FLABBY AND INJECTIVE OBJECTS IN TOPOSES

INGO BLECHSCHMIDT

Abstract. We introduce a general notion of *flabby objects* in elementary toposes and study their basic properties. In the special case of localic toposes, this notion reduces to the common notion of flabby sheaves, yielding a site-independent characterization of flabby sheaves. Continuing a line of research started by Roswitha Harting, we use flabby objects to show that an internal notion of injective objects coincides with the corresponding external notion, in stark contrast with the situation for projective objects. As an application, we give internal characterizations of sheaf cohomology and show that higher direct images can be understood as internal cohomology.

Rough draft, comments welcome! iblech@speicherleck.de

XXX: introduction

1. Flabby sheaves

A sheaf F on a topological space (or locale) X is *flabby* (flasque) if and only if all restriction maps $F(X) \to F(U)$ are surjective. The following properties of flabby sheaves render them fundamental to the theory of sheaf cohomology:

(1) Let $(U_i)_i$ be an open covering of X. A sheaf F on X is flabby if and only if all of its restrictions $F|_{U_i}$ are flabby as sheaves on U_i.
(2) Let $f : X \to Y$ be a continuous map. If F is a flabby sheaf on X, then $f_*(F)$ is a flabby sheaf on Y.
(3) Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves of modules.
  (a) If F is flabby, then this sequence is also exact as a sequence of presheaves.
  (b) If F and H are flabby, then so is G.
  (c) If F and G are flabby, then so is H.
(4a) Any sheaf can be embedded into a flabby sheaf.
(4b) Any sheaf of modules can be embedded into a flabby sheaf of modules.

Since we want to develop an analogous theory for flabby objects in elementary toposes, it is worthwhile to analyze the logical and set-theoretic commitments which are required to establish these properties. The standard proofs of properties (1), (3a), (3b) and (3c) require Zorn's lemma to construct maximal extensions of given sections. The standard proof of property (4b) requires the law of excluded middle, to ensure that the Godement construction actually yields a flabby sheaf. Properties (2) and (4a) can be verified purely intuitionistically.

There is an alternative definition of flabbiness, to be introduced below, which is equivalent to the usual one in presence of Zorn's lemma and which requires different commitments: For the alternative definition, properties (1), (3b) and (4a) can be verified purely intuitionistically, and property (2) can be verified purely intuitionistically for open injections f. There is a substitute for property (3a) which can be verified purely intuitionistically. We do not know whether property (4b) can be established purely intuitionistically, but we give a rudimentary analysis in Section 6.

Both definitions can be generalized to yield notions of flabby objects in elementary toposes; but for toposes which are not localic, the two resulting notions will differ, and only the one obtained from the alternative definition is stable under pullback and can be characterized in the internal language. We therefore adopt in this paper the alternative one as the official definition.

Definition 1.1. A sheaf F on a topological space (or locale) X is *flabby* if and only if for all opens U and all sections $s \in F(U)$, there is an open covering $X = \bigcup_i U_i$ such that, for all i, the section s can be extended to a section on $U \cup U_i$.

If F is a flabby sheaf in the traditional sense, then F is obviously also flabby in the sense of Definition 1.1 – singleton coverings will do. Conversely, let F be a flabby sheaf in the sense of Definition 1.1. Let $s \in F(U)$ be a local section. Zorn's lemma implies that there is a maximal extension $s' \in F(U')$. By assumption, there is an open covering $X = \bigcup_i U_i$ such that, for all i, the section s' can be extended to $U' \cup U_i$. Since s' is maximal, $U' \cup U_i = U'$ for all i. Therefore $X = \bigcup_i U_i \subseteq U'$; hence s' is a global section, as desired.

We remark that unlike the traditional definition of flabbiness, Definition 1.1 exhibits flabbiness as a manifestly local notion.

2. Flabby sets

We intend this section to be applied in the internal language of an elementary topos; we will speak about sets and maps between sets, but intend our arguments to be applied to objects and morphisms in toposes. We will therefore be careful to reason purely intuitionistically. We adopt the terminology of [4] regarding subterminals and subsingletons: A subset $K \subseteq X$ is *subterminal* if and only if any given elements are equal ($\forall x, y \in K.\ x = y$), and it is a *subsingleton* if and only if there is an element $x \in X$ such that $K \subseteq \{x\}$. Any subsingleton is trivially subterminal, but the converse might fail.

Definition 2.1. A set X is *flabby* if and only if any subterminal subset of X is a subsingleton, that is, if and only if for any subset $K \subseteq X$ such that $\forall x, y \in K.\ x = y$, there exists an element $x \in X$ such that $K \subseteq \{x\}$.

