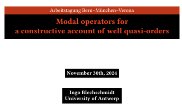
2024-12-01

Modal operators for a constructive account of well quasi-orders



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Def. Let $(X, \leq)$ be a quasi-order.

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Natural numbers

Prop. $(\mathbb{N}, \leq)$ is well.

Proof. Let $\alpha : \mathbb{N} \to \mathbb{N}$. By LEM, there is a minimum $\alpha\, i$. Set $j := i + 1$. $\square$

offensive?

$7, \quad 4, \quad 3, \quad 1, \quad 8, \quad 2, \quad \ldots$

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The presented proof rests on the law of excluded middle and hence cannot immediately be interpreted as a program for finding suitable indices $i < j$. However, constructive proofs are also possible (for instance by induction on the value of a given term of the sequence, see Constructive combinatorics of Dickson's Lemma by Iosif Petrakis for several fine quantitative results). And even more: There is a procedure for regarding this proof—and many others in the theory of well quasi-orders—as *blueprints* for more informative constructive proofs. This shall be our motto for today:

Do not take classical proofs literally, instead ask which constructive proofs they are blueprints for.

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Assuming LEM and DC, …

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The displayed stability results, along with several others, provide a flexible toolbox for constructing new well quasi-orders from given ones. However, with the classical formulation of *well*, renamed "well$_\infty$" on the next slide, these results are inherently classical.

In Higman's lemma, the set X^* of finite lists of elements of X is equipped with the following ordering: We have $x_0 \dots x_{n-1} \leq y_0 \dots y_{m-1}$ iff there is an increasing injection $f : \{0, \dots, n-1\} \to \{0, \dots, m-1\}$ such that $x_i \leq y_{f(i)}$ for all $i < n$.

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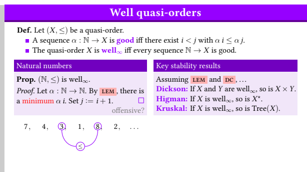

The dependence of the theory on well quasi-orders on classical transfinite methods is already present in one of basic observations of this theory:

Lemma. Let X be well$_\infty$. Let $\alpha : \mathbb{N} \to X$. Then there is an increasing subsequence $\alpha\, i_0 \leq \alpha\, i_1 \leq \ldots$.

Proof. Let $K := \{n \in \mathbb{N} \mid \neg\exists m > n.\, \alpha\, n \leq \alpha\, m\}$ the set of indices of those terms which cannot appear as the first component of a good pair. If K is in bijection with $\mathbb{N}$, there is a subsequence $\alpha\, k_0 \leq \alpha\, k_1 \leq \ldots$ with $k_0, k_1, \ldots \in K$. As X is well$_\infty$, this sequence is good, a contradiction.

Hence K is not in bijection with $\mathbb{N}$. Assuming LEM, it is hence bounded by a number N, and (again with LEM), for every index $a > N$ there is an index $b > a$ such that $\alpha\, a \leq \alpha\, b$. Thus, assuming DC, every number $i_0 > N$ is a suitable starting index for an infinite increasing subsequence. □

The appeal to dependent choice can be removed by always picking the smallest possible next index in $\mathbb{N} \setminus K$, doable by yet another invocation of LEM, but the result remains fundamentally noneffective—in the special case $X = (\{0, 1\}, =)$, the statement of the lemma implies the infinite pigeonhole principle.

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The assertion "$\mathrm{Good} \mid [\,]$" is pronounced "Good bars the empty list", and is defined as follows: Let B be a predicate on $X^\star$. Then $B \mid \sigma$ is inductively generated by the following two clauses.

1. If $B\sigma$, then $B \mid \sigma$.
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Here σx denotes the concatenation of the list σ with the element x. The accompanying induction principle is the following: Let Q be a predicate on X^* such that, for all $\sigma \in X^*$, $B\sigma \Rightarrow Q\sigma$ and $(\forall x \in X.\ Q(\sigma x)) \Rightarrow Q\sigma$. Then, for all $\sigma \in X^*$: If $B \mid \sigma$, then $Q\sigma$.

