University of Augsburg

Doctoral Thesis

Using the internal language of toposes in algebraic geometry

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Contents

PART I

Basics

1. Introduction

Internal language of toposes. A *topos* is a category which shares certain categorical properties with the category of sets; the archetypical example is the category of sets, and the most important example for the purposes of this thesis is the category of set-valued sheaves on a topological space.

Any topos $\mathcal{E}$ supports an *internal language*. This is a device which allows one to *pretend* that the objects of $\mathcal{E}$ are plain sets and that the morphisms are plain maps between sets, even if in fact they are not. For instance, consider a morphism $\alpha : X \to Y$ in $\mathcal{E}$. From the *internal point of view*, this looks like a map between sets, and we can formulate the condition that this map is surjective; we write this as

$$\mathcal{E} \models \forall y\!:\!Y.\ \exists x\!:\!X.\ \alpha(x) = y.$$

The appearance of the colons instead of the usual element signs reminds us that this expression is not to be taken literally – X and Y are objects of $\mathcal{E}$ and thus not necessarily sets. The definition of the internal language is made in such a way so that the meaning of this internal statement is that α is an epimorphism. Similarly, the translation of the internal statement that α is injective is that α is a monomorphism.

Furthermore, we can *reason* with the internal language. There is a metatheorem to the effect that if some statement φ holds from the internal point of view of a topos $\mathcal{E}$ and if φ logically implies some further statement ψ, then ψ holds in $\mathcal{E}$ as well. As a simple example, consider the elementary fact that the composition of surjective maps is surjective. Interpreting this statement in the internal language of $\mathcal{E}$, we obtain the more abstract result that the composition of epimorphisms in $\mathcal{E}$ is epic.

There is, however, a slight caveat to this metatheorem. Namely, the internal language of a topos is in general only *intuitionistic*, not *classical*. This means that internally, one can not use the law of excluded middle ($\varphi \vee \neg\varphi$), the law of double negation elimination ($\neg\neg\varphi \Rightarrow \varphi$), or the axiom of choice. For instance, one rendition of the axiom of choice is that any vector space is free. But it need not be the case that a vector space internal to a topos is free as seen from the internal perspective: By the technique explained in this thesis, this would imply the absurd statement that any sheaf of modules on a reduced scheme is locally free.

The restriction to intuitionistic reasoning is not as confining as it might first appear. We will discuss its practical consequences below (on page 20).

Algebraic geometry. We apply the internal language of toposes to algebraic geometry in two different ways, corresponding to the two different toposes associated to a scheme X: the *little Zariski topos* which is just the topos $\mathrm{Sh}(X)$ of set-valued sheaves on X, and the *big Zariski topos* which we introduce below.

The internal language of the little Zariski topos can be applied as follows. The structure sheaf $\mathcal{O}_X$ of a scheme X is a sheaf of rings in that the sets of local sections carry ring structures and these ring structures are compatible with restriction. From the internal point of view of $\mathrm{Sh}(X)$, the structure sheaf $\mathcal{O}_X$ looks much simpler: It looks just like a plain ring (and not a sheaf of rings). Similarly, a sheaf of $\mathcal{O}_X$-modules looks just like a plain module over that ring.

This allows to import notions and facts from basic linear and commutative algebra into the sheaf setting. For instance, it turns out that a sheaf of $\mathcal{O}_X$-modules is of finite type if and only if, from the internal perspective, it is finitely generated as an $\mathcal{O}_X$-module. Now consider the following fact of linear algebra: If in a short exact sequence of modules the two outer ones are finitely generated, then the middle one is too. The usual proof of this fact is intuitionistically acceptable and can thus be interpreted in the internal language. It then *automatically* yields the following more advanced proposition: If in a short exact sequence of sheaves of $\mathcal{O}_X$-modules the two outer ones are of finite type, then the middle one is too.

This example was not special: *Any (intuitionistically valid) theorem about modules yields a corresponding theorem about sheaves of modules.*

The internal language machinery thus allows us to understand the basic notions and statements of scheme theory as notions and statements of linear and commutative algebra, interpreted in a suitable sheaf topos. This brings conceptual clarity and reduces technical overhead.

In Section 2, we explain how the internal language machinery works, and then develop in Part II a *dictionary* between common notions of scheme theory and corresponding notions of algebra. Once built, this dictionary can be used arbitrarily often. We stress that no in-depth knowledge of topos theory or categorical logic is necessary to apply this apparatus.

Two highlights of our approach are the following. If X is a reduced scheme, the internal universe of $\mathrm{Sh}(X)$ has the peculiar feature that $\mathcal{O}_X$ is Noetherian and a field, even if X is not locally Noetherian and (as will almost always be the case) the local rings $\mathcal{O}_{X,x}$ are not fields. Linear and commutative algebra over $\mathcal{O}_X$ are therefore particularly simple from the internal point of view. For instance, Grothendieck's generic freeness lemma, which is usually proved using a somewhat involved series of reduction steps, admits a short, easy, and conceptual proof with this technique.

To briefly indicate a part of this, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules of finite type. A basic version of Grothendieck's generic freeness lemma then states that $\mathcal{F}$ is locally free on some dense open subset of X; this fact is stated in Vakil's lecture notes as an "important hard exercise" [76, Exercise 13.7.K]. In fact, this proposition is just the interpretation of the following basic statement of intuitionistic linear algebra in the sheaf topos: Any finitely generated vector space is *not not* free. The proof of this statement is entirely straightforward.[1]

The second highlight is that we can shed light on the phenomenon that sometimes, truth of a property at a point x spreads to some open neighbourhood of x; and in particular that sometimes, truth of a property at the generic point spreads to some dense open subset. For instance, if the stalk of a sheaf of finite type is zero at some point, the sheaf is even zero on some open neighbourhood; but this spreading does not occur for general sheaves which may fail to be of finite type.

We formalize this by introducing a *modal operator* $\Box$ into the internal language, such that the internal statement $\Box\varphi$ means that φ holds on some open neighbourhood of x. Furthermore, we introduce a simple operation on formulas, the $\Box$-*translation* $\varphi \mapsto \varphi^{\Box}$, such that $\varphi^{\Box}$ means that φ holds at the point x. This translation is defined on a purely syntactical level. The question whether truth at x spreads to

[1] Intuitionistically, the statement that any finitely generated vector space is *free* is stronger than the doubly negated version and can not be shown. It would imply that any sheaf of finite type is not only locally free on some dense open subset, but locally free on the entire space. We discuss this example in more detail in Section 5 and in particular in Lemma 5.9. A proof of Grothendieck's generic freeness lemma in its full form is given in Section 11.5.

truth on a neighbourhood can then be formulated in the following way: Does $\varphi^\Box$ intuitionistically imply $\Box\varphi$?

This allows to deal with the question in a simpler, more logical way, with the technicalities of sheaves blinded out. We also give a metatheorem which covers a wide range of cases. Namely, spreading occurs for all those properties which can be formulated in the internal language without using "$\Rightarrow$", "$\forall$", and "$\neg$".

To take up the example above, consider the property of a module $\mathcal{F}$ being the zero module. In the internal language, it can be formulated as $(\forall x : \mathcal{F}.\ x = 0)$. Because of the appearance of "$\forall$", the metatheorem is not applicable to this statement. But if $\mathcal{F}$ is of finite type, there are generators $x_1, \ldots, x_n : \mathcal{F}$ from the internal point of view, and the condition can be reformulated as $x_1 = 0 \wedge \cdots \wedge x_n = 0$; the metatheorem is applicable to this statement.

Synthetic algebraic geometry. All of the applications mentioned above employ the little Zariski topos of the base scheme X, the topos of sheaves on the underlying topological space of X. Its internal language simplifies the treatment of sheaves of rings and modules over X, but the treatment of *schemes* over X is simplified only a little bit: From the internal point of view of $\mathrm{Sh}(X)$, a morphism $T \to X$ of schemes looks like a morphism $T \to \mathrm{pt}$. Therefore relative scheme theory is turned into absolute scheme theory (over the ring $\mathcal{O}_X$), but it still requires the machinery of locally ringed spaces.

The internal language of the *big* Zariski topos of X allows for a more far-reaching change of perspective. It incorporates Grothendieck's functor-of-points philosophy in order to cast modern algebraic geometry, relative to the arbitrary base scheme X, in a naive *synthetic* language reminiscient of the classical Italian school.

The synthetic approach is best explained by contrasting it with the usual approach to scheme theory, which is to layer it upon some standard form of set theory: to give a scheme means to firstly give a set of points; then to describe a topology on this set; and finally to equip the resulting space with a local sheaf of rings. Basic objects of study in algebraic geometry, such as closed subschemes of projective spaces, are in this way encoded using a large amount of machinery.

There is also a somewhat lesser used, but philosophically rewarding and more "economical" approach within set theory: Grothendieck's functorial approach. In this account of scheme theory, to give a scheme means to give a functor from the category of commutative rings to the category of sets. For instance, the Fermat scheme is given by the functor

$$A \longmapsto \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\},$$

that is by a *scheme* in the colloquial sense for prescribing a set of solutions for any ring.

This approach requires fewer preparations and involves only objects of intrinsic interest to algebraic geometry: A-valued points, where A ranges over all rings. These tend to be better behaved, for instance in that the set of A-valued points of a product of schemes is isomorphic to the product of the sets of A-valued points, and are more fundamental from a geometric point of view. In contrast, the set-theoretical points of a scheme in the approach using locally ringed spaces actually parameterize irreducible closed subsets, not points in an intuitive sense.

However, the description of basic objects can still be somewhat involved in the functorial approach. For instance, while the functor associated to projective n-space is given on fields by the simple expression

$$\begin{aligned} K \longmapsto\ & \text{the set of lines through the origin in } K^{n+1} \\ & \cong \{[x_0 : \cdots : x_n] \mid x_i \neq 0 \text{ for some } i\}, \end{aligned}$$

on general rings it is given by

$$A \longmapsto \text{the set of quotients } A^n \twoheadrightarrow P, \text{ where } P \text{ is projective, modulo isomorphism.}$$

On the one hand, typically only field-valued points admit a simple description. On the other hand, the A-valued points for more general rings A are crucial in order to impart a meaningful sense of cohesion on the field-valued points and therefore can't simply be dropped.[2]

We can resolve the tension by incorporating an automatic management of the *stage of definition*, the rings A such that we're considering A-valued points, into our language. Such a language is provided by the internal language of the big Zariski topos. It allows for the Fermat scheme to be given by the naive expression

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

and for projective n-space to be given by either of the expressions

$$\text{the set of lines through the origin in } (\underline{\mathbb{A}}^1)^{n+1} \quad \text{or} \quad \{[x_0 : \cdots : x_n] \mid x_i \neq 0 \text{ for some } i\}.$$

This is not a specialized trick to give short descriptions of some schemes: Like with the internal universe of any topos, the full power of intuitionistic logic is available to reason about the objects constructed in this way.

We can thus add an approach to the list of ways of giving a rigorous foundation to algebraic geometry, the synthetic approach which layers scheme theory not upon a classical set theory, but rather directly encodes schemes as sets and morphisms of schemes as maps of sets in the nonclassical universe provided by the big Zariski topos of a base scheme. We can therefore use a simple, element-based language to talk about schemes.

This is similar to synthetic approaches to other fields of mathematics, such as differential geometry [49], domain theory [42], computability theory [10], and more recently and very successfully homotopy theory [75]. The synthetic approaches allow in each case to encode the objects of study directly as (nonclassical) sets, with geometric, domain-theoretic, computability-theoretic, or homotopic structure being automatically provided for.

The implicit algebro-geometric structure has visible consequences on the internal universe of the big Zariski topos and endows it with a distinctive algebraic flavor. For instance, the statement "*any* map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial function" holds from the internal point of view. This is also a property which sets the internal universe of the big Zariski topos apart from the toposes studied in synthetic differential geometry.

If one is content with building upon classical scheme theory, the big Zariski topos $\mathrm{Zar}(X)$ of a base scheme X can be constructed as the topos of sheaves on the Grothendieck site Sch/X of X-schemes.[3] Explicitly, an object of $\mathrm{Zar}(X)$ is a

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[2]For instance, let $\underline{\mathbb{A}}^1 : A \mapsto A$ be the functor associated to the affine line. Yoneda's lemma guarantees that the set of morphisms $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ in the functor category $[\mathrm{Ring}, \mathrm{Set}]$ is in canonical bijection with the set $\mathbb{Z}[U]$, as one would expect: Algebraic functions $\mathbb{A}^1 \to \mathbb{A}^1$ should be given by polynomials. (The discussion could also be relativized so that the answer is the polynomial ring $k[U]$, where k is some base field.) However, if we calculate the set of morphisms in $[\mathrm{Field}, \mathrm{Set}]$ we obtain $\int_{K \in \mathrm{Field}} \mathrm{Hom}(K, K)$, a set which contains pathological functions such as some which permute the elements of the prime fields in arbitrary ways.

[3]Some care is needed in order to avoid set-theoretical issues of size. We discuss this fine point in Section 15. If one is interested in foundational questions and doesn't simply want to use the big Zariski topos in order to employ its convenient internal language, one can rest assured that there's a way to construct it without resorting to classical scheme theory.

functor $F : (\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition with respect to Zariski coverings: If $T = \bigcup_i U_i$ is a cover of an X-scheme T by open subsets, the diagram

$$F(T) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

should be an equalizer diagram. A premier example of an object of $\mathrm{Zar}(X)$ is the functor $\underline{Y}$ of points associated to an X-scheme Y, mapping an X-scheme T to $\mathrm{Hom}_X(T, Y)$. It satisfies the gluing condition since one can glue morphisms of schemes in the Zariski topology.

The object $\underline{\mathbb{A}}^1$ which already appeared is the functor of points of the affine line over X, the X-scheme $\mathbb{A}^1_X := X \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{Z}[U]$. Its value on an X-scheme T is

$$\underline{\mathbb{A}}^1_X(T) = \mathrm{Hom}_X(T, \mathbb{A}^1_X) \cong \mathrm{Hom}_{\operatorname{Spec} \mathbb{Z}}(T, \operatorname{Spec} \mathbb{Z}[U]) \cong \Gamma(T, \mathcal{O}_T).$$

This object has a canonical structure as a ring object in $\mathrm{Zar}(X)$. In fact, from the internal point of view of $\mathrm{Zar}(X)$, it is a local ring and even a field in the sense that nonzero elements are invertible. In the case $X = \operatorname{Spec} \mathbb{Z}$, this was first observed by Kock [50]. At the same time, it is not a reduced ring – a feat possible only in an intuitionistic context. This curious interplay is quite important, since the sets

$$\{x : \underline{\mathbb{A}}^1_X \mid x = 0\} \quad \text{and} \quad \{x : \underline{\mathbb{A}}^1_X \mid x^2 = 0\}$$

should and do describe two different X-schemes: the first is isomorphic to X while the second is an infinitesimal thickening of X, the vanishing scheme of U^2 in $\mathbb{A}^1_X$. In contrast, the sets $\{x : \underline{\mathbb{A}}^1_X \mid x \neq 0\}$ and $\{x : \underline{\mathbb{A}}^1_X \mid x^2 \neq 0\}$ should and do coincide. By the field property, both conditions are equivalent to x being invertible.

Modal operators are useful in the big topos setting as well. For instance, there is a modal operator $\Box_{\text{ét}}$ in the big Zariski topos such that the internal statement $\Box_{\text{ét}}\varphi$ roughly means that φ holds on an étale covering and such that the translated formula $\varphi^{\Box_{\text{ét}}}$ means that φ holds in the *big étale topos* familiar from étale cohomology. In this way, we can access the internal universe of the big étale topos from within the big Zariski topos. The ring $\underline{\mathbb{A}}^1_X$ enjoys additional properties when studied in the étale topos, for instance it is separably closed.

Limitations. The internal language is *local*, in the sense that if $X = \bigcup_i U_i$ is an open covering and an internal statement holds in the sheaf toposes $\mathrm{Sh}(U_i)$, it holds in $\mathrm{Sh}(X)$ as well. On the one hand, this property is very useful. But on the other hand, it causes an inherent limitation of the internal language: Global properties of sheaves of modules like “being generated by global sections”, “being ample”, or “having vanishing sheaf cohomology” and global properties of schemes like “being quasicompact” can *not* be expressed in the internal language.

Thus for global considerations, the internal language of $\mathrm{Sh}(X)$ is only useful in that local subparts can be simplified. Also, some global features reflect themselves in certain metaproperties of the internal language. For instance, a scheme is quasicompact if and only if the internal language has a weak version of the so-called disjunction property of mathematical logic (Section 8).

The locality limitation only refers to locality with respect to the base scheme. For instance, the little and big Zariski toposes of X *can* distinguish between affine and projective n-space over X, even though these are locally isomorphic.

The internal languages of both toposes can be used on a case-by-case basis, employing them as part of longer arguments in the context of ordinary scheme theory where it’s useful to do so. However, if one wants to stay solely in one of the provided internal universes and not use ordinary scheme theory at all, then one will of course run into the further limitation that internal scheme theory, as put forward in this thesis, is only developed to a small amount.

Introductory literature. This text is intended to be self-contained, requiring only basic knowledge of scheme theory. In particular, we assume no prior familiarity with topos theory or formal logic. But if the interested reader is so inclined, she will find a gentle introduction to topos theory in an article by Leinster [52]. Standard references for the internal language of a topos include the book of Mac Lane and Moerdijk [56, Chapter VI], the book of Borceux [19, Chapter 6], and Part D of Johnstone's Elephant [44]. In the 1970s, there was a flurry of activity on applications of the internal language. An article by Mulvey [61] of this time gives a very accessible introduction to the topic, culminating in an internal proof of the Serre–Swan theorem (with just one external ingredient needed).

Related work. The internal language of toposes was applied to algebraic geometry before. For instance, Wraith used it to construct (and verify the universal property of) the little étale topos of a scheme by internally developing the theory of strict henselization [82]. However, to the best of my knowledge, systematically building a dictionary between external and internal notions has not been attempted before, and the use of modal operators to study the spreading of properties from points to neighbourhoods seems to be new as well.

Brandenburg put forward a related program of internalization in his PhD thesis [21]. However, he internalizes constructions of algebraic geometry not in toposes, but in tensor categories. There is some overlap in working out precise universal properties, particularly when dealing with the big Zariski topos.

In other branches of mathematics, the internal language of toposes is used as well. For instance, there is an ongoing effort in mathematical physics to understand quantum mechanical systems from an internal point of view: To any quantum mechanical system, one can associate a so-called Bohr topos containing an internal mirror image of the system. This mirror image looks like a system of classical mechanics from the internal perspective, and therefore tools like Gelfand duality can be used to construct an internal phase space for the system [22, 41].

In stochastics, the usefulness of an internal language was recently stressed by Tao [71]. Such a language makes the common notational practice of dropping the explicit dependence of the value $X(\omega)$ of a random variable on the sample ω completely rigorous and simplifies the basic theory. Tao also highlighted how a suitable language can be used to simplify "ε/δ management" in analysis [70]. Furthermore, there is a topos-theoretic approach to measure theory, in which the sheaf of measurable real functions on a σ-algebra looks like the ordinary set of real numbers from an internal point of view [43]; this has applications in noncommutative geometry [40].

Intuitionistic methods have found many applications in computer science. Recently, the internal language of a topos of trees and a suitable modal operator was used to study guarded recursion, encompassing, for instance, an internal Banach fixed-point theorem [15].

In constructive mathematics, the internal language of toposes is routinely used to obtain models of intuitionistic theories fulfilling certain anti-classical axioms. For instance, there are toposes in which the axiom "any map $\mathbb{R} \to \mathbb{R}$ is continuous" (appropriately formulated) holds [49, 60] and toposes in which the Church–Turing thesis "any map $\mathbb{N} \to \mathbb{N}$ is computable" holds (certain realizability toposes). The internal language can also be used to extract computational content out of classical constructions. To cite just one recent example, Mannaa and Coquand used it to implement algorithms for working with the algebraic closure of an arbitrary field of characteristic zero [57].

One way this thesis contributes to the program of constructive mathematics is that intuitionistic mathematics gains new areas of application. For instance, the constructive account of the theory of Krull dimension was originally developed to remove Noetherian hypotheses, extract computational meaning, and simplify proofs [32, 29]. It can now also be used to reason about the dimension of schemes, since the topological dimension of a scheme X coincides with the Krull dimension of the structure sheaf $\mathcal{O}_X$ regarded as an ordinary ring from the internal perspective of $\mathrm{Sh}(X)$ (Section 3.4).

We obtained a second contribution to constructive mathematics as a byproduct of deducing transfer principles which relate a module over a ring A with its induced quasicoherent sheaf on $\operatorname{Spec} A$: Using the internal language of the little Zariski topos we can algorithmically turn certain non-constructive arguments concerning prime ideals into constructive ones. We discuss this in Section 11.4; it is related to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others [33, 27].

Caramello uses topos theory to build bridges between different mathematical subjects, in a certain precise sense [24, 23]. She exploits that toposes can admit presentations by different sites. Our contribution is certainly related to her grand research program in spirit, but since we focus only on specific presentations of a few specific toposes associated to schemes, there is as yet no direct technical connection.

XXX: mention and explain: Mulvey/Burden, Vickers, Awodey, Coquand, ...

XXX: further work, ...

XXX: Mention insights on relative spectrum, Mulvey's "obscure statement", ...

XXX: Mention that the internal language unlocks new intrinsic characterizations and descriptions, which would otherwise be too unwieldy to formulate or think about.

Notational convention. Occasionally, when quantifying, we use colons instead of element signs not only in internal statements, but also in external ones; particularly when we want to stress that a discussion takes place in an intuitionistic context.

2. The internal language of a sheaf topos

At its heart, the internal language of a topos provides a coherent way of translating any mentions of set-theoretical elements to *generalized elements*, carefully keeping track of and adapting the stage of definition. We want to illustrate this with a simple example before giving the formal definition.

A map $f : X \to Y$ of sets is injective if and only if

$$\forall x, x' \in X.\ f(x) = f(x') \Longrightarrow x = x'. \tag{1}$$

This condition can not only be interpreted in Set, but in any category $\mathcal{C}$ whose objects are structured sets and whose morphisms are maps between the underlying sets. If we want to go beyond such kind of categories, we have to restate the condition in purely category-theoretic language:

$$\forall (1 \xrightarrow{x} X), (1 \xrightarrow{x'} X).\ f \circ x = f \circ x' \Longrightarrow x = x'. \tag{2}$$

This condition makes sense in all categories which contain a terminal object 1, and is equivalent to condition (1) in the case $\mathcal{C} = \mathrm{Set}$. This has a deeper reason: The one-element set $1 = \{\star\}$ is a *separator* of Set, that is objects of Set are uniquely determined by their *global elements*, morphisms from the terminal object.

However, in categories in which the terminal object is not a separator, condition (2) is not very meaningful. This is for instance the case if $\mathcal{C}$ is the category $\mathrm{Sh}(X)$ of set-valued sheaves on a topological space X. Global elements of a sheaf $\mathcal{F}$ are in natural one-to-one correspondence with global sections $s \in \mathcal{F}(X)$ (hence the name), whereby condition (2) only states that f is *injective on global sections*. Since many interesting sheaves admit no or only few global sections, this statement is typically not very substantial.

A basic tenet of category theory is therefore to not only refer to global elements $1 \to X$, but also to *generalized elements* $A \to X$, where A ranges over all objects. The domain A is called the *stage of definition* in this context. Bearing this principle in mind, a better translation of the injectivity condition is the statement

$$\forall \text{objects } A \text{ in } \mathcal{C}.\ \forall (A \xrightarrow{x} X), (A \xrightarrow{x'} X) \text{ in } \mathcal{C}.\ \ f \circ x = f \circ x' \implies x = x'. \tag{3}$$

This statement expresses that f is a monomorphism and therefore correctly captures the structural essence of injectivity.

Unlike this manual translation guided by trial and error and categorical philosophy, the internal language provides a purely mechanical translation scheme. It is fully formal, can be analyzed rigorously, works smoothly with arbitrarily convoluted statements, and most importantly can be trusted to support *reasoning*: If a statement formulated in a naive element-based language intuitionistically implies a further such statement, then the translation of the former implies the translation of the latter.

The power of the internal language doesn't unfold in basic situations like with the example above, where one can easily translate statements and even proofs by hand. It unfolds when considering more complex statements. For instance, the short proof of Grothendieck's generic freeness lemma promised in the introduction rests on the internal statement "any ideal of $\mathcal{O}_{\mathrm{Spec}(R)}[U_1, \ldots, U_n]$ is *not not* finitely generated", where R is a reduced ring. For the proof of Grothendieck's generic freeness lemma it's not necessary to actually perform the translation of this statement into external language, but for definiteness we display the translation here nevertheless:

> For any element $f \in R$ and any (not necessarily quasicoherent) sheaf of ideals $\mathcal{J} \hookrightarrow \mathcal{O}_{\mathrm{Spec}(R)}[U_1, \ldots, U_n]|_{D(f)}$: If
> for any element $g \in R$ the condition that
> the sheaf $\mathcal{J}$ is of finite type on $D(g)$
> implies that $g = 0$,
> then $f = 0$.

This statement is obviously quite convoluted, and its proof is even more so; therefore it probably wouldn't occur to one to base a proof of Grothendieck's generic freeness lemma on this statement. The internal language is thus of real use here. We'll expand on this example in Section 3.9 and in Section 11.5.[4]

2.1. Internal statements. Let X be a topological space. Later, X will be the underlying space of a scheme. The meaning of internal statements is given by a set of rules, the *Kripke–Joyal semantics* of the topos of sheaves on X.

Definition 2.1. The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ over U is given by the rules listed in Table 1, recursively in the structure of φ. In a *formula over U* there may appear sheaves defined on U as domains of quantifications, U-sections of sheaves as terms, and

[4] The statement can be proven by hand, but it's much simpler to only verify the case $n = 0$ (and even reduce this case to simple other properties which $\mathcal{O}_{\mathrm{Spec}(R)}$ enjoys from the internal point of view) and then to apply Hilbert's basis theorem. Hilbert's basis theorem is famous for admitting only a nonconstructive proof, and nonconstructive proofs can't be translated by the internal language machinery; but this is only true for the conclusion "any ideal is finitely generated". The intuitionistically weaker conclusion "any ideal is *not not* finitely generated" does admit a constructive proof, and is all what's needed here.

$U \models s = t : \mathcal{F}$	$:\Longleftrightarrow$	$s\vert_U = t\vert_U \in \Gamma(U, \mathcal{F})$
$U \models s \in \mathcal{G}$	$:\Longleftrightarrow$	$s\vert_U \in \Gamma(U, \mathcal{G})$ ($\mathcal{G}$ a subsheaf of $\mathcal{F}$, s a section of $\mathcal{F}$)
$U \models \top$	$:\Longleftrightarrow$	$U = U$ (always fulfilled)
$U \models \bot$	$:\Longleftrightarrow$	$U = \emptyset$
$U \models \varphi \wedge \psi$	$:\Longleftrightarrow$	$U \models \varphi$ and $U \models \psi$
$U \models \bigwedge_{j \in J} \varphi_j$	$:\Longleftrightarrow$	for all $j \in J$: $U \models \varphi_j$ (J an index set)
$U \models \varphi \vee \psi$	$:\Longleftrightarrow$	$\cancel{U \models \varphi \text{ or } U \models \psi}$ there exists a covering $U = \bigcup_i U_i$ such that for all i: $U_i \models \varphi$ or $U_i \models \psi$
$U \models \bigvee_{j \in J} \varphi_j$	$:\Longleftrightarrow$	$\cancel{U \models \varphi_j \text{ for some } j \in J}$ (J an index set) there exists a covering $U = \bigcup_i U_i$ such that for all i: $U_i \models \varphi_j$ for some $j \in J$
$U \models \varphi \Rightarrow \psi$	$:\Longleftrightarrow$	$\cancel{U \models \varphi \text{ implies } U \models \varphi}$ for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$
$U \models \forall s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	for all sections $s \in \Gamma(V, \mathcal{F})$ on open $V \subseteq U$: $V \models \varphi(s)$
$U \models \exists s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	$\cancel{\text{there exists a section } s \in \Gamma(U, \mathcal{F}) \text{ such that } U \models \varphi(s)}$ there exists an open covering $U = \bigcup_i U_i$ such that for all i: there exists $s_i \in \Gamma(U_i, \mathcal{F})$ such that $U_i \models \varphi(s_i)$
$U \models \forall \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	for all sheaves $\mathcal{F}$ on open $V \subseteq U$: $V \models \varphi(\mathcal{F})$
$U \models \exists \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	there exists an open covering $U = \bigcup_i U_i$ such that for all i: there exists a sheaf $\mathcal{F}_i$ on U_i such that $U_i \models \varphi(\mathcal{F}_i)$

TABLE 1. The Kripke–Joyal semantics of a sheaf topos.

morphisms of sheaves on U as function symbols. If $V \subseteq U$ is an open subset, then formulas over U can be pulled back to formulas over V. The symbols "$\top$" and "$\bot$" denote truth and falsehood, respectively. The universal and existential quantifiers come in two flavors: for bounded and unbounded quantification. The translation of $U \models \neg\varphi$ does not have to be separately defined, since negation can be expressed using other symbols: $\neg\varphi :\equiv (\varphi \Rightarrow \bot)$. If we want to emphasize the particular topos, we write

$$\mathrm{Sh}(X) \models \varphi \quad :\Longleftrightarrow \quad X \models \varphi.$$

Remark 2.2. The last two rules in Table 1, concerning *unbounded quantification*, are not part of the classical Kripke–Joyal semantics. They are part of Mike Shulman's stack semantics [67], a slight extension. They are needed so that we can formulate universal properties in the internal language.

Example 2.3. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is a monomorphism of sheaves if and only if, from the internal perspective, α is simply an injective map:

$$X \models \ulcorner \alpha \text{ is injective} \urcorner$$

$$\Longleftrightarrow X \models \forall s : \mathcal{F}.\ \forall t : \mathcal{F}.\ \alpha(s) = \alpha(t) \Rightarrow s = t$$

$$\Longleftrightarrow \text{for all open } U \subseteq X, \text{ sections } s \in \Gamma(U, \mathcal{F}):$$

for all open $V \subseteq U$, sections $t \in \Gamma(V, \mathcal{F})$:

$$V \models \alpha(s) = \alpha(t) \Rightarrow s = t$$

$\iff$ for all open $U \subseteq X$, sections $s \in \Gamma(U, \mathcal{F})$:

for all open $V \subseteq U$, sections $t \in \Gamma(V, \mathcal{F})$:

for all open $W \subseteq V$:

$$\alpha_W(s|_W) = \alpha_W(t|_W) \text{ implies } s|_W = t|_W$$

$\iff$ for all open $U \subseteq X$, sections $s, t \in \Gamma(U, \mathcal{F})$:

$$\alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U$$

$\iff$ α is a monomorphism of sheaves

The corner quotes "$\ulcorner \ldots \urcorner$" indicate that translation into formal language is left to the reader. Similarly, α is an epimorphism of sheaves if and only if, from the internal perspective, α is a surjective map. Notice that injectivity and surjectivity are notions of a simple element-based language. The Kripke–Joyal semantics takes care to properly handle *all* sections, not only global ones.

The rules are not all arbitrary. They are finely concerted to make the following two propositions true, which are crucial for a proper appreciation of the internal language.

Proposition 2.4 (Locality of the internal language)**.** *Let $U = \bigcup_i U_i$ be covered by open subsets. Let φ be a formula over U. Then*

$$U \models \varphi \qquad \textit{iff} \qquad U_i \models \varphi \textit{ for each } i.$$

Proof. Induction on the structure of φ. Note that the canceled rules would make this proposition false. ☐

As a corollary, one may restrict the open coverings and universal quantifications in the the definition of the Kripke–Joyal semantics (Table 1) to open subsets of some basis of the topology. For instance, if X is a scheme, one may restrict to affine open subsets.

Furthermore, the proposition shows that the internal language is monotone in the following sense: If $U \models \varphi$, and V is an open subset of U, then $V \models \varphi$. (This follows by applying the proposition to the trivial covering $U = V \cup U$.)

Proposition 2.5 (Soundness of the internal language)**.** *If a formula φ implies a further formula ψ in intuitionistic logic, then $U \models \varphi$ implies $U \models \psi$.*

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that on each U_i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in Appendix 23. ☐

In particular, if a formula ψ has an unconditional intuitionistic proof, then $U \models \psi$.

The restriction to intuitionistic logic is really necessary at this point. We will encounter many examples of classically equivalent internal statements whose translations using the Kripke–Joyal semantics are wildly different. To anticipate just one example, the statement

$$X \models \ulcorner \mathcal{F} \text{ is finite free} \urcorner,$$

referring to a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, means that $\mathcal{F}$ is finite locally free. The statement

$$X \models \neg\neg(\ulcorner \mathcal{F} \text{ is finite free} \urcorner)$$

instead means that $\mathcal{F}$ is finite locally free on a dense open subset of X.

In particular, our treatment of modal operators to understand spreading of properties from points to neighbourhoods depends on having the ability to make finer distinctions – distinctions which are not visible in classical logic. In Section 2.4 there is a discussion of what the restriction to intuitionistic logic amounts to in practice.

Because of the multitude of quantifiers, literal translations of internal statements can sometimes get slightly unwieldy. There are simplification rules for certain often-occuring special cases:

Proposition 2.6.

$$\begin{aligned}
U \models \forall s\,{:}\,\mathcal{F}.\ \forall t\,{:}\,\mathcal{G}.\ \varphi(s,t) \quad &\Longleftrightarrow \quad \textit{for all open } V \subseteq U, \\
&\qquad \textit{sections } s \in \Gamma(V,\mathcal{F}),\ t \in \Gamma(V,\mathcal{G}){:}\ V \models \varphi(s,t) \\
U \models \forall s\,{:}\,\mathcal{F}.\ \varphi(s) \Rightarrow \psi(s) \quad &\Longleftrightarrow \quad \textit{for all open } V \subseteq U,\ \textit{sections } s \in \Gamma(V,\mathcal{F}){:} \\
&\qquad V \models \varphi(s) \textit{ implies } V \models \psi(s) \\
U \models \exists! s\,{:}\,\mathcal{F}.\ \varphi(s) \quad &\Longleftrightarrow \quad \textit{for all open } V \subseteq U, \\
&\qquad \textit{there is exactly one section } s \in \Gamma(V,\mathcal{F}) \textit{ with:} \\
&\qquad V \models \varphi(s)
\end{aligned}$$

Proof. Straightforward. By way of example, we prove the existence claim in the "only if" direction of the last rule. (This rule formalizes the saying "unique existence implies global existence".) By definition of $\exists!$, it holds that

$$U \models \exists s\,{:}\,\mathcal{F}.\ \varphi(s) \qquad \text{and} \qquad U \models \forall s,t\,{:}\,\mathcal{F}.\ \varphi(s) \wedge \varphi(t) \Rightarrow s = t.$$

Let $V \subseteq U$ be an arbitrary open subset. Then there exist local sections $s_i \in \Gamma(V_i, \mathcal{F})$ such that $V_i \models \varphi(s_i)$, where $V = \bigcup_i V_i$ is an open covering. By the locality of the internal language, on intersections it holds that $V_i \cap V_j \models \varphi(s_i)$, so by the uniqueness assumption, it follows that the local sections agree on intersections. They therefore glue to a section $s \in \Gamma(V, \mathcal{F})$. Since $V_i \models \varphi(s)$ for all i, the locality of the internal language allows us to conclude that $V \models \varphi(s)$. □

Remark 2.7. Note that $\mathrm{Sh}(X) \models \neg\varphi$ is in general a much stronger statement than merely saying that $\mathrm{Sh}(X) \models \varphi$ does not hold: The former always implies the latter (unless $X = \emptyset$, in which case *any* internal statement is true), but the converse does not hold: The former statement means that $U = \emptyset$ is the *only* open subset on which φ holds, that is that φ holds *nowhere*. In contrast, the statement $\mathrm{Sh}(X) \not\models \varphi$ only means that φ does *not hold everywhere*.

2.2. Internal constructions. The Kripke–Joyal semantics defines the interpretation of internal *statements*. The interpretation of internal *constructions* is given by the following definition.

Definition 2.8. The interpretation of an internal construction T is denoted by $[\![T]\!] \in \mathrm{Sh}(X)$ and given by the following rules.

- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \times \mathcal{G}]\!]$ is the categorical product of $\mathcal{F}$ and $\mathcal{G}$ (i. e. their product as presheaves).

- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \amalg \mathcal{G}]\!]$ is the categorical coproduct of $\mathcal{F}$ and $\mathcal{G}$, i.e. the sheafification of the presheaf $U \mapsto \Gamma(U, \mathcal{F}) \amalg \Gamma(U, \mathcal{G})$.
- If $\mathcal{F}$ is a sheaf, the interpretation $[\![\mathcal{P}(\mathcal{F})]\!]$ of the power set construction is the sheaf given by
$$U \subseteq X \text{ open} \quad \longmapsto \quad \{\mathcal{G} \hookrightarrow \mathcal{F}|_U\},$$
i.e. sections on an open set U are subsheaves of $\mathcal{F}|_U$ (either literally or isomorphism classes of arbitrary monomorphisms into $\mathcal{F}|_U$).
- If $\mathcal{F}$ is a sheaf and $\varphi(s)$ is a formula containing a free variable $s : \mathcal{F}$, the interpretation $[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]$ is given by the subpresheaf of $\mathcal{F}$ defined by
$$U \subseteq X \text{ open} \quad \longmapsto \quad \{s \in \Gamma(U, \mathcal{F}) \mid U \models \varphi(s)\}.$$
Note that by the locality of the internal language, this presheaf is in fact a sheaf.

The definition is made in such a way that, from the internal perspective, the constructions enjoy their expected properties. For instance, it holds that
$$\mathrm{Sh}(X) \models \big(\forall x : [\![\{s : \mathcal{F} \mid \varphi(s)\}]\!].\ \psi(x)\big) \Longleftrightarrow \big(\forall x : \mathcal{F}.\ \varphi(x) \Rightarrow \psi(x)\big).$$
We gloss over several details here. See [44, Section D4.1] for a proper treatment.

Morphisms can internally be constructed by appealing to the *principle of unique choice*: Let $\varphi(s, t)$ be a formula with free variables of type $s : \mathcal{F}$, $t : \mathcal{G}$. Assume
$$\mathrm{Sh}(X) \models \forall s : \mathcal{F}.\ \exists! t : \mathcal{G}.\ \varphi(s, t).$$
Then there is one and only one morphism $\alpha : \mathcal{F} \to \mathcal{G}$ of sheaves such that for any local section $s \in \Gamma(U, \mathcal{F})$, $\mathrm{Sh}(X) \models \varphi(s, \alpha(s))$. This follows from the meaning of unique existence with the Kripke–Joyal semantics (Proposition 2.6).

An important application is showing that two sheaves $\mathcal{F}$ and $\mathcal{G}$ are isomorphic (usually as objects with more structure, for instance sheaves of modules). To this end, it suffices to give a formula $\varphi(s, t)$ satisfying, in addition to the condition above, the condition $\mathrm{Sh}(X) \models \forall t : \mathcal{G}.\ \exists! s : \mathcal{F}.\ \varphi(s, t)$, expressing that the induced morphism α is a bijective map from the internal perspective. Note that this implies the statement
$$\mathrm{Sh}(X) \models \exists \alpha : \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}).\ \ulcorner \alpha \text{ is bijective} \urcorner,$$
but this statement is strictly weaker: Its interpretation with the Kripke–Joyal semantics is that the sheaves $\mathcal{F}$ and $\mathcal{G}$ are *locally* isomorphic.

2.3. Geometric formulas and constructions. In formal and categorical logic so-called geometric formulas play a special role. They are named that way because, in a sense which can be made precise, their meaning is preserved under pullback with geometric morphisms.

Definition 2.9. A formula is *geometric* if and only if it consists only of
$$= \quad \in \quad \top \quad \bot \quad \wedge \quad \vee \quad \bigvee \quad \exists,$$
but not "$\bigwedge$" nor "$\Rightarrow$" nor "$\forall$" (and thus not "$\neg$" either, since negation is defined using "$\Rightarrow$"). A *geometric implication* is a formula of the form
$$\forall \cdots \forall.\ (\cdots) \Rightarrow (\cdots)$$
with the bracketed subformulas being geometric.

The *parameters* of a formula φ are the sheaves being quantified over, sections of sheaves appearing as terms, and morphisms of sheaves appearing as function symbols in φ. We say that a formula φ holds *at a point* $x \in X$ if and only if the formula obtained by substituting all parameters in φ with their stalks at x holds in the usual mathematical sense.

Lemma 2.10. *Let $x \in X$ be a point. Let φ be a geometric formula (over some open neighbourhood V of x). Then φ holds at x if and only if there exists an open neighbourhood $U \subseteq X$ of x (contained in V) such that φ holds on U.*

Proof. This is a very general instance of the phenomenon that sometimes, truth at a point spreads to truth on a neighbourhood. It can be proven by induction on the structure of φ, but we will give a more conceptual proof later (Corollary 6.32). $\square$

This lemma is in fact a very useful metatheorem. We will properly discuss its significance in Section 6.7. For now, we just use it to prove a simple criterion for the internal truth of a geometric implication; we will apply this criterion many times.

Corollary 2.11. *A geometric implication holds on X if and only if it holds at every point of X.*

Proof. For notational simplicity, we consider a geometric implication of the form

$$\forall s : \mathcal{F}.\ \varphi(s) \Rightarrow \psi(s).$$

For the "only if" direction, assume that this formula holds on X and let $x \in X$ be an arbitrary point. Let $s_x \in \mathcal{F}_x$ be the germ of an arbitrary local section s of $\mathcal{F}$ and assume that $\varphi(s)$ holds at x. Then by the lemma, it follows that $\varphi(s)$ holds on some open neighbourhood of x. By assumption, $\psi(s)$ holds on this neighbourhood as well. Again by the lemma, $\psi(s)$ holds at x.

For the "if" direction, assume that the geometric implication holds at every point. Let $U \subseteq X$ be an arbitrary open subset and let $s \in \Gamma(U, \mathcal{F})$ be a local section such that $\varphi(s)$ holds on U. By the lemma and the locality of the internal language, to show that $\psi(s)$ holds on U, it suffices to show that $\psi(s)$ holds at every point of U. This is clear, since again by the lemma, $\varphi(s)$ holds at every point of U. $\square$

Example 2.12. Injectivity and surjectivity are geometric implications (surjectivity can be spelled $\forall y : \mathcal{G}.\ (\top \Rightarrow \exists x : \mathcal{F}.\ \alpha(x) = y)$). Thus the corollary gives a deeper reason for the well-known fact that a morphism of sheaves is a monomorphism resp. an epimorphism if and only if it is stalkwise injective resp. surjective.

A construction is *geometric* if and only if it commutes with pullback under arbitrary geometric morphisms. We do not want to discuss the notion of geometric morphisms here; suffice it to say that calculating the stalk at a point $x \in X$ is an instance of such a pullback. Among others, the following constructions are geometric:

- finite product: $(\mathcal{F} \times \mathcal{G})_x \cong \mathcal{F}_x \times \mathcal{G}_x$
- finite coproduct: $(\mathcal{F} \amalg \mathcal{G})_x \cong \mathcal{F}_x \amalg \mathcal{G}_x$
- arbitrary coproduct: $(\coprod_i \mathcal{F}_i)_x \cong \coprod_i (\mathcal{F}_i)_x$
- set comprehension with respect to a *geometric* formula φ:

$$[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]_x \cong \{[s] \in \mathcal{F}_x \mid \varphi(s) \text{ holds at } x\}$$

- free module: $(\mathcal{R}\langle\mathcal{F}\rangle)_x \cong \mathcal{R}_x\langle\mathcal{F}_x\rangle$ ($\mathcal{R}$ a sheaf of rings, $\mathcal{F}$ a sheaf of sets)
- localization of a module: $\mathcal{F}[\mathcal{S}^{-1}]_x \cong \mathcal{F}_x[\mathcal{S}_x^{-1}]$

Note that compatibility with taking stalks is not sufficient for geometricity. It is just the most easily visualized requirement. The following constructions are not in general geometric:

- arbitrary product
- set comprehension with respect to a non-geometric formula
- powerset
- internal Hom: $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})_x \ncong \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$

2.4. Appreciating intuitionistic logic. The principal (and only) difference between classical and intuitionistic logic is that in classical logic, the axioms schemes of *excluded middle* and *double negation elimination* are added.

$$\varphi \vee \neg\varphi \qquad\qquad \neg\neg\varphi \Rightarrow \varphi$$

A classically trained mathematician might legitimately wonder why one should drop these axioms: Are they not obviously true? The pragmatic answer to this question is that the translations of these axioms with the Kripke–Joyal semantics are, except for uninteresting special cases of the base space X, plainly false – irrespective of one's philosophical convictions. Therefore the internal language is in general only sound with respect to intuitionistic logic and not with respect to classical logic. Concretely, there is the following proposition.

Proposition 2.13. *The internal language of a* T_1*-space* X *is* Boolean*, i. e. it verifies the classical axiom schemes displayed above, if and only if* X *is discrete. The internal language of an irreducible or locally Noetherian scheme* X *is Boolean if and only if* X *has dimension* ≤ 0.

Proof. The internal language of $\mathrm{Sh}(X)$ is Boolean if and only if for any open subset $U \subseteq X$ it holds that U is the only dense open subset of U. This can be checked manually, by using the definition of the Kripke–Joyal semantics, but we'll be able to give a more conceptual proof later (Lemma 6.18). The first claim is then an exercise in point-set topology, while the second is more difficult (Corollary 3.15). □

However, there is also a more satisfying answer, which furthermore illuminates how to intuitively picture intuitionistic mathematics. Namely, when doing intuitionistic mathematics, we use the same formal symbols as classically, but with *a different intended meaning.* For instance, the classical reading of an existential statement like $\exists x : A.\ \varphi(x)$ is that there exists some element $x : A$ with the property $\varphi(x)$. In contrast, its intuitionistic reading is that such an element can actually be *constructed*, i. e. explicitly given in some form. This is a much stronger statement. Classically, a proof that it is *not* the case that such an element does *not* exist – formally $\neg\neg\exists x : A.\ \varphi(x)$ (or, equivalently even in intuitionistic mathematics, $\neg\forall x : A.\ \neg\varphi(x)$) – suffices to demonstrate the existential statement; this is not so in intuitionistic mathematics.

Similarly, the intuitionistic meaning of a disjunction $\varphi \vee \psi$ is not only that one of the disjuncts is true, but that one can explicitly state which case holds. It is in general not enough to show that it is impossible that both φ and ψ fail.

In this picture, it is obvious that one should not adopt the law of excluded middle or the principle of double negation elimination as axioms. Note that we do not *reject* those axioms in the sense of postulating their converses either, we simply don't use them. Therefore any intuitionistically true result is also true classically. In fact, for some special instances, these two classical axioms do hold intuitionistically. For example, any natural number is zero or is not zero – this is not a triviality, but can be proven by induction.[5]

A consequence of not adopting these axioms is that proofs by contradiction are not generally justified; they are intuitionistically acceptable only for those statements which can be proven to be true or false. Note that a proof of a *negated formula* is not the same as a proof by contradiction. For instance, the usual proof

[5] The analogous statement about real numbers cannot be shown. Intuitively, for a number given by a decimal expansion starting with $0.0000\ldots$ one cannot decide whether the string of zeros will continue indefinitely or whether eventually a non-zero digit will occur. This argument can be made rigorous. The analogous statement about algebraic numbers *can* be proven; the information contained in a witness of algebraicity (a monic polynomial which the given number is a zero of) suffices to make the case distiction [59, Chapter VI.1, p. 140].

that $\sqrt{2}$ is not rational is intuitionistically perfectly fine: From the assumption that $\sqrt{2}$ is rational one deduces a contradiction ($\bot$). This is exactly the definition of $\neg(\ulcorner\sqrt{2}$ is rational$\urcorner)$.

A more positive consequence of not adopting the law of excluded middle and the principle of double negation elimination is that intuitionistically, we can make *finer distinctions*. For instance, for a formula φ, the doubly negated formula $\neg\neg\varphi$ ("*not not* φ") is a certain kind of weakening of φ: If φ holds, then $\neg\neg\varphi$ does as well, while the converse can not be shown in general.[6] An example from everday life runs as follows: If in the morning you can't find the key for your appartment, but you know that it must hide somewhere since you used it to open the door in the evening before, you intuitionistically know $(\neg\neg\exists x.\ \ulcorner$the key is at position $x\urcorner)$, but you cannot claim the unnegated proposition. One cannot model this distinction with pure classical logic.

Double negation also has a concrete geometric meaning with the Kripke–Joyal semantics. Namely, $X \models \neg\neg\varphi$ holds if and only if there is a dense open subset U of X such that $U \models \varphi$. This is of course a weaker statement than $X \models \varphi$. In Section 6, we will discuss this fact and other *modal operators* in more detail. For instance, there is a similarly defined modal operator $\Box$ such that $X \models \Box\varphi$ if and only if there is an open neighbourhood U of a given point x such that $U \models \varphi$. Also there is a different operator $\Box$ such that $X \models \Box\varphi$ if and only if φ holds on a scheme-theoretically dense open subset.

For future reference, note that if $\varphi \Rightarrow \psi$, then also $\neg\neg\varphi \Rightarrow \neg\neg\psi$; and note that weakening twice has no further effect, i. e. $\neg\neg\neg\neg\varphi \Leftrightarrow \neg\neg\varphi$.[7]

A classical mathematician might then ask which classical results are valid intuitionistically. The answer is that in linear and commutative algebra, most of the basic theorems stay valid, provided one exercises some caution in formulating them (for instance, one should not arbitrarily weaken assumptions by introducing double negations). This is because the proofs of these statements are usually direct; if intuitionistically unacceptable case distictions do occur, they can often be eliminated by streamlining the proof.

Consider as a simple example the proposition that the kernel of a linear map is a linear subspace. The case distiction "either the kernel consists just of the zero vector, in which case the claim is trivial, or otherwise ..." is not intuitionistically acceptable, but it can be entirely dispensed with: The proof for the general case works in the special case just as well.

Finally, we should clarify the status of the axiom of choice. This axiom, which is strictly speaking not part of classical logic, but of a classical set theory, is not accepted in an intuitionistic context: By *Diaconescu's theorem*, it implies the law of excluded middle in presence of the other axioms of set theory.

Standard references for intuitionistic algebra are a textbook by Mines, Richman and Ruitenburg [59] and a textbook by Lombardi [54], the standard reference for intuitionistic analysis is a book by Bishop and Bridges [16]. Further explanations and pointers to relevant literature can be found in an expository article and a recorded lecture by Bauer [12, 11]. A recent survey of intuitionistic logic from a historical and logical point of view is [58].

Remark 2.14. For ease of exposition, we work in a classical metatheory. This means that we allow ourselves to occasionally use the law of excluded middle and the axiom of choice when reasoning *about* the internal language. In particular,

[6]A detailed proof of the correct implication goes as follows: Assume φ. We are to show $\neg\neg\varphi$, i. e. $(\neg\varphi \Rightarrow \bot)$. So assume $\neg\varphi$, we are to show $\bot$. Since φ and $\varphi \Rightarrow \bot$, $\bot$ indeed follows.

[7]In fact, negating thrice is the same as negating once: Assume $\neg\neg\neg\varphi$. We are to show $\neg\varphi$. So assume φ, we are to show $\bot$. Since φ, $\neg\neg\varphi$. By $\neg\neg\neg\varphi$, $\bot$ follows.

we have the theory of schemes as commonly presented at our disposal. But we should note that this concession is really a cop out, and that it would be better to develop an intuitionistic theory of schemes. If this were done, one could extend our approach to understand morphisms of schemes from an internal point of view – a morphism $Y \to X$ would internally look like a morphism $Y \to \mathrm{pt}$. See Section 12 for details.

PART II

The little Zariski topos

3. Sheaves of rings

Recall that a *sheaf of rings* can be categorically described as a sheaf of sets $\mathcal{R}$ together with maps of sheaves $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, $- : \mathcal{R} \to \mathcal{R}$, and global elements $0, 1$ such that certain axioms hold. For instance, the axiom on the commutativity of addition is rendered in diagrammatic form as follows:

$$\begin{tikzcd} \mathcal{R} \times \mathcal{R} \arrow[rr, "\text{swap}"] \arrow[dr, "+"'] & & \mathcal{R} \times \mathcal{R} \arrow[dl, "+"] \\ & \mathcal{R} & \end{tikzcd}$$

From the internal perspective, a sheaf of rings looks just like a plain ring. This is the content of the following proposition.

Proposition 3.1. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of sets on X. Let $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $- : \mathcal{R} \to \mathcal{R}$ be maps of sheaves and let $0, 1$ be global elements of $\mathcal{R}$. Then these data define a sheaf of rings if and only if, from the internal perspective, these data fulfill the usual equational ring axioms.*

Proof. We only discuss the commutativity axiom. The internal statement

$$\mathrm{Sh}(X) \models \forall x, y : \mathcal{R}.\ x + y = y + x$$

means that for any open subset $U \subseteq X$ and any local sections $x, y \in \Gamma(U, \mathcal{R})$, it holds that $x + y = y + x \in \Gamma(U, \mathcal{R})$. This is precisely the external commutativity condition. $\square$

Lemma 3.2. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of rings on X. Let f be a global section of $\mathcal{R}$. Then the following statements are equivalent:*

(1) *f is invertible from the internal point of view, i. e.* $\mathrm{Sh}(X) \models \exists g : \mathcal{R}.\ fg = 1$.
(2) *f is invertible in all stalks $\mathcal{R}_x$.*
(3) *f is invertible in* $\Gamma(X, \mathcal{R})$.

Proof. Since invertibility is a geometric implication, the equivalence of the first two statements is clear. Also, it is obvious that the third statement implies the other two. For the remaining direction, note that the uniqueness of inverses in rings can be proven intuitionistically. Therefore, if f is invertible from the internal point of view, it actually holds that

$$\mathrm{Sh}(X) \models \exists! g : \mathcal{R}.\ fg = 1.$$

Since unique internal existence implies global existence (Proposition 2.6), this shows that the first statement implies the third. $\square$

3.1. Reducedness.

Recall that a scheme X is *reduced* if and only if all stalks $\mathcal{O}_{X,x}$ are reduced rings. Since the condition on a ring R to be reduced is a geometric implication,

$$\forall s : R.\ \Big(\bigvee_{n \geq 0} s^n = 0 \Big) \Longrightarrow s = 0,$$

we immediately obtain the following characterization of reducedness in the internal language:

Proposition 3.3. *A scheme X is reduced iff, from the internal point of view, the ring $\mathcal{O}_X$ is reduced.*

3.2. Locality. Recall the usual definition of a local ring: a ring possessing exactly one maximal ideal. This is a so-called *higher-order condition* since it involves quantification over subsets. It is also not of a geometric form. Therefore, for our purposes, it is better to adopt the following elementary definition of a local ring.

Definition 3.4. A *local ring* is a ring R such that $1 \neq 0$ in R and for all $x, y : R$

$$x + y \text{ invertible} \quad \Longrightarrow \quad x \text{ invertible} \ \vee \ y \text{ invertible}.$$

In classical logic, it is an easy exercise to show that this definition is equivalent to the usual one. In intuitionistic logic, we would need to be more precise in order to even state the question of equivalence, since intuitionistically, the notion of a maximal ideal bifurcates into several non-equivalent notions.[8] This is a common phenomenon in intuitionistic mathematics: Classically equivalent notions may bifurcate into related but inequivalent notions intuitionistically, each having a unique character and yielding slightly different theories.

Proposition 3.5. *In the internal language of a scheme X (or a locally ringed space), the ring $\mathcal{O}_X$ is a local ring.*

Proof. The stated locality condition is a conjunction of two geometric implications (the first one being $1 = 0 \Rightarrow \bot$, the second being the displayed one) and holds on each stalk. □

Remark 3.6. When first exposed to locally ringed spaces, one might ask why the requirement is that the *stalks* $\mathcal{O}_{X,x}$ are local rings, instead of the easier-to-define sets of sections $\mathcal{O}_X(U)$. This question has of course a good geometric answer. Using the internal language, it also has a purely formal answer: The requirement that the stalks are local rings is precisely the requirement that the ring $\mathcal{O}_X$ is a local ring from the perspective of the internal language of X.

3.3. Field properties. From the internal point of view, the structure sheaf $\mathcal{O}_X$ of a scheme X is *almost* a field, in the sense that any element which is not invertible is nilpotent. This is a genuine property of schemes, not shared with arbitrary locally ringed spaces. It is also a specific feature of the internal universe: Neither the local rings $\mathcal{O}_{X,x}$ nor the rings of local sections $\Gamma(U, \mathcal{O}_X)$ have this property in general.

Proposition 3.7. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Rightarrow \ulcorner s \text{ nilpotent} \urcorner.$$

Proof. By the locality of the internal language and since X can be covered by open affine subsets, it is enough to show that for any affine scheme $X = \operatorname{Spec} A$ and any global function $s \in \Gamma(X, \mathcal{O}_X) = A$ it holds that

$$X \models \neg(\ulcorner s \text{ invertible} \urcorner) \quad \text{implies} \quad X \models \ulcorner s \text{ nilpotent} \urcorner.$$

The meaning of the antecedent is that any open subset on which s is invertible is empty. This implies in particular that the standard open subset $D(s)$ is empty. This means that s is an element of any prime ideal of A, thus nilpotent, and therefore implies the a priori weaker statement $X \models \ulcorner s \text{ nilpotent} \urcorner$ (which would allow s to have different indices of nilpotency on an open covering). □

[8]For instance, should a maximal ideal $\mathfrak{m}$ be such that if $\mathfrak{n}$ is any ideal with $\mathfrak{m} \subseteq \mathfrak{n} \subsetneq (1)$, then $\mathfrak{m} = \mathfrak{n}$? Or should the condition be that if $\mathfrak{n}$ is any ideal with $\mathfrak{m} \subseteq \mathfrak{n}$, then $\mathfrak{m} = \mathfrak{n}$ or $\mathfrak{n} = (1)$? Intuitionistically, the latter condition is stronger than the former.

Remark 3.8. In classical logic, the statement "not invertible implies nilpotent" is equivalent to "any element is invertible or nilpotent". However, in intuitionistic logic, the latter is strictly stronger than the former. We will see in the next section (Corollary 3.14) that the structure sheaf of a scheme fulfills the latter condition if and only if the scheme is zero-dimensional (or empty).

Corollary 3.9. *Let X be a scheme. If X is reduced, the ring $\mathcal{O}_X$ is a field from the internal point of view, in the sense that*

$$\mathrm{Sh}(X) \models \forall s\,{:}\,\mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible}\urcorner) \Rightarrow s = 0.$$

Conversely, if $\mathcal{O}_X$ is a field in this internal sense, then X is reduced.

Proof. We can prove this purely in the internal language: It suffices to give an intuitionistic proof of the fact that a local ring which satisfies the condition of the previous proposition fulfills the stated field condition if and only if it is reduced. This is straightforward. $\square$

This field property is very useful. We will put it to good use when giving a simple proof of the fact that $\mathcal{O}_X$-modules of finite type on a reduced scheme are locally free on a dense open subset (Lemma 5.9). Note that the field property only holds in the precise form as stated; the classically equivalent condition that any element is invertible or zero is intuitionistically stronger. This is an instance of the already remarked upon phenomenon of intuitionistic bifurcation of notions.