In the presence of the law of excluded middle, a set is flabby if and only if it is inhabited. This characterization is a constructive taboo:

Proposition 2.2. *If any inhabited set is flabby, then the law of excluded middle holds.*

Proof. Let φ be a truth value. The set $X := \{0\} \cup \{1 \mid \varphi\} \subseteq \{0, 1\}$ is inhabited by 0 and contains 1 if and only if φ holds. Let K be the subterminal $\{1 \mid \varphi\} \subseteq X$. Flabbiness of X implies that there exists an element $x \in X$ such that $K \subseteq \{x\}$. We have $x = 0$ or $x = 1$. The first case entails $\neg\varphi$. The second case entails $1 \in X$, so φ. □

Let $\mathcal{P}_{\leq 1}(X)$ be the set of subterminals of X.

Proposition 2.3. *A set X is flabby if and only if the canonical map $X \to \mathcal{P}_{\leq 1}(X)$ which sends an element x to the singleton set $\{x\}$ is final.*

Proof. By definition. □

The set $\mathcal{P}_{\leq 1}(X)$ of subterminals of X can be interpreted as the set of *partially-defined elements* of X. In this view, the empty subset is the maximally undefined element and a singleton is a maximally defined element. A set is flabby if and only if any of its partially-defined elements can be refined to an honest element.

Definition 2.4. (1) A set X is *injective* if and only if, for any injection $i : A \to B$, any map $f : A \to X$ can be extended to a map $B \to X$.
(2) An R-module M is *injective* if and only if, for any linear injection $i : A \to B$ between R-modules, any linear map $f : A \to M$ can be extended to a linear map $B \to M$.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed] \\ X & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed] \\ M & \end{tikzcd}$$

In the presence of the law of excluded middle, a set is injective if and only if it is inhabited. In the presence of the axiom of choice, an abelian group is injective (as a $\mathbb{Z}$-module) if and only if it is divisible. Injective sets and modules have been intensively studied before [**XXX**]; the following properties are well-known:

Proposition 2.5. (1) *Any set can be embedded into an injective set.*
(2) *Any injective module is also injective as a set.*
(3) *Assuming the axiom of choice, any module can be embedded into an injective module.*

Proof. (1) One can check that, for instance, the full powerset $\mathcal{P}(X)$ and the set of subterminals $\mathcal{P}_{\leq 1}(X)$ are each injective. [**XXX**]
(2) The forgetful functor from modules to sets possesses a left exact left adjoint. More explicitly, if $i : A \to B$ is an injective map between sets and if $f : A \to M$ is an arbitrary map, then the induced map $R\langle A\rangle \to R\langle B\rangle$ between free modules is also injective, the given map f lifts to a linear map $R\langle A\rangle \to M$, and an R-linear extension $R\langle B\rangle \to M$ induces an extension $B \to M$ of f.
(3) One verifies that any abelian group can be embedded into a divisible abelian group. By the axiom of choice, divisible abelian groups are injective. The result for modules over arbitrary rings then follows purely formally [**XXX**]. □

Proposition 2.6. *Any injective set is flabby.*

Proof. Let X be an injective set. Let $K \subseteq X$ be a subterminal. The inclusion $f : K \to X$ extends along the injection $K \to 1 = \{\star\}$ to a map $1 \to X$. The unique image x of that map has the property that $K \subseteq \{x\}$. □

Corollary 2.7. *Any set can be embedded into a flabby set.*

Proof. Immediate by Proposition 2.5(1) and Proposition 2.6. □

A further corollary of Proposition 2.6 is that the statement "any inhabited set is injective" is a constructive taboo: If any inhabited set is injective, then any inhabited set is flabby, thus the law of excluded middle follows by Proposition 2.2.

Proposition 2.8. *Any singleton set is flabby. The cartesian product of flabby sets is flabby.*

Proof. Immediate. □

Subsets of flabby sets are in general not flabby, as else any set would be flabby in view of Corollary 2.7.

Proposition 2.9.
(1) *Let I be an injective set. Let T be an arbitrary set. Then the set I^T of maps from T to I is flabby.*
(2) *Let I be an injective R-module. Let T be an arbitrary R-module. Then the set $\mathrm{Hom}_R(T, I)$ of linear maps from T to I is flabby.*

Proof. We first cover the case of sets. Let $K \subseteq I^T$ be a subterminal. We consider the injectivity diagram

$$\begin{tikzcd} T' \arrow[r, hook] \arrow[d] & T \arrow[dl, dashed] \\ I & \end{tikzcd}$$

where T' is the subset $\{s \in T \mid K \text{ is inhabited}\} \subseteq T$ and the solid vertical map sends $s \in T'$ to $g(s)$, where g is an arbitrary element of K. This association is well-defined. Since I is injective, a dotted lift as indicated exists. If K is inhabited, this lift is an element of K.

