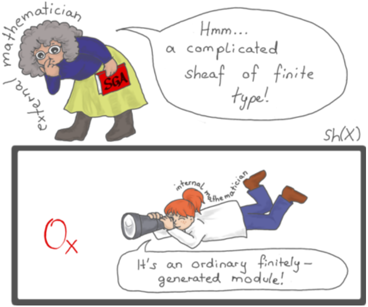


Using the internal language of toposes in algebraic geometry

Ingo Blechschmidt
University of Augsburg

Special PSSL 99
in honor of Jirka Adámek's retirement

March 13th, 2016

Outline

1. Basics
   - What is a scheme?
   - What is a topos?
   - What is the internal language?
2. Building and using a dictionary
3. Quasicoherence of sheaves of modules
4. Spreading of properties
5. The relative and internal spectrum

What is a scheme?

- A **manifold** is a space which is **locally isomorphic** to some open subset of some $\mathbb{R}^n$.
- A **scheme** is a space which is **locally isomorphic** to the **spectrum of some (commutative) ring**:

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

- By **space** we mean: topological space X equipped with a local sheaf $\mathcal{O}_X$ of rings.

What is a topos?

Formal definition

A **topos** is a category which has finite limits, is cartesian closed and has a subobject classifier.

Motto

A topos is a category sufficiently rich to support an **internal language**.

Examples

- Set: category of sets
- $\mathrm{Sh}(X)$: category of set-valued sheaves on a space X

What is the internal language?

Let $\mathcal{E}$ be a topos. Then we can define the meaning of

$$\mathcal{E} \models \varphi \qquad (\text{“}\varphi \text{ holds in } \mathcal{E}\text{”})$$

for formulas φ over $\mathcal{E}$ using the **Kripke–Joyal semantics**.

externally	internally to $\mathcal{E}$
object of $\mathcal{E}$	set/type
morphism in $\mathcal{E}$	map of sets
monomorphism	injective map
epimorphism	surjective map
ring object	ring
module object	module

If φ implies ψ **intuitionistically**, then $\mathcal{E} \models \varphi$ implies $\mathcal{E} \models \psi$.

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf of sets	set/type
morphism of sheaves	map of sets
monomorphism	injective map
epimorphism	surjective map
sheaf of rings	ring
sheaf of modules	module
sheaf of finite type	finitely generated module
finite locally free sheaf	finite free module
coherent sheaf	coherent module
tensor product of sheaves	tensor product of modules
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

Building a dictionary

Understand notions of algebraic geometry as notions of algebra internal to $\mathrm{Sh}(X)$.

externally	internally to $\mathrm{Sh}(X)$
sheaf o	
morph	
monon	
epimo	
sheaf o	
sheaf o	
sheaf o	
finite l	
coherer	
tensor	s
rank function	minimal number of generators
sheaf of rational functions	total quotient ring of $\mathcal{O}_X$

MISCONCEPTIONS ABOUT K_X

by Steven L. Kleiman

There are three common misconceptions about the sheaf K_X of meromorphic functions on a ringed space X: (1) that K_X can be defined as the sheaf associated to the presheaf of total fraction rings,

(*) $$U \mapsto \Gamma(U, O_X)_{tot},$$

see [EGA IV$_4$, 20.1.3, p. 227] and [1, (3.2), p. 137]; (2) that the stalks $K_{X,x}$ are equal to the total fraction rings $(O_{X,x})_{tot}$, see [EGA IV$_4$, 20.1.1 and 20.1.3, pp. 226-7]; and (3) that if X is a scheme and $U = \mathrm{Spec}\,(A)$ is

Using the dictionary

Let X be a scheme. Employ its **small Zariski topos**: $\mathrm{Sh}(X)$.

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.

$$\Downarrow$$

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

Using the dictionary

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

Any sheaf of modules of finite type on a reduced scheme is locally free *on a dense open subset*.

Ravi Vakil: “Important hard exercise” (13.7.K).

