USING THE INTERNAL LANGUAGE OF TOPOSES IN ALGEBRAIC GEOMETRY

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ABSTRACT. There are several important topoi associated to a scheme, for instance the petit and gros Zariski topoi. These come with an internal mathematical language which closely resembles the usual formal language of mathematics, but is "local on the base scheme":

For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a certain condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

These expository notes give an introduction to this topic and show how the internal point of view can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry. No prior knowledge about topos theory and formal logic is assumed.

CONTENTS

1. INTRODUCTION

2. KRIPKE–JOYAL SEMANTICS

Let X be a topological space. Later, X will be the underlying space of a scheme.

Definition 2.1 (Kripke–Joyal semantics of a sheaf topos)**.** The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ is given by the following rules, recursively in the structure of φ:

$$\begin{array}{lll}
U \models f = g : \mathcal{F} & :\Longleftrightarrow & f|_U = g|_U \in \Gamma(U, \mathcal{F}) \\
U \models \varphi \wedge \psi & :\Longleftrightarrow & U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi & :\Longleftrightarrow & \cancel{U \models \varphi \text{ or } U \models \psi} \\
 & & \text{there exists a covering } U = \bigcup_i U_i \text{ s.th. for all } i\text{:} \\
 & & \qquad U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi & :\Longleftrightarrow & \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall f : \mathcal{F}.\ \varphi(f) & :\Longleftrightarrow & \text{for all sections } f \in \Gamma(V, \mathcal{F}), V \subseteq U\text{: } V \models \varphi(f) \\
U \models \exists f : \mathcal{F}.\ \varphi(f) & :\Longleftrightarrow & \cancel{\text{there exists a section } f \in \Gamma(U, \mathcal{F}) \text{ s.th. } U \models \varphi(f)} \\
 & & \text{there exists a covering } U = \bigcup_i U_i \text{ s.th. for all } i\text{:} \\
 & & \qquad \text{there exists } f_i \in \Gamma(U_i, \mathcal{F}) \text{ s.th. } U_i \models \varphi(f_i) \\
U \models \forall \mathcal{F}.\ \varphi(\mathcal{F}) & :\Longleftrightarrow & \text{for all sheaves } \mathcal{F} \text{ on } V,\ V \subseteq U\text{: } V \models \varphi(\mathcal{F}) \\
U \models \exists \mathcal{F}.\ \varphi(\mathcal{F}) & :\Longleftrightarrow & \text{there exists a covering } U = \bigcup_i U_i \text{ s.th. for all } i\text{:} \\
 & & \qquad \text{there exists a sheaf } \mathcal{F}_i \text{ on } U_i \text{ s.th. } U_i \models \varphi(\mathcal{F}_i)
\end{array}$$

Remark 2.2. The last two rules, concerning *unbounded quantification*, are not part of the classical Kripke–Joyal semantics, but instead of Mike Shulman's stack semantics [**?**], a slight extension. They are needed so that we can formulate universal properties in the internal language.

The rules are not all arbitrary. They are finely concerted to make the following propositions true, which are crucial for a proper appreciation of the internal language.

Proposition 2.3 (Locality of the internal language)**.** *Let $U = \bigcup_i U_i$ be covered by open subsets. Let φ be a formula. Then*

$$U \models \varphi \qquad \textit{iff} \qquad U_i \models \varphi \textit{ for each } i.$$

Proof. Induction on the structure of φ. Note that the canceled rules would make this proposition false. □

Proposition 2.4 (Soundness of the internal language)**.** *If a formula φ implies a further formula ψ in intuitionistic logic, then*

$$U \models \varphi \qquad \textit{implies} \qquad U \models \psi.$$

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that on each U_i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in [?, D1.3.1]. □

- geometric formulas
- geometric constructions
- simplification rules
- notation: $\mathrm{Sh}(X) \models \varphi$ means $X \models \varphi$
- remark: do not confuse $\mathrm{Sh}(X) \models \neg\varphi$ with $\mathrm{Sh}(X) \not\models \varphi$
- first steps: invertibility, nilpotency (needed later)