The same kind of argument applies to the case of modules. If K is a subterminal of $\mathrm{Hom}_R(T, I)$, we define T' to be the submodule $\{s \in T \mid s = 0 \text{ or } K \text{ is inhabited}\}$ and consider the analogous injectivity diagram, where the solid vertical map $f : T' \to I$ is now defined by cases: Let $s \in T'$. If $s = 0$, then we set $f(s) = 0$; if K is inhabited, then we set $f(s) := g(s)$, where g is an arbitrary element of K. This association is again well-defined, and a dotted lift yields the desired element of $\mathrm{Hom}_R(T, I)$. □

Proposition 2.9 can be used to give an alternative proof of Proposition 2.6 and to generalize Proposition 2.6 to modules: If I is an injective set, then the set $I^1 \cong I$ is flabby. If I is an injective module, then the set $\mathrm{Hom}_R(R, I) \cong I$ is flabby.

Lemma 2.10.
(1) *Let I be an injective set. Let $i : A \to B$ be an injection. Let $f : A \to I$ be an arbitrary map. Then the set of extensions of f to B is flabby.*
(2) *Let I be an injective R-module. Let $i : A \to B$ be a linear injection. Let $f : A \to I$ be an arbitrary linear map. Then the set of linear extensions of f to B is flabby.*

Proof. For the first claim, we set $X := \{\bar{f} \in I^B \mid \bar{f} \circ i = f\}$. Let $K \subseteq X$ be a subterminal. We consider the injectivity diagram

$$\begin{tikzcd} i[A] \cup B' \arrow[r, hook] \arrow[d, "g"'] & B \arrow[dl, dashed] \\ I & \end{tikzcd}$$

where B' is the set $\{s \in B \mid K \text{ is inhabited}\}$ and the solid vertical arrow g is defined in the following way: Let $s \in i[A] \cup B'$. If $s \in i[A]$, we set $g(s) := f(a)$, where $a \in A$

is an element such that $s = i(a)$. If $s \in B'$, we set $g(s) := \bar{f}(s)$, where $\bar{f}$ is any element of K. These prescriptions determine a well-defined map.

Since I is injective, there exists a dotted map rendering the diagram commutative. This map is an element of X. Furthermore, if K is inhabited, then this map is an element of K.

The proof of the second claim is similar. We set $X := \{\bar{f} \in \mathrm{Hom}_R(B, I) \,|\, \bar{f} \circ i = f\}$. Let $K \subseteq X$ be a subterminal. We consider the injectivity diagram

$$\begin{tikzcd} i[A] + B' \arrow[r, hook] \arrow[d, "g"'] & B \arrow[dl, dashed, bend left] \\ I & \end{tikzcd}$$

where B' is the submodule $\{t \in B \,|\, t = 0 \text{ or } K \text{ is inhabited}\} \subseteq B$ and the solid vertical arrow g is defined in the following way: Let $s \in i[A] + B'$. Then $s = i(a) + t$ for an element $a \in A$ and an element $t \in B'$. Since $t \in B'$, $t = 0$ or K is inhabited. If $t = 0$, we set $g(s) := f(a)$. If K is inhabited, we set $g(s) := f(a) + \bar{f}(s)$, where $\bar{f}$ is any element of K. These prescriptions determine a well-defined map.

Since I is injective, there exists a dotted map rendering the diagram commutative. This map is an element of X. Furthermore, if K is inhabited, then this map is an element of K. $\square$

Proposition 2.11. *Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be a short exact sequence of modules. Let $s \in M''$. If M' is flabby, then the set of preimages of s under p is flabby.*

Proof. Let $X := \{u \in M \,|\, p(u) = s\}$. Let $K \subseteq X$ be a subterminal. Since p is surjective, there is an element $u_0 \in X$. The translated set $K - u_0 \subseteq M$ is still a subterminal, and its preimage under i is as well. Since M' is flabby, there is an element $v \in M'$ such that $i^{-1}[K - u_0] \subseteq \{v\}$. We verify that $K \subseteq \{u_0 + i(v)\}$.

Thus let $u \in K$ be given. Then $p(u - u_0) = 0$, so by exactness the set $i^{-1}[K - u_0]$ is inhabited. It therefore contains v. Thus $i(v) \in K - u_0$. Since $K = \{u\}$, it follows that $i(v) = u - u_0$, so $u \in \{u_0 + i(v)\}$ as claimed. $\square$

Proposition 2.12. *Let $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be a short exact sequence of modules. If M' and M'' are flabby, so is M.*

Proof. Let $K \subseteq M$ be a subterminal. Then its image $p[K] \subseteq M''$ is a subterminal as well. Since M'' is flabby, there is an element $s \in M''$ such that $p[K] \subseteq \{s\}$.

