathbb{T}}$ proves $\sigma \urcorner$ is false from the internal point of view of Set[$\mathbb{T}$]. However, it is not the case that the statement $\ulcorner 1 + 1 = 0$ in $U_\mathbb{T} \urcorner$ is false from the internal point of view. In fact, this statement holds in a nontrivial slice of Set[$\mathbb{T}$], the open subtopos coinciding with the classifying topos of the theory of rings of characteristic two.

Intuitively, the problem is that while the meaning of $\ulcorner \underline{\mathbb{T}} \vdash \sigma \urcorner$ is fixed, the meaning of $\ulcorner \sigma$ holds for $U_\mathbb{T} \urcorner$ can vary with the slice. This problem can be solved by passing from $\underline{\mathbb{T}}$ to a varying theory, the internal theory $\underline{\mathbb{T}}/U_\mathbb{T}$ defined below. Unlike $\underline{\mathbb{T}}$, this theory is not the pullback of an external geometric theory.

Definition 1.1. Let $\mathbb{T}$ be a geometric theory. The theory $\underline{\mathbb{T}}/U_\mathbb{T}$ is the geometric theory internal to Set[$\mathbb{T}$] which arises from $\underline{\mathbb{T}}$ by adding additional constant symbols e_x, one for each element $x : U_\mathbb{T}$, axioms $(\top \vdash f(e_{x_1}, \ldots, e_{x_n}) = e_{f(x_1,\ldots,x_n)})$ for each function symbol f and n-tuple of elements of $U_\mathbb{T}$, and axioms $(\top \vdash R(e_{x_1}, \ldots, e_{x_n}))$ for each relation symbol R and n-tuple $(e_{x_1}, \ldots, e_{x_n})$ such that $R(e_{x_1}, \ldots, e_{x_n})$ holds for $U_\mathbb{T}$.

[1] Here $\underline{\mathbb{T}}$ is the internal geometric theory induced by $\mathbb{T}$, obtained by pulling back the set of sorts, the set of function symbols and so on along the geometric morphism Set[$\mathbb{T}$] $\to$ Set. More details will be given in Section **??**. The corner quotes indicate that for sake of readability, the translation into formal language is to be carried out by the reader.

From the point of view of $\mathrm{Set}[\mathbb{T}]$, a model of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is a model of $\underline{\mathbb{T}}$ equipped with a $\underline{\mathbb{T}}$-homomorphism from $U_{\mathbb{T}}$. In particular, the identity $(U_{\mathbb{T}} \to U_{\mathbb{T}})$ is a model of $\underline{\mathbb{T}}/U_{\mathbb{T}}$.

Theorem 1.2. *Let $\mathbb{T}$ be a geometric theory. Then, internally to* $\mathrm{Set}[\mathbb{T}]$*:*

> *A geometric sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ holds for $U_{\mathbb{T}}$ if and only if $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves σ.*

Theorem 1.3. *Let $\mathbb{T}$ be a geometric theory. Let α be a first-order formula over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) *The formula α holds for $U_{\mathbb{T}}$.*
(2) *The formula α is provable in first-order intuitionistic logic modulo the axioms of $\mathbb{T}$ and the additional axioms*

$$\sigma \quad \Longleftrightarrow \quad \underline{\mathbb{T}}/U \vdash' \sigma$$

where σ ranges over all geometric sequents.

2. Background on classifying toposes

3. Proofs of the main theorems

4. The special case of Horn theories

5. Applications

References

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