Using the internal language of toposes in algebraic geometry

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Summary

With the internal language of toposes, we can

- express sheaf-theoretic concepts in a simple, element-based language and thus understand them in a more conceptual way,
- mechanically obtain corresponding sheaf-theoretic theorems for any (intuitionistic) theorem of linear and commutative algebra, and
- understand which properties spread from points to neighbourhoods.

What is a topos?

A *topos* is a category which has finite limits, is cartesian closed and has a subobject classifier. More simply, a topos is a category which has similar properties to the category of sets.

Important examples of toposes are the category of sets and the category of sheaves on a topological space.

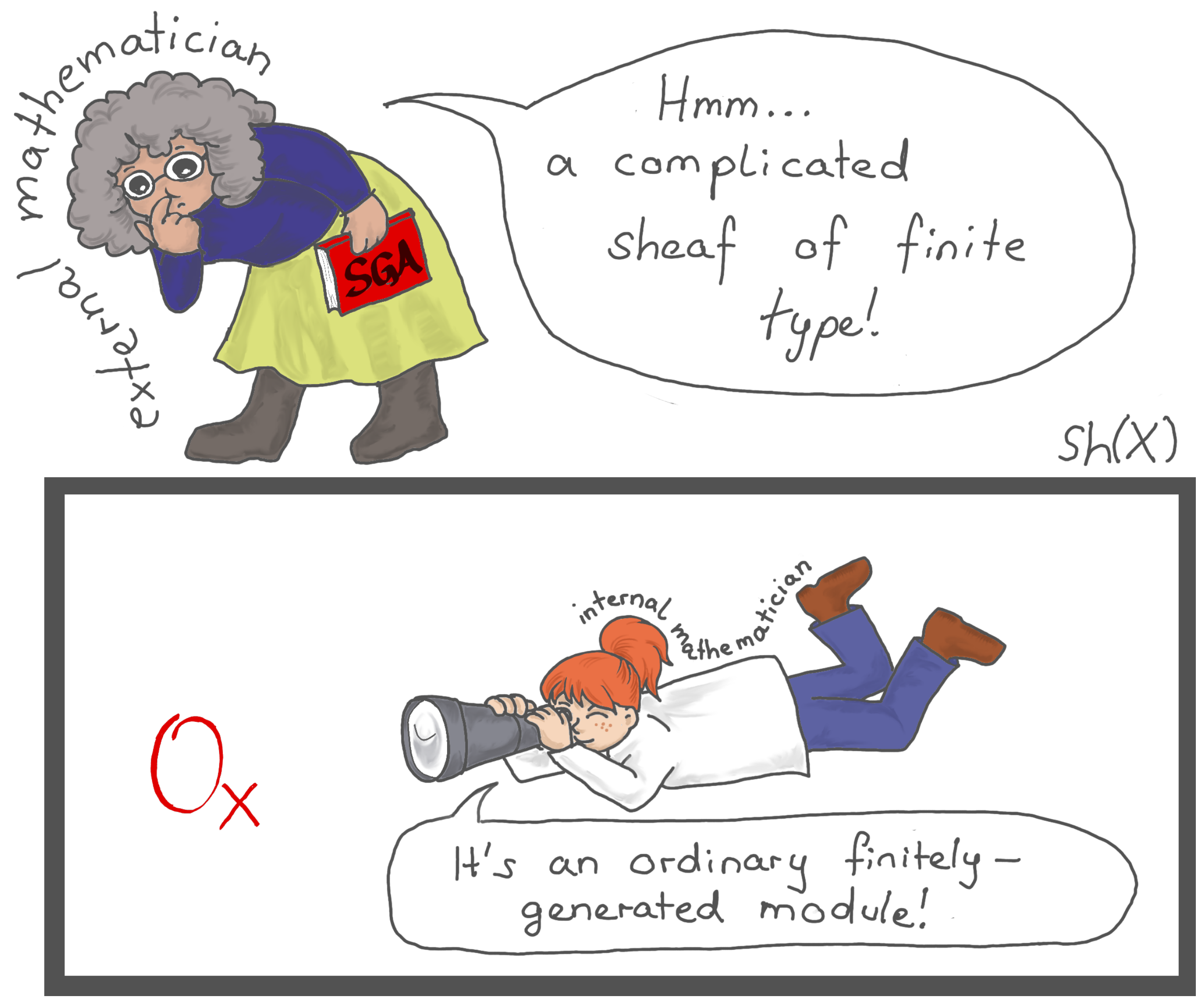
What is the internal language?