Since p is surjective, there is an element $u_0 \in M$ such that $p(u_0) = s$.

The preimage $i^{-1}[K - u_0] \subseteq M'$ is a subterminal. Since M' is flabby, there exists an element $v \in M'$ such that $i^{-1}[K - u_0] \subseteq \{v\}$.

Thus $K \subseteq \{u_0 + i(v)\}$. $\square$

Noticeably missing here is XXX ...

3. Flabby objects

Definition 3.1. An object X of an elementary topos $\mathcal{E}$ is *flabby* if and only if the statement "X is a flabby set" holds in the stack semantics of $\mathcal{E}$.

This definition amounts to the following: An object X of an elementary topos $\mathcal{E}$ is flabby if and only if, for any monomorphism $K \to A$ and any morphism $K \to X$, there exists an epimorphism $B \to A$ and a morphism $B \to X$ such that the following diagram commutes.

$$\begin{tikzcd} K \times_A B \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed, bend left] \\ X & \end{tikzcd}$$

Instead of referencing arbitrary stages $A \in \mathcal{E}$, one can also just reference the generic stage: Let $\mathcal{P}_{\leq 1}(X)$ denote the *object of subterminals* of X; this object is a certain subobject of $\mathcal{P}(X) = [X, \Omega_\mathcal{E}]_\mathcal{E}$, the powerobject of X. The subobject K_0 of $X \times \mathcal{P}_{\leq 1}(X)$ classified by the evaluation morphism $X \times \mathcal{P}_{\leq 1}(X) \to X \times \mathcal{P}(X) \to \Omega_\mathcal{E}$ is the *generic subterminal* of X. The object X is flabby if and only if there exists an epimorphism $B \to \mathcal{P}_{\leq 1}(X)$ and a morphism $B \to X$ such that the following diagram commutes.

$$\begin{tikzcd} K_0 \times_{\mathcal{P}_{\leq 1}(X)} B \arrow[r, hook] \arrow[d] & \mathcal{P}_{\leq 1}(X) \arrow[dl, dashed, bend left] \\ X & \end{tikzcd}$$

Proposition 3.2. *Let X and T be objects of an elementary topos $\mathcal{E}$.*

(1) *If X is flabby, so is $X \times T$ as an object of $\mathcal{E}/T$.*
(2) *The converse holds if the unique morphism $T \to 1$ is an epimorphism.*

Proof. This holds for any property which can be defined in the stack semantics [6, Lemma 7.3]. □

Proposition 3.3. *Let F be a sheaf on a topological space X (or a locale). Then F is flabby as a sheaf if and only if F is flabby as an object of the sheaf topos $\mathrm{Sh}(X)$.*

Proof. The proof is routine; we only verify the "only if" direction. Let F be flabby as a sheaf. It suffices to verify the definining condition for stages of the form $A = \mathrm{Hom}(\cdot, U)$, where U is an open of X. A monomorphism $K \to A$ then amounts to an open $V \subseteq U$ (the union of all opens on which K is inhabited). A morphism $K \to F$ amounts to a section $s \in F(V)$. Since F is flabby as a sheaf, there is an open covering $X = \bigcup_{i \in I} V_i$ such that, for all i, the section s can be extended to a section s_i of $V \cup V_i$. The desired epimorphism is $B := \coprod_i \mathrm{Hom}(\cdot, (V \cup V_i) \cap U) \to A$, and the desired morphism $B \to X$ is given by the sections $s_i|_{(V \cup V_i) \cap U}$.

As stated, the argument in the previous paragraph requires the axiom of choice to pick the extensions s_i; this can be avoided by a standard trick of expanding the index set of the coproduct to include the choices: We redefine $B := \coprod_{(i,t) \in I'} \mathrm{Hom}(\cdot, (V \cup V_i) \cap U)$, where $I' = \{(i \in I, t \in F(V \cup V_i)) \mid t|_V = s\}$ and define the morphism $B \to X$ on the (i,t)-summand by $t|_{(V \cup V_i) \cap U}$. □

Proposition 3.4. *Let X be a flabby object of a localic topos $\mathcal{E}$. If Zorn's lemma is available in the metatheory, then X possesses a global element (a morphism $1 \to X$).*

Proof. This is a restatement of the discussion following Definition 1.1. □

Proposition 3.5. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. If f_* preserves epimorphisms, then f_* preserves flabby objects.*

