A CONSTRUCTIVE KNASTER–TARSKI PROOF OF THE UNCOUNTABILITY OF THE REALS

INGO BLECHSCHMIDT AND MATTHIAS HUTZLER

Abstract. We give an uncountability proof of the reals which relies on their order completeness instead of their sequential completeness. We use neither a form of the axiom of choice nor the law of excluded middle, therefore the proof applies to the MacNeille reals in any flavor of constructive mathematics. The proof leans heavily on Levy's unusual proof of the uncountability of the reals.

One way to verify the uncountability of the reals is as follows. We first observe that the usual diagonalization technique shows that the powerset of the naturals is uncountable. We then show that this powerset is in bijection with the reals. In constructive mathematics, more precisely the kind of mathematics which can be carried out in any topos or the kind of mathematics formalizable in Intuitionistic Zermelo–Fraenkel set theory, the first step is still valid, while the second might fail. This failure may occur for any of the several possible flavors of the reals such as the Cauchy reals, the Dedekind reals or the MacNeille reals (which all coincide in classical mathematics, but might differ in constructive mathematics). Therefore a different approach is needed.

This note gives a constructive proof that one of these flavors, the MacNeille reals, is uncountable. To the best of our knowledge, this is the first result in that direction. However, it is just a baby step towards an understanding whether any of the more interesting flavors of the reals can constructively be shown to be uncountable, a problem posed to us by Andrej Bauer who we gratefully acknowledge. The proof presented here is made possible because the MacNeille reals – unlike the Cauchy or Dedekind reals – can constructively be shown to be (conditionally) order complete [1, Lemma D4.7.7].

Sensibilities of constructive mathematics aside, the proof presented here is interesting because it uses only the order completeness of the reals, not their sequential completeness, and because it puts the Knaster–Tarski fixed point theorem to good use. This fixed point theorem is fundamental to theoretical computer science, but appears to be seldomly used in classical analysis. The proof is an adaptation of Levy's unusual proof [2].

Theorem. *Let $f : \mathbb{N} \to \mathbb{R}$ be a map. Then there is a number $x_0 \in \mathbb{R}$ such that for no $n_0 \in \mathbb{N}$, $f(n_0) = x_0$.*

Proof. The map

$$g : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto \sup_M \sum_{n \in M} 2^{-n},$$

where M ranges over all those (Bishop-)finite subsets of $\mathbb{N}$ such that $f[M] < x$, is well-defined (because the sets the suprema are taken of are inhabited by zero and bounded from above by 2), monotone, has a postfixpoint ($0 \leq g(0)$), and has an upper bound for all of its postfixpoints (if $x \leq g(x)$, then $x \leq 2$). By the Knaster–Tarski fixed point theorem, it therefore has a greatest postfixpoint x_0.

If $x_0 = f(n_0)$ for a number $n_0 \in \mathbb{N}$, then for any finite subset M of $\mathbb{N}$ such that $f[M] < x_0$,

$$\sum_{n \in M} 2^{-n} + 2^{-n_0} = \sum_{n \in M \cup \{n_0\}} 2^{-n} \leq g(x_0 + 2^{-n_0}),$$

hence $g(x_0 + 2^{-n_0}) \geq g(x_0) + 2^{-n_0} \geq x_0 + 2^{-n_0}$. Thus $x_0 + 2^{-n_0}$ is a greater postfixpoint than x_0, a contradiction. $\square$

It is possible to unwind the application of the Knaster–Tarski fixed point theorem to obtain an entirely elementary proof of uncountability. This unwinding makes the impredicative nature of the proof manifest.

Second proof. We consider the same map $g : \mathbb{R} \to \mathbb{R}$ as in the first proof. Let x_0 be the supremum of the set $A := \{x \in \mathbb{R} \,|\, x \leq g(x)\}$; this supremum exists because A is inhabited (by zero) and bounded from above (by 2).

If $x_0 = f(n_0)$ for a number $n_0 \in \mathbb{N}$, then $x_0 - 2^{-n_0}$ is an upper bound for A, contradicting the fact that x_0 is the least upper bound of A: Let $x \in A$. If $x > x_0 - 2^{-n_0}$, then $g(x + 2^{-n_0}) \geq g(x) + 2^{-n_0} \geq x + 2^{-n_0}$ (where the first inequality is as in the first proof, exploiting that $x \leq x_0$ by definition of x_0), hence $x + 2^{-n_0} \in A$, thus $x + 2^{-n_0} \leq x_0$, a contradiction. Hence $x \leq x_0 - 2^{-n_0}$. $\square$

References

[1] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Oxford University Press, 2002.

[2] E. Levy. "An unusual proof that the reals are uncountable". arXiv.org:0901.0446 [math.HO]. 2009.

Università di Verona, Department of Computer Science, Strada le Grazie 15, 37134 Verona, Italy

E-mail address: ingo.blechschmidt@univr.it

Universität Augsburg, Institut für Mathematik, Universitätsstr. 14, 86159 Augsburg, Germany

E-mail address: matthias.ralph.hutzler@student.uni-augsburg.de