3. Sheaves of rings

3.1. **Reducedness.** Recall that a scheme X is *reduced* if and only if all stalks $\mathcal{O}_{X,x}$ are reduced rings. Since the condition on a ring R to be reduced is a geometric implication,

$$\forall s : R.\ s^2 = 0 \Longrightarrow s = 0,$$

we immediately obtain the following characterization of reducedness in the internal language:

Proposition 3.1. *A scheme X is reduced iff, from the internal point of view, the ring $\mathcal{O}_X$ is reduced.*

3.2. **Locality.** Recall the usual definition of a local ring: a ring possessing exactly one maximal ideal. This is a higher-order condition and in particular not of a geometric form. Therefore, for our purposes, it's better to adopt the following elementary definition of a local ring.

Definition 3.2. A *local ring* is a ring R such that $1 \neq 0$ in R and for all $x, y \in R$

$$x + y \text{ invertible} \quad \Longrightarrow \quad x \text{ invertible } \vee\ y \text{ invertible}.$$

In classical logic, it's an easy exercise to show that this definition is equivalent to the usual one. In intuitionistic logic, we would need to be more precise in order to even state the question of equivalence, since intuitionistically, the notion of a maximal ideal bifurcates into several non-equivalent notions.

Proposition 3.3. *In the internal language of a scheme X (or a locally ringed space), the ring $\mathcal{O}_X$ is a local ring.*

Proof. The stated locality condition is a conjunction of two geometric implications (the first one being $1 = 0 \Rightarrow \bot$, the second being the displayed one) and holds on each stalk. □

3.3. **Field properties.** From the internal point of view, the structure sheaf $\mathcal{O}_X$ of a scheme X is *almost* a field, in the sense that any element which is not invertible is nilpotent. This is a genuine property of schemes, not shared with general locally ringed spaces.

Proposition 3.4. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Rightarrow \ulcorner s \text{ nilpotent} \urcorner.$$

Proof. By the locality of the internal language and since X can be covered by open affine subsets, it's enough to show that for any affine scheme $X = \operatorname{Spec} A$ and global function $s \in \Gamma(X, \mathcal{O}_X) = A$ it holds that

$$X \models \neg(\ulcorner s \text{ invertible} \urcorner) \quad \text{implies} \quad X \models \ulcorner s \text{ nilpotent} \urcorner.$$

The meaning of the antecedent is that any open subset on which s is invertible is empty. So in particular, the standard open subset $D(s)$ is empty. Therefore s is an element of any prime ideal of A and thus nilpotent. This implies the a priori weaker statement $X \models \ulcorner s \text{ nilpotent}\urcorner$ (which would allow s to have different indices of nilpotency on an open covering). □

Corollary 3.5. *Let X be a scheme. If X is reduced, the ring $\mathcal{O}_X$ is a field from the internal point of view, in the sense that*

$$\mathrm{Sh}(X) \models \forall s\!:\!\mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible}\urcorner) \Rightarrow s = 0.$$

The converse holds as well.

Proof. We can prove this purely in the internal language: It suffices to give an intuitionistic proof of the fact that a local ring which satisfies the condition of the previous proposition fulfills the stated field condition if and only if it is reduced. This is straightforward. □

This field property is very useful. We will put it to good use when giving a simple proof of the fact that $\mathcal{O}_X$-modules of finite type on a reduced scheme are locally free on a dense open subset (proposition **??**).

- Remark that intuitionistically, the notion of a field bifurcates into several inequivalent notions
- discreteness

4. Sheaves of modules

- of finite type, of finite presentation, coherent
- basic lemmas
- flatness
- important hard exercise

5. Upper semicontinuous functions

5.1. **Interlude on natural numbers.** In classical logic, the natural numbers are complete in the sense that any inhabited set of natural numbers possesses a minimal element. This statement can not be proven intuitionistically – intuitively, this is because one cannot explicitly pinpoint the (classically existing) minimal element of an arbitrary inhabited set. In intuitionistic logic, this principle can be salvaged in two essentially different ways: either be strengthening the premise, or by weakening the conclusion.

