USING THE INTERNAL LANGUAGE OF TOPOSES IN ALGEBRAIC GEOMETRY

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Abstract. There are several important topoi associated to a scheme, for instance the petit and gros Zariski topoi. These come with an internal mathematical language which closely resembles the usual formal language of mathematics, but is "local on the base scheme":

For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a certain condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

These expository notes give an introduction to this topic and show how the internal point of view can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry. No prior knowledge about topos theory and formal logic is assumed.

Contents

1. Introduction

2. Kripke–Joyal semantics

Let X be a topological space. Later, X will be the underlying space of a scheme.

Definition 2.1. The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ is given by the following rules, recursively in the structure of φ:

$$\begin{array}{lcl}
U \models f = g : \mathcal{F} & :\Longleftrightarrow & f|_U = g|_U \in \Gamma(U, \mathcal{F}) \\
U \models \top & :\Longleftrightarrow & U = U \text{ (always fulfilled)} \\
U \models \bot & :\Longleftrightarrow & U = \emptyset \\
U \models \varphi \wedge \psi & :\Longleftrightarrow & U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi & :\Longleftrightarrow & \cancel{U \models \varphi \text{ or } U \models \psi} \\
& & \text{there exists a covering } U = \bigcup_i U_i \text{ s.th. for all } i\text{:} \\
& & \qquad U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi & :\Longleftrightarrow & \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall f : \mathcal{F}.\ \varphi(f) & :\Longleftrightarrow & \text{for all sections } f \in \Gamma(V, \mathcal{F})\text{, open } V \subseteq U\text{: } V \models \varphi(f) \\
U \models \exists f : \mathcal{F}.\ \varphi(f) & :\Longleftrightarrow & \cancel{\text{there exists a section } f \in \Gamma(U, \mathcal{F}) \text{ s.th. } U \models \varphi(f)} \\
& & \text{there exists an open covering } U = \bigcup_i U_i \text{ s.th. for all } i\text{:} \\
& & \qquad \text{there exists } f_i \in \Gamma(U_i, \mathcal{F}) \text{ s.th. } U_i \models \varphi(f_i) \\
U \models \forall \mathcal{F}.\ \varphi(\mathcal{F}) & :\Longleftrightarrow & \text{for all sheaves } \mathcal{F} \text{ on } V\text{, open } V \subseteq U\text{: } V \models \varphi(\mathcal{F}) \\
U \models \exists \mathcal{F}.\ \varphi(\mathcal{F}) & :\Longleftrightarrow & \text{there exists an open covering } U = \bigcup_i U_i \text{ s.th. for all } i\text{:} \\
& & \qquad \text{there exists a sheaf } \mathcal{F}_i \text{ on } U_i \text{ s.th. } U_i \models \varphi(\mathcal{F}_i)
\end{array}$$

The translation of $U \models \neg\varphi$ does not have to be defined, since negation can be expressed using other symbols: $\neg\varphi :\equiv (\varphi \Rightarrow \bot)$. Analogously to the rules for $\wedge$ and $\vee$, we can define rules for arbitrary set-indexed conjuctions ($\bigwedge_{j \in J}$) and disjunctions ($\bigvee_{j \in J}$). If we want to emphasize the particular topos, we write

$$\mathrm{Sh}(X) \models \varphi \quad :\Longleftrightarrow \quad X \models \varphi.$$

Remark 2.2. The last two rules, concerning *unbounded quantification*, are not part of the classical Kripke–Joyal semantics, but instead of Mike Shulman's stack semantics [**?**], a slight extension. They are needed so that we can formulate universal properties in the internal language.

Example 2.3. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is a monomorphism of sheaves if and only if, from the internal perspective, α is simply an injective map:

$$\begin{aligned}
& X \models \ulcorner \alpha \text{ is injective} \urcorner \\
\Longleftrightarrow\ & X \models \forall s, t \colon \mathcal{F}.\ \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F})\text{:} \\
& \qquad U \models \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F})\text{:} \\
& \qquad \text{for all open } V \subseteq U\text{:} \\
& \qquad\qquad \alpha_V(s|_V) = \alpha_V(t|_V) \text{ implies } s|_V = t|_V \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F})\text{:} \\
& \qquad \alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U \\
\Longleftrightarrow\ & \alpha \text{ is a monomorphism of sheaves}
\end{aligned}$$

The corner quotes "$\ulcorner \ldots \urcorner$" indicate that translation into formal language is left to the reader. Similarly, α is an epimorphism of sheaves if and only if, from the internal perspective, α is a surjective map.

The rules are not all arbitrary. They are finely concerted to make the following propositions true, which are crucial for a proper appreciation of the internal language.

Proposition 2.4 (Locality of the internal language)**.** *Let $U = \bigcup_i U_i$ be covered by open subsets. Let φ be a formula. Then*

$$U \models \varphi \qquad \textit{iff} \qquad U_i \models \varphi \textit{ for each } i.$$

Proof. Induction on the structure of φ. Note that the canceled rules would make this proposition false. ☐

Proposition 2.5 (Soundness of the internal language)**.** *If a formula φ implies a further formula ψ in intuitionistic logic, then $U \models \varphi$ implies $U \models \psi$.*

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that on each U_i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in [**?**, D1.3.1]. ☐

Because of the multitude of quantifiers, literal translations of internal statements can sometimes get slightly unwieldy. There are simplification rules for certain often-occuring special cases:

Proposition 2.6.

$$U \models \forall s\!:\!\mathcal{F}.\ \forall t\!:\!\mathcal{G}.\ \varphi(s,t) \iff \textit{for all open } V \subseteq U, \textit{ sections } s \in \Gamma(V,\mathcal{F}),\ t \in \Gamma(V,\mathcal{G})\textit{: } V \models \varphi(s,t)$$

$$U \models \forall s\!:\!\mathcal{F}.\ \varphi(s) \Rightarrow \psi(s) \iff \textit{for all open } V \subseteq U, \textit{ sections } s \in \Gamma(V,\mathcal{F})\textit{: } V \models \varphi(s) \textit{ implies } V \models \psi(s)$$

$$U \models \exists! s\!:\!\mathcal{F}.\ \varphi(s) \iff \textit{for all open } V \subseteq U, \textit{ there is exactly one section } s \in \Gamma(V,\mathcal{F}) \textit{ with: } V \models \varphi(s)$$

Proof. Straightforward. By way of example, we prove the existence claim in the "only if" direction of the last rule. (Note that this rule formalizes the saying "unique existence is global existence".) By definition of $\exists!$, it holds that

$$U \models \exists s\!:\!\mathcal{F}.\ \varphi(s)$$

and

$$U \models \forall s,t\!:\!\mathcal{F}.\ \varphi(s) \wedge \varphi(t) \Rightarrow s = t.$$