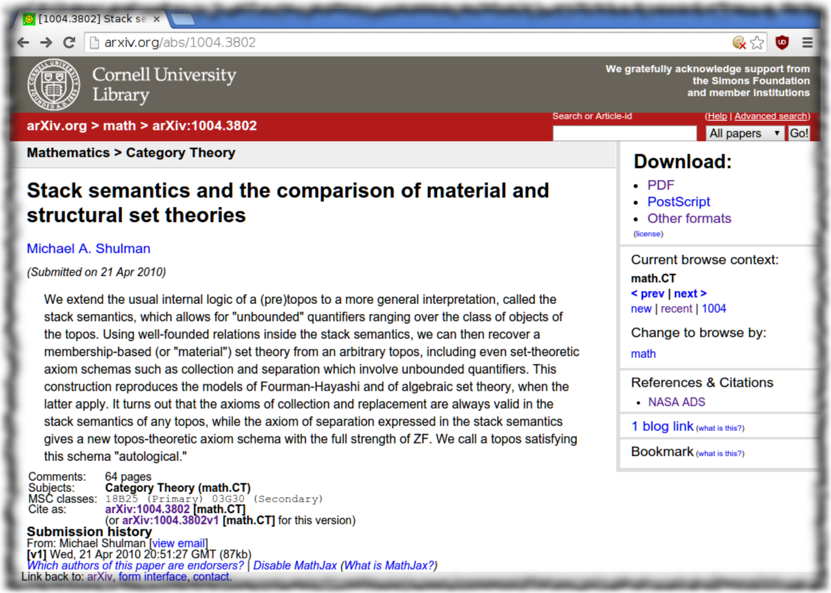
The objective

Understand notions and statements of **algebraic geometry** as notions and statements of (intuitionistic) **commutative algebra** internal to suitable **toposes**.

Further examples:

- Characterizing quasicoherence internally
- Understanding spreading of properties in a logical way
- Constructing the relative spectrum internally

Praise for Mike Shulman



[1004.3802] Stack se ×

arxiv.org/abs/1004.3802

Cornell University Library

We gratefully acknowledge support from the Simons Foundation and member institutions

arXiv.org > math > arXiv:1004.3802

Search or Article-id (Help | Advanced search)

All papers Go!

Mathematics > Category Theory

Stack semantics and the comparison of material and structural set theories

Michael A. Shulman

(Submitted on 21 Apr 2010)

We extend the usual internal logic of a (pre)topos to a more general interpretation, called the stack semantics, which allows for "unbounded" quantifiers ranging over the class of objects of the topos. Using well-founded relations inside the stack semantics, we can then recover a membership-based (or "material") set theory from an arbitrary topos, including even set-theoretic axiom schemas such as collection and separation which involve unbounded quantifiers. This construction reproduces the models of Fourman-Hayashi and of algebraic set theory, when the latter apply. It turns out that the axioms of collection and replacement are always valid in the stack semantics of any topos, while the axiom of separation expressed in the stack semantics gives a new topos-theoretic axiom schema with the full strength of ZF. We call a topos satisfying this schema "autological."

Comments: 64 pages
Subjects: Category Theory (math.CT)
MSC classes: 18B25 (Primary) 03G30 (Secondary)
Cite as: arXiv:1004.3802 [math.CT]
(or arXiv:1004.3802v1 [math.CT] for this version)

Submission history
From: Michael Shulman [view email]
[v1] Wed, 21 Apr 2010 20:51:27 GMT (87kb)

Which authors of this paper are endorsers? | Disable MathJax (What is MathJax?)
Link back to: arXiv, form interface, contact.

Download:

- PDF
- PostScript
- Other formats

(license)

Current browse context:
math.CT
< prev | next >
new | recent | 1004

Change to browse by:
math

References & Citations
- NASA ADS

1 blog link (what is this?)

Bookmark (what is this?)

A curious property

Let X be a scheme. Internally to $\mathrm{Sh}(X)$,

any non-invertible element of $\mathcal{O}_X$ is nilpotent.