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With bar induction, well$_{\text{ind}}$ $\Leftarrow$ well$_\infty$.
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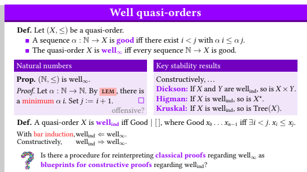
Is there a procedure for reinterpreting **classical proofs** regarding well$_\infty$ as **blueprints for constructive proofs** regarding well$_{\text{ind}}$?



The original notion well$_\infty$:

- ✓ short and simple
- ✓ constructively satisfied for the main examples (but only because of the theory around well$_{\text{ind}}$)
- ✓ concise abstract proofs (albeit employing transfinite methods)
- ✗ main results not constructively attainable
- ✗ philosophically strenuous by the quantification over all sequences
- ✗ not stable under "change of base"—a forcing extension of the universe may well contain more sequences than the base universe
- ✗ negative (universal) condition

The constructive substitute well$_{\text{ind}}$:

- ✓ main results constructive
- ✓ stable under change of base
- ✓ positive (existential) condition
- ✗ proofs intriguing, but also somewhat alien, not just some trivial reshuffling of the classical arguments, classical sequence language cannot be used

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With bar induction, well$_{\text{ind}}$ $\Leftarrow$ well$_\infty$.
Constructively, well$_{\text{ind}}$ $\Rightarrow$ well$_\infty$. Moreover, if X is well$_{\text{ind}}$, then …

- for every *partial* function α, if $\forall n.\, \neg\neg(\alpha\, n\!\downarrow)$, then $\neg\neg\exists i < j.\, \alpha\, i\!\downarrow \wedge\, \alpha\, j\!\downarrow \wedge\, \alpha\, i \leq \alpha\, j$.
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Constructively, the notion well$_{\text{ind}}$ is much stronger than well$_\infty$, as it ensures goodness (in an appropriate sense) of sequence-like entities which are not actually honest maps $\mathbb{N} \to X$.

For partial maps α, by $\alpha\, n \downarrow$ we mean that α is defined on the input n. If LEM is available, then a partial map such that $\neg\neg(\alpha\, n \downarrow)$ for all $n \in \mathbb{N}$ is already a total map, but without LEM the hypothesis well$_\infty$ does not have anything to say about such a partially-defined sequence.

If DC is available, then every multivalued map contains a singlevalued map, but again without DC the hypothesis well$_\infty$ does not have anything to say about multivalued sequences.

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Central insight: A quasi-order X is well$_{\text{ind}}$ iff $\Box \forall \alpha : \mathbb{N} \to X.\, \exists i < j.\, \alpha\, i \leq \alpha\, j$.

It turns out that these entities are, or give rise to, actual maps $\mathbb{N} \to X$—but in a forcing extension of the universe.

Forcing originated in set theory to construct new models for set theory from given ones, in order to explore the range of set-theoretic possibility. For instance, by forcing we can construct models of zfc validating the continuum hypothesis and also models which falsify it.

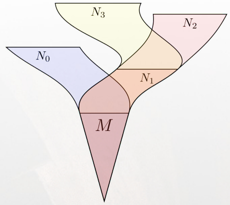
We here refer to a simplification of original forcing which is useful in a constructive metatheory. At its core, every forcing extension is just a formula and proof translation of a certain form. For instance, there is a forcing extension validating LEM even if the base universe does not; this forcing extension is not a deep mystery, for a statement holds in that forcing extension iff its double negation translation holds in the base universe and it is well-known that the double negation translation of LEM is an intuitionistic tautology.

Here is a set of slides on constructive forcing, and Section 4 of this joint paper with Peter Schuster contains a written summary of constructive forcing.

The modal multiverse of constructive forcing

Def. A statement φ holds …

- **everywhere** ($\Box\varphi$) iff it holds in every topos (over the current base).
- **somewhere** ($\Diamond\varphi$) iff it holds in some positive topos.
- **proximally** ($\diamondsuit\varphi$) iff it holds in some positive overt topos.



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By *topos*, we mean *Grothendieck topos.* In constructive forcing, a "forcing extension of the base universe" is exactly the same thing as a Grothendieck topos.

A particular member of the rich and varied landscape of toposes is the *trivial topos*, in which every statement whatsoever holds. By restricting to positive toposes, we exclude this special case.