The observation that the structure sheaf is (almost) a field is attributed by Tierney to Mulvey [73, p. 209]. Tierney also states that "its precise significance is still somewhat obscure" (ibid). We think that it's significant as a special case of the following more general proposition, which says that we can deduce a certain unconditional statement from the premise that, under the assumption that some element $f\,{:}\,\mathcal{O}_X$ is invertible, an element $s\,{:}\,\mathcal{O}_X$ is zero. This is interesting on its own, but will be of particular importance in understanding quasicoherence from the internal point of view (Section 9) and interpreting the relative spectrum as an internal spectrum (Section 12).

Proposition 3.10. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall f\,{:}\,\mathcal{O}_X.\ \forall s\,{:}\,\mathcal{O}_X.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0.$$

Proof. It is enough to show that for any affine scheme $X = \operatorname{Spec} A$ and any global functions $f, s \in A$ such that

$$X \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0),$$

it holds that $X \models \bigvee_{n \geq 0} f^n s = 0$. This indeed follows, since by assumption such a function s is zero on $D(f)$, i. e. s is zero as an element of $A[f^{-1}]$. $\square$

Proposition 3.7 follows from this proposition by setting $s := 1$.

3.4. Krull dimension. Recall that the *Krull dimension* of a ring is usually defined as the supremum of the lengths of strictly ascending chains of prime ideals. As with the classical definition of a local ring, this definition does not lead to a well-behaved notion in an intuitionistic context. Furthermore, it is a higher-order condition, so interpreting it with the Kripke–Joyal semantics is a bit unwieldy.

Luckily, there is an elementary definition of the Krull dimension which works intuitionistically and which is classically equivalent to the usual notion. It was found by Coquand and Lombardi, building upon work by Joyal and Español [32, 29], and can be used to give a short proof that $\dim k[X_1, \ldots, X_n] = n$, where k is a field [28].

Definition 3.11. Let R be a ring. A *complementary sequence* for a sequence $(a_0, \ldots, a_n)$ of elements of R is a sequence $(b_0, \ldots, b_n)$ such that the following inclusions of radical ideals hold:

$$\begin{cases} \sqrt{(1)} & \subseteq & \sqrt{(a_0, b_0)} \\ \sqrt{(a_0 b_0)} & \subseteq & \sqrt{(a_1, b_1)} \\ \sqrt{(a_1 b_1)} & \subseteq & \sqrt{(a_2, b_2)} \\ & \vdots & \\ \sqrt{(a_{n-1} b_{n-1})} & \subseteq & \sqrt{(a_n, b_n)} \\ \sqrt{(a_n b_n)} & \subseteq & \sqrt{(0)} \end{cases}$$

The ring R is *of Krull dimension* $\leq n$ if and only if for any sequence $(a_0, \ldots, a_n)$ there exists a complementary sequence. (The ring R is trivial if and only if it is of Krull dimension ≤ -1.)

Note that unlike the usual definition, this definition posits only a condition on elements and not on ideals. It is thus of a simpler logical form. (The radical ideals appear only for convenience. We will dispose of them in the proof of Proposition 3.13.) Also note that we do not define the Krull dimension of a ring as some natural number (this is intuitionistically not possible for general rings). Instead, we only define what it means for the Krull dimension to be less than or equal to a given natural number.

For the following, no intuition about the definition is needed; however, we feel that some motivation might be of use. Recall that we can picture inclusions of radical ideals geometrically by considering standard open subsets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}$: The inclusion $\sqrt{(f)} \subseteq \sqrt{(g, h)}$ holds if and only if $D(f) \subseteq D(g) \cup D(h)$, and intersections are calculated by products, i. e. $D(f) \cap D(g) = D(fg)$.

The condition that $(b_0, \ldots, b_n)$ is complementary to $(a_0, \ldots, a_n)$ thus means that $D(a_0)$ and $D(b_0)$ cover all of $\operatorname{Spec} R$; that their intersection is covered by $D(a_1)$ and $D(b_1)$; that in turn their intersection is covered by $D(a_2)$ and $D(b_2)$; $\ldots$; and that finally, the intersection of $D(a_n)$ and $D(b_n)$ is empty.

For the special case $n = 0$, the condition that R is of Krull dimension ≤ 0 means that for any element a_0 there exists an element b_0 such that $D(a_0)$ and $D(b_0)$ cover $\operatorname{Spec} R$ and are disjoint.

The definition of the Krull dimension can be written in such a way as to mimic the definition of the inductive Menger–Urysohn dimension of topological spaces [32, Section 1].

Theorem 3.12. *Let R be a ring.*

(1) *In classical logic, the ring R is of Krull dimension $\leq n$ if and only if its Krull dimension as usually defined using chains of prime ideals is less than or equal to n.*
(2) *If the ring R is of Krull dimension $\leq n$, the radical of any finitely generated ideal is equal to the radical of some ideal which can be generated by $n + 1$ elements. This holds intuitionistically, and there is an explicit algorithm for computing the reduced set of generators from the given ones. (Kronecker's theorem)*

Proof. See [32, Theorem 1.2] for the first statement. The proof relies on the observation that $\dim R \leq n$ if and only if $\dim R[S_x^{-1}] \leq n - 1$ for all $x \in R$, where $S_x = x^{\mathbb{N}}(1 + xR) \subseteq R$. We put the second statement only to demonstrate that the definition of the Krull dimension is constructively sensible. It follows from the identity $\sqrt{(x, a_0, \ldots, a_n)} = \sqrt{(a_0 - xb_0, \ldots, a_n - xb_n)}$, where $(b_0, \ldots, b_n)$ is a complementary sequence for $(a_0, \ldots, a_n)$. ☐

We can apply the constructive theory of Krull dimension to the structure sheaf $\mathcal{O}_X$ of a scheme X as follows. Note that the condition that a scheme X has dimension exactly n (in the usual sense using ascending chains of closed irreducible subsets) is not local – the dimension may vary on an open cover; therefore it is not possible to characterize this condition in the internal language. However, the condition that the dimension of X is less than or equal to n *is* local, thus there is hope that it can be internalized. And indeed, this is the case.

Proposition 3.13. *Let X be a scheme. Then:*

$$\dim X \leq n \quad \iff \quad \mathrm{Sh}(X) \models \ulcorner \mathcal{O}_X \text{ is of Krull dimension } \leq n \urcorner$$

Proof. A condition of the form "$\sqrt{(f)} \subseteq \sqrt{(g,h)}$" like in the constructive definition of the Krull dimension is not a geometric formula when taken on face value. However, it is equivalent to a geometric condition, namely to

$$\exists a, b : \mathcal{O}_X.\ \bigvee_{m \geq 0} f^m = ag + bh.$$

Therefore the condition $\ulcorner \mathcal{O}_X$ is of Krull dimension $\leq n \urcorner$ is (equivalent to) a geometric implication and thus holds internally if and only if it holds at every point $x \in X$. This in turn means that the Krull dimension of any stalk $\mathcal{O}_{X,x}$ is less than or equal to n. This is equivalent to the (Krull) dimension of X being less than or equal to n. □

We will state and prove a generalization of this lemma about the dimension of closed subschemes later, as Lemma 10.9.

If X is a reduced scheme, we have seen in Corollary 3.9 that $\mathcal{O}_X$ is a field from the internal perspective, in the sense that non-invertible elements are zero. But fields are well-known to be of Krull dimension zero. Why is this not a contradiction to the proposition just proven? Intuitionistically, the notion of a field bifurcates into several non-equivalent notions:

(1) "Any element which is not invertible is zero."
(2) "Any element which is not zero is invertible."
(3) "Any element is either zero or invertible."

Only fields in the sense (3) are automatically of Krull dimension zero. Fields in the weaker senses can have higher Krull dimension, as exhibited by the structure sheaf of reduced schemes with positive dimension.

For the following corollary, note that if a scheme X is not empty, $\dim X \leq 0$ is equivalent to $\dim X = 0$.

Corollary 3.14. *Let X be a scheme. Then:*

$$\dim X \leq 0 \quad \iff \quad \mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \ulcorner s \text{ inv.} \urcorner \vee \ulcorner s \text{ nilpotent} \urcorner.$$

If furthermore X is reduced, this is further equivalent to $\mathcal{O}_X$ being a field in the strong sense that any element of $\mathcal{O}_X$ is invertible or zero.

Proof. By the proposition and the fact that $\mathcal{O}_X$ is a local ring from the internal perspective, this is an immediate consequence of interpreting the following standard fact of ring theory in the internal language of $\mathrm{Sh}(X)$: A local ring R is of Krull dimension ≤ 0 if and only if any element of R is invertible or nilpotent.

It is well-known that this holds classically; to make sure that it holds intuitionistically as well (so that it can be used in the internal universe), we give a proof of the "only if" direction. Let $a : R$ be arbitrary. By assumption on the Krull dimension, there exists an element $b : R$ such that $\sqrt{(1)} \subseteq \sqrt{(a,b)}$ and $\sqrt{(ab)} = \sqrt{(0)}$. The latter means that ab is nilpotent. Since R is local, the former implies that a is invertible or that b is invertible. In the first case, we are done. In the second case, it follows that a is nilpotent, so we are done as well. □

As a further corollary note the curious fact that the classicality of the internal language of $\mathrm{Sh}(X)$, where X is a scheme, is tightly coupled with the properties of the ring $\mathcal{O}_X$: Internally, the law of excluded middle and the principle of double negation elimination are "almost equivalent" to the Krull dimension of $\mathcal{O}_X$ being ≤ 0.

Corollary 3.15. *Let X be a scheme. If the internal language of* $\mathrm{Sh}(X)$ *is Boolean, then* $\dim X \leq 0$*. The converse holds if X is irreducible or locally Noetherian.*

Proof. We show that any element of $\mathcal{O}_X$ is invertible or nilpotent, therefore verifying the hypothesis of the previous corollary. Let $s : \mathcal{O}_X$ be given. By assumption, either s is invertible or s is not invertible. In the latter case s is nilpotent by Proposition 3.7.

We defer the converse direction to Proposition 7.19 since we don't want to interrupt the exposition here with a certain necessary technical condition. □

3.5. Integrality. In intuitionistic logic, the notion of an integral domain bifurcates into several inequivalent notions. The following two are important for our purposes:

Definition 3.16. A ring R is an *integral domain in the weak sense* if and only if $1 \neq 0$ in R and

$$\forall x, y : R.\ xy = 0 \Longrightarrow (x = 0) \vee (y = 0).$$

A ring R is an *integral domain in the strong sense* if and only if $1 \neq 0$ in R and

$$\forall x : R.\ x = 0 \vee \ulcorner x \text{ is regular} \urcorner,$$

where $\ulcorner x$ is regular$\urcorner$ means that $xy = 0$ implies $y = 0$ for any $y : R$.

For the following result, recall that a scheme X (or a ringed space) is *integral at a point $x \in X$* if and only if $\mathcal{O}_{X,x}$ is an integral domain (in either sense, since we have adopted a classical metatheory).

Proposition 3.17. *Let X be a ringed space. Then:*

1. *X is integral at all points if and only if, internally, $\mathcal{O}_X$ is an integral domain in the weak sense.*
2. *If X is even a locally Noetherian scheme, then $\mathcal{O}_X$ is an integral domain in the weak sense iff it is an integral domain in the strong sense from the internal point of view.*

Proof. The condition on a ring to be an integral domain in the weak sense is a conjunction of two geometric implications, "$1 = 0 \Rightarrow \bot$" and the implication displayed in the definition. Therefore the first statement is obvious.

For the second statement, note that the condition on a function $f \in \Gamma(U, \mathcal{O}_X)$ to be regular from the internal perspective is open: It holds at a point $x \in U$ if and only if it holds on some open neighbourhood of x. We will give a proof of this specific feature of locally Noetherian schemes later on, when we have developed appropriate machinery to do so easily (Proposition 7.4). In any case, this openness property was the essential ingredient for the equivalence between "holding internally" and "holding at every point" (Corollary 2.11). Therefore $\mathcal{O}_X$ is an integral domain in the strong sense from the internal point of view if and only if all local rings $\mathcal{O}_{X,x}$ are integral domains. By the first statement, this is equivalent to $\mathcal{O}_X$ being an integral domain in the weak sense from the internal point of view. □

We record the following lemma for later use. The proof presented here is already simple, but a more conceptual proof is also possible (see Section 11.3).

Lemma 3.18. *Let $X = \operatorname{Spec} A$ be an affine scheme. Let $f \in A$. Then f is a regular element of A if and only if f is a regular element of $\mathcal{O}_X$ from the internal perspective.*

Proof. The Kripke–Joyal translation of internal regularity is:

> For any (without loss of generality: standard) open subset $U \subseteq X$ and any function $g \in \Gamma(U, \mathcal{O}_X)$, $fg = 0$ in $\Gamma(U, \mathcal{O}_X)$ implies $g = 0$ in $\Gamma(U, \mathcal{O}_X)$.

So the "if" direction is clear (use $U := X$). For the "only if" direction, note that $\Gamma(U, \mathcal{O}_X)$ is a localization of A and that regular elements remain regular in localizations. □

3.6. Bézout property. Recall that a *Bézout ring* is a ring in which any finitely generated ideal is a principal ideal. In intuitionistic mathematics, this is a better notion than that of a principal ideal ring: The requirement that *any* ideal is a principal ideal is far too strong. Intuitively, this is because without any given generators to begin with, one cannot hope to explicitly pinpoint a principal generator. One can (provably) not even verify this property for the ring $\mathbb{Z}$.[9]

Proposition 3.19. *Let X be a scheme (or a ringed space).*

(1) *$\mathcal{O}_X$ is a Bézout ring from the internal perspective if and only if all rings $\mathcal{O}_{X,x}$ are Bézout rings.*
(2) *$\mathcal{O}_X$ is such that, from the internal perspective, of any two elements, one divides the other, if and only if all rings $\mathcal{O}_{X,x}$ are such.*

Proof. Both properties can be formulated as geometric implications:

$$(1) \quad \forall f, g : \mathcal{O}_X.\ \top \Rightarrow \exists d : \mathcal{O}_X.\ (\exists a, b : \mathcal{O}_X.\ d = af + bg) \wedge (\exists u : \mathcal{O}_X.\ f = ud) \wedge (\exists v : \mathcal{O}_X.\ g = vd)$$

$$(2) \quad \forall f, g : \mathcal{O}_X.\ \top \Rightarrow (\exists u : \mathcal{O}_X.\ f = ug) \vee (\exists u : \mathcal{O}_X.\ g = uf)$$

□

Corollary 3.20. *Let X be a Dedekind scheme, i. e. a locally Noetherian normal scheme of dimension ≤ 1. Then, from the internal perspective, any matrix over $\mathcal{O}_X$ can be put into Smith canonical form, i. e. is equivalent to a (rectangular) diagonal matrix with diagonal entries $a_1 | a_2 | \cdots | a_n$ successively dividing each other.*

Proof. It is well-known that such a scheme has principal ideal domains as local rings $\mathcal{O}_{X,x}$. For local domains, the Bézout condition is equivalent to the property that of any two elements, one divides the other. Therefore all local rings have this property, and by the previous proposition, the internal ring $\mathcal{O}_X$ has it as well. The statement thus follows from interpreting the following fact of linear algebra in the internal universe: Let R be a ring such that of any two elements, one divides the other. Then any matrix over R can be put into Smith canonical form.

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[9] Assume that any ideal of $\mathbb{Z}$ is finitely generated. Let φ be an arbitrary statement; we want to intuitionistically deduce $\varphi \vee \neg\varphi$. Consider the ideal $\mathfrak{a} := \{x \in \mathbb{Z} \,|\, (x = 0) \vee \varphi\} \subseteq \mathbb{Z}$. The definition is such that φ holds if and only if $\mathfrak{a}$ contains an element other than zero; and that $\neg\varphi$ holds if and only if zero is the only element of $\mathfrak{a}$. By assumption, $\mathfrak{a}$ is finitely generated. Since $\mathbb{Z}$ is a Bézout ring, it is therefore even principal: $\mathfrak{a} = (x_0)$ for some $x_0 \in \mathbb{Z}$. Even intuitionistically we have $(x_0 = 0) \vee (x_0 \neq 0)$ (for the natural numbers, this can be proven by induction). In the first case, it follows that $\mathfrak{a}$ contains only zero; in the second case, it follows that $\mathfrak{a}$ contains an element other than zero. Thus $\neg\varphi \vee \varphi$.
This kind of reasoning is called *exhibiting a Brouwerian counterexample*. The definition of $\mathfrak{a}$ may look slightly dubious, considering that φ does not depend on x; but we will see that such definitions actually have a clear geometric meaning – they can be used to define extensions of sheaves by zero in the internal language (Lemma 10.2).

The usual proof of this fact is indeed intuitionistically acceptable: Let a matrix over R be given. By induction, one can show that for any finite family of ring elements, one divides all the others. So some matrix entry is a factor of all the others. We can put this entry to the upper left by row and column transformations and then kill the other entries of the first row and the first column. After these operations, it is still the case that the entry in the first row and column is a factor of all other entries. Continuing in this fashion, we obtain a diagonal matrix. Its diagonal entries already fulfill the divisibility condition and thus do not have to be sorted. $\square$

Note that phrases such as "if by chance the entry in the upper left divides all the others, we can directly proceed with the next step; otherwise, some other entry must be a factor of all entries, so ..." may not be included in a proof which is intended to be intuitionistically acceptable. Those phrases assume that one may make the case distinction that for any two ring elements x, y, either x divides y or not. Fortunately, those case distinctions are in fact superfluous.

A consequence of the corollary is that internally to the sheaf topos of a Dedekind scheme, the usual structure theorem on finitely presented $\mathcal{O}_X$-modules is available. We will exploit this in Lemma 4.21, where we give an internal proof of the fact that on Dedekind schemes, torsion-free $\mathcal{O}_X$-modules are locally free.

3.7. Normality. We will discuss the property of a ring to be *normal*, i. e. to be integrally closed in its total field of fractions, in Section 7.3, after giving an internal characterization of the sheaf of rational functions.

3.8. Special properties of constant sheaves of rings. Let R be an ordinary ring and $\underline{R}$ the associated sheaf of locally constant R-valued functions on a topological space. If R is reduced, local, or a field, then $\underline{R}$ is so as well, from the internal point of view.

We will prove this in greater generality: Appropriately formulated, a constant sheaf $\underline{R}$ has some property φ from the internal point of view if and only if R has the property φ externally (Lemma 11.1).

3.9. Noetherian properties. Recall the usual notion of a Noetherian ring: Any sequence $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots$ of ideals should stabilize, i. e. there should exist a natural number n such that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$.

Intuitionistically, this definition has two problems. Firstly, without the axiom of dependent choice, it is often not possible to construct a *sequence* of ideals: Often, it is only possible to show that there *exists* a suitable ideal $\mathfrak{a}_{n+1}$ depending on $\mathfrak{a}_n$. But since in general there is no canonical choice for this successor ideal, the axiom of dependent choice would be required to collect those into a sequence, i. e. a function from $\mathbb{N}$ to the set of ideals.

Secondly, the conclusion that the sequence stabilizes is too strong. Intuitionistically, one cannot even show that a weakly descending sequence of natural numbers stabilizes in this sense; the statement that one could is equivalent to the *limited principle of omniscience for* $\mathbb{N}$. Intuitionistically, it is only true that a weakly descending sequence $a_0 \geq a_1 \geq \cdots$ of natural numbers eventually *halts* in the sense that there exists an index n such that $a_n = a_{n+1}$ (but $a_{n+1} > a_{n+2}$ is allowed).[10]

We give two constructively inequivalent notions of Noetherian rings. The first one is of independent constructive interest and enjoys the property that the structure

[10]Classically, the following three statements on a ring are equivalent: (1) Every ascending chain of ideals stabilizes. (2) Every ascending chain of finitely generated ideals stabilizes. (3) Every ascending chain of finitely generated ideals halts.

sheaf of a scheme X satisfies the Noetherian condition from the internal point of view of $\mathrm{Sh}(X)$ if and only if X is locally Noetherian.

The second one is quite weak from a constructive point of view, but still interesting from a geometric point of view and useful enough to derive nontrivial consequences. It is satisfied by the structure sheaf of any (not necessarily locally Noetherian) reduced scheme.

Processly Noetherian rings.

Definition 3.21. Let M be a partially ordered set. An *ascending process with values in M* consists of an initial value $x_0 \in M$ and a function $f : M \to \mathcal{P}(M)$ such that:

- For any $x \in M$ and any $y \in f(x)$, $x \preceq y$.
- The set $f(x_0)$ is inhabited.
- For any $x_1 \in f(x_0)$, the set $f(x_1)$ is inhabited.
- For any $x_1 \in f(x_0)$ and any $x_2 \in f(x_1)$, the set $f(x_2)$ is inhabited.
- And so on.

Such a process *halts* if and only if there exists a step n and elements $x_1, \ldots, x_n$ such that $x_{i+1} \in f(x_i)$ for $i = 0, \ldots, n-1$ and such that $x_n \in f(x_n)$. The set M satisfies the *ascending process condition* if and only if all ascending processes with values in M halt.

Intuitively, we picture $f(x)$ as the set of all possible results of running the process for a single step, starting with the value x. This set could be a singleton, but it may also contain more than one element, for instance if the process cannot provide the next value in a canonical way. Instead of arbitrarily choosing a definitive value for its result, the process may instead collect all the possible values in the set $f(x)$.

Definition 3.22. A ring A is *processly Noetherian* if and only if the set of finitely generated ideals in A satisfies the ascending process condition.

An ascending chain of elements $a_0 \preceq a_1 \preceq \cdots$ in a partially ordered set gives rise to an ascending process by setting $x_0 := a_0$ and $f(x) := \{y \mid \exists n.\ x = a_n \wedge y = a_{n+1}\}$. (This process halts iff there is an index n such that $a_n = a_{n+1}$.) Conversely, the axiom of dependent choice would allow to construct an ascending chain from an ascending process. In the presence of this axiom, for instance in a classical context, a ring is therefore processly Noetherian if and only if it is Noetherian in the usual sense.

The notion of a processly Noetherian ring works well in an intuitionistic context: Important rings such as $\mathbb{Z}$ and more generally $\mathcal{O}_K$ for any algebraic number field K are processly Noetherian, and matrices over Bézout rings which are integral domains in the weak sense and processly Noetherian can be put into Smith canonical form.

Richman also studied Noetherian rings in a constructive context without dependent choice [65]. His notion of *ascending tree condition* is equivalent to our ascending process condition. His condition emphasizes the branching nature of a non-deterministic computation, while ours emphasizes the step-for-step picture of computation.

There are three reasons why we did not define a ring to be processly Noetherian if and only if the set of all (not only finitely generated) ideals satisfies the ascending process condition. Firstly, this stricter condition excludes rings as $\mathbb{Z}$.[11] Secondly, restricting to finitely generated ideals in this context is a well-established procedure in

[11] The main ingredient in the proof that $\mathbb{Z}$ is Noetherian is that any ideal of $\mathbb{Z}$ is a principal ideal, since (looking at the prime factor decomposition) one can give explicit bounds on the length of strictly ascending chains of principal ideals. However, as detailed in the footnote on page 29, constructively one cannot show that every ideal of $\mathbb{Z}$ is a principal ideal; one can only verify

constructive mathematics [59, 65] and suffices for the applications of the Noetherian condition one typically expects. Thirdly, our definition provides a link to the external condition on a scheme to be locally Noetherian, as shown by the following proposition.

XXX: Adapt the proof of the proposition to conform to the new (corrected) definition.

Proposition 3.23. *A scheme X is locally Noetherian if and only if the ring $\mathcal{O}_X$ is processly Noetherian from the internal point of view.*

Proof. We only prove the "only if" direction. We may assume that $X = \operatorname{Spec} A$ is affine with A a Noetherian ring and that internally, we are given an ascending process on the set of finitely generated ideals of $\mathcal{O}_X$. Externally, this is a morphism $\underline{\mathbb{N}} \to \mathcal{P}(\mathcal{M})$ where $\mathbb{N}$ is the constant sheaf with value $\mathbb{N}$ and U-sections of $\mathcal{M}$ are finite type ideal sheaves of $\mathcal{O}_X|_U$.

Since $X \models \ulcorner f(0)$ is inhabited$\urcorner$, there exists an open covering $X = \bigcup_i U_i$ and finite type ideal sheaves $\mathcal{I}_i \hookrightarrow \mathcal{O}_X|_{U_i}$ such that $U_i \models \mathcal{I}_i \in f(0)$. Without loss of generality, we may assume that the open sets U_i are standard open sets $D(f_i)$ and that the covering is finite. Since the sheaves $\mathcal{I}_i$ are quasicoherent (being of finite type, they are images of morphisms of the form $(\mathcal{O}_X|^n_{U_i} \to \mathcal{O}_X|_{U_i})$, they correspond to ideals $J_i \subseteq A[f_i^{-1}]$. Note for future reference that for $D(g) \subseteq D(f_i)$, the restricted ideal $\mathcal{I}_i|_{D(g)}$ corresponds to the extension of J_i in the further localized ring $A[g^{-1}]$.

For each i, $D(f_i) \models \exists \mathfrak{a} \in f(1).\ \mathcal{I}_i \subseteq \mathfrak{a}$. Thus there exists an open covering $D(f_i) = \bigcup_j D(g_{ij})$ and finite type ideal sheaves $\mathcal{I}_{ij} \hookrightarrow \mathcal{O}_X|_{D(g_{ij})}$; these correspond to ideals $J_{ij} \subseteq A[g_{ij}^{-1}]$ such that $J_i \subseteq J_{ij}$ (where we have suppressed the localization morphism $A[f_i^{-1}] \to A[g_{ij}^{-1}]$ in the notation). Equivalently, writing $J_i' := A \cap J_i$ and $J_{ij}' := A \cap J_{ij}$ for the contractions, we have the inclusions $J_i' \subseteq J_{ij}'$ of ideals of A.

Continuing in this fashion, we obtain a tree of ideals $J'_{i_1 \cdots i_n}$. Each path in this tree is a chain of ascending ideals and thus stabilizes since A is Noetherian. Since only finitely many subtrees branch off at each node, there appear only finitely many distinct ideals in this tree (this is an application of the graph-theoretical *König's lemma*).

There thus exists a natural number n such that $J'_{i_1 \cdots i_n} = J'_{i_1 \cdots i_n i_{n+1}}$ for all appropriate indices $i_1, \ldots, i_n, i_{n+1}$. For this number n, the internal statement $X \models \exists \mathfrak{a} \in f(n) \cap f(n+1)$ holds; we leave further details to the reader. ☐

Remark 3.24. The proof shows that, internally speaking, even the set of all quasicoherent ideals (instead of merely the finitely generated ones) fulfills the ascending process condition, if the base scheme is locally Noetherian. We have not taken this property as the definition of a processly Noetherian ring since it is a notion not usually studied in constructive mathematics (compare Remark 9.9).

XXX: Give examples made possible by the internal Noetherianity XXX: How about: Any quasicoherent submodule of a module of finite type is of finite type as well.

Weakly Noetherian rings. Classically, there is a characterization of Noetherian rings which doesn't involve ascending sequences: A ring is Noetherian if and only if any of its ideals is finitely generated. We mentioned in the footnote on page 29 that this condition is far too strong from a constructive point of view; not even the ring $\mathbb{Z}$ verifies it. However, it can be weakened to yield a constructively sensible notion:

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that finitely generated ideals are principal. Geometrically, ideals which are not finitely generated correspond to sheaves of ideals which may fail to be quasicoherent.

Definition 3.25. A ring A is *weakly Noetherian* if and only if any ideal of A is *not not* finitely generated. A module M is *weakly Noetherian* if and only if any submodule of M is *not not* finitely generated.

Example 3.26. There is an intuitionistic proof that the ring $\mathbb{Z}$ is weakly Noetherian: Let $\mathfrak{a} \subseteq \mathbb{Z}$ be any ideal. Under the assumption that either there exists a nonzero element in $\mathfrak{a}$ or not, the ideal $\mathfrak{a}$ is *not not* finitely generated, even *not not* principal: For in the first case, a minimal element d of $\mathfrak{a} \cap \mathbb{N}^+$ (which *not not* exists) witnesses $\mathfrak{a} = (d)$. In the second case the ideal $\mathfrak{a}$ is the zero ideal. Since the assumption is *not not* satisfied, the ideal $\mathfrak{a}$ is *not not not not* finitely generated, so *not not* finitely generated. (We remark on this proof scheme on page 46.)

Theorem 3.27. *Let A be a weakly Noetherian ring. Then the polynomial algebra $A[X]$ is weakly Noetherian as well, intuitionistically.*

Proof. Classically, this is precisely the statement of Hilbert's basis theorem, whose usual accounts do not care about the sensibilities of constructive mathematics. However, a careful reading of for instance the proof given in [6, Theorem 7.5] shows that the theorem holds intuitionistically as stated. □

Lemma 3.28. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. Intuitionistically, the module M is weakly Noetherian if and only if M' and M'' are.*

Proof. The usual proof applies. □

Proposition 3.29. *Let X be an arbitrary reduced scheme (not necessarily locally Noetherian). Then $\mathcal{O}_X$ is weakly Noetherian from the internal point of view of* $\mathrm{Sh}(X)$.

Proof. By Corollary 3.9, the ring $\mathcal{O}_X$ fulfills a suitable field condition from the internal point of view. Therefore it suffices to give an intuitionistic proof of the following statement: Let k be a ring such that any element of k which is not invertible is zero. Then any ideal of k is *not not* finitely generated.

Let $\mathfrak{a} \subseteq k$ be an arbitrary ideal. We have $\neg\neg(1 \in \mathfrak{a} \vee 1 \not\in \mathfrak{a})$. Therefore $\neg\neg(\mathfrak{a} = (1) \vee \mathfrak{a} = (0))$. Thus $\mathfrak{a}$ is *not not* finitely generated (even *not not* principal). □

The fact that $\mathcal{O}_X$ is weakly Noetherian from the internal point of view will allow for a quick proof of Grothendieck's generic freeness lemma in Section 11.5. The translation of the statement that $\mathcal{O}_X[U_1, \ldots, U_n]$ is weakly Noetherian was displayed on page 14.

4. Sheaves of modules

From the internal perspective, a sheaf of $\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of rings, looks just like a plain module over the plain ring $\mathcal{R}$. This is proven just as the correspondence between sheaf of rings and internal rings (Proposition 3.1).

XXX: talk about modules over constant sheaves of rings as well

4.1. Finite local freeness. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *finite locally free* if and only if there exists a covering of X by open subsets U such that on each such U, the restricted module $\mathcal{F}|_U$ is isomorphic as an $\mathcal{O}_X|_U$-module to $(\mathcal{O}_X|_U)^n$ for some natural number n (which may depend on U).

Proposition 4.1. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is finite locally free if and only if, from the internal perspective, $\mathcal{F}$ is a finite free module, i. e.*

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \ulcorner \mathcal{F} \cong (\mathcal{O}_X)^n \urcorner,$$

or more elementarily

$$\mathrm{Sh}(X) \models \bigvee_{n\geq 0} \exists x_1,\ldots,x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1,\ldots,a_n : \mathcal{O}_X.\ x = \sum_i a_i x_i.$$

Proof. By the expression "$(\mathcal{O}_X)^n$" in the internal language we mean the internally constructed object $\mathcal{O}_X \times \cdots \times \mathcal{O}_X$ with its componentwise $\mathcal{O}_X$-module structure. This coincides with the sheaf $(\mathcal{O}_X)^n$ as usually understood.

It is clear that the two stated internal conditions are equivalent, since the corresponding proof in linear algebra is intuitionistically acceptable. The equivalence with the external notion of finite local freeness follows because the interpretation of the first condition with the Kripke–Joyal semantics is the following: There exists a covering of X by open subsets U such that for each such U, there exists a natural number n and a morphism of sheaves $\varphi : \mathcal{F}|_U \to (\mathcal{O}_X|_U)^n$ such that

$$U \models \ulcorner\varphi \text{ is } \mathcal{O}_X\text{-linear}\urcorner \quad\text{and}\quad U \models \ulcorner\varphi \text{ is bijective}\urcorner.$$

The first subcondition means that φ is a morphism of sheaves of $\mathcal{O}_X|_U$-modules and the second one means that φ is an isomorphism of sheaves. □

XXX: algebras of finite type, ...

4.2. Finite type, finite presentation, coherence. Recall the conditions of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or a ringed space) to be of finite type, of finite presentation, and to be coherent:

- $\mathcal{F}$ is *of finite type* if and only if there exists a covering of X by open subsets U such that for each such U, there exists an exact sequence
$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$
of $\mathcal{O}_X|_U$-modules.
- $\mathcal{F}$ is *of finite presentation* if and only if there exists a covering of X by open subsets U such that for each such U, there exists an exact sequence
$$(\mathcal{O}_X|_U)^m \longrightarrow (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$
- $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type and the kernel of any $\mathcal{O}_X|_U$-linear morphism $(\mathcal{O}_X|_U)^n \to \mathcal{F}|_U$, where $U \subseteq X$ is any open subset, is of finite type.

The following proposition gives translations of these definitions into the internal language.

Proposition 4.2. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then:*

- *$\mathcal{F}$ is of finite type if and only if $\mathcal{F}$, considered as an ordinary module from the internal perspective, is finitely generated, i. e. if*
$$\mathrm{Sh}(X) \models \bigvee_{n\geq 0} \exists x_1,\ldots,x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1,\ldots,a_n : \mathcal{F}.\ x = \sum_i a_i x_i.$$
- *$\mathcal{F}$ is of finite presentation if and only if $\mathcal{F}$ is a finitely presented module from the internal perspective, i. e. if*
$$\mathrm{Sh}(X) \models \bigvee_{n,m\geq 0} \ulcorner\text{there is a short exact sequence } \mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0\urcorner.$$

- *$\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is a coherent module from the internal perspective, i. e. if*

$$\begin{aligned}\mathrm{Sh}(X) \models\ & \ulcorner \mathcal{F} \text{ is finitely generated} \urcorner \wedge \\ & \bigwedge_{n \geq 0} \forall \varphi : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \ulcorner \ker \varphi \text{ is finitely generated} \urcorner.\end{aligned}$$

Proof. Straightforward – the translations of the internal statements using the Kripke–Joyal semantics are precisely the corresponding external statements. □

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *generated by global sections* if and only if there exist global sections $s_i \in \Gamma(X, \mathcal{F})$ such that for any $x \in X$, the stalk $\mathcal{F}_x$ is generated by the germs of the s_i. This condition is of course not local on the base. Therefore there cannot exist a formula φ such that for any space X and any $\mathcal{O}_X$-module $\mathcal{F}$ it holds that $\mathcal{F}$ is generated by global sections if and only if $\mathrm{Sh}(X) \models \varphi(\mathcal{F})$. But still, global generation can be characterized by a mixed internal/external statement:

Proposition 4.3. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is generated by global sections if and only if there exist global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that*

$$\mathrm{Sh}(X) \models \forall x : \mathcal{F}.\ \bigvee\nolimits_{J = \{i_1, \ldots, i_n\} \subseteq I \text{ finite}} \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_j a_j x_{i_j}.$$

Furthermore, $\mathcal{F}$ is generated by finitely many global sections if and only if there exist global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{F})$ such that

$$\mathrm{Sh}(X) \models \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_j a_j x_j.$$

Proof. The given internal statements are geometric implications, their validity can thus be checked stalkwise. □

4.3. Tensor product and flatness. Recall that the tensor product of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X (or a ringed space) is usually constructed as the sheafification of the presheaf

$$U \subseteq X \text{ open} \quad \longmapsto \quad \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$

From the internal point of view, $\mathcal{F}$ and $\mathcal{G}$ look like ordinary modules, so that we can consider their tensor product as usually constructed in commutative algebra, as a certain quotient of the free module on the elements of $\mathcal{F} \times \mathcal{G}$:

$$\mathcal{O}_X \langle x \otimes y \mid x : \mathcal{F}, y : \mathcal{G} \rangle / R,$$

where R is the submodule generated by

$$\begin{gathered}(x + x') \otimes y - x \otimes y - x' \otimes y, \\ x \otimes (y + y') - x \otimes y - x \otimes y', \\ (sx) \otimes y - s(x \otimes y), \\ x \otimes (sy) - s(x \otimes y)\end{gathered}$$

with $x, x' : \mathcal{F}$, $y, y' : \mathcal{G}$, $s : \mathcal{O}_X$. This internal construction gives rise to the same sheaf of modules as the externally defined tensor product:

Proposition 4.4. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the internally constructed tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ coincides with the external one.*

Proof. Since the proof of the corresponding fact of commutative algebra is intuitionistically acceptable, the internally defined tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ has the following universal property: For any $\mathcal{O}_X$-module H, any $\mathcal{O}_X$-bilinear map $\mathcal{F} \times \mathcal{G} \to H$ uniquely factors over the canonical map $\mathcal{F} \times \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Interpreting this property with the Kripke–Joyal semantics, we see that the internally constructed tensor product has the following external property: For any open subset $U \subseteq X$ and any $\mathcal{O}_X|_U$-module $\mathcal{H}$ on U, any $\mathcal{O}_X|_U$-bilinear morphism $\mathcal{F}|_U \times \mathcal{G}|_U \to \mathcal{H}$ uniquely factors over the canonical morphism $\mathcal{F}|_U \times \mathcal{G}|_U \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U$.

In particular, for $U = X$, this property is well-known to be the universal property of the externally constructed tensor product. Therefore the claim follows. $\square$

A description of the stalks of the tensor product follows purely by considering the logical form of the construction:

Corollary 4.5. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the stalks of the tensor product coincide with the tensor products of the stalks: $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.*

Proof. We constructed the tensor product using the following operations: product of two sets, free module on a set, quotient module with respect to a submodule; submodule generated by a set of elements given by a geometric formula. All of these operations are geometric, so the tensor product construction is geometric as well (see Section 2.3). Hence taking stalks commutes with performing the construction. $\square$

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if and only if all stalks $\mathcal{F}_x$ are flat $\mathcal{O}_{X,x}$-modules. We can characterize flatness in the internal language.

Proposition 4.6. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat if and only if, from the internal perspective, $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Recall that flatness of an A-module M can be characterized without reference to tensor products by the following condition (using suggestive vector notation): For any natural number p, any p-tuple $m : M^p$ of elements of M and any p-tuple $a : A^p$ of elements of A, it should hold that

$$a^T m = 0 \implies \bigvee_{q \geq 0} \exists n : M^q, B : A^{p \times q}.\ Bn = m \wedge a^T B = 0.$$

The equivalence of this condition with tensoring being exact holds intuitionistically as well [59, Theorem III.5.3]. This formulation of flatness has the advantage that it is the conjunction of geometric implications (one for each $p \geq 0$); therefore it holds internally if and only if it holds at any point. $\square$

4.4. Support. Recall that the *support* of an $\mathcal{O}_X$-module $\mathcal{F}$ is the subset $\operatorname{supp} \mathcal{F} := \{x \in X \mid \mathcal{F}_x \neq 0\} \subseteq X$. If $\mathcal{F}$ is of finite type, this set is closed, since its complement is then open by a standard lemma. (We will give an internal proof of this fact in Lemma 6.39.)

Proposition 4.7. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the interior of the complement of the support of $\mathcal{F}$ can be characterized as the largest open subset of X on which the internal statement $\mathcal{F} = 0$ holds.*

Proof. For any open subset $U \subseteq X$, it holds that:

$$\begin{aligned} & U \subseteq \operatorname{int}(X \setminus \operatorname{supp} \mathcal{F}) \\ \iff & U \subseteq X \setminus \operatorname{supp} \mathcal{F} \\ \iff & U \subseteq \{x \in X \mid \forall s \in \mathcal{F}_x.\ s = 0\} \\ \iff & U \models \forall s\,{:}\,\mathcal{F}.\ s = 0 \\ \iff & U \models \ulcorner \mathcal{F} = 0 \urcorner \end{aligned}$$

The second to last equivalence is because "$\forall s\,{:}\,\mathcal{F}.\ s = 0$" is a geometric implication and can thus be checked stalkwise. □

Remark 4.8. The support of a sheaf of *sets* $\mathcal{F}$ is defined as the subset $\{x \in X \mid \mathcal{F}_x \text{ is not a singleton}\}$. A similar proof shows that the interior of its complement can be characterized as the largest open subset of X where the internal statement $\ulcorner \mathcal{F}$ is a singleton$\urcorner$ holds.

4.5. Torsion. Let R be a ring. Recall that the *torsion submodule* M_{tors} of an R-module M is defined as

$$M_{\text{tors}} := \{x : M \mid \exists a : R.\ \ulcorner a \text{ regular} \urcorner \wedge ax = 0\} \subseteq M.$$

This definition is meaningful even if R is not an integral domain. An R-module M is *torsion-free* if and only if M_{tors} is the zero submodule; an R-module M is a *torsion module* if and only if $M_{\text{tors}} = M$.

Recall also that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an integral scheme X, there is a unique subsheaf $\mathcal{F}_{\text{tors}} \subseteq \mathcal{F}$ with the property that $\Gamma(U, \mathcal{F}_{\text{tors}}) = \Gamma(U, \mathcal{F})_{\text{tors}}$ for all affine open subsets $U \subseteq X$. The content of the following proposition is that internally constructing the torsion submodule of $\mathcal{F}$, regarded as a plain module from the internal perspective, gives exactly this subsheaf. There is therefore no harm in using the same notation "$\mathcal{F}_{\text{tors}}$" for the result of the internal construction.

Proposition 4.9. *Let X be an integral scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $U = \operatorname{Spec} A \subseteq X$ be an affine open subset. Let $s \in \Gamma(U, \mathcal{F})$ be a local section. Then*

$$s \in \Gamma(U, \mathcal{F})_{\text{tors}} \quad \textit{if and only if} \quad U \models s \in \mathcal{F}_{\text{tors}}.$$

Proof. The "only if" direction is trivial in view of Lemma 3.18: If s is a torsion element of $\Gamma(U, \mathcal{F})$, there exists a regular element $a \in \Gamma(U, \mathcal{O}_X)$ such that $as = 0$. By the lemma, this element is regular from the internal perspective as well, so $U \models \ulcorner a \text{ regular} \urcorner \wedge as = 0$.

For the "if" direction, we may assume that there exists an open covering $X = \bigcup_i U_i$ by standard open subsets $U_i = D(f_i)$ such that there are sections $a_i \in \Gamma(U_i, \mathcal{O}_X) = A[f_i^{-1}]$ with $U_i \models \ulcorner a_i \text{ regular} \urcorner \wedge a_i s = 0$. Without loss of generality, we may assume that the denominators of the a_i's are ones, that the f_i are finite in number, and that the f_i are regular (i. e. nonzero, since A is an integral domain). By Lemma 3.18, the a_i are regular in $A[f_i^{-1}]$ and by regularity of the f_i also regular in A. Therefore their product $\prod_i a_i \in A$ is regular in A as well and annihilates s. □

Proposition 4.10. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then $(\mathcal{F}_{\text{tors}})_x = (\mathcal{F}_x)_{\text{tors}}$.*

Proof. This would be obvious if the condition on an element $s : \mathcal{F}$ to belong to $\mathcal{F}_{\text{tors}}$ were a geometric formula. Because of the universal quantifier, it is not:

$$s \in \mathcal{F}_{\text{tors}} \quad \iff \quad \exists a : \mathcal{O}_X.\ (\forall b : \mathcal{O}_X.\ ab = 0 \Rightarrow b = 0) \wedge as = 0.$$

But since X is assumed to be locally Noetherian, regularity is an open property nonetheless (see Proposition 7.4 for an internal proof of this fact). Thus the claim still follows, just like in the proof of Proposition 3.17. □

4.6. Internal proofs of common lemmas.

Lemma 4.11. *Let X be a scheme (or a ringed space). Let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{H}$ are of finite type, so is $\mathcal{G}$; similarly, if $\mathcal{F}$ and $\mathcal{H}$ are finite locally free, so is $\mathcal{G}$.

Proof. From the internal perspective, we are given a short exact sequence of modules with the outer ones being finitely generated (resp. finite free) and we have to show that the middle one is finitely generated (resp. finite free) as well. It is well-known that this follows; and since the usual proof of this fact is intuitionistically acceptable, we are done. □

Note that the proof works very generally, in the context of arbitrary ringed spaces, and is still very simple. This is common to proofs using the internal language. Particular features of schemes enter only at clearly recognizable points, for instance when an internal property specific to the structure sheaf of schemes is used (such as in Proposition 3.7).

Lemma 4.12. *Let X be a scheme (or a ringed space).*

- *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If two of the three modules are coherent, so is the third.*
- *Let $\mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules such that $\mathcal{F}$ is of finite type and $\mathcal{G}$ is coherent. Then its kernel is of finite type as well.*
- *If $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module and $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module, the $\mathcal{O}_X$-modules $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ are coherent as well.*

Proof. These statements follow directly from interpreting the corresponding standard proofs of commutative algebra in the internal language. For those standard proofs, see for instance the lecture notes of Ravi Vakil [76, Section 13.8], where they are given as a series of exercises. □

Lemma 4.13. *Let X be a scheme (or locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of finite locally free $\mathcal{O}_X$-modules. Then the kernel of α is finite locally free as well.*

Proof. It suffices to give an intuitionistic proof of the following statement: The kernel of a matrix over a local ring, which as a linear map is surjective, is finite free.

Let $M : R^{n\times m}$ be such a matrix. Since by the surjectivity assumption some linear combination of the columns is e_1 (the first canonical basis vector), some linear combination of the entries of the first row of M is 1. By locality of R, at least one entry of the first row is invertible. By applying appropriate column and row transformations, we may therefore assume that M is of the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \widetilde{M} & \\ 0 & & & \end{array}\right)$$

with the submatrix $\widetilde{M}$ fulfilling the same condition as M. Continuing in this way, it follows that $m \geq n$ and that we may assume that M is of the form

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right).$$

The kernel of such a matrix is obviously freely generated by the canonical basis vectors corresponding to the zero columns. In particular, the rank of the kernel is $m - n$. □

Remark 4.14. The internal language machinery gives no reason to believe that the dual statement is true, i. e. that the cokernel of a monomorphism of finite locally free $\mathcal{O}_X$-modules is finite locally free. This would follow from an intuitionistic proof of the statement that the cokernel of an injective map between finite free modules over a local ring is finite free. But this statement is of course false (consider the exact sequence $0 \longrightarrow \mathbb{Z}_{(2)} \xrightarrow{\cdot 2} \mathbb{Z}_{(2)} \longrightarrow \mathbb{F}_2 \longrightarrow 0$ of $\mathbb{Z}_{(2)}$-modules).

Lemma 4.15. *Let X be a scheme (or locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of finite locally free $\mathcal{O}_X$-modules of the same rank. Then α is an isomorphism.*

Proof. It suffices to give an intuitionistic proof of the following statement: A square matrix over a local ring, which as a linear map is surjective, is invertible.

This follows from the proof of the previous lemma, since it shows that the kernel of such a matrix is finite free of rank zero. □

Remark 4.16. The conclusion of Lemma 4.15 also holds if X is only assumed to be a ringed space. To show this, it suffices to give an intuitionistic proof of the following statement: A square matrix over a (not necessarily local) ring, which as a linear map is surjective, is invertible. Such a matrix A possesses a right inverse. Therefore $\det A$ is invertible. Thus A is invertible with inverse $(\det A)^{-1} \cdot \operatorname{ad} A$.

Lemma 4.17. *Let X be a scheme (or a ringed space). Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Then for the closures of the supports there holds the equation* $\operatorname{cl} \operatorname{supp} \mathcal{G} = \operatorname{cl} \operatorname{supp} \mathcal{F} \cup \operatorname{cl} \operatorname{supp} \mathcal{H}$.

Proof. Switching to complements, we have to prove that

$$\operatorname{int}(X \setminus \operatorname{supp} \mathcal{G}) = \operatorname{int}(X \setminus \operatorname{supp} \mathcal{F}) \cap \operatorname{int}(X \setminus \operatorname{supp} \mathcal{H}).$$

By Proposition 4.7, it suffices to prove

$$\mathrm{Sh}(X) \models (\mathcal{G} = 0 \iff \mathcal{F} = 0 \land \mathcal{H} = 0);$$

this is a basic observation in linear algebra, valid intuitionistically. □

Of course, a stronger version of this lemma – about the supports themselves instead of their closures – is easily proven without using the internal language. We included this example only for illustrative purposes.

Lemma 4.18. *Let X be a scheme (or locally ringed space). Let $\mathcal{L}$ be a line bundle on X, i. e. an $\mathcal{O}_X$-module locally free of rank 1. Let $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L})$ be global sections. Then these sections globally generate $\mathcal{F}$ if and only if*

$$\mathrm{Sh}(X) \models \bigvee_i \ulcorner \alpha(s_i) \text{ is invertible for some isomorphism } \alpha : \mathcal{L} \to \mathcal{O}_X \urcorner.$$

Proof. It suffices to give an intuitionistic proof of the following fact: Let R be a local ring. Let L be a free R-module of rank 1. Let $s_1, \ldots, s_n : L$ be given elements. Then L is generated as an R-module by these elements if and only if for some i, the image of s_i under some isomorphism $L \to R$ is invertible.

Note that the choice of such an isomorphism does not matter, since any two such isomorphisms $\alpha, \beta : L \to R$ differ by a unit of R: $\alpha(x) = \alpha(\beta^{-1}(1)) \cdot \beta(x)$ for any $x : L$, and $\alpha(\beta^{-1}(1)) \cdot \beta(\alpha^{-1}(1)) = 1$ in R.

For the "if" direction, we have that some $\alpha(s_i)$ is a generator of R. Since α is an isomorphism, it follows that s_i generates L, and thus that in particular, the family $s_1, \ldots, s_n$ generates L.

For the "only if" direction, we have that the unit of R can be expressed as a linear combination of the $\alpha(s_i)$, where $\alpha : L \to R$ is some isomorphism (whose existence is assured by the assumption on the rank of L). Since R is a local ring, it follows that one of the summands and thus one of the $\alpha(s_i)$ is invertible. $\square$

Remark 4.19. Note that the canonical ring homomorphism $\mathcal{O}_{X,x} \twoheadrightarrow k(x)$ is local. Therefore a germ in $\mathcal{O}_{X,x}$ is invertible if and only if its image in $k(x)$ is not zero. From this one can conclude that global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{F})$ generate $\mathcal{F}$ if and only if, for any point $x \in X$, the images $s_i \in \mathcal{F}|_x$ in the fibers do not vanish simultaneously.

Lemma 4.20. *Let X be a scheme (or a ringed space). Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of rank 1. Then $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.*

Proof. Recall that the dual is defined by $\mathcal{L}^\vee := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. Since "$\mathcal{H}$om" looks like "Hom" from the internal point of view, the dual sheaf $\mathcal{L}^\vee$ looks just like the ordinary dual module. However, to prove the claim, it does *not* suffice to give an intuitionistic proof of the following fact of linear algebra: "Let L be a free R-module of rank 1. Then there exists an isomorphism $L^\vee \otimes_R L \to R$." Since the interpretation of "$\exists$" using the Kripke–Joyal semantics is local existence, this would only show that $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L}$ is *locally* isomorphic to $\mathcal{O}_X$.

Instead, we have to actually *write down* (i. e. explicitly give) a linear map in the internal language – not using the assumption that L is free of rank 1, as this would introduce an existential quantifier again (see Section 2.2). So we have to prove the following fact: Let L be an R-module. Then there explicitly exists a linear map $L^\vee \otimes_R L \to R$ such that this map is an isomorphism if L is free of rank 1.

This is done as usual: Define $\alpha : L^\vee \otimes_R L \to R$ by $\lambda \otimes x \mapsto \lambda(x)$. Since L is free of rank 1, there is an isomorphism $L \cong R$. Precomposing α with the induced isomorphism $R^\vee \otimes_R R \to L^\vee \otimes_R L$, we obtain the linear map $R^\vee \otimes_R R \to R$ given by the same term: $\lambda \otimes x \mapsto \lambda(x)$. One can check that an inverse is given by $x \mapsto \mathrm{id}_R \otimes x$. $\square$

Lemma 4.21. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

1. *Assume X to be a locally Noetherian scheme. Then $\mathcal{F}$ is torsion-free (meaning $\mathcal{F}_{\mathrm{tors}} = 0$) if and only if all stalks $\mathcal{F}_x$ are torsion-free.*
2. *The quotient sheaf $\mathcal{F}/\mathcal{F}_{\mathrm{tors}}$ is torsion-free and the torsion submodule $\mathcal{F}_{\mathrm{tors}}$ is a torsion module.*
3. *The dual sheaf $\mathcal{F}^\vee$ is torsion-free.*
4. *If $\mathcal{F}$ is reflexive (meaning that the canonical morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism), it is torsion-free.*
5. *If $\mathcal{F}$ is finite locally free, it is reflexive.*
6. *Assume X to be a Dedekind scheme and $\mathcal{F}$ to be of finite presentation. If $\mathcal{F}$ is torsion-free, then it is finite locally free.*

Proof. The first statement follows from the observation that $(\mathcal{F}_{\mathrm{tors}})_x = (\mathcal{F}_x)_{\mathrm{tors}}$ (Proposition 4.10). All the others follow simply by interpreting the corresponding facts of linear algebra in the internal universe. For concreteness, we give intuitionistic proofs of the last three statements.

So let M be an reflexive R-module. We have to show that M is torsion-free. To this end, let an element $x : M$ and a regular element $a : R$ such that $ax = 0$ be given. For any $\vartheta : M^\vee$, it follows that $\vartheta(x) = 0$, since $a\vartheta(x) = \vartheta(ax) = \vartheta(0) = 0$ and a is regular. Thus the image of x under the canonical map $M \to M^{\vee\vee}$ is zero. By reflexivity, this implies that x is zero.

For statement (5), we have to prove that R-modules of the form R^n are reflexive. This is obvious, the required inverse map is $(R^n)^{\vee\vee} \to R^n,\ \lambda \mapsto \sum_i \lambda(\vartheta_i)$ where $\vartheta_i : R^n \to R,\ (x_j)_j \mapsto x_i$.

In view of Corollary 3.20 we can put matrices over $\mathcal{O}_X$ into Smith canonical form if X is a Dedekind scheme. Therefore it suffices to give an intuitionistic proof of the following fact: Let R be an integral domain in the strong sense such that matrices over R can be put into Smith canonical form. Let M be a finitely presented torsion-free R-module. Then M is finite free.

This goes as follows: Since M is finitely presented, it is the cokernel of some matrix. Without loss of generality, we may assume that it is a diagonal matrix, so M is isomorphic to some (finite) direct sum $\bigoplus_i R/(a_i)$. Since M is torsion-free, all the summands $R/(a_i)$ are torsion-free as well. Since R is an integral domain in the strong sense, this holds if and only if the a_i are either zero or invertible. Thus $R/(a_i)$ is isomorphic to R or to the zero module. In any case, $R/(a_i)$ is finite free and therefore M is finite free as well. $\square$

5. Upper semicontinuous functions

5.1. Interlude on natural numbers. In classical logic, the natural numbers are complete in the sense that any inhabited set of natural numbers possesses a minimal element. This statement can not be proven intuitionistically – intuitively, this is because one cannot explicitly pinpoint the (classically existing) minimal element of an arbitrary inhabited set;[12] see below for a sheaf-theoretic interpretation.

In intuitionistic logic, the completeness principle can be salvaged in two essentially different ways: either by strengthening the premise, or by weakening the conclusion.

Lemma 5.1. *Let $U \subseteq \mathbb{N}$ be an inhabited subset of the natural numbers.*

(1) *Assume U to be* detachable*, i. e. assume that for any natural number n, either $n \in U$ or $n \notin U$. Then U possesses a minimal element.*
(2) *In any case, U does* not not *possess a minimal element.*

Proof. The first statement can be proven by induction on the witness of inhabitation, i. e. the given number n such that $n \in U$. We omit further details, since we will not need this statement in our applications.

For the second statement, we give a careful proof since logical subtleties matter. To simplify the exposition, we assume that U is upward-closed, i. e. that any number larger than some element of U lies in U as well. Any subset can be closed in this way (by considering $\{n \in \mathbb{N} \mid \exists m \in U.\ n \geq m\}$) and a minimal element of the closure will be a minimal element for U as well.

We induct on the number $n \in U$ given by the assumption that U is inhabited. In the case $n = 0$ we are done since 0 is a minimal element of U. For the induction step $n \to n+1$, the intuitionistically valid double negation of the law of excluded middle gives
$$\neg\neg(n \in U \vee n \notin U).$$
Because of the tautologies $(\varphi \Rightarrow \psi) \Rightarrow (\neg\neg\varphi \Rightarrow \neg\neg\psi)$ and $\neg\neg\neg\neg\varphi \Rightarrow \neg\neg\varphi$ (see Section 2.4), it suffices to show that $n \in U \vee n \notin U$ implies the conclusion. So assume $n \in U \vee n \notin U$. If $n \in U$, then U does *not not* possess a minimal element by the induction hypothesis. If $n \notin U$, then $n+1$ is a minimal element (and so, in

[12] Let φ be an arbitrary formula. Assuming that any inhabited subset of the natural numbers possesses a minimal element, we want to show that $\varphi \vee \neg\varphi$. Define the subset $U := \{n \in \mathbb{N} \mid (n = 1) \vee \varphi\} \subseteq \mathbb{N}$, which surely is inhabited by $1 \in U$. So by assumption, there exists a number $z \in \mathbb{N}$ which is the minimum of U. We have $z = 0$ or $z > 0$. If $z = 0$, we have $0 \in U$, so $(0 = 1) \vee \varphi$, so φ holds. If $z > 0$, then $\neg\varphi$ holds: If φ were true, zero would be an element of U, contradicting the minimality of z.

particular, U does *not not* possess a minimal element): If m is any element of U, we have $m \geq n+1$ or $m \leq n$. In the first case, we're done. In the second case, it follows that $n \in U$ because U is upward-closed and so we obtain a contradiction. From this contradiction we can trivially deduce $m \geq n+1$ as well. □

If we want to work with a complete partially ordered set (poset) of natural numbers in intuitionistic logic, we have to construct a suitable completion. The idea of the following definition is to encode numbers as the (not necessarily existing) minimum of inhabited upward-closed subsets.

Definition 5.2. The *completed poset of natural numbers* is the set $\widehat{\mathbb{N}}$ of all inhabited upward-closed subsets of $\mathbb{N}$, ordered by reverse inclusion. The elements of $\widehat{\mathbb{N}}$ are called *generalized natural numbers.*

Lemma 5.3. *The completed poset of natural numbers is the least poset containing* $\mathbb{N}$ *and possessing minima of arbitrary inhabited subsets.*

Proof. The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \quad \longmapsto \quad \uparrow(n) := \{m \in \mathbb{N} \,|\, m \geq n\}.$$

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

We omit the proof of the universal property. □

Remark 5.4. In classical logic, the map $\widehat{\mathbb{N}} \to \mathbb{N}$, $U \mapsto \min U$ is a well-defined isomorphism of partially ordered sets. In fact, it is the inverse of the canonical embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$. In intuitionistic logic, this embedding is still injective, but it can not be shown to be surjective: It is only the case that any element of $\widehat{\mathbb{N}}$ does *not not* possess a preimage (by Lemma 5.1).

5.2. A geometric interpretation. We are interested in the completed natural numbers for the following reason: A generalized natural number in the topos of sheaves on a topological space X is the same as an upper semicontinuous function $X \to \mathbb{N}$.