ON THE SPECTRUM OF A RINGED TOPOS **209**

For completeness, two further remarks should be added to this treatment of the spectrum. One is that in **E** the canonical map $A \to \Gamma_*(LA)$ is an isomorphism—i.e., the representation of A in the ring of "global sections" of LA is complete. The second, due to Mulvey in the case $\mathbf{E} = \mathbf{S}$, is that in Spec(**E**, A) the formula

$$\neg(x \in U(LA)) \Rightarrow \exists n(x^n = 0)$$

is valid. This is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas. In any case, calculations such as these are easier from the point of view of the Heyting algebra of radical ideals of A, and hence will be omitted here.

Miles Tierney. On the spectrum of a ringed topos. 1976.

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Diamond_f\text{-sheaf for any } f : \mathcal{O}_X,$$
$$\text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

Quasicoherence

Let X be a scheme. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module.

Then $\mathcal{E}$ is quasicoherent if and only if, internally to $\mathrm{Sh}(X)$,

$$\mathcal{E}[f^{-1}] \text{ is a } \Diamond_f\text{-sheaf for any } f : \mathcal{O}_X,$$
$$\text{where } \Diamond_f \varphi :\equiv (f \text{ invertible} \Rightarrow \varphi).$$

In particular: If $\mathcal{E}$ is quasicoherent, then internally

$$(f \text{ invertible} \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0$$

for any $f : \mathcal{O}_X$ and $s : \mathcal{E}$.

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator $\Diamond$. Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond,$$

$\Diamond : \Omega_\mathcal{E} \to \Omega_\mathcal{E}$

where the translation $\varphi \mapsto \varphi^\Diamond$ is given by:

$$\begin{aligned}
(s = t)^\Diamond &:\equiv \Diamond(s = t) \\
(\varphi \wedge \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \wedge \psi^\Diamond) \\
(\varphi \vee \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \vee \psi^\Diamond) \\
(\varphi \Rightarrow \psi)^\Diamond &:\equiv \Diamond(\varphi^\Diamond \Rightarrow \psi^\Diamond) \\
(\forall x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\forall x : X.\, \varphi^\Diamond(x)) \\
(\exists x : X.\, \varphi(x))^\Diamond &:\equiv \Diamond(\exists x : X.\, \varphi^\Diamond(x))
\end{aligned}$$

The $\Diamond$-translation

Let $\mathcal{E}_\Diamond \hookrightarrow \mathcal{E}$ be a subtopos given by a local operator $\Diamond$.
Then

$$\mathcal{E}_\Diamond \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \varphi^\Diamond.$$

$\Diamond : \Omega_\mathcal{E} \to \Omega_\mathcal{E}$

Let X be a scheme. Depending on $\Diamond$, $\mathrm{Sh}(X) \models \Diamond\varphi$ means that φ holds on …

- … a dense open subset.
- … a schematically dense open subset.
- … a given open subset U.
- … an open subset containing a given closed subset A.
- … an open neighbourhood of a given point $x \in X$.

Can tackle the question "$\varphi^\Diamond \stackrel{?}{\Rightarrow} \Diamond\varphi$" logically.

The absolute spectrum

Let A be a commutative ring (in Set).

Is there a **free local ring** $A \to A'$ over A?

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{R} \\
& \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

No, if we restrict to Set.

Yes, if we allow a change of topos: Then $A \to \mathcal{O}_{\operatorname{Spec} A}$ is the universal localization.

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \text{topological space of the prime ideals of } A$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\operatorname{Spec} A := \cancel{\text{topological space of the prime ideals of } A}$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned}\operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \text{topological space of the prime filters of } A\end{aligned}$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned} \operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \cancel{\text{topological space of the prime filters of } A} \end{aligned}$$

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned}\operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \cancel{\text{topological space of the prime filters of } A} \\ &:= \text{locale of the prime filters of } A\end{aligned}$$

Define the frame of opens of $\operatorname{Spec} A$ to be the frame of radical ideals in A.