Let $V \subseteq U$ be an arbitrary open subset. Then there exist local section $s_i \in \Gamma(V_i,\mathcal{F})$ such that $V_i \models \varphi(s_i)$, where $V = \bigcup_i V_i$ is an open covering. By the locality of the internal language, on intersections it holds that $V_i \cap V_j \models \varphi(s_i)$, so by the uniqueness assumption, it follows that the local sections agree on intersections. They therefore glue to a section $s \in \Gamma(V,\mathcal{F})$. Since $V_i \models \varphi(s)$ for any i, the locality of the internal language allows us to conclude that $V \models \varphi(s)$. $\square$

Remark 2.7. Note that $\mathrm{Sh}(X) \models \neg\varphi$ is in general a much stronger statement that merely supposing that $\mathrm{Sh}(X) \models \varphi$ does not hold: The former always implies the latter (unless $X = \emptyset$, in which case *any* internal statement is true), but the converse does not hold: The former statement means that $U = \emptyset$ is the *only* open subset on which φ holds.

2.1. **Geometric formulas and constructions.** In categorical logic, so-called geometric formulas play a special role, because their meaning is preserved under pullback with geometric morphisms.

Definition 2.8. A formula is *geometric* if and only if it consists only of

$$= \quad \top \quad \bot \quad \wedge \quad \vee \quad \bigvee \quad \exists,$$

but not "$\bigwedge$" nor "$\Rightarrow$" nor "$\forall$" (and thus not "$\neg$" either, since this is defined using "$\Rightarrow$"). A *geometric implication* is a formula of the form

$$\forall \cdots \forall.\ (\cdots) \Rightarrow (\cdots)$$

with the bracketed subformulas being geometric.

We say that a formula φ holds *at a point* $x \in X$ if and only if the formula obtained by substituting all parameters in φ (e. g. sheaves being quantified over) with their stalks at x holds in the usual mathematical sense.

Lemma 2.9. *Let $x \in X$ be a point. Let φ be a geometric formula. Then φ holds at x if and only if there exists a neighbourhood $U \subseteq X$ of x such that φ holds on U.*

Proof. This is a very general instance of the phenomenom that sometimes, truth at a point spreads to truth on a neighbourhood. It can be proven by induction on the structure of φ, but we will give a more conceptual proof later (lemma XXX). □

Corollary 2.10. *A geometric implication holds on X if and only if it holds at every point of X.*

Proof. For notational simplicity, we consider a geometric implication of the form

$$\forall s : \mathcal{F}.\ \varphi(s) \Rightarrow \psi(s).$$

For the "only if" direction, assume that this formula holds on X and let $x \in X$ be an arbitrary point. Let $s_x \in \mathcal{F}_x$ be the germ of an arbitrary local section s of $\mathcal{F}$ and assume that $\varphi(s)$ holds at x. Then by the lemma, it follows that $\varphi(s)$ holds on some open neighbourhood of x. By assumption, $\psi(s)$ holds on this neighbourhood as well. Again by the lemma, $\psi(s)$ holds at x.

For the "if" direction, assume that the geometric implication holds at every point. Let $U \subseteq X$ be an arbitrary open subset and let $s \in \Gamma(U, \mathcal{F})$ be a local section such that $\varphi(s)$ holds on U. By the lemma and the locality of the internal language, to show that $\psi(s)$ holds on U, it suffices to show that $\psi(s)$ holds at every point of U. This is clear, since again by the lemma, $\varphi(s)$ holds at every point of U. □

- remark on how, even though injectivity and surjectivity are notions of an element-based language, the Kripke–Joyal semantics manages to incorporate *all* elements, not only global ones
- remark that since injectivity and surjectivity are geometric implications, monos/epis are stalkwise injective/surjective (and vice versa)
- geometric constructions
- crash course on intuitionistic logic
- first steps: invertibility, nilpotency (needed later)
- somewhere, the external interpretation of power sets has to be explained (needed for instance for completed natural numbers and □-sheaves)

3. Sheaves of rings

3.1. **Reducedness.** Recall that a scheme X is *reduced* if and only if all stalks $\mathcal{O}_{X,x}$ are reduced rings. Since the condition on a ring R to be reduced is a geometric implication,

$$\forall s : R.\ s^2 = 0 \Longrightarrow s = 0,$$

we immediately obtain the following characterization of reducedness in the internal language:

Proposition 3.1. *A scheme X is reduced iff, from the internal point of view, the ring $\mathcal{O}_X$ is reduced.*

3.2. **Locality.** Recall the usual definition of a local ring: a ring possessing exactly one maximal ideal. This is a higher-order condition and in particular not of a geometric form. Therefore, for our purposes, it's better to adopt the following elementary definition of a local ring.

Definition 3.2. A *local ring* is a ring R such that $1 \neq 0$ in R and for all $x, y \in R$

$$x + y \text{ invertible} \quad \Longrightarrow \quad x \text{ invertible} \ \vee\ y \text{ invertible}.$$

In classical logic, it's an easy exercise to show that this definition is equivalent to the usual one. In intuitionistic logic, we would need to be more precise in order to even state the question of equivalence, since intuitionistically, the notion of a maximal ideal bifurcates into several non-equivalent notions.

Proposition 3.3. *In the internal language of a scheme X (or a locally ringed space), the ring $\mathcal{O}_X$ is a local ring.*

Proof. The stated locality condition is a conjunction of two geometric implications (the first one being $1 = 0 \Rightarrow \bot$, the second being the displayed one) and holds on each stalk. □

3.3. **Field properties.** From the internal point of view, the structure sheaf $\mathcal{O}_X$ of a scheme X is *almost* a field, in the sense that any element which is not invertible is nilpotent. This is a genuine property of schemes, not shared with general locally ringed spaces.

Proposition 3.4. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall s \colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible}\urcorner) \Rightarrow \ulcorner s \text{ nilpotent}\urcorner.$$

Proof. By the locality of the internal language and since X can be covered by open affine subsets, it's enough to show that for any affine scheme $X = \operatorname{Spec} A$ and global function $s \in \Gamma(X, \mathcal{O}_X) = A$ it holds that

$$X \models \neg(\ulcorner s \text{ invertible}\urcorner) \quad \text{implies} \quad X \models \ulcorner s \text{ nilpotent}\urcorner.$$

The meaning of the antecedent is that any open subset on which s is invertible is empty. So in particular, the standard open subset $D(s)$ is empty. Therefore s is an element of any prime ideal of A and thus nilpotent. This implies the a priori weaker statement $X \models \ulcorner s \text{ nilpotent}\urcorner$ (which would allow s to have different indices of nilpotency on an open covering). □

Corollary 3.5. *Let X be a scheme. If X is reduced, the ring $\mathcal{O}_X$ is a field from the internal point of view, in the sense that*

$$\mathrm{Sh}(X) \models \forall s \colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible}\urcorner) \Rightarrow s = 0.$$

The converse holds as well.