Lemma 5.5. *Let X be a topological space. The sheaf $\widehat{\mathbb{N}}$ of generalized natural numbers on X is canonically isomorphic to the sheaf of upper semicontinuous $\mathbb{N}$-valued functions on X.*

Proof. When referring to the natural numbers in the internal language, we actually refer to the constant sheaf $\underline{\mathbb{N}}$ on X. (This is because the sheaf $\underline{\mathbb{N}}$ fulfills the axioms of a natural numbers object, cf. [56, Section VI.1].) Recall that its sections on an open subset $U \subseteq X$ are continuous functions $U \to \mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology.

Therefore, a section of $\widehat{\mathbb{N}}$ on an open subset $U \subseteq X$ is given by a subsheaf $\mathcal{A} \hookrightarrow \underline{\mathbb{N}}|_U$ such that

$$U \models \exists n \,{:}\, \underline{\mathbb{N}}.\ n \in \mathcal{A} \quad \text{and} \quad U \models \forall n, m \,{:}\, \underline{\mathbb{N}}.\ n \geq m \wedge n \in \mathcal{A} \Rightarrow m \in \mathcal{A}.$$

Since these conditions are geometric implications, they are satisfied if and only if any stalk $\mathcal{A}_x$ is an inhabited upward-closed subset of $\underline{\mathbb{N}}_x \cong \mathbb{N}$. The association

$$x \in X \quad \longmapsto \quad \min\{n \in \mathbb{N} \,|\, n \in \mathcal{A}_x\}$$

thus defines a map $X \to \mathbb{N}$. This map is indeed upper semicontinuous, since if $n \in \mathcal{A}_x$, there exists an open neighbourhood V of x such that the constant function with value n is an element of $\Gamma(V, \mathcal{A})$ and therefore $n \in \mathcal{A}_y$ for all $y \in V$.

Conversely, let $\alpha : U \to \mathbb{N}$ be a upper semi-continous function. Then

$$V \subseteq U \text{ open} \quad \longmapsto \quad \{f : V \to \mathbb{N} \mid f \text{ continuous, } f \geq \alpha \text{ on } V\}$$

is a subobject of $\underline{\mathbb{N}}|_U$ which internally is inhabited and upward-closed. Further details are left to the reader. □

Under the correspondence given by the lemma, locally *constant* functions map exactly to the (image of the) *ordinary* internal natural numbers (in the completed natural numbers). In a similar vein, the sheaf given by the internal construction of the set of *all* upward-closed subsets of the natural numbers (not only the inhabited ones) is canonically isomorphic to the sheaf of upper semicontinuous functions with values in $\mathbb{N} \cup \{+\infty\}$.

Note that the correspondence can be used to understand classical facts about upper semicontinuous functions as features of intuitionistic number theory. For instance, it is well-known that any upper semicontinuous $\mathbb{N}$-valued function on an arbitrary topological space is locally constant on a dense open subset. This can be explained as follows: The generalized natural number associated to such a function is *not not* an ordinary natural number from the internal point of view. Since "not not" translates to "holding on a dense open subset" (Proposition 6.4), it follows that there is a dense open subset on which the function corresponds to an ordinary internal natural number, i. e. is locally constant.

5.3. The upper semicontinuous rank function. Recall that the rank of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or locally ringed space) at a point $x \in X$ is defined as the $k(x)$-dimension of the vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. If we assume that $\mathcal{F}$ is of finite type around x, this dimension is finite and equals the minimal number of elements needed to generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module by Nakayama's lemma.

In the internal language, we can define an element of $\widehat{\mathbb{N}}$ by

$$\operatorname{rank} \mathcal{F} := \min\{n \in \mathbb{N} \mid \ulcorner\text{there is a gen. family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner\} \in \widehat{\mathbb{N}}.$$

If the module $\mathcal{F}$ is finite locally free, it will be a finite free module from the internal point of view and the rank defined in this way will be an actual natural number (see below); but in general, the rank is really an element of the completion.

Proposition 5.6. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type on a scheme X (or locally ringed space). Under the correspondence given by the Lemma 5.5, the internally defined rank maps to the rank function of $\mathcal{F}$.*

Proof. We have to show that for any point $x \in X$ and natural number n, there exists a generating family for $\mathcal{F}_x$ consisting of n elements if and only if there exists an open neighbourhood U of x such that

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

The "if" direction is obvious. For the "only if" direction, consider (liftings to local sections of a) generating family $s_1, \ldots, s_n$ of $\mathcal{F}_x$. Since $\mathcal{F}$ is of finite type, there also exist sections $t_1, \ldots, t_m$ on some neighbourhood V of x which generate any stalk $\mathcal{F}_y$, $y \in V$. Since the t_i can be expressed as a linear combination of the s_j in $\mathcal{F}_x$, the same is true on some open neighbourhood $U \subseteq V$ of x. On this neighbourhood, the s_j generate any stalk $\mathcal{F}_y$, $y \in U$, so by geometricity we have

$$U \models \ulcorner s_1, \ldots, s_n \text{ generate } \mathcal{F}\urcorner. \qquad \square$$

Remark 5.7. Once we understand when properties holding at a point spread to neighbourhoods, we will be able to give a simpler proof of the proposition (see Lemma 6.41).

Lemma 5.8. *Let X be a scheme (or a locally ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. If $\mathcal{F}$ is finite locally free, its rank function is locally constant. The converse holds if X is a reduced scheme.*

Proof. The rank function is locally constant if and only if internally, the rank of $\mathcal{F}$ is an actual natural number. Also recall that the structure sheaf fulfills a certain field condition if X is a reduced scheme (Corollary 3.9). Therefore it suffices to give a proof of the following fact of intuitionistic linear algebra: Let R be a local ring. Let M be a finitely generated R-module. If M is finite free, its rank is an actual natural number. The converse holds if R fulfills the field condition that any element which is not invertible is zero.

So assume that such a module M is finite free. Then it is isomorphic to R^n for some actual natural number n; by the internal proof in Lemma 4.13, the rank of M is therefore this number n (for any surjection $R^m \twoheadrightarrow R^n$ it holds that $m \geq n$).

Conversely, assume that the rank of M is an actual natural number. Then there exists a minimal generating family $x_1, \ldots, x_n : M$. We can verify that this family is indeed linearly independent (and thus a basis, demonstrating that M is finite free): Let $\sum_i a_i x_i = 0$ with $a_i : R$. If any a_i were invertible, the family $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ would too generate M, contradicting the minimality. So each a_i is not invertible. By the field property of R, each a_i is zero. ☐

Lemma 5.9. *Let X be a reduced scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then $\mathcal{F}$ is finite locally free on a dense open subset.*

Proof. Since "dense open" translates to "not not" in the internal language (Proposition 6.4), it suffices to give an intuitionistic proof of the following fact: Let R be a local ring which fulfills an appropriate field condition. Let M be a finitely generated R-module. Then M is *not not* finite free.

By Remark 5.4, the rank of such a module M is *not not* an actual natural number. By the last part of the previous proof, it thus follows that M is *not not* finite free. ☐

Remark 5.10. Note that besides basics on natural numbers in an intuitionistic setting and some dictionary terms ("reduced", "finite locally free", "finite type", "dense open"), this proof does not depend on any further tools. In particular, Nakayama's lemma and facts about semicontinuous functions do not enter. For the (more complex) standard proof of this fact, see for instance [76], where the claim is dubbed an "important hard exercise" (Exercise 13.7.K).

5.4. The upper semicontinuous dimension function. Recall that the dimension of a topological space X at a point $x \in X$ is defined as the infimum

$$\dim_x X := \inf\{\dim U \mid U \text{ open neighbourhood of } x\}.$$

The map $X \to \mathbb{N} \cup \{+\infty\},\ x \mapsto \dim_x X$ is upper semicontinuous and thus corresponds to an internal generalized (possibly unbounded) natural number. The following proposition shows that this number has an explicit description.

Proposition 5.11. *Let X be a scheme. Then the upper semicontinuous function associated to the internal number "Krull dimension of $\mathcal{O}_X$" is the dimension function $x \mapsto \dim_x X$.*

Proof. Internally, we define the Krull dimension of $\mathcal{O}_X$ as the infimum over all natural numbers n such that $\mathcal{O}_X$ is of Krull dimension $\leq n$. This infimum need not exist in the natural numbers, of course; so we really mean the upward-closed set $\mathcal{A}$ of all those numbers. (It is inhabited if and only if, from the external perspective, the dimension of X is locally finite. In this case, it defines a generalized natural number.)

We thus have to show for any point $x \in X$:

$$\inf\{n \in \mathbb{N} \cup \{+\infty\} \mid n \in \mathcal{A}_x\} = \dim_x X.$$

The condition on n can be expressed as follows, where we write "$\underline{n}$" to denote the constant function with value n:

$$\begin{aligned}
& n \in \mathcal{A}_x \\
\Longleftrightarrow\ & \text{for some open neighbourhood } U \text{ of } x,\ \underline{n} \in \Gamma(U, \mathcal{A}) \\
\Longleftrightarrow\ & \text{for some open neighbourhood } U \text{ of } x, \\
& \qquad U \models \ulcorner \mathcal{O}_X \text{ is of Krull dimension} \leq n \urcorner \\
\Longleftrightarrow\ & \text{for some open neighbourhood } U \text{ of } x, \\
& \qquad \dim U \leq n
\end{aligned}$$

The second equivalence follows from the external description of internally-defined subsheaves given in Section 2.2. We thus have:

$$\inf\{n \mid n \in \mathcal{A}_x\} = \inf\{\dim U \mid U \text{ open neighbourhood of } x\} = \dim_x X. \qquad \square$$

6. Modalities

Philosophers and logicians do not only study what is *true*, but also what is *known*, what is *believed*, what is *possible*, and so on. Such *modalities* are absent from the usual mathematical practice. However, it turns out that a specific kind of such modalities plays a role in understanding when properties spread from points to neighbourhoods.

Briefly, this is because for any point x of a topological space X, there exists a modal operator $\Box$ such that for any formula φ of the internal language of the sheaf topos $\mathrm{Sh}(X)$, the internal statement $\Box\varphi$ means that φ holds on some open neighbourhood of the given point x. In this way, we can reduce sheaf-theoretic questions to questions of modal intuitionistic (non-sheafy) mathematics.

The techniques developed in this section also enable us to use the internal language of $\mathrm{Sh}(X)$ to talk about sheaves on *subspaces* of X (and more general *sublocales* of X).

Topological interpretations of modal logic were studied before, for instance by Awodey and Kishida [8]. However, they study a different kind of modal operators, not corresponding to the Lawvere–Tierney topologies of topos theory, and pursue different goals.

6.1. Basics on truth values and modal operators.

Definition 6.1. The *set of truth values* Ω is the powerset of the singleton set $1 := \{\star\}$, where $\star$ is a formal symbol.

In classical logic, any subset of $\{\star\}$ is either empty or inhabited, so that Ω contains exactly two elements, the empty set ("false") and $\{\star\}$ ("true"). But in intuitionistic logic, this can not be shown; indeed, if we interpret the definition in the topos of sheaves on a space X, we obtain a (large) sheaf Ω with

$$U \subseteq X \text{ open} \quad \longmapsto \quad \Gamma(U, \Omega) = \{V \subseteq U \mid V \text{ open}\}.$$

(This is because by definition of Ω as the power object of the terminal sheaf 1, sections of Ω on an open subset U correspond to subsheaves $\mathcal{F} \hookrightarrow 1|_U$, and those are given by the greatest open subset $V \subseteq U$ such that $\Gamma(V, \mathcal{F})$ is inhabited.) Obviously, in general, this sheaf has many sections, in particular more than the binary coproduct $1 \amalg 1$ (unless any open subset of X is also closed).

The *truth value* of a formula φ is by definition the subset $\{x \in 1 \mid \varphi\} \in \Omega$, where "$x$" is a fresh variable not appearing in φ. This subset is inhabited if and only if φ holds and is empty if and only if $\neg\varphi$ holds. Conversely, we can associate to a subset $F \subseteq 1$ the proposition $\ulcorner F$ is inhabited$\urcorner$.

By the above description of Ω in a sheaf topos $\mathrm{Sh}(X)$, the interpretation of the truth value of a formula φ in the internal language of $\mathrm{Sh}(X)$ is a certain open subset of X. Tracing the definitions, we see that this open subset is precisely the largest open subset on which φ holds, i. e. the union of all open subsets $U \subseteq X$ such that $U \models \varphi$.

Under the correspondence of formulas with truth values, logical operations like $\wedge$ and $\vee$ map to set-theoretic operations like $\cap$ and $\cup$ – for instance, we have

$$\{x \in 1 \,|\, \varphi\} \cap \{x \in 1 \,|\, \psi\} = \{x \in 1 \,|\, \varphi \wedge \psi\}.$$

This justifies a certain abuse of notation: We will sometimes treat elements of Ω as propositions and use logical instead of set-theoretic connectives. In particular, if φ and ψ are elements of Ω, we will write "$\varphi \Rightarrow \psi$" to mean $\varphi \subseteq \psi$; "$\bot$" to mean $\emptyset$; and "$\top$" to mean 1.

Definition 6.2. A *modal operator* (or *Lawvere–Tierney topology*) is a map $\Box : \Omega \to \Omega$ such that for all $\varphi, \psi \in \Omega$,

(1) $\varphi \Longrightarrow \Box\varphi$,
(2) $\Box\Box\varphi \Longrightarrow \Box\varphi$,
(3) $\Box(\varphi \wedge \psi) \Longleftrightarrow \Box\varphi \wedge \Box\psi$.

The intuition is that $\Box\varphi$ is a certain weakening of φ, where the precise meaning of "weaker" depends on the modal operator. By the second axiom, weakening twice is the same as weakening once.

In classical logic, where $\Omega = \{\bot, \top\}$, there are only two modal operators: the identity map and the constant map with value $\top$. Both of these are not very interesting: The identity operator does not weaken propositions at all, while the constant operator weakens every proposition to the trivial statement $\top$.

In intuitionistic logic, there can potentially exist further modal operators. For applications to algebraic geometry, the following four operators will have a clear geometric meaning and be of particular importance:

(1) $\Box\varphi :\equiv (\alpha \Rightarrow \varphi)$, where α is a fixed proposition.
(2) $\Box\varphi :\equiv (\varphi \vee \alpha)$, where α is a fixed proposition.
(3) $\Box\varphi :\equiv \neg\neg\varphi$ (the *double negation modality*).
(4) $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition.

Lemma 6.3. *Any modal operator $\Box$ is monotonic, i. e. if $\varphi \Rightarrow \psi$, then $\Box\varphi \Rightarrow \Box\psi$. Furthermore, there holds a modus ponens rule: If $\Box\varphi$ holds, and φ implies $\Box\psi$, then $\Box\psi$ holds as well.*

Proof. Assume $\varphi \Rightarrow \psi$. This is equivalent to supposing $\varphi \wedge \psi \Leftrightarrow \varphi$. We are to show that $\Box\varphi \Rightarrow \Box\psi$, i. e. that $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box\varphi$. This follows since by the third axiom on a modal operator, we have $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box(\varphi \wedge \psi)$, and $\Box$ respects equivalence of propositions.

For the second statement, consider that if $\varphi \Rightarrow \Box\psi$, by monotonicity and the second axiom on a modal operator it follows that $\Box\varphi \Rightarrow \Box\Box\psi \Rightarrow \Box\psi$. □

The modus ponens rule justifies the following proof scheme: When trying to show, given that some boxed statement $\Box\varphi$ holds, that some further boxed statement $\Box\psi$ holds, we may give a proof of $\Box\psi$ under the stronger assumption φ. Symbolically:

$$(\Box\varphi \Rightarrow \Box\psi) \Longleftrightarrow (\varphi \Rightarrow \Box\psi).$$

6.2. Geometric meaning. Let X be a topological space. As discussed above, an open subset $U \subseteq X$ defines an internal truth value (a global section of the sheaf Ω). We also denote it by "U", such that

$$V \models U \quad \Longleftrightarrow \quad V \subseteq U$$

for any open subset $V \subseteq X$. (Shortcutting the various intermediate steps, this can also be taken as a definition of "$V \models U$".) If $A \subseteq X$ is a closed subset, there is thus an internal truth value A^c corresponding to the open subset $A^c = X \setminus A$. If $x \in X$ is a point, we define "$!x$" to denote the truth value corresponding to $\operatorname{int}(X \setminus \{x\})$, such that

$$V \models\, !x \quad \Longleftrightarrow \quad V \subseteq \operatorname{int}(X \setminus \{x\}) \quad \Longleftrightarrow \quad x \notin V.$$

Proposition 6.4. *Let $U \subseteq X$ be a fixed open and $A \subseteq X$ be a fixed closed subset. Let $x \in X$. Then, for any open subset $V \subseteq X$, it holds that:*

$$\begin{aligned}
&V \models (U \Rightarrow \varphi) && \Longleftrightarrow && V \cap U \models \varphi. \\
&V \models (\varphi \vee A^c) && \Longleftrightarrow && \text{there is an open subset } W \subseteq V \\
&&&&& \qquad \text{containing } A \cap V \text{ such that } W \models \varphi. \\
&V \models \neg\neg\varphi && \Longleftrightarrow && \text{there is a dense open subset } W \subseteq V \text{ s.th. } W \models \varphi. \\
&V \models ((\varphi \Rightarrow\, !x) \Rightarrow\, !x) && \Longleftrightarrow && x \notin V \text{ or there is an open neighbourhood } W \subseteq V \\
&&&&& \qquad \text{of } x \text{ such that } W \models \varphi.
\end{aligned}$$

Proof. (1) Omitted.

(2) Let $V \models \varphi \vee A^c$. Then there exists an open covering $V = \bigcup_i V_i$ such that for each i, $V_i \models \varphi$ or $V_i \subseteq A^c$. Let $W \subseteq V$ be the union of those V_i such that $V_i \models \varphi$. Then $W \models \varphi$ by the locality of the internal language and $A \cap V \subseteq W$.

Conversely, let $W \subseteq V$ be an open subset containing $A \cap V$ such that $W \models \varphi$. Then $V = W \cup (V \cap A^c)$ is an open covering attesting $V \models \varphi \vee A^c$.

(3) For the "only if" direction, let $W \subseteq V$ be the largest open subset on which φ holds, i. e. the union of all open subsets of V on which φ holds. For the "if" direction, we may assume that the given set W is also the largest open subset on which φ holds (by enlarging W if necessary). The claim then follows by the following chain of equivalences:

$$\begin{aligned}
&V \models \neg\neg\varphi \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open.} \Big(\forall Z \subseteq Y \text{ open.} (Z \models \varphi) \Rightarrow Z = \emptyset\Big) \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open.} \Big(\forall Z \subseteq Y \text{ open. } Z \subseteq W \Rightarrow Z = \emptyset\Big) \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open. } Y \cap W = \emptyset \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &W \text{ is dense in } V.
\end{aligned}$$

(4) Straightforward, since the interpretation of the internal statement with the Kripke–Joyal semantics is

$$\forall Y \subseteq V \text{ open.} \Big(\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow x \notin Z\Big) \Longrightarrow x \notin Y. \qquad \square$$

6.3. The subspace associated to a modal operator. Any modal operator $\Box : \Omega \to \Omega$ in the sheaf topos of X induces on global sections a map

$$j : \mathcal{T}(X) \to \mathcal{T}(X),$$

Modal operator	associated nucleus	$j(V) = X$ iff ...	subspace
$\Box\varphi :\equiv (U \Rightarrow \varphi)$	$j(V) = \operatorname{int}(U^c \cup V)$	$U \subseteq V$	U
$\Box\varphi :\equiv (\varphi \vee A^c)$	$j(V) = V \cup A^c$	$A \subseteq V$	A
$\Box\varphi :\equiv \neg\neg\varphi$	$j(V) = \operatorname{int}(\operatorname{cl}(V))$	V is dense in X	smallest dense sublo- cale of X
$\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$	$j(V) = X \setminus \operatorname{cl}\{x\}$, if $x \notin V$ $j(V) = X$, if $x \in V$	$x \in V$	$\{x\}$

Table 2. List of important modal operators and their associated nuclei (notation as in Proposition 6.4).

where $\mathcal{T}(X) = \Gamma(X, \Omega)$ is the set of open subsets of X. Explicitly, it is given by

$$\begin{aligned} j(U) &= \text{largest open subset of } X \text{ on which } \Box U \text{ holds} \\ &= \bigcup \{V \subseteq X \mid V \text{ open}, \ V \models \Box U\}. \end{aligned}$$

By the axioms on a modal operator, the map j fulfills similar such axioms: For any open subsets $U, V \subseteq X$,

(1) $U \subseteq j(U)$,
(2) $j(j(U)) \subseteq j(U)$,
(3) $j(U \cap V) = j(U) \cap j(V)$.

Such a map is called a *nucleus* on $\mathcal{T}(X)$. Table 2 lists the nuclei associated to the four modal operators of Proposition 6.4.

Any nucleus j defines a subspace X_j of X, to be described below, with a small caveat: In general, the subspace X_j can not be realized as a topological subspace, but only as a so-called *sublocale*; the notion of a locale is a slight generalization of the notion of a topological space, in which an underlying set of points is not part of the definition. Instead, a locale is simply given by a lattice of arbitrary *opens* satisfying some axioms – these opens may, but do not necessarily have to, be sets of points. Sheaf theory carries over to locales essentially unchanged, since the notions of presheaves and sheaves only refer to open sets and coverings, but not points. Accessible introductions to the theory of locales include two notes by Johnstone [45, 46].

Definition 6.5. Let j be a nucleus on $\mathcal{T}(X)$. Then the sublocale X_j of X is given by the lattice of opens $\mathcal{T}(X_j) := \{U \in \mathcal{T}(X) \mid j(U) = U\}$.

If j is induced by a modal operator $\Box$, we also write "$X_\Box$" for X_j. In three of the four cases listed in Table 2, the sublocale $X_\Box$ can indeed be realized as a topological subspace. The only exception is the sublocale $X_{\neg\neg}$ associated to the double negation modality. It can also be described as the *smallest dense sublocale* of X; this is obviously a genuine locale-theoretic notion, since there is (in general) no smallest dense topological subspace (consider $\mathbb{R}$ and its dense subsets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$).

The inclusion $i : X_j \hookrightarrow X$ can not in general be described on the level of points, since X_j might not be realizable as a topological subspace. But for sheaf-theoretic purposes, it suffices to describe i on the level of opens. This is done as follows:

$$i^{-1} : \mathcal{T}(X) \longrightarrow \mathcal{T}(X_j), \qquad U \longmapsto j(U).$$

Thus we can relate the toposes of sheaves on X_j and X by the usual pullback and pushforward functors.

$$i^{-1}\mathcal{F} = \text{sheafification of } (U \mapsto \operatorname{colim}_{U \preceq i^{-1}V} \Gamma(V, \mathcal{F}))$$
$$i_*\mathcal{G} = (U \mapsto \Gamma(i^{-1}U, \mathcal{G})) = (U \mapsto \Gamma(j(U), \mathcal{G}))$$

As familiar from honest topological subspace inclusions, the pushforward functor $i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X)$ is fully faithful and the composition $i^{-1} \circ i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X_j)$ is (canonically isomorphic to) the identity.

6.4. Internal sheaves and sheafification. It turns out that the image of the pushforward functor $i_* : \mathrm{Sh}(X_\Box) \to \mathrm{Sh}(X)$, where $\Box$ is a modal operator in $\mathrm{Sh}(X)$, can be explicitly described. Namely, it consists exactly of those sheaves which from the internal point of view are so-called $\Box$-*sheaves*, a notion explained below.

Furthermore, if we identify $\mathrm{Sh}(X_\Box)$ with its image in $\mathrm{Sh}(X)$, the pullback functor is given by an internal sheafification process with respect to the modality $\Box$. Thus the external situation of pushforward/pullback translates to forget/sheafify. This broadens the scope of the internal language of $\mathrm{Sh}(X)$: It can not only be used to talk about sheaves on X in a simple, element-based language, but also to talk about sheaves on arbitrary subspaces of X.

To describe the notion of $\Box$-sheaves and related ones, we switch to the internal perspective and thus forget that we're working over a base space X; we are simply given a modal operator $\Box : \Omega \to \Omega$ and have to take care that our proofs are intuitionistically acceptable. A reference for the material in this subsection is a preprint by Fer-Jan de Vries [81].[13]

Recall that a set S is a *subsingleton* if and only if $\forall x, y : S.\ x = y$, and that a set S is a *singleton* if and only if it is a subsingleton and inhabited (i. e. $\exists x : S.\ \top$); this amounts to $\exists! x : S.\ \top$.

Definition 6.6. A set F is $\Box$-*separated* if and only if

$$\forall x, y : F.\ \Box(x = y) \Longrightarrow x = y.$$

A set F is a $\Box$-*sheaf* if and only if it is $\Box$-separated and

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton} \urcorner) \Longrightarrow \exists x : F.\ \Box(x \in S).$$

The two conditions can be combined: A set F is a $\Box$-sheaf if and only if

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton} \urcorner) \Longrightarrow \exists! x : F.\ \Box(x \in S).$$

XXX: explain how to read these definitions

Definition 6.7. The *plus construction* of a set F with respect to $\Box$ is the set

$$F^+ := \{S \subseteq F \mid \Box(\ulcorner S \text{ is a singleton} \urcorner)\}/\sim,$$

where the equivalence relation is defined by $S \sim T :\Leftrightarrow \Box(S = T)$. There is a canonical map $F \to F^+$ given by $x \mapsto [\{x\}]$. The $\Box$-*sheafification* of a set F is the set F^{++}.

If F is $\Box$-separated, then for any subset $S \subseteq F$ it holds that

$$\Box(\ulcorner S \text{ is a singleton} \urcorner) \quad \Longleftrightarrow \quad \ulcorner S \text{ is a subsingleton} \urcorner \wedge \Box(\ulcorner S \text{ is inhabited} \urcorner).$$

[13] Note that on page 5 of that preprint there is a slight typing error: Fact 2.1(i) gives the characterization of j-closedness, not j-denseness. The correct characterization of j-denseness in that context is $\forall b \in B.\ j(b \in A)$.

Remark 6.8. The topos of *pre*sheaves on a topological space X admits an internal language as well [56, Section VI.7, discussion after Theorem 1]. In it, there exists a modal operator $\Box$ reflecting the topology of X. A presheaf on X is separated in the usual sense if, from the internal perspective of $\mathrm{PSh}(X)$, it is $\Box$-separated; and it is a sheaf if, from the internal perspective, it is a $\Box$-sheaf. Furthermore, the $\Box$-sheafification of a presheaf (considered as a set from the internal perspective) coincides with the usual sheafification.

Example 6.9. Any singleton set is a $\Box$-sheaf. The empty set is always $\Box$-separated (trivially) and is a $\Box$-sheaf if and only if $\Box\bot \Rightarrow \bot$.

We will see geometric examples of $\Box$-sheaves in further sections. For instance, on an integral or locally Noetherian scheme X, the structure sheaf $\mathcal{O}_X$ is $\neg\neg$-separated and its $\neg\neg$-sheafification is the sheaf $\mathcal{K}_X$ of rational functions (Proposition 7.9).

Lemma 6.10. *For any set F, it holds that:*

(1) *F^+ is $\Box$-separated.*
(2) *The canonical map $F \to F^+$ is injective if and only if F is $\Box$-separated.*
(3) *If F is $\Box$-separated, then F^+ is a $\Box$-sheaf.*
(4) *If F is a $\Box$-sheaf, then the canonical map $F \to F^+$ is bijective.*

Let $\mathrm{Sh}_\Box(\mathrm{Set})$ be the full subcategory of Set *consisting of the $\Box$-sheaves. Then it holds that:*

(5) *The functor $(\_\_)^+ : \mathrm{Set} \to \mathrm{Set}$ is left exact.*
(6) *The functor $(\_\_)^{++} : \mathrm{Set} \to \mathrm{Sh}_\Box(\mathrm{Set})$ is left exact and left adjoint to the forgetful functor $\mathrm{Sh}_\Box(\mathrm{Set}) \to \mathrm{Set},\ F \mapsto F$.*

Proof. These are all straightforward, and in fact simpler than their classical counterparts, since there are no colimit constructions which would have to be dealt with. $\Box$

Remark 6.11. As is to be expected from the familiar inclusion of sheaves in presheaves on topological spaces, the forgetful functor $\mathrm{Sh}_\Box(\mathrm{Set}) \to \mathrm{Set}$ does not in general preserve colimits. It is instructive to see why epimorphisms in $\mathrm{Sh}_\Box(\mathrm{Set})$ need not be epimorphisms in Set: A map $f : A \to B$ between $\Box$-sheaves is an epimorphism in $\mathrm{Sh}_\Box(\mathrm{Set})$ if and only if

$$\forall y{:}B.\ \Box(\exists x{:}X.\ f(x) = y),$$

i. e. preimages do not need to exist, it suffices for them to "$\Box$-exist". (Using results about the $\Box$-translation, to be introduced below, this characterization will be obvious.) This condition is intuitionistically weaker than the condition that f is an epimorphism in Set, i. e. that f is surjective. Compare this to the failure of the forgetful functor $\mathrm{Sh}(X) \to \mathrm{PSh}(X)$ to preserve epimorphisms: A morphism of sheaves does not need to have preimages for any local section in order to be an epimorphism. Instead, it suffices for any local section to *locally* have preimages.

Proposition 6.12. *Let X be a topological space. Let $\Box$ be a modal operator in $\mathrm{Sh}(X)$. Let $i : X_\Box \hookrightarrow X$ be the inclusion of the associated sublocale. Corestricting the pushforward functor $i_* : \mathrm{Sh}(X_\Box) \to \mathrm{Sh}(X)$ to its essential image, it induces an equivalence $\mathrm{Sh}(X_\Box) \simeq \mathrm{Sh}_\Box(\mathrm{Sh}(X))$ between the category of sheaves on $X_\Box$ and the category of $\Box$-sheaves in $\mathrm{Sh}(X)$.*

Proof. For the further development of the theory, we need the statement of this proposition, but not the proof, which really is routine in dealing with subtoposes and modal operators. Nevertheless, a proof goes like follows: Combine Example A4.6.2(a) and Theorem C1.4.7 of [44] and note that for a topos of sheaves on a locale Y, it holds that $\mathcal{T}(Y) = \Gamma(Y, \Omega_{\mathrm{Sh}(Y)})$, and that the subobject classifier of $\mathrm{Sh}_\Box(\mathrm{Sh}(X))$ is $\{\varphi : \Omega_{\mathrm{Sh}(X)} \,|\, \Box\varphi \Leftrightarrow \varphi\}$. $\Box$

Remark 6.13. It's possible to rewrite the sheaf condition in the following form. A set F is $\Box$-separated if and only if, for any truth value $\varphi : \Omega$ such that $\Box\varphi$, the canonical map

$$F \longrightarrow F^{\varphi},$$

which maps an element $x : F$ to the constant map $\varphi \to X$ with value x (where φ is considered as a subset of the terminal set 1), is injective. The set F is a $\Box$-sheaf if and only if furthermore this map is surjective for all such truth values.

6.5. Sheaves for the double negation modality. Recall that if $\Box$ is the modal operator associated to a sub*space* Y of a topological space X, then the sheaves on X which are $\Box$-sheaves are easy to describe: These are precisely the sheaves in the essential image of the pushforward functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. For the double negation modality, the same is true, only that Y is then the perhaps unfamiliar *smallest dense sublocale* of X.

The following proposition gives a characterization of $\neg\neg$-separated presheaves and $\neg\neg$-sheaves in explicit terms.

Proposition 6.14. *Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X. Then:*

(1) *$\mathcal{F}$ is $\neg\neg$-separated if and only if any two local sections of $\mathcal{F}$, which are defined on a common domain and which agree on a dense open subset of their domain, are already equal.*
(2) *$\mathcal{F}$ is a $\neg\neg$-sheaf if and only if it is $\neg\neg$-separated and for any open subset $U \subseteq X$ and any open subset $V \subseteq U$ dense in U, any V-section of $\mathcal{F}$ extends to a U-section of $\mathcal{F}$.*
(3) *If $\mathcal{F}$ is $\neg\neg$-separated, the sections of $\mathcal{F}^+$ on an open subset $U \subseteq X$ can be described by pairs (V, s), where V is a dense open subset of U and s is a section of $\mathcal{F}$ on V. Two such pairs $(V, s), (V', s')$ determine the same element in $\Gamma(U, \mathcal{F}^+)$ if and only if s and s' agree on $V \cap V'$.*

Proof. The first statement is obvious from the definition of $\neg\neg$-separatedness (Definition 6.6 for $\Box = \neg\neg$) and the geometric interpretation of double negation (Proposition 6.4).

For the second statement, we need to show that if $\mathcal{F}$ is $\neg\neg$-separated, $\mathcal{F}$ has the extension property if and only if

$$\mathrm{Sh}(X) \models \forall \mathcal{S} : \mathcal{P}(\mathcal{F}).\ \ulcorner \mathcal{S} \text{ is a subsingleton} \urcorner \wedge \neg\neg(\ulcorner \mathcal{S} \text{ is inhabited} \urcorner) \Longrightarrow \\ \exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S}).$$

Note that a section $\mathcal{S} \in \Gamma(U, \mathcal{P}(\mathcal{F}))$ which internally is a subsingleton and *not not* inhabited is precisely a subsheaf $\mathcal{S} \hookrightarrow \mathcal{F}|_U$ such that all stalks $\mathcal{S}_x$, $x \in U$ are subsingletons and such that for some dense open subset $V \subseteq U$, the stalks $\mathcal{S}_x$, $x \in V$ are inhabited. This is precisely the datum of a section of $\mathcal{F}$ defined on some dense open subset of U: Consider the gluing of the unique germs in $\mathcal{S}_x$ for those points x such that $\mathcal{S}_x$ is inhabited. (Conversely, a section $s \in \Gamma(V, \mathcal{F})$ defines a subsheaf $\mathcal{S}$ by setting $\Gamma(W, \mathcal{S}) := \{s|_W \mid W \subseteq V\}$.)

In view of this explicit description and the observation that the asserted existence ("$\exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S})$") is actually a question of unique existence, the second statement follows.

For the third statement, one can check that the presheaf on X defined by

$$U \subseteq X \text{ open} \quad \longmapsto \quad \{(V, s) \mid V \subseteq U \text{ dense open},\ s \in \Gamma(V, \mathcal{F})\}/\sim$$

is in fact a sheaf (with respect to the topology of X), internally a $\neg\neg$-sheaf, and that it has the universal property of the $\neg\neg$-sheafification of $\mathcal{F}$. $\Box$

The conditions (1) and (2) of the previous proposition can be summarized as follows: A sheaf $\mathcal{F}$ on a topological space is a $\neg\neg$-sheaf if and only if, for any open subset $U \subseteq X$, the restriction map $\Gamma(\operatorname{int}\operatorname{cl} U, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is bijective [43, Lemma 36].

In the case that X contains a *generic point*, that is a point $\xi \in X$ such that $\operatorname{cl}\{\xi\} = X$, we can describe the sublocale $X_{\neg\neg}$ in very explicit terms: In this case, it coincides with the subspace $\{\xi\}$. Such a point exists and is unique if X is an irreducible scheme and need not exist otherwise.

Lemma 6.15. *Let X be a topological space and $\xi \in X$ a point such that $\operatorname{cl}\{\xi\} = X$. Then the modal operator $\Box :\equiv ((\_\_ \Rightarrow\, !\xi) \Rightarrow\, !\xi)$ coincides with the double negation modality and $X_{\neg\neg} = \{\xi\}$ as sublocales of X.*

Proof. The semantics of the formula $!\xi$ was defined by the equivalence

$$U \models\, !\xi \quad\Longleftrightarrow\quad \xi \not\in U.$$

By the assumption on ξ, this is equivalent to requiring $U = \emptyset$. Thus for any open subset U the formulas $!\xi$ and $\bot$ have the same meaning; they are therefore logically equivalent from the internal point of view. The given modal operator thus simplifies to

$$\Box\varphi \quad\equiv\quad ((\varphi \Rightarrow\, !\xi) \Rightarrow\, !\xi) \quad\Leftrightarrow\quad ((\varphi \Rightarrow \bot) \Rightarrow \bot) \quad\Leftrightarrow\quad \neg\neg\varphi.$$

The second claim follows Table 2. ☐

Corollary 6.16. *Let X be a topological space and $\xi \in X$ a point such that $\operatorname{cl}\{\xi\} = X$. Since $X_{\neg\neg} = \{\xi\}$, the category of $\neg\neg$-sheaves in $\operatorname{Sh}(X)$ coincides with the category of sheaves on $\{\xi\}$ and can therefore be identified with the category of sets. Under this identification,*

1. *sheafifying an object $\mathcal{F} \in \operatorname{Sh}(X)$ with respect to the double negation modality (i. e. pulling back to $X_{\neg\neg}$) is the same as calculating its generic stalk $\mathcal{F}_\xi$ and*
2. *pushing forward a set M along $X_{\neg\neg} \hookrightarrow X$ is the same as calculating the constant sheaf associated to M.*

Proof. The first statement follows because pulling back to $X_{\neg\neg}$ is the same as pulling back to $\{\xi\}$. The pushforward of a set M, considered as a sheaf on $X_{\neg\neg}$, to X is explicitly given by

$$U \quad\longmapsto\quad \begin{cases} M, & \text{if } U \neq \emptyset, \\ \{\star\}, & \text{else.} \end{cases}$$

We omit the routine verification that this sheaf coincides with the constant sheaf $\underline{M}$ associated to M. ☐

The following technical lemma will occasionally be handy. It is an internal reflection of the fact that an open subset of an affine scheme can always be written as the union of standard open subsets. We will generalize it to schemes which are not necessarily integral in Section 7 (see Lemma 7.18).

Lemma 6.17. *Let X be an integral scheme. Let φ be any formula over X. Then*

$$\operatorname{Sh}(X) \models \neg\neg\varphi \Longrightarrow \exists f : \mathcal{O}_X.\ \neg\neg(\ulcorner f \text{ inv.}\urcorner) \wedge (\ulcorner f \text{ inv.}\urcorner \Rightarrow \varphi).$$

Proof. We may assume that X is the spectrum of an integral domain A and that there is a dense open subset $U \subseteq X$ on which φ holds. The open set U may be covered by standard open subsets $D(f_i)$; since X is irreducible, at least one of these is itself dense. We may take this f_i as the sought f. ☐

We can now also follow up on a promise made earlier and prove the following somewhat tangential lemma.

Lemma 6.18. *Let X be a topological space. The internal language of* $\mathrm{Sh}(X)$ *is Boolean if and only if for any open subset $U \subseteq X$ it holds that U is the only dense open subset of U.*

Proof. That the internal language of $\mathrm{Sh}(X)$ is Boolean amounts to

$$\mathrm{Sh}(X) \models \forall \varphi : \Omega.\ \neg\neg\varphi \Rightarrow \varphi.$$

This is equivalent to the external statement that for any open subset $U \subseteq X$ and for any open subset $V \subseteq U$ it holds that: If V is dense in U, then V is equal to U. □

6.6. The □-translation. There is certain well-known transformation $\varphi \mapsto \varphi^{\neg\neg}$ on formulas, the *double negation translation*, with the following curious property: A formula φ is derivable in classical logic if and only if its translation $\varphi^{\neg\neg}$ is derivable in intuitionistic logic. The translation $\varphi^{\neg\neg}$ is obtained from φ by putting "$\neg\neg$" before any subformula, i. e. before any "$\exists$" and "$\forall$", around any logical connective, and around any atomic statement ("$x = y$", "$x \in A$"). For instance, the double negation translation of "f is surjective" is

$$\neg\neg\forall y : Y.\ \neg\neg\exists x : X.\ \neg\neg f(x) = y.$$

We will describe a slight generalization of the double negation translation, the $\Box$-translation for any modal operator $\Box$. It will be pivotal for using the internal language of a space X to express internal statements about sheaves defined on subspaces of X. The $\Box$-translation has been studied in other contexts before [1, 36]. To the best of my knowledge, this application – expressing the internal language of subtoposes in the internal language of the ambient topos – is new.

Definition 6.19. The $\Box$-*translation* is recursively defined as follows.

$$\begin{aligned}
(f = g)^\Box &:\equiv \Box(f = g) \\
(x \in A)^\Box &:\equiv \Box(x \in A) \\
\top^\Box &:\equiv \Box\top \quad (\Leftrightarrow \top) \\
\bot^\Box &:\equiv \Box\bot \\
(\varphi \wedge \psi)^\Box &:\equiv \Box(\varphi^\Box \wedge \psi^\Box) & (\textstyle\bigwedge_i \varphi_i)^\Box &:\equiv \Box(\textstyle\bigwedge_i \varphi_i^\Box) \\
(\varphi \vee \psi)^\Box &:\equiv \Box(\varphi^\Box \vee \psi^\Box) & (\textstyle\bigvee_i \varphi_i)^\Box &:\equiv \Box(\textstyle\bigvee_i \varphi_i^\Box) \\
(\varphi \Rightarrow \psi)^\Box &:\equiv \Box(\varphi^\Box \Rightarrow \psi^\Box) \\
(\forall x : X.\ \varphi)^\Box &:\equiv \Box(\forall x : X.\ \varphi^\Box) & (\forall X.\ \varphi)^\Box &:\equiv \Box(\forall X.\ \varphi^\Box) \\
(\exists x : X.\ \varphi)^\Box &:\equiv \Box(\exists x : X.\ \varphi^\Box) & (\exists X.\ \varphi)^\Box &:\equiv \Box(\exists X.\ \varphi^\Box)
\end{aligned}$$

Definition 6.20. A formula φ is $\Box$-*stable* if and only if $\Box\varphi$ implies φ.

Lemma 6.21.
(1) *Formulas in the image of the $\Box$-translation are $\Box$-stable, i. e. for any formula φ it holds that $\Box(\varphi^\Box) \Longrightarrow \varphi^\Box$.*
(2) *In the definition of the $\Box$-translation, one may omit the boxes printed in gray.*

Proof. The first statement is obvious, since one of the axioms on a modal operator demands that $\Box\Box\varphi \Rightarrow \Box\varphi$ for any formula φ. The second statement follows by an induction on the formula structure. By way of example, we prove the case for "$\Rightarrow$":

$$\begin{aligned}
&(\varphi \Rightarrow \psi)^{\Box \text{ with the gray parts}} \\
\Longleftrightarrow\ &\Box(\varphi^{\Box \text{ with the gray parts}} \Rightarrow \psi^{\Box \text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box \text{ with the gray parts}} \Rightarrow \psi^{\Box \text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box \text{ without the gray parts}} \Rightarrow \psi^{\Box \text{ without the gray parts}}) \\
\Longleftrightarrow\ &(\varphi \Rightarrow \psi)^{\Box \text{ without the gray parts}}
\end{aligned}$$

The first step is by definition; the second by $\Box$-stability of $\psi^{\Box \text{ with the gray parts}}$ and the intuitionistic tautology $\Box(\alpha \Rightarrow \beta) \Leftrightarrow (\alpha \Rightarrow \beta)$ for $\Box$-stable formulas β; the third by induction hypothesis; the fourth by definition. $\Box$

Lemma 6.22. *The $\Box$-translation is sound with respect to intuitionistic logic: Assume that there exists an intuitionistic proof of an implication $\varphi \Rightarrow \psi$. Then there is also an intuitionistic proof of the translated implication $\varphi^\Box \Rightarrow \psi^\Box$.*

Proof. This follows by an induction on the structure of intuitionistic proofs. We have to verify that we can mirror any inference rule of intuitionistic logic in the translation. For instance, one of the disjunction rules justifies the following proof scheme: In order to prove $\varphi \vee \psi \Rightarrow \chi$, it suffices to give proofs of $\varphi \Rightarrow \chi$ and $\psi \Rightarrow \chi$. We have to justify the translated proof scheme: In order to prove $(\varphi \vee \psi)^\Box \Rightarrow \chi^\Box$, it suffices to give proofs of $\varphi^\Box \Rightarrow \chi^\Box$ and $\psi^\Box \Rightarrow \chi^\Box$.

So assume that proofs of the two implications are given. Further assume $(\varphi \vee \psi)^\Box$, i. e. $\Box(\varphi^\Box \vee \psi^\Box)$. We want to show $\chi^\Box$. Since this is a $\Box$-stable statement, we may assume that in fact $\varphi^\Box \vee \psi^\Box$ holds. Then the claim is obvious by the two given proofs.

The cases for the other rules (see Appendix 23 for a list) are similar and left to the reader. $\Box$

Remark 6.23. The reader well-versed in formal logic will have noticed that we are mixing syntax and semantics here. The proper way to state the lemma would be to formally adjoin a box operator to the language of intuitionistic logic, governed by three inference rules which are modeled on the three axioms on a modal operator. This formal box operator could then be instantiated by any concrete modal operator $\Box : \Omega \to \Omega$.

Soundness of the $\Box$-translation is important for the following reason. If φ and φ' are equivalent formulas, we are accustomed to be able to freely substitute φ by φ' anywhere we want. Since a modal operator $\Box$ is semantically defined as a map $\Omega \to \Omega$, it is trivially justified that $\Box\varphi$ and $\Box\varphi'$ are equivalent: The formulas φ and φ' give rise to the *same* element $\{x \in 1 \,|\, \varphi\} = \{x \in 1 \,|\, \varphi'\}$ of Ω, and therefore their images under $\Box$ are equal as well.

However, it is *not* clear and in fact wrong in general that the translated formulas $\varphi^\Box$ and $(\varphi')^\Box$ are equivalent. This follows only if the soundness lemma can be applied (two times, once for each direction). We should stress that to apply this lemma, it is not enough to merely *know* that φ and φ' are equivalent; instead, there has to be an intuitionistic proof of this equivalence. This is really a stronger requirement, since an equivalence $\varphi \Leftrightarrow \varphi'$ might hold in a particular model, i. e. in the internal language of some particular topos, without possessing an intuitionistic proof, i. e. holding in any topos. We give an explicit example of this situation below (Example 6.38).

Lemma 6.24. *Let φ be a formula such that for any subformulas ψ appearing as antecedents of implications, it holds that $\psi^\Box \Rightarrow \Box\psi$. (In particular, this condition* *is satisfied if there are no "⇒" signs in φ or if φ is a geometric formula.) Then* $\Box\varphi \Rightarrow \varphi^\Box$.

Proof. We prove this by an induction on the formula structure. All cases except for "⇒" are obvious. For this case, assume $\Box(\psi \Rightarrow \chi)$; we are to show that $(\psi^\Box \Rightarrow \chi^\Box)$. Since this is a $\Box$-stable statement, we can in fact assume that $(\psi \Rightarrow \chi)$. We then have

$$\psi^\Box \Longrightarrow \Box\psi \Longrightarrow \Box\chi \Longrightarrow \chi^\Box,$$

with the first step being by the requirement on antecedents, the second by the monotonicity of $\Box$, and the third by the induction hypothesis. □

Lemma 6.25. *Let φ be a geometric formula. Then $\varphi^\Box \Leftrightarrow \Box\varphi$.*

Proof. The "⇐" direction is by Lemma 6.24. The "⇒" direction is an induction on the formula structure. By way of example, we verify the case about "$\bigvee$". So assume $\Box(\bigvee_i \varphi_i^\Box)$; we are to show that $\Box(\bigvee_i \varphi_i)$. Since this is a boxed statement, we may in fact assume $\bigvee_i \varphi_i^\Box$, so for some index j, it holds that $\varphi_j^\Box$. By the induction hypothesis, it follows that $\Box\varphi_j$. By $\varphi_j \Rightarrow \bigvee_i \varphi_i$ and the monotonicity of $\Box$, it follows that $\Box(\bigvee_i \varphi_i)$. □

Note that an analogous argument for infinite conjunctions is not valid: Assume $(\bigwedge_i \varphi_i)^\Box$. So for all j, $\varphi_j^\Box$ holds. By the induction hypothesis, $\Box\varphi_j$ holds for any j. But from this we may not deduce $\Box \bigwedge_i \varphi_i$, since the axioms on a modal operator only require commutativity with finite conjunctions. This failure also has a geometric interpretation, for instance in the special case $\Box = \neg\neg$: Given dense open subsets U_i on which formulas φ_i hold, we may not conclude that there exists a single dense open subset U on which all the formulas φ_i hold.

Remark 6.26. In the special case that $\Box$ is the double negation modality, Lemma 6.25 holds with slightly weaker hypotheses: Namely, implications may occur in φ, provided that for their antecedents ψ it holds that $\psi \Rightarrow \psi^\Box$. This is because for the double negation modality, the formula $\Box(\psi \Rightarrow \chi)$ is equivalent to $\psi \Rightarrow \Box\chi$. (In general, for an arbitrary modality, only the former implies the latter, but not vice versa.) The case for "⇒" in the inductive proof then goes as follows: Assume $(\psi \Rightarrow \chi)^\Box$. Then $\psi \Rightarrow \psi^\Box \Rightarrow \chi^\Box \Rightarrow \Box\chi$, so $\Box(\psi \Rightarrow \chi)$.

Lemma 6.27. *Let φ, φ', ψ be formulas. Assume that:*
- *The formula φ' is geometric. (More generally, it suffices for $(\varphi')^\Box$ to imply $\Box\varphi'$.)*
- *There is an intuitionistic proof that φ and φ' are equivalent under the (only) hypothesis ψ.*
- *Both $\Box\psi$ and $\psi^\Box$ hold.*

Then $\varphi^\Box \Rightarrow \Box\varphi$.

Proof. Assume $\varphi^\Box$. Since $\psi^\Box$, $(\varphi \wedge \psi)^\Box$. Because the $\Box$-translation is sound with respect to intuitionistic logic (Lemma 6.22) it follows that $(\varphi')^\Box$. As φ' is geometric, it follows that $\Box\varphi'$. Since $\Box\psi$ holds, it follows that $\Box\varphi$. □

Example 6.28. Let M be an R-module. The statement that M is zero is not geometric: $\varphi :\equiv (\forall x\,{:}\,M.\ x = 0)$. But if M is generated by some finite family $x_1, \ldots, x_n : M$, then φ is equivalent to the statement $\varphi' :\equiv (x_1 = 0 \wedge \cdots \wedge x_n = 0)$ which is geometric; and there is an intuitionistic proof of this equivalence. Since no implication signs occur in $\psi :\equiv \ulcorner M$ is generated by $x_1, \ldots, x_n \urcorner$, Lemma 6.27 is applicable and shows that $\varphi^\Box$ implies $\Box\varphi$. This example will gain geometric meaning in Lemma 6.39.

Lemma 6.29. *For the modality $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition, the $\Box$-translation of the law of excluded middle holds. In particular, this applies to the double negation modality $\Box = \neg\neg$, where $\alpha = \bot$.*

Proof. We are to show that $(\varphi \vee \neg\varphi)^\Box$, i.e. that

$$((\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)) \Longrightarrow \alpha) \Longrightarrow \alpha.$$

So assume that the antecedent holds. If $\varphi^\Box$ holds, then in particular $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α hold. Therefore it follows that $(\varphi^\Box \Rightarrow \alpha)$. This implies $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α. $\Box$

6.7. Truth at stalks vs. truth on neighbourhoods. We now state the crucial property of the $\Box$-translation. Recall that "$X_\Box$" denotes the sublocale of X induced by $\Box$ (Definition 6.5).

Theorem 6.30. *Let X be a topological space. Let $\Box$ be a modal operator in $\mathrm{Sh}(X)$. Let φ be a formula over X. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(X_\Box) \models \varphi,$$

where on the right hand side, all parameters occuring in φ were pulled back to $X_\Box$ along the inclusion $X_\Box \hookrightarrow X$.

We have not yet explicitly stated the Kripke–Joyal semantics for a sheaf topos over a locale, which $X_\Box$ is in general. The definition is exactly the same as in the case for sheaf toposes over a topological space, only that any mention of "open sets" has to be substituted by the more general "opens" and any mention of the union operator "$\bigcup$" has to be interpreted by the supremum operator in the lattice of opens of the locale. For $X_\Box$, this is $\sup U_i = j(\bigcup_i U_i)$. Before giving a proof of the theorem, we want to discuss some of its consequences.

Corollary 6.31. *Let X be a topological space.*

(1) *Let $U \subseteq X$ be an open subset and let $\Box\varphi :\equiv (U \Rightarrow \varphi)$. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(U) \models \varphi.$$

(2) *Let $A \subseteq X$ be a closed subset and let $\Box\varphi :\equiv (\varphi \vee A^c)$. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(A) \models \varphi.$$

(3) *Let $\Box\varphi :\equiv \neg\neg\varphi$. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(X_{\neg\neg}) \models \varphi.$$

(4) *Let $x \in X$ be a point and let $\Box\varphi :\equiv ((\varphi \Rightarrow {!x}) \Rightarrow {!x})$. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \varphi \textit{ holds at } x.$$

Proof. Combine Theorem 6.30 and Table 2. $\Box$

We want to discuss the last case of the corollary in more detail. Let x be a point of a topological space X and let φ be a formula. Let $\Box$ be the modal operator given in the corollary. Then φ *holds at x* if and only if, from the internal perspective of $\mathrm{Sh}(X)$, the translated formula $\varphi^\Box$ holds; and φ *holds on some open neighbourhood of x* if and only if, from the internal perspective, the formula $\Box\varphi$ holds.

Thus the question whether the truth of φ at the point x spreads to some open neighbourhood can be formulated in the following way:

Does $\varphi^\Box$ imply $\Box\varphi$ in the internal language of $\mathrm{Sh}(X)$?

Phrased this way, technicalities like appropriately shrinking open neighbourhoods are blinded out. A purposefully trivial example to illustrate this is the following. Let X be a scheme (or a ringed space). Let $f, g \in \Gamma(X, \mathcal{O}_X)$ be global functions. Suppose that the germs of f and g are zero in some stalk $\mathcal{O}_{X,x}$; we want to show that they are zero on a common open neighbourhood of x.

Usual proof. Since the germ of f vanishes in $\mathcal{O}_{X,x}$, there is an open neighbourhood U_1 of x such that $f|_{U_1} = 0$ in $\Gamma(U_1, \mathcal{O}_X)$. Since furthermore the germ of g vanishes in the same stalk, there exists an open neighbourhood U_2 of x such that $g|_{U_2} = 0$. The intersection of both neighbourhoods is still an open neighbourhood of x; on this it holds that f and g both vanish. $\square$

Proof in the internal language. We may suppose that $(f = 0 \wedge g = 0)^\Box$, that is $\Box(f = 0) \wedge \Box(g = 0)$, and have to prove that $\Box(f = 0 \wedge g = 0)$. (To this end, we could simply invoke the third axiom on a modal operator, but we want to stay close to the given external proof.) So by assumption, both $\Box(f = 0)$ and $\Box(g = 0)$ hold. Since our goal is to prove a boxed statement, we may in fact assume that $f = 0$ and $g = 0$. Thus $f = 0 \wedge g = 0$. $\square$

By using the internal language with its modal operators, we can thus reduce basic facts of scheme theory which deal with stalks and neighbourhoods to facts of algebra in a *modal intuitionistic context*. As with using the internal language in its basic form without modalities, this brings conceptual clarity and reduced technical overhead. There are, however, two more distinctive advantages. Firstly, many internal proofs do not require specific properties of the modal operator and thus work with any modal operator. By interpreting such a proof using different operators, one obtains an entire family of external statements without any additional work (see Lemma 6.39 for an example).

Secondly, the following corollary gives a general metatheorem which is applicable to a wide range of cases. It allows to decide whether spreading will occur (or is likely not to occur) simply by looking at the *logical form* of the statement in question.

Corollary 6.32. *Let X be a topological space. Let φ be a formula. If φ is geometric, truth of φ at a point $x \in X$ implies truth of φ on some open neighbourhood of x, and vice versa.*

Proof. By the purely logical lemmas of the previous section, it holds that $\varphi^\Box \Leftrightarrow \Box\varphi$. $\square$

Corollary 6.33. *Let X be a topological space. Let φ be a formula. If φ is geometric, the property "φ holds at a point $x \in X$" is open.*

Proof. This is just a reformulation of the previous corollary: If φ holds at a point $x \in X$, it holds on some open neighbourhood U of x as well. Going back to stalks, it follows that φ holds at every point of U. $\square$

Example 6.34. Let X be a scheme (or a ringed space). Since the condition for a function $f : \mathcal{O}_X$ to be nilpotent is geometric (it is $\bigvee_{n \geq 0} f^n = 0$), nilpotency of f at a point is equivalent to nilpotency on some open neighbourhood.

Combined with Lemma 6.27, this metatheorem is quite useful. We will illustrate it with many examples in the next subsection.

An important special case of spreading from stalks to neighbourhoods is the case of spreading from the generic point (should it exist) to a dense open subset. Whether this occurs can be phrased by Lemma 6.15 as follows:

Does $\varphi^{\neg\neg}$ imply $\neg\neg\varphi$ in the internal language of $\mathrm{Sh}(X)$*?*

This question is a question of ordinary (non-modal) intuitionistic algebra.

Example 6.35. We have seen in Remark 6.11 that a morphism $f : A \to B$ in $\mathrm{Sh}(X_\Box) \simeq \mathrm{Sh}_\Box(\mathrm{Sh}(X))$ is an epimorphism if and only if $\mathrm{Sh}(X) \models \forall y : B.\ \Box(\exists x : X.\ f(x) = y)$. We can now understand a simple proof of this fact:

$$\begin{aligned}
&f \text{ is an epimorphism in } \mathrm{Sh}_\Box(\mathrm{Sh}(X)) \\
\Longleftrightarrow\ &\mathrm{Sh}_\Box(\mathrm{Sh}(X)) \models \ulcorner f \text{ is surjective}\urcorner \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models (\ulcorner f \text{ is surjective}\urcorner)^\Box \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \forall y : B.\ \Box(\exists x : X.\ \Box(f(x) = y)) \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \forall y : B.\ \Box(\exists x : X.\ f(x) = y).
\end{aligned}$$

The ultimate equivalence is by Lemma 6.25, applied to the geometric subformula "$\exists x : X.\ f(x) = y$".

Remark 6.36. Theorem 6.30 can also be motivated by purely logical considerations. Namely, one can check that interpreting a formula φ by $\mathrm{Sh}(X) \models \varphi^\Box$ gives rise to a model of intuitionistic logic – if φ intuitionistically implies ψ, then $\mathrm{Sh}(X) \models \varphi^\Box$ implies $\mathrm{Sh}(X) \models \psi^\Box$. In categorical logic, it is therefore a natural question whether there exists a topos $\mathcal{E}$ such that $\mathcal{E} \models \varphi$ if and only if $\mathrm{Sh}(X) \models \varphi^\Box$. Theorem 6.30 gives an affirmative answer to this question, explicitly stating that $\mathcal{E} := \mathrm{Sh}(X_\Box)$ is such a topos.