For positive toposes $\mathcal{E}$, a geometric implication holds in $\mathcal{E}$ iff it holds in the base universe. For positive overt toposes $\mathcal{E}$, we even have that a bounded first-order formula holds in $\mathcal{E}$ iff it holds in the base. Hence, for the purpose of verifying a bounded first-order assertion about the base, we can freely pass to a positiv overt topos with problem-adapted better higher-order properties (such as that some uncountable set from the base now appears countable, or that an infinite sequence whose existence is predicted by failing dependent choice now actually exists).

Here is a rough early draft of a preprint with more details about the modal multiverse.

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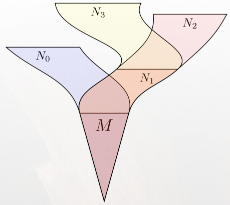


1 A quasiorder is well$_{\text{ind}}$ iff *everywhere*, every sequence is good.

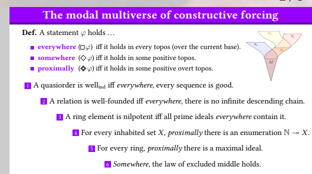
2 A relation is well-founded iff *everywhere*, there is no infinite descending chain.

3 A ring element is nilpotent iff all prime ideals *everywhere* contain it.

4 For every inhabited set X, *proximally* there is an enumeration $\mathbb{N} \twoheadrightarrow X$.

5 For every ring, *proximally* there is a maximal ideal.

6 *Somewhere*, the law of excluded middle holds.


The modal multiverse of constructive forcing
Def. A statement φ holds …
everywhere ($\Box\varphi$) iff it holds in every topos (over the current base).
somewhere ($\Diamond\varphi$) iff it holds in some positive topos.
proximally ($\diamonddiamond\varphi$) iff it holds in some positive overt topos.
1 A quasiorder is well$_{\text{ind}}$ iff everywhere, every sequence is good.
2 A relation is well-founded iff everywhere, there is no infinite descending chain.
3 A ring element is nilpotent iff all prime ideals everywhere contain it.
4 For every inhabited set X, proximally there is an enumeration $\mathbb{N} \twoheadrightarrow X$.
5 For every ring, proximally there is a maximal ideal.
6 Somewhere, the law of excluded middle holds.

The idea to study the modal multiverse of toposes in a principled manner was proposed by Alexander Oldenziel in 2016. *Foreshadowed by:*

- 1984 André Joyal, Miles Tierney. “An extension of the Galois theory of Grothendieck”.
- 1987 Andreas Blass. “Well-ordering and induction in intuitionistic logic and topoi”.
- 2010s Milly Maietti, Steve Vickers. Ongoing work on arithmetic universes.
- 2011 Joel David Hamkins. “The set-theoretic multiverse”.
- 2013 Shawn Henry. “Classifying topoi and preservation of higher order logic by geometric morphisms”.

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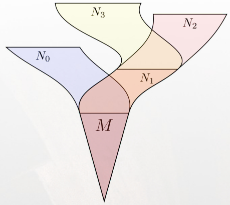

1 A quasiorder is well$_{\text{ind}}$ iff *everywhere*, every sequence is good.

2 A relation is well-founded iff *everywhere*, there is no infinite descending chain.

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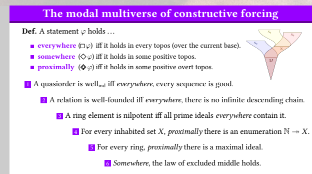

With the modal language we seek to provide an accessible and modular framework for constructivization results.

For instance, conservativity of classical logic over intuitionistic logic for geometric implications (known under various names such as Barr's theorem, Friedman's trick, escaping the continuation monad, ...) is packaged up by the observation that *somewhere*, the law of excluded middle holds.

Another example: In the community around Krull's lemma, it is well-known that we can constructively infer that a given ring element $x \in A$ is nilpotent from knowing that it is contained in the *generic prime ideal* of A. This entity is not actually an honest prime ideal of the ring A in the base universe, but a certain combinatorial notion (efficiently dealt with using *entailment relations*). Constructive forcing allows us to reify the generic prime ideal as an actual prime ideal in a suitable forcing extension.