Proof. We can prove this purely in the internal language: It suffices to give an intuitionistic proof of the fact that a local ring which satisfies the condition of the previous proposition fulfills the stated field condition if and only if it is reduced. This is straightforward. □

This field property is very useful. We will put it to good use when giving a simple proof of the fact that $\mathcal{O}_X$-modules of finite type on a reduced scheme are locally free on a dense open subset (proposition **??**).

The following proposition says that one can deduce a certain unconditional statement from the premise that an element $s \colon \mathcal{O}_X$ is zero under the assumption that some further element $f \colon \mathcal{O}_X$ is invertible.

Proposition 3.6. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall f \colon \mathcal{O}_X.\ \forall s \colon \mathcal{O}_X.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0.$$

Proof. It's enough to show that for any affine scheme $X = \operatorname{Spec} A$ and global functions $f, s \in A$ such that

$$X \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0),$$

it holds that $X \models \bigvee_{n \geq 0} f^n s = 0$. This is obvious, since by assumption such a function s is zero on $D(f)$, i. e. s is zero as an element of $A[f^{-1}]$. □

- Remark that intuitionistically, the notion of a field bifurcates into several inequivalent notions
- discreteness

4. Sheaves of modules

4.1. **Finite type, finite presentation, coherence.** Recall the conditions of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or ringed space) to be of finite type, of finite presentation and to be coherent:

- $\mathcal{F}$ is *of finite type* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence
$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$
of $\mathcal{O}_X|_U$-modules.
- $\mathcal{F}$ is *of finite presentation* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence
$$(\mathcal{O}_X|_U)^m \longrightarrow (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$
- $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type and the kernel of any $\mathcal{O}_X|_U$-linear morphism $(\mathcal{O}_X|_U)^n \to \mathcal{F}|_U$, $U \subseteq X$ any open subset, is of finite type.

The following proposition gives translations of these definitions into the internal language.

Proposition 4.1. *Let X be a scheme (or ringed locale). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then:*

(1) *$\mathcal{F}$ is of finite type if and only if $\mathcal{F}$, considered as an ordinary module from the internal perspective, is finitely generated, i. e. if*
$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{F}.\ x = \sum_i a_i x_i.$$
(2) *$\mathcal{F}$ is of finite presentation if and only if $\mathcal{F}$ is a finitely presented module from the internal perspective, i. e. if ...*
(3) *$\mathcal{F}$ is coherent if and only if ...*

4.2. **Tensor product.** Recall that the tensor product of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X (or ringed space) is usually constructed as the sheafification of the presheaf

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$

From the internal point of view, $\mathcal{F}$ and $\mathcal{G}$ look like ordinary modules, so that we can consider their tensor product as usually constructed in commutative algebra, as a certain quotient of a free module on the elements of $\mathcal{F} \times \mathcal{G}$:

$$\mathcal{O}_X \langle x \otimes y \mid x : \mathcal{F}, y : \mathcal{G} \rangle / R,$$

where R is the submodule generated by

$$\begin{gathered}(x+x')\otimes y - x\otimes y - x'\otimes y,\\ x\otimes(y+y') - x\otimes y - x\otimes y',\\ (sx)\otimes y - s(x\otimes y),\\ x\otimes(sy) - s(x\otimes y)\end{gathered}$$

with $x, x' : \mathcal{F}$, $y, y' : \mathcal{G}$, $s : \mathcal{O}_X$. This internal construction will give rise to the same sheaf of modules as the externally defined tensor product:

Proposition 4.2. *Let X be scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the internally constructed tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ coincides with the external one.*

Proof. Since the proof of the corresponding fact of commutative algebra is intuitionistic, the internally defined tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ fulfills the following universal property: For any $\mathcal{O}_X$-module $\mathcal{H}$, any $\mathcal{O}_X$-bilinear map $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ uniquely factors over the canonical morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Interpreting this property with the Kripke–Joyal semantics, we see that the internally constructed tensor product fulfills the following external property: For any open subset $U \subseteq X$ and any $\mathcal{O}_X|_U$-module $\mathcal{H}$ on U, any $\mathcal{O}_X|_U$-bilinear map $\mathcal{F}|_U \times \mathcal{G}|_U \to \mathcal{H}$ uniquely factors over the canonical morphism $\mathcal{F} \times \mathcal{G} \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U$.

In particular, for $U = X$, this property is well-known to be the universal property satisfied by the externally constructed tensor product. Therefore the claim follows. □

By the internal construction, a description of the stalks of the tensor product follows purely by considering the logical form of the construction:

Corollary 4.3. *Let X be scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the stalks of the tensor product coincide with the tensor products of the stalks: $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.*

Proof. We constructed the tensor product using the following operations: product of two sets, free module on a set, quotient module with respect to a submodule; submodule generated by a set of elements given by a geometric formula. All of these operations are geometric, so the tensor product construction is geometric as well. Hence taking stalks commutes with performing the construction. □

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if and only if all stalks $\mathcal{F}_x$ are flat $\mathcal{O}_{X,x}$-modules. We can characterize flatness in the internal language.

Proposition 4.4. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat if and only if, from the internal perspective, $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Recall that flatness of an A-module M can be characterized without reference to tensor products by the following condition (using suggestive vector notation): For any natural number p, any p-tuple $m : M^p$ of elements of M and any p-tuple $a : A^p$ of elements of A,

$$a^T m = 0 \implies \bigvee_{q \geq 0} \exists n : M^q, B : A^{p \times q}.\ Bn = m \wedge a^T B = 0.$$

This formulation of flatness has the advantage that it is the conjunction of geometric implications; therefore it holds internally if and only if it holds at any stalk. □

- of finite type, of finite presentation, coherent
- basic lemmas: finite type in exact sequences, filtered colimits, flatness, ...
- important hard exercise
- torsion (check Liu p. 174)

5. Upper semicontinuous functions

5.1. **Interlude on natural numbers.** In classical logic, the natural numbers are complete in the sense that any inhabited set of natural numbers possesses a minimal element. This statement can not be proven intuitionistically – intuitively, this is because one cannot explicitly pinpoint the (classically existing) minimal element of an arbitrary inhabited set. In intuitionistic logic, this principle can be salvaged in two essentially different ways: either be strengthening the premise, or by weakening the conclusion.

Lemma 5.1. *Let $U \subseteq \mathbb{N}$ be an inhabited subset of the natural numbers.*

(1) *Assume U to be* detachable*, i. e. assume that for any natural number n, either $n \in U$ or $n \notin U$. Then U possesses a minimal element.*
(2) *In any case, U does* not not *possess a minimal element.*

Proof. (1) By induction on the witness of inhabitation, i. e. the given number n such that $n \in U$. Details omitted, since we will not need this statement.