Proof. Let $X \in \mathcal{F}$ be a flabby object. Let $k : K \to A$ be a monomorphism in $\mathcal{E}$ and let $x : K \to f_*(X)$ be an arbitary morphism. Without loss of generality, we may assume that A is the terminal object 1 of $\mathcal{E}$. Then $f^*(k) : f^*(K) \to 1$ is a monomorphism in $\mathcal{F}$ and the adjoint transpose $x^t : f^*(K) \to X$ is a morphism in $\mathcal{F}$. Since X is flabby, there is an epimorphism $B \to 1$ in $\mathcal{F}$ and a morphism $y : B \to X$ such that the morphism $f^*(K) \times B \to X$ factors over y. Hence x factors over $f_*(y) : f_*(B) \to f_*(X)$. We conclude because the morphism $f_*(B) \to f_*(1)$ is an epimorphism by assumption. □

The assumption on f_* of Proposition 3.5 is for instance satisfied if f is a local geometric morphism or if f is induced by an open continuous injection between topological spaces. XXX

Definition 3.6. An object I of an elementary topos $\mathcal{E}$ is *externally injective* if and only if for any monomorphism $A \to B$ in $\mathcal{E}$, the canonical map $\mathrm{Hom}_{\mathcal{E}}(B, I) \to \mathrm{Hom}_{\mathcal{E}}(A, I)$ is surjective. It is *internally injective* if and only if for any monomorphism $A \to B$ in $\mathcal{E}$, the canonical morphism $[B, I] \to [A, I]$ between Hom objects is an epimorphism in $\mathcal{E}$.

If R is a ring in an elementary topos $\mathcal{E}$, a similar definition can be given for R-modules in $\mathcal{E}$, referring only to the set respectively the object of linear maps. The condition for an object to be internally injective can be rephrased in various ways. The following proposition lists five of these conditions. The equivalence of the first four is due to Harting [3].

Proposition 3.7. *Let $\mathcal{E}$ be an elementary topos. Then the following statements about an object $I \in \mathcal{E}$ are equivalent.*

(1) *I is internally injective.*
(1') *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object $p^*(I)$ has property (1) as an object of $\mathcal{E}/A$.*
(2) *The functor $[\cdot, I] : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ maps monomorphisms in $\mathcal{E}$ to morphisms for which any global element of the target locally (after change of base along an epimorphism) possesses a preimage.*
(2') *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object $p^*(I)$ has property (2) as an object of $\mathcal{E}/A$.*
(3) *The statement "I is an injective set" holds in the stack semantics of $\mathcal{E}$.*

Proof. The implications (1) ⇒ (2), (1') ⇒ (2'), (1') ⇒ (1) and (2') ⇒ (2) are trivial.

The equivalence (1') ⇔ (3) follows directly from the interpretation rules of the stack semantics.

The implication (2) ⇒ (2') employs the extra left adjoint $p_! : \mathcal{E}/A \to \mathcal{E}$ of $p^* : \mathcal{E} \to \mathcal{E}/A$ (which maps an object $(X \to A)$ to X), as in the usual proof that injective sheaves remain injective when restricted to smaller open subsets: We have that $p_* \circ [\cdot, p^*(I)]_{\mathcal{E}/A} \cong [\cdot, I]_{\mathcal{E}} \circ p_!$, the functor $p_!$ preserves monomorphisms, and one can check that p_* reflects the property that global elements locally possess preimages. Details are in [**harting**].[1]

The implication (2') ⇒ (1') follows by performing an extra change of base, since any non-global element becomes a global element after a suitable change of base. □

[1] Harting formulates the statement for abelian group objects, and has to assume that $\mathcal{E}$ contains a natural numbers object to ensure the existence of an abelian version of $p_!$.

Let R be a ring in $\mathcal{E}$. Then the analogue of Proposition 3.7 holds for R-modules in $\mathcal{E}$, if $\mathcal{E}$ is assumed to have a natural numbers object. The extra assumption is needed in order to construct the left adjoint $p_! : \mathrm{Mod}_{\mathcal{E}/A}(R \times A) \to \mathrm{Mod}_{\mathcal{E}}(R)$. Phrased in the internal language, this adjoint maps a family $(M_a)_{a\in A}$ of R-modules to the direct sum $\bigoplus_{a\in A} M_a$. Details on this construction, phrased in the language of sets but interpretable in the internal language, can for instance be found in [5, page 54].

Somewhat surprisingly, and in stark contrast with the situation for internally projective objects (which are defined dually), internal injectivity coincides with external injectivity for localic toposes.