Proof of Theorem 6.30. A fancy proof goes as follows. First, one shows intuitionistically that for a modal operator $\Box$ in Set, it holds that

$$\mathrm{Set} \models \varphi^\Box \quad\Longleftrightarrow\quad \mathrm{Sh}_\Box(\mathrm{Set}) \models \varphi.$$

This can be done by an easy and nontechnical induction on the structure of formulas φ. Then one interprets this result in the sheaf topos $\mathrm{Sh}(X)$:

$$\begin{aligned}
&\mathrm{Sh}(X) \models \varphi^\Box && \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \ulcorner \mathrm{Set} \models \varphi^\Box \urcorner && \text{by idempotency} \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \ulcorner \mathrm{Sh}_\Box(\mathrm{Set}) \models \varphi \urcorner && \text{by the first step} \\
\Longleftrightarrow\ &\mathrm{Sh}_\Box(\mathrm{Sh}(X)) \models \varphi && \text{by idempotency} \\
\Longleftrightarrow\ &\mathrm{Sh}(X_\Box) \models \varphi && \text{since } \mathrm{Sh}_\Box(\mathrm{Sh}(X)) \simeq \mathrm{Sh}(X_\Box)
\end{aligned}$$

By *idempotency*, we mean that internally employing the Kripke–Joyal semantics to interpret doubly-internal statements is the same as using the Kripke–Joyal semantics once. However, we do not want to discuss this here any further; some details can be found in the original article on the stack semantics [67, Lemma 7.20], but the lemma given there is not general enough to justify the second use of idempotency above. For this, one would have to extend the stack semantics to support internal statements about locally internal categories like $\mathrm{Sh}(X_\Box) \hookrightarrow \mathrm{Sh}(X)$ (which then look like locally small categories from the internal point of view). This is worthwhile for other reasons too, but shall not be pursued here.

Therefore, we give a more explicit proof. By induction, we are going to prove that for any open subset $U \subseteq X$ and any formula φ over U, it holds that

$$U \models_X \varphi^\Box \quad\Longleftrightarrow\quad j(U) \models_{X_\Box} \varphi,$$

where the internal statements are to be interpreted by the Kripke–Joyal semantics of X and $X_\Box$ respectively and j is the nucleus associated to $\Box$. We may assume that any sheaves occuring in φ as domains of quantifications are in fact $\Box$-sheaves; we justify this with a separate lemma below.

The cases $\varphi \equiv \top$, $\varphi \equiv (\psi \wedge \chi)$, and $\varphi \equiv \bigwedge_i \psi_i$ are trivial. For $\varphi \equiv \bot$, the claim is that $U \models_X \Box\bot$ if and only if $j(U) \models_{X_\Box} \bot$. The former means $U \subseteq j(\emptyset)$ and the latter means $j(U) \leq \sup \emptyset = j(\emptyset)$, so the claim follows from the first two axioms on a nucleus. $\square$

XXX: include proof for other cases

Lemma 6.37. *Let $\Box$ be a modal operator. Let φ be a formula. Let $\psi :\equiv \varphi^\Box$ be the $\Box$-translation of φ. Let ψ' be the formula obtained from ψ by substituting any occuring domain of quantification by its $\Box$-sheafification, as syntactically defined in Definition 6.7. Then ψ and ψ' are intuitionistically equivalent.*

Proof. For any formula φ, we denote by "$\varphi^\boxplus$" the result of first applying the $\Box$-translation to φ and then substituting any set F occuring in φ as a domain of quantification by the plus construction F^+. Recall that for any such F there is a canonical map $F \to F^+,\ x \mapsto [\{x\}]$. We are going to show by induction that for any formula $\varphi(x_1, \ldots, x_n)$ in which elements $x_i : F_i$ may occur as terms, it holds that $\varphi^\Box(x_1, \ldots, x_n)$ is equivalent to $\varphi^\boxplus([\{x_1\}], \ldots, [\{x_n\}])$. This suffices to prove the lemma.

The cases for

$$\top \quad \bot \quad \wedge \quad \bigwedge \quad \vee \quad \bigvee \quad \Longrightarrow$$

are trivial. The cases for unbounded "$\forall$" and "$\exists$" are trivial as well. The case for "$=$" is slightly more interesting; let $\varphi(x, y) \equiv (x = y)$. Then we are to show that $\varphi^\Box(x, y) \equiv \Box(x = y)$ (equality in some set F) is equivalent to $\varphi^\boxplus([\{x\}], [\{y\}]) \equiv \Box([\{x\}] = [\{y\}])$ (equality in F^+). This follows by the definition of the plus construction. The case for "$\in$" is similar.

Let $\varphi \equiv (\exists x : F.\ \psi(x))$, where we have dropped further variables occuring in ψ for simplicity. Then we are to show that $\varphi^\Box \equiv \Box(\exists x : F.\ \psi^\Box(x))$ is equivalent to $\varphi^\boxplus \equiv \Box(\exists \bar{x} : F^+.\ \psi^\boxplus(\bar{x}))$. The "only if" direction is trivial (set $\bar{x} := [\{x\}]$). For the "if" direction, we may assume that there exists $\bar{x} : F^+$ such that $\psi^\boxplus(\bar{x})$, since we want to prove a boxed statement. By definition of the plus construction, it holds that $\Box(\ulcorner \bar{x}$ is a singleton$\urcorner)$. So, again since we want to prove a boxed statement, we may assume that $\bar{x}$ is actually a singleton. Therefore there exists $x : F$ such that $\bar{x} = [\{x\}]$ and that $\psi^\boxplus([\{x\})$ holds. By the induction hypothesis, it follows that $\psi^\Box(x)$. From this the claim follows.

The case for "$\forall$" is similar. $\square$

Example 6.38. Let X be a scheme. Let f be a global function on X. Let $\varphi :\equiv \neg(\ulcorner f$ inv.$\urcorner)$ and $\varphi' :\equiv \ulcorner f$ nilpotent$\urcorner$. Then, by Proposition 3.10, we have $\mathrm{Sh}(X) \models (\varphi \Leftrightarrow \varphi')$. But in general, this does not imply that $\mathrm{Sh}(X) \models (\varphi^\Box \Leftrightarrow (\varphi')^\Box)$. Consider for instance the modal operator given by $\Box\alpha :\equiv ((\alpha \Rightarrow !x) \Rightarrow !x)$ associated to a point $x \in X$. Then $\mathrm{Sh}(X) \models (\varphi^\Box \Leftrightarrow (\varphi')^\Box)$ means that the equivalence $\varphi \Leftrightarrow \varphi'$ holds at the point x. This is false for $X = \operatorname{Spec} \mathbb{Z}$, $f = 2$, and $x = (2)$, since in the local ring $\mathcal{O}_{X,x} = \mathbb{Z}_{(2)}$, the element f is not invertible while also not being nilpotent.

6.8. Internal proofs of common lemmas.

Lemma 6.39. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.*

- *Let $x \in X$ be a point. Then the stalk $\mathcal{F}_x$ is zero if and only if $\mathcal{F}$ is zero on some open neighbourhood of x.*
- *Let $A \subseteq X$ be a closed subset. Then the restriction $\mathcal{F}|_A$ (i. e. the pullback of $\mathcal{F}$ to A) is zero if and only if $\mathcal{F}$ is zero on some open subset X containing A.*

Proof. Both statements are simply internalizations of Example 6.28, using the modal operators $\Box = (\_\_ \vee A^c)$ and $\Box = ((\_\_ \Rightarrow !x) \Rightarrow !x)$. $\square$

Remark 6.40. Note that the proposition fails if one drops the hypothesis that $\mathcal{F}$ is of finite type. Indeed, in this case one cannot reformulate the condition that $\mathcal{F}$ is zero in a geometric way.

In a remark after the proof of Proposition 5.6, we promised to present a simpler proof of it once we would have developed the theory for doing so. We can now follow up on this promise.

Lemma 6.41. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point. Let n be a natural number. Then the following statements are equivalent:*

1. *There exists a generating family for $\mathcal{F}_x$ consisting of n elements.*
2. *There exists an open neighbourhood U of x such that*

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

Proof. Using the modal operator $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow\ !x) \Rightarrow\ !x)$, we have to show that the following statements in the internal language are equivalent:

(1) $\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner^\Box$.
(2) $\Box(\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner)$.

By Lemma 6.24, the second statement implies the first – note that in a formal spelling of the statement in quotes,

$$\exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i, \tag{4}$$

no implication signs occur. To show the converse direction, we may assume that there is a generating family $y_1, \ldots, y_m : \mathcal{F}$ for $\mathcal{F}$ (since $\mathcal{F}$ is, externally speaking, of finite type). Then the $\Box$-translation of the statement that the y_i generate $\mathcal{F}$ holds as well (again by Lemma 6.24). Since there is an intuitionistic proof of

$$\begin{aligned}&\ulcorner y_1, \ldots, y_m \text{ generate } \mathcal{F}\urcorner \Longrightarrow \\ &\qquad\qquad \big(\ulcorner\text{there exist } x_1, \ldots, x_n : \mathcal{F} \text{ which generate } \mathcal{F}\urcorner \Longleftrightarrow \\ &\qquad\qquad\qquad\qquad \exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m \times n}.\ \ulcorner \vec{y} = A\vec{x}\urcorner\big),\end{aligned}$$

we can substitute the non-geometric formula (4) by the geometric formula

$$\exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m \times n}.\ \ulcorner \vec{y} = A\vec{x}\urcorner$$

(Lemma 6.27). Thus the claim follows. ☐

Lemma 6.42. *Let X be a scheme (or a ringed space). Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be of finite type and assume that $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for some point $x \in X$. Then α is an epimorphism on some open neighbourhood of x.*

Proof. In the presence of generators $y_1, \ldots, y_n : \mathcal{G}$, the non-geometric surjectivity condition $(\forall y : \mathcal{G}.\ \exists x : \mathcal{F}.\ \alpha(x) = y)$ can be reformulated in a geometric way: $\bigwedge_{i=1}^n \exists x : \mathcal{F}.\ \alpha(x) = y_i$. Thus the claim follows by Lemma 6.27. ☐

Lemma 6.43. *Let $i : A \hookrightarrow X$ be a closed immersion of schemes (or ringed spaces). Let $\mathcal{F}$ be an $\mathcal{O}_A$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. Let $\Box$ be the modal operator defined by $\Box\varphi :\equiv (\varphi \vee A^c)$. From the internal perspective, we have a surjective ring homomorphism $i^\sharp : \mathcal{O}_X \to \mathcal{O}_A$, where we omit the forgetful functor i_* from $\Box$-sheaves to arbitrary sets in the notation, and an $\mathcal{O}_A$-module $\mathcal{F}$. Furthermore, we may assume that $\mathcal{F}$ is a $\Box$-sheaf. We can regard $\mathcal{F}$ as an $\mathcal{O}_X$-module by $i^\sharp$.

Note that $A^c \Rightarrow (\mathcal{F} = 0)$, by $\Box$-separatedness of $\mathcal{F}$.

We are to show that $\mathcal{F}$ is a finitely generated $\mathcal{O}_X$-module if and only if the $\Box$-translation of "$\mathcal{F}$ is a finitely generated $\mathcal{O}_A$-module" holds. In explicit terms, we have to show the equivalence of the following statements:

(1) $\bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_i i^\sharp(a_i) x_i$.
(2) $\Box(\bigvee_{n \geq 0} \Box(\exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \Box(\exists b_1, \ldots, b_n : \mathcal{O}_A.\ \Box(x = \sum_i b_i x_i))))$.

It is clear that the first statement implies the second. For the converse direction, we just have to repeatedly use the observation that $\Box\varphi$ implies $\varphi \vee (\mathcal{F} = 0)$ (once for each occurence of $\Box$). So in each step, we either obtain the statement we want or may assume that $\mathcal{F}$ is the trivial module, in which case any subclaim trivially follows. By surjectivity of $i^\sharp$, we may write any $b : \mathcal{O}_A$ as $b = i^\sharp(a)$ for some $a : \mathcal{O}_X$. □

Lemma 6.44. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. Then $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.*

Proof. It suffices to give an intuitionistic proof of the following fact: The construction $\mathrm{Hom}_R(M, \_\_)$ is geometric if M is a finitely presented R-module. So assume that M is the cokernel of a presentation matrix $(a_{ij}) : R^{n \times m}$. Then we can calculate the Hom with any R-module N as

$$\mathrm{Hom}_R(M, N) \cong \Big\{ x : N^n \ \Big|\ \bigwedge_{j=1}^{m} \sum_{i=1}^{n} a_{ij} x_i = 0 : N \Big\},$$

and this construction is patently geometric, as a set comprehension with respect to a geometric formula. □

Lemma 6.45. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $x \in X$. Then the stalk $\mathcal{F}_x$ is a finite free $\mathcal{O}_{X,x}$-module if and only if $\mathcal{F}$ is finite locally free on some open neighbourhood of x.*

Proof. The internal statement that $\mathcal{F}$ is a finite free module is not geometric:

$$\bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i.$$

But it can equivalently be reformulated as

$$\bigvee_{n \geq 0} \exists \alpha : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n).\ \exists \beta : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \alpha \circ \beta = \mathrm{id} \wedge \beta \circ \alpha = \mathrm{id}.$$

This reformulation is geometric, therefore it holds at x if and only if it holds on some open neighbourhood of x. The claim follows since, by the previous proposition, taking stalks commutes with calculating $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \_\_)$ resp. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \_\_)$; thus the pulled back formula indeed expresses that $\mathcal{F}_x$ is finite free as an $\mathcal{O}_{X,x}$-module. □

Lemma 6.46. *Let X be an integral scheme with generic point ξ. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a torsion module if and only if its generic stalk $\mathcal{F}_\xi$ vanishes.*

Proof. The generic stalk vanishes if and only if the internal statement "$(\mathcal{F} = 0)^{\neg\neg}$" holds. Therefore it suffices to give an intuitionistic proof of the following internal statement: The module $\mathcal{F}$ is torsion if and only if any element of $\mathcal{F}$ is *not not* zero.

For the "only if" direction, let $x : \mathcal{F}$ be an arbitrary element. Since $\mathcal{F}$ is a torsion module, there exists a regular element $a : \mathcal{O}_X$ such that $ax = 0$. Since X is reduced, regularity is equivalent to not-not-invertibility. Since we want to verify the $\neg\neg$-stable statement "$\neg\neg(x = 0)$", we may in fact assume that a is invertible. Then $x = 0$ obviously follows.

For the "if" direction, let $x : \mathcal{F}$ be an arbitrary element; by assumption, x is *not not* zero. Since X is integral, Lemma 6.17 is applicable. Therefore there exists an element $a : \mathcal{O}_X$ such that a is *not not* invertible and such that invertibility of a implies $x = 0$. Since $\mathcal{F}$ is quasicoherent, for some natural number n it holds that $a^n x = 0$ (Theorem 9.3 below). Since a is *not not* invertible, it is regular (see

Lemma 7.7 below for a short and self-contained proof), and therefore a^n is regular. So $x \in \mathcal{F}_{\mathrm{tors}}$. □

By simply using a different modal operator than "*not not*", we will – without any additional work – obtain a more general form of this lemma, applicable to non-integral schemes (see Lemma 7.20).

- general explanation of modalities (as for instance in philosophy)
- explain that for some modal operators, the □-translation of the law of excluded middle is valid; explain consequences
- spreading of properties from stalk to neighbourhood: give many examples
- give proof of the expressions for the nuclei listed in the table

7. Rational functions and Cartier divisors

7.1. The sheaf of rational functions. Recall that the sheaf $\mathcal{K}_X$ of rational functions on a scheme X (or a ringed space) can be defined as the sheaf associated to the presheaf

$$U \subseteq X \text{ open} \quad \longmapsto \quad \Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S})^{-1}],$$

where $\Gamma(U, \mathcal{S})$ is the multiplicative set of those sections of $\mathcal{O}_X$ on U, which are regular in each stalk $\mathcal{O}_{X,x}$, $x \in U$. Recall also that there are some wrong definitions in the literature [48].

Using the internal language, we can give a simpler definition of $\mathcal{K}_X$. Recall that we can associate to any ring R its total quotient ring, i. e. its localization at the multiplicative subset of regular elements. Since from the internal perspective $\mathcal{O}_X$ is an ordinary ring, we can associate to it its total quotient ring $\mathcal{O}_X[\mathcal{S}^{-1}]$, where $\mathcal{S}$ is internally defined by the formula

$$\mathcal{S} := \{s : \mathcal{O}_X \mid \ulcorner s \text{ is regular} \urcorner\} \subseteq \mathcal{O}_X.$$

Externally, this ring is the sheaf $\mathcal{K}_X$.

Proposition 7.1. *Let X be a scheme (or a ringed space). The sheaf of rings defined in the internal language by localizing $\mathcal{O}_X$ at its set of regular elements is (canonically isomorphic to) the sheaf $\mathcal{K}_X$ of rational functions.*

Proof. Internally, the ring $\mathcal{O}_X[\mathcal{S}^{-1}]$ has the following universal property: For any ring R and any homomorphism $\mathcal{O}_X \to R$ which maps the elements of $\mathcal{S}$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}] \to R$ which renders the evident diagram commutative.

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr] \arrow[dr] & & R \\ & \mathcal{O}_X[\mathcal{S}^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

The translation using the Kripke–Joyal semantics gives the following universal property: For any open subset $U \subseteq X$, any sheaf of rings $\mathcal{R}$ on U and any homomorphism $\mathcal{O}_X|_U \to \mathcal{R}$ which maps all elements of $\Gamma(V, \mathcal{S})$, $V \subseteq U$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}]|_U \to \mathcal{R}$ which renders the evident diagram commutative. It is well-known that the sheaf $\mathcal{K}_X$ as usually defined has this universal property as well. □

Proposition 7.2. *Let X be a scheme (or a ringed space). Then the stalks of $\mathcal{K}_X$ are given by*

$$\mathcal{K}_{X,x} = \mathcal{O}_{X,x}[\mathcal{S}_x^{-1}].$$

The elements of $\mathcal{S}_x$ are exactly the germs of those local sections which are regular not only in $\mathcal{O}_{X,x}$, but in all rings $\mathcal{O}_{X,y}$ where y ranges over some open neighbourhood of x (depending on the section).

Proof. Since localization is a geometric construction, the first statement is made entirely trivial by our framework. The second statement follows since

$$\Gamma(U, \mathcal{S}) = \{s \in \Gamma(U, \mathcal{O}_X) \mid U \models \ulcorner s \text{ is regular}\urcorner\}$$

and regularity is a geometric implication, so that $U \models \ulcorner s$ is regular$\urcorner$ if and only if the germ s_y is regular in $\mathcal{O}_{X,y}$ for all $y \in U$. □

Remark 7.3. Speaking internally, the multiplicative set $\mathcal{S}$ is saturated. Therefore an element $s/t : \mathcal{K}_X$ is invertible in $\mathcal{K}_X$ if and only if the numerator s belongs to $\mathcal{S}$, i. e. is an regular element of $\mathcal{O}_X$.

7.2. Regularity of local functions. It is well-known that on a locally Noetherian scheme, regularity spreads from stalks to neighbourhoods, i. e. a section of $\mathcal{O}_X$ is regular in $\mathcal{O}_{X,x}$ if and only if it is regular on some open neighbourhood of x. This fact has a simple proof in the internal language.

Proposition 7.4. *Let X be a locally Noetherian scheme. Let $s \in \Gamma(U, \mathcal{O}_X)$ be a local function on X. Let $x \in U$. Then the following statements are equivalent:*

(1) *The section s is regular in $\mathcal{O}_{X,x}$.*
(2) *The section s is regular in all local rings $\mathcal{O}_{X,y}$ where y ranges over some open neighbourhood of x.*

Proof. Let $\Box$ be the modal operator defined by $\Box\varphi :\equiv ((\varphi \Rightarrow {!x}) \Rightarrow {!x})$. By Corollary 6.31, we are to show that the following statements of the internal language are equivalent:

(1) $(\ulcorner s \text{ is regular}\urcorner)^\Box$, i. e. $\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow \Box(t = 0)$.
(2) $\Box(\ulcorner s \text{ is regular}\urcorner)$, i. e. $\Box(\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow t = 0)$.

It is clear that the second statement implies the first – in fact, this is true without any assumptions on X: Let $t : \mathcal{O}_X$ be such that $st = 0$. Since we want to prove the boxed statement $\Box(t = 0)$, we may assume that s is regular and prove $t = 0$. This is immediate. (This direction also follows simply by examining the logical form and applying Lemma 6.24.)

For the converse direction, consider the annihilator of s, i. e. the ideal

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X.$$

This ideal satisfies the quasicoherence condition (Example 9.6), thus I is a quasicoherent submodule of a finitely generated module. Since X is locally Noetherian, it follows that I is finitely generated as well, say by $x_1, \ldots, x_n : I$. By assumption, each generator $x_i : I$ fulfills $\Box(x_i = 0)$. Since we want to prove a boxed statement, we may in fact assume $x_i = 0$. Thus $I = (0)$ and the assertion that s is regular follows. □

Note that the proof critically depends on the ideal I being finitely generated, since a modal operator need only commute with finite conjuctions. Intuitively, each time we use the modus ponens rule $\Box\varphi \wedge (\varphi \Rightarrow \psi) \Rightarrow \Box\psi$, we restrict to a smaller open neighbourhood of x. Since infinite intersections of open sets need not be open, we cannot expect an infinitary modus ponens rule to hold.

Corollary 7.5. *Let X be a locally Noetherian scheme. Then the stalks $\mathcal{K}_{X,x}$ of the sheaf of rational functions are given by the total quotient rings of the local rings $\mathcal{O}_{X,x}$.*

Proof. Combine Proposition 7.2 and Proposition 7.4. □

7.3. Normality. Recall that a ring R is *normal* if and only if it is integrally closed in its total quotient ring. Recall also that a scheme X (or a ringed space) is *normal* if and only if all rings $\mathcal{O}_{X,x}$ are normal.

Proposition 7.6. *A locally Noetherian scheme is normal if and only if the ring $\mathcal{O}_X$ is normal from the internal perspective.*

Proof. The condition of normality can be put into a form which is almost a geometric implication:

$$\begin{aligned}&\forall s, t : \mathcal{O}_X.\ \Big(\ulcorner t \text{ regular}\urcorner \wedge \\ &\qquad (\exists a_0, \ldots, a_{n-1} : \mathcal{O}_X.\ s^n + a_{n-1} t s^{n-1} + \cdots + a_1 t^{n-1} s + a_0 t^n = 0) \Longrightarrow \\ &\qquad\qquad\qquad\qquad\qquad\qquad \exists u : \mathcal{O}_X.\ s = ut\Big).\end{aligned}$$

The only non-geometric subpart is the condition on t to be regular. However, by Proposition 7.4, for the purposes of comparing its truth at points vs. on neighbourhoods, it behaves just like a geometric formula. Therefore the claim follows. □

7.4. Geometric interpretation of rational functions. Recall that on integral schemes, rational functions (i. e. sections of $\mathcal{K}_X$) are the same thing as regular functions defined on dense open subsets. This amounts to saying that $\mathcal{K}_X$ *is the* $\neg\neg$-*sheafification of* $\mathcal{O}_X$ (see Proposition 6.14). We want to rederive this result, as far as possible in the internal language, and generalize it to arbitrary (not necessarily locally Noetherian) schemes.

Lemma 7.7. *Let X be a reduced scheme. Then:*

(1) *$\mathcal{O}_X$ is $\neg\neg$-separated.*
(2) *Internally, an element $s : \mathcal{O}_X$ is regular if and only if it is* not not *invertible.*

Proof. Recall from Corollary 3.9 that

$$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible}\urcorner) \Leftrightarrow s = 0. \tag{5}$$

From this we can deduce that $\mathcal{O}_X$ is $\neg\neg$-separated: Assume $\neg\neg(s = 0)$ for $s : \mathcal{O}_X$. If s were invertible, we would have $\neg\neg(1 = 0)$ and thus $\perp$. Therefore s is not invertible and thus zero.

For the "only if" direction of the second statement, note that a regular element is not zero (if it were, then the true statement $0 \cdot 0 = 0 \cdot 1$ would imply the false statement $0 = 1$) and thus *not not* invertible (by the contrapositive of equivalence (5)). For the "if" direction, let $st = 0$ in $\mathcal{O}_X$. Since s is *not not* invertible, it follows that t is *not not* zero. Since $\mathcal{O}_X$ is $\neg\neg$-separated, this implies that t really is zero. □

For the following, we need two technical conditions. Say that an affine scheme $\operatorname{Spec} A$ has property $(\star)$ if and only if:

> Every open dense subset $U \subseteq \operatorname{Spec} A$ contains a *standard open* dense subset.

Say that $\operatorname{Spec} A$ has property $(\star\star)$ if and only if:

> Every open scheme-theoretically dense subset $U \subseteq \operatorname{Spec} A$ contains a *standard open* scheme-theoretically dense subset.

The first condition is satisfied if A is an irreducible ring (i. e. if $\operatorname{Spec} A$ is irreducible) or more generally if A contains only finitely many minimal prime ideals. Both conditions are satisfied if A is integral or if A is Noetherian; for convenience, we give a proof in the latter case.

Proposition 7.8. *Let A be a Noetherian ring. Then* $\operatorname{Spec} A$ *has properties* $(\star)$ *and* $(\star\star)$.

Proof. Recall that, under the Noetherian hypothesis, an open subset of $\operatorname{Spec} A$ is dense if and only if it contains all minimal prime ideals and that it is scheme-theoretically dense if and only if it contains all associated prime ideals. There are only a finite number of these prime ideals. Therefore the claim is reduced to the following statement:

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be a finite number of points of an open subset $U \subseteq \operatorname{Spec} A$. Then there exists a standard open subset $D(f) \subseteq U$ which also contains these points.

The proof of this statement is an easy application of the prime avoidance lemma. $\square$

Proposition 7.9. *Let X be a reduced scheme. Assume that X can be covered by open affine subsets which have property $(\star)$. For instance, this condition is satisfied if X is integral, the set of irreducible components is locally finite, or if X is locally Noetherian. Then $\mathcal{K}_X$ is the $\neg\neg$-sheafification of $\mathcal{O}_X$.*

Proof. We first show that $\mathcal{K}_X$ is $\neg\neg$-separated, so assume $\neg\neg(a/s = 0)$ for $a/s : \mathcal{K}_X$. Since $\mathcal{K}_X$ is obtained from $\mathcal{O}_X$ by localizing at regular elements, the fraction a/s vanishes in $\mathcal{K}_X$ if and only if $a = 0$ in $\mathcal{O}_X$. Thus it follows that $\neg\neg(a = 0)$ in $\mathcal{O}_X$ and therefore $a = 0$ in $\mathcal{O}_X$; in particular, $a/s = 0$ in $\mathcal{K}_X$.

We defer the proof that $\mathcal{K}_X$ is a $\neg\neg$-sheaf to the end and first verify the universal property of $\neg\neg$-sheafification. So let G be a $\neg\neg$-sheaf and let $\alpha : \mathcal{O}_X \to G$ be a map. We can define an extension $\bar{\alpha} : \mathcal{K}_X \to G$ in the following way: Let $f : \mathcal{K}_X$. Define the subsingleton $S := \{x : G \mid \exists b : \mathcal{O}_X.\ f = b/1 \wedge x = \alpha(b)\} \subseteq G$. Since f can be written in the form a/s with s *not not* invertible, it follows that S is *not not* inhabited. Since G is a $\neg\neg$-sheaf, there exists a unique $x : G$ such that $\neg\neg(x \in S)$. We declare $\bar{\alpha}(f)$ to be this x. It is straightforward to check that the composition $\mathcal{O}_X \to \mathcal{K}_X \to G$ equals α and that $\bar{\alpha}$ is unique with this property.

Up to this point, the proof did not need that X is a scheme – it was enough for X to be a ringed space such that equivalence (5) holds and such that $\neg(0 = 1)$ in $\mathcal{O}_X$. Only now, in showing that $\mathcal{K}_X$ is a $\neg\neg$-sheaf, the scheme condition enters. To this end, we first reformulate the sheaf condition in a way such that it only refers to $\mathcal{O}_X$, not $\mathcal{K}_X$: The quotient ring $\mathcal{K}_X$ is a $\neg\neg$-sheaf if and only if

$$\begin{aligned} \mathrm{Sh}(X) \models \forall T \subseteq \mathcal{O}_X.\ \ulcorner T \text{ subsingleton}\urcorner \wedge \neg\neg(\ulcorner T \text{ inhabited}\urcorner) \Longrightarrow \\ \exists a, b : \mathcal{O}_X.\ \ulcorner b \text{ regular}\urcorner \wedge \neg\neg(b^{-1}a \in T). \end{aligned}$$

This is done just as in the proof of Theorem 9.3. **XXX: reorder qcoh before this because of the reference?** Note that “b^{-1}” refers to the inverse of b which indeed exists in a doubly negated context, since b is assumed regular. More explicitly, we should write

$$\neg\neg(\exists c : \mathcal{O}_X.\ bc = 1 \wedge ca \in T) \quad \text{instead of} \quad \neg\neg(b^{-1}a \in T).$$

To verify the Kripke–Joyal interpretation of the rewritten sheaf condition, let an affine open subset $U = \operatorname{Spec} A \subseteq X$ (having property $(\star)$) and a subsheaf $T \hookrightarrow \mathcal{O}_X|_U$ be given such that T is internally a subsingleton and *not not* inhabited. We may glue the unique germs in the inhabited stalks of T to obtain a section $s \in \Gamma(V, \mathcal{O}_X)$ where $V \subseteq U$ is a dense open subset. Since U has property $(\star)$, we may assume that $V = D(f)$ is a standard open subset. Because V is dense and A is reduced, the function f is a regular element of A. Since $\Gamma(V, \mathcal{O}_X) = A[f^{-1}]$, we can write $s = a/f^n$ with $a \in A$ and $n \geq 0$.

By Lemma 3.18, the function $b := f^n$ is also regular as an element of $\mathcal{O}_U$ from the internal point of view. Note that b is invertible on V, since $V = D(f) \subseteq D(b)$. It follows that on the dense open subset $V \subseteq U$, the sections s and $b^{-1}a$ agree. This observation concludes the proof. $\square$

Corollary 7.10. *Let X be a reduced scheme admitting a cover by affine open subschemes with property $(\star)$. Then $\mathcal{K}_X$ is the result of pulling back $\mathcal{O}_X$ to the sublocale $X_{\neg\neg}$ and then pushing forward again. If X is irreducible with generic point ξ, then $\mathcal{K}_X$ is the constant sheaf associated to the set $\mathcal{O}_{X,\xi}$.*

Proof. Recall from Section 6.4 that pulling back to $X_{\neg\neg}$ is equivalent to sheafifying with respect to the double negation modality; and that pushing forward is equivalent to forgetting the sheaf property. Therefore the first statement holds.

For the second statement, recall from Lemma 6.15 that the sublocale $X_{\neg\neg}$ is given by the subspace $\{\xi\}$; that the sheafification functor $\mathrm{Sh}(X) \to \mathrm{Sh}(\{\xi\}) \simeq \mathrm{Set}$ is given by calculating the stalk at ξ; and that the inclusion functor $\mathrm{Set} \simeq \mathrm{Sh}(\{\xi\}) \hookrightarrow \mathrm{Sh}(X)$ is given by the constant sheaf construction. $\square$

If X is a general scheme (not necessarily reduced), we can describe $\mathcal{K}_X$ in a similar way as a sheafification of $\mathcal{O}_X$; specifically, it is the sheafification with respect to the modal operator defined by

$$\widehat{\Box}\varphi :\equiv \ulcorner \mathcal{O}_X \text{ is } (\varphi \Rightarrow \underline{\quad})\text{-separated}\urcorner$$

in the internal language of $\mathrm{Sh}(X)$, i. e.

$$\widehat{\Box}\varphi :\equiv \forall s : \mathcal{O}_X.\ (\varphi \Rightarrow s = 0) \Rightarrow s = 0.$$

This modal operator has an explicit scheme-theoretic description.

Lemma 7.11. *Let U be an open subset of a scheme X. Then* $\mathrm{Sh}(X) \models \widehat{\Box}U$ *if and only if U is scheme-theoretically dense in X.*

Proof. We have the following chain of equivalences.

$$\begin{aligned}
&X \models \widehat{\Box}U \\
\Longleftrightarrow\ &\ulcorner \mathcal{O}_X \text{ is } (U \Rightarrow \underline{\quad})\text{-separated}\urcorner \\
\Longleftrightarrow\ &X \models \ulcorner \mathcal{O}_X \to \mathcal{O}_X^+ \text{ is injective}\urcorner \\
&\qquad\qquad (\text{where the plus construction is wrt. the modality } (U \Rightarrow \underline{\quad})) \\
\Longleftrightarrow\ &X \models \ulcorner \mathcal{O}_X \to \mathcal{O}_X^{++} \text{ is injective}\urcorner \\
&\qquad\qquad (\text{by the factorization } \mathcal{O}_X \to \mathcal{O}_X^+ \to \mathcal{O}_X^{++}) \\
\Longleftrightarrow\ &\text{the canonical morphism } \mathcal{O}_X \to j_*\mathcal{O}_U \text{ (with } j : U \hookrightarrow X) \text{ is injective} \\
\Longleftrightarrow\ &U \text{ is scheme-theoretically dense in } X.
\end{aligned}$$ $\square$

Using the internal language of a scheme, talking about scheme-theoretically dense open subsets is therefore just as easy as talking about ordinary topologically dense open subsets; the difference simply amounts to using the modal operator "$\widehat{\Box}$" instead of "*not not*".

Proposition 7.12. *Let X be a ringed space. Then:*

(1) *The operator $\widehat{\Box}$ fulfills the axioms on a modal operator.*
(2) *$\mathcal{O}_X$ is $\widehat{\Box}$-separated.*
(3) *$\mathcal{K}_X$ is $\widehat{\Box}$-separated.*
(4) *Internally, it holds that $\widehat{\Box}(\ulcorner f \text{ inv.}\urcorner)$ implies that f is regular for any $f : \mathcal{O}_X$.*

Suppose furthermore that X is a scheme. Then:

(5) *The converse in (4) holds.*
(6) *If X can be covered by open affine subschemes with property $(\star\star)$, then $\mathcal{K}_X$ is the $\widehat{\Box}$-sheafification of $\mathcal{O}_X$.*

Proof. The first four properties are entirely formal; we thus skip over some details. For the first property, we verify the second axiom on a modal operator. So we assume $\widehat{\Box}\widehat{\Box}\varphi$ and have to show $\widehat{\Box}\varphi$. To this end, let $s : \mathcal{O}_X$ be arbitrary such that $\varphi \Rightarrow (s = 0)$; we have to prove that $s = 0$. If $\mathcal{O}_X$ were separated with respect to the modal operator $(\varphi \Rightarrow \_\_)$, it would follow that $s = 0$. So unconditionally it holds that $\widehat{\Box}\varphi \Rightarrow (s = 0)$. Since by assumption $\mathcal{O}_X$ is $(\widehat{\Box}\varphi \Rightarrow \_\_)$-separated, the claim follows.

For the second property, let $s : \mathcal{O}_X$ be arbitrary such that $\widehat{\Box}(s = 0)$. Obviously it holds that $(s = 0) \Rightarrow (s = 0)$. Thus, since $\mathcal{O}_X$ is separated with respect to $((s = 0) \Rightarrow \_\_)$, it follows that $s = 0$. The proof of the third property is similar.

For the fourth property, assume $\widehat{\Box}(\ulcorner f \text{ inv.}\urcorner)$ and let $h : \mathcal{O}_X$ be arbitrary such that $fh = 0$. Then, trivially, it holds that $\ulcorner f \text{ inv.}\urcorner \Rightarrow h = 0$. Since $\mathcal{O}_X$ is separated with respect to $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$, it follows that $h = 0$.

We may now suppose that X is a scheme. To verify the fifth property, let a regular element $f : \mathcal{O}_X$ be given. We have to show that $\mathcal{O}_X$ is separated with respect to the modality $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$. So assume that $\ulcorner f \text{ inv.}\urcorner \Rightarrow (s = 0)$ for some $s : \mathcal{O}_X$. By Proposition 3.10 it follows that $f^n s = 0$ for some natural number n. Since f is regular, we may conclude that $s = 0$.

The verification of the universal property of $\mathcal{K}_X$ is done analogously as in the case that X is reduced: For the proof of Proposition 7.9, it was critical that regular elements of $\mathcal{O}_X$ are *not not* invertible. We now need (and have) that regular elements of $\mathcal{O}_X$ are $\widehat{\Box}(\ulcorner\text{invertible}\urcorner)$.

Thus it only remains to verify that $\mathcal{K}_X$ is a $\widehat{\Box}$-sheaf. We may again imitate the proof of Proposition 7.9; using the same notation, we may now suppose that V is a standard open subset such that $U \models \widehat{\Box}V$ (previously, we supposed that $U \models \neg\neg V$). The proof that the denominator b is regular (as seen from the internal perspective, as an element of $\mathcal{O}_U$) now goes as follows: We have $V \subseteq D(b)$. Therefore $U \models \widehat{\Box}V$ implies $U \models \widehat{\Box}(\ulcorner b \text{ inv.}\urcorner)$. By the fourth property, it follows that $U \models \ulcorner b \text{ is regular}\urcorner$. ☐

Remark 7.13. The modal operator $\widehat{\Box}$ is the largest (weakest) operator such that $\mathcal{O}_X$ is $\widehat{\Box}$-separated, i. e. if $\Box$ is any modal operator such that $\mathcal{O}_X$ is $\Box$-separated, then $\Box\varphi \Rightarrow \widehat{\Box}\varphi$ for any proposition φ.

In the special case that X is a reduced scheme, Proposition 7.12 recovers the result of Proposition 7.9:

Proposition 7.14. *Let X be a scheme. Then $\widehat{\Box}\varphi \Rightarrow \neg\neg\varphi$ for any formula φ. The converse holds if X is reduced, so that in this case the modal operator $\widehat{\Box}$ coincides with the double negation modality.*

Proof. Let φ be an arbitrary formula and assume $\widehat{\Box}\varphi$. Note that $\neg\varphi$ is equivalent to $\varphi \Rightarrow (1 = 0)$. Since by assumption $\mathcal{O}_X$ is separated with respect to the $(\varphi \Rightarrow \_\_)$-modality, this in turn is equivalent to $1 = 0 : \mathcal{O}_X$, i. e. to $\bot$. Thus $\neg\neg\varphi$.

For the converse direction, let $\varphi \Rightarrow (s = 0)$ for some $s : \mathcal{O}_X$; we have to show that in fact $s = 0$. Since by assumption $\neg\neg\varphi$, it follows that s is *not not* zero. Since X is reduced, $\mathcal{O}_X$ is $\neg\neg$-separated, so this implies that s is really zero. ☐

As a corollary, we can reprove the following basic lemma about scheme-theoretical density.

Lemma 7.15. *Let U be an open subset of a scheme X. If U is scheme-theoretically dense, then U is also dense in the plain topological sense. The converse holds if X is reduced.*

Proof. The set U is scheme-theoretically dense if and only if $\mathrm{Sh}(X) \models \widehat{\Box} U$ and is dense if and only if $\mathrm{Sh}(X) \models \neg\neg U$. Therefore the claim follows from the previous proposition. $\Box$

Proposition 7.16. *Let X be a scheme admitting a cover of open affine subsets with property ($\star\star$). Then $\mathcal{K}_X$ is the result of pulling back $\mathcal{O}_X$ to the sublocale $X_{\widehat{\Box}}$ associated to the modal operator $\widehat{\Box}$ and then pushing forward again. If X is locally Noetherian, this sublocale is the subspace of associated points in X.*

In formulas, the proposition says that the canonical map

$$\mathcal{K}_X \longrightarrow i_* i^{-1} \mathcal{O}_X$$

is an isomorphism, where $i : X_{\widehat{\Box}} \hookrightarrow X$ is the inclusion of the sublocale $X_{\widehat{\Box}}$. This result requires a cover with property ($\star\star$), but no Noetherian hypothesis.

Proof. The first statement follows trivially by the results of Section 6.4 and the fact that $\mathcal{K}_X$ is the $\widehat{\Box}$-sheafification of $\mathcal{O}_X$.

For the second statement, we need to verify that the nucleus $j_{\mathrm{Ass}(\mathcal{O}_X)}$ associated to the subspace of associated points coincides with the nucleus $j_{\widehat{\Box}}$ associated to the modal operator $\widehat{\Box}$. Recall from Subsection 6.3 that the latter is given by

$$\begin{aligned} j_{\widehat{\Box}}(U) &= \text{largest open subset of } X \text{ on which } \widehat{\Box} U \text{ holds} \\ &= \bigcup \{V \subseteq X \mid V \text{ open}, \ V \models \widehat{\Box} U\} \end{aligned}$$

and note that the former is given by

$$j_{\mathrm{Ass}(\mathcal{O}_X)}(U) = \bigcup \{V \subseteq X \mid V \text{ open}, \ V \cap \mathrm{Ass}(\mathcal{O}_X) \subseteq U\}.$$

This is a general fact of locale theory, not depending on particular properties of $\mathrm{Ass}(\mathcal{O}_X)$. To verify this, one needs to check that $j_{\mathrm{Ass}(\mathcal{O}_X)}$ is indeed a nucleus and that the canonical map

$$\{U \in \mathcal{T}(X) \,|\, j_{\mathrm{Ass}(\mathcal{O}_X)}(U) = U\} \longrightarrow \mathcal{T}(\mathrm{Ass}(\mathcal{O}_X)), \ U \longmapsto \mathrm{Ass}(\mathcal{O}_X) \cap U$$

is an isomorphism of frames with inverse given by $\mathrm{Ass}(\mathcal{O}_X) \cap U \mapsto j_{\mathrm{Ass}(\mathcal{O}_X)}(U)$.

The equivalence thus follows from a standard result on the set of associated points on locally Noetherian schemes:

$$\begin{aligned} & V \cap \mathrm{Ass}(\mathcal{O}_X) \subseteq U \\ \Longleftrightarrow\ & \mathrm{Ass}(\mathcal{O}_V) \subseteq U \\ \Longleftrightarrow\ & U \cap V \text{ is scheme-theoretically dense in } U \\ & \qquad \text{(this step requires the Noetherian assumption)} \\ \Longleftrightarrow\ & V \models \widehat{\Box} U. \end{aligned}$$

$\Box$

Lemma 7.17. *Let X be a scheme admitting a cover of open affine subsets with property ($\star\star$). Let $j : U \hookrightarrow X$ be the inclusion of an open subset containing the sublocale $X_{\widehat{\Box}}$. (If X is locally Noetherian, this is equivalent to requiring that U contains $\mathrm{Ass}(\mathcal{O}_X)$.) Then the canonical morphism $\mathcal{K}_X \to j_* \mathcal{K}_U$ is an isomorphism.*

Proof. Write $i : X_{\widehat{\Box}} \hookrightarrow X$ and $i' : X_{\widehat{\Box}} \hookrightarrow U$ for the inclusions. By the previous proposition, the sheaf $\mathcal{K}_X$ is given by $i_* i^{-1} \mathcal{O}_X$. Similarly, the sheaf $j_* \mathcal{K}_U$ is given by $j_* i'_* i'^{-1} j^{-1} \mathcal{O}_X$. The claim follows since $j \circ i' = i$. $\Box$

Lemma 7.18. *Let X be a scheme admitting a cover by affine open subschemes with property ($\star$) respectively ($\star\star$). Let φ be any formula over X. Then*

$$\mathrm{Sh}(X) \models \neg\neg\varphi \Longrightarrow \exists f : \mathcal{O}_X.\ \neg\neg(\ulcorner f \text{ inv.}\urcorner) \wedge (\ulcorner f \text{ inv.}\urcorner \Rightarrow \varphi)$$

respectively

$$\mathrm{Sh}(X) \models \widehat{\Box}\varphi \Longrightarrow \exists f : \mathcal{O}_X .\, \widehat{\Box}(\ulcorner f \text{ inv.}\urcorner) \wedge (\ulcorner f \text{ inv.}\urcorner \Rightarrow \varphi).$$

Proof. The proof of Lemma 6.17 carries over, *mutatis mutandis.* ☐

Proposition 7.19. *Let X be a scheme of dimension ≤ 0 such that the set of irreducible components is locally finite or such that X is locally Noetherian. Then the internal language of* $\mathrm{Sh}(X)$ *is Boolean. (The converse holds as well and was already stated as Corollary 3.15.)*

Proof. It suffices to verify the principle of double negation elimination, since the law of excluded middle is equivalent to it.[14] So let φ be an arbitrary formula and assume $\neg\neg\varphi$. By the previous lemma there exists an element $f : \mathcal{O}_X$ such that f is *not not* invertible and such that $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \varphi)$. Since $\dim X \leq 0$, this element is invertible or nilpotent (Corollary 3.14). In the first case, we are done. In the second case, some power f^n is zero and therefore in particular *not not* zero. Since f is *not not* invertible, this implies that *not not* $1 = 0$. On the other hand $1 \neq 0$, so we obtain a contradiction; from this contradiction φ trivially follows. ☐

Lemma 7.20. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a torsion module if and only if the restriction of $\mathcal{F}$ to* $\mathrm{Ass}(\mathcal{O}_X)$ *vanishes.*

Proof. By Proposition 7.16 and Lemma 7.18 it suffices to repeat the proof of Lemma 6.46 with "*not not*" substituted by "$\widehat{\Box}$". ☐

7.5. Cartier divisors. Let X be a scheme (or a ringed space). Recall that a *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^\times/\mathcal{O}_X^\times$. This sheaf can be constructed internally, with the same notation: It is the quotient of the group of invertible elements of the ring $\mathcal{K}_X$ by the subgroup of invertible elements of the ring $\mathcal{O}_X$. So an arbitrary section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ is internally of the form $[s/t]$ with $s, t : \mathcal{O}_X$ being regular elements; this is a simpler description than the usual external one as a family $(f_i)_i$ of functions $f_i \in \Gamma(U_i, \mathcal{K}_X^\times)$ such that $f_i^{-1}|_{U_i \cap U_j} \cdot f_j|_{U_i \cap U_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$ for all i, j.

We can sketch the basic theory of Cartier divisors completely from the internal perspective. In accordance with common practice, we write the group operation of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ (which is induced by multiplication of elements in $\mathcal{K}_X^\times$) additively.

Definition 7.21. A Cartier divisor is *effective* if and only if, from the internal perspective, it can be written in the form $[s/1]$ with $s : \mathcal{O}_X$ being a regular element.

Thus a Cartier divisor $[s/t]$ is effective if and only if s is an $\mathcal{O}_X$-multiple of t.

Definition 7.22. A Cartier divisor D is *principal* if and only if there exists a global section $f \in \Gamma(X, \mathcal{K}_X^\times)$ such that internally, $D = [f]$. Two Cartier divisors are *linearly equivalent* if and only if their difference is a principal divisor.

Note that decidedly, principality is a global notion: For *any* divisor D it is true that locally there exists sections f of $\mathcal{K}_X^\times$ such that $D = [f]$.

Definition 7.23. The *line bundle associated to a Cartier divisor D* is the $\mathcal{O}_X$-submodule

$$\mathcal{O}_X(D) := \{g \in \mathcal{K}_X \mid gD \in \mathcal{O}_X\} = D^{-1}\mathcal{O}_X \subseteq \mathcal{K}_X$$

[14]This is a standard fact of intuitionistic logic. Assume that the principle of double negation elimination holds. We want to verify the law of excluded middle, so let an arbitrary formula φ be given. Even intuitionistically it holds that $\neg\neg(\varphi \vee \neg\varphi)$. By double negation elimination it follows that $\varphi \vee \neg\varphi$.

of $\mathcal{K}_X$. Here we are abusing language for "$gD \in \mathcal{O}_X$" to mean that $gf \in \mathcal{O}_X$ if $D = [f]$ with $f : \mathcal{K}_X$; and for "$D^{-1}\mathcal{O}_X$" to mean $f^{-1}\mathcal{O}_X$. This condition respectively submodule does not depend on the representative f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^\times$.

The submodule $\mathcal{O}_X(D)$ is indeed locally free of rank 1, since internally f^{-1} gives a one-element basis. Note that D is effective if and only if $\mathcal{O}_X(-D)$ is a subset of $\mathcal{O}_X$ from the internal perspective (this comparison makes sense, since $\mathcal{O}_X(-D)$ and $\mathcal{O}_X$ are both canonically embedded in $\mathcal{K}_X$). In this case, we can define the *support* of D to be the closed subscheme of X associated to the sheaf of ideals $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$.

Definition 7.24. The *Cartier divisor associated to a free $\mathcal{O}_X$-submodule $\mathcal{L} \subseteq \mathcal{K}_X$ of rank 1* is $D := [f^{-1}]$, where $f : \mathcal{K}_X$ is the unique element of some one-element basis of $\mathcal{L}$.

The basis element $f : \mathcal{K}_X$ does indeed lie in $\mathcal{K}_X^\times$: Write $f = s/t$ with $s, t : \mathcal{O}_X$. It suffices to show that s is a regular element of $\mathcal{O}_X$. So let $h : \mathcal{O}_X$ such that $sh = 0$ in $\mathcal{O}_X$. Then in particular $hf = 0$ in $\mathcal{K}_X$. By linear independence, it follows that $h = 0$ in $\mathcal{K}_X$ and thus $h = 0$ in $\mathcal{O}_X$.

Furthermore, the associated divisor does not depend on the choice of f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^\times$: If $f\mathcal{O}_X = g\mathcal{O}_X \subseteq \mathcal{K}_X$, then there exist $u, v : \mathcal{O}_X$ such that $fu = g$ and $gv = f$ in $\mathcal{K}_X$. It follows that $uv = fuvf^{-1} = gvf^{-1} = ff^{-1} = 1$ in $\mathcal{K}_X$ and thus in $\mathcal{O}_X$, by injectivity of the canonical map $\mathcal{O}_X \to \mathcal{K}_X$. Therefore u and v are elements of $\mathcal{O}_X^\times$.

Lemma 7.25. *Let D and D' be divisors on X. Then $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \cong \mathcal{O}_X(D + D')$.*

Proof. The wanted morphism of sheaves $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D') \to \mathcal{O}_X(D + D')$ is given by multiplication. That this is well-defined and an isomorphism can be checked from the internal point of view, where the claims are obvious. □

Proposition 7.26. *The association $D \mapsto \mathcal{O}_X(D)$ defines a one-to-one correspondence between Cartier divisors on X and rank-one submodules of $\mathcal{K}_X$. This correpondence descends to a one-to-one correspondence between Cartier divisiors up to linear equivalance and rank-one submodules of $\mathcal{K}_X$ up to isomorphism (as abstract $\mathcal{O}_X$-modules, ignoring their embedding into $\mathcal{K}_X$).*

Proof. The first statement is obvious from the definitions. For the second statement, it suffices to show that $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$ if and only if D is principal. An isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ gives a global section $f \in \mathcal{K}_X^\times$ (by considering the image of the unit element) such that internally, $D = [f^{-1}]$; this shows that D is principal. The converse is similar. □

For the following definition, recall that we can localize an $\mathcal{O}_X$-module $\mathcal{L}$ away from the set $\mathcal{S} \subseteq \mathcal{O}_X$ of regular elements to obtain a $\mathcal{K}_X$-module $\mathcal{L}[S^{-1}]$.

Definition 7.27. Let $f : \mathcal{L}[\mathcal{S}^{-1}]$ be a rational section of a line bundle $\mathcal{L}$ on X. Assume that "f is nontrivial", that is multiplication by f is an injective map $\mathcal{O}_X \to \mathcal{L}[\mathcal{S}^{-1}]$. Then the *associated divisor* of f is $\operatorname{div}(f) := [\psi(s)/t]$, where $f = s/t$ with $s : \mathcal{L}$ and $t : \mathcal{O}_X$ and $\psi : \mathcal{L} \to \mathcal{O}_X$ is an isomorphism.

One can check that $\psi(s)$ is a regular element of $\mathcal{O}_X$; this statement is in fact equivalent to the multiplication map $\mathcal{O}_X \to \mathcal{L}[\mathcal{S}^{-1}]$ being injective. Furthermore one can check that $[\psi(s)/t]$ does not depend on the choice of s, t, and ψ.

Proposition 7.28. *Let $f : \mathcal{L}[\mathcal{S}^{-1}]$ be a nontrivial rational section of a line bundle $\mathcal{L}$ on X. Then multiplication by f induces an isomorphism $\mathcal{O}_X(\operatorname{div}(f)) \to \mathcal{L}$.*

Proof. The isomorphism should map a rational function g to $g \cdot f$. This is a priori an element of $\mathcal{L}[\mathcal{S}^{-1}]$; we have to check that it can be regarded as an element of $\mathcal{L}$. Just as in the definition of $\operatorname{div}(f)$, write $f = s/t$ and fix an isomorphism $\psi : \mathcal{L} \to \mathcal{O}_X$. Write $g = (t/\psi(s)) \cdot h$ for some function $h : \mathcal{O}_X$. Then $g \cdot f = sh/\psi(s) = h\psi^{-1}(1)$, since $s = \psi^{-1}(\psi(s)) = \psi(s) \cdot \psi^{-1}(1)$. The element $h\psi^{-1}(1)$ can indeed be considered as an element of $\mathcal{L}$.

Injectivity of the map $\mathcal{O}_X(\operatorname{div}(f)) \to \mathcal{L}$ is by the nontriviality of f. For surjectivity, note that $(t/\psi(s)) \cdot \psi(v)$ is a preimage to $v : \mathcal{L}$, since $(t/\psi(s)) \cdot \psi(v) \cdot f = \psi(v)\psi(s)\psi^{-1}(1)/\psi(s) = v$. □

Proposition 7.29. *Let $\mathcal{L}$ be a line bundle on X. Assume that $\mathcal{L}$ can be embedded into $\mathcal{K}_X$. Then $\mathcal{L}$ possesses a nontrivial rational section.*

Proof. Let $i : \mathcal{L} \to \mathcal{K}_X$ denote the given injection. Let (v) be an one-element basis for $\mathcal{L}$. Write $i(v) = s/t$. Then s is regular, since $hs = 0$ implies $i(hv) = 0$ and thus $h = 0$, for any $h : \mathcal{O}_X$. Therefore $f := tv/s$ is a well-defined element of $\mathcal{L}[\mathcal{S}^{-1}]$. Furthermore it is nontrivial in the desired sense: If $h \cdot (tv/s) = 0$, then $htv = 0$, thus $ht = 0$ and $h = 0$.

It remains to check that f is independent of the choice of v and of the representation $i(v) = s/t$; else we defined only local sections which might not glue to a single nontrivial rational section (externally speaking). This is obvious. □

Proposition 7.30. *Let D be an effective divisor on X. Then the complement of its support is scheme-theoretically dense.*

Proof. The complement of the support of D, that is $D(\mathcal{O}_X(-D))$, is the truth value associated to the statement "$1 \in \mathcal{O}_X(-D)$". By Lemma 7.11, we therefore have to verify that $\mathcal{O}_X$ is separated with respect to the modal operator $(1 \in \mathcal{O}_X(-D) \Rightarrow \underline{\ \ })$.

Let $s : \mathcal{O}_X$ be given such that $1 \in \mathcal{O}_X(-D) \Rightarrow s = 0$; we have to show that $s = 0$. Writing $D = [f/1]$ where $f : \mathcal{O}_X$ is a regular element, this condition is equivalent to $\ulcorner f \text{ inv.} \urcorner \Rightarrow s = 0$. By Proposition 3.10 it follows that $f^n s = 0$ for some $n \geq 0$. Since f is regular, we may cancel f^n in this equation. □

Proposition 7.31. *Assume that X is an integral scheme. Then any line bundle on X is (uncanonically) a submodule of $\mathcal{K}_X$.*

Proof. Let ξ be the generic point of X and let $\Box := \neg\neg$ denote the modal operator such that internal sheafification with respect to $\Box$ is the same as pulling back to $\{\xi\}$ and then pushing forward to X again (see Section 6.5). Let $\mathcal{L}$ be a line bundle on X. Since $\mathcal{L}_\xi \cong \mathcal{O}_{X,\xi}$ (uncanonically), there is some injection $\mathcal{L}_\xi \to \mathcal{K}_{X,\xi}$; this corresponds internally to an injection $\mathcal{L}^{++} \to \mathcal{K}_X^{++}$. Since $\mathcal{K}_X$ is already a $\Box$-sheaf (see Proposition 7.9) and $\mathcal{L}$ is $\Box$-separated (being isomorphic to $\mathcal{O}_X$), we have the global injection

$$\mathcal{L} \hookrightarrow \mathcal{L}^{++} \hookrightarrow \mathcal{K}_X^{++} \xrightarrow{(\cong)^{-1}} \mathcal{K}_X. \qquad \Box$$

- "$\operatorname{div}(g) + D \geq 0$"

8. Compactness and metaproperties

8.1. Quasicompactness. As stated in the introduction, quasicompactness of a space can not be detected by the internal language: There cannot exist a formula φ such that a topological space is quasicompact if and only if $\mathrm{Sh}(X) \models \varphi$, since the latter is always a local property on X while quasicompactness is not. However, quasicompactness can be characterized by a *metaproperty* of the internal language.

This result is best stated in a way which does not explicitly refer to a notion of finiteness. So recall that quasicompactness of a topological space X can be phrased in the following way: For any directed set I and any monotone family $(U_i)_{i\in I}$ of open subsets, if $X = \bigcup_i U_i$ then $X = U_i$ for some $i \in I$. As usual, a *directed set* is an inhabited partially ordered set such that for any two elements there exists a common upper bound. A family $(U_i)_{i\in I}$ is *monotone* if and only if $i \preceq j$ implies $U_i \subseteq U_j$.

Proposition 8.1. *Let X be a topological space. Then X is quasicompact if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any directed set I and any monotone family $(\varphi_i)_{i\in I}$ of formulas over X,*

$$\mathrm{Sh}(X) \models \bigvee_{i\in I} \varphi_i \quad \textit{implies} \quad \textit{for some } i \in I,\ \mathrm{Sh}(X) \models \varphi_i.$$

The monotonicity condition means that $\mathrm{Sh}(X) \models (\varphi_i \Rightarrow \varphi_j)$ for any $i \preceq j$ in I.

Stated more succintly, a topological space X is quasicompact if and only if "$\mathrm{Sh}(X) \models$" commutes with directed "$\bigvee_{i\in I}$"'s.

Proof. For the "only if" direction, let such a family of formulas be given. Declare U_i to be the largest open subset of X where φ_i holds. Then by assumption, the U_i form a monotone family and cover X. By quasicompactness of X, some single U_i covers X as well, such that the corresponding formula φ_i holds on X.

For the "if" direction, note that a monotone family (U_i) of open subsets induces a monotone family of formulas by defining $\varphi_i :\equiv U_i$. This correspondence is such that $\mathrm{Sh}(X) \models \bigvee_i \varphi_i$ holds if and only if $X = \bigcup_i U_i$ and such that $\mathrm{Sh}(X) \models \varphi_i$ if and only if $X = U_i$. With these observations the claim is obvious. □

Example 8.2. Let X be a quasicompact scheme (or quasicompact ringed space). Let $f \in \Gamma(X, \mathcal{O}_X)$ be a global function. Endow the set of natural numbers with the usual ordering. Then the family of formulas given by $(f^n = 0)_{n\in\mathbb{N}}$ is monotone. Thus, if it internally holds that f is nilpotent, then f is nilpotent as an element of $\Gamma(X, \mathcal{O}_X)$ as well.