(2) We give a careful proof since logical subtleties matter. To simplify the exposition, we assume that U is upward-closed, i. e. that any number larger than some element of U lies in U as well. Any subset can be closed in this way (by considering $\{n \in \mathbb{N} \mid \exists m \in U.\ n \geq m\}$) and a minimal element of the closure will be a minimal element for U as well.

We induct on the number $n \in U$ given by the assumption that U is inhabited. In the case $n = 0$ we are done since 0 is a minimal element of U. For the induction step $n \to n + 1$, the weak law of excluded middle gives

$$\neg\neg(n \in U \vee n \notin U).$$

If we can show that $n \in U \vee n \notin U$ implies the conclusion, we're done by XXX. So assume $n \in U \vee n \notin U$. If $n \in U$, then U does not not possess a minimal element by the induction hypothesis. If $n \notin U$, then $n + 1$ is a minimal element (and so, in particular, U does not not possess a minimal element): For if m is any element of U, we have $m \geq n+1$ or $m \leq n$. In the first case, we're done. In the second case, it follows that $n \in U$ because U is upward-closed and so we obtain a contradiction. From this contradiction we can deduce $m \geq n + 1$. □

If we want to work with a complete set of natural numbers in intuitionistic logic, we have to construction a completion.

Definition 5.2. The partially ordered set of *completed natural numbers* is the set $\widehat{\mathbb{N}}$ of all inhabited upward-closed subsets of $\mathbb{N}$, ordered by reverse inclusion.

Lemma 5.3. *The poset of completed natural numbers is the least partially ordered set containing $\mathbb{N}$ and possessing minima of arbitrary inhabited subsets.*

Proof. The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \longmapsto \uparrow(n) := \{m \in \mathbb{N} \mid m \geq n\}.$$

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

The proof of the universal property is left to the reader. $\square$

Remark 5.4. In classical logic, the map $\widehat{\mathbb{N}} \to \mathbb{N}$, $U \mapsto \min U$ is a well-defined isomorphism of partially ordered sets.

5.2. **A geometric interpretation.** We are interested in the completed natural numbers for the following reason: A completed natural number of the topos of sheaves on a topological space X is the same as an upper semicontinuous function $X \to \mathbb{N}$.

Lemma 5.5. *Let X be a topological space. The sheaf $\widehat{\mathbb{N}}$ of completed natural numbers on X is canonically isomorphic to the sheaf of upper semicontinuous $\mathbb{N}$-valued functions on X.*

Proof. When referring to the natural numbers in the internal language, we actually refer to the constant sheaf $\underline{\mathbb{N}}$ on X. (This is because the sheaf $\underline{\mathbb{N}}$ fulfills the axioms of a natural numbers object, cf. [**?**, XXX].) Recall that its sections on an open subset $U \subseteq X$ are continuous functions $U \to \mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology.

Therefore, a section of $\widehat{\mathbb{N}}$ on an open subset $U \subseteq X$ is given by a subsheaf $\mathcal{A} \hookrightarrow \underline{\mathbb{N}}|_U$ such that

$$U \models \exists n : \underline{\mathbb{N}}.\ n \in \mathcal{A} \quad \text{and} \quad U \models \forall n, m : \underline{\mathbb{N}}.\ n \geq m \wedge n \in \mathcal{A} \Rightarrow m \in \mathcal{A}.$$

Since these conditions are geometric, they are satisfied if and only if any stalk $\mathcal{A}_x$ is an inhabited upward-closed subset of $\underline{\mathbb{N}}_x \cong \mathbb{N}$. The association

$$x \in X \longmapsto \min\{n \in \mathbb{N} \mid n \in \mathcal{A}_x\}$$

thus defines a map $X \to \mathbb{N}$. This map is indeed upper semicontinuous, since if $n \in \mathcal{A}_x$, there exists a neighbourhood V of x such that the constant function with value n is an element of $\Gamma(V, \mathcal{A})$ and therefore $n \in \mathcal{A}_y$ for all $y \in V$.

Conversely, let $\alpha : U \to \mathbb{N}$ be a upper semi-continous function. Then

$$V \subseteq X \text{ open} \longmapsto \{f : V \to \mathbb{N} \mid f \text{ continuous, } f \geq \alpha \text{ on } V\}$$

is a subobject of $\underline{\mathbb{N}}|_U$ which internally is inhabited and upward-closed. Further details are left to the reader. $\square$

Under the correspondence given by the lemma, locally *constant* functions map exactly to the (image of the) *ordinary* internal natural numbers (in the completed natural numbers).

Remark 5.6. In a similar vein, the sheaf given by the internal construction of the set of *all* upward-closed subsets of the natural numbers (not only the inhabited ones) is canonically isomorphic to the sheaf of upper semicontinuous functions with values in $\mathbb{N} \cup \{+\infty\}$.

5.3. **The upper semicontinuous rank function.** Recall that the rank of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or locally ringed space) at a point $x \in X$ is defined as the $k(x)$-dimension of the vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. If we assume that $\mathcal{F}$ is of finite type around x, this dimension is finite and equals the minimal number of elements needed to generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module (by Nakayama's lemma).

In the internal language, we can define an element of $\widehat{\mathbb{N}}$ by

$$\operatorname{rank} \mathcal{F} := \min\{n \in \mathbb{N} \mid \ulcorner\text{there is a gen. family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner\} \in \widehat{\mathbb{N}}.$$

If $\mathcal{F}$ is locally finitely free, it will be a finitely free module from the internal point of view and the rank defined in this way will be an actual natural number; but in general, the rank is really an element of the completion.

Proposition 5.7. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type on a scheme X (or locally ringed space). Under the correspondence given by the previous lemma, the internally defined rank maps to the rank function of $\mathcal{F}$.*

Proof. We have to show that for any point $x \in X$ and natural number n, there exists a generating family for $\mathcal{F}_x$ consisting of n elements if and only if there exists a neighbourhood U of x such that

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

The "if" direction is obvious. For the "only if" direction, consider (liftings to local sections of a) generating family $s_1, \ldots, s_n$ of $\mathcal{F}_x$. Since $\mathcal{F}$ is of finite type, there also exist sections $t_1, \ldots, t_m$ on some neighbourhood V of x which generate any stalk $\mathcal{F}_y$, $y \in V$. Since the t_i can be expressed as a linear combination of the s_j in $\mathcal{F}_x$, the same is true on some open neighbourhood $U \subseteq V$ of x. On this neighbourhood, the s_j generate any stalk $\mathcal{F}_y$, $y \in U$, so we have

$$U \models \ulcorner s_1, \ldots, s_n \text{ generate } \mathcal{F}\urcorner. \qquad \square$$

Remark 5.8. Once we understand when properties holding at a stalk spread to a neighbourhood, we will be able to give a simpler proof of the proposition (see XXX).