Theorem 3.8. *Let I be an object of an elementary topos $\mathcal{E}$. If I is externally injective, then I is also internally injective. The converse holds if $\mathcal{E}$ is localic and Zorn's lemma is available.*

Proof. For the "only if" direction, let I be an object which is externally injective. Then I satisfies Condition (2) in Proposition 3.7, even without having to pass to covers.

For the "if" direction, let I be an internally injective object. Let $i : A \to B$ be a monomorphism in $\mathcal{E}$ and let $f : A \to I$ be an arbitrary morphism. We want to show that there exists an extension $B \to I$ of f along i. To this end, we consider the object of such extensions, defined by the internal expression

$$F := \{\bar{f} \in [B, I] \mid \bar{f} \circ i = f\}.$$

Global elements of F are extensions of the kind we are looking for. By Lemma 2.10, this object is flabby. By Proposition 3.4, it has a global element. ☐

The analogue of Theorem 3.8 for modules holds as well, if $\mathcal{E}$ is assumed to have a natural numbers object. The proof carries over word for word, only referencing Lemma 2.10(2) instead of Lemma 2.10(1). It seems that Harting was not aware of this, even though she did show that injectivity of sheaves of modules over topological spaces is a local notion [**XXX**], as she (mistakenly) states in [2, page 233] that "the notions of injectivity and internal injectivity do not coincide" for modules.

Since we were careful in Section 2 to use the law of excluded middle or the axiom of choice only where needed, most results of that section carry over to flabby and internally injective objects. Specifically, we have:

Scholium 3.9. *For any elementary topos $\mathcal{E}$:*

1. *Any object can be embedded into an internally injective object.*
2. *(If $\mathcal{E}$ has a natural numbers object.) The underlying unstructured object of an internally injective module is internally injective.*
3. *Any internally injective object is flabby.*
4. *Any object can be embedded into a flabby object.*
5. *The terminal object is flabby. The product of flabby objects is flabby.*
6. *Let I be an internally injective object. Let T be an arbitrary object. Then $[T, I]$ is a flabby object.*
7. *(If $\mathcal{E}$ has a natural numbers object.) Let I be an internally injective R-module. Let T be an arbitrary R-module. Then $[T, I]_R$, the subobject of the internal Hom consisting only of the linear maps, is a flabby object.*
8. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules in $\mathcal{E}$. If M' and M'' are flabby objects, so is M.*

Proof. The analogous statements were established purely intuitionistically in Section 2, and the stack semantics is sound with respect to intuitionistic logic. □

Scholium 3.10. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-modules in an elementary topos $\mathcal{E}$. Let M' be a flabby object. If $\mathcal{E}$ is localic and Zorn's lemma is available, the induced sequence $0 \to \Gamma(M') \to \Gamma(M) \to \Gamma(M') \to 0$ of $\Gamma(R)$-modules is exact, where $\Gamma(X) = \mathrm{Hom}_{\mathcal{E}}(1, X)$.*

Proof. We only have to verify exactness at $\Gamma(M'')$, so let $s \in \Gamma(M')$. Interpreting Proposition 2.11 in $\mathcal{E}$, we see that the object of preimages of s is flabby. Since $\mathcal{E}$ is localic, this object is a flabby sheaf; since Zorn's lemma is available, it possesses a global element. Such an element is the desired preimage of s in $\Gamma(M)$. □

If $\mathcal{E}$ is not necessarily localic or Zorn's lemma is not available, only a weaker substitute for Scholium 3.10 is available: Given $s \in \Gamma(M'')$, the object of preimages of s is flabby. XXX

Remark 3.11. A direct generalization of the traditional notion of a flabby sheaf, as opposed to our reimagining in Definition 1.1, to elementary toposes is the following. An object X of an elementary topos $\mathcal{E}$ is *strongly flabby* if and only if, for any monomorphism $K \to 1$ in $\mathcal{E}$, any morphism $K \to X$ lifts to a morphism $1 \to X$.

One can verify, purely intuitionistically, that a sheaf F on a space T is flabby in the traditional sense if and only if F is a strongly flabby object of $\mathrm{Sh}(T)$.

The notion of strongly flabby objects is, however, not stable under base change and therefore cannot be characterized in the internal language. A specific example is the G-set G (with the translation action), considered as an object of the topos BG of G-sets, where G is a nontrivial group. This object is not strongly flabby, since the morphism $\emptyset \to G$ does not lift, but its pullback to the slice $BG/G \simeq \mathrm{Set}$ is (assuming the law of excluded middle).