Proposition 8.3. *Let X be a topological space. Let $K \subseteq X$ be an open subset which is* locally quasicompact *in the sense that there exists an open covering $X = \bigcup_j U_j$ such that each $K \cap U_j$ is quasicompact. Then the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any directed set I and monotone family $(\varphi_i)_{i\in I}$ of formulas over X it holds that*

$$\mathrm{Sh}(X) \models \big(K \Rightarrow \bigvee_i \varphi_i\big) \quad \textit{implies} \quad \mathrm{Sh}(X) \models \bigvee_i (K \Rightarrow \varphi_i).$$

If additionally for any open subset $V \subseteq X$ the set $K \cap V$ is locally quasicompact in V, the following stronger and purely internal statement holds:

$$\mathrm{Sh}(X) \models \big(K \Rightarrow \bigvee_i \varphi_i\big) \Longrightarrow \bigvee_i (K \Rightarrow \varphi_i).$$

Proof. Assume that $\mathrm{Sh}(X) \models (K \Rightarrow \bigvee_i \varphi_i)$. This is equivalent to $K \models \bigvee_i \varphi_i$. By the locality of the internal language, it follows that $K \cap U_j \models \bigvee_i \varphi_i$ for each j. Since $K \cap U_j$ is quasicompact, it follows by the previous proposition that there exists an index $i_j \in I$ such that $K \cap U_j \models \varphi_{i_j}$. This is equivalent to $U_j \models (K \Rightarrow \varphi_{i_j})$. In particular, it holds that $U_j \models \bigvee_i (K \Rightarrow \varphi_i)$. Since this is true for any j, it follows that $X \models \bigvee_i (K \Rightarrow \varphi_i)$, again by the locality of the internal language.

The second statement is a corollary of the first one. □

Example 8.4. Any retrocompact subset of a scheme is locally quasicompact in the sense of the proposition.

Example 8.5. Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ be a global function. Then the open set $D(f) = \{x \in X \mid f_x \text{ is invertible in } \mathcal{O}_{X,x}\}$ is locally quasicompact in the sense of the proposition, even in the stronger sense: Let $V \subseteq X$ be any open set. Consider a covering $V = \bigcup_i U_i$ by open affine subsets $U_i = \operatorname{Spec} A_i$. Then $D(f) \cap U_i \cong \operatorname{Spec} A_i[f^{-1}]$ is quasicompact.

From this example it will trivially follow that the nilradical $\sqrt{(0)} \subseteq \mathcal{O}_X$ of a scheme and indeed the radical of any quasicoherent ideal sheaf is quasicoherent (Example 9.7). This example is also pivotal for giving a simple description of the quasicoherator (Proposition 9.11), which in turn is needed for an internal understanding of the relative spectrum (Section 12).

Remark 8.6. In applications, the open set K of the proposition is often given as the largest open subset on which some formula ψ holds. (For instance, in the previous example, K was given by the formula $\ulcorner f$ is invertible in $\mathcal{O}_X \urcorner$.) Then the conclusion of the proposition is that *assuming that ψ holds commutes with directed disjunctions.*

8.2. Locality. A stronger condition on a topological space X than quasicompactness is locality: A topological space is *local* if and only if for any open covering $X = \bigcup_i U_i$ (not necessarily directed) a certain single subset U_i covers X as well. For instance, the spectrum of a ring A is local if and only if A is a local ring. Locality has the following characterization as a metaproperty of $\mathrm{Sh}(X)$.

Proposition 8.7. *Let X be a topological space. Then X is local if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any set I and any family $(\varphi_i)_{i \in I}$ of formulas over X, it holds that*

$$\mathrm{Sh}(X) \models \bigvee_{i \in I} \varphi_i \quad \textit{implies} \quad \textit{for some } i \in I,\ \mathrm{Sh}(X) \models \varphi_i.$$

In this case, the internal language has additionally the following (weaker) metaproperty: For any sheaf $\mathcal{F}$ on X and any formula $\varphi(s)$ containing a variable $s : \mathcal{F}$, it holds that

$$\mathrm{Sh}(X) \models \exists s : \mathcal{F}.\ \varphi(s) \quad \textit{implies} \quad \textit{for some } s \in \Gamma(X, \mathcal{F}),\ \mathrm{Sh}(X) \models \varphi(s).$$

Proof. The proof of the first part is very similar to the proof of the previous proposition. For the "only if" direction of the second part, note that the antecedent implies that there exist local section $s_i \in \Gamma(U_i, \mathcal{F})$ such that $U_i \models \varphi(s_i)$ for some open covering $X = \bigcup_i U_i$. By locality of X, one such U_i suffices to cover X; so the corresponding section s_i is actually a global section and verifies $X \models \varphi(s_i)$. □

Remark 8.8. The second metaproperty stated in the proposition is indeed weaker than the condition that X is local. For instance, let X be a space consisting of two discrete points. Then $\mathrm{Sh}(X)$ has the second metaproperty, but X is not local.

8.3. Irreducibility. In intuitionistic logic, De Morgan's law $\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$ is not generally justified; therefore we can't use it when working internally to the topos of sheaves on a general scheme X. The following proposition demonstrates that if X is irreducible, the law does hold.

Proposition 8.9. *A topological space X is irreducible if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any formulas φ and ψ*

$$\mathrm{Sh}(X) \models \neg(\varphi \wedge \psi) \quad \textit{implies} \quad \mathrm{Sh}(X) \models \neg\varphi \textit{ or } \mathrm{Sh}(X) \models \neg\psi,$$

and not $\mathrm{Sh}(X) \models \bot$. Furthermore, in this case the following internal logical principle holds:

$$\mathrm{Sh}(X) \models \forall \alpha, \beta \in \Omega.\ \neg(\alpha \wedge \beta) \Rightarrow (\neg\alpha \vee \neg\beta).$$

Proof. The statement "$\mathrm{Sh}(X) \models \neg(\varphi \wedge \psi)$" means that $U \cap V = \emptyset$, where U and V are the largest open subsets on which φ respectively ψ hold. The disjunction "$\mathrm{Sh}(X) \models \neg\varphi$ or $\mathrm{Sh}(X) \models \neg\psi$" means that $U = \emptyset$ or $V = \emptyset$. And "$\mathrm{Sh}(X) \models \bot$" is equivalent to $X = \emptyset$.

Therefore, if X is irreducible, then the internal language has the claimed metaproperty. The converse can be seen by instantiating φ and ψ with the formulas associated to given open subsets having empty intersection. It then follows that one of these formulas is false in the internal language; thus the associated subset is empty.

The stated internal logical principle holds since nonempty open subsets of irreducible spaces are irreducible. □

8.4. Internal proofs of common lemmas.

Lemma 8.10. *Let X be an irreducible reduced scheme. Then all local rings $\mathcal{O}_{X,x}$ are integral domains.*

Proof. It suffices to give a proof of the following statement: Let R be a local ring such that elements which are not invertible are nilpotent. Further assume that R is reduced. Then R is an integral domain in the weak sense.

This proof may, additionally to the rules of intuitionistic logic, use the classical axiom given by Proposition 8.9.

So let arbitrary elements $x, y : R$ with $xy = 0$ be given. Then it is not the case that x and y are both invertible: If they were, their product xy would be invertible as well, contradicting $1 \neq 0$. By the classicality principle, it follows that x is not invertible or that y is not invertible. Thus x or y is nilpotent and therefore zero. □

- basic lemmas: filtered colimits, flatness, ...

9. Quasicoherent sheaves of modules

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space X is *quasicoherent* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^J \longrightarrow (\mathcal{O}_X|_U)^I \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules, where I and J are arbitrary sets (which may depend on U).

If X is indeed a scheme, quasicoherence can also be characterized in terms of inclusions of distinguished open subsets of affines: An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any open affine subscheme $U = \operatorname{Spec} A$ of X and any function $f \in A$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}), \ \tfrac{s}{f^n} \longmapsto f^{-n} s|_{D(f)}$$

is an isomorphism of $A[f^{-1}]$-modules. Here $D(f) \subseteq U$ denotes the standard open subset $\{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$. Both conditions can be internalized.

Proposition 9.1. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \exists I, J \text{ lc. } \ulcorner\text{there exists an exact sequence } \mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner.$$

The "lc" indicates that when interpreting this internal statement with the Kripke–Joyal semantics, I and J should only be instantiated with locally constant *sheaves.*

Proof. We only sketch the proof. The translation of the internal statement is that there exists a covering of X by open subsets U such that for each such U, there exist sets I, J and an exact sequence

$$(\mathcal{O}_X|_U)^{\underline{J}} \longrightarrow (\mathcal{O}_X|_U)^{\underline{I}} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

where $\underline{I}$ and $\underline{J}$ are the constant sheaves associated to I respectively J. The term "$(\mathcal{O}_X|_U)^{\underline{I}}$" refers to the internally defined free $\mathcal{O}_X$-module with basis the elements of $\underline{I}$. By exploiting that $\underline{I}$ is a discrete set from the internal point of view (i. e. any two elements are either equal or not), one can show that this is the same as $(\mathcal{O}_X|_U)^I$; similarly for J. With this observation, the statement follows. $\square$

Remark 9.2. The restriction to locally constant sheaves is really necessary: The internal statement $\mathrm{Sh}(X) \models \exists I, J.\ \ulcorner$there exists an exact sequence $\mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner$ is true for *any* $\mathcal{O}_X$-module $\mathcal{F}$. This is because the usual proof of the fact that any module admits a resolution by (not necessarily finite) free modules is intuitionistically acceptable and thus also valid in the internal universe.

We don't think that there is a useful internal characterization of locally constant sheaves. The alternative internal condition given by the following theorem does not need such a characterization.

Theorem 9.3. *Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if, from the internal perspective, for any $f : \mathcal{O}_X$, the localized module $\mathcal{F}[f^{-1}]$ is a sheaf for the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$.*

In detail, the internal condition is that for any $f : \mathcal{O}_X$, it holds that

$$\forall s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow s = 0$$

and for any subsingleton $\mathcal{S} \subseteq \mathcal{F}[f^{-1}]$ it holds that

$$(\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner \mathcal{S} \text{ inhabited}\urcorner) \Longrightarrow \exists s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{S}).$$

Unlike with the internalizations of finite type, finite presentation and coherence, this condition is *not* a standard condition of commutative algebra. In fact, in classical logic, this condition is always satisfied – for trivial logical reasons if f is invertible, and because $\mathcal{F}[f^{-1}]$ is the zero module if f is not invertible (since f is nilpotent then, by Proposition 3.7).

That this condition in not known in commutative algebra is to be expected: Quasicoherence is a condition on sheaves of modules, ensuring that they are locally isomorphic to sheaves of the form $M^\sim$, where M is a plain module. But in commutative algebra, one *only* studies plain modules (and not sheaves of modules). The quasicoherence condition is imported into the realm of commutative algebra only by the internal language.

We give the proof of the theorem below, after first discussing some examples and consequences. The proof will explain the origin of this condition.

Example 9.4. The zero $\mathcal{O}_X$-module is quasicoherent, since (it and) all localizations of it are singleton sets from the internal perspective and thus $\Box$-sheaves for any modal operator $\Box$ (Example 6.9).

Corollary 9.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subseteq \mathcal{F}$ be a submodule. Then $\mathcal{G}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \forall s : \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{G}) \Longrightarrow \bigvee_{n \geq 0} f^n s \in \mathcal{G}.$$

Proof. We can give a purely internal proof. Let $f : \mathcal{O}_X$. Since subpresheaves of separated sheaves are separated, the module $\mathcal{G}[f^{-1}]$ is in any case separated with respect to the modal operator $\Box$ with $\Box\varphi :\equiv (\ulcorner f \text{ inv.}\urcorner \Rightarrow \varphi)$.

Now suppose that $\mathcal{G}$ is quasicoherent. Let $f : \mathcal{O}_X$. Let $s : \mathcal{F}$ and assume that if f were invertible, s would be an element of $\mathcal{G}$. Define the subsingleton $S := \{t : \mathcal{G}[f^{-1}] \mid \ulcorner f \text{ inv.}\urcorner \wedge t = s/1\}$. Then S would be inhabited by $s/1$ if f were invertible. Since $\mathcal{G}[f^{-1}]$ is a $\Box$-sheaf, it follows that there exists an element u/f^n of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be the case that $u/f^n = s/1 \in \mathcal{G}[f^{-1}] \subseteq \mathcal{F}[f^{-1}]$. Since $\mathcal{F}[f^{-1}]$ is $\Box$-separated, it follows that it actually holds that $u/f^n = s/1 \in \mathcal{F}[f^{-1}]$. Therefore there exists $m : \mathbb{N}$ such that $f^m f^n s = f^m u \in \mathcal{F}$. Thus $f^{m+n} s$ is an element of $\mathcal{G}$.

For the converse direction, assume that $\mathcal{G}$ fulfills the stated condition. Let$f : \mathcal{O}_X$. Let $S \subseteq \mathcal{G}[f^{-1}]$ be a subsingleton which would be inhabited if f were invertible. By regarding S as a subset of $\mathcal{F}[f^{-1}]$, it follows that there exists an element $u/f^n \in \mathcal{F}[f^{-1}]$ such that, if f were invertible, u/f^n would be an element of S. In particular, u would be an element of $\mathcal{G}$. By assumption it follows that there exists $m : \mathbb{N}$ such that $f^m u \in G$. Thus $(f^m u)/(f^m f^n)$ is an element of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be an element of S. $\Box$

Example 9.6. Let X be a scheme and s be a global section of $\mathcal{O}_X$. Then the annihilator of s, i. e. the sheaf of ideals internally defined by the formula

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X$$

is quasicoherent. To prove this in the internal language it suffices to verify the condition of the proposition. So let $f : \mathcal{O}_X$ and $t : \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.}\urcorner \Rightarrow t \in I$, i. e. assume that if f were invertible, then st would be zero. By Proposition 3.10 it follows that $f^n st = 0$ for some $n : \mathbb{N}$, i. e. that $f^n t \in I$.

Example 9.7. Let X be a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasicoherent ideal sheaf. Then the radical of $\mathcal{I}$, internally definable as

$$\sqrt{\mathcal{I}} := \Big\{ s : \mathcal{O}_X \;\Big|\; \bigvee_{n \geq 0} s^n \in \mathcal{I} \Big\},$$

is quasicoherent as well: Let $f : \mathcal{O}_X$ and $s : \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \sqrt{\mathcal{I}}$, i. e. assume that if f were invertible, some power s^n would be an element of $\mathcal{I}$. Since *assuming that f is invertible commutes with directed disjunctions* (Example 8.5), it follows that for some natural number n, it holds that $\ulcorner f \text{ inv.}\urcorner \Rightarrow s^n \in \mathcal{I}$. By quasicoherence of $\mathcal{I}$, we may deduce that for some natural number m, it holds that $f^m s^n \in \mathcal{I}$. Thus $fs \in \sqrt{\mathcal{I}}$.

Proposition 9.8. *Let X be a scheme of dimension ≤ 0. Then any $\mathcal{O}_X$-module is quasicoherent.*

Proof. By Corollary 3.14, any element $f : \mathcal{O}_X$ is invertible or nilpotent. Therefore the quasicoherence condition of Theorem 9.3 is trivially satisfied for any $\mathcal{O}_X$-module. $\Box$

Remark 9.9. In general intuitionistic mathematics – not inside the internal universe of a scheme – the notion of quasicoherence as given by the internal condition of Theorem 9.3 does not seem to be very interesting: For many important rings, there are few quasicoherent modules in this sense. For instance, let M be a module over a ring R in which every element is invertible or not invertible. (The ring $\mathbb{Z}$ is such a ring.) Then M is quasicoherent if and only if for any $f : R$ which is not invertible, the localized module $M[f^{-1}]$ is the zero module, i. e. any element of M is annihilated by some power f^n. As a concrete example, any $\mathbb{Z}$-submodule of $\mathbb{Z}$ which contains a nonzero element fails to be quasicoherent.

Proof of Theorem 9.3. By the well-known characterization of quasicoherence in terms of inclusions of distinguished open subsets, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}),\ s/f^n \longmapsto f^{-n} s|_{D(f)} \tag{6}$$

is bijective. We will see that this map is injective for all such U and f if and only if from the internal perspective, for any $f : \mathcal{O}_X$, the set $\mathcal{F}[f^{-1}]$ is a separated presheaf with respect to the modal operator ($\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_$); and we will see that in this case, the map is additionally surjective for all such U and f if the full sheaf condition is fulfilled.

Since the sheaf $\mathcal{F}[f^{-1}]$ does not appear in the stated characterization, we will first reformulate the separatedness and the sheaf condition in terms of $\mathcal{F}$ instead of $\mathcal{F}[f^{-1}]$. To this end, note that the separatedness condition is equivalent to

$$\forall f : \mathcal{O}_X.\ \forall s : \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0 : \mathcal{F}) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0 : \mathcal{F}. \tag{7}$$

The equivalence can easily be proven in the internal language. The sheaf condition is equivalent to the conjunction of the separatedness condition and

$$\forall f : \mathcal{O}_X.\ \forall \mathcal{K} \subseteq \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner K \text{ is a singleton}\urcorner) \Longrightarrow \bigvee_{n \geq 0} \exists s : \mathcal{F}.\ \ulcorner f \text{ inv.}\urcorner \Rightarrow f^{-n} s \in \mathcal{K}. \tag{8}$$

In one direction, a set $\mathcal{S} \subseteq \mathcal{F}[f^{-1}]$ is given; construct $K := \{s : \mathcal{F} \mid s/1 \in \mathcal{S}\} \subseteq \mathcal{F}$. In the other direction, a set $\mathcal{K} \subseteq \mathcal{F}$ is given; construct $S := \{s : \mathcal{F}[f^{-1}] \mid \exists s' : \mathcal{F}.\ s' \in \mathcal{K} \wedge s = s'/1\} \subseteq \mathcal{F}[f^{-1}]$. The remaining details can easily be filled in.

We now interpret the internal statement (7) with the Kripke–Joyal semantics. Using the simplification rules, the external meaning is that for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$ the following condition is satisfied: For any section $s \in \Gamma(U, \mathcal{F})$ it should hold that

$$U \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \quad \text{implies} \quad U \models \bigvee_{n \geq 0} f^n s = 0.$$

The antecedent is equivalent to saying that s is zero in $\Gamma(D(f), \mathcal{F})$. The consequent is (by quasicompactness of U, see Example 8.2) equivalent to saying that for some $n \geq 0$, the section $f^n s$ is zero in $\Gamma(U, \mathcal{F})$, i.e. that s is zero in $\Gamma(U, \mathcal{F})[f^{-1}]$. So this condition is precisely the injectivity of the canonical map (6).

The external meaning of statement (8) is that for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$ the following condition is satisfied: For any subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ it should hold that

$$U \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner \mathcal{K} \text{ is a singleton}\urcorner) \quad \text{implies} \quad U \models \bigvee_{n \geq 0} \exists s : \mathcal{F}.\ \ulcorner f \text{ inv.}\urcorner \Rightarrow f^{-n} s \in \mathcal{K}.$$

Given the injectivity of the canonical map (6) (for any affine open subset, not only U), this condition is equivalent to its surjectivity: To see that surjectivity is sufficient, let a subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ verifying the antecedent be given. Since $\mathcal{K}|_{D(f)}$ is a singleton sheaf, we can consider its unique section $u \in \Gamma(D(f), \mathcal{K}) \subseteq \Gamma(D(f), \mathcal{F})$. By surjectivity, there exists a preimage, i.e. a fraction $s/f^n \in \Gamma(U, \mathcal{F})[f^{-1}]$ such that $u = f^{-n} s|_{D(f)}$ in $\Gamma(D(f), \mathcal{F})$. Thus $U \models f^{-n} s \in \mathcal{K}$ holds and the consequent is verified.

To see that surjectivity is necessary, let a section $u \in \Gamma(D(f), \mathcal{F})$ be given. Define a subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ by setting $\Gamma(V, \mathcal{K}) := \{u|_V \mid V \subseteq D(f)\}$. Then $\mathcal{K}$ verifies the antecedent. Thus the consequent holds: There exists an open covering $U = \bigcup_i U_i$ such that for each i, there exists a natural number n_i and a section $s_i \in \Gamma(U_i, \mathcal{F})$ such that $f^{-n_i} s_i = u$ on $U_i \cap D(f)$. Without loss of generality, we may assume that the U_i are distinguished open subsets $D(g_i) \subseteq U$; that they are finite in number; and that the natural numbers n_i agree with each other and thus equal some number n. Since $s_i = s_j$ in $\Gamma(U_i \cap U_j \cap D(f), \mathcal{F})$, injectivity of the canonical map (6) (on the affine set $U_i \cap U_j = D(g_i g_j)$) implies that $s_i = s_j$ in $\Gamma(U_i \cap U_j, \mathcal{F})[f^{-1}]$. Thus for any indices i, j there exists a natural number m_{ij} such that $f^{m_{ij}} s_i = f^{m_{ij}} s_j$ in $\Gamma(U_i \cap U_j, \mathcal{F})$. We may assume that the numbers m_{ij} equal some common number m; thus the local sections $f^m s_i$ glue to a section $s \in \Gamma(U, \mathcal{F})$. The sought preimage of u is the fraction s/f^{n+m}, since $f^{-(n+m)} s|_{D(f)}$ equals u in $\Gamma(D(f), \mathcal{F})$ (as this is true on the covering $D(f) = \bigcup_i (D(f) \cap U_i)$). □

For applications in Section 12 about interpreting the relative spectrum as an internal spectrum, we want to specialize to radical ideal sheaves. In particular, we want to describe the *quasicoherator* – the left adjoint to the inclusion of the quasicoherent radical ideals in the poset of all radical ideals – in simple terms.

Proposition 9.10. *Let X be a scheme. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a radical ideal.*

(1) *The ideal $\mathcal{I}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X .\ (\ulcorner s \text{ inv.} \urcorner \Rightarrow s \in \mathcal{I}) \Rightarrow s \in \mathcal{I}.$$

(2) *The reflection of $\mathcal{I}$ in the poset of quasicoherent radical ideals is the sheaf $\overline{\mathcal{I}}$ given by the internal expression*

$$\overline{\mathcal{I}} := \{ s : \mathcal{O}_X \mid \ulcorner s \text{ inv.} \urcorner \Rightarrow s \in \mathcal{I} \}.$$

Proof. Both claims can be verified by purely internal reasoning. The first claim is a straightforward calculation using the characterization given in Corollary 9.5. We discuss the second one in more detail.

Firstly, it's obvious that $\overline{\mathcal{I}}$ contains $\mathcal{I}$ and that $\overline{\mathcal{I}}$ is a radical ideal. To verify that $\overline{\mathcal{I}}$ is quasicoherent, let $s : \mathcal{O}_X$ be given such that, if s were invertible, then s would be an element of $\overline{\mathcal{I}}$. Symbolically, we have

$$\ulcorner s \text{ inv.} \urcorner \Longrightarrow (\ulcorner s \text{ inv.} \urcorner \Rightarrow s \in \mathcal{I}),$$

which of course implies

$$\ulcorner s \text{ inv.} \urcorner \Longrightarrow s \in \mathcal{I}.$$

This is precisely the condition for s to be an element of $\overline{I}$.

To verify that the construction $\mathcal{I} \mapsto \overline{\mathcal{I}}$ is really left adjoint to the inclusion, let a quasicoherent radical ideal $\mathcal{J}$ be given such that $\mathcal{I} \subseteq \mathcal{J}$. We have to show that $\overline{\mathcal{I}} \subseteq \mathcal{J}$. This is straightforward. □

For arbitrary $\mathcal{O}_X$-algebras $\mathcal{A}$, the description of the quasicoherator for radical ideals of $\mathcal{A}$ is more involved, but still sufficiently explicit for the applications in Section 12.

Proposition 9.11. *Let X be a scheme. Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Then the reflection of a radical ideal $\mathcal{I} \subseteq \mathcal{A}$ in the poset of quasicoherent radical ideals of $\mathcal{A}$ is given by the internal expression*

$$\overline{\mathcal{I}} := \bigcup_{n \geq 0} \mathcal{I}_n,$$

where $(\mathcal{I}_n)$ is the family of radical ideals defined recursively by

$$\mathcal{I}_0 := \mathcal{I},$$

$$\mathcal{I}_{n+1} := \text{the radical ideal generated by } \{ fs \mid f : \mathcal{O}_X, s : \mathcal{A}, (\ulcorner f \text{ inv.} \urcorner \Rightarrow s \in \mathcal{I}_n) \}.$$

Proof. We argue internally. The set $\overline{\mathcal{I}}$ contains $\mathcal{I}$ and is a radical ideal, as an ascending union of radical ideals. To verify that $\overline{\mathcal{I}}$ is quasicoherent, let $f : \mathcal{O}_X$ and $s : \mathcal{A}$ be given such that, if f were invertible, then s would be an element of $\overline{\mathcal{I}}$. This means that we have

$$\ulcorner f \text{ inv.} \urcorner \Longrightarrow \bigvee_{n \geq 0} s \in \mathcal{I}_n.$$

Since assuming that f is invertible commutes with directed disjunctions (Example 8.5), there is a natural number n such that

$$\ulcorner f \text{ inv.}\urcorner \Longrightarrow s \in \mathcal{I}_n.$$

Therefore $fs \in \mathcal{I}_{n+1} \subseteq \overline{\mathcal{I}}$.

Finally, to verify that the construction $\mathcal{I} \mapsto \overline{\mathcal{I}}$ is indeed left adjoint to the inclusion of the quasicoherent radical ideals in all radical ideals, let a quasicoherent radical ideal $\mathcal{J}$ be given such that $\mathcal{I} \subseteq \mathcal{J}$. By induction we can show that $\mathcal{I}_n \subseteq \mathcal{J}$ for all natural numbers n. Therefore $\overline{\mathcal{I}} \subseteq \mathcal{J}$. □

Remark 9.12. If the goal was to close a given radical ideal under the condition

$$\forall s : \mathcal{A}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I}) \Longrightarrow fs \in \mathcal{I},$$

where $f : \mathcal{O}_X$ is a fixed element, no infinite iteration would be necessary. The closure would in this case simply be given by

$$\overline{\mathcal{I}}^f := \text{the radical ideal generated by the set } \{fs \mid s : \mathcal{A}, (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I})\}.$$

There is also a purely formal description of the reflector, given by

$$\mathcal{I} \longmapsto \bigcap\{\mathcal{J} \subseteq \mathcal{A} \mid \mathcal{J} \text{ is a quasicoherent radical ideal such that } \mathcal{I} \subseteq \mathcal{J}\}.$$

Verifying that this construction has the universal property of the reflector is straightforward. However, it is not sufficiently concrete for calculations. In particular, we don't see a way to prove the following corollary without the explicit description given by Proposition 9.11.

Corollary 9.13. *Let X be a scheme. Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Let $\mathcal{I}$ and $\mathcal{J}$ be radical ideals of $\mathcal{A}$. Then $\overline{\mathcal{I} \cap \mathcal{J}} = \overline{\mathcal{I}} \cap \overline{\mathcal{J}}$.*

Proof. The claim is not purely formal. As a left adjoint, the reflector preserves arbitrary suprema (as a map from the poset of all radical ideals into the poset of all quasicoherent radical ideals); but the claim is that it preserves (finite) intersections.

Since the reflector is monotone, it is clear that $\overline{\mathcal{I} \cap \mathcal{J}} \subseteq \overline{\mathcal{I}} \cap \overline{\mathcal{J}}$.

To verify the converse direction, we show by induction that $\mathcal{I}_n \cap \mathcal{J}_m \subseteq \overline{\mathcal{I} \cap \mathcal{J}}$ for all natural numbers n and m. The base case is trivial, since $\mathcal{I}_0 \cap \mathcal{J}_0 = \mathcal{I} \cap \mathcal{J}$. For the induction step let $x \in \mathcal{I}_{n+1} \cap \mathcal{J}_m$. Then $x^\ell = \sum_i f_i s_i$ for some natural number ℓ and elements $f_i : \mathcal{O}_X$, $s_i : \mathcal{A}$ such that $\ulcorner f_i \text{ inv.}\urcorner \Rightarrow s_i \in \mathcal{I}_n$. In particular we have $\ulcorner f_i \text{ inv.}\urcorner \Rightarrow s_i x \in \mathcal{I}_n \cap \mathcal{J}_m$, so by the induction hypothesis $\ulcorner f_i \text{ inv.}\urcorner \Rightarrow s_i x \in \overline{\mathcal{I} \cap \mathcal{J}}$. This implies $f_i s_i x \in \overline{\mathcal{I} \cap \mathcal{J}}$, since $\overline{\mathcal{I} \cap \mathcal{J}}$ is quasicoherent. Therefore $x^{\ell+1} \in \overline{\mathcal{I} \cap \mathcal{J}}$ and thus $x \in \overline{\mathcal{I} \cap \mathcal{J}}$. □

Remark 9.14. If in the situation of Proposition 9.11 the algebra $\mathcal{A}$ is not quasicoherent, the construction $\mathcal{I} \mapsto \overline{\mathcal{I}}$ is still left adjoint to the inclusion of the radical ideal sheaves which satisfy the (then somewhat unmotivated) internal condition given in Corollary 9.5 in the poset of all radical ideal sheaves. Also Corollary 9.13 remains valid. This is even the case if X is an arbitrary ringed space; in this case, the proofs of Proposition 9.11 and Corollary 9.5 have to be modified, since then we may not suppose that assuming that an element of $\mathcal{O}_X$ is invertible commutes with directed disjunctions.

Instead, the reflector $\mathcal{I} \mapsto \overline{\mathcal{I}}$ has to be characterized by

$$\overline{\mathcal{I}} := \text{least fixed point of } P \text{ above } \mathcal{I},$$

where P is the monotone operator on the set of radical ideals which takes a radical ideal $\mathcal{I}$ to the radical ideal generated by $\{fs \mid f : \mathcal{O}_X, s : \mathcal{A}, (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I})\}$. The existence of these fixed points is guaranteed by the Knaster–Tarski theorem, which is intuitionistically valid in the version we need [13].

The following proof scheme is useful for verifying properties of the least fixed point. Let $\varphi(\mathcal{J})$ be a statement on radical ideals $\mathcal{J}$ such that $\varphi(\sup_i \mathcal{J}_i) \Leftrightarrow \bigvee_i \varphi(\mathcal{J}_i)$ for every family $(\mathcal{J}_i)_i$ of radical ideals. If

$$\varphi(P(\mathcal{J})) \Longrightarrow \varphi(\mathcal{J})$$

for all radical ideals $\mathcal{J}$ containing $\mathcal{I}$, then $\varphi(\overline{\mathcal{I}}) \Rightarrow \varphi(\mathcal{I})$. This proof scheme is a special case of the following more general scheme, which is also sometimes needed for reasoning about the least fixed point.

Let L be a complete partial order. Let α be a map from the set of radical ideals to L such that $\alpha(\sup_i \mathcal{J}_i) = \sup_i \alpha(\mathcal{J}_i)$ for every family $(\mathcal{J}_i)_i$ of radical ideals. If

$$\alpha(P(\mathcal{J})) \preceq \alpha(\mathcal{J})$$

for all radical ideals $\mathcal{J}$ containing $\mathcal{I}$, then $\alpha(\overline{\mathcal{I}}) \preceq \alpha(\mathcal{I})$.

Remark 9.15. The reflector can also be given by the formula

$$\overline{\mathcal{I}} = \bigcap_{\mathcal{J}} \Big(\mathcal{J} : \bigcap_{f : \mathcal{O}_X} (\mathcal{J} : \overline{\mathcal{I}}^f) \Big),$$

where $\overline{\mathcal{I}}^f$ is as in Remark 9.12 and the first intersection is indexed by all radical ideals $\mathcal{J} \subseteq \mathcal{A}$. This identity follows by the description of $\overline{\mathcal{I}}$ as a least fixed point and the explicit formula for the least fixed point from the proof of its existence [13]. It also follows from the observation that the operation $\mathcal{J} \mapsto \overline{\mathcal{J}}$ is the nucleus associated to the intersection of the sublocales given by the nuclei $\mathcal{J} \mapsto \overline{\mathcal{J}}^f$, which in turn is evident from the description of the relative spectrum as a classifying locale given in Proposition 12.13.

10. Subschemes

10.1. Sheaves on open and closed subspaces. It is well-known that sheaves defined on open or closed subspaces of a topological space X can be related with certain sheaves on X, by using appropriate extension functors. We can define these functors and show their basic properties in the internal language. Recall from Section 6.2 that we have defined a formula "U" for any open subset $U \subseteq X$ such that $V \models U$ if and only if $V \subseteq U$.

Lemma 10.1. *Let X be a topological space. Let $j : U \hookrightarrow X$ be the inclusion of an open subspace. Then there is a canonical functor $j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ called* extension by the empty set *with the following properties:*

1. *The functor $j_!$ is left adjoint to the restriction functor $j^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$.*
2. *The composition $j^{-1} \circ j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(U)$ is (canonically isomorphic to) the identity.*
3. *The essential image of $j_!$ consists of exactly those sheaves on X whose stalks are empty at all points of U^c. For those sheaves $\mathcal{F}$ it holds that $j_! j^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Internally, for a set $\mathcal{F}$, we can define $j_!(\mathcal{F})$ simply to be the set comprehension

$$j_!(\mathcal{F}) := \{x : \mathcal{F} \mid U\}.$$

Externally, the sections of the thus defined sheaf on an open subset $V \subseteq X$ are given by $\{x \in \Gamma(V, \mathcal{F}) \mid V \subseteq U\}$, i.e. all of $\Gamma(V, \mathcal{F})$ if $V \subseteq U$ and the empty set otherwise. With this short internal description, all of the stated properties can be easily verified in the internal language.

For instance, recall that internally the functor j^{-1} is given by sheafifying with respect to the modal operator $\Box :\equiv (U \Rightarrow \_\_)$. Thus, to show the second statement, we have to give a bijection $(j_!(\mathcal{F}))^{++} \to \mathcal{F}$ for any $\Box$-sheaf $\mathcal{F}$. (This map has to be given explicitly, to not only show a weaker statement about a local isomorphism – see Section 2.2). To this end, we can use the composition

$$(j_!(\mathcal{F}))^{++} \hookrightarrow \mathcal{F}^{++} \xrightarrow{(\cong)^{-1}} \mathcal{F},$$

where the first map is injective since sheafifying is exact. It is also surjective, since the $\Box$-translation of the statement ⌜$j_!(\mathcal{F}) \to \mathcal{F}$ is surjective⌝ holds: For any element $x : \mathcal{F}$, it holds that $\Box$(⌜x possesses a preimage⌝).

For the third property, note that a sheaf $\mathcal{F}$ on X fulfills the stated condition on stalks if and only if, from the internal perspective, it holds that $U \Rightarrow$ ⌜$\mathcal{F}$ is inhabited⌝. We omit further details. $\Box$

Lemma 10.2. *Let X be a ringed space. Let $j : U \hookrightarrow X$ be the inclusion of an open subspace. Then there is a canonical functor $j_! : \mathrm{Mod}_U(\mathcal{O}_U) \to \mathrm{Mod}_X(\mathcal{O}_X)$ called* extension by zero *with the following properties:*

1. *The functor $j_!$ is left adjoint to the restriction functor $j^{-1} : \mathrm{Mod}_X(\mathcal{O}_X) \to \mathrm{Mod}_U(\mathcal{O}_U)$.*
2. *The composition $j^{-1} \circ j_! : \mathrm{Mod}_U(\mathcal{O}_U) \to \mathrm{Mod}_U(\mathcal{O}_U)$ is (canonically isomorphic to) the identity.*
3. *The essential image of $j_!$ consists of exactly those $\mathcal{O}_X$-modules whose stalks are zero at all points of U^c. For those sheaves $\mathcal{F}$ it holds that $j_! j^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Internally, a sheaf of modules on $\mathcal{O}_U$ is simply a module on $\mathcal{O}_X^{++}$ which is a $\Box$-sheaf, where $\Box :\equiv (U \Rightarrow \_\_)$. The suitable internal definition for the extension by zero of such a module $\mathcal{F}$ is

$$j_!(\mathcal{F}) := \{x : \mathcal{F} \mid (x = 0) \vee U\}.$$

With this description, all necessary verifications are easy. Note that an $\mathcal{O}_X$-module $\mathcal{F}$ fulfills the stated condition on stalks if and only if internally, it holds that $\forall x : \mathcal{F}.\ ((x = 0) \vee U)$. $\Box$

Lemma 10.3. *Let X be a topological space. Let $i : A \hookrightarrow X$ be the inclusion of a closed subspace. The essential image of the inclusion $i_* : \mathrm{Sh}(A) \to \mathrm{Sh}(X)$ consists of exactly those sheaves whose support is a subset of A. For those sheaves $\mathcal{F}$ it holds that $i_* i^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Recall that the modal operator associated to A is $\Box\varphi :\equiv (\varphi \vee A^c)$, and that by Section 6.4 the essential image of i_* consists of exactly those sheaves which are $\Box$-sheaves from the internal perspective. Let $\mathcal{F}$ be a sheaf on X. Then it holds that

$$\operatorname{supp}\mathcal{F} \subseteq A \quad\Longleftrightarrow\quad A^c \subseteq X \setminus \operatorname{supp}\mathcal{F} \quad\Longleftrightarrow\quad A^c \subseteq \operatorname{int}(X \setminus \operatorname{supp}\mathcal{F}).$$

Since the interior of the complement of $\operatorname{supp}\mathcal{F}$ can be characterized as the largest open subset of X on which the internal statement "$\mathcal{F}$ is a singleton" holds (Remark 4.8), the condition on the support is fulfilled if and only if

$$\mathrm{Sh}(X) \models (A^c \Rightarrow \ulcorner\mathcal{F} \text{ is a singleton}\urcorner).$$

We thus have to show that this internal condition is equivalent to $\mathcal{F}$ being a $\Box$-sheaf. For the "if" direction, assume A^c. Then the empty subset $S \subseteq \mathcal{F}$ trivially verifies the condition that $\Box$(⌜S is a singleton⌝). There thus exists an element $x : \mathcal{F}$ (such that $\Box(x \in S)$). If we're given a further element $y : \mathcal{F}$, it trivially holds that $\Box(x = y)$. By $\Box$-separatedness, it thus follows that $x = y$. Thus $\mathcal{F}$ is the singleton $\{x\}$. The proof of the "only if" direction is similar.

The second statement says that internally, sheafifying a $\Box$-sheaf with respect to the modal operator $\Box$ and then forgetting that the result is a $\Box$-sheaf amounts to doing nothing. This is obvious. $\square$

10.2. Closed subschemes. Let X be a ringed space. Recall that an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ defines a closed subset $V(\mathcal{I}) = \{x \in X \,|\, \mathcal{I}_x \neq (1) \subseteq \mathcal{O}_{X,x}\}$, a sheaf of rings $\mathcal{O}_X/\mathcal{I}$, and a ringed space $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ where $\mathcal{O}_{V(\mathcal{I})}$ is the pullback of $\mathcal{O}_X/\mathcal{I}$ to $V(\mathcal{I})$. In the internal universe, we can reify $V(\mathcal{I})$ by giving a modal operator $\Box$ such that externally, the subspace $X_\Box$ coincides with $V(\mathcal{I})$.

Proposition 10.4. *Let X be a ringed space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. Then:*

(1) *The subspace of X associated to the modal operator $\Box$ defined by $\Box\varphi :\equiv (\varphi \vee (1 \in \mathcal{I}))$ is $V(\mathcal{I})$.*
(2) *The support of $\mathcal{O}_X/\mathcal{I}$ is exactly $V(\mathcal{I})$.*
(3) *The canonical morphism $i : V(\mathcal{I}) \to X$ is a closed immersion of ringed spaces.*

Proof. For any open subset $U \subseteq X$, it holds that $U \models 1 \in \mathcal{I}$ if and only if $U \subseteq D(\mathcal{I}) = X \setminus V(\mathcal{I})$. Thus $D(\mathcal{I})$ can be characterized as the largest open subset on which "$1 \in \mathcal{I}$" holds. According to Table 2 on page 48, the stated modal operator thus defines the subspace $D(\mathcal{I})^c$, i. e. $V(\mathcal{I})$.

For the second statement, note that since $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings, its support is closed. Therefore the largest open subset of X where the internal statement "$\mathcal{O}_X/\mathcal{I} = 0$" holds is the complement of the support (Proposition 4.7). Since $D(\mathcal{I})$ is the largest open subset where the internal statement "$\mathcal{I} = (1)$" holds, it suffices to show that internally, $\mathcal{O}_X/\mathcal{I} = 0$ if and only if $\mathcal{I} = (1)$. This is obvious.

The topological part of the third statement is clear. For the ring-theoretic part, we have to show that the canonical ring homomorphism $\mathcal{O}_X \to i_*\mathcal{O}_{V(\mathcal{I})}$, that is the canonical projection $\mathcal{O}_X \to \mathcal{O}_X/(\mathcal{I})$, is an epimorphism of sheaves. This is obvious. $\square$

By Lemma 10.3, the sheaf $\mathcal{O}_X/\mathcal{I}$ is thus a $\Box$-sheaf from the internal perspective.

Proposition 10.5. *Let X be a locally ringed space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. Then the ringed space $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ is locally ringed as well.*

Proof. We have to show that

$$\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is a local ring} \urcorner.$$

By Theorem 6.30, this is equivalent to

$$\mathrm{Sh}(X) \models (\ulcorner \mathcal{O}_X/\mathcal{I} \text{ is a local ring} \urcorner)^{\Box},$$

where $\Box$ is the modal operator given by $\Box\varphi :\equiv (\varphi \vee (1 \in \mathcal{I}))$. We therefore have to give an intuitionistic proof of the fact

$$\forall x, y : \mathcal{O}_X/\mathcal{I}.\ \ulcorner x + y \text{ inv.} \urcorner \Longrightarrow \Box(\ulcorner x \text{ inv.} \urcorner \vee \ulcorner y \text{ inv.} \urcorner).$$

So let $x = [s], y = [t] : \mathcal{O}_X/\mathcal{I}$ such that $x + y$ is invertible in $\mathcal{O}_X/\mathcal{I}$. This means that there exists $u : \mathcal{O}_X$ and $v : \mathcal{I}$ such that $us + ut + v = 1$ in $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a local ring, it follows that us, ut, or v is invertible. In the first two cases, it follows that x respectively y are invertible in $\mathcal{O}_X/\mathcal{I}$. In the third case, it follows that $1 \in \mathcal{I}$ and thus any boxed statement is trivially true. $\square$

If X is a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal sheaf, it is well-known that the locally ringed space $V(\mathcal{I})$ is a scheme if and only if $\mathcal{I}$ is quasicoherent. We cannot give an internal proof of this fact since we lack an internal characterization of being a scheme.

Lemma 10.6. *Let X be a scheme (or a ringed space). Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. The ringed space $V(\mathcal{I})$ is reduced if and only if, from the internal perspective of* $\mathrm{Sh}(X)$*, the ideal $\mathcal{I}$ is a radical ideal.*

Proof. The following chain of equivalences holds:

$$\begin{aligned}
&\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is a reduced ring} \urcorner \\
\Longleftrightarrow\ &\mathrm{Sh}(V(\mathcal{I})) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_{V(\mathcal{I})}.\ s^n = 0 \Longrightarrow s = 0 \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \big(\bigwedge_{n \geq 0} \forall s : \mathcal{O}_X/\mathcal{I}.\ s^n = 0 \Rightarrow s = 0\big)^{\Box} \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X/\mathcal{I}.\ s^n = 0 \Rightarrow \Box(s = 0) \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X.\ s^n \in \mathcal{I} \Rightarrow \Box(s \in \mathcal{I}) \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X.\ s^n \in \mathcal{I} \Rightarrow s \in \mathcal{I} \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \ulcorner \mathcal{I} \text{ is a radical ideal} \urcorner
\end{aligned}$$

In the second-to-last step, we used that $\Box(s \in \mathcal{I}) \equiv ((s \in \mathcal{I}) \vee (1 \in \mathcal{I}))$ implies $s \in \mathcal{I}$. This is trivial in both cases of the disjunction. □

Lemma 10.7. *Let X be a scheme (or a ringed space).*

(1) *There exists a reduced closed sub-ringed space $X_{\mathrm{red}} \hookrightarrow X$ having the same underlying topological space as X with the following universal property: Any morphism $Y \to X$ of (ringed or locally ringed) spaces such that Y is reduced factors uniquely over the closed immersion $X_{\mathrm{red}} \hookrightarrow X$.*
(2) *Let $A \subseteq X$ be a closed subset. Then there exists a structure of a reduced closed ringed subspace on A with a similar universal property.*

Proof. For the first statement, let $\mathcal{N} \subseteq \mathcal{O}_X$ be the nilradical of $\mathcal{O}_X$. This can internally be simply defined by $\mathcal{N} := \sqrt{(0)} = \{s : \mathcal{O}_X \mid \bigvee_{n \geq 0} s^n = 0\}$. Define X_{red} as the closed subspace associated to this ideal sheaf. This ringed space is reduced by the previous lemma. If X is a scheme, then quasicoherence of $\mathcal{N}$ (which is necessary and sufficient for X_{red} to be a scheme) can be shown internally (Example 9.7). The proof of the universal property can also be done in the internal language, by using that the well-known fact of locale theory that the category of locales over X is equivalent to internal locales in $\mathrm{Sh}(X)$; but we do not want to discuss this further.

For the second statement, internally define the ideal $\mathcal{I} := \sqrt{\{s : \mathcal{O}_X \mid s = 0 \vee A^c\}} \subseteq \mathcal{O}_X$. Then $1 \in \mathcal{I}$ if and only if A^c, thus by Proposition 10.4 the closed ringed subspace defined by $\mathcal{I}$ has A as underlying topological space. It is reduced since $\mathcal{I}$ is a radical ideal. □

Lemma 10.8. *Let X be a scheme of dimension $\leq n$. Let $V(\mathcal{I}) \hookrightarrow X$ be a closed subscheme which is locally cut out by a regular equation. Then* $\dim V(\mathcal{I}) \leq n - 1$.

Proof. By Proposition 3.13, it suffices to give an intuitionistic proof of the following fact of dimension theory: Let A be an arbitrary ring of dimension $\leq n$. Let $I = (s) \subseteq A$ be an ideal which is generated by a regular element $s : A$. Then the $\Box$-translation of "A/I is of dimension $\leq n - 1$" holds. In fact, we can show that A/I really is of dimension $\leq n - 1$; since no implication signs occur in a formal rendering of "being of dimension $\leq n - 1$", Lemma 6.24 is applicable and implies that this a stronger statement.

For this, let a sequence $([a_0], \ldots, [a_{n-1}])$ of elements in A/I be given. We can lift and extend this sequence to the sequence $(a_0, \ldots, a_{n-1}, s)$ of elements of A. Since $\dim A \leq n$, there exists a complementary sequence $(b_0, \ldots, b_{n-1}, b_n)$. Since s is regular, the inclusion $\sqrt{(sb_n)} \subseteq \sqrt{(0)}$ given by the definition of complementarity implies that b_n is nilpotent. Thus we have that $\sqrt{(a_{n-1}b_{n-1})} \subseteq \sqrt{(s, b_n)} = \sqrt{(s)}$ in A, which translates to $\sqrt{([a_{n-1}][b_{n-1}])} \subseteq \sqrt{(0)}$ in A/I. Therefore $([b_0], \ldots, [b_{n-1}])$ is a complementary sequence to $([a_0], \ldots, [a_{n-1}])$ in A/I. $\square$

Lemma 10.9. *Let X be a scheme. Let $\mathcal{I}$ be a sheaf of $\mathcal{O}_X$-modules. Then:*

$$\dim V(\mathcal{I}) \leq n \quad \Longleftrightarrow \quad \mathrm{Sh}(X) \models \ulcorner \mathcal{O}_X/\mathcal{I} \text{ is of Krull dimension } \leq n \urcorner.$$

Proof. By Proposition 3.13, the condition $\dim V(\mathcal{I}) \leq n$ is equivalent to

$$\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is of Krull dimension } \leq n \urcorner.$$

By Theorem 6.30 this is equivalent to

$$\mathrm{Sh}(X) \models (\ulcorner \mathcal{O}_X/\mathcal{I} \text{ is of Krull dimension } \leq n \urcorner)^{\Box},$$

where $\Box$ is the modal operator given by $\Box\varphi := (\varphi \vee (1 \in \mathcal{I}))$. The claimed equivalence then follows by Lemma 6.24 (for "$\Leftarrow$") and by direct inspection similar to the proof of Lemma 6.43 (for "$\Rightarrow$"). $\square$

- open subschemes
- Koszul resolution

11. Transfer principles

Let M be an A-module. A natural question is how properties of M relate to properties of the induced quasicoherent sheaf $M^\sim$ on $\operatorname{Spec} A$. For instance it is well-known that

- M is finitely generated iff $M^\sim$ is of finite type,
- M is flat over A iff $M^\sim$ is flat over $\mathcal{O}_{\operatorname{Spec} A}$, and
- M is torsion iff $M^\sim$ is a torsion sheaf.

Using the internal language of the little Zariski topos of $\operatorname{Spec} A$, we can give a simple, conceptual, and uniform explanation of these equivalences. Namely, from the internal point of view, the module $M^\sim$ is obtained from the constant sheaf $\underline{M}$ by localizing at the *generic filter*, a particular multiplicative subset to be introduced below, and the set M and the sheaf $\underline{M}$ share the same properties (by Lemma 11.1 below).

This makes it obvious that, for instance, properties which are stable under localization pass from M to $M^\sim$.

11.1. Internal properties of constant sheaves.

Lemma 11.1. *Let φ be a formula in which arbitrary sets and elements may occur as parameters. Let X be a topological space and let $U \subseteq X$ be an open subset. Then*

$$U \models \varphi \quad \textit{iff} \quad (U \textit{ inhabited} \Rightarrow \varphi).$$

Note that we are abusing notation on the left hand side: The parameters of φ, which are sets and elements, must be read as the induced constant sheaves and constant functions (sections of that sheaves). Unbounded quantifiers have to be read as ranging only over locally constant sheaves, not all sheaves.

Proof. By induction on the structure of φ. By way of example, we give the argument in the case that $\varphi \equiv (a = b)$, where a and b are elements of some set M. Then $U \models \varphi$ means by definition that the constant functions $U \to M$ with value a respectively b coincide. This is equivalent to saying that a and b coincide if U is inhabited. $\square$

The lemma in particular implies that constant sheaves enjoy several classical properties from the internal point of view, even though the internal language only supports intuitionistic reasoning in general. For instance, for a constant sheaf $\underline{M}$ it holds that

$$\mathrm{Sh}(X) \models \forall x, y : \underline{M}.\ \neg\neg(x = y) \Rightarrow x = y$$

and even

$$\mathrm{Sh}(X) \models \forall x, y : \underline{M}.\ x = y \vee x \neq y.$$

11.2. The generic filter. Let A be a ring.

Definition 11.2. A *filter* of A is a subset $F \subseteq A$ such that

- $0 \notin F$,
- $1 \in F$,
- $x + y \in F \Longrightarrow (x \in F) \vee (y \in F)$, and
- $xy \in F \Longleftrightarrow (x \in F) \wedge (y \in F)$

for all $x, y : A$.

In classical logic, the complement of a prime ideal is a filter and furthermore every filter is of such a form. In constructive mathematics however, it is useful to axiomatize complements of prime ideals directly, avoiding negations. Intuitionistically, since De Morgan's law $\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$ is not available, one can neither show that the complement of a prime ideal is a filter nor that the complement of a filter is a prime ideal.

A filter is in particular a multiplicative subset. Inverting the elements of a filter results in a local ring, while intuitionistically the localization of a ring at a prime ideal cannot in general be verified to be local.

Definition 11.3. The *generic filter* $\mathcal{F}$ is the subsheaf of $\underline{A}$ on $\operatorname{Spec} A$ given by

$$\Gamma(U, \mathcal{F}) := \{f : U \to A \mid f(\mathfrak{p}) \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in U\}.$$

Proposition 11.4.

(1) *Let $f \in A$ and $x \in A$. Then $D(f) \models x \in \mathcal{F}$ if and only if $f \in \sqrt{(x)}$.*
(2) *The stalk $\mathcal{F}_\mathfrak{p}$ at a point $\mathfrak{p} \in \operatorname{Spec} A$ is in canonical bijection with $A \setminus \mathfrak{p}$.*
(3) *From the internal point of view of $\mathrm{Sh}(\operatorname{Spec} A)$, the generic filter is indeed a filter of $\underline{A}$.*

Proof. By definition $D(f) \models x \in \mathcal{F}$ means that $x \notin \mathfrak{p}$ for all prime ideals $\mathfrak{p}$ with $f \notin \mathfrak{p}$. This is well-known to be equivalent to $f \in \sqrt{(x)}$.

For the claim about stalks, note that the canonical map $\mathcal{F}_\mathfrak{p} \to A \setminus \mathfrak{p}$ sending a germ $[f]$ to $f(\mathfrak{p})$ is invertible with inverse being the map which sends an element $x \notin \mathfrak{p}$ to the germ of the constant function with value x (defined on $D(x)$).

Regarding the third statement we only verify the axiom regarding sums, the other verifications being easier. Interpreting this axiom with the Kripke–Joyal semantics and restricting without loss of generality to open subsets where given locally constant functions are constant, let elements $x, y \in A$ be given such that $D(f) \models x + y \in \mathcal{F}$. By the first statement $f \in \sqrt{(x + y)}$. Therefore $D(f) \subseteq D(x) \cup D(y)$, and on $D(x)$ it holds that $x \in \mathcal{F}$ and on $D(y)$ it holds that $y \in \mathcal{F}$. $\square$

The significance of the generic filter is given by the following proposition.

Proposition 11.5. *From the internal point of view of $\mathrm{Sh}(\operatorname{Spec} A)$,*

(1) *the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is the localization of the constant sheaf $\underline{A}$ at the generic filter: $\mathcal{O}_{\operatorname{Spec} A} = \underline{A}[\mathcal{F}^{-1}]$, and*
(2) *the quasicoherent sheaf of modules $M^\sim$ associated to an A-module M is the localization of the constant sheaf $\underline{M}$ at the generic filter.*

Proof. Ignoring the ring respectively module structure, the second statement is more general; therefore we prove this one. We didn't discuss the case of quotients in Section 2.2. However it should be perspicuous that the interpretation of $\underline{M}[\mathcal{F}^{-1}]$ is defined as the colimit of $\mathcal{E} \twoheadrightarrow \underline{M} \times \mathcal{F}$, taken in the category of sheaves on $\operatorname{Spec} A$, where $\mathcal{E}$ is the subsheaf of $\mathcal{F} \times (\underline{M} \times \mathcal{F}) \times (\underline{M} \times \mathcal{F})$ given by $\mathcal{E}(U) := \{(s,(x,t),(y,u)) \mid sux = sty\}$.

This colimit can be obtained as the sheafification of the similarly defined presheaf colimit $\mathcal{E}' \twoheadrightarrow \underline{M}_{\text{pre}} \times \mathcal{F}$, where $\underline{M}_{\text{pre}}$ is the constant *presheaf* associated to M. On an open subset U this presheaf colimit is simply the localization $\Gamma(U, \underline{M}_{\text{pre}})[\Gamma(U,\mathcal{F})^{-1}] = M[\Gamma(U,\mathcal{F})^{-1}]$. In the special case that $U = D(f)$ is a standard open subset, Proposition 11.4(a) shows that this module is canonically isomorphic to $M[f^{-1}]$. The quasicoherent sheaf $M^\sim$ of modules admits the same description. ☐

Recognizing $\mathcal{O}_{\operatorname{Spec} A}$ as a localization of $\underline{A}$ fits nicely into the following abstract algebraic motivation for schemes: Does the ring A admit a *universal localization*, i. e. a homomorphism $A \to A'$ into a local ring such that every homomorphism $A \to B$ into a local ring factors via a local map over $A \to A'$? Intuitively speaking, can we localize a ring at all prime ideals at once, or equivalently at all filters at once? The answer is *no* in general,[15] but always *yes* if we are willing to change the topos in which we look for a solution: The universal localization of A is given by the ring $\mathcal{O}_{\operatorname{Spec} A}$ in the topos $\operatorname{Sh}(\operatorname{Spec} A)$; this ring is constructed by localizing $\underline{A}$ at the generic filter, a filter which exists in $\operatorname{Sh}(\operatorname{Spec} A)$ but not in Set.

We expand on this point of view in Section 12 on the relative spectrum.

For transferring properties of $M^\sim$ to M, the following metatheorem is crucial.

Proposition 11.6. *Let $\mathcal{I}$ be an ideal in $\underline{A}$ such that, for all inhabited open subsets $U \subseteq \operatorname{Spec} A$ and elements $x \in A$, the set $\Gamma(X,\mathcal{I})$ contains the constant function with value x if $\Gamma(U,\mathcal{I})$ does. Then*

$$D(f) \models \ulcorner \mathcal{I} \cap \mathcal{F} \text{ is inhabited} \urcorner \quad \textit{implies} \quad \textit{for some } n \geq 0,\ D(f) \models f^n \in \mathcal{I}.$$

Lemma 11.1 gives a simple and purely syntactical criterion for the hypothesis on $\mathcal{I}$: It suffices for $\mathcal{I}$ to be internally defined by an expression of the form $\{a : \underline{A} \mid \varphi(a)\}$, where φ is a formula which refers only to constant sheaves.

The metatheorem reflects the following well-known fact of classical ring theory: If an ideal meets every filter (that is, the complement of every prime ideal), it is the unit ideal. In this formulation the statement can't be proven intuitionistically; the occurence of *every filter* has to be replaced by *generic filter*. Intuitively, the generic filter is a reification of the abstract idea of an "arbitrary filter", a filter about which nothing is known except that it satisfies the filter axioms.

Proof. Let $D(f) \models \ulcorner \mathcal{I} \cap \mathcal{F} \text{ is inhabited} \urcorner$. Then there exists an open cover $D(f) = \bigcup_i D(f_i)$ and elements $x_i \in A$ such that $D(f_i) \models x_i \in \mathcal{F}$ and $D(f_i) \models x_i \in \mathcal{I}$. By Proposition 11.4 we have that $f_i \in \sqrt{(x_i)}$ and therefore $D(f_i) \models f_i^{m_i} \in \mathcal{I}$ for some $m_i \geq 0$. We may assume that all the $D(f_i)$ are inhabited and that the exponents m_i are all equal to some number m. The assumption on $\mathcal{I}$ implies $D(f) \models f_i^m \in \mathcal{I}$ for all i. By a standard argument we can write $f^n = \sum_i a_i f_i^m$ for some coefficients a_i; thus $D(f) \models f^n \in \mathcal{I}$. ☐

[15]Assume that the universal localization A' of a ring A exists as an ordinary ring in Set. Then any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A are equal: Let $s \notin \mathfrak{p}$. Since s is invertible in the local ring $A_\mathfrak{p}$ and the map $A' \to A_\mathfrak{p}$ induced by $A \to A_\mathfrak{p}$ is local, it is also invertible in A'. Therefore the image of s in $A_\mathfrak{q}$ is invertible as well. Thus $s \notin \mathfrak{q}$.

Remark 11.7. The stronger statement

$$D(f) \models (\ulcorner \mathcal{I} \cap \mathcal{F} \text{ is inhabited} \urcorner \Rightarrow \bigvee_{n \geq 0} (f^n \in \mathcal{I}))$$

does not hold in general. Indeed, consider the example $f := 1$ and $\mathcal{I} := [\![(g)]\!] := [\![\{a : \underline{A} \mid \exists b : \underline{A}.\ a = bg\}]\!]$, where g is a fixed element of A which is not nilpotent and not invertible. Since $D(g) \models g \in \mathcal{I} \cap \mathcal{F}$, the stronger statement would imply $D(g) \models 1 \in \mathcal{I}$. By Lemma 11.1, this is equivalent to g being invertible in A.