The modal multiverse of constructive forcing

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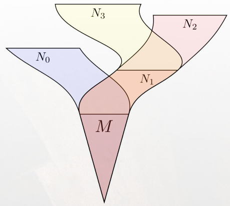

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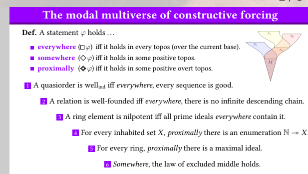

Here is an example how the modal language helps us to give a constructive proof (for well$_{\text{ind}}$) of Dickson's lemma. In a similar vein Higman's lemma and Kruskal's theorem can be proven; with the exception of a preparatory first paragraph the original proofs can be copied word for word.

Prop. Let X and Y be well$_{\text{ind}}$ quasi-orders. Then $X \times Y$ is well.

Lemma. Let $\alpha = (\beta, \gamma) : \mathbb{N} \to X \times Y$ be a sequence in an arbitrary topos. We need to show that α is good. It suffices to prove that *somewhere*, α is good, as goodness is a geometric implication (in fact even a geometric formula). Without loss of generality, we may suppose LEM as it holds *somewhere*.

Hence there is an infinite increasing subsequence

$$\beta\, k_0 \leq \beta\, k_1 \leq \ldots.$$

As Y is well$_{\text{ind}}$, the sequence $\gamma\, k_0, \gamma\, k_1, \ldots$ is good, so there exist $i < j$ with $\gamma\, k_i \leq \gamma\, k_j$. Since we also have $\beta\, k_i \leq \beta\, k_j$, we are done. □

Def. A quasi-order X is **well$_{\mathbf{impl}}$** iff (approximately) for every monotone predicate B,

$$\text{if } B \mid_{\text{incr}} [\,], \text{ then } B \mid [\,].$$

"Assume that no matter how the empty list evolves to an *increasing* list σ, eventually $B\sigma$. Then no matter how the empty list evolves to an *arbitrary* list τ, eventually $B\tau$."

Answering a question by Berardi–Buriola–Schuster

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They recognized that their definition implies well$_{\text{ind}}$ (and hence well$_\infty$), and posed the converse as an open question.

The generating clauses for $B \mid_{\text{incr}} \sigma$, for increasing finite lists σ, are the following:

1. If $B\sigma$, then $B \mid_{\text{incr}} \sigma$.
2. If $B \mid_{\text{incr}} \sigma x$ for all $x \in X$ such that σx is increasing, then $B \mid_{\text{incr}} \sigma$.

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Proof. Trivially Good $\mid_{\text{incr}} [\,]$, hence Good $\mid [\,]$ by assumption. □

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Proof. Let $x_0, x_1, \ldots$ be an infinite sequence (in an arbitrary topos). *Somewhere,* LEM holds and implies that there is an increasing infinite subsequence $x_{i_0}, x_{i_1}, \ldots$. *There* we have, by assumption, a finite prefix with $B\,x_{i_0} \ldots x_{i_n}$. As B is monotone, *there* we also have $B\,x_0 x_1 \ldots x_{i_n}$. So *somewhere* there is a finite prefix validating B. Hence there actually is a finite prefix validating B. □

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The modal language unlocked the displayed positive answer to their question. As the modal framework is fully constructive, the proof can in principle be unrolled to yield a direct (but likely longer) argument not involving constructive forcing, giving an explicit algorithm of type

$$\text{Good} \mid [\,] \times B \mid_{\text{incr}} [\,] \longrightarrow B \mid [\,]$$

for transforming given witnessing trees. This unwinding is work in progress.

```
data _|_ {X : Set} (P : List X → Set) : List X → Set where
  now   : {σ : List X} → P σ → P | σ
  later : {σ : List X} → ((x : X) → P | (x :: σ)) → P | σ

module _ (X : Set) (_≼_ : X → X → Set) where
  Good : List X → Set
  Good σ = ∃[ i ] ∃[ j ] (i < j × lookup σ i ≼ lookup σ j)

  Well-∞ : Set
  Well-∞ = (α : ℕ → X) → ∃[ n ] Good (take n α)

  Well-ind : Set
  Well-ind = Good | []
U:**-  wqo.agda     Bot   L29   <N>   (Agda:Checked +5)

U:%*-  *All Done*   All   L1    <M>   (AgdaInfo)
```

Agda formalization in progress.

Modal operators for a constructive account of well quasi-orders

Agda formalization in progress.