The absolute spectrum, internalized

Let A be a commutative ring in a topos $\mathcal{E}$.

To construct the **free local ring** over A, give a constructive account of the spectrum:

$$\begin{aligned}\operatorname{Spec} A &:= \cancel{\text{topological space of the prime ideals of } A} \\ &:= \cancel{\text{topological space of the prime filters of } A} \\ &:= \text{locale of the prime filters of } A\end{aligned}$$

Define the frame of opens of $\operatorname{Spec} A$ to be the frame of radical ideals in A.

This gives an internal description of Monique Hakim's spectrum functor $\mathrm{RT} \to \mathrm{LRT}$.

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe $\underline{\mathbf{Spec}}_X\,\boldsymbol{\mathcal{A}}$, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\,\mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe **$\underline{\mathrm{Spec}}_X\, \mathcal{A}$**, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\, \mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X\, \mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

The relative spectrum

Let X be a scheme and $\mathcal{O}_X \xrightarrow{\varphi} \mathcal{A}$ be a quasicoherent algebra. Can we describe $\underline{\mathbf{Spec}}_X\,\boldsymbol{\mathcal{A}}$, a scheme over X, internally?

Desired universal property:

$$\mathrm{Hom}_{\mathrm{Sch}/X}(T, \underline{\mathrm{Spec}}_X\,\mathcal{A}) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_T)$$

for all X-schemes $T \xrightarrow{\mu} X$.

Solution: Define internally the frame of $\underline{\mathrm{Spec}}_X\,\mathcal{A}$ to be the frame of those radical ideals $I \subseteq \mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \forall s : \mathcal{A}.\, (f \text{ invertible in } \mathcal{O}_X \Rightarrow s \in I) \Longrightarrow fs \in I.$$

Its **points** are those prime filters G of $\mathcal{A}$ such that

$$\forall f : \mathcal{O}_X.\, \varphi(f) \in G \Longrightarrow f \text{ invertible in } \mathcal{O}_X.$$

The relative spectrum, reformulated

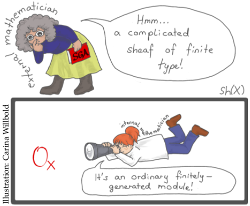
Let $B \to A$ be an algebra in topos.

Is there a **free local and local-over-*B* ring** $A \to A'$ over A?

$$\begin{tikzcd}
B \arrow[r] \arrow[rr, bend left, "\text{local}"] \arrow[dr, bend right, "\text{local}"'] & A \arrow[r] \arrow[d] & \overset{\text{local}}{R} \\
& \underset{\text{local}}{A'} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

Form limits in the category of **locally ringed locales** by **relocalizing** the corresponding limit in ringed locales.

Understand notions and statements of algebraic geometry as notions and statements of algebra internal to appropriate toposes.



- Simplify proofs and gain conceptual understanding.
- Understand relative geometry as absolute geometry.
- Develop a synthetic account of scheme theory.
- Contribute to constructive algebra.

http://tiny.cc/topos-notes

spreading of properties, general transfer principles, applications to constructive algebra, quasicoherence, internal Cartier divisors, pullback along immersions = internal sheafification, scheme dimension = internal Krull dimension of $\mathcal{O}_X$, dense = not not, modal operators, relative spectrum, other toposes, étale topology, group schemes = groups, ...

You should totally look up:

The Adventures of Sheafification Man

More on the internal language

More generally, for an object U of a topos $\mathcal{E}$, we define the meaning of

$$U \models \varphi \qquad (\varphi \text{ holds on } U).$$

Writing "$\mathcal{E} \models \varphi$" is then an abbreviation for "$1 \models \varphi$", where "1" denotes the terminal object of $\mathcal{E}$.

In addition to soundness with respect to intuitionistic logic, the internal language has the following two important properties:

- **Monotonicity:** If $p : V \to U$ is an arbitrary morphism and $U \models \varphi$, then also $V \models \varphi$.
- **Locality:** If $p : V \to U$ is an epimorphism and $V \models \varphi$, then also $U \models \varphi$.