Remark 11.8. Recall from Proposition 7.1 that the sheaf $\mathcal{K}_{\operatorname{Spec} A}$ of rational functions can internally by obtained by localizing $\mathcal{O}_{\operatorname{Spec} A}$ at the set of regular elements. Since $\mathcal{O}_{\operatorname{Spec} A}$ is itself a localization, the sheaf $\mathcal{K}_{\operatorname{Spec} A}$ is therefore obtained by a two-step process. It can also be obtained in a single step by localizing $\underline{A}$ at $\mathcal{T}$, where $\mathcal{T}$ is the subsheaf of $\underline{A}$ defined by

$$\Gamma(U, \mathcal{T}) = \{f : U \to A \mid f(\mathfrak{p}) \text{ is regular in } A_\mathfrak{p} \text{ for all } \mathfrak{p} \in U\}.$$

This subsheaf is characterized by the property that, for all $f \in A$ and $x \in A$, $D(f) \models x \in \mathcal{T}$ if and only if x is regular in $A[f^{-1}]$.

11.3. Internal proofs of common lemmas.

Lemma 11.9. *Let A be a ring. Then A is reduced if and only if the scheme* $\operatorname{Spec} A$ *is reduced.*

Proof. By Proposition 3.3 the scheme $\operatorname{Spec} A$ is reduced if and only if $\mathcal{O}_{\operatorname{Spec} A}$ is a reduced ring from the internal point of view of $\mathrm{Sh}(\operatorname{Spec} A)$.

For the "only if" direction assume that A is reduced. Then $\underline{A}$ is reduced as well, by Lemma 11.1. Since localizations of reduced rings are reduced (and this fact has an intuitionistic proof), in particular $\mathcal{O}_{\operatorname{Spec} A} = \underline{A}[\mathcal{F}^{-1}]$ is reduced.

For the "if" direction let $x \in A$ be an element such that $x^n = 0$. Since $\mathcal{O}_{\operatorname{Spec} A} = \underline{A}[\mathcal{F}^{-1}]$ is reduced from the internal point of view, the element x is zero in that ring, that is

$$\mathrm{Sh}(\operatorname{Spec} A) \models \exists s : \mathcal{F}.\ sx = 0.$$

Therefore the ideal internally defined by

$$\mathcal{I} := \{a : \underline{A} \mid ax = 0\}$$

meets the generic filter. By Proposition 11.6 it follows that $\mathrm{Sh}(\operatorname{Spec} A) \models 1 \in \mathcal{I}$. By Lemma 11.1 this is equivalent to $1 \cdot x = 0$ as elements of A. ☐

Note that the "if" direction also admits a shorter proof, by simply considering the Kripke–Joyal interpretation of $\mathrm{Sh}(\operatorname{Spec} A) \models \ulcorner \mathcal{O}_{\operatorname{Spec} A}$ is reduced$\urcorner$ and using $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong A$. We included the given proof to give a simple example of the mixed internal/external reasoning with the generic filter. In a similar way we could reprove Lemma 3.18, that is the statement that a ring element $f \in A$ is regular in A if and only if, from the internal point of view, it is regular in $\mathcal{O}_{\operatorname{Spec} A}$.

Lemma 11.10. *Let M be an A-module. Then $M^\sim$ is of finite type if and only if M is finitely generated.*

Proof. First assume that M is finitely generated over A. Then $\underline{M}$ is finitely generated over $\underline{A}$, by Lemma 11.1. Since localizations of finitely generated modules are finitely generated (over the localized ring), the module $M^\sim = \underline{M}[\mathcal{F}^{-1}]$ is finitely generated from the internal point of view. By Proposition 4.2 this means that $M^\sim$ is of finite type from the external point of view.

For the "only if" direction, we assume that $M^\sim$ is finitely generated over $\mathcal{O}_{\operatorname{Spec} A}$ from the internal point of view and have to verify that M is finitely generated over A. So it holds that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \underline{M}[\mathcal{F}^{-1}].\ \ulcorner\text{the } x_i \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner.$$

Since multiplying a generating family by an unit results again in a generating family, we have in fact that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \underline{M}.\ \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner$$

or equivalently

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{n \geq 0, x_1, \ldots, x_n \in M} \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner.$$

Since this is a directed disjunction and $\operatorname{Spec} A$ is quasicompact, Proposition 8.1 is applicable and shows that there exists a natural number $n \geq 0$ and elements $x_1, \ldots, x_n \in M$ such that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner.$$

We claim that these x_i also span M as an A-module. So let $x \in M$ be arbitrary. By elementary linear algebra we can deduce that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \exists s \in \mathcal{F}.\ \exists a_1, \ldots, a_n : \underline{A}.\ sx = \sum_i a_i x_i.$$

Therefore the ideal internally defined by

$$\mathcal{I} := \{ s : \underline{A} \mid \exists a_1, \ldots, a_n : \underline{A}.\ sx = \textstyle\sum_i a_i x_i \}$$

meets the generic filter. Proposition 11.6 shows that $\mathrm{Sh}(\operatorname{Spec} A) \models 1 \in \mathcal{I}$, that is x is an element of the A-span of the x_i. □

Remark 11.11. If $M^\sim$ can be generated by $\leq n$ elements over $\mathcal{O}_{\operatorname{Spec} A}$ from the internal point of view, it needn't be the case that M can be generated by $\leq n$ elements over A. It is instructive to see where the appropriately modified version of the above proof fails: In this case we still have

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{x_1, \ldots, x_n \in M} \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner,$$

but this disjunction is no longer directed.

Lemma 11.12. *Let X be a scheme. Then kernels and cokernels of morphisms between quasicoherent $\mathcal{O}_X$-modules are quasicoherent.*

Proof. We may assume that $X = \operatorname{Spec} A$ is affine. A morphism between quasicoherent $\mathcal{O}_X$-modules is of the form $\underline{\varphi}[\mathcal{F}^{-1}] : \underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$, where $\varphi : M \to N$ is a linear map between A-modules. Since taking constant shaves and localization are exact, we have the chain of isomorphisms

$$(\underline{\ker(\varphi)})[\mathcal{F}^{-1}] = (\ker(\underline{\varphi}))[\mathcal{F}^{-1}] = \ker(\underline{\varphi}[\mathcal{F}^{-1}]),$$

and similarly for the cokernel. □

11.4. An application to constructive mathematics. The generic filter has a practical application in constructive mathematics. Recall that intuitionistically prime and maximal ideals don't work very well, since one often needs the axiom of choice or related set-theoretical principles in dealing with them. This is unfortunate, since prime and maximal ideals are very useful in some situations. For example:

- To verify that a ring element is nilpotent, it suffices to verify that it is an element of every prime ideal. For instance, this is calculationally simpler when proving that the coefficients of a nilpotent polynomial are themselves nilpotent.
- To verify that there is an relation of the form $1 = p_1 f_1 + \cdots + p_m f_m$ among polynomials $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ where K is an algebraically closed field, it suffices to show that the f_i don't have a common zero.

One could of course simply switch to classical logic in this case. However this might not be desirable, as a constructive proof would contain more information: For instance, if we have classically proven that an element x is an element of every prime ideal, then we know that some power x^n is zero. But from such a proof we can't directly read off any upper bound on n.

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes all non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all filters") by "for the generic filter".

This was already explored by Coquand, Coste, Lombardi, Roy, and others under the theme of *dynamical methods in algebra* [33, 27]. Here we show how one can use the generic filter, as reified by a sheaf in the little Zariski topos, to achieve similar effects.

Proposition 11.13. *Let M and N be A-modules. Let $\alpha : M \to N$ be a linear map. The interpretations of the statements in the second column of Table 3 in the internal language of* $\mathrm{Sh}(\operatorname{Spec} A)$ *are intuitionistically equivalent to the statements given in the third column.*

Proof. To demonstrate the technique we verify the first and the last claim. To make the following proofs constructive we would have to define $\operatorname{Spec} A$ and its sheaf topos in a constructive fashion, not using prime ideals. This can be done, by constructing $\operatorname{Spec} A$ as a locale instead of a topological space (see for instance Section 12.2 and [82, p. 743f.]), but we won't discuss details here.

The interpretation of $\mathrm{Sh}(\operatorname{Spec} A) \models x \notin \mathcal{F}$ by the Kripke–Joyal semantics is that $D(f) \models x \in \mathcal{F}$ implies $D(f) = \emptyset$ for all $f \in A$. By Proposition 11.4(a) this is equivalent to

$$\forall f \in A.\ f \in \sqrt{(x)} \Rightarrow f \in \sqrt{(0)},$$

that is the statement that x is nilpotent in A.

Assume that $\alpha : M \to N$ is surjective. By Lemma 11.1 the induced map $\underline{M} \to \underline{N}$ is surjective from the internal point of view. Since localization preserves surjectivity, also the map $\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.

Conversely, assume that $\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective from the internal point of view. To verify that $\alpha : M \to N$ is surjective, let $y \in N$. The assumption implies that the ideal internally defined by

$$\mathcal{I} := \{s : \underline{A} \mid \exists x : \underline{A}.\ sy = \underline{\alpha}(x)\}$$

meets the generic filter. By Proposition 11.6 this implies that $\mathrm{Sh}(\operatorname{Spec} A) \models 1 \in \mathcal{I}$, that is there exists an element $x \in A$ such that $\alpha(x) = y$. $\square$

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are fin. gen. over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is fin. gen. over $\underline{A}[\mathcal{F}^{-1}]$.	M is fin. gen. over A.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

TABLE 3. Substituting the use of prime ideals by the generic filter.

XXX: Discuss "finitely generated" example in more detail.

The sheaf-theoretical approach using the generic filter is different from the dynamical methods in the following aspect. We have to reword classical arguments using (the generic) filter instead of (the generic) prime ideal. Depending on the situation this might be a nuisance. One might be tempted to employ the complement of the generic filter, but this is only an ideal, not a prime ideal from the internal point of view.[16] **XXX: Give counterexample.**

11.5. An internal proof of Grothendieck's generic freeness lemma. The goal of this subsection is to give a simple proof of Grothendieck's generic freeness lemma in the following general form.

Theorem 11.14. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. Then there is a dense open subset $U \subseteq \operatorname{Spec}(A)$ such that over U,*

1. *$B^\sim$ is finitely presented as an $\mathcal{O}_{\operatorname{Spec}(A)}$-algebra,*
2. *$M^\sim$ is of finite presentation over $B^\sim$, and*
3. *$M^\sim$ is (not necessarily finite) locally free as an $\mathcal{O}_{\operatorname{Spec}(A)}$-module.*

The usual proofs of Grothendieck's generic freeness lemma proceed using a series of reduction steps which are arguably not very memorable or straightforward, see for instance [69, Tag 051Q] or [68]. In particular, there doesn't seem to be a published proof which tackles the Noetherian and non-Noetherian cases in one go. Employing the internal language, Grothendieck's generic freeness lemma can be proved in a simple, conceptual, and constructive way without any reduction steps.

This section was prompted by a MathOverflow thread [26] and greatly benefited from discussions with Brandenburg.

Proof of Theorem 11.14. It suffices to prove that, from the internal point of view of $\operatorname{Sh}(\operatorname{Spec}(A))$, it's *not not* the case that

1. $B^\sim$ is of finite presentation over $\mathcal{O}_{\operatorname{Spec}(A)}$,
2. $M^\sim$ is finitely presented as a $B^\sim$-module, and
3. $M^\sim$ is (not necessarily finite) free over $\mathcal{O}_{\operatorname{Spec}(A)}$.

[16] One can check that the complement of $\mathcal{F}$ in $\underline{A}$ is the subsheaf $\mathcal{P}$ defined by $\Gamma(U, \mathcal{P}) := \{f : U \to A \mid f(\mathfrak{p}) \in \mathfrak{p} \text{ for all } \mathfrak{p} \in U\}$ and that $D(f) \models x \in \mathcal{P}$ if and only if fx is nilpotent. This can be used to show that the statement $\operatorname{Sh}(\operatorname{Spec} A) \models \forall x, y : \underline{A}.\ xy \in \mathcal{P} \Rightarrow x \in \mathcal{P} \vee y \in \mathcal{P}$ is false in general.

$x^0y^7v_1$	$x^1y^7v_1$	$x^2y^7v_1$	$x^3y^7v_1$	$x^4y^7v_1$	$x^5y^7v_1$	$x^6y^7v_1$	$x^7y^7v_1$
$x^0y^6v_1$	$x^1y^6v_1$	$x^2y^6v_1$	$x^3y^6v_1$	$x^4y^6v_1$	$x^5y^6v_1$	$x^6y^6v_1$	$x^7y^6v_1$
$x^0y^5v_1$	$x^1y^5v_1$	$x^2y^5v_1$	$x^3y^5v_1$	$x^4y^5v_1$	$x^5y^5v_1$	$x^6y^5v_1$	$x^7y^5v_1$
$x^0y^4v_1$	$x^1y^4v_1$	$x^2y^4v_1$	$x^3y^4v_1$	$x^4y^4v_1$	$x^5y^4v_1$	$x^6y^4v_1$	$x^7y^4v_1$
$x^0y^3v_1$	$x^1y^3v_1$	$x^2y^3v_1$	$x^3y^3v_1$	$x^4y^3v_1$	$x^5y^3v_1$	$x^6y^3v_1$	$x^7y^3v_1$
$x^0y^2v_1$	$x^1y^2v_1$	$x^2y^2v_1$	$x^3y^2v_1$	$x^4y^2v_1$	$x^5y^2v_1$	$x^6y^2v_1$	$x^7y^2v_1$
$x^0y^1v_1$	$x^1y^1v_1$	$x^2y^1v_1$	$x^3y^1v_1$	$x^4y^1v_1$	$x^5y^1v_1$	$x^6y^1v_1$	$x^7y^1v_1$
$x^0y^0v_1$	$x^1y^0v_1$	$x^2y^0v_1$	$x^3y^0v_1$	$x^4y^0v_1$	$x^5y^0v_1$	$x^6y^0v_1$	$x^7y^0v_1$

FIGURE 1. A single step in the iterative process used in the proof of Theorem 11.14, in the special case $n = 2, m = 1$. The hatched cells indicate vectors which have already been removed from the generating family. The vector in the red cell was found to be expressible as a linear combination of vectors with smaller index (blue cells). It is therefore about to be removed, along with the vectors in all cells to the top and to the right of the red cell.

Since $B^\sim$ is finitely generated as an $\mathcal{O}_{\mathrm{Spec}(A)}$-algebra, it is isomorphic to an algebra of the form $\mathcal{O}_{\mathrm{Spec}(A)}[X_1, \ldots, X_n]/\mathfrak{a}$ for some number $n \geq 0$ and some ideal $\mathfrak{a}$. By Proposition 3.29 and Theorem 3.27, the ring $\mathcal{O}_{\mathrm{Spec}(A)}[X_1, \ldots, X_n]$ is weakly Noetherian. Therefore $\mathfrak{a}$ is *not not* finitely generated, showing that $B^\sim$ is *not not* of finite presentation over $\mathcal{O}_{\mathrm{Spec}(A)}$.

Similarly, the module $M^\sim$ is of the form $(B^\sim)^m/U$ for some number $m \geq 0$ and some submodule U. Since $(B^\sim)^m$ is weakly Noetherian as a direct sum of weakly Noetherian modules, the submodule U is *not not* finitely generated. Thus $M^\sim$ is *not not* a finitely presented $B^\sim$-module.

The basic idea to show that $M^\sim$ is *not not* free over $\mathcal{O}_{\mathrm{Spec}(A)}$ is as follows. Since $\mathcal{O}_{\mathrm{Spec}(A)}$ is a field in the sense that noninvertible elements are zero, minimal generating families are already linearly independent; we observed this in the proof of Lemma 5.8. By the finiteness hypotheses, the module $M^\sim$ admits a countable generating family. It's *not not* the case that either one of these vectors can be expressed as a linear combination of the others, or not. In the second case we're

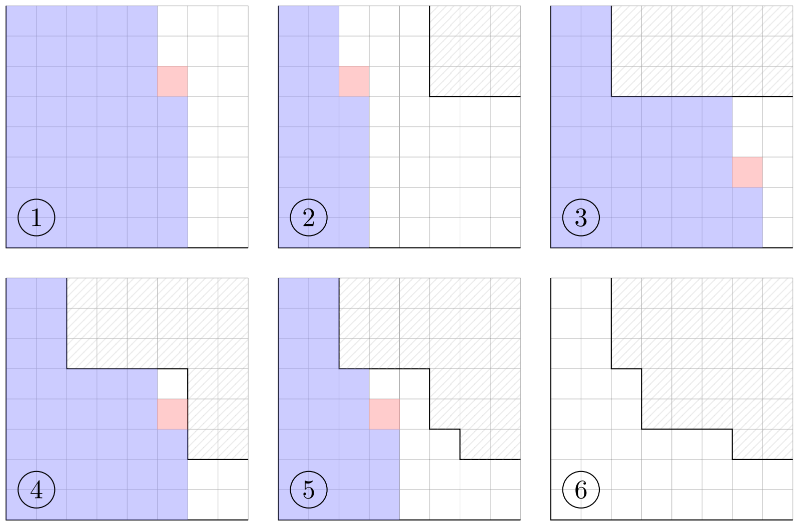

FIGURE 2. The iterative process used in the proof of Theorem 11.14, in the special case $n = 2, m = 1$. The process terminates after reducing the generating family a finite number of times.

done; in the first case, we remove the redundant vector and continue in the same fashion.

However, if we shrink the given generating family in this naive fashion, the process may not terminate in finitely many steps. In a classical context, Zorn's lemma could be used to iterate the process transfinitely and eventually obtain a minimal generating family, but Zorn's lemma is not available in the internal universe of the little Zariski topos. We therefore have to pick the vectors we'll remove in a more systematic fashion.

Let $(x_1, \ldots, x_n)$ be a generating family for $B^\sim$ as an $\mathcal{O}_{\mathrm{Spec}(A)}$-algebra and let $(v_1, \ldots, v_m)$ be a generating family for $B^\sim$ as a $B^\sim$-module. We endow the set

$$I := \{(j, i_1, \ldots, i_n) \mid j \in \{1, \ldots, m\}, i_1, \ldots, i_n \in \{0, 1, \ldots\}\}$$

with the lexicographic order. We choose the family $(x_1^{i_1} \cdots x_n^{i_n} v_j)_{j,i_1,\ldots,i_n}$ as the starting point of the shrinking process. In each step, we use that it's *not not* the case that

- either one of the vectors of the generating family can be expressed as a linear combination of vectors in the family with a smaller index,
- or not.

In the second case, the generating family is linearly independent: For any linear combination summing to zero, we can show that all coefficients are zero, beginning with the coefficient which is paired with the vector of greatest index.

Figure 1 illustrates our action in the first case. We remove the redundant vector $x_1^{i_1} \cdots x_n^{i_n} v_j$ *and also any vector with greater powers of the* $x_1, \ldots, x_n$ from the generating family. The resulting family will still be a generating family, since the linear combination witnessing the redundancy of $x_1^{i_1} \cdots x_n^{i_n} v_j$ successively gives rise to linear combinations witnessing the redundancy of the vectors $x_1^{\geq i_1} \cdots x_n^{\geq i_n} v_j$; we maintain the invariant that any member of the starting generating family can be expressed as a linear combination of vectors of the current generating family with smaller or equal index.

As indicated in Figure 2, this process terminates after finitely many steps. This fact is related to the fact that the ordinal ω^n is well-founded. $\square$

Since the given internal proof was (necessarily) intuitionistically valid, the internal language machinery is intuitionistically valid, and the construction of the spectrum can be set up in an intuitionistically sensible way (Section 12), an intuitionistic external proof not employing the topos machinery can be extracted from the given argument. The resulting proof will verify Grothendieck's generic freeness lemma in the following form.

Theorem 11.15. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. Assume that the only element $f \in A$ such that*

(1) *$B[f^{-1}]$ is of finite presentation over $A[f^{-1}]$,*
(2) *$M[f^{-1}]$ is finitely presented as a $B[f^{-1}]$-module, and*
(3) *$M[f^{-1}]$ is free over $A[f^{-1}]$*

is $f = 0$. Then $A = 0$.

In classical logic, this form implies Grothendieck's generic freeness lemma in its more abstract formulation by a routine argument: Let $U \subseteq \operatorname{Spec}(A)$ be the union over all standard open subsets $D(f)$ such that the statements (1), (2), and (3) in Theorem 11.15 hold. The statements (1), (2), and (3) of Theorem 11.14 hold on this open subset, therefore it remains to show that U is dense.

So let a nonempty open subset V of $\operatorname{Spec}(A)$ be given. This contains a standard open subset $D(g) \subseteq V$ such that g is not nilpotent. Therefore the localized ring $A[g^{-1}]$ is not zero. Thus the conclusion of Theorem 11.15 is not satisfied. Since we assume classical logic, there is a nonzero element $f \in A[g^{-1}]$ such that statements (1), (2), and (3) in Theorem 11.15 hold for $A[g^{-1}][f^{-1}]$, $B[g^{-1}][f^{-1}]$, and $M[g^{-1}][f^{-1}]$. Writing $f = h/g^n$, we see that $U \cap V$ contains the nonzero open subset $D(gh)$.

We refrain from giving the resulting explicit proof of Theorem 11.15 here, but will report on it in the future [18]. A part of the proof was included by Brandenburg in a paper of his [20].

Remark 11.16. There is no hope that there is an intuitionistic proof of Grothendieck's generic freeness lemma in the form of Theorem 11.14 even if the spectrum is constructed in an intuitionistically sensible way, since there is the following Brouwerian counterexample. Let φ be an arbitrary statement. Then the $\mathbb{Z}$-module $M := \mathbb{Z}/\mathfrak{a}$, where $\mathfrak{a} := \{x \in \mathbb{Z} \mid (x = 0) \vee \varphi\}$ as in Footnote 9 on page 29, is finitely generated. By assumption, there exists a nonzero element $f \in \mathbb{Z}$ such that $M[f^{-1}]$ is a finite free module over $A[f^{-1}]$ of some rank n. If $n = 0$, then $f^m \in \mathfrak{a}$ for some $m \geq 0$, therefore φ holds. If $n \geq 1$, then $\neg\varphi$ holds, since φ would imply $\mathfrak{a} = \mathbb{Z}$ and therefore $M[f^{-1}] = 0$. Since $n = 0 \vee n \geq 1$, it follows that $\varphi \vee \neg\varphi$.

12. Relative spectrum

Recall that if $\mathcal{A}$ is a quasicoherent $\mathcal{O}_X$-algebra on a scheme X, one can construct the *relative spectrum* $\underline{\operatorname{Spec}}_X \mathcal{A}$ by appropriately gluing the spectra $\operatorname{Spec}\Gamma(U, \mathcal{A})$ where U ranges over the affine opens of X. This relative spectrum comes equipped with a canonical morphism $\underline{\operatorname{Spec}}_X \mathcal{A} \to X$.

From the internal point of view of $\operatorname{Sh}(X)$, the sheaf $\mathcal{A}$ looks just like a plain algebra, to which therefore the usual (absolute) spectrum construction can be applied. One could hope that this construction yields the relative spectrum.

In this section, we discuss generalities on how to make sense of this internal construction; we show that this proposed construction is too naive and doesn't yield the relative spectrum; we give a refined internal construction which does yield the relative spectrum, discuss its relation to the naive construction, and phrase it in topos-theoretic terms; and we deduce, as an application, a description of limits in the category of locally ringed spaces. We also cover the relative Proj construction.

In much of the following, it's not actually necessary that X is a scheme and $\mathcal{A}$ is a quasicoherent algebra. If X is not a scheme or $\mathcal{A}$ is not quasicoherent, then $\underline{\mathrm{Spec}}_X(\mathcal{A})$ might fail to be a scheme and can of course not be constructed by gluing usual spectra, but it still exists as a more general kind of space and still verifies a meaningful universal property. We give details on this generalization below.

12.1. Internal locales. Let X be a topological space (or a locale). A fundamental fact in the theory of locales is that there is a canonical equivalence between the category of *locales over* X – that is locales Y equipped with a morphism $Y \to X$ – and *internal locales in* $\mathrm{Sh}(X)$ [46, p. 49]. An internal locale in a topos $\mathcal{E}$ is given by an object L of $\mathcal{E}$ (the internal lattice of opens of the locale) together with a binary relation $(\preceq) \hookrightarrow L \times L$ such that the axioms on a locale hold from the internal point of view. (For our purposes, we do not need a precise wording of these axioms.)

The equivalence is described as follows: A locale $f : Y \to X$ over X induces an internal locale $I(Y)$ with object of opens given by $\mathcal{T}(I(Y)) := f_*\Omega_{\mathrm{Sh}(Y)} \in \mathrm{Sh}(X)$, where f_* is the pushforward functor and $\Omega_{\mathrm{Sh}(Y)}$ is the object of truth values in the topos of sheaves on Y. Conversely, an internal locale given by an internal frame $\mathcal{L} \in \mathrm{Sh}(X)$ induces an (external) locale $E(\mathcal{L})$ with lattice of opens given by $\mathcal{T}(E(\mathcal{L})) := \Gamma(X, \mathcal{L})$. This comes equipped with a canonical morphism $Y \to X$ of locales which we do not need to describe explicitly [44, Section C1.6].

As a special case, the internalization of the trivial locale $\mathrm{id} : X \to X$ over X has as lattice of opens the object $\mathrm{id}_*\Omega_{\mathrm{Sh}(X)} = \Omega_{\mathrm{Sh}(X)} = \mathcal{P}(1)$. This is precisely the lattice of opens of the one-point space. Thus $I(X) \cong \mathrm{pt}$. This illustrates the intuition behind working internally in $\mathrm{Sh}(X)$: From the perspective of $\mathrm{Sh}(X)$, the space X looks like the one-point space (even if in fact it is not).

One can associate to an internal locale T in a topos $\mathcal{E}$ a topos of internal sheaves on it: $\mathrm{Sh}_{\mathcal{E}}(T)$. The correspondence is made in such a way that the topos of sheaves on a locale Y over X is equivalent to the topos of sheaves on the internal locale $I(Y)$: $\mathrm{Sh}(Y) \simeq \mathrm{Sh}_{\mathrm{Sh}(X)}(I(Y))$.

There is no similarly nice correspondence between topological spaces over X and internal topological spaces in $\mathrm{Sh}(X)$ [44, Corollary C1.6.7]. This is one of the reasons why locales are better suited for working internally and for switching between internal and external perspectives.

For verification of properties of such sheaves, the *idempotency* of the internal language is useful: If φ is a formula over Y, then

$$\mathrm{Sh}(Y) \models \varphi \qquad \text{if and only if} \qquad \mathrm{Sh}(X) \models \ulcorner \mathrm{Sh}(I(Y)) \models \varphi \urcorner.$$

Here we're abusing notation in two ways. Firstly, the formula φ has to be appropriately interpreted in the expression "$\mathrm{Sh}(I(Y)) \models \varphi$". Secondly, the expression "$\mathrm{Sh}(I(Y))$" doesn't actually refer to the category $\mathrm{Sh}_{\mathrm{Sh}(X)}(I(Y))$, but to the *locally internal* category induced by the canonical geometric morphism $\mathrm{Sh}_{\mathrm{Sh}(X)}(I(Y)) \to \mathrm{Sh}(X)$. We give some details on this point in Section 16. However, in the situations encountered in this section, the meaning will always be reasonably clear.

12.2. The spectrum of a ring as a locale. Recall that the spectrum of a ring A is usually constructed as the set

$$\mathrm{Spec}\, A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

endowed with a certain topology and a sheaf of rings $\mathcal{O}_{\mathrm{Spec}\, A}$. From an intuitionistic (and thus internal) point of view, this construction does not work well: Prime ideals are intuitionistically much more elusive than classically, where one can appeal to Zorn's lemma to obtain maximal (and thus prime) ideals. More to the point, one cannot show that this construction of the spectrum as a topological space verifies the expected universal property, namely

$$\mathrm{Hom}_{\mathrm{LRS}}(X, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X))$$

for all locally ringed spaces X (or some variant of this property involving more general kinds of spaces).

On the other hand, the lattice of opens of $\operatorname{Spec} A$ admits a simple description not requiring the notion of prime ideals:

$$\mathcal{T}(\operatorname{Spec} A) \cong \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \text{ is a radical ideal}\}.$$

An open subset $U \subseteq \operatorname{Spec} A$ corresponds to the radical ideal $\{h \in A \mid D(h) \subseteq U\}$ (so in particular, the open subset $D(f)$ corresponds to the radical ideal $\sqrt{(f)}$); conversely, a radical ideal $\mathfrak{a}$ corresponds to the open subset $\bigcup_{h \in \mathfrak{a}} D(h)$.

Thus, in an intuitionistic context, we will construct the spectrum of a ring A as a locale, not as a topological space, and adopt the following definition.

Definition 12.1. The *spectrum* $\operatorname{Spec}(A)$ of a ring A is the locale whose lattice of opens is the lattice of radical ideals of A. We endow it with the structure sheaf $\mathcal{O}_{\operatorname{Spec}(A)} := \underline{A}[\mathcal{F}^{-1}]$, where $\mathcal{F}$ is the generic filter as described in Section 11.2.

This construction has the expected universal property, namely that it is adjoint to the global functions functor:

$$\mathrm{Hom}_{\mathrm{LRL}}(X, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Here, "LRL" refers to the category of *locally ringed locales*, i. e. locales X equipped with a sheaf of rings $\mathcal{O}_X$ such that from the internal point of view of $\mathrm{Sh}(X)$, the ring $\mathcal{O}_X$ is a local ring. A morphism $Y \to X$ of locally ringed locales consists of a locale morphism $f : Y \to X$ and a morphism $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ of sheaves of rings on Y such that, from the internal point of view of $\mathrm{Sh}(Y)$, the ring homomorphism $f^\sharp$ is a local homomorphism. The notion of a locally ringed locale is thus a straightforward generalization of that of a locally ringed space.

Schemes are usually regarded as locally ringed spaces, not as locally ringed locales. However, in a classical context where the axiom of choice is available, schemes are *sober* topological spaces [69, Tag 01IS]. For sober topological spaces, the passage from the space to its induced locale (forgetting the set of points and only keeping the frame of open subsets) doesn't lose information: The category of sober topological spaces with arbitrary continuous maps embeds into the category of locales as a full subcategory. Therefore the category of schemes can just as well be viewed as a full subcategory of the category of locally ringed locales.

The importance of a locale-theoretic approach to spectra of rings, especially in relative situations, has also been stressed by Lurie [55, p. 37].

Points of the locale-theoretic spectrum. Constructing the spectrum as a locale instead of a topological space sidesteps any issues with prime ideals, since points are not a defining ingredient of a locale. However, points are still meaningful as a *derived concept*: A point of locale X is a morphism $1 \to X$, where 1 is the terminal locale, the locale corresponding to the one-point topological space with lattice of opens $\mathcal{P}(1) = \Omega$. Therefore it's still an interesting question what the points of the locale $\operatorname{Spec}(A)$ look like.

Proposition 12.2. *Let A be a ring. Then the points of the locale* $\operatorname{Spec}(A)$ *are in canonical one-to-one correspondence with the filters of A (as in Definition 11.2), even intuitionistically.*

Proof. The points of a locale X are in canonical one-to-one correspondence with the *completely prime filters* of $\mathcal{T}(X)$, subsets $K \subseteq \mathcal{T}(X)$ which are upward-closed, downward-directed, and have the property that, whenever a supremum of a set $M \subseteq \mathcal{T}(X)$ is contained in K, then so is some element of M.

Such a completely prime filter $K \subseteq \mathcal{T}(\operatorname{Spec}(A))$ corresponds to the ring-theoretic filter

$$F := \{s : A \mid \sqrt{(s)} \in K\} \subseteq A,$$

and a ring-theoretic filter $F \subseteq A$ corresponds to the completely prime filter

$$K := \{\mathfrak{a} : \mathcal{T}(\operatorname{Spec}(A)) \mid \mathfrak{a} \cap F \text{ is inhabited}\}.$$

We omit the required routine verifications. □

In classical logic, where complementation yields a one-to-one correspondence between filters and prime ideals, the points of $\operatorname{Spec}(A)$ are therefore in canonical bijection with the prime ideals of A, just as one would expect.

Observing that intuitionistically the points of the locale $\operatorname{Spec}(A)$ are filters, not prime ideals, one might wonder: Is the locale-theoretic approach really necessary? Wouldn't it suffice to define $\operatorname{Spec}(A)$ as the topological space of filters of A? Indeed, for some time this was believed [51, Section 3]; however, this hope turned out to be too naive: Joyal gave an explicit example of a nontrivial ring in a certain topos without any filters [73, pp. 200f.], thus showing that the construction can't have the expected universal property and that therefore a true pointfree approach as provided by lattice theory/locale theory [30], topos theory, or formal topology [66] is necessary to construct the spectrum in an intuitionistic context.[17]

The spectrum as a classifying locale. The fact that the points of $\operatorname{Spec}(A)$ are in canonical one-to-one correspondence with the filters of A is a shadow of a more general fact. Namely, for any locale X (and in fact any topos), maps $X \to \operatorname{Spec}(A)$ are in canonical one-to-one correspondence with the internal filters of A in $\operatorname{Sh}(X)$, that is subsheaves of the constant sheaf $\underline{A}$ satisfying the filter axioms from the point of view of the internal language of $\operatorname{Sh}(X)$: The locale $\operatorname{Spec}(A)$ is the *classifying locale of the theory of filters of A*.

The fact about the points of $\operatorname{Spec}(A)$ can be recovered from this observation as follows. A point of $\operatorname{Spec}(A)$ is a map $1 \to \operatorname{Spec}(A)$ and therefore corresponds to a subsheaf of the constant sheaf $\underline{A}$ satisfying the filter axioms from the point of view of $\operatorname{Sh}(1)$. Since $\operatorname{Sh}(1) \simeq \mathrm{Set}$, such a subsheaf amounts to a subset of A satisfying the filter axioms.

The notion of classifying locales provides a pleasant way of approaching the problem of constructing a space of models of a *propositional geometric theory* (in the case of the spectrum the theory of filters), simultaneously streamlining the usual topological approach and generalizing it to work in an intuitionistic context: Instead of first constructing the *set* of models (filters of A) and then manually endowing this set with a suitable topology (the Zariski topology), one can simply consider the *locale* of models, that is the classifying locale of the theory. Its sets of points coincides with the set of models of the topological approach, but the locale is not determined by its sets of points, facilitating a better behavior in contexts where the points might be elusive.

Put more concisely, the topological space of filters doesn't work well in an intuitionistic context, but the locale of filters does.

A lucid expository account of the theory of classifying locales can be found in a survey article by Vickers [79].

[17]When following reference [73], note that Tierney calls "primes" what we call "filters". Joyal's example was none other than the ring $\underline{\mathbb{A}}^1_S$ in $\operatorname{Zar}(S)$ in the special case $S = \operatorname{Spec}(\mathbb{Z})$.

Remark 12.3. For comparison with a refined geometric theory discussed below, we describe the geometric theory of filters of A here explicitly. It has one atomic proposition "$s \in F$" for each element $s : A$, and its axioms are given by the following axiom schemes:

(1) $\top \vdash 1 \in F$
(2) $st \in F \dashv\vdash s \in F \wedge t \in F$ (two axioms for each $s, t : A$)
(3) $0 \in F \vdash \bot$
(4) $s + t \in F \vdash s \in F \vee t \in F$ (one axiom for each $s, t : A$)

A trivial case. For later use, we study the question when the spectrum is the one-point space. The answer is well-known classically, but since we want to use this result in an internal context, we have to give an intuitionistic proof.

Lemma 12.4. *Let A be a ring. Its spectrum is a one-point space (as a locale) if and only if $1 \neq 0$ in A any element of A is nilpotent or invertible.*

Proof. The locale $\operatorname{Spec} A$ is a one-point space if and only if the unique continuous map $\operatorname{Spec}(A) \to \mathrm{pt}$ of locales is an isomorphism. This is the case if and only if the canonical frame homomorphism

$$\begin{array}{rcl} \Omega = \mathcal{P}(1) & \longrightarrow & \mathcal{T}(\operatorname{Spec} A) \\ \varphi & \longmapsto & \mathfrak{a}_\varphi := \sup\{\sqrt{(1)} \mid \varphi\} = \{x : A \mid \ulcorner x \text{ nilpotent} \urcorner \vee \varphi\} \end{array}$$

is surjective and reflects the ordering (and is therefore automatically injective). If $1 = 0$ in A, this homomorphism is not injective, since $\bot$ and $\top$ get both mapped to $\sqrt{(0)}$. For the rest of the proof, we'll therefore assume that $1 \neq 0$ in A.

Under this assumption, the homomorphism reflects the ordering: If $\mathfrak{a}_\varphi \subseteq \mathfrak{a}_\psi$, then $(1 \in \mathfrak{a}_\varphi) \Rightarrow (1 \in \mathfrak{a}_\psi)$. Since the unit of A is not nilpotent, this amounts to $\varphi \Rightarrow \psi$.

The homomorphism is surjective if and only if for any radical ideal $\mathfrak{a} \subseteq A$, it holds that $\mathfrak{a} = \{x : A \mid \ulcorner x \text{ nilpotent} \urcorner \vee \varphi\}$ for some proposition φ. By considering the condition "$1 \in \mathfrak{a}$", it follows that this proposition φ must be equivalent to the proposition "$1 \in \mathfrak{a}$" (if it is at all possible to write $\mathfrak{a}$ in such a way).

So the map is surjective if and only if for any radical ideal $\mathfrak{a} \subseteq A$ and any element x of A it holds that

$$x \in \mathfrak{a} \quad \Longleftrightarrow \quad \ulcorner x \text{ nilpotent} \urcorner \vee (1 \in \mathfrak{a}).$$

The "if" direction always holds. If any element of A is nilpotent or invertible, the "only if" direction holds as well (for any $\mathfrak{a}$ and any x). Conversely, if the "only if" direction holds, then any element of A is nilpotent or invertible. This follows by considering the radical ideal $\sqrt{(f)}$ for an element $f : A$. ☐

Remark 12.5. The structure sheaf $\mathcal{O}_X$ of a scheme fulfills almost, but not quite, the condition given in Lemma 12.4: By Proposition 3.7, it has the property that any element which is not invertible is nilpotent. In classical logic, this statement is equivalent to the statement that every element is nilpotent or invertible. However, intuitionistically the former is a weaker statement than the latter. This observation entails that the internally constructed spectrum does *not* coincide with the relative spectrum, and that instead a refined approach is necessary. Section 12.4 is devoted to studying this difference.

12.3. Digression: Further topologies on the set of prime ideals. The Zariski topology is not the only interesting topology on the set of prime ideals. For instance, the constructible topology and the flat topology studied by Tarizadeh [72] too have their uses. While the contents of Section 12.2 are well-known, the locale-theoretic approach to these variants of the spectrum and their universal properties appear to not have been studied much.

The universal properties given in the following two propositions should be compared with the following way of phrasing the universal property of the ordinary locale-theoretic spectrum. The usual phrasing employs the categories RL and LRL of (locally) ringed locales, therefore emphasizing the spatial character. But the dual categories $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$ can be used just as well; since the morphisms in $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$ go in the direction of the ring-theoretic parts, they can be thought of as the category of *all* rings respectively *all* local rings, where "all" refers to the fact that these categories don't only include the (local) rings in Set, but the (local) rings in arbitrary localic sheaf toposes.

Formulated using $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$, and adopting the notation to suppress mention of the involved spaces (instead of the involved sheaves of rings), the universal property of $\mathrm{Spec}(A)$ reads as follows: For any local ring $\mathcal{O}_Y$ over any locale Y,

$$\mathrm{Hom}_{\mathrm{LRL}^{\mathrm{op}}}(\mathcal{O}_{\mathrm{Spec}(A)}, \mathcal{O}_Y) \cong \mathrm{Hom}_{\mathrm{RL}^{\mathrm{op}}}(A, \mathcal{O}_Y).$$

The morphism $A \to \mathcal{O}_{\mathrm{Spec}(A)}$ in $\mathrm{RL}^{\mathrm{op}}$ is therefore the *universal localization* of A.

Proposition 12.6. *Let A be a ring. The locale given by the space of prime ideals of A with the flat topology is the classifying locale of prime ideals of A. Equipped with $\underline{A}/\mathcal{P}$ as structure sheaf, where $\mathcal{P}$ is the generic prime ideal, it is the universal way of mapping A to an integral domain in the weak sense (as defined in Section 3.5).*

Proposition 12.7. *Let A be a ring. The locale given by the space of prime ideals of A with the constructible topology is the classifying locale of detachable prime ideals (or equivalently detachable filters) of A. Equipped with $\underline{A}/\mathcal{P}$ as structure sheaf, where $\mathcal{P}$ is the generic prime ideal, it is the universal way of mapping A to an integral domain in the strong sense. Equipped with $\underline{A}[\mathcal{F}^{-1}]$, where $\mathcal{F}$ is the generic filter, is is the universal way of mapping A to a local ring in which invertibility is decidable.*

In constructive mathematics, a subset $U \subseteq A$ is *detachable* if and only if for every element $a : A$, either $a \in U$ or $a \notin U$. While intuitionistically the complement of a filter might fail to be a prime ideal and the complement of a prime ideal might fail to be a filter, the complement of a detachable filter is a detachable prime ideal, and vice versa.

XXX: write down proof

12.4. The relative spectrum as an ordinary spectrum from the internal point of view. Let X be a scheme and $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Since $\mathcal{A}$ looks like a plain algebra from the internal perspective of $\mathrm{Sh}(X)$, we can consider its internally defined spectrum. This is a locale internal to $\mathrm{Sh}(X)$; we might hope that its externalization is precisely the relative spectrum of $\mathcal{A}$ (considered as a locale):

$$E(\mathrm{Spec}\,\mathcal{A}) \overset{?}{\cong} \underline{\mathrm{Spec}}_X\,\mathcal{A}.$$

However, this turns out to be too naive. The locale $E(\mathrm{Spec}(\mathcal{A}))$ is equipped with a map to X, being an externalization of a locale internal to $\mathrm{Sh}(X)$, and it is equipped with a sheaf of rings (because we can transport the internally defined structure sheaf along the equivalence $\mathrm{Sh}_{\mathrm{Sh}(X)}(\mathrm{Spec}(\mathcal{A})) \simeq \mathrm{Sh}(E(\mathrm{Spec}(A)))$. Furthermore, this sheaf of rings is local, since we know

$$\mathrm{Sh}(X) \models \ulcorner \mathrm{Sh}(\mathrm{Spec}(\mathcal{A})) \models \ulcorner \mathcal{O}_{\mathrm{Spec}(\mathcal{A})} \text{ is a local ring} \urcorner \urcorner$$

which by idempotency of the internal language is equivalent to

$$\mathrm{Sh}(E(\mathrm{Spec}(\mathcal{A}))) \models \ulcorner \mathcal{O}_{\mathrm{Spec}(\mathcal{A})} \text{ is a local ring} \urcorner.$$

However, the map $E(\mathrm{Spec}(\mathcal{A})) \to X$ is only part of a morphism of ringed locales, not of locally ringed locales (even though domain and codomain happen to be locally ringed): Internally, the morphism $(\operatorname{Spec}(\mathcal{A}), \mathcal{O}_{\operatorname{Spec}(\mathcal{A})}) \to (\mathrm{pt}, \mathcal{O}_X)$ of ringed locales, which is defined using the $\mathcal{O}_X$-algebra structure of $\mathcal{A}$, is not a morphism of locally ringed locales (even though domain and codomain happen to be locally ringed).

In contrast, the true relative spectrum $\underline{\operatorname{Spec}}_X(\mathcal{A})$ is equipped with a morphism of locally ringed locales to X.

It's illuminating to compare the different universal properties of $E(\operatorname{Spec}(\mathcal{A}))$ and $\underline{\operatorname{Spec}}_X(\mathcal{A})$. There is a canonical morphism $E(\operatorname{Spec}(\mathcal{A})) \to E(\operatorname{Spec}(\mathcal{O}_X))$ of locally ringed locales (the externalization of the canonical morphism $\operatorname{Spec}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{O}_X)$ given by the $\mathcal{O}_X$-algebra structure of $\mathcal{A}$), but in general, the locales $E(\operatorname{Spec}(\mathcal{O}_X))$ and X are not isomorphic.

As we justify below, the externalization of the internally defined spectrum has the universal property

$$\operatorname{Hom}_{\mathrm{LRL}/E(\operatorname{Spec}\mathcal{O}_X)}(Y, E(\operatorname{Spec}\mathcal{A})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mu_*\mathcal{O}_Y)$$

for all locally ringed locales Y over $E(\operatorname{Spec}\mathcal{O}_X)$. Here, μ is the structure morphism $Y \to \operatorname{Spec}\mathcal{O}_X$, $E(\operatorname{Spec}\mathcal{O}_X)$ is the locally ringed locale associated to the internally defined spectrum of $\mathcal{O}_X$, and $\mathrm{LRL}_{\mathrm{Sh}(X)}$ is the category of locally ringed locales internal to $\mathrm{Sh}(X)$. In contrast, the relative spectrum has the different universal property

$$\operatorname{Hom}_{\mathrm{LRL}/X}(Y, \underline{\operatorname{Spec}}_X \mathcal{A}) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mu_*\mathcal{O}_Y)$$

for all locally ringed locales Y over X.[18] The crucial difference is that in general, the internally defined locally ringed locale $\operatorname{Spec}\mathcal{O}_X$ does *not* coincide with the internal locally ringed locale $(\mathrm{pt}, \mathcal{O}_X)$ (which is simply $(X, \mathcal{O}_X)$ from the external point of view). More succinctly, the functor $E \circ \operatorname{Spec}$ is an adjoint to the pushforward-of-sheaf-of-functions functor $\mathrm{LRL}/E(\operatorname{Spec}\mathcal{O}_X) \to \mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}}$, while the relative spectrum functor is an adjoint to the analogous functor $\mathrm{LRL}/X \to \mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}}$.

The universal property of $E(\operatorname{Spec}(\mathcal{A}))$ can be determined as follows. From the internal point of view of $\mathrm{Sh}(X)$, the locally ringed locale $E(\operatorname{Spec}(\mathcal{A}))$ looks like the ordinary locale-theoretic spectrum $\operatorname{Spec}(\mathcal{A})$ and therefore has the universal property

$$\operatorname{Hom}_{\mathrm{LRL}}(Y, \operatorname{Spec}(\mathcal{A})) \cong \operatorname{Hom}_{\mathrm{Ring}}(\mathcal{A}, \Gamma(Y, \mathcal{O}_Y))$$

for any locally ringed locale Y.[19] If we restrict the right-hand side to the set of $\mathcal{O}_X$-algebra homomorphisms, the left-hand side restricts to the set of morphisms $Y \to \operatorname{Spec}(\mathcal{A})$ of locally ringed locales over the locally ringed locale $\operatorname{Spec}(\mathcal{O}_X)$. So we have

$$\operatorname{Hom}_{\mathrm{LRL}/\operatorname{Spec}(\mathcal{O}_X)}(Y, \operatorname{Spec}(\mathcal{A})) \cong \operatorname{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \Gamma(Y, \mathcal{O}_Y)).$$

This discussion took place in the internal universe of $\mathrm{Sh}(X)$. Externally, the displayed universal property implies that for any locally ringed locale $\mu : Y \to X$ over $E(\operatorname{Spec}(\mathcal{O}_X))$,

$$\operatorname{Hom}_{\mathrm{LRL}/E(\operatorname{Spec}(\mathcal{O}_X))}(Y, E(\operatorname{Spec}(\mathcal{A}))) \cong \operatorname{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_Y),$$

as claimed above.

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[18] If X is a scheme and $\mathcal{A}$ is quasicoherent, this universal property is well-known, even though it's usually only stated for schemes Y over X instead of general locally ringed locales over X. In any case, we take this universal property as the definition of what the relative spectrum should be.

[19] Externally, this implies that for any locally ringed locale over the underlying locale of X (that is, for any locale Y equipped with a morphism $\mu : Y \to X$ and a local sheaf of rings), we have

$$\operatorname{Hom}_{\mathrm{LR}(\mathrm{L}/X)}(Y, E(\operatorname{Spec}(\mathcal{A}))) \cong \operatorname{Hom}_{\mathrm{Ring}_{\mathrm{Sh}(X)}}(\mathcal{A}, \mu_*\mathcal{O}_Y).$$

Definition 12.8. Let R be a ring. Let A be an R-algebra. The *local spectrum* of A over R is the locale $\mathrm{Spec}(A|R)$ with lattice of opens given by

$$\begin{aligned}\mathcal{T}(\mathrm{Spec}(A|R)) := \{\mathfrak{a} \subseteq A \,|\, &\mathfrak{a} \text{ is a radical ideal such that} \\ &\forall f : R.\ \forall s : A.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathfrak{a}) \Rightarrow fs \in \mathfrak{a}\}.\end{aligned}$$

We'll equip the local spectrum with the structure of a locally ringed locale below. It is this refined construction which correctly internalizes the relative spectrum:

Theorem 12.9. *Let X be a scheme (or a locally ringed locale). Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. Then the externalization $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))$ coincides with $\underline{\mathrm{Spec}}_X(\mathcal{A})$ as locally ringed locales over X.*

Before giving the proof, we want to clarify some details of the construction.

Firstly, the base ring R directly enters the construction. This is in contrast to the usual spectrum: If A is an R-algebra, the construction of $\mathrm{Spec}(A)$ does not depend on the R-algebra structure of A. The algebra structure only enters in the construction of a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$.

Secondly, in the case that X is a scheme and $\mathcal{A}$ is a quasicoherent $\mathcal{O}_X$-algebra, we can compare the externalization of $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$ with the result of the construction of $\underline{\mathrm{Spec}}_X(\mathcal{A})$ by gluing spectra:

Proposition 12.10. *Let X be a scheme. Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Then $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))$ coincides with $\underline{\mathrm{Spec}}_X(\mathcal{A})$ as locales over X.*

Proof. The condition

$$\forall f : \mathcal{O}_X.\ \forall s : \mathcal{A}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathfrak{a}) \Longrightarrow fs \in \mathfrak{a}$$

appearing in Definition 12.8 is precisely the internal quasicoherence condition of Corollary 9.5 (slightly simplified in view that $\mathfrak{a}$ is a radical ideal). The sections of the sheaf $[\![\mathcal{T}(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))]\!]$ on an open subset $U \subseteq X$ are therefore precisely the quasicoherent sheaves of radical ideals $\mathfrak{a} \hookrightarrow \mathcal{A}|_U$. Let $\pi : \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ be the canonical morphism. If U is affine, then

$$\pi^{-1}U \cong \underline{\mathrm{Spec}}_X(\mathcal{A}) \times_X U \cong \underline{\mathrm{Spec}}_U(\mathcal{A}|_U) \cong \mathrm{Spec}(\Gamma(U, \mathcal{A}))$$

is affine as well and

$$\begin{aligned}\Gamma(U, \mathcal{T}(I(\underline{\mathrm{Spec}}_X(\mathcal{A})))) &= \Gamma(U, \pi_*\Omega_{\underline{\mathrm{Spec}}_X(\mathcal{A})}) = \Omega_{\underline{\mathrm{Spec}}_X(\mathcal{A})}(\pi^{-1}U) \\ &\cong \text{set of open subsets of } \pi^{-1}U \\ &\cong \text{set of open subsets of } \mathrm{Spec}(\Gamma(U, \mathcal{A})) \\ &\cong \text{set of radical ideals of } \Gamma(U, \mathcal{A}) \\ &\cong \text{set of quasicoherent sheaves of radical ideals of } \mathcal{A}|_U \\ &\cong \Gamma(U, [\![\mathcal{T}(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))]\!]).\end{aligned}$$

Therefore $I(\underline{\mathrm{Spec}}_X(\mathcal{A}))$ is canonically isomorphic to $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$ as locales internal to $\mathrm{Sh}(X)$. Expressed externally: The relative spectrum $\underline{\mathrm{Spec}}_X(\mathcal{A})$ coincides with the externalization of $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$ as locales over X, as claimed. ☐

Thirdly, the partial order $\mathcal{T}(\mathrm{Spec}(A|R))$ is indeed a frame. A quick way to verify this is to recognize that it is related to the frame of opens of $\mathrm{Spec}(A)$ by the formula

$$\mathcal{T}(\mathrm{Spec}(A|R)) = \{\mathfrak{a} : \mathcal{T}(\mathrm{Spec}(A)) \,|\, \mathfrak{a} = \overline{\mathfrak{a}}\},$$

where $(\mathfrak{a} \mapsto \overline{\mathfrak{a}})$ is the quasicoherator described in Remark 9.14. Since the quasicoherator satisfies the axioms on a nucleus, this formula exhibits $\mathrm{Spec}(A|R)$ as a sublocale of $\mathrm{Spec}(A)$. In particular, suprema are calculated in $\mathcal{T}(\mathrm{Spec}(A|R))$ by applying the quasicoherator to the suprema calculated in $\mathcal{T}(\mathrm{Spec}(A))$. We denote the inclusion $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$ by "i".

Lastly, it's interesting to know the points of $\mathrm{Spec}(A|R)$, even though these don't determine $\mathrm{Spec}(A)$.

Definition 12.11. Let R be a ring. Let $\varphi : R \to A$ be an algebra. A filter $F \subseteq A$ *lies over the filter of units* if and only if $\varphi^{-1}F \subseteq R^\times$, that is if

$$\varphi(r) \in F \Longrightarrow r \text{ is invertible in } R$$

for all $r : R$. (The reverse inclusion "$\varphi^{-1}F \supseteq R^\times$" holds automatically.)

This definition will mostly be used in situations where the ring R is local, in which case the subset $R^\times$ is actually a filter and the phrase "filter of units" is therefore justified.

It's illuminating to consider Definition 12.11 in a classical context, even though the use case we have in mind is to apply it in the internal language of the little Zariski topos of a base scheme. Classically, a filter F lies over the filter of units if and only if $\varphi^{-1}\mathfrak{p} \supseteq R \setminus R^\times$, where $\mathfrak{p} := F^c = A \setminus F$ is the prime ideal associated to F. If R is local, the set $R \setminus R^\times$ is the unique maximal ideal $\mathfrak{m}$ of R. Thus F lies over the filter of units if and only if $\mathfrak{p}$ lies over the maximal ideal.

Proposition 12.12. *Let R be a ring. Let $\varphi : R \to A$ be an R-algebra. Then the points of* $\mathrm{Spec}(A|R)$ *are intuitionistically in canonical one-to-one correspondence with those filters of A which lie over the filter of units.*

Proof. The correspondence outlined in Proposition 12.2 can be adapted to the situation at hand. A completely prime filter $K \subseteq \mathcal{T}(\mathrm{Spec}(A|R))$ corresponds to the ring-theoretic filter

$$F := \{s : A \mid \overline{\sqrt{(s)}} \in K\}$$

and a ring-theoretic filter F corresponds to the completely prime filter

$$K := \{\mathfrak{a} : \mathcal{T}(\mathrm{Spec}(A|R)) \mid \mathfrak{a} \cap F \text{ is inhabited}\}.$$

It's instructive to perform some of the necessary verifications, to see how the quasicoherator is used, even though Proposition 12.13 will subsume this correspondence.

The filter F corresponding to K has the displayed property for the following reason. Let $\varphi(r) \in F$. We want to verify that r is invertible in R. Under the assumption that r is invertible in R, it's trivial that 1 is an element of

$$\begin{aligned} \mathfrak{a} &:= \sup\{\sqrt{(1)} \mid r \text{ is invertible in } R\} \\ &= \{s : A \mid s \text{ is nilpotent or } r \text{ is invertible in } R\} \in \mathcal{T}(\mathrm{Spec}(A)). \end{aligned}$$

Therefore, without any assumption on r, we have that $r \cdot 1 = \varphi(r)$ is an element of $\overline{\mathfrak{a}}$ and therefore $\overline{\sqrt{(\varphi(r))}} \subseteq \overline{\mathfrak{a}}$. Since K is upward-closed, it follows that $\overline{\mathfrak{a}} \in K$. Since $\overline{\mathfrak{a}}$ is the supremum of the set $\{\sqrt{(1)} \mid r \text{ is invertible}\}$ in $\mathcal{T}(\mathrm{Spec}(A|R))$ and K is completely prime, it follows that this set is inhabited. Thus r is invertible in R.

The set K corresponding to a ring-theoretic filter F is completely prime for the following reason. Let $\sup_i \mathfrak{a}_i = \overline{\sqrt{\sum_i \mathfrak{a}_i}} \in K$. Then $\overline{\sqrt{\sum_i \mathfrak{a}_i}} \cap F$ is inhabited. By the special assumption on F, the intersection $\sqrt{\sum_i \mathfrak{a}_i} \cap F$ is inhabited as well: In the case that X is a scheme, this follows easily using the description of the quasicoherator given in Proposition 9.11. In the general case, we use the proof scheme outlined in Remark 9.14 – using the notation of that remark, if $P(\mathfrak{b}) \cap F$ is inhabited, then $\mathfrak{b} \cap F$ is as well.

A short calculation using the filter axioms then shows that there exists an index i such that $\mathfrak{a}_i \cap F$ is inhabited. □

Proposition 12.13. *Let R be a ring. Let $\varphi : R \to A$ be an algebra. Then* $\operatorname{Spec}(A|R)$ *is the classifying locale of the theory of filters of A which lie over the filter of units, that is of the geometric theory with atomic propositions "$s \in F$" for $s : A$ and axioms given by the following axiom schemes:*

(1) $\top \vdash 1 \in F$
(2) $st \in F \dashv\vdash s \in F \wedge t \in F$ *(two axioms for each $s, t : A$)*
(3) $0 \in F \vdash \bot$
(4) $s + t \in F \vdash s \in F \vee t \in F$ *(one axiom for each $s, t : A$)*
(5) $\varphi(r) \in F \vdash \bigvee\{\top \mid r \text{ is invertible in } R\}$ *(one axiom for each $r : R$)*

Proof. The frame of the classifying locale of the given theory T is the free frame on generators "$s \in F$" for $s : A$ subject to the relations given by the axioms of the theory. More explicitly, it's the Lindenbaum algebra $L(T)$ of the theory, so its elements are the formulas of the theory up to provable equivalence and the ordering is defined by $[\varphi] \preceq [\psi] :\Leftrightarrow (\varphi \vdash \psi)$. We want to verify that this frame is isomorphic to $\mathcal{T}(\operatorname{Spec}(A|R))$.

We define a frame homomorphism $L(T) \to \mathcal{T}(\operatorname{Spec}(A|R))$ by sending the generators $[s \in F]$ to the radical ideal $\overline{\sqrt{(s)}}$. This respects the relations and therefore gives a well-defined map. The map is surjective, since a preimage to $\mathfrak{a} : \mathcal{T}(\operatorname{Spec}(A|R))$ is $[\bigvee_{s \in \mathfrak{a}}(s \in F)]$. To verify that it is an isomorphism of frames, we therefore only have to verify that it reflects the ordering.