6. Rational functions and Cartier divisors

6.1. **The sheaf of rational functions.** Recall that the sheaf $\mathcal{K}_X$ of rational functions on a scheme X (or ringed space) can be defined as the sheaf associated to the presheaf

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S})^{-1}],$$

where $\Gamma(U, \mathcal{S})$ is the multiplicative set of those sections of $\mathcal{O}_X$ on U, which are regular in each stalk $\mathcal{O}_{X,x}$, $x \in U$. Recall also there are some wrong definitions in the literature [**?**].

Using the internal language, we can give a simpler definition of $\mathcal{K}_X$. Recall that we can associate to any ring R its total quotient ring, i. e. its localization at the multiplicative subset of regular elements. Since from the internal perspective $\mathcal{O}_X$ is an ordinary ring, we can associate to it its total quotient ring $\mathcal{O}_X[\mathcal{S}^{-1}]$, where $\mathcal{S}$ is internally defined by the formula

$$\mathcal{S} := \{s : \mathcal{O}_X \mid \ulcorner s \text{ is regular}\urcorner\} \subseteq \mathcal{O}_X.$$

Externally, this ring is the sheaf $\mathcal{K}_X$.

Proposition 6.1. *Let X be a scheme (or a ringed space). The sheaf of rings defined in the internal language by localizing $\mathcal{O}_X$ at its set of regular elements is (canonically isomorphic to) the sheaf $\mathcal{K}_X$ of rational functions.*

Proof. Internally, the ring $\mathcal{O}_X[\mathcal{S}^{-1}]$ fulfills the following universal property: For any ring R and any homomorphism $\mathcal{O}_X \to R$ which maps the elements of $\mathcal{S}$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}] \to R$ which makes the evident diagram commute.

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr] \arrow[dr] & & R \\ & \mathcal{O}_X[\mathcal{S}^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

The translation using the Kripke–Joyal semantics gives the following universal property: For any open subset $U \subseteq X$, any sheaf of rings $\mathcal{R}$ on U and any homomorphism $\mathcal{O}_X|_U \to \mathcal{R}$ which maps all elements of $\Gamma(V, \mathcal{S})$, $V \subseteq U$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}]|_U \to \mathcal{R}$ which makes the evident diagram commute. It is well-known [**?**] that the sheaf $\mathcal{K}_X$ as usually defined satisfies this universal property as well. □

Proposition 6.2. *Let X be a scheme (or ringed space). Then the stalks of $\mathcal{K}_X$ are given by*

$$\mathcal{K}_{X,x} = \mathcal{O}_{X,x}[\mathcal{S}_x^{-1}].$$

The elements of $\mathcal{S}_x$ are exactly the germs of those local sections which are regular not only in $\mathcal{O}_{X,x}$, but in all rings $\mathcal{O}_{X,y}$ where y ranges over some neighbourhood of x (depending on the section).

Proof. Since localization is a geometric construction, the first statement is entirely trivial. The second statement follows since

$$\Gamma(U, \mathcal{S}) = \{ s \in \Gamma(U, \mathcal{O}_X) \,|\, U \models \ulcorner s \text{ is regular} \urcorner \}$$

and regularity is a geometric implication, so that $U \models \ulcorner s \text{ is regular} \urcorner$ if and only iff the germ s_y is regular in $\mathcal{O}_{X,y}$ for all $y \in U$. □

6.2. **Regularity of local functions.** It is well known that on a locally Noetherian scheme, regularity spreads from stalks to neighbourhoods, i. e. a section of $\mathcal{O}_X$ is regular in $\mathcal{O}_{X,x}$ if and only if it is regular on some neighbourhood on x. This fact has a simple proof in the internal language:

Proposition 6.3. *Let X be a locally Noetherian scheme. Let $s \in \Gamma(U, \mathcal{O}_X)$ be a local function on X. Let $x \in U$. Then the following statements are equivalent:*

1. *The section s is regular in $\mathcal{O}_{X,x}$.*
2. *The section s is regular in all local rings $\mathcal{O}_{X,y}$ where y ranges over some neighbourhood of x.*

Proof. Let $\Box$ be the modal operator defined by $\Box(\varphi) :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$. By XXX, we are to show that the following statements of the internal language are equivalent:

1. $(\ulcorner s \text{ is regular} \urcorner)^{\Box}$, i. e. $\forall t \colon \mathcal{O}_X.\ st = 0 \Rightarrow \Box(t = 0)$.
2. $\Box(\ulcorner s \text{ is regular} \urcorner)$, i. e. $\Box(\forall t \colon \mathcal{O}_X.\ st = 0 \Rightarrow t = 0)$.

It is clear that the second statement implies the first – in fact, this is true without any assumptions on X: Let $t : \mathcal{O}_X$ be such that $st = 0$. Since we want to prove the boxed statement $\Box(t = 0)$, we may assume that s is regular and prove $t = 0$. This follows by definition.

For the converse direction, consider the annihilator of s, i. e. the ideal

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X.$$

This ideal satisfies the quasicoherence condition (example 9.4), thus I is a quasicoherent submodule of a finitely generated module. Since X is locally Noetherian, it follows that I is finitely generated as well. By assumption, each generator $x_i \in I$ fulfills $\Box(x_i = 0)$. Since we want to prove a boxed statement, we may in fact assume $x_i = 0$. Thus $I = (0)$ and the assertion, that s is regular, follows. $\Box$

Corollary 6.4. *Let X be a locally Noetherian scheme. Then the stalks $\mathcal{K}_{X,x}$ of the sheaf of rational functions are given by the total quotient rings of the local rings $\mathcal{O}_{X,x}$.*

6.3. Geometric interpretation of the sheaf of rational functions.

- on reduced schemes, $\mathcal{K}_X$ is the sheaf of meromorphic functions
- show $\mathcal{K}_X = j_*(\mathcal{O}_X)$?
- internal definition of Cartier divisors
- correspondence between Cartier divisors and sub-$\mathcal{O}_X$-modules of $\mathcal{K}_X$

7. Relative spectrum

- ...

8. Modalities

8.1. Basics on truth values and modal operators.

Definition 8.1. The *set of truth values* Ω is the powerset of the singleton set $1 := \{\star\}$, where $\star$ is a formal symbol.