4. Characterizing cohomology

XXX

5. Flabby objects in the effective topos

The notion of flabby objects originates from the notion of flabby shaves and is therefore closely connected to Grothendieck toposes. It is therefore instructive to study flabby objects in elementary toposes which are not Grothendieck toposes, away from their original conceptual home. We begin this study with establishing the following observations on flabby objects in the effective topos. We follow the terminology of [**hyland:effective-topos**].

Proposition 5.1. *Let X be a flabby object in the effective topos. Let $f : X \to X$ be a morphism. If X is effective, the statement "f has a fixed point" holds in the effective topos.*

Proposition 5.2. *Assuming the law of excluded middle in the metatheory, any $\neg\neg$-separated module in the effective topos can be embedded into a flabby module.*

The intuitive reason for why Proposition 5.1 holds is the following. Let X be a flabby object in the effective topos. Then there is a procedure which computes for any subterminal $K \subseteq X$ an element x_K such that $K \subseteq \{x_K\}$. This element might not depend extensionally on K, but this fine point is not important for this discussion. Let now $f : X \to X$ be a morphism. We construct the self-referential subset $K := \{f(x_K)\}$; the formal proof below will indicate how this can be done. Then $K \subseteq \{x_K\}$, so $f(x_K) = x_K$.

A corollary of Proposition 5.1 is that the trivial module is the only flabby module in the effective topos whose underlying unstructured object is an effective set: Given such a flabby module M, let $v \in M$ be an arbitrary element. Then the morphism $x \mapsto v + x$ has a fixed point; thus $v + x = x$ for some element x, and hence $v = 0$.

It is the self-referentiality which makes the proof of Proposition 5.1 work, but the blame for paucity of flabby objects in the effective topos is to put on the realizers for statements of the form "$K = K$", where $(=)$ is the nonstandard equality predicate of the powerobject $\mathcal{P}(X)$. A procedure witnessing flabbiness has to compute a reflexivity realizer for a suitable element x_K from a reflexivity realizer for a given element K. However, such realizers are not very informative. Metaphorically speaking, a procedure witnessing flabbiness has to conjure elements out of thin air.

This problem does not manifest with objects X which are not effective sets. Reflexivity realizers for these objects are themselves not very informative; a procedure witnessing flabbiness therefore only has to turn one kind of non-informative realizers into another kind. The flabby modules featuring in the proof of Proposition 5.2 will accordingly not be effective sets.

Proof of Proposition 5.1. For any Turing machine e, let $v_e : |X| \to \Sigma$ be the nonstandard predicate given by

$$\begin{aligned} v_e(x) = \{m \in \mathbb{N} \,|\, &\text{there is an element } x_0 \in |X| \text{ such that} \\ &e \text{ terminates with an element of } [\![x_0 = x_0]\!] \text{ and } m \in [\![x = x_0]\!]\} \end{aligned}$$

and let $K_e \in \Sigma^{|X| \times \Sigma}$ be the nonstandard predicate given by

$$K_e(x, u) = [\![(x = x) \wedge (u \leftrightarrow v_e(x))]\!].$$

One can explicitly construct a realizer a_e of the statement "$K_e = K_e$", where $(=)$ is the nonstandard equality predicate of the object $\mathcal{P}_{\leq 1}(X)$ of subterminals of X. This is where the assumption that X is effective is important; without it, we could only verify "$K_e = K_e$" where $(=)$ is the nonstandard equality predicate of the full powerobject $\mathcal{P}(X)$.

Since X is flabby, there is a realizer r for the statement "$\forall K \in \mathcal{P}_{\leq 1}(X).\ \exists x \in X.\ \forall y \in X.\ (y \in K \Rightarrow y = x)$". Let s be a realizer for the statement "$\forall x \in X.\ \exists y \in Y.\ y = f(x)$". Let e be the particular Turing machine which proceeds as follows:

1. Simulate r on input a_e in order to obtain a realizer $b \in [\![x = x]\!]$ for some $x \in |X|$.
2. Simulate s on input b in order to obtain a realizer $c \in [\![f(x) = f(x)]\!]$.
3. Output c.

The description of the machine e makes use of the number e coding it; the recursion theorem yields a general reason why this self-referentiality is possible. Here we can even do without this theorem, since a close inspection of the construction of a_e shows that a_e is actually independent of e. This should not come as a surprise, as reflexivity realizers of $\mathcal{P}(X)$ and $\mathcal{P}_{\leq 1}(X)$ are known to be not very informative.