By the axiom schemes (1) and (2), any formula of T is provably equivalent to a formula of the form $\bigvee_i(s_i \in F)$. It therefore suffices to verify that, for any families $(s_i)_i$ and $(t_j)_j$ such that $\overline{\sqrt{(s_i)_i}} \subseteq \overline{\sqrt{(t_j)_j}}$, the sequent $\bigvee_i(s_i \in F) \vdash \bigvee_j(t_j \in F)$ is derivable. We'll show more generally: If $\mathfrak{a}$ and $\mathfrak{b}$ are radical ideals such that $\overline{\mathfrak{a}} \subseteq \overline{\mathfrak{b}}$, then $\bigvee_{s \in \mathfrak{a}}(s \in F) \vdash \bigvee_{t \in \mathfrak{b}}(t \in F)$. This follows from the following chain of deductions:

$$\bigvee_{s \in \mathfrak{a}}(s \in F) \vdash \bigvee_{s \in \overline{\mathfrak{a}}}(s \in F) \vdash \bigvee_{s \in \overline{\mathfrak{b}}}(s \in F) \vdash \bigvee_{s \in \mathfrak{b}}(s \in F).$$

All but the final step are trivial. The final step is an application of the general proof scheme outlined in Remark 9.14. In the notation of that remark, we set $\alpha(\mathcal{J}) := [\bigvee_{s \in \mathcal{J}}(s \in F)]$ and exploit that, if $s \in P(\mathcal{J})$, then $s \in F \vdash \bigvee_{t \in \mathcal{J}}(t \in F)$. This is because s can be written as $s^n = \sum_j a_j f_j u_j$ such that, for each j, if f_j is invertible in R then $u_j \in \mathcal{J}$, and we have the following chain of deductions.

$$\begin{aligned} s \in F &\vdash s^n \in F \\ &\vdash \bigvee_j (t_j f_j u_j \in F) \\ &\vdash \bigvee_j (\varphi(f_j) \in F \wedge u_j \in F) \\ &\vdash \bigvee_j \Big(\bigvee\{\top \mid f_j \text{ invertible in } R\} \wedge u_j \in F\Big) \\ &\vdash \bigvee_j \bigvee\{(u_j \in F) \mid f_j \text{ invertible in } R\} \\ &\vdash \bigvee_{t \in \mathcal{J}} (t \in F). \end{aligned}$$

□

Lemma 12.14. *Let R be a local ring. Let $\varphi : R \to A$ be an R-algebra. Then, intuitionistically, the locale* $\operatorname{Spec}(A|R)$ *carries a canonical structure as a locally ringed locale over* (pt, R) *and has the following universal property: For any locally* *ringed locale* $(Y, \mathcal{O}_Y)$ *over* (pt, R)*,*

$$\mathrm{Hom}_{\mathrm{LRL}/(\mathrm{pt},R)}(Y, \mathrm{Spec}(A|R)) \cong \mathrm{Hom}_{\mathrm{Alg}(R)}(A, \Gamma(Y, \mathcal{O}_Y)).$$

Proof. Since $\mathrm{Spec}(A|R)$ is a sublocale of $\mathrm{Spec}(A)$, we can equip $\mathrm{Spec}(A|R)$ with the restriction of $\mathcal{O}_{\mathrm{Spec}(A)}$ to $\mathrm{Spec}(A|R)$ as the structure sheaf:

$$\mathcal{O}_{\mathrm{Spec}(A|R)} := i^{-1}\mathcal{O}_{\mathrm{Spec}(A)} = i^{-1}(\underline{A}[\mathcal{F}^{-1}]) \cong (i^{-1}\underline{A})[(i^{-1}\mathcal{F})^{-1}] \cong \underline{A}[(i^{-1}\mathcal{F})^{-1}].$$

The generic filter $\mathcal{F}$ was described in Section 11.2. The penultimate isomorphism is because localizing is a geometric construction. Since locality of a ring is a geometric implication, this structure sheaf is indeed a local sheaf of rings. Thus $\mathrm{Spec}(A|R)$ is a locally ringed locale.

Next, we have to describe a morphism $(\mathrm{Spec}(A|R), \mathcal{O}_{\mathrm{Spec}(A|R)}) \to (\mathrm{pt}, R)$. Locale-theoretically, this morphism is given by the unique map $! : \mathrm{Spec}(A|R) \to \mathrm{pt}$. The ring-theoretic part is given by the composition

$$!^{-1}R = \underline{R} \longrightarrow \underline{A} \longrightarrow \underline{A}[(i^{-1}\mathcal{F})^{-1}] = \mathcal{O}_{\mathrm{Spec}(A|R)}.$$

This homomorphism of rings which happen to be local is indeed a local homomorphism, that is, it reflects invertibility. More precisely,

$$\mathrm{Spec}(A|R) \models \forall f : \underline{R}.\ \ulcorner \underline{\varphi}(f) \text{ is inv. in } \mathcal{O}_{\mathrm{Spec}(A|R)} \urcorner \Rightarrow \ulcorner f \text{ is inv. in } \underline{R} \urcorner.$$

Denoting the modal operator associated to the sublocale inclusion $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$ by "$\Box$", this statement is equivalent to

$$\mathrm{Spec}(A) \models (\forall f : \underline{R}.\ \underline{\varphi}(f) \in \mathcal{F} \Rightarrow \ulcorner f \text{ is inv. in } \underline{R} \urcorner)^{\Box}$$

by Theorem 6.30 and Lemma 6.22. To verify this, let $s : A$ and $f : R$ be given such that $\sqrt{(s)} \models \varphi(f) \in \mathcal{F}$, that is, $s \in \sqrt{(\varphi(f))}$. We are to show that $\sqrt{(s)} \models \Box(\ulcorner f \text{ is invertible in } \underline{R} \urcorner)$.

The largest open in $\mathrm{Spec}(A)$ on which $\ulcorner f$ is invertible in $\underline{R}\urcorner$ holds is

$$\begin{aligned} \mathfrak{a} &:= \sup\{\sqrt{(1)} \mid f \text{ is invertible in } R\} \\ &= \{t : A \mid t \text{ is nilpotent or } f \text{ is invertible in } R\} \in \mathcal{T}(\mathrm{Spec}(A)), \end{aligned}$$

by Lemma 11.1. Under the assumption that f is invertible in R, trivially $1 \in \mathfrak{a}$. Therefore, without any assumptions on f, we have that $\varphi(f) \in \overline{\mathfrak{a}}$. Thus $\sqrt{(\varphi(f))} \subseteq \overline{\mathfrak{a}}$ and therefore $\sqrt{(\varphi(f))} \models \Box(\ulcorner f \text{ is invertible in } \underline{R}\urcorner)$. Since $\sqrt{(s)} \subseteq \sqrt{(\varphi(f))}$, the monotonicity of the internal language implies $\sqrt{(s)} \models \Box(\ulcorner f \text{ is invertible in } \underline{R}\urcorner)$.

Finally, we verify the universal property. Let Y be a locally ringed locale over (pt, R) and let a morphism $A \to \Gamma(Y, \mathcal{O}_Y)$ of R-algebras be given. We like this data to uniquely induce a morphism $Y \to \mathrm{Spec}(A|R)$ of locally ringed locales over (pt, R).

To obtain a locale-theoretic map $f : Y \to \mathrm{Spec}(A|R)$, by Proposition 12.13 we need to specify a filter of $\underline{A}$ in $\mathrm{Sh}(Y)$ which lies over the filter of units. The given morphism $A \to \Gamma(Y, \mathcal{O}_Y)$ induces a morphism $\alpha : \underline{A} \to \mathcal{O}_Y$ in $\mathrm{Sh}(Y)$. Since $\mathcal{O}_Y$ is a local ring, the subsheaf $\mathcal{O}_Y^\times$ is a filter. Its preimage $F := \alpha^{-1}\mathcal{O}_Y^\times$ is the sought filter of $\underline{A}$. It lies over the filter of units because the composition $\underline{R} \to \underline{A} \to \mathcal{O}_Y$ is local. By the general theory, the pullback of the generic filter in $\mathrm{Sh}(\mathrm{Spec}(A|R))$ to $\mathrm{Sh}(Y)$ along f is F.

The ring-theoretic part of the sought morphism $Y \to \mathrm{Spec}(A|R)$ of locally ringed locales over (pt, R) is the canonical homomorphism

$$f^{-1}\mathcal{O}_{\mathrm{Spec}(A|R)} = f^{-1}(\underline{A}[(i^{-1}\mathcal{F})^{-1}]) = \underline{A}[F^{-1}] \longrightarrow \mathcal{O}_Y$$

of local rings.

This finishes the description of the construction. We omit further verifications that the construction works as claimed. $\Box$

Remark 12.15. The modal operator $\Box$ associated to the inclusion $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$ can be defined in the internal language of $\mathrm{Sh}(\mathrm{Spec}(A))$. Namely, it's the smallest operator such that the $\Box$-translated statement

$$(\ulcorner\text{the morphism } \underline{R} \to \mathcal{O}_{\mathrm{Spec}(A)} \text{ is local}\urcorner)^{\Box}$$

holds. It is thus the smallest operator such that for any $f : \underline{R}$ with $\underline{\varphi}(f) \in \mathcal{F}$, $\Box(\ulcorner f \text{ is invertible in } \underline{R}\urcorner)$. The sublocale $\mathrm{Spec}(A|R)$ is therefore the largest sublocale of $\mathrm{Spec}(A)$ on which the morphism $\underline{R} \to \mathcal{O}_{\mathrm{Spec}(A)}$ is local.

Proof of Theorem 12.9. Follows immediately by interpreting the intuitionistic proof of Lemma 12.14 in the internal language of $\mathrm{Sh}(X)$, applied to $R := \mathcal{O}_X$ and $A := \mathcal{A}$. Then "$(\mathrm{pt}, \mathcal{O}_X)$" actually refers to the locally ringed locale $(X, \mathcal{O}_X)$ and "$\Gamma(Y, \mathcal{O}_Y)$" refers to $\mu_*\mathcal{O}_Y$, where $\mu : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a locally ringed locale over $(X, \mathcal{O}_X)$. $\Box$

Theorem 12.9 settles the question how the the little Zariski topos of $\underline{\mathrm{Spec}}_X(\mathcal{A})$ looks like from the internal point of view of $\mathrm{Sh}(X)$. A related question is how the big Zariski topos looks like. We give the answer in Theorem 16.4.

12.5. Comparing the different spectrum constructions. For rings and algebras, there are at least the following spectrum constructions.

- The ordinary spectrum of a ring, possibly realized as a locale instead of a topological space in order to work in an intuitionistic setting: $\mathrm{Ring}^{\mathrm{op}} \to \mathrm{LRS}$ or $\mathrm{Ring}^{\mathrm{op}} \to \mathrm{LRL}$
- The local spectrum of an algebra: $\mathrm{Alg}(R)^{\mathrm{op}} \to \mathrm{LRL}/(\mathrm{pt}, R)$
- Gillam's spectrum of a sheaf of algebras [37]: $\mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}} \to \mathrm{LRS}/(X, \mathcal{O}_X)$
- Hakim's spectrum of a ringed topos [38], yielding a locally ringed topos: $\mathrm{RT} \to \mathrm{LRT}$.
- Cole's general framework for spectrum constructions [25] (also reported on at [47, Theorem 6.58])

These are related as follows.

As described in Section 12.4, the ordinary spectrum construction can not only be applied to rings, but also to sheaves of rings and indeed ring objects internal to arbitrary elementary toposes equipped with a natural numbers object, by employing the internal language. Applied to a ring $\mathcal{O}$ internal to such a topos $\mathcal{E}$, it yields a locally ringed locale internal to $\mathcal{E}$, or equivalently a locally ringed localic topos internal to $\mathcal{E}$. Externally, this corresponds to a locally ringed topos which is equipped with a localic geometric morphism to $\mathcal{E}$.

The ordinary spectrum construction can therefore be used to turn a ringed topos $(\mathcal{E}, \mathcal{O})$ (with a natural numbers object) into a locally ringed topos (which will be equipped with a morphism of ringed toposes to $(\mathcal{E}, \mathcal{O})$, but which will, even if $\mathcal{O}$ happens to be a local ring, not be equipped with a morphism of locally ringed toposes to $(\mathcal{E}, \mathcal{O})$).

By comparing the universal properties one sees that this kind of internal application of the ordinary spectrum construction coincides with the result of Hakim's spectrum construction. In fact, it can be interpreted as a simultaneous simplification and generalization of Hakim's construction: It's simpler, since it's just the familiar spectrum construction and no explicit site calculations are required; and it's more general, since Hakim's construction only applies to ringed Grothendieck toposes whereas the internally-performed construction of the ordinary spectrum applies to ringed elementary toposes with natural numbers object.

Gillam's spectrum coincides with internally performing the construction of the local spectrum, with the caveat that Gillam's construction starts with and yields a locally ringed space, whereas ours starts with and yields a locally ringed locale.[20] More precisely:

For a locale Y, let Y_P be the topological space of points of Y, and for a topological space T, let T_L be the induced locale. Let $(X, \mathcal{O}_X)$ be a sober locally ringed topological space. Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. Then we have a morphism $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)) \to X_L$ of locally ringed locales. Since $X \cong (X_L)_P$, there is an induced morphism $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))_P \to X$ of locally ringed spaces. The adjunction $(\_\_)_L \dashv (\_\_)_P$ relating locales and topological spaces then yields, for any locally ringed space $\mu : Y \to X$ over X,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{LRS}/X}(Y, E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))_P) &\cong \mathrm{Hom}_{\mathrm{LRL}/X_L}(Y_L, E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))) \\ &\cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_Y).\end{aligned}$$

This is precisely the universal property which Gillam's spectrum enjoys.

Cole's framework for spectrum constructions is sufficiently general to encompass both the ordinary spectrum and the local spectrum, and by extension Hakim's spectrum and Gillam's spectrum. As is well-known, the ordinary spectrum can be obtained from Cole's framework by applying it to the geometric theory $\mathbb{S}$ of rings, its quotient theory $\mathbb{T}$ of local rings, and the admissible class $\mathbb{A}$ of local homomorphisms (notation as in [47, Theorem 6.58]). The local spectrum can be obtained by applying it to the geometric theory $\mathbb{S}$ of $\mathcal{O}_X$-algebras, its quotient theory $\mathbb{T}$ of local $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$, and the admissible class of local homomorphisms. For this to make sense, one has to interpret Cole's framework in the internal language of $\mathrm{Sh}(X)$, since there are no external geometric theories of (local) $\mathcal{O}_X$-algebras.

In general, the local spectrum doesn't coincide with the usual spectrum and Gillam's spectrum doesn't coincide with Hakim's spectrum. However, if the base space is a scheme of dimension ≤ 0, they do coincide.

Proposition 12.16. *Let X be a scheme. Then $E(\mathrm{Spec}(\mathcal{O}_X)) \cong X$ as locales over X if and only if $\dim X \leq 0$.*

Proof. The externalization of $\mathrm{Spec}\,\mathcal{O}_X$ coincides with X if and only if from the internal point of view, the locale $\mathrm{Spec}\,\mathcal{O}_X$ coincides with the one-point locale. By interpreting Lemma 12.4 in the internal language of $\mathrm{Sh}(X)$, it follows that this is the case if and only if

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \ulcorner f \text{ nilpotent} \urcorner \vee \ulcorner f \text{ invertible} \urcorner.$$

(Internally, it always holds that $\neg(1 = 0)$ in $\mathcal{O}_X$, even if X happens to be the empty scheme. Therefore the lemma is indeed applicable.) By Corollary 3.14, this condition is equivalent to the dimension of X being less than or equal to zero (i. e. to X being empty or having dimension exactly zero). □

Corollary 12.17. *Let X be a scheme. Then the relative spectrum of $\mathcal{O}_X$-algebras can be calculated by the internal spectrum (instead of the internal local spectrum) if and only if $\dim X \leq 0$.*

Proof. The externalization of the internal spectrum of arbitrary $\mathcal{O}_X$-algebras $\mathcal{A}$ coincides with the relative spectrum if and only if it coincides in the special case $\mathcal{A} = \mathcal{O}_X$. This is apparent by the universal properties of both constructions. Thus the claim follows from Proposition 12.16. □

Which construction is more fundamental, the ordinary spectrum of a ring or the local spectrum of an algebra? The ordinary spectrum $\mathrm{Spec}(A)$ can be

[20] More generally, the local spectrum construction can be applied to any algebra over a local ring $\mathcal{O}$ internal to an elementary topos $\mathcal{E}$ with a natural numbers objects and yields a locally ringed topos equipped with a morphism of locally ringed toposes to $(\mathcal{E}, \mathcal{O})$.

expressed as the local spectrum $\mathrm{Spec}(A^\sim|\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})})$, where $A^\sim$ is the induced quasicoherent algebra on $\mathrm{Sh}(\mathrm{Spec}(\mathbb{Z}))$. This fact is well-known in the alternate form "$\underline{\mathrm{Spec}}_{\mathrm{Spec}(\mathbb{Z})}(A^\sim) \cong \mathrm{Spec}(A)$".

Fast and loose reasoning as follows could lead one to believe that it's similarly possible to express the local spectrum as an ordinary spectrum. Let R be a local ring. Let $\varphi : R \to A$ be an algebra. The points of $\mathrm{Spec}(A|R)$ are those filters $F \subseteq A$ such that $\varphi^{-1}F = R^\times$. Illicitly assuming classical logic, the points of $\mathrm{Spec}(A|R)$ are in canonical one-to-one correspondence with those prime ideals $\mathfrak{p} \subseteq A$ such that $\varphi^{-1}\mathfrak{p} = \mathfrak{m}_R$. The points of $\mathrm{Spec}(A|R)$ are therefore in canonical one-to-one correspondence with the points of $\mathrm{Spec}(A \otimes_R k)$, where $k = R/\mathfrak{m}_R$ is the residue field of R. Therefore $\mathrm{Spec}(A|R)$ and $\mathrm{Spec}(A \otimes_R k)$ might coincide.

However, we have the following negative result.[21]

Proposition 12.18. *In general, the local spectrum of an algebra can't be expressed as an ordinary spectrum.*

Proof. It is well-known that the ordinary spectrum is always quasicompact. The local spectrum, however, can fail to be quasicompact. A quick way to see this is to notice that, if that was the case, the locale-theoretic part of the projection morphism $\underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ would always be a proper map of locales [77].

There's also a more direct way of seeing this, which in fact proves a slightly stronger statement. Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. From the internal point of view of $\mathrm{Sh}(X)$, the local spectrum $\mathrm{Spec}(\mathcal{O}_X[f^{-1}]|\mathcal{O}_X) \hookrightarrow \mathrm{Spec}(\mathcal{O}_X|\mathcal{O}_X) \cong \mathrm{pt}$ is the open sublocale of pt corresponding to the truth value of "f is invertible". Explicitly, the frame of opens of $\mathrm{Spec}(\mathcal{O}_X[f^{-1}]|\mathcal{O}_X)$ is isomorphic to $\{\psi : \Omega \mid \psi \Rightarrow f \text{ is invertible}\}$.

The ordinary spectrum always has the Frobenius reciprocity property, being quasicompact. In contrast, the locale $\mathrm{Spec}(\mathcal{O}_X[f^{-1}]|\mathcal{O}_X)$ has this property if and only if f is nilpotent or invertible. ☐

Finally, we want to restate the universal properties of the ordinary spectrum and the local spectrum in ring-theoretic language, employing the dual categories $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$, as in Section 12.3.

Let A be a ring. The morphism $A \to \mathcal{O}_{\mathrm{Spec}(A)}$ in $\mathrm{RL}^{\mathrm{op}}$ (the ring-theoretic part of the canonical morphism $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \to (\mathrm{Set}, A)$) is the *universal localization* of A: The ring $\mathcal{O}_{\mathrm{Spec}(A)}$ is local, and for any morphism $A \to \mathcal{B}$ into a local ring $\mathcal{B}$ (over any locale), there is a unique local morphism $\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{B}$ rendering the

[21] Intuitionistically, it's still true that the prime ideals of a quotient ring $A/\mathfrak{p}$ are in one-to-one correspondence with those prime ideals of A which contain $\mathfrak{p}$. However, the analogous statement "filters of A/F correspond to those filters of A which are contained in F" can't be shown intuitionistically, if A/F is defined as A/F^c. However, informally speaking, this failure is not the fault of the statement, but of the definition of A/F. The definition raises red flags from an intuitionistic point of view, since not F, but only its complement F^c enters the construction.
The statement can be salvaged by defining "A/F" to mean the set A equipped with a new *apartness relation* defined by $a \# b :\Leftrightarrow a - b \in F$. (A basic example for a ring-with-apartness-relation is the field of real numbers equipped with $x \# y :\Leftrightarrow \exists q \in \mathbb{Q}.\ |x - y| \geq q > 0$.) A filter G of this ring-with-apartness-relation A is by definition a subset $G \subseteq A$ which verifies the filter axioms and which is *open with respect to the apartness relation* in that for any elements $a, b : A$, the implication $a \in G \Rightarrow (b \in G) \vee (a \# b)$ holds.
This construction provides one of several motivations for developing the theory of rings using apartness relations and anti-ideals; one can even define the spectrum of a ring-with-apartness-relation. However, we'll not pursue these ideas further here.

diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{\mathcal{B}} \\
& \underset{\text{local}}{\mathcal{O}_{\mathrm{Spec}(A)}} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

commutative. In contrast, the universal property of the local spectrum is as follows. Let R be a ring. Let A be an R-algebra. The morphism $A \to \mathcal{O}_{\mathrm{Spec}(A|R)}$ is the universal way of turning A into a local ring *which is local over* R: The ring $\mathcal{O}_{\mathrm{Spec}(A|R)}$ is local, the composition $R \to A \to \mathcal{O}_{\mathrm{Spec}(A|R)}$ is local, and for any morphism $A \to \mathcal{B}$ into a local ring (over any locale) such that the composition $R \to A \to \mathcal{B}$ is local, there is a unique local morphism $\mathcal{O}_{\mathrm{Spec}(A|R)} \to \mathcal{B}$ such that the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[rrr, bend left, "\text{local}"] \arrow[drr, bend right, "\text{local}"'] & A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{\mathcal{B}} \\
& & \underset{\text{local}}{\mathcal{O}_{\mathrm{Spec}(A|R)}} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

commutes.

Remark 12.19. It's possible to state the universal property of the structure sheaf of the big Zariski topos of a ring A, more precisely of the canonical morphism $(\mathrm{Zar}(A), \underline{\mathbb{A}}^1_{\mathrm{Spec}\,A}) \to (\mathrm{Set}, A)$ of ringed toposes, in a similar manner, employing the dual categories $\mathrm{RT}^{\mathrm{op}}$ and $\mathrm{LRT}^{\mathrm{op}}$ of the categories of (locally) ringed toposes. However, unlike the universal property of the spectrum, this universal property looks slightly odd from an algebraic point of view: For any morphism $A \to \mathcal{B}$ into a local ring (over any topos $\mathcal{E}$), there is a unique bijective homomorphism $\underline{\mathbb{A}}^1_{\mathrm{Spec}\,A} \to \mathcal{B}$ rendering the diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{\mathcal{B}} \\
& \underset{\text{local}}{\underline{\mathbb{A}}^1_{\mathrm{Spec}\,A}} \arrow[ur, dashed, "\text{bijective}"'] &
\end{tikzcd}$$

commutative. By "bijective" we mean that the ring-theoretic part $f^\sharp : f^{-1}\underline{\mathbb{A}}^1_{\mathrm{Spec}\,A} \to \mathcal{B}$ of the morphism $f : (\mathcal{E}, \mathcal{B}) \to (\mathrm{Zar}(A), \underline{\mathbb{A}}^1_{\mathrm{Spec}\,A})$ is bijective as seen from the internal point of view of $\mathcal{E}$.

12.6. The spectrum of the generic ring. Let Set[Ring] be the classifying topos of the theory of rings; explicitly, it's the topos of preshaves on $\mathrm{Ring}_{\mathrm{fp}}^{\mathrm{op}}$, the dual of the category of finitely presented rings. This topos contains the *generic ring* U (explicitly the presheaf $R \mapsto R$): any ring in any topos is the pullback of U along a suitable geometric morphism.

Let Set[LocRing] be the classifying topos of the theory of local rings. Explicitly, it's the big Zariski topos $\mathrm{Zar}(\mathrm{Spec}(\mathbb{Z}))$ (built using one of the *parsimonious sites*, as described in Section 15). This topos contains the *generic local ring* U': any local ring in any topos is the pullback of U' along a suitable geometric morphism.

Let A be a ring. By the universal property of Set[Ring], there is a geometric morphism $g : \mathrm{Set} \to \mathrm{Set[Ring]}$ such that $g^{-1}U \cong A$. Since U' is in particular a ring, again by the universal property of Set[Ring], there is a geometric morphism $f : \mathrm{Set}[\mathrm{LocRing}] \to \mathrm{Set}[\mathrm{Ring}]$ such that $f^{-1}U \cong U'$. By the universal property of Set[LocRing], the topos of sheaves over the spectrum of A admits a geometric morphism g' to Set[LocRing] such that $(g')^{-1}U' \cong \mathcal{O}_{\mathrm{Spec}(A)}$.

The resulting solid diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[dr, dotted, "h"] \arrow[drr, dotted, bend left, "\tilde{g}"] \arrow[ddr, dotted, bend right, "\tilde{f}"'] & & \\
& \mathrm{Sh}(\mathrm{Spec}(A)) \arrow[r, "g'"] \arrow[d, "f'"'] & \mathrm{Set}[\mathrm{LocRing}] \arrow[d, "f"] \\
& \mathrm{Set} \arrow[r, "g"'] \arrow[ur, phantom, "\Nearrow\eta"] & \mathrm{Set}[\mathrm{Ring}]
\end{tikzcd}$$

commutes up to a non-invertible natural transformation η; under the equivalence

$$\begin{aligned}
&\text{category of geometric morphisms } \mathrm{Sh}(\mathrm{Spec}(A)) \to \mathrm{Set}[\mathrm{Ring}] \simeq \\
&\qquad\qquad\qquad\qquad \text{category of ring objects in } \mathrm{Sh}(\mathrm{Spec}(A))
\end{aligned}$$

this transformation corresponds to the non-invertible localization homomorphism $\underline{A} \to \underline{A}[\mathcal{F}^{-1}] = \mathcal{O}_{\mathrm{Spec}(A)}$. It is folklore that this square is a lax pullback square in the 2-category of Grothendieck toposes (for instance, this is reported on at [5]); however, this is not true.

Given a topos $\mathcal{E}$ together with geometric morphisms $\tilde{f} : \mathcal{E} \to \mathrm{Set}$ and $\tilde{g} : \mathcal{E} \to \mathrm{Set}[\mathrm{LocRing}]$ and a natural transformation $\tilde{\eta} : \tilde{f}^{-1} \circ g^{-1} \Rightarrow \tilde{g}^{-1} \circ f^{-1}$ (these data correspond to a local ring $\mathcal{O}_{\mathcal{E}}$ in $\mathcal{E}$ together with a ring homomorphism $\varphi : \underline{A} \to \mathcal{O}_{\mathcal{E}}$), there is a canonical geometric morphism $h : \mathcal{E} \to \mathrm{Sh}(\mathrm{Spec}(A))$ (determined by requiring that $h^{-1}\mathcal{F} \cong \mathcal{F}_0 := \varphi^{-1}[\mathcal{O}_{\mathcal{E}}^{\times}]$), and this morphism renders the lower left triangle commutative up to a natural isomorphism, but it renders the upper right triangle commutative only up to a non-invertible natural transformation (corresponding to the non-invertible ring homomorphism $\underline{A}[\mathcal{F}_0^{-1}] \to \mathcal{O}_E$).

The observation that the square is not a lax pullback is joint with Peter Arndt and Matthias Hutzler. The observation raises two questions: What is the lax pullback (which exists by general theory), if it's not Sh(Spec(A))? And how can Sh(Spec(A)) be described as a pullback? The following two propositions answer these questions. The geometric morphism $\mathrm{Set} \to \mathrm{Set}[\mathrm{Ring}]$ which they implicitly refer to is the morphism g mentioned above.

Proposition 12.20. *Let A be a ring. The lax pullback* $(\mathrm{Set} \Rightarrow_{\mathrm{Set}[\mathrm{Ring}]} \mathrm{Set}[\mathrm{LocRing}])$ *is the big Zariski topos of* $\mathrm{Spec}(A)$ *(built using one of the parsimonious sites, as described in Section 15).*

Proof. The claim can be checked by hand, but it's more instructive to employ the general theory of classifying toposes. In the situation

$$\begin{tikzcd}
(\mathrm{Set}[T] \Rightarrow_{\mathrm{Set}[T_0]} \mathrm{Set}[T']) \arrow[r] \arrow[d] & \mathrm{Set}[T'] \arrow[d, "f"] \\
\mathrm{Set}[T] \arrow[r, "g"'] \arrow[ur, phantom, "\Nearrow\eta"] & \mathrm{Set}[T_0],
\end{tikzcd}$$

where T_0, T, and T' are arbitrary geometric theories, the lax pullback classifies the geometric theory whose models consist of a model M of T, a model N of T', and a homomorphism $G(M) \to F(N)$ of T_0-models. The constructions G and F are given by the geometric morphisms g and f:

Any object of Set$[T]$ can be obtained by geometric constructions from U_T, the universal model of T in Set$[T]$. In particular, the pullback $g^{-1}U_{T_0}$, which is a model of T_0, can be obtained by geometric constructions from U_T. Therefore the geometric morphism g displays a way to turn the generic model of T into a model of T_0 using only geometric constructions. The same constructions can be applied to any model M of T, yielding a model $G(M)$ of T_0.

In the concrete situation at hand, the theory T is the empty theory (admitting in any topos a unique model M), the theory T' is the theory of local rings, and T_0 is the theory of rings. The T_0-model $G(M)$ is the ring A. The T_0-model $F(N)$ of a local ring N is the underlying ring of N.

Therefore the lax pullback (Set $\Rightarrow_{\mathrm{Set[Ring]}}$ Set[LocRing]) classifies ring homomorphisms $A \to R$ where R is a local ring, that is, local A-algebras. It's well-known that Zar(Spec(A)) classifies these as well. $\square$

Proposition 12.21. *Let A be a ring. The pullback of the spectrum of the generic ring along* Set $\to$ Set[Ring] *is the spectrum of A.*

Proof. There are two related ways of making the statement precise. Firstly, the spectrum of the generic ring U can be interpreted as a (locally ringed) locale internal to Set[Ring]. Locales can be pulled back along geometric morphisms (even though the pullback of a frame along a geometric morphism typically fails to be a frame) [80]. In this way Spec(U) pulls back to a locale internal to Set, that is an ordinary external locale. The claim is that this locale is canonically isomorphic to Spec(A).

A second way to interpret the statement of the proposition is to regard the spectrum of the generic ring as a localic geometric morphism with codomain Set[Ring]. The claim is then that the diagram

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Spec}(A)) \arrow[r] \arrow[d] & \mathrm{Sh}_{\mathrm{Set[Ring]}}(\mathrm{Spec}(U)) \arrow[d] \\
\mathrm{Set} \arrow[r, "g"'] & \mathrm{Set[Ring]}
\end{tikzcd}$$

is a pullback diagram in the 2-category of toposes.

Using the language of classifying locales and classifying toposes, both claims are easy to establish. The pulled-back locale (or topos) classifies the pulled-back geometric theory [80, Corollary 5.4]. Since the description of the theory which Spec(U) classifies – the theory of filters of U – is itself geometric, the pulled-back theory is the theory of filters of $g^{-1}U \cong A$.[22] $\square$

Proposition 12.22.
(1) *Let A be an R-algebra. The local spectrum* Spec($A|R$) *is the pullback of* Spec($U''|R$)*, where U'' is the* generic R-algebra *contained in the classifying topos $\mathcal{E}$ of R-algebras, along the geometric morphism* Set $\to \mathcal{E}$ *given by A.*
(2) *Let X be a scheme (or a locally ringed locale). Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. The relative spectrum* $\underline{\mathrm{Spec}}_X(\mathcal{A})$ *is the pullback of* Spec($U''|\mathcal{O}_X$)*, where U'' is the generic $\mathcal{O}_X$-algebra contained in the classifying* Sh(X)*-topos $\mathcal{E}$ of $\mathcal{O}_X$-algebras, along the geometric morphism* Sh(X) $\to \mathcal{E}$ *given by $\mathcal{A}$.*

Proof. Straightforward modification of the proof of Proposition 12.21. $\square$

Remark 12.23. The big Zariski topos Zar(Spec(A)) can be obtained as the pullback of the big Zariski topos of the generic ring U, if both toposes are understood to be defined using the parsimonious sites as described in Section 15.

[22] In the notation of [80, Section 5], the theory of filters of U is represented by a GRD-system with $G = U$ and $R = 1 \amalg U^2 \amalg U^2 \amalg 1 \amalg U^2$ (one summand for each axiom scheme).

12.7. Limits in the category of locally ringed locales. The category of ringed locales has small limits, by the naive construction. For instance, the fiber product $X \times_Z Y$ of ringed locales is given by the fiber product of the underlying locales and the structure sheaf $\pi_X^{-1}\mathcal{O}_X \otimes_{\pi_Z^{-1}\mathcal{O}_Z} \pi_Y^{-1}\mathcal{O}_Y$. More generally, the limit of a small diagram of ringed locales is given by the limit L of the underlying locales and the colimit of the pulled-back structure sheaves (calculated in the category of sheaves of rings on L).

However, when applied to a diagram of locally ringed locales, the ringed locale which this simple construction yields is in general not locally ringed. This can be nicely understood from the internal point of view: Let R be a local ring. Let $R \to A$ and $R \to B$ be local R-algebras which are furthermore local over R. Then the tensor product $A \otimes_R B$ is in general not a local ring. Indeed, this fails even in the easiest case, where all rings involved are fields: The rings $\mathbb{R}$ and $\mathbb{C}$ are local, and the inclusion $\mathbb{R} \to \mathbb{C}$ is local, but $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X]/(X^2+1) \cong \mathbb{C}[X]/(X^2+1) \cong \mathbb{C} \times \mathbb{C}$ is not.

The following proposition explains that the true limit in the category of locally ringed locales is obtained by *relocalizing* the limit in the category of ringed locales.

Proposition 12.24. *The category of locally ringed locales has all small limits.*

Proof. For notational simplicity, we describe how products in the category of locally ringed locales can be constructed. The general case is entirely analogous.

Let X and Y be locally ringed locales. Their product P as ringed locales has two defects: Firstly, it's not locally ringed. Secondly, the ring-theoretic parts of the projection morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ aren't local, that is, don't reflect invertibility.

The first issue could be solved by constructing, internally to $\mathrm{Sh}(P)$, the ordinary spectrum of $\mathcal{O}_P$. From the external point of view, this would yield a locally ringed locale equipped with morphisms of ringed, but not of locally ringed, locales to X and Y.

To solve both issues, we need to employ a refined spectrum construction, similar to the modification required by the internal account of the relative spectrum: Internally to $\mathrm{Sh}(P)$, we construct the classifying locale of the theory of those filters of $\mathcal{O}_P$ which simultaneously lie over the filter of units of $\pi_X^{-1}\mathcal{O}_X$ and which lie over the filter of units of $\pi_Y^{-1}\mathcal{O}_Y$. This locale is a sublocale of $\mathrm{Spec}(\mathcal{O}_P)$, the largest such that the morphisms to $(\mathrm{pt}, \pi_X^{-1}\mathcal{O}_X)$ and to $(\mathrm{pt}, \pi_Y^{-1}\mathcal{O}_Y)$ are morphisms of locally ringed locales.

The externalization of the internal locally ringed locale obtained in this way is the sought product of X and Y in the category of locally ringed locales. $\square$

Remark 12.25. The category of locally ringed locales embeds as a (non-full) coreflective subcategory into the category of ringed locales; the coreflector maps a ringed locale $(X, \mathcal{O}_X)$ to the externalization of $\mathrm{Spec}(\mathcal{O}_X)$ (constructed internally to $\mathrm{Sh}(X)$). However, as is familiar in situations where the embedding is not full [3], it's in general not the case that limits in LRL are calculated by applying the coreflector to the limit calculated in RL. Employing the language of the proof of Proposition 12.24, applying the coreflector only solves the first issue, but not the second.

It's instructive to determine the points of limits in LRL, even though a locale is of course not determined by its points. For instance, the construction in Proposition 12.24 shows that the points of the product $X \times Y$ of locally ringed locales in LRL are in canonical one-to-one correspondence with tuples (x, y, F), where x is a point of X, y is a point of y, and F is a filter of $\mathcal{O}_{X,x} \otimes_{\mathbb{Z}} \mathcal{O}_{Y,y}$ which lies over the filter of units of $\mathcal{O}_{X,x}$ and of $\mathcal{O}_{Y,y}$. In classical logic, those tuples are in canonical one-to-one correspondence with tuples $(x, y, \mathfrak{p})$, where x and y are as before and $\mathfrak{p}$ is a prime ideal of $k(x) \otimes_{\mathbb{Z}} k(y)$.

Similarly, points of the fiber product $X \times_Z Y$ are in canonical one-to-one correspondence with tuples (x, y, F), where x is a point of X and y is a point of y such that both map to the same point z of Z, and F is a filter of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ lying over the filter of units of $\mathcal{O}_{X,x}$ and of $\mathcal{O}_{Y,y}$ (and therefore automatically of $\mathcal{O}_{Z,z}$). In classical logic, those tuples are in canonical one-to-one correspondence with tuples $(x, y, \mathfrak{p})$, where x and y are as before and $\mathfrak{p}$ is a prime ideal of $k(x) \otimes_{k(z)} k(y)$.

Remark 12.26. By the adjunction $(\underline{\quad})_L \dashv (\underline{\quad})_P$ relating locales and topological spaces, limits of locally ringed spaces which happen to be sober can be calculated by regarding them as locally ringed locales by $(\underline{\quad})_L$, calculating their limit in LRL, and taking the associated topological space of the limit by $(\underline{\quad})_P$.

Small diagrams of arbitrary locally ringed spaces admit limits as well. Indeed, the proof of Proposition 12.24 was adapted from Gillam's proof of this fact [37, Corollary 5].

12.8. Relative Proj construction. Similar issues as with the relative spectrum arise with the Proj construction: The standard definition of the Proj construction as a topological space of homogeneous prime ideals gives rise to a space which can't intuitionistically be shown to satisfy the expected universal property. The construction has to be reimagined as a locale instead of a topological space. A certain sublocale of this locale then yields the relative Proj construction when interpreted in the internal language of the little Zariski topos of a base scheme (or a locally ringed locale).

Definition 12.27. The *Proj construction* of an $\mathbb{N}$-graded ring S is the locale with frame of opens given by

$$\begin{aligned}\mathcal{T}(\mathrm{Proj}(S)) := \{\mathfrak{a} \subseteq S \mid \mathfrak{a} \text{ is a homogeneous radical ideal such that} \\ \forall x : S.\ xS_+ \subseteq \mathfrak{a} \Rightarrow x \in \mathfrak{a}\},\end{aligned}$$

where $S_+ = \bigoplus_{i>0} S_i$ is the irrelevant ideal.

A quick way to see that the partial order $\mathcal{T}(\mathrm{Proj}(S))$ is a frame is to recognize that it's the frame of opens of a sublocale of $\mathrm{Spec}(S)$. The associated nucleus $j : \mathcal{T}(\mathrm{Spec}(S)) \to \mathcal{T}(\mathrm{Spec}(S))$ is given by

$$j(\mathfrak{a}) := (\sqrt{\mathfrak{a}^h} : S_+),$$

where $\mathfrak{a}^h$ is the homogenization of $\mathfrak{a}$, the ideal of S generated by all homogeneous components of the elements of $\mathfrak{a}$. Since $\mathfrak{a} \subseteq \mathfrak{a}^h \subseteq \sqrt{\mathfrak{a}^h} \subseteq j(\mathfrak{a})$, a radical ideal $\mathfrak{a}$ is an element of $\mathcal{T}(\mathrm{Proj}(S))$ if and only if $\mathfrak{a} = j(\mathfrak{a})$.

One way to derive this definition is to start, within a classical context, with the general expression for the nucleus associated to the subspace of $\mathrm{Spec}(S)$ consisting of those prime ideals which are homogeneous and don't contain S_+, and then rewrite this expression to not refer to prime ideals.

Definition 12.28. A filter $F \subseteq S$ in an $\mathbb{N}$-graded ring S is *homogeneous* if and only if, for any element $a : S$, the filter F contains a if it contains at least one of the homogeneous components of a. It *meets the irrelevant ideal* if and only if $F \cap S_+$ is inhabited.

In classical logic, a subset is a homogeneous filter meeting the irrelevant ideal if and only if its complement is a homogeneous prime ideal not containing the irrelevant ideal. Intuitionistically, neither direction can be shown.

Proposition 12.29. *Let S be an $\mathbb{N}$-graded ring. Then* $\mathrm{Proj}(S)$ *is the classifying locale of any of the following geometric theories.*

(1) *The theory of homogeneous filters of S meeting the irrelevant ideal, that is the theory of Remark 12.3 supplemented by the following two axiom schemes:*
- $\bigvee_i (a_i \in F) \vdash a \in F$ *(one axiom for each decomposition $a = \sum_i a_i$ of an element of S into homogeneous components)*
- $\top \vdash \bigvee_{a \in S_+} (a \in F)$ *(one axiom)*

(2) *The theory given by one atomic proposition "$a \in F_i$" for each homogeneous element a of degree i in S and axioms given by the following axiom schemes:*
- $\top \vdash 1 \in F_0$ *(one axiom)*
- $st \in F_{i+j} \dashv\vdash s \in F_i \wedge t \in F_j$ *(two axioms for each $i, j \geq 0$, $s \in S_i$, $t \in S_j$)*
- $0 \in F_i \vdash \bot$ *(one axiom for each $i \geq 0$)*
- $s + t \in F_i \vdash s \in F_i \vee t \in F_i$ *(one axiom for each $i \geq 0$, $s, t \in A_i$)*
- $\top \vdash \bigvee_{i \geq 1} \bigvee_{a \in S_i} (a \in F_i)$ *(one axiom)*

(3) *The same theory as in (2), but with atomic propositions only for homogeneous elements of degree ≥ 1 and without the first axiom "$\top \vdash 1 \in F_0$".*

Proof. That $\mathrm{Proj}(S)$ coincides with the classifying locale of the theory given in (1), can be verified by a direct calculation. By the general theory, the nucleus associated to the quotient theory given in (1) maps a radical ideal $\mathfrak{a} : \mathcal{T}(\mathrm{Spec}(S))$ to the least fixed point above of $\mathfrak{a}$ of the map

$$\mathfrak{b} \longmapsto \mathfrak{b} \vee \bigvee_{a\,:\,S} \Big(\sqrt{(a_i)_i} \cap \big(\sqrt{(a)} \to \mathfrak{b} \big) \Big) \vee \Big(\sqrt{(a)_{a \in S_+}} \to \mathfrak{b} \Big),$$

where $(\mathfrak{c} \to \mathfrak{b}) = (\mathfrak{b} : \mathfrak{c})$ is the Heyting implication and "$\vee$" is the join in $\mathcal{T}(\mathrm{Spec}(S))$. We omit the intermediate steps of the calculation.

The theories given in (1) and in (2) are bi-interpretable. The interpretation of the atomic propositions "$a \in F_i$" of theory (2) using the signature of theory (1) is "$a \in F$". Verifying the axioms is straightforward. Conversely, the interpretation of "$a \in F$" in the signature of theory (2) is "$\bigvee_i (a_i \in F_i)$", where $a = \sum_i a_i$ is the decomposition into homogeneous components. For verifying the axioms, one needs the lemma that

$$\bigvee_i (s_i \in F_i) \wedge \bigvee_j (t_j \in F_j) \ \dashv\vdash \ \bigvee_n \Big(\sum_{i+j=n} s_i t_j \in F_n \Big)$$

is derivable in theory (2), for any decompositions $s = \sum_i s_i$ and $t = \sum_j t_j$ of elements of S into homogeneous components. In the guise "$\sqrt{(s_i)_i} \cap \sqrt{(t_j)_j} = \sqrt{(\sum_{i+j=n} s_i t_j)_n}$" this is a familiar fact on the content of polynomials [9, Proposition 1].

Also theories (2) and (3) are bi-interpretable. The interpretation of "$a \in F_0$" in the signature of theory (3) is "$\bigvee_{i \geq 1} \bigvee_{h \in S_i} (ha \in F_i)$". □

Corollary 12.30. *Let S be an $\mathbb{N}$-graded ring. The points of* $\mathrm{Proj}(S)$ *are in canonical one-to-one correspondence with the homogeneous filters of S meeting the irrelevant ideal.*

Proof. Points of $\mathrm{Proj}(S)$ are given by models of the theory of homogeneous filters of S meeting the irrelevant ideal in Set. □

Remark 12.31. The same presentation as in Proposition 12.29(3) has been used to construct $\mathrm{Proj}(S)$ not as a locale, but as a distributive lattice [31].

Definition 12.32. Let S be an $\mathbb{N}$-graded ring. The *generic homogeneous filter meeting the irrelevant ideal* is the subsheaf $\mathcal{F} \hookrightarrow \underline{S}$ over $\mathrm{Proj}(S)$ generated by the sections a over $D_+(a) := j(\sqrt{(a)})$.

Equivalently, the generic homogeneous filter meeting the irrelevant ideal is the pullback of the generic filter in $\mathrm{Sh}(\mathrm{Spec}(S))$ to $\mathrm{Sh}(\mathrm{Proj}(S))$.

Definition 12.33. Let S be an $\mathbb{N}$-graded ring. The structure sheaf of $\mathrm{Proj}(S)$ is the homogeneous localization $\underline{S}[\mathcal{F}^{-1}]_0$ of the ring $\underline{S}$ at the generic homogeneous filter meeting the irrelevant ideal, that is the degree-zero part of $\underline{S}[\mathcal{F}^{-1}]$. The *tilde construction* of a graded S-module M is $M^\sim := \underline{M}[\mathcal{F}^{-1}]_0$.

The locally ringed locale $\mathrm{Proj}(S)$ and the tilde construction defined in this way enjoy their familiar properties. For instance, we have the following lemma.

Lemma 12.34. *Let S be an $\mathbb{N}$-graded ring.*

1. *Let $f : S$ be homogeneous of degree $d \geq 1$. Then $D_+(h) \cong \mathrm{Spec}(S[f^{-1}]_0)$.*
2. *Assume that S is generated as an S_0-algebra by S_1. Let M and N be graded S-modules. Then $M^\sim \otimes_{\mathcal{O}_{\mathrm{Proj}(S)}} N^\sim \cong (M \otimes_S N)^\sim$.*
3. *Under the same assumption as in (2), the twisting sheaves $\mathcal{O}(m) := (S(m))^\sim$ are finite locally free of rank 1.*

Proof. For the first statement, it suffices to verify that the theories of homogeneous filters of S meeting the irrelevant ideal and containing h and of filters of $S[f^{-1}]_0$ are bi-interpretable. It's slightly more convenient to use the presentation given by Proposition 12.29(2) for the former theory.

The interpretation of "$q \in F$" for $q : S[f^{-1}]_0$ in the signature of the theory given by Proposition 12.29(2) is

$$\bigvee \{(x \in F_{di}) \mid q = x/f^i \text{ for some } x : S,\ i \geq 0\}.$$

Conversely, the interpretation of "$a \in F_i$" in the signature of the theory of filters of $S[f^{-1}]_0$ is "$x^d/h^i \in F$".

The second statement follows from the calculation

$$\begin{aligned} M^\sim \otimes_{\mathcal{O}_{\mathrm{Proj}(S)}} N^\sim &= \underline{M}[\mathcal{F}^{-1}]_0 \otimes_{\underline{S}[\mathcal{F}^{-1}]_0} \underline{N}[\mathcal{F}^{-1}]_0 \\ &\cong (\underline{M} \otimes_{\underline{S}} \underline{N})[\mathcal{F}^{-1}]_0 \cong (\underline{M \otimes_S N})[\mathcal{F}^{-1}]_0 = (M \otimes_S N)^\sim. \end{aligned}$$

The first isomorphism maps $x/s \otimes y/t$ to $(x \otimes y)/(st)$. By the assumption that S is generated as an S_0-algebra by S_1, the generic filter contains a homogeneous element h of degree 1 from the internal point of view of $\mathrm{Sh}(\mathrm{Proj}(S))$. Therefore the map has an inverse sending $(a \otimes b)/u$, where a and b are homogeneous of degrees i and j, to $(h^j a)/u \otimes b/h^j$. The second isomorphism is because the tensor product is a geometric construction and therefore commutes with constructing the constant sheaf.

For the proof of the third statement, we show that $(S(m))^\sim$ is a finite free module of rank 1 from the internal point of view. We again use that the generic filter contains a homogeneous element $h : \underline{S}$ of degree 1 from the internal point of view. Such an element allows to define an isomorphism $\mathcal{O}_{\mathrm{Proj}(S)} = \underline{S}[\mathcal{F}^{-1}]_0 \to \underline{S(m)}[\mathcal{F}^{-1}]_0 = \mathcal{O}(m)$ by mapping x/s to $(h^m x)/s$ if $m \geq 0$ and to $x/(h^{-m} s)$ otherwise. ☐

Definition 12.35. Let R be a ring. Let S be an $\mathbb{N}$-graded R-algebra. The *local Proj construction* of S over R is the sublocale $\mathrm{Proj}(S|R)$ of $\mathrm{Proj}(S)$ with frame of opens given by

$$\mathcal{T}(\mathrm{Proj}(S|R)) := \{\mathfrak{a} : \mathcal{T}(\mathrm{Proj}(S)) \mid \forall f : R.\ \forall s : S.\ (\ulcorner f \text{ inv.} \urcorner \Rightarrow s \in \mathfrak{a}) \Rightarrow fs \in \mathfrak{a}\}$$

and with the pullback of $\mathcal{O}_{\mathrm{Proj}(S)}$ as the structure sheaf.

Proposition 12.36. *Let R be a ring. Let S be an $\mathbb{N}$-graded R-algebra. Then the local Proj construction* $\mathrm{Proj}(S|R)$ *is the classifying locale of the theory of homogeneous filters of S meeting the irrelevant ideal and lying over the filter of units.*

Proof. Direct calculation similar to the proof of Proposition 12.29. □

Since pullback and localization commute, the structure sheaf of $\mathrm{Proj}(S|R)$ can also be described as $\underline{S}[\mathcal{F}^{-1}]_0$, where by abuse of notation we mean by "$\mathcal{F}$" the pullback of the generic filter on $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S|R)$. This filter has the special property

$$\mathrm{Sh}(\mathrm{Proj}(S|R)) \models \forall r : \underline{R}.\ r \in \mathcal{F} \Rightarrow \ulcorner r \text{ inv. in } \underline{R} \urcorner.$$

Theorem 12.37. *Let X be a scheme (or a locally ringed locale). Let $\mathcal{S}$ be an $\mathbb{N}$-graded $\mathcal{O}_X$-algebra. Then the externalization $E(\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X))$ coincides with the relative Proj construction $\underline{\mathrm{Proj}}_X(\mathcal{S})$ as locally ringed locales over X.*

Proof. For simplicity, we assume that $\mathcal{S}$ is generated as an $\mathcal{S}_0$-algebra by $\mathcal{S}_1$. In this case, the expected universal property of the relative Proj construction is that it's a locally ringed locale over X such that, for all locally ringed locales $\mu : Y \to X$ over X, the set $\mathrm{Hom}_{\mathrm{LRL}/X}(Y, \underline{\mathrm{Proj}}_X(\mathcal{S}))$ is canonically isomorphic (by pullback of the standard such datum on $\underline{\mathrm{Proj}}_X(\mathcal{S})$) to the set of pairs $(\mathcal{L}, \psi)$ such that

- $\mathcal{L}$ is a line bundle on Y and
- $\psi : \mu^*\mathcal{S} \to \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$ is a graded morphism of $\mathcal{O}_Y$-algebras such that the degree-1 part of ψ is a surjective morphism $\mu^*\mathcal{S}_1 \to \mathcal{L}$

modulo equivalence. For instance, it is known that this property is satisfied in the case that X is a scheme and $\mathcal{S}$ is quasicoherent [69, Tag 01O4].

We verify that $E(\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X))$ enjoys the same property, even if X is not a scheme or $\mathcal{S}$ is not quasicoherent. For the rest of the proof, we switch to the internal universe of $\mathrm{Sh}(X)$.

The local Proj construction is a locally ringed locale over $(\mathrm{pt}, \mathcal{O}_X)$ by the unique morphism $! : \mathrm{Proj}(\mathcal{S}|\mathcal{O}_X) \to \mathrm{pt}$ of locales and by the canonical morphism $!^\sharp : \underline{\mathcal{O}_X} \to \underline{\mathcal{S}}_0 \to \underline{S}[\mathcal{F}^{-1}]_0 = \mathcal{O}_{\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)}$ of local rings.

As the standard datum on $\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$, we choose the line bundle $\mathcal{O}(1)$ (pulled back to $\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$) together with the canonical morphism $!^*\mathcal{S} \to \oplus_{n \geq 0} \mathcal{O}(1)^{\otimes n}$.

Let Y be a locally ringed locale over $(\mathrm{pt}, \mathcal{O}_X)$. Let a pair $(\mathcal{L}, \psi)$ be given. In the internal language of $\mathrm{Sh}(Y)$, we define a filter by the formula

$$\mathcal{F}' := \{ s : \underline{S} \mid \ulcorner \text{there exists } i \text{ such that } (\psi(s_i \otimes 1)) \text{ is a basis of } \mathcal{L}^{\otimes i} \urcorner \} \subseteq \underline{S},$$

where s_i refers to the homogeneous component of s of degree i. Since $\mathcal{L}^{\otimes i}$ is finite free of rank 1, a one-element family in $\mathcal{L}^{\otimes i}$ is a basis if and only if it's a generating family. This observation can be repeatedly used to verify that $\mathcal{F}'$ is homogeneous, meets the irrelevant ideal, and lies over the filter of units. Since $\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$ is the classifying locale of such filters (Proposition 12.36), we obtain a morphism $f : Y \to \mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$ of locales which is unique with the property that $f^{-1}\mathcal{F} = \mathcal{F}'$.

To obtain a morphism $Y \to \mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$ of locally ringed locales, it remains to define a morphism $f^\sharp : f^{-1}\mathcal{O}_{\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)} = \underline{S}[\mathcal{F}'^{-1}]_0 \to \mathcal{O}_Y$. A canonical choice is

$$x/s \mapsto \ulcorner \text{ the coefficient of } \psi(x \otimes 1) \text{ with respect to the basis } (\psi(s \otimes 1)) \urcorner.$$

We omit further verifications. □

13. Higher direct images and other derived functors

13.1. Flabby sheaves. Recall that a sheaf $\mathcal{F}$ of sets on a topological space (or a locale) X is *flabby* if and only if, for any open subset $U \subseteq X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Flabbiness of a sheaf is a local property, even though it doesn't seem like that at first sight: If the restrictions $\mathcal{F}|_{U_i}$ of $\mathcal{F}$ to the members of an open covering $X = \bigcup_i U_i$ are flabby, then the verification that $\mathcal{F}$ is flabby can't proceed as follows. "Let $s \in \mathcal{F}(U)$ be an arbitrary section. Since each $\mathcal{F}|_{U_i}$ is flabby, the section $s|_{U \cap U_i}$ extends to a section on U_i." The reason is that the individual extensions obtained in this way might not glue.

A correct proof employs Zorn's lemma in a typical way, considering a maximal extension and then verifying that the subset this maximal extension is defined on is all of X.

Since flabbiness is a local property, it's not unreasonable to expect that flabbiness can be characterized in the internal language. The following proposition shows that this is indeed the case.

Proposition 13.1. *Let $\mathcal{F}$ be a sheaf of sets on a topological space X (or a locale). Then the following statements are equivalent:*

(1) *$\mathcal{F}$ is flabby.*

(2) *"Any section of $\mathcal{F}$ can be locally extended": For any open subset $U \subseteq X$ and any section $s \in \mathcal{F}(U)$ there is an open covering $X = \bigcup_i V_i$ such that, for each i, there is an extension of s to $U \cup V_i$ (that is, a section $s' \in \mathcal{F}(U \cup V_i)$ such that $s'|_U = s$).*

    (If X is a space instead of a locale, this can be equivalently formulated as follows: For any open subset $U \subseteq X$, any section $s \in \mathcal{F}(U)$, and any point $x \in X$, there is an open neighbourhood V of x and an extension of s to $U \cup V$.)

(3) *From the point of view of the internal language of $\mathrm{Sh}(X)$, for any subsingleton $K \subseteq \mathcal{F}$ there exists an element $s : \mathcal{F}$ such that $s \in K$ if K is inhabited. More precisely,*

$$\mathrm{Sh}(X) \models \forall K \subseteq \mathcal{F}.\ (\forall s, s' : K.\ s = s') \Longrightarrow \exists s : \mathcal{F}.\ (K \textit{ is inhabited} \Rightarrow s \in K).$$

(4) *The canonical map $\mathcal{F} \to \mathcal{P}_{\leq 1}(\mathcal{F}), s \mapsto \{s\}$ is final from the internal point of view, that is*

$$\mathrm{Sh}(X) \models \forall K : \mathcal{P}_{\leq 1}(\mathcal{F}).\ \exists s : \mathcal{F}.\ K \subseteq \{s\},$$

where $\mathcal{P}_{\leq 1}(\mathcal{F})$ is the object of subsingletons of $\mathcal{F}$.

Proof. The implication "(1) $\Rightarrow$ (2)" is trivial. The converse direction uses a typical argument with Zorn's lemma, considering a maximal extension. The equivalence "(2) $\Leftrightarrow$ (3)" is routine, using the Kripke–Joyal semantics to interpret the internal statement. Condition 4 is a straightforward reformulation of Condition 3. $\square$

Condition 2 of the proposition is, unlike the standard definition of flabbiness, manifestly local. Also its equivalence with Condition 3 and Condition 4 is intuitionistically valid; therefore one might consider to adopt Condition 2 as the definition of flabbiness.

The object $\mathcal{P}_{\leq 1}(\mathcal{F})$ of subsingletons of $\mathcal{F}$ can be interpreted as the object of *partially-defined elements* of $\mathcal{F}$. In this view, the empty subset is the maximally undefined element and a singleton is a maximally defined element. The proposition shows that $\mathcal{F}$ is flabby if and only if any such partially-defined element can be refined to an honest element of $\mathcal{F}$.

13.2. Injective sheaves. Recall that an object I of a category $\mathcal{C}$ is *injective* if and only if, for any monomorphism $X \to Y$ in $\mathcal{C}$ and any morphism $X \to I$, there is a lifting such that the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[dl, dashed] \\ I & \end{tikzcd}$$

commutes. Equivalently, an object I is injective if and only if the Hom functor $\mathrm{Hom}_{\mathcal{C}}(\underline{\quad}, I) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ maps monomorphisms in $\mathcal{C}$ to surjective maps. This general definition is often specialized to one of these cases: to the category of modules over a ring, to the category of set-valued sheaves on a topological space, and to the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.

The definition is seldomly applied in the category of sets, since in a classical context it's easy to show that a set is injective if and only if it's inhabited, thereby completely settling the question which objects are injective in a trivial manner.

The question is more interesting in an intuitionistic setting, since intuitionistically one cannot prove that inhabited sets are injective [2]; but one can still verify that any set embeds into an injective set: The powerset $\mathcal{P}(X)$ and even the smaller set $\mathcal{P}_{\leq 1}(X)$ of subsingletons of a given set X are injective. We verify this in the proof of Lemma 13.8.