Lemma 5.1. *Let $U \subseteq \mathbb{N}$ be an inhabited subset of the natural numbers.*

(1) *Assume U to be* detachable*, i. e. assume that for any natural number n, either $n \in U$ or $n \notin U$. Then U possesses a minimal element.*
(2) *In any case, U does* not not *possess a minimal element.*

Proof. (1) By induction on the witness of inhabitation, i. e. the given number n such that $n \in U$. Details omitted, since we will not need this statement.
(2) We give a careful proof since logical subtleties matter. To simplify the exposition, we assume that U is upward-closed, i. e. that any number larger than some element of U lies in U as well. Any subset can be closed in this way (by considering $\{n \in \mathbb{N} \mid \exists m \in U.\ n \geq m\}$) and a minimal element of the closure will be a minimal element for U as well.

We induct on the number $n \in U$ given by the assumption that U is inhabited. In the case $n = 0$ we are done since 0 is a minimal element of U. For the induction step $n \to n+1$, the weak law of excluded middle gives

$$\neg\neg(n \in U \vee n \notin U).$$

If we can show that $n \in U \vee n \notin U$ implies the conclusion, we're done by XXX. So assume $n \in U \vee n \notin U$. If $n \in U$, then U does not not possess a minimal element by the induction hypothesis. If $n \notin U$, then $n+1$ is a minimal element (and so, in particular, U does not not possess a minimal element): For if m is any element of U, we have $m \geq n+1$ or $m \leq n$. In the first case, we're done. In the second case, it follows that $n \in U$ because U is upward-closed and so we obtain a contradiction. From this contradiction we can deduce $m \geq n+1$. □

If we want to work with a complete set of natural numbers in intuitionistic logic, we have to construction a completion.

Definition 5.2. The partially ordered set of *completed natural numbers* is the set $\widehat{\mathbb{N}}$ of all inhabited upward-closed subsets of $\mathbb{N}$, ordered by reverse inclusion.

Lemma 5.3. *The poset of completed natural numbers is the least partially ordered set containing* $\mathbb{N}$ *and possessing minima of arbitrary inhabited subsets.*

Proof. The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \longmapsto \uparrow(n) := \{m \in \mathbb{N} \mid m \geq n\}.$$

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

The proof of the universal property is left to the reader. □

Remark 5.4. In classical logic, the map $\widehat{\mathbb{N}} \to \mathbb{N}$, $U \mapsto \min U$ is a well-defined isomorphism of partially ordered sets.

5.2. **A geometric interpretation.** We are interested in the completed natural numbers for the following reason: A completed natural number of the topos of sheaves on a topological space X is the same as an upper semicontinuous function $X \to \mathbb{N}$.

Lemma 5.5. *Let X be a topological space. The sheaf $\widehat{\mathbb{N}}$ of completed natural numbers on X is canonically isomorphic to the sheaf of upper semicontinuous* $\mathbb{N}$*-valued functions on X.*

Proof. When referring to the natural numbers in the internal language, we actually refer to the constant sheaf $\underline{\mathbb{N}}$ on X. (This is because the sheaf $\underline{\mathbb{N}}$ fulfills the axioms of a natural numbers object, cf. [**?**, XXX].) Recall that its sections on an open subset $U \subseteq X$ are continuous functions $U \to \mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology.

Therefore, a section of $\widehat{\mathbb{N}}$ on an open subset $U \subseteq X$ is given by a subsheaf $\mathcal{A} \hookrightarrow \underline{\mathbb{N}}|_U$ such that

$$U \models \exists n : \underline{\mathbb{N}}.\ n \in \mathcal{A} \quad \text{and} \quad U \models \forall n, m : \underline{\mathbb{N}}.\ n \geq m \wedge n \in \mathcal{A} \Rightarrow m \in \mathcal{A}.$$

Since these conditions are geometric, they are satisfied if and only if any stalk $\mathcal{A}_x$ is an inhabited upward-closed subset of $\underline{\mathbb{N}}_x \cong \mathbb{N}$. The association

$$x \in X \longrightarrow \min\{n \in \mathbb{N} \mid n \in \mathcal{A}_x\}$$

thus defines a map $X \to \mathbb{N}$. This map is indeed upper semicontinuous, since if $n \in \mathcal{A}_x$, there exists a neighbourhood V of x such that the constant function with value n is an element of $\Gamma(V, \mathcal{A})$ and therefore $n \in \mathcal{A}_y$ for all $y \in V$.