Passing a_e to r yields a reflexivity realizer of some element $x_{K_e} \in |X|$. Therefore the Turing machine e does terminate, with a reflexivity realizer for $f(x_{K_e})$. Thus the statement "$f(x_{K_e}) \in K$" is realized; hence "$f(x_{K_e}) = x_{K_e}$" is as well. □

Proof of Proposition 5.2. Let $(\Gamma \dashv \Delta) : \mathrm{Set} \to \mathrm{Eff}$ be the inclusion of the double-negation sheaves. For a $\neg\neg$-separated module M in the effective topos, the canonical morphism $M \to \Delta(\Gamma(M))$ is a monomorphism; the set $\Gamma(M)$ is flabby by virtue of being inhabited; and Δ preserves flabby objects by Proposition 3.5. □

6. Conclusion

We originally set out to develop an intuitionistic account of Grothendieck's sheaf cohomology. Čech methods can be carried out constructively, and XXX(Barakat), but it appears that there is not a general framework for sheaf cohomology which works in an intuitionistic metatheory.

The main obstacle preventing Grothendieck's theory of derived functors to be interpreted constructively is its reliance on injective resolutions. It is known that in the absence of the axiom of choice, much less in a purely intuitionistic context, there might not be any nontrivial injective abelian group [1].

In principle, this problem could be remedied by employing flabby resolutions instead of injective ones. There are, however, two problems with this suggestion. Firstly, we needed Zorn's lemma to show that flabby sheaves are acyclic for the global sections functor (Proposition XXX). This problem might be mitigated by relying on the substitute property XXX. The more serious problem is that it is an open question whether one can show, purely intuitionistically, that any sheaf of modules embeds into a flabby sheaf of modules. The following is known about this problem:

(1) There is a purely intuitionistic proof that any sheaf of sets embeds into a flabby sheaf of sets (Scholium 3.9(4)).
(2) The existence of enough flabby modules, and even the existence of enough injective modules, is *not* a constructive taboo, that is, these statements do not entail a classical principle like the law of excluded middle or the principle of omniscience. This is because assuming the axiom of choice, any Grothendieck topos has enough injective (and therefore flabby) modules.
(3) There is a way of embedding any module into a flabby module if one is prepared to ignore set-theoretical difficulties. Namely, let M be an R-module. Inductively construct a collection T of terms by the following clauses: $0 \in T$ (where 0 is a formal symbol); if $t, s \in T$, then $t + s \in T$; if $t \in T$ and $r \in R$, then $rt \in T$; if $x \in M$, then $\underline{x} \in T$; if $K \subseteq T$ is a subterminal, then $\varepsilon_K \in T$. Let $(\sim)$ be the finest equivalence relation on T such that $t + (s + u) \sim (t + s) + u$, $t + s \sim s + t$, $t + 0 \sim t \sim 0 + t$, $0t \sim 0$, $1t \sim t$, $r(t + s) \sim rt + rs$, $(r + r')t \sim rt + r't$ and such that $\underline{0} \sim 0$, $\underline{x + y} \sim \underline{x} + \underline{y}$, $\underline{rx} \sim r\underline{x}$, and such that $\varepsilon_{\{t\}} \sim t$ for all terms t. Then the quotient $T/{\sim}$ is a flabby R-module into which M embeds. However, it is not clear that $T/{\sim}$ is a set.
(4) There appears to be some tension regarding the effective topos: Proposition 5.2 shows that at least $\neg\neg$-separated modules in the effective topos always embed into flabby modules, assuming the law of excluded middle

in the metatheory, while Proposition 5.1 shows that no nontrivial effective module is flabby.

We currently believe that it is not possible to give a constructive account of a global cohomology functor. However, XXX.

References

[1] A. Blass. "Injectivity, projectivity, and the axiom of choice". In: *Trans. Amer. Math. Soc.* 255 (1979), pp. 31–59.

[2] R. Harting. "A remark on injectivity of sheaves of abelian groups". In: *Archiv der Mathematik* 36.1 (1981), pp. 230–233.

[3] R. Harting. "Locally injective abelian groups in a topos". In: *Comm. Algebra* 11.4 (1983), pp. 349–376.

[4] A. Kock. "Algebras for the partial map classifier monad". In: *Category Theory. Proceedings of the International Conference held in Como, Italy, July 22–28, 1990.* Ed. by A. Carboni, M. Pedicchio, and G. Rosolini. Vol. 1488. Lecture Notes in Math. Springer, 1991, pp. 262–278.

[5] R. Mines, F. Richman, and W. Ruitenburg. *A Course in Constructive Algebra.* Universitext. Springer, 1988.

[6] M. Shulman. "Stack semantics and the comparison of material and structural set theories". 2010. URL: https://arxiv.org/abs/1004.3802.

Università di Verona, Department of Computer Science, Strada le Grazie 15, 37134 Verona, Italy

Email address: iblech@speicherleck.de