In classical logic, any subset of $\{\star\}$ is either empty or inhabited, so that Ω contains exactly two elements, the empty set ("false") and $\{\star\}$ ("true"). But in intuitionistic logic, this can not be shown; indeed, if we interpret the definition in the topos of sheaves on a space X, we obtain a sheaf Ω with

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \Omega) = \{V \subseteq U \mid V \text{ open}\}.$$

(This is because by definition of Ω as the power object of the terminal sheaf 1, sections of Ω on an open subset U correspond to subsheaves $\mathcal{F} \hookrightarrow 1|_U$, and those are given by the greatest open subset $V \subseteq U$ such that $\Gamma(V, \mathcal{F})$ is inhabited.)

The *truth value* of a formula φ is by definition the subset $\{x \in 1 \mid \varphi\} \in \Omega$, where "$x$" is a fresh variable not appearing in φ. This subset is inhabited if and only if φ holds and is empty if and only if $\neg\varphi$ holds. Conversely, we can associate to a subset $F \subseteq 1$ the formula $\ulcorner F$ is inhabited$\urcorner$.

Under this correspondence of formulas with truth values, logical operations like $\wedge$ and $\vee$ map to set-theoretic operations like $\cap$ and $\cup$ – for instance, we have

$$\{x \in 1 \mid \varphi\} \cap \{x \in 1 \mid \psi\} = \{x \in 1 \mid \varphi \wedge \psi\}.$$

This justifies a certain abuse of notation: We will sometimes treat elements of Ω as propositions and use logical instead of set-theoretic connectives. In particular, if φ

and ψ are elements of Ω, we will write "$\varphi \Rightarrow \psi$" to mean $\varphi \subseteq \psi$; "$\perp$" to mean $\emptyset$; and "$\top$" to mean 1.

Definition 8.2. A *modal operator* is a map $\Box : \Omega \to \Omega$ such that for all $\varphi, \psi \in \Omega$,

(1) $\varphi \Longrightarrow \Box\varphi$,
(2) $\Box\Box\varphi \Longrightarrow \Box\varphi$,
(3) $\Box(\varphi \wedge \psi) \Longleftrightarrow \Box\varphi \wedge \Box\psi$.

The intuition is that $\Box\varphi$ is a certain weakening of φ, where the precise meaning of "weaker" depends on the modal operator. By the second axiom, weakening twice is the same as weakening once.

In classical logic, where $\Omega = \{\perp, \top\}$, there are only two modal operators: the identity function and the constant function with value $\top$. Both of these are not very interesting: The identity operator does not weaken propositions at all, while the constant operator weakens every proposition to the trivial statement $\top$.

In intuitionistic logic, there can potentially exist further modal operators. For applications to algebraic geometry, the following four operators will have a clear geometric meaning and be of particular importance:

(1) $\Box\varphi :\equiv (\alpha \Rightarrow \varphi)$, where α is a fixed proposition.
(2) $\Box\varphi :\equiv (\varphi \vee \alpha)$, where α is a fixed proposition.
(3) $\Box\varphi :\equiv \neg\neg\varphi$ (the *double negation modality*).
(4) $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition.

Lemma 8.3. *Any modal operator $\Box$ is monotonic, i. e. if $\varphi \Rightarrow \psi$, then $\Box\varphi \Rightarrow \Box\psi$. Furthermore, there holds a modus ponens rule: If $\Box\varphi$ holds, and φ implies $\Box\psi$, then $\Box\psi$ holds as well.*

Proof. Assume $\varphi \Rightarrow \psi$. This is equivalent to supposing $\varphi \wedge \psi \Leftrightarrow \varphi$. We are to show that $\Box\varphi \Rightarrow \Box\psi$, i. e. that $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box\varphi$. The statement follows since by the third axiom on a modal operator, we have $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box(\varphi \wedge \psi)$.

For the second statement, consider that if $\varphi \Rightarrow \Box\psi$, by monotonicity and the second axiom on a modal operator it follows that $\Box\varphi \Rightarrow \Box\Box\psi \Rightarrow \Box\psi$. $\Box$

The modus ponens rule justifies the following proof scheme: In showing that a boxed statement $\Box\psi$ holds given that a further boxed statement $\Box\varphi$ holds, we may assume that indeed φ holds.

8.2. **Geometric meaning.** Let X be a topological space. As discussed above, an open subset $U \subseteq X$ defines an internal truth value (a global section of the sheaf Ω) also denoted by "U" such that

$$V \models U \quad \Longleftrightarrow \quad V \subseteq U$$

for any open subset $V \subseteq X$. (Shortcutting the various intermediate steps, this can also be taken as a definition of "$V \models U$".) If $A \subseteq X$ is a closed subset, there is thus an internal truth value A^c corresponding to the open subset $A^c = X \setminus A$. If $x \in X$ is a point, we define "$!x$" to denote the truth value corresponding to $\operatorname{int}(X \setminus \{x\})$, such that

$$V \models\, !x \quad \Longleftrightarrow \quad V \subseteq \operatorname{int}(X \setminus \{x\}) \quad \Longleftrightarrow \quad x \notin V.$$

Proposition 8.4. *Let $U \subseteq X$ be a fixed open and $A \subseteq X$ be a fixed closed subset. Let $x \in X$. Then, for any open subset $V \subseteq X$, it holds that:*

$$\begin{aligned}
V \models (U \Rightarrow \varphi) \quad &\Longleftrightarrow \quad V \cap U \models \varphi. \\
V \models (\varphi \vee A^c) \quad &\Longleftrightarrow \quad \text{there is an open subset } W \subseteq V \\
&\qquad\quad \text{containing } A \cap V \text{ s. th. } W \models \varphi. \\
V \models \neg\neg\varphi \quad &\Longleftrightarrow \quad \text{there is a dense open subset } W \subseteq V \text{ s. th. } W \models \varphi. \\
V \models ((\varphi \Rightarrow !x) \Rightarrow !x) \quad &\Longleftrightarrow \quad x \notin V \text{ or there is an open neighbourhood } W \subseteq V \\
&\qquad\quad \text{of } x \text{ s. th. } W \models \varphi.
\end{aligned}$$

Proof. (1) Omitted.

(2) Let $V \models \varphi \vee A^c$. Then there exists an open covering $V = \bigcup_i V_i$ such that for each i, $V_i \models \varphi$ or $V_i \subseteq A^c$. Let $W \subseteq V$ be the union of those V_i such that $V_i \models \varphi$. Then $W \models \varphi$ by the locality of the internal language and $A \cap V \subseteq W$.

Conversely, let $W \subseteq V$ be an open subset containing $A \cap V$ such that $W \models \varphi$. Then $V = W \cup (V \cap A^c)$ is an open covering attesting $V \models \varphi \vee A^c$.