Conversely, let $\alpha : U \to \mathbb{N}$ be a upper semi-continous function. Then

$$V \subseteq X \longmapsto \{f : V \to \mathbb{N} \mid f \text{ continuous, } f \geq \alpha \text{ on } V\}$$

is a subobject of $\underline{\mathbb{N}}|_U$ which internally is inhabited and upward-closed. Further details are left to the reader. ☐

Under the correspondence given by the lemma, locally *constant* functions map exactly to the (image of the) *ordinary* internal natural numbers (in the completed natural numbers).

Remark 5.6. In a similar vein, the sheaf given by the internal construction of the set of *all* upward-closed subsets of the natural numbers (not necessarily only the inhabited ones) is canonically isomorphic to the sheaf of upper semicontinuous functions with values in $\mathbb{N} \cup \{+\infty\}$.

5.3. **The upper semicontinuous rank function of an $\mathcal{O}_X$-module of finite type.** Recall that the rank of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or locally ringed space) at a point $x \in X$ is defined as the $k(x)$-dimension of the vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. If we assume that $\mathcal{F}$ is of finite type around x, this dimension is finite and equals the minimal number of elements needed to generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module (by Nakayama's lemma).

In the internal language, we can define an element of $\widehat{\mathbb{N}}$ by

$$\operatorname{rank} \mathcal{F} := \min\{n \in \mathbb{N} \mid \ulcorner\text{there is a gen. family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner\} \in \widehat{\mathbb{N}}.$$

If $\mathcal{F}$ is locally finitely free, it will be a finitely free module from the internal point of view and the rank defined in this way will be an actual natural number; but in general, the rank is really an element of the completion.

Proposition 5.7. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type on a scheme X (or locally ringed space). Under the correspondence given by the previous lemma, the internally defined rank maps to the rank function of $\mathcal{F}$.*

Proof. We have to show that for any point $x \in X$ and natural number n, there exists a generating family for $\mathcal{F}_x$ consisting of n elements if and only if there exists a neighbourhood U of x such that

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

The "if" direction is obvious. For the "only if" direction, consider (liftings to local sections of a) generating family $s_1, \ldots, s_n$ of $\mathcal{F}_x$. Since $\mathcal{F}$ is of finite type, there also exist sections $t_1, \ldots, t_m$ on some neighbourhood V of x which generate any stalk $\mathcal{F}_y$, $y \in V$. Since the t_i can be expressed as a linear combination of the s_j in $\mathcal{F}_x$, the same is true on some open neighbourhood $U \subseteq V$ of x. On this neighbourhood, the s_j generate any stalk $\mathcal{F}_y$, $y \in U$, so we have

$$U \models \ulcorner s_1, \ldots, s_n \text{ generate } \mathcal{F}\urcorner.$$ ☐

6. Rational functions and Cartier divisors

- internal definition of K_X
- internal definition of Cartier divisors
- correspondence between Cartier divisors and sub-O_X-modules of K_X

7. Relative spectrum

- ...

8. Modalities

- negneg
- spreading of properties from stalk to neighbourhood
- internal sheafification

9. Unsorted

- "functoriality"
- Kähler differentials
- completion of the natural numbers, rank function
- closed and open subschemes
- reduced closed subscheme
- Koszul resolution
- meta properties, uses (e.g. nilpotent on stalks iff globally nilpotent, some lemmas about limits of modules)
- locally small categories
- big Zariski topos
- open/closed immersions
- morphisms of schemes...
- proper maps...
- limits and colimits...
- related work: Mulvey/Burden, Wraith, Vickers, the Bohr topos crew, Awodey, ...

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