The internal language of a topos $\mathcal{E}$ allows us to construct objects and morphisms of the topos, formulate statements about them, and prove such statements in a *naive element-based* language. The translation of internal statements and proofs into external ones is facilitated by an easy mechanical procedure, the *Kripke–Joyal semantics*. *Special case:* The language of the topos of sets is the usual formal mathematical language.

external point of view	internal point of view
objects of $\mathcal{E}$	sets
morphisms of $\mathcal{E}$	maps of sets
monomorphisms in $\mathcal{E}$	surjective maps
epimorphisms in $\mathcal{E}$	injective maps

The small Zariski topos

Let X be a scheme. Let $\mathrm{Sh}(X)$ be the small Zariski topos, i. e. the topos of set-valued sheaves on X. From the point of view of $\mathrm{Sh}(X)$, the structure sheaf $\mathcal{O}_X$ looks like an *ordinary ring* (instead of a sheaf of rings), and sheaves of $\mathcal{O}_X$-modules look like *ordinary modules* on that ring.



Basic example

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. It is well-known that if $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, then $\mathcal{F}$ is as well.

A sheaf is of finite type if and only if, internally, it is a finitely generated module. Therefore the proposition follows *at once* by interpreting the analogous statement of intuitionistic linear algebra in the little Zariski topos: Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

We can thus recognize notions and statements of scheme theory as notions and statements of non-sheafy linear algebra. *Caveat:* Non-intuitionistic proofs by contradiction can not be interpreted with the internal language.

Locally free sheaves

Let X be a reduced scheme. The structure sheaf $\mathcal{O}_X$ looks like a *field* from the internal point of view. Recall that neither the rings of local sections nor the stalks are fields.

Let $\mathcal{F}$ be a finite type sheaf of $\mathcal{O}_X$-modules. Then it is well-known that $\mathcal{F}$ is locally free on a dense open subset of X. (Important hard exercise in Ravi Vakil's notes.)

This follows *at once* from the following easy proposition of intuitionistic linear algebra: Let M be a finitely generated vector space. Then M is *not not* finite free.

Rational functions

The sheaf $\mathcal{K}_X$ of rational functions can internally simply be defined as the total quotient ring of $\mathcal{O}_X$.

Spreading of properties

The following metatheorem covers a wide range of cases: Let φ be a property which can be formulated without using $\Rightarrow, \neg, \forall$. Then φ holds at a point if and only if it holds on some open neighbourhood of the point.

For instance, a sheaf of modules $\mathcal{F}$ is zero if and only if, from the internal perspective, "$\forall x \in \mathcal{F} \colon x = 0$". Because of the "$\forall$", a stalk may be zero without the sheaf being zero on a neighbourhood.

But if $\mathcal{F}$ is of finite type, the condition can be reformulated using generators as "$x_1 = 0 \wedge \cdots \wedge x_n = 0$". The meta-theorem is applicable to this statement and thus a stalk is zero if and only if $\mathcal{F}$ is zero on a neighbourhood.

Dictionary of external vs. internal notions

Expository notes are available at `http://tiny.cc/topos` (work in progress).

Tensor product of sheaves = internal ordinary tensor product, internal Cartier divisors, quasicoherent sheaves = internal ordinary modules satisfying an interesting condition, more metatheorems about spreading of properties, pullback along immersions = internal sheafification, relative spectrum = internal spectrum, scheme dimension = Krull dimension of $\mathcal{O}_X$, dense = not not, further modal operators, other toposes, group schemes = groups, …

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