(3) For the "only if" direction, let $W \subseteq V$ be the largest open subset on which φ holds, i. e. the union of all open subsets of V on which φ holds. For the "if" direction, we may assume that the given W is also the largest open subset on which φ holds (by enlarging W if necessary). The claim then follows by the following chain of equivalences:

$$\begin{aligned}
&V \models \neg\neg\varphi \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open. } [\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow Z = \emptyset] \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open. } [\forall Z \subseteq Y \text{ open. } Z \subseteq W \Rightarrow Z = \emptyset] \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open. } Y \cap W = \emptyset \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &W \text{ is dense in } V.
\end{aligned}$$

(4) Straightforward, since the interpretation of the internal statement with the Kripke–Joyal semantics is

$$\forall Y \subseteq V \text{ open. } [\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow x \notin Z] \Longrightarrow x \notin Y. \qquad \square$$

Any modal operator $\Box : \Omega \to \Omega$ in the sheaf topos of X induces on global sections a map

$$j : \mathrm{Op}(X) \to \mathrm{Op}(X),$$

where $\mathrm{Op}(X) = \Gamma(X, \Omega)$ is the set of open subsets of X. By the axioms on a modal operator, the map j fulfills similar axioms: For any open subsets $U, V \subseteq X$,

(1) $U \subseteq j(U)$,
(2) $j(j(U)) \subseteq j(U)$,
(3) $j(U \cap V) = j(U) \cap j(V)$.

Such a map is called a *nucleus* on $\mathrm{Op}(X)$. Table 1 lists the nuclei associated to the four modal operators of proposition 8.4.

Modal operator	associated nucleus: $j(V) = \cdots$	$j(V) = X$ iff ...
$\Box\varphi :\equiv (U \Rightarrow \varphi)$	$\mathrm{int}(U^c \cup V)$	$U \subseteq V$
$\Box\varphi :\equiv (\varphi \vee A^c)$	$V \cup A^c$	$A \subseteq V$
$\Box\varphi :\equiv \neg\neg\varphi$	$\mathrm{int}(\mathrm{cl}(V))$	V is dense in X
$\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$	$\mathrm{int}(\mathrm{cl}(V \cap \mathrm{cl}\{x\}) \cup (X \setminus \mathrm{cl}\{x\}))$ $= \begin{cases} X \setminus \mathrm{cl}\{x\}, & \text{if } x \notin V \\ X, & \text{if } x \in V \end{cases}$	$x \in V$

Table 1. List of important modal operators and their associated nuclei (notation as in proposition 8.4).

8.3. **The $\Box$-translation.** There is certain well-known transformation $\varphi \mapsto \varphi^{\neg\neg}$ on formulas, the *double negation translation*, with the following curious property: A formula φ is derivable in classical logic if and only if its translation $\varphi^{\neg\neg}$ is derivable in intuitionistic logic. The translation $\varphi^{\neg\neg}$ is obtained from φ by putting "$\neg\neg$" before any subformula, i. e. before any "$\exists$" and "$\forall$", around any logical connective and around any atomic statement ("$x = y$", "$x \in A$").

We will describe a slight generalization of the double negation translation, the $\Box$-translation for any modal operator $\Box$. It has the following crucial property:

Proposition 8.5. *Let X be a topological space. Let $\Box$ be a modal operator in* $\mathrm{Sh}(X)$*. Let φ be a formula. Then*

$$\mathrm{Sh}(X) \models \varphi^{\Box} \quad \textit{iff} \quad \mathrm{Sh}(X_{\Box}) \models \varphi.$$

Here, "$X_{\Box}$" denotes the subspace (really "sublocale") of X defined by $\Box$; we will explain this notion further down. Of particular importance is the case $\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$ where $x \in X$ is a point, since for this modal operator, the proposition specializes to

$$\mathrm{Sh}(X) \models \varphi^{\Box} \quad \text{iff} \quad \varphi \text{ holds at } x.$$

Thus the question whether truth of a proposition φ at a point x spreads to a neighbourhood of x can be formulated in the following way: *Does $\varphi^{\Box}$ imply $\Box\varphi$?* We can give a general answer to this question.

Definition 8.6. The $\Box$-*translation* is recursively defined as follows.

$$\begin{aligned}
(f = g)^{\Box} &:\equiv \Box(f = g) \\
(x \in A)^{\Box} &:\equiv \Box(x \in A) \\
\top^{\Box} &:\equiv \Box\top \quad (\Leftrightarrow \top) \\
\bot^{\Box} &:\equiv \Box\bot \\
(\varphi \wedge \psi)^{\Box} &:\equiv \Box(\varphi^{\Box} \wedge \psi^{\Box}) & (\textstyle\bigwedge_i \varphi_i)^{\Box} &:\equiv \Box(\textstyle\bigwedge_i \varphi_i^{\Box}) \\
(\varphi \vee \psi)^{\Box} &:\equiv \Box(\varphi^{\Box} \vee \psi^{\Box}) & (\textstyle\bigvee_i \varphi_i)^{\Box} &:\equiv \Box(\textstyle\bigvee_i \varphi_i^{\Box}) \\
(\varphi \Rightarrow \psi)^{\Box} &:\equiv \Box(\varphi^{\Box} \Rightarrow \psi^{\Box}) \\
(\forall x\,{:}\,X.\ \varphi)^{\Box} &:\equiv \Box(\forall x\,{:}\,X.\ \varphi^{\Box}) & (\forall X.\ \varphi)^{\Box} &:\equiv \Box(\forall X.\ \varphi^{\Box}) \\
(\exists x\,{:}\,X.\ \varphi)^{\Box} &:\equiv \Box(\exists x\,{:}\,X.\ \varphi^{\Box}) & (\exists X.\ \varphi)^{\Box} &:\equiv \Box(\exists X.\ \varphi^{\Box})
\end{aligned}$$

Lemma 8.7. (1) *Formulas in the image of the $\Box$-translation are $\Box$-stable, i. e. for any formula φ it holds that $\Box(\varphi^\Box) \Longrightarrow \varphi^\Box$.*
(2) *In the definition of the $\Box$-translation, one may omit the boxes printed in gray.*

Lemma 8.8. *…, then $\Box\varphi \Rightarrow \varphi^\Box$.*

Lemma 8.9. *…, then $\varphi^\Box \Rightarrow \Box\varphi$.*

- general explanation of modalities (as for instance in philosophy)
- introduce the notation $X_\Box$; explain the relationship between $\mathrm{Sh}(X_\Box)$ and $\mathrm{Sh}(X)$ (given by pushforward/pullback resp. forget/sheafify from the internal perspective)
- introduce the $\Box$-translation; give basic lemmas
- explain that $\mathrm{Sh}(X) \models \varphi^\Box$ iff $\mathrm{Sh}(X_\Box) \models \varphi$; specialize to the four most important modal operators ($\neg\neg$, $U \Rightarrow \_\_$, $\_\_ \vee A^c$, $(\_\_ \Rightarrow !x) \Rightarrow !x$)
- explain that for some modal operators, the $\Box$-translation of the law of excluded middle is valid; explain consequences
- spreading of properties from stalk to neighbourhood: give general metatheorem for geometric statements; give many examples
- internal sheafification? To what extent is a description necessary? Contemplate what I want to say about the geometric meaning of $\mathcal{K}_X$

9. Quasicoherent sheaves of modules

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space X is *quasicoherent* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^J \longrightarrow (\mathcal{O}_X|_U)^I \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules, where I and J are arbitrary sets (which may depend on U).