The rules of the Kripke–Joyal semantics

In the special case that $\mathcal{E} = \mathrm{Sh}(X)$ is the topos of sheaves on a topological space (or locale) X, the rules of the Kripke–Joyal semantics look as follows. We tersely write "$U \models \varphi$" instead of "$\mathrm{Hom}(\_, U) \models \varphi$ for open subsets $U \subseteq X$.

$$
\begin{array}{lcl}
U \models f = g : \mathcal{F} & :\Longleftrightarrow & f|_U = g|_U \in \mathcal{F}(U) \\
U \models \varphi \wedge \psi & :\Longleftrightarrow & U \models \varphi \text{ and } U \models \psi \\
U \models \varphi \vee \psi & :\Longleftrightarrow & \cancel{U \models \varphi \text{ or } U \models \psi} \\
 & & \text{there exists a covering } U = \bigcup_i U_i \text{ s. th. for all } i\text{:} \\
 & & \qquad U_i \models \varphi \text{ or } U_i \models \psi \\
U \models \varphi \Rightarrow \psi & :\Longleftrightarrow & \text{for all open } V \subseteq U\text{: } V \models \varphi \text{ implies } V \models \psi \\
U \models \forall f : \mathcal{F}.\, \varphi(f) & :\Longleftrightarrow & \text{for all sections } f \in \mathcal{F}(V), V \subseteq U\text{: } V \models \varphi(f) \\
U \models \exists f : \mathcal{F}.\, \varphi(f) & :\Longleftrightarrow & \text{there exists a covering } U = \bigcup_i U_i \text{ s. th. for all } i\text{:} \\
 & & \qquad \text{there exists } f_i \in \mathcal{F}(U_i) \text{ s. th. } U_i \models \varphi(f_i)
\end{array}
$$

Translating internal statements I

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\mathrm{Sh}(X) \models \ulcorner \alpha \text{ is injective} \urcorner$$

$$\iff \mathrm{Sh}(X) \models \forall s\colon \mathcal{F}.\, \forall t\colon \mathcal{F}.\, \alpha(s) = \alpha(t) \Rightarrow s = t$$

$\iff$ for all open $U \subseteq X$, sections $s \in \mathcal{F}(U)$:
for all open $V \subseteq U$, sections $t \in \mathcal{F}(V)$:
for all open $W \subseteq V$:
$\alpha_W(s|_W) = \alpha_W(t|_W)$ implies $s|_W = t|_W$

$\iff$ for all open $U \subseteq X$, sections $s, t \in \mathcal{F}(U)$:
$\alpha_U(s|_U) = \alpha_U(t|_U)$ implies $s|_U = t|_U$

$\iff$ α is a monomorphism of sheaves

Translating internal statements II

Let X be a topological space (or locale) and let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$\mathrm{Sh}(X) \models \ulcorner \alpha \text{ is surjective} \urcorner$$

$$\iff \mathrm{Sh}(X) \models \forall t\,{:}\,\mathcal{G}.\, \exists s\,{:}\,\mathcal{F}.\, \alpha(s) = t$$

$\iff$ for all open $U \subseteq X$, sections $t \in \mathcal{G}(U)$:

there exists an open covering $U = \bigcup_i U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that:

$$\alpha|_{U_i}(s_i) = t|_{U_i}$$

$\iff$ α is an epimorphism of sheaves

Translating internal statements III

Let X be a topological space (or locale) and let $s, t \in \mathcal{F}(X)$ be global sections of a sheaf $\mathcal{F}$ on X. Then:

$$\begin{aligned}
& \mathrm{Sh}(X) \models \neg\neg(s = t) \\
\iff & \mathrm{Sh}(X) \models ((s = t) \Rightarrow \bot) \Rightarrow \bot \\
\iff & \text{for all open } U \subseteq X \text{ such that} \\
& \qquad \text{for all open } V \subseteq U \text{ such that} \\
& \qquad\qquad s|_V = t|_V, \\
& \qquad \text{it holds that } V = \emptyset, \\
& \text{it holds that } U = \emptyset \\
\iff & \text{there exists a dense open set } W \subseteq X \text{ such that } s|_W = t|_W
\end{aligned}$$