If X is indeed a scheme, quasicoherence can also be characterized in terms of inclusions of distinguished open subsets of affines: An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any open affine subscheme $U = \operatorname{Spec} A$ of X and any function $f \in A$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}), \ \tfrac{s}{f^n} \longmapsto f^{-n} s|_{D(f)}$$

is an isomorphism of $A[f^{-1}]$-modules. Here $D(f) \subseteq U$ denotes the standard open subset $\{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$. Both conditions can be internalized.

Proposition 9.1. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \exists I, J \text{ lc. } \ulcorner\text{there exists an exact sequence } \mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner.$$

The "lc" indicates that when interpreting this internal statement with the Kripke–Joyal semantics, I and J should only be instantiated with locally constant *sheaves.*

Proof. We only sketch the proof. The translation of the internal statement is that there exists a covering of X by open subsets U such that for each such U, there exist sets I, J and an exact sequence

$$(\mathcal{O}_X|_U)^{\underline{J}} \longrightarrow (\mathcal{O}_X|_U)^{\underline{I}} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

where $\underline{I}$ and $\underline{J}$ are the constant sheaves associated to I respectively J. The term "$(\mathcal{O}_X|_U)^{\underline{I}}$" refers to the internally defined free $\mathcal{O}_X$-module with basis the elements of $\underline{I}$. By exploiting that $\underline{I}$ is a discrete set from the internal point of view (i. e. any two elements are either equal or not), one can show that this is the same as $(\mathcal{O}_X|_U)^I$; similarly for J. With this observation, the statement follows. ☐

In practice, the internal condition given by the proposition is not very useful, since at the moment, we do not know of any internal characterization of locally constant sheaves. The internal condition given by the following proposition does not have this defect.

Proposition 9.2. *Let X be scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if, from the internal perspective, the localized module $\mathcal{F}[f^{-1}]$ is a sheaf for the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$ for any $f : \mathcal{O}_X$.*

In detail, the internal condition is that for any $f : \mathcal{O}_X$, it holds that

$$\forall s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow s = 0$$

and for any subsingleton $S \subseteq \mathcal{F}[f^{-1}]$ it holds that

$$(\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner S \text{ inhabited}\urcorner) \Longrightarrow \exists s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in S).$$

Unlike with the internalizations of finite type, finite presentation and coherence, this condition is *not* a standard condition of commutative algebra. In fact, in classical logic, this condition is always satisfied – for trivial logical reasons if f is invertible and because $\mathcal{F}[f^{-1}]$ is the zero module if f is not invertible (since then, it's nilpotent). This is to be expected: *Any* module M in commutative algebra is quasicoherent in the sense that the associated sheaf of modules $M^\sim$ is quasicoherent.

The proof will explain the origin of this condition.

Proof of proposition 9.2. ... ☐

Corollary 9.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subseteq \mathcal{F}$ be a submodule. Then $\mathcal{G}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \forall s : \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{G}) \Longrightarrow \bigvee_{n \geq 0} f^n s \in \mathcal{G}.$$

Proof. We can give a purely internal proof. Let $f : \mathcal{O}_X$. Since subpresheaves of separated sheaves are separated, the module $\mathcal{G}[f^{-1}]$ is in any case separated with respect to the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$.

Now suppose that $\mathcal{G}$ is quasicoherent. Let $f : \mathcal{O}_X$. Let $s : \mathcal{F}$ and assume that if f were invertible, s would be an element of $\mathcal{G}$. Define the subsingleton $S := \{t : \mathcal{G}[f^{-1}] \mid \ulcorner f \text{ inv.}\urcorner \wedge t = s/1\}$. Then S would be inhabited by $s/1$ if f were invertible. Since $\mathcal{G}[f^{-1}]$ is a sheaf, it follows that there exists an element u/f^n of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be the case that $u/f^n = s/1 \in \mathcal{G}[f^{-1}] \subseteq \mathcal{F}[f^{-1}]$. Since $\mathcal{F}[f^{-1}]$ is separated, it follows that it actually holds that $u/f^n = s/1 \in \mathcal{F}[f^{-1}]$. Therefore there exists $m : \mathbb{N}$ such that $f^m f^n s = f^m u \in \mathcal{F}$. Thus $f^{m+n} s$ is an element of $\mathcal{G}$.

For the converse direction, assume that $\mathcal{G}$ fulfills the stated condition. Let $f : \mathcal{O}_X$. Let $S \subseteq \mathcal{G}[f^{-1}]$ be a subsingleton which would be inhabited if f were invertible. By regarding S as a subset of $\mathcal{F}[f^{-1}]$, it follows that there exists an element $u/f^n \in \mathcal{F}[f^{-1}]$ such that, if f were invertible, u/f^n would be an element of S. In particular, u would be an element of $\mathcal{G}$. By assumption it follows that there exists $m : \mathbb{N}$ such that $f^m u \in G$. Thus $(f^m u)/(f^m f^n)$ is an element of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be an element of S. $\square$

Example 9.4. Let X be a scheme and s be a global section of $\mathcal{O}_X$. Then the annihilator of s, i. e. the sheaf of ideals internally defined by the formula

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X$$

is quasicoherent. To prove this in the internal language, it suffices to verify the condition of the proposition. So let $f : \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.} \urcorner \Rightarrow t \in I$, i. e. assume that if f were invertible, st would be zero. By proposition 3.6 it follows that $f^n st = 0$ for some $n : \mathbb{N}$, i. e. that $f^n t \in I$.

- is the condition good enough to show that modules of finite type are quasicoherent? To show that cokernels are quasicoherent?
- discussion meaning of the sheaf condition in external language
- give more examples: $\sqrt{(0)}$, (h), ...
- Noetherian hypotheses: for example, that any quasicoherent submodule of a module of finite type is of finite type as well

10. Unsorted

- "functoriality"
- Kähler differentials
- closed and open subschemes
- reduced closed subscheme
- Koszul resolution
- meta properties, uses (e.g. nilpotent on stalks iff globally nilpotent, some lemmas about limits of modules)
- locally small categories
- big Zariski topos
- open/closed immersions
- morphisms of schemes...
- proper maps...
- limits and colimits...
- related work: Mulvey/Burden, Wraith, Vickers, the Bohr topos crew, Awodey, ...

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