Spreading from points to neighbourhoods

All of the following lemmas have a short, sometimes trivial proof. Let $\mathcal{F}$ be a sheaf of finite type on a ringed space X. Let $x \in X$. Let $A \subseteq X$ be a closed subset. Then:

1. $\mathcal{F}_x = 0$ iff $\mathcal{F}|_U = 0$ for some open neighbourhood of x.
2. $\mathcal{F}|_A = 0$ iff $\mathcal{F}|_U = 0$ for some open set containing A.
3. $\mathcal{F}_x$ can be generated by n elements iff this is true on some open neighbourhood of x.
4. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.
5. $\mathcal{F}$ is torsion iff $\mathcal{F}_\xi$ vanishes (assume X integral and $\mathcal{F}$ quasicoherent).
6. $\mathcal{F}$ is torsion iff $\mathcal{F}|_{\mathrm{Ass}(\mathcal{O}_X)}$ vanishes (assume X locally Noetherian and $\mathcal{F}$ quasicoherent).

The smallest dense sublocale

Let X be a reduced scheme satisfying a technical condition. Let $i : X_{\neg\neg} \to X$ be the inclusion of the smallest dense sublocale of X.

Then $i_* i^{-1} \mathcal{O}_X \cong \mathcal{K}_X$.

- This is a highbrow way of saying "rational functions are regular functions which are defined on a dense open subset".
- Another reformulation is that $\mathcal{K}_X$ is the sheafification of $\mathcal{O}_X$ with respect to the $\neg\neg$-modality.
- There is a generalization to nonreduced schemes.

Transfer principles

Let M be an A-module. How do M and the sheaf $M^\sim$ on $\operatorname{Spec} A$ relate?

Observe that $M^\sim \cong \underline{M}[\mathcal{F}^{-1}]$ is the localization of M at the **generic prime filter** and that M shares all first-order properties with the constant sheaf of modules $\underline{M}$. Therefore:

$M^\sim$ inherits all those properties of M which are **stable under localization**.

Examples: finitely generated, free, flat, …

A converse holds as well, suitably formulated.

Applications in algebra

Let A be a commutative ring. The internal language of $\mathrm{Sh}(\operatorname{Spec} A)$ allows you to say "without loss of generality, we may assume that A is local", even constructively.

The kernel of any matrix over a principial ideal domain is finitely generated.

$$\Downarrow$$

The kernel of any matrix over a Prüfer domain is finitely generated.

Hilbert's program in algebra

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides, for instance exhibiting explicit witnesses. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all prime filters") by "for the generic prime filter".

More specifically, simply interpret a given proof using prime filters in $\mathrm{Sh}(\operatorname{Spec} A)$ and let it refer to $\mathcal{F} \hookrightarrow \underline{A}$.

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

This is related (in a few cases equivalent) to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others. Their approach is more versatile.

The big Zariski topos

Let X be a scheme. The **big Zariski topos** is the topos of sheaves on Sch/X with respect to the Zariski topology. From its point of view, …

- … X-schemes look just like sets,
- … $\mathbb{P}^n_X$ is given by the naive expression

$$\{(x_0,\ldots,x_n) \,|\, x_1 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\text{rescaling}),$$

- … the cotangent “bundle” of an X-scheme T is

$$\text{the set of maps } \Delta \to \underline{T},$$

  where $\Delta = \{\varepsilon \in \underline{\mathbb{A}}^1_X \,|\, \varepsilon^2 = 0\}$.
- … affinity is a “double dual condition”, and
- … the étale topology is the coarsest topology $\Diamond$ s. th.

$$\forall f : \underline{\mathbb{A}}^1_X[T].\, f \text{ is monic separable} \Rightarrow \Diamond(\exists t : \underline{\mathbb{A}}^1.\, f(t) = 0).$$