For a cartesian closed category $\mathcal{C}$, there is also the notion of an *internally injective* object. This is an object I such that the internal Hom functor $[\underline{\quad}, I] : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ maps monomorphisms in $\mathcal{C}$ to epimorphisms. In the specical case that $\mathcal{C}$ is a elementary topos with a natural numbers object, such as the topos of set-valued sheaves on a space, this condition can be rephrased in several ways. The following proposition lists five of these conditions. The equivalence of the first four is due to Harting [39].

Proposition 13.2. *Let $\mathcal{E}$ be an elementary topos with a natural numbers object. Then the following statements about an object $I \in \mathcal{E}$ are equivalent.*

(1) *I is internally injective.*
(2) *The functor $[\underline{\quad}, I] : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ maps monomorphisms in $\mathcal{E}$ to morphisms for which any global element of the target locally (after change of base along an epimorphism) possesses a preimage.*
(3) *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object p^*I has property (1) as an object of $\mathcal{E}/A$.*
(4) *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object p^*I has property (2) as an object of $\mathcal{E}/A$.*
(5) *From the point of view of the internal language of $\mathcal{E}$, the object I is injective.*[23]

Proof. The implications "(1) $\Rightarrow$ (2)", "(3) $\Rightarrow$ (4)", "(3) $\Rightarrow$ (1)", and "(4) $\Rightarrow$ (2)" are trivial.

The equivalence "(3) $\Leftrightarrow$ (5)" follows directly from the interpretation rules of the stack semantics.

The implication "(2) $\Rightarrow$ (4)" employs the extra left adjoint $p_! : \mathcal{E}/A \to \mathcal{E}$ of $p^* : \mathcal{E} \to \mathcal{E}/A$ (which maps an object $(X \to A)$ to X), as in the usual proof that injective sheaves remain injective when restricted to smaller open subsets: We have that $p_* \circ [\underline{\quad}, p^*I]_{\mathcal{E}/A} \cong [\underline{\quad}, I]_{\mathcal{E}} \circ p_!$, the functor $p_!$ preserves monomorphisms, and one can check that p_* reflects the property that global elements locally possess preimages. Details are in [39, Thm. (1)1].

The implication "(4) $\Rightarrow$ (3)" follows by performing an extra change of base, since any non-global element becomes a global element after a suitable change of base. ☐

[23] In Section 2, we have only introduced the internal language for sheaf toposes. The general definition is in [67, Section 7].

Somewhat surprisingly, and in stark contrast with the situation for internally projective objects (which are defined dually), internal injectivity coincides with external injecticity for sheaf toposes over spaces.

Theorem 13.3. *Let X be a topological space (or a locale). An object $\mathcal{I} \in \mathrm{Sh}(X)$ is injective if and only if it is internally injective.*

Proof. For the "only if" direction, let $\mathcal{I}$ be an injective sheaf of sets. Then $\mathcal{I}$ satisfies Condition (2) in Proposition 13.2, even without having to pass to covers.

For the "if" direction, let $\mathcal{I}$ be an internally injective object. Let $m : \mathcal{E} \to \mathcal{F}$ be a monomorphism in $\mathrm{Sh}(X)$ and let $k : \mathcal{E} \to \mathcal{I}$ be an arbitrary morphism. We want to show that there exists an extension $\mathcal{F} \to \mathcal{I}$ of k along m. To this end, we consider the sheaf defined by the internal expression

$$\mathcal{G} := [\![\{k' : [\mathcal{F}, \mathcal{I}] | k' \circ m = k\}]\!].$$

Global sections of $\mathcal{G}$ are extensions of the kind we're looking for. Therefore it suffices to show that $\mathcal{G}$ is flabby. We do this by verifying Condition (3) of Proposition 13.1 in the internal language of $\mathrm{Sh}(X)$.

Let $K \subseteq \mathcal{G}$ be a subsingleton. We consider the injectivity diagram

$$\begin{tikzcd} m[\mathcal{E}] \cup \mathcal{F}' \arrow[r, hook] \arrow[d] & \mathcal{F} \arrow[dl, dashed] \\ \mathcal{I}, & \end{tikzcd}$$

where $\mathcal{F}' := \{s : \mathcal{F} \,|\, K \text{ is inhabited}\}$ and the solid vertical arrow is defined in the following way: It should map an element $s \in \mathcal{F}'$ to $k'(s)$, where k' is any element of K; and it should map an element $m[u] \in m[\mathcal{E}]$ to $k(u)$. These prescriptions determine a well-defined map.

Since $\mathcal{I}$ is injective from the internal point of view we're taking up here, there exists a dotted map rendering the diagram commutative. This map is an element of $\mathcal{G}$. Furthermore, this map is an element of K, if K is inhabited. $\square$

Theorem 13.4. *Let $(X, \mathcal{O}_X)$ be a ringed topological space (or a ringed locale). An $\mathcal{O}_X$-module $\mathcal{I}$ is injective if and only if it is internally injective.*

Proof. Proposition 13.2 can be adapted from sheaves to sets to sheaves of modules, with the same proof.

The proof of Theorem 13.3 can be adopted as well. It suffices to change "$m[\mathcal{E}] \cup \mathcal{F}'$" to "$m[\mathcal{E}] + \mathcal{F}''$", where $\mathcal{F}'' := \{s : \mathcal{F} \,|\, s = 0 \text{ or } K \text{ is inhabited}\}$. $\square$

XXX: Remark that proof requires AxC and that it's a unique feature of sheaf toposes. Counterexample?

13.3. Internal proofs of common lemmas.

Lemma 13.5. *A sheaf of sets or a sheaf of modules is injective if and only if it is locally injective.*

Proof. By Theorem 13.3 respectively Theorem 13.4, injectivity can be characterized in the internal language. Any such property is local. $\square$

Lemma 13.6. *Let X be a topological space (or a locale).*

1. *Let $\mathcal{I}$ be an injective sheaf of sets over X. Let $\mathcal{F}$ be an arbitrary sheaf of sets. Then $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{I})$ is flabby.*
2. *Let $\mathcal{I}$ be an injective sheaf of modules over some sheaf $\mathcal{O}_X$ of rings over X. Let $\mathcal{F}$ be an arbitrary sheaf of modules. Then $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})$ is flabby.*

Proof. We cover the case of sheaf of sets first. By Theorem 13.3 and Proposition 13.1, it suffices to give an intuitionistic proof of the following statement: If I is an injective set and F is an arbitrary set, then partially defined elements of the set $[F, I]$ of all maps $F \to I$ can be refined to honest elements.

Thus let a subsingleton $K \subseteq [F, I]$ be given. We consider the injectivity diagram

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & F \arrow[dl, dashed] \\ I & \end{tikzcd}$$

where F' is the subset $\{s : F \mid K \text{ is inhabited}\} \subseteq F$ and the solid vertical map sends $s \in F'$ to $f(s)$, where f is an arbitrary element of K. This association is well-defined. Since I is injective, a dotted lift as indicated exists. If K is inhabited, this lift is an element of K.

The same kind of argument applies to the case of sheaves of modules, relying on Theorem 13.4 and defining F' as the submodule "$\{s : F \mid s = 0 \text{ or } K \text{ is inhabited}\}$". $\square$

Corollary 13.7. *Injective sheaves of sets and injective sheaves of modules are flabby.*

Proof. Follows from the previous lemma by considering the special cases $\mathcal{F} := 1$ respectively $\mathcal{F} := \mathcal{O}_X$. $\square$

Lemma 13.8. *Let X be a topological space (or a locale). Any sheaf of sets over X can be embedded into an injective (therefore flabby) sheaf of sets.*

Proof. By Proposition 13.1, it suffices to give an intuitionistic proof of the following statement: Any set F can be embedded into an injective set.

As already indicated there at least two simple ways that F can be embedded into an injective set: by embedding F in its powerset $\mathcal{P}(F)$ or by embedding F in $\mathcal{P}_{\leq 1}(F)$, the set of subsingletons of F. For conciseness, we only verify that $\mathcal{P}_{\leq 1}(F)$ is injective.

So let $m : A \hookrightarrow B$ be an injective map and let $k : A \to \mathcal{P}_{\leq 1}(F)$ be an arbitrary map. Then we can extend k to a map $k' : B \to \mathcal{P}_{\leq 1}(F)$ by defining for $y : B$

$$\begin{aligned} k'(y) &:= \bigcup k[m^{-1}[\{y\}]] \\ &= \{s : F \mid s \in k(x) \text{ for some } x \in A \text{ such that } m(x) = y\}. \end{aligned} \qquad \square$$

Remark 13.9. The *Godement construction* is a well-known way of embedding an inhabited sheaf of sets $\mathcal{F}$ into an injective sheaf, namely embedding it into the sheaf of not necessarily continuous sections of the étale space of $\mathcal{F}$:

$$U \subseteq X \quad \longmapsto \quad \prod_{x \in U} \mathcal{F}_x.$$

The sheaf $\mathcal{P}_{\leq 1}(\mathcal{F})$ does not coincide with this construction. Instead by Definition 2.8, it is the sheaf with

$$U \subseteq X \quad \longmapsto \quad \{\langle V, s\rangle \mid V \subseteq U \text{ open}, s \in \mathcal{F}(V)\}.$$

It's not possible to describe the Godement construction in the internal language of $\mathrm{Sh}(X)$, since in the Godement construction the underlying set of X enters. But the sheaf topos of X doesn't remember this set. For instance, if X is an inhabited indiscrete topological spac, then $\mathrm{Sh}(X)$ is equivalent to Set.

Remark 13.10. It's not known to me whether it's possible to intuitionistically prove that any module can be embedded into a module which satisfies the internal flabbiness criterion of Proposition 13.1. This would give an internal proof that any sheaf of modules can be embedded into a flabby sheaf of modules. The naive candidates don't work: The set $\mathcal{P}_{\leq 1}(F)$ doesn't admit a canonical module structure, and the free module over that set is not flabby in general.

Since by the Godement construction the statement is true in many models of intuitionistic logic, the sheaf toposes over topological spaces, and furthermore the proof that the Godement construction yields a flabby sheaf is intuitionistically valid,[24] it's not unreasonable to believe that such an intuitionistic proof is possible.

On the other hand, it's certainly not possible to intuitionistically prove that any module can be embedded into an injective module, since it's consistent with Zermelo–Fraenkel set theory that no nontrivial injective abelian groups exist [17].

Lemma 13.11. *Let X be a ringed space (or a ringed locale). Let $0 \to \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{E}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{E}'$ is flabby, then the induced sequence*

$$0 \longrightarrow \Gamma(X, \mathcal{E}') \longrightarrow \Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{E}'') \longrightarrow 0$$

is exact.

Proof. Since taking global sections is left exact (being a right adjoint functor), it suffices to verify that the map $\Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{E}'')$ is surjective. We'll do this by showing, in the internal language of $\mathrm{Sh}(X)$, that the sheaf of preimages of a given global section $s \in \Gamma(X, \mathcal{E}'')$ is flabby and therefore has a global section.

In the internal language, this sheaf has the description $F := \{u : \mathcal{E} \mid \beta(u) = s\}$. To verify the internal condition of Proposition 13.1, let a subsingleton $K \subseteq F$ be given. Since β is surjective, there is a preimage $u_0 \in F$. The translated set $K - u_0 \subseteq \mathcal{E}$ is still a subsingleton, and its preimage under α is as well. By the assumption on $\mathcal{E}'$, there is an element $v : \mathcal{E}$ such that $v \in \alpha^{-1}[K - u_0]$ if $K - u_0$ is inhabited. We'll now verify that $u_0 + \alpha(v) \in K$ if K is inhabited.

So assume that K is inhabited. Then $K - u_0$ is as well. Since the image of its unique element under β is zero and the given sequence is exact, the set $\alpha^{-1}[K - u_0]$ is inhabited as well. Therefore $v \in \alpha^{-1}[K - u_0]$. Thus $u_0 + \alpha(v) \in K$. ☐

Lemma 13.12. *Let X be a ringed space (or a ringed locale). Let $0 \to \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{E}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{E}'$ and $\mathcal{E}''$ are flabby, then $\mathcal{E}$ is flabby as well.*

Proof. We verify the condition of Proposition 13.1 in the internal language of $\mathrm{Sh}(X)$.

Let $K \subseteq \mathcal{E}$ be a subsingleton. Then its image $\beta[K] \subseteq \mathcal{E}''$ is a subsingleton as well. Since partial elements of $\mathcal{E}''$ can be refined to honest elements, there is an element $s : \mathcal{E}''$ such that $\beta[K] \subseteq \{s\}$.

Since β is surjective, there is an element $t_0 : \mathcal{E}$ such that $\beta(t_0) = s$.

The preimage $\alpha^{-1}[K - t_0] \subseteq \mathcal{E}'$ is a subsingleton. This partial element can be refined to an honest element, that is there exists an element $u : \mathcal{E}'$ such that $\alpha^{-1}[K - t_0] \subseteq \{u\}$.

The partial element K can thereby refined to the honest element $t := t_0 + \alpha(u)$. ☐

XXX: Higher direct images
XXX: Ext, Tor

[24]In order for the Godement construction to work in a intuitionistic metatheory, one has to tweak its definition a little bit. Instead of mapping an open subset U to $\prod_{x \in U} \mathcal{F}_x$, one has to map U to $\prod_{x \in U} \mathcal{P}_{\leq 1}(\mathcal{F}_x)$. This has the added advantage that it works even if $\mathcal{F}$ is not inhabited.

PART III

The big Zariski topos

The preceding part demonstrated that working in the internal universe of the little Zariski topos of a scheme S, the topos of sheaves on S, is useful for simplifying local work on S. The basic tenet was that sheaves of modules look just like plain modules and that theorems of intuitionistic algebra yield theorems about sheaves.

But the little Zariski topos is not particularly suited for dealing with *schemes* over S. For this, we need a related topos. For the scope of this introduction only, we employ the following slightly problematic definition which we'll fix in Section 15. We'll keep the base scheme S fixed throughout this part.

14. Basics

Definition 14.1 (provisional)**.** The *big Zariski topos* $\mathrm{Zar}(S)$ of a scheme S is the topos of sheaves on the Grothendieck site Sch/S of schemes over S.

Explicitly, an object of $\mathrm{Zar}(S)$ is a functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition with respect to ordinary Zariski coverings: If $X = \bigcup_i U_i$ is a cover of an S-scheme X by open subsets, the canonical diagram

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

should be an equalizer diagram.

Internal language. Just like the topos of sheaves on a topological space or on a locale admits an internal language, so does the big Zariski topos. The necessary modifications of the Kripke–Joyal semantics (Definition 2.1) are straightforward. Instead of defining recursively the meaning of "$U \models \varphi$" for open subsets $U \subseteq S$, we define the meaning of "$T \models \varphi$" for S-schemes T and slightly rewrite the rules for implication and universal quantification. Instead of

$$\begin{array}{rcl} U \models \varphi \Rightarrow \psi & :\Longleftrightarrow & \text{for all open } V \subseteq U\text{:} \\ & & \qquad V \models \varphi \text{ implies } V \models \psi \\ U \models \forall s\!:\!\mathcal{F}.\ \varphi(s) & :\Longleftrightarrow & \text{for all sections } s \in \Gamma(V, \mathcal{F}) \text{ on open } V \subseteq U\text{:} \\ & & \qquad V \models \varphi(s) \end{array}$$

they have to read as follows.

$$\begin{array}{rcl} T \models \varphi \Rightarrow \psi & :\Longleftrightarrow & \text{for all morphisms } T' \to T \text{ in } \mathrm{Sch}/S\text{:} \\ & & \qquad T' \models \varphi \text{ implies } T' \models \psi \\ T \models \forall s\!:\!F.\ \varphi(s) & :\Longleftrightarrow & \text{for all morphisms } T' \to T \text{ in } \mathrm{Sch}/S \text{ and} \\ & & \quad \text{all sections } s \in \Gamma(T', F)\text{:} \\ & & \qquad T' \models \varphi(s) \end{array}$$

The analogs of Proposition 2.4 and Proposition 2.5 are true for the internal language of the big Zariski topos:

Proposition 14.2. *Let T be an S-scheme and φ be a formula over T.*

(1) *If $T \models \varphi$ and if there is an intuitionistic proof that φ implies a further formula ψ, then $T \models \psi$.*
(2) *Let $T' \to T$ be a morphism of S-schemes. If $T \models \varphi$, then $T' \models \varphi$.*
(3) *If $T = \bigcup_i T_i$ is an open covering and if $T_i \models \varphi$ for all i, then $T \models \varphi$.*

Proof. The proofs of Proposition 2.4 and Proposition 2.5 carry over. □

When working with the internal language of the little Zariski topos, we often used the fact that if a formula holds on some open subset U, then it also holds on all open subsets contained in U. Proposition 14.2(2) states a stronger version of this: All properties which can be expressed using the internal language of the big Zariski topos are automatically *stable under base change.*

Important objects in the big Zariski topos. It's convenient to introduce notation for objects which often appear when working with the big Zariski topos.

Let X be an S-scheme. Its functor of points, which maps an S-scheme T to $\mathrm{Hom}_S(T, X)$, is an object of $\mathrm{Zar}(S)$. We denote it by "$\underline{X}$".

From the internal point of view of $\mathrm{Zar}(S)$, such a functor $\underline{X}$ looks like a single set. It can be pictured as the "set of points of X", where "point" doesn't mean "point of the underlying topological space of X", but rather "T-point of X", where T varies over all S-schemes. The internal language of the big Zariski topos hides any explicit mentions of the stage T; it is therefore a device for reifying the multitude of points of X, defined on varying stages, as a single entity.

Particularly important is $\underline{\mathbb{A}}^1_S$, the functor of points of the affine line over S. The object $\underline{S}$ is the terminal object in $\mathrm{Zar}(S)$. This fits into the philosophy: From the point of view of the big Zariski topos, the base scheme should simply look like a point. The functor of points of $S \amalg S$ looks like a two-element set from the internal point of view.

Let $\mathcal{F}$ be a sheaf of sets on S. For reasons explained in Section 16, we denote by "$\pi^{-1}(\mathcal{F})$" the induced sheaf on Sch/S mapping an S-scheme $(f : T \to S)$ to $\Gamma(T, f^{-1}(\mathcal{F}))$.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules. We denote by "$\mathcal{F}^{\mathrm{Zar}}$" the induced sheaf on Sch/S mapping an S-scheme $(f : T \to S)$ to $\Gamma(T, f^*(\mathcal{F}))$.

A first example illustrating the Kripke–Joyal translation rules. Since all the sets $\underline{\mathbb{A}}^1_S(T) \cong \Gamma(T, \mathcal{O}_T)$ carry ring structures, the object $\underline{\mathbb{A}}^1_S$ can be endowed with a canonical structure as a ring object in $\mathrm{Zar}(S)$. For a particular S-scheme T, the ring $\underline{\mathbb{A}}^1_S(T)$ isn't necessarily a field, but the system of these rings, conceptualized as a single entity from the internal point of view, does satisfy a field axiom. In the case $S = \operatorname{Spec} \mathbb{Z}$, this was first observed by Kock [50].

Proposition 14.3. *The ring $\underline{\mathbb{A}}^1_S$ is a field from the internal point of view of* $\mathrm{Zar}(S)$*, in the sense that*

$$\mathrm{Zar}(S) \models \forall f : \underline{\mathbb{A}}^1_S.\ \neg(f = 0) \Rightarrow \ulcorner f \text{ inv.} \urcorner.$$

Proof. According to the Kripke–Joyal semantics of $\mathrm{Zar}(S)$, we have to show that for any S-scheme T and any function $f \in \Gamma(T, \mathcal{O}_T)$ the statement $T \models \neg(f = 0)$ implies $T \models \ulcorner f \text{ inv.} \urcorner$. The antecedent states, for any T-scheme T', that if the pullback of f to T' vanishes, T' is then the empty scheme.

As with the analogous statement of the little Zariski topos (Lemma 3.2), the consequent means that f is invertible in $\Gamma(T, \mathcal{O}_T)$.

The claim follows by considering the particular T-scheme $T' := V(f)$. Since f vanishes on $V(f)$, this subscheme is empty and therefore its complement $D(f)$ is all of T. □

The field property can be interpreted as follows. A function f not being the zero function does not imply that it's invertible. But if f is *universally nonzero* in that the only scheme such that pullback of f to that scheme vanishes is the empty scheme, then f is indeed invertible.

15. On the proper choice of a big Zariski site

Unlike with the construction of the little Zariski topos, set-theoretical issues of size arise when constructing the big Zariski topos. These can be solved in several different manners, yielding toposes which are not equivalent, and actually differ in some important aspects, but otherwise enjoy very similar properties.

Naive approach. Some authors construct the big Zariski topos of S as the topos of sheaves over the site Sch/S of all schemes over S. This option is quite attractive since the Yoneda embedding $\mathrm{Sch}/S \to \mathrm{Sh}(\mathrm{Sch}/S)$, which sends an S-scheme to its functor of points, is fully faithful, therefore the internal language of $\mathrm{Sh}(\mathrm{Sch}/S)$ can distinguish arbitrary schemes.

However, since Sch/S is not essentially small, forming the sheaf topos is not possible in plain Zermelo–Fraenkel set theory.

Since it's still possible to meaningfully speak of individual functors $(\mathrm{Sch}/S)^{\mathrm{op}} \to S$, we can attach a Kripke–Joyal semantics to $\mathrm{Sh}(\mathrm{Sch}/S)$, as long as we keep in mind that $\mathrm{Sh}(\mathrm{Sch}/S)$ might not contain a subobject classifier and might not be cartesian closed. From the internal point of view, powersets and function sets might therefore not exist.

Using Grothendieck universes. We could also assume the existence of a Grothendieck universe $\mathcal{U}$ containing S and construct $\mathrm{Zar}(S)$ as the topos of sheaves over the small site $\mathrm{Sch}_{\mathcal{U}}/S$, the category of S-schemes contained in U.

By the *comparison lemma* **??**, we could also construct $\mathrm{Zar}(S)$ as the topos of sheaves over $\mathrm{Aff}_{\mathcal{U}}/S$, the category of S-schemes in U which are affine (as absolute schemes), and obtain an equivalent topos.

In this case, the Yoneda functor $\mathrm{Sch}/S \to \mathrm{Zar}(S)$ might not be faithful, but the restricted Yoneda functor $\mathrm{Sch}_{\mathcal{U}}/S \to \mathrm{Zar}(S)$ will.

Approach of the Stacks Project. The Stacks Project proposes a more nuanced approach, namely expanding a given set M_0 of schemes containing S to a superset M which is closed (up to isomorphism) under several constructions [69, Tag 000H]: fiber products, countable coproducts, domains of open and closed immersions and of morphisms of finite type, spectra of local rings $\mathcal{O}_{X,x}$, spectra of residue fields, and others.

The Stacks Project then defines $\mathrm{Zar}(S)$ as $\mathrm{Sh}(\mathrm{Sch}_M/S)$, where Sch_M/S is the small category of S-schemes in M, or equivalently as $\mathrm{Sh}(\mathrm{Aff}_M/S)$. This approach has the advantage that one doesn't have to assume the existence of a Grothendieck universe; the *partial universe* M can be constructed entirely within ZFC set theory using transfinite recursion.

Employing parsimonious sites. From a topos-theoretical point of view, it's natural to settle for an even more parsimonious site: the site $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ consisting of the S-schemes which are locally of finite presentation over S, or equivalently the essentially small site $(\mathrm{Aff}/S)_{\mathrm{lfp}}$ of the S-schemes which are locally of finite presentation over S and affine (as absolute schemes).[25]

[25]It's not reasonable to restrict to the even smaller site consisting of the finitely presented S-schemes, since open immersions can fail to be finitely presented. We want the site used to construct $\mathrm{Zar}(S)$ to be closed under domains of open immersions, for instance to facilitate a comparison with the little Zariski topos $\mathrm{Sh}(S)$, whose site does contain all open subsets of S. Furthermore, since a

In the special case that $S = \operatorname{Spec}(A)$ is affine, this site is the dual of the category of finitely presented A-algebras; in this case the topos-theoretic points of the resulting topos are precisely the local A-algebras, and moreover, the resulting topos is the classifying topos of the theory of local A-algebras, such that for any Grothendieck topos $\mathcal{E}$, geometric morphisms $\mathcal{E} \to \operatorname{Sh}((\mathrm{Aff}/S)_{\mathrm{lfp}})$ correspond to local A-algebras internal to $\mathcal{E}$.

In contrast, the toposes arising when using the larger sites have categories of points which contain further objects in addition to all local A-algebras; and no simple description of the theory they classify is known.

A further advantage of these parsimonious sites is that they don't require arbitrary choices of a starting set M_0 or a way of expanding M_0 to a sufficiently ample set M of schemes.

However, the parsimonious sites also have a serious disadvantage, namely that with them, the Yoneda functor is only fully faithful when restricted to $(\mathrm{Sch}/S)_{\mathrm{lfp}}$. For instance, in the case $S = \operatorname{Spec}(\mathbb{Z})$, the schemes $\operatorname{Spec}(\mathbb{Q})$ and the empty scheme have isomorphic functors of points, whereby $\operatorname{Spec}(\mathbb{Q})$ looks like the empty set from the internal point of view.[26]

In the following, we do not commit to a single one of these options for resolving the set-theoretical size issues, but rather keep any of them in mind. This approach will sometimes necessitate phrases such as "for any S-scheme T contained in the site used to define $\operatorname{Zar}(S)$", which might seem awkward to a topos-theorist when taken out of context, since the site used to construct a Grothendieck topos is not at all uniquely determined by the resulting topos.

We will indicate the few places where the choice of site makes a difference.

16. Relation between the big and little Zariski toposes

The big Zariski topos $\operatorname{Zar}(S)$ is a topos over the little Zariski topos $\operatorname{Sh}(S)$ in that there is a canonical geometric morphism

$$\pi : \operatorname{Zar}(S) \longrightarrow \operatorname{Sh}(S)$$

with direct and inverse image parts given by

$$\pi_* E = E|_{\operatorname{Sh}(S)} \qquad \text{and} \qquad \pi^{-1}\mathcal{F} = ((T \xrightarrow{f} S) \mapsto \Gamma(T, f^{-1}\mathcal{F})).$$

Since π^{-1} is fully faithful, this geometric morphism is connected; and furthermore, it is a local geometric morphism (a further right adjoint $\pi^!$ which is fully faithful exists).

By general results on local geometric morphisms, the adjoint pair $(\pi_* \dashv \pi^!)$ is a geometric morphism which is right inverse to π and which exhibits $\operatorname{Sh}(S)$ as a subtopos of $\operatorname{Zar}(S)$, similarly to how Set is a subtopos of a sheaf topos over a local topological space. In this context, it's customary to introduce notation for the idempotent monad $\sharp$ and the idempotent comonad $\flat$ arising from the adjoint triple $\pi^{-1} \dashv \pi_* \dashv \pi^!$:

$$\sharp E = \pi^!(E|_{\operatorname{Sh}(S)}) \qquad \text{and} \qquad \flat E = \pi^{-1}(E|_{\operatorname{Sh}(S)}).$$

In the case that $S = \operatorname{Spec}(A)$ is an affine scheme and we employ one of the parsimonious sites to construct $\operatorname{Zar}(S)$, it's well-known that $\operatorname{Sh}(S)$ classifies local localizations of A and that $\operatorname{Zar}(S)$ classifies arbitrary local A-algebras. On points, the morphism π sends a local A-algebra $\varphi : A \to R$ to the local localization $A \to A[(\varphi^{-1}[R^\times])^{-1}]$, and its right inverse sends a local localization $A \to A[F^{-1}]$ to itself.

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finitely presented S-scheme might not admit an open covering by finitely presented S-schemes which are affine (as absolute schemes), the toposes $\operatorname{Sh}((\mathrm{Sch}/S)_{\mathrm{fp}})$ and $\operatorname{Sh}((\mathrm{Aff}/S)_{\mathrm{fp}})$ can differ.

[26]**XXX: explicit proof**

16.1. Recovering the big Zariski topos from the little Zariski topos. What does $\mathrm{Zar}(S)$ classify in the case that S is an arbitrary scheme? We don't know a nontautologous answer to this question, but we can answer a related one: What does $\mathrm{Zar}(S)$ classify as seen from the internal point of view of $\mathrm{Sh}(S)$?

To make sense of this question, we employ a slight extension of Shulman's stacks semantics which allows to refer to locally internal categories [63] over a base topos $\mathcal{E}$ from the internal language. Using this extension, a locally internal category over $\mathcal{E}$ looks like a locally small category from the internal point of view of $\mathcal{E}$. In particular, a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ gives rise to a locally internal category (which over an object $A \in \mathcal{E}$ is given by the $\mathcal{E}/A$-enriched category $\mathcal{F}/f^{-1}A$) which will look like an ordinary topos from the internal point of view of $\mathcal{E}$.

For instance, the trivial $\mathcal{E}$-topos $\mathcal{E}$ will look like Set and the slice topos $\mathcal{E}/X$ will look like Set/X from the internal point of view of $\mathcal{E}$.

Theorem 16.1. *In the situation that the site used to construct* $\mathrm{Zar}(S)$ *is one of the parsimonious sites, the big Zariski topos* $\mathrm{Zar}(S)$ *is, from the internal point of view of* $\mathrm{Sh}(S)$*, the classifying topos of the theory of local* $\mathcal{O}_S$*-algebras which are local over* $\mathcal{O}_S$*.*

For an arbitrary topos $\mathcal{F}$ over Set, the concept of an "$\mathcal{O}_S$-algebra in $\mathcal{F}$" doesn't make any sense – in contrast to the concept of an A-algebra in $\mathcal{F}$, which can either be defined as a ring homomorphism $\underline{A} \to R$ in $\mathcal{F}$ (where $\underline{A}$ is the pullback of $A \in \mathrm{Set}$ to $\mathcal{F}$) or as a ring object which is equipped with an A-indexed family of endomorphisms satisfying suitable axioms. However, for a $\mathrm{Sh}(S)$-topos $f : \mathcal{F} \to \mathrm{Sh}(S)$, the concept of an $\mathcal{O}_S$-algebra in $\mathcal{F}$ is meaningful: It's a ring homomorphism $f^{-1}\mathcal{O}_S \to R$ in $\mathcal{F}$.

Similarly, there is no absolute "geometric theory of $\mathcal{O}_S$-algebras". However, there is a geometric theory of $\mathcal{O}_S$-algebras *internal to* $\mathrm{Sh}(S)$. Theorem 16.1 should be viewed in this light.

The proviso "local over $\mathcal{O}_S$" is as in the discussion of the relative spectrum from the internal point of view (Section 12).

Proof of Theorem 16.1. We have to verify that, from the point of view of $\mathrm{Sh}(S)$, the topos $\mathrm{Zar}(S)$ contains a canonical local and local-over-$\mathcal{O}_S$ $\mathcal{O}_S$-algebra and that for any Grothendieck topos $\mathcal{F}$, pulling back this canonical algebra yields an equivalence between the category of geometric morphisms $\mathcal{F} \to \mathrm{Zar}(S)$ and the category of local and local-over-$\mathcal{O}_S$ $\mathcal{O}_S$-algebras in $\mathcal{F}$.

The canonical local and local-over-$\mathcal{O}_S$ $\mathcal{O}_S$-algebra in $\mathrm{Zar}(S)$ is the algebra $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$. Indeed, the ring $\underline{\mathbb{A}}^1_S$ is local and the homomorphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ is local, since its restriction to any sheaf topos $\mathrm{Sh}(X)$, where $f : X \to S$ is an S-scheme contained in the site used to define $\mathrm{Zar}(S)$, is local: It's the morphism $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$.

We now want to verify the universal property, which expressed internally to $\mathrm{Sh}(S)$ reads as

$$\mathrm{Hom}(\mathcal{E}, \mathrm{Zar}(S)) \simeq \text{category of local and local-over-}\mathcal{O}_S\ \mathcal{O}_S\text{-algebras in } \mathcal{E}.$$

Externally, this means that for any open subset $U \subseteq S$ and any topos $\mathcal{E}$ over $\mathrm{Sh}(S)/\underline{U}$,

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Sh}(S)/\underline{U}}(\mathcal{E}, \mathrm{Zar}(S)/\pi^{-1}\underline{U}) \simeq \\ &\qquad \text{category of local and local-over-}\pi^{-1}\mathcal{O}_S\ \pi^{-1}\mathcal{O}_S\text{-algebras in } \mathcal{E}.\end{aligned}$$

We will verify this equivalence in the case that $S = \mathrm{Spec}(A)$ is affine and that $S = U$. This suffices to establish the theorem, since $\mathrm{Sh}(S)/\underline{U} \simeq \mathrm{Sh}(U)$, $\mathrm{Zar}(S)/\pi^{-1}\underline{U} \simeq \mathrm{Zar}(U)$, and since the internal language is local.

So let $f : \mathcal{E} \to \mathrm{Sh}(\mathrm{Spec}(A))$ be a $\mathrm{Sh}(\mathrm{Spec}(A))$-topos. By the universal property of $\mathrm{Zar}(\mathrm{Spec}(A))$ as the classifying topos of local A-algebras, a geometric morphism $g :$

$\mathcal{E} \to \mathrm{Zar}(\mathrm{Spec}(A))$ is uniquely determined by a local A-algebra $\varphi : \underline{A} \to \mathcal{B}$ in $\mathcal{E}$. By the universal property of $\mathrm{Sh}(\mathrm{Spec}(A))$ as the classifying topos of local localizations of A, the composition $\pi \circ g : \mathcal{E} \to \mathrm{Sh}(\mathrm{Spec}(A))$ is uniquely determined by the local localization $\underline{A} \to g^{-1}\pi^{-1}\mathcal{O}_{\mathrm{Spec}(A)} = g^{-1}(\flat\underline{\mathbb{A}}^1_S)$ in $\mathcal{E}$.

In the composition

$$\underline{A} \longrightarrow \flat\underline{\mathbb{A}}^1_S \longrightarrow \underline{\mathbb{A}}^1_S,$$

the first morphism is a local localization and the second morphism is local. Since these properties can be formulated as geometric implications,[27] they are preserved by the functor g^{-1}. Since furthermore such a factorization is unique, the localization $\underline{A} \to g^{-1}(\flat\underline{\mathbb{A}}^1_S)$ which determines $\pi \circ g$ coincides with the localization $\underline{A}[(\varphi^{-1}((\underline{\mathbb{A}}^1_S)^\times))^{-1}]$. Referring directly to the involved filters, the filter $g^{-1}\mathcal{F}$ which determines $\pi \circ g$ (where $\mathcal{F}$ is the generic filter of $\underline{A}$ in $\mathrm{Sh}(\mathrm{Spec}(A))$) coincides with the filter $\varphi^{-1}((\underline{\mathbb{A}}^1_S)^\times)$. This explains the first equivalence in the chain

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Sh}(\mathrm{Spec}(A))}(\mathcal{E}, \mathrm{Zar}(\mathrm{Spec}(A))) \\
\simeq\ &\text{category of local algebras } \varphi : \underline{A} \to \mathcal{B} \text{ in } \mathcal{E} \text{ such that } \varphi^{-1}\mathcal{B}^\times = f^{-1}\mathcal{F} \\
\simeq\ &\text{category of local algebras } \psi : f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{B} \text{ in } \mathcal{E} \text{ such that } \psi \text{ is local.}
\end{aligned}$$

The second equivalence maps an algebra φ to $\underline{A}[(\varphi^{-1}\mathcal{B}^\times)^{-1}] \to \mathcal{B}$; conversely, an algebra ψ is mapped to the composition $\underline{A} \to f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \xrightarrow{\psi} \mathcal{B}$. □

Similarly to how Theorem 16.1 shows how the big Zariski topos of S looks like from the point of view of $\mathrm{Sh}(S)$, it's possible to give an internal description of what the big Zariski topos of an arbitrary relative spectrum over S looks like. We state and verify such a description in Theorem 16.4.

16.2. Recovering the little Zariski topos from the big Zariski topos. Theorem 16.1 shows that $\mathrm{Zar}(S)$ can be reconstructed from $\mathrm{Sh}(S)$ (and its structure sheaf $\mathcal{O}_S$). Similarly, it's possible to reconstruct $\mathrm{Sh}(S)$ from $\mathrm{Zar}(S)$ (and the canonical morphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$).

Theorem 16.2. *In the situation that the site used to construct* $\mathrm{Zar}(S)$ *is one of the parsimonious sites, the little Zariski topos* $\mathrm{Sh}(S)$ *is the largest subtopos of* $\mathrm{Zar}(S)$ *where the canonical morphism* $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ *is an isomorphism.*

In other words, the little Zariski topos is the largest subtopos $\mathcal{E} \hookrightarrow \mathrm{Zar}(S)$ such that $\mathrm{Zar}(S) \models (\ulcorner \flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S \text{ is bijective}\urcorner)^\Box$ (where $\Box$ is the modal operator corresponding to the subtopos), that is that the pullback of the canonical morphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ to $\mathcal{E}$ is an isomorphism.

In the case that $S = \mathrm{Spec}(A)$ is affine, we also have the ring $\underline{A}$ in $\mathrm{Zar}(S)$ available. In this case the condition is equivalent to

$$\mathrm{Zar}(S) \models \ulcorner \underline{A} \to \underline{\mathbb{A}}^1_S \text{ is a localization}\urcorner^\Box,$$

since in the composition $\underline{A} \to \flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ the first morphism is a localization.

Proof of Theorem 16.2. The little Zariski topos is a subtopos of the big Zariski topos via the right inverse s of $\pi : \mathrm{Zar}(S) \to \mathrm{Sh}(S)$, the geometric morphism $(\pi_* \dashv \pi^!)$. The

[27] A ring homomorphism $\alpha : R \to T$ is a localization (that is, isomorphic to the canonical localization morphism $R \to R[S^{-1}]$ for some multiplicative subset S) if and only if the canonical comparison morphism $R[(\alpha^{-1}T^\times)^{-1}] \to T$ is bijective. This is the case if and only if

$$\forall y : T.\ \exists x : R.\ \exists s : R.\ \ulcorner \alpha(s) \text{ inv.}\urcorner \land y = \alpha(s)^{-1}x \qquad \text{and}$$
$$\forall x : R.\ \alpha(x) = 0 \Rightarrow \exists s : R.\ \ulcorner \alpha(s) \text{ inv.}\urcorner \land sx = 0.$$

pullback of $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ to $\mathrm{Sh}(S)$ is therefore the morphism $(\flat\underline{\mathbb{A}}^1_S)|_{\mathrm{Sh}(S)} \to \underline{\mathbb{A}}^1_S|_{\mathrm{Sh}(S)}$, that is $\mathcal{O}_S \to \mathcal{O}_S$, which is an isomorphism.

Let $f : \mathcal{E} \hookrightarrow \mathrm{Zar}(S)$ be any subtopos such that the pullback of $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ to $\mathcal{E}$ is an isomorphism. We want to verify that f factors over the inclusion $s : \mathrm{Sh}(S) \hookrightarrow \mathrm{Zar}(S)$.

$$\begin{tikzcd} \mathcal{E} \arrow[rr, hook, "f"] \arrow[dr, dashed] & & \mathrm{Zar}(S) \\ & \mathrm{Sh}(S) \arrow[ur, hook, "s"'] & \end{tikzcd}$$

A candidate for a morphism $\mathcal{E} \to \mathrm{Sh}(S)$ witnessing this factorization is the composition $\pi \circ f$. It remains to show that $s \circ (\pi \circ f) = f$. Both $s \circ (\pi \circ f)$ and f are morphisms of $\mathrm{Sh}(S)$-toposes, where $\mathcal{E}$ is regarded as a $\mathrm{Sh}(S)$-topos by the composition $\pi \circ f$. By the universal property of the big Zariski topos given in Theorem 16.1, they are therefore uniquely determined by the $\mathcal{O}_S$-algebra they classify.

The morphism $s \circ (\pi \circ f)$ classifies the $\mathcal{O}_S$-algebra $f^{-1}\pi^{-1}s^{-1}\underline{\mathbb{A}}^1_S = f^{-1}(\flat\underline{\mathbb{A}}^1_S)$. The morphism f classifies the $\mathcal{O}_S$-algebra $f^{-1}\underline{\mathbb{A}}^1_S$. Since $f^{-1}(\flat\underline{\mathbb{A}}^1_S) \to f^{-1}\underline{\mathbb{A}}^1_S$ is an isomorphism, these algebras coincide. □

16.3. Change of base. Let $f : X \to S$ be a morphism of schemes. In any of the situations that

(1) the parsimonious sites are used to construct the big Zariski toposes and f is locally of finite presentation, or
(2) the same (Grothendieck or partial) universe is used for constructing both Zariski toposes and both X and S are contained in the universe,

the morphism f induces an essential geometric morphism $\mathrm{Zar}(X) \to \mathrm{Zar}(S)$ which we again denote by "f". Explicitly, the big Zariski toposes are related by the adjoint triple $f_! \dashv f^{-1} \dashv f_*$ with

$$\begin{aligned} f_*F &= ((T \xrightarrow{g} S) \mapsto F(T \times_S X)), \\ f^{-1}E &= ((T \xrightarrow{g} X) \mapsto F(T \xrightarrow{g} X \xrightarrow{f} S)), \\ f_!F &= ((T \xrightarrow{g} S) \mapsto \coprod_{h:T\to X} F(T \xrightarrow{h} X)). \end{aligned}$$

In situation (2), the well-definedness of these functors is trivial. In situation (1), the well-definedness rests on the lemma that an S-morphism $h : T \to X$ is locally of finite presentation if T and X are locally of finite presentation over S [69, Tag 02FV].

The objects of $\mathrm{Zar}(S)$ listed on page 122 pull back as expected:

- Let Y be an S-scheme. Then $f^{-1}\underline{Y} = \underline{Y \times_S X}$, by the universal property of the fiber product.
- In particular, $f^{-1}\underline{\mathbb{A}}^1_S = \underline{\mathbb{A}}^1_X$, since $\mathbb{A}^1_S \times_S X = \mathbb{A}^1_X$.
- Let $\mathcal{F}$ be a sheaf of sets on S. Then $f^{-1}\pi_S^{-1}\mathcal{F} = \pi_X^{-1}f^{-1}\mathcal{F}$.
- Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules. Then $f^{-1}\mathcal{F}^{\mathrm{Zar}} = (f^*\mathcal{F})^{\mathrm{Zar}}$.

The functors $f_! \dashv f^{-1}$ induce the equivalence

$$\mathrm{Zar}(X) \simeq \mathrm{Zar}(S)/\underline{X},$$

which is very useful for the purposes of synthetic algebraic geometry. Explicitly, this equivalence is described by

$$\begin{aligned} F &\longmapsto (f_!F \to f_!1), \\ ((T \xrightarrow{g} X) \mapsto \{s \in (f^{-1}E)(T) \mid \alpha(s) = g\} &\longleftarrow (E \xrightarrow{\alpha} \underline{X}). \end{aligned}$$

From the internal point of view of $\mathrm{Zar}(S)$, the big Zariski topos of X is therefore simply $\mathrm{Set}/\underline{X}$, the category of $\underline{X}$-indexed families of sets or equivalently the category of sheaves on $\underline{X}$ considered as a *discrete* locale. This fits nicely with the philosophy "S-schemes are plain unstructured sets from the internal point of view of $\mathrm{Zar}(S)$".

A consequence for the internal language is that if $\varphi(x)$ is a formula over $\mathrm{Zar}(S)$ with a free variable $x : X$, then

$$\mathrm{Zar}(S) \models \forall x : X.\ \varphi(x) \qquad \text{iff} \qquad \mathrm{Zar}(X) \models \varphi(\Delta),$$

where $\Delta : \underline{X} \to \underline{X \times_S X}$ is the diagonal morphism.[28]

In contrast, for the little Zariski toposes, there is no similarly simple description. From the internal point of view of $\mathrm{Sh}(S)$, the topos $\mathrm{Sh}(X)$ looks like the topos of sheaves over a locale which is not discrete, and the topos $\mathrm{Zar}(X)$ doesn't even look like a topos of sheaves over an arbitrary locale (discrete or not).

Remark 16.3. Some care is needed when dealing with the modalities $\flat$ and $\sharp$, since they are not compatible with change of base. If $f : X \to S$ is a morphism of schemes, then in general $f^{-1}(\flat E) \not\cong \flat(f^{-1}E)$, since

$$\begin{aligned} f^{-1}(\flat E) &= ((T \xrightarrow{g} X) \mapsto \Gamma(T, g^{-1}f^{-1}(E|_{\mathrm{Sh}(S)}))), \quad \text{but} \\ \flat(f^{-1}E) &= ((T \xrightarrow{g} X) \mapsto \Gamma(T, g^{-1}(E|_{\mathrm{Sh}(X)}))). \end{aligned}$$

A special case in which the canonical morphism $f^{-1}(\flat E) \to \flat(f^{-1}E)$ is an isomorphism is when f is an open immersion.

16.4. The big Zariski topos of a relative spectrum.

Theorem 16.4. *Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_S$-algebra. In the situation that the parsimonious sites are used for constructing big Zariski toposes, the big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is, from the internal point of view of $\mathrm{Sh}(S)$, the classifying topos of the theory of local $\mathcal{A}$-algebras which are local over $\mathcal{O}_S$.*

Proof. The proof is similar to the proof of Theorem 16.1. Let $X = \underline{\mathrm{Spec}}_S(\mathcal{A})$ and $f : X \to S$ be the canonical morphism. The big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is a $\mathrm{Sh}(S)$-topos by the composition $\mathrm{Zar}(\underline{\mathrm{Spec}}_S(\mathcal{A})) \to \mathrm{Zar}(S) \to \mathrm{Sh}(S)$. The pullback of $\mathcal{O}_S$ along this geometric morphism is $f^{-1}(\flat\underline{\mathbb{A}}^1_S)$. A canonical $\mathcal{O}_S$-algebra in $\mathrm{Zar}(\underline{\mathrm{Spec}}_S(\mathcal{A}))$ is therefore

$$f^{-1}(\flat\underline{\mathbb{A}}^1_S) \longrightarrow \flat\underline{\mathbb{A}}^1_X \longrightarrow \underline{\mathbb{A}}^1_X.$$

This algebra is indeed local and local over $f^{-1}(\flat\underline{\mathbb{A}}^1_S)$.

For verifying the universal property, it suffices to restrict to the case that $S = \mathrm{Spec}(R)$ is affine, as in the proof of Theorem 16.1, and consider a geometric morphism $f : \mathcal{E} \to \mathrm{Sh}(S)$. In this case $\mathcal{A} = A^\sim$ and $X = \underline{\mathrm{Spec}}_S(\mathcal{A}) = \mathrm{Spec}(A)$. Let $\alpha : R \to A$ be the structure morphism of A. We then have the chain of equivalences

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{Sh}(S)}(\mathcal{E}, \mathrm{Zar}(X)) \\ &\simeq \text{cat. of local algebras } \varphi : \underline{A} \to \mathcal{B} \text{ in } \mathrm{Zar}(X) \text{ such that } \underline{\alpha}^{-1}\varphi^{-1}\mathcal{B}^\times = f^{-1}\mathcal{F} \\ &\simeq \text{cat. of local algebras } \psi : f^{-1}\mathcal{A} \to \mathcal{B} \text{ such that } f^{-1}\mathcal{O}_S \to f^{-1}\mathcal{A} \to \mathcal{B} \text{ is local.} \end{aligned}$$

The first equivalence maps a geometric morphism $g : \mathcal{E} \to \mathrm{Zar}(X)$ to $\underline{A} \to g^{-1}\underline{\mathbb{A}}^1_X$. The second equivalence acts as follows. Given an algebra $\varphi : \underline{A} \to \mathcal{B}$ such

[28]For an arbitrary object X in a topos $\mathcal{E}$, one can show that $\mathcal{E} \models \forall x : X.\ \varphi(x)$ if and only if φ holds for the *generic element* of X, which is the generalized element $\mathrm{id}_X : X \to X$ defined on stage X. This is equivalent to XXX

that $\underline{\alpha}^{-1}\varphi^{-1}\mathcal{B}^\times = f^{-1}\mathcal{F}$, we can factor $\underline{R} \to \underline{A} \to \mathcal{B}$ uniquely as a localization $\underline{R} \to C$ followed by a local homomorphism $C \to \mathcal{B}$. By the condition on filters, the localization C is isomorphic to $f^{-1}\mathcal{O}_S$. From the description $\mathcal{A} = \underline{A}[\mathcal{F}^{-1}]$ it is apparent that $\underline{A} \to \mathcal{B}$ factors over $\underline{A} \to f^{-1}\mathcal{A}$. In this way, we obtain morphisms $f^{-1}\mathcal{O}_S \to f^{-1}\mathcal{A} \to \mathcal{B}$. □

The only reason why we have supposed that $\mathcal{A}$ is quasicoherent in the statement of Theorem 16.4 is because else $\underline{\mathrm{Spec}}_S(\mathcal{A})$ might fail to be a scheme, whereby the notion "big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$" is not defined.

In fact, we propose the following definition: If $(X, \mathcal{O}_X)$ is an arbitrary locally ringed locale (or even a locally ringed topos), then the big Zariski topos of X should be the classifying $\mathrm{Sh}(X)$-topos of the theory (internal to $\mathrm{Sh}(X)$) of local $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$. The following proposition shows that this definition is consistent with Theorem 16.1 and with Theorem 16.4.

Proposition 16.5. *Let $\mathcal{A}$ be an $\mathcal{O}_S$-algebra. The following constructions, performed internally to* $\mathrm{Sh}(S)$*, yield canonically equivalent toposes:*

(1) *Constructing first the local spectrum $X := \mathrm{Spec}(\mathcal{A}|\mathcal{O}_S)$ and then, internally to $\mathrm{Sh}_{\mathrm{Sh}(S)}(X)$, the classifying topos of the theory of $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$.*
(2) *Constructing the classifying topos of the theory of $\mathcal{A}$-algebras which are local over $\mathcal{O}_S$.*

If furthermore $\mathcal{A}$ is finitely presented as an $\mathcal{O}_S$-algebra from the internal point of view of $\mathrm{Sh}(S)$*, then the following construction yields the same result as well:*

(3) *Constructing first the big Zariski topos of S as the classifying topos of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$ and then constructing, internally to that topos, the slice topos over $[\mathcal{A}^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}$.*

Proof. If S is indeed a scheme, as is supposed throughout this part, and $\mathcal{A}$ is quasicoherent, then all three constructions yield the big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ (defined using one of the parsimonious sites). For the first construction, this is by Theorem 12.9 and Theorem 16.1; for the second construction, this is by Theorem 16.4; and for the third construction, this is by Theorem 16.1, Proposition 18.4, and the description of the slice topos in Section 16.3. However, the claim also holds if $\mathcal{A}$ is not quasicoherent or if S is an arbitrary locally ringed locale, and it's instructive to see the proof in this more general situation.

We work in the internal universe of $\mathrm{Sh}(S)$. Let $\mathcal{E}$ be an arbitrary (Grothendieck) topos. Then $\mathcal{E}$-valued points of the three toposes are given by:

(1) a filter $F \subseteq \mathcal{A}$ lying over the filter of units of $\mathcal{O}_S$ together with a local $\mathcal{A}_F$-algebra R which is local over $\mathcal{A}_F$
(2) a local $\mathcal{A}$-algebra which is local over $\mathcal{O}_S$
(3) a local $\mathcal{O}_S$-algebra R which is local over $\mathcal{O}_S$ together with an element of the stalk of $[\mathcal{A}^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}$ at the point corresponding to R

In the case that $\mathcal{A}$ is finitely presented, the stalk appearing in description (3) is canonically isomorphic to the set of R-algebra homomorphisms $\mathcal{A} \otimes_{\mathcal{O}_S} R \to R$, as discussed in Lemma 6.44.

With these descriptions, the equivalence is immediate. For instance, a datum $(F \subseteq \mathcal{A}, \mathcal{A}_F \to R)$ as in description (1) gives rise to the datum $(\mathcal{O}_S \to \mathcal{A}_F \to R)$ as in description (2). Conversely, the structure morphism of a datum as in description (2) can be factored as a localization followed by a local homomorphism to yield a datum as in (1). □

17. Sheaves of rings, algebras, and modules

XXX: locally free, ...

17.1. Quasicoherence.

Definition 17.1. An R-module E is *synthetically quasicoherent* if and only if, for any finitely presented R-algebra A, the canonical R-algebra homomorphism

$$E \otimes_R A \longrightarrow [\mathrm{Spec}(A), E] = [[A, R]_{\mathrm{Alg}(R)}, E]$$

which maps a pure tensor $x \otimes f$ to $(\varphi \mapsto \varphi(f)x)$ is bijective. Here and in the following, the set $[\mathrm{Spec}(A), E]$ is the set of all maps $\mathrm{Spec}(A) \to E$, and $[A, R]_{\mathrm{Alg}(R)}$ is the set of all R-algebra homomorphisms $A \to R$.

This definition has the following interpretation. The codomain of the displayed canonical map is the set of all E-valued functions on $\mathrm{Spec}(A)$. Elements of $E \otimes_R A$ induce such functions; these induced functions can reasonably be called "algebraic". In a synthetic context, there should be no other E-valued functions as these algebraic ones, and different algebraic expressions should yield different functions. This is precisely what the postulated bijectivity expresses.

Theorem 17.2. *Let $E \in \mathrm{Zar}(S)$ be an $\underline{\mathbb{A}}^1_S$-module. If E is quasicoherent, that is of the form $(\mathcal{E}_0)^{\mathrm{Zar}}$ for some quasicoherent $\mathcal{O}_S$-module $\mathcal{E}_0$, then E is synthetically quasicoherent from the internal point of view of* $\mathrm{Zar}(S)$*. The converse holds in any of the following situations:*

1. *The site used to construct* $\mathrm{Zar}(S)$ *is one of the parsimonious sites.*
2. *The base scheme S is concentrated (quasicompact and quasiseparated) and the functor E maps directed limits of inverse systems of S-schemes with affine transition morphisms to colimits in* Set.

Proof. To verify that E is synthetically quasicoherent, we have to verify a condition for $\underline{\mathbb{A}}^1_S$-algebras A in any slice $\mathrm{Zar}(S)/\underline{T}$. If such an algebra is finitely presented from the internal point of view, then there is a covering $T = \bigcup_i T_i$ such that the each of the restrictions of the algebra to the T_i is of the form $(\mathcal{A}_0)^{\mathrm{Zar}}$ for some finitely presented $\mathcal{O}_{T_i}$-algebra $\mathcal{A}_0$. Without loss of generality, we will just assume that A itself is of the form $(\mathcal{A}_0)^{\mathrm{Zar}}$ for a finitely presented $\mathcal{O}_S$-algebra $\mathcal{A}_0$.

By Proposition 18.4, the internal expression $\mathrm{Spec}(A)$ is the functor of points of $\underline{\mathrm{Spec}}_S\, \mathcal{A}_0$. For any S-scheme $f : T \to S$ contained in the site used to define $\mathrm{Zar}(S)$, we consider the fiber product

$$\begin{tikzcd} \underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0) \arrow[r, "f'"] \arrow[d, "p'"'] & \underline{\mathrm{Spec}}_S\, \mathcal{A}_0 \arrow[d, "p"] \\ T \arrow[r, "f"'] & S. \end{tikzcd}$$

Since $\underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0) \to S$ is contained in the site (for any of our admissible sites), we may conclude using the following chain of isomorphisms:

$$\begin{aligned}
[\mathrm{Spec}(A), E](T) &\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T}, [\mathrm{Spec}(A), E]) \cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times \mathrm{Spec}(A), E) \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T \times_S \underline{\mathrm{Spec}}_S\, \mathcal{A}_0}, E) \cong E(\underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0)) \\
&\cong \Gamma(\underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0), (p')^* f^*\mathcal{E}_0) \cong \Gamma(T, (p')_*(p')^* f_*\mathcal{E}_0) \\
&\cong \Gamma(T, f^*\mathcal{E}_0 \otimes_{\mathcal{O}_T} f^*\mathcal{A}_0) \cong \Gamma(T, (\mathcal{E}_0 \otimes_{\mathcal{O}_S} \mathcal{A}_0)^{\mathrm{Zar}}) \\
&\cong \Gamma(T, (\mathcal{E}_0)^{\mathrm{Zar}} \otimes_{\underline{\mathbb{A}}^1_S} (\mathcal{A}_0)^{\mathrm{Zar}}) \cong \Gamma(T, E \otimes_{\underline{\mathbb{A}}^1_S} A).
\end{aligned}$$

The antepenultimate isomorphism is because pullback of modules in $\mathrm{Sh}(S)$ to modules in $\mathrm{Sh}(T)$ commutes with tensor product. The penultimate isomorphism is because pullback of a sheaf in $\mathrm{Sh}(S)$ to a sheaf in $\mathrm{Zar}(S)$ commutes with tensor product (Lemma **??**).

For the converse direction, we first verify that the restrictions $E|_{\mathrm{Sh}(T)}$ to the little Zariski topos of each S-scheme T contained in the site used to define $\mathrm{Zar}(S)$ are quasicoherent $\mathcal{O}_T$-modules. For this, we employ the quasicoherence criterion of Theorem 9.3: For any open affine subset $T' \subseteq T$ and any function $h \in \Gamma(T', \mathcal{O}_T)$ we verify that the canonical morphism

$$E|_{\mathrm{Sh}(T)}[h^{-1}] \longrightarrow j_*(E|_{\mathrm{Sh}(D(h))}) \tag{†}$$

is an isomorphism, where $j : D(h) \hookrightarrow T'$ denotes the inclusion. This follows from the assumption of synthetic quasicoherence by considering the $\underline{\mathbb{A}}^1_S$-module $A := \underline{\mathbb{A}}^1_S[h^{-1}]$ (in the slice $\mathrm{Zar}(S)/\underline{T'}$): This expresses that the canonical morphism

$$E \otimes_{\underline{\mathbb{A}}^1_S} \underline{\mathbb{A}}^1_S[h^{-1}] \longrightarrow [\mathrm{Spec}(A), E] \tag{‡}$$

is an isomorphism (of $\underline{\mathbb{A}}^1_S$-modules in $\mathrm{Zar}(S)/\underline{T'}$). Restricting the domain to $\mathrm{Sh}(T')$ yields the sheaf $E|_{\mathrm{Sh}(T')} \otimes_{\mathcal{O}_{T'}} \mathcal{O}_{T'}[h^{-1}]$, since restricting commutes with the geometric constructions "forming the tensor product" and "localizing away from h". Since $\mathrm{Spec}(A)$ is the functor of points of $D(h)$, restricting the codomain to $\mathrm{Sh}(T')$ yields the sheaf $j_*(E|_{\mathrm{Sh}(D(h))})$. The canonical morphism (†) which we want to recognize as an isomorphism is therefore the restriction of the canonical morphism (‡) which we know to be an isomorphism.

A natural candidate for an quasicoherent $\mathcal{O}_S$-module $\mathcal{E}_0$ with $E \cong (\mathcal{E}_0)^{\mathrm{Zar}}$ is $\mathcal{E}_0 := E|_{\mathrm{Sh}(S)}$. We'll show that this is indeed true. Let $f : T \to S$ be any S-scheme contained in the site used to define $\mathrm{Zar}(S)$. We assume for the time being that f is of finite presentation and affine, so $T \cong \underline{\mathrm{Spec}}_S\, \mathcal{A}_0$ for some finitely presented $\mathcal{O}_S$-algebra $\mathcal{A}_0$. We want to verify that the canonical morphism

$$f^*(E|_{\mathrm{Sh}(S)}) \longrightarrow E|_{\mathrm{Sh}(T)} \tag{§}$$

is an isomorphism. Since the functor f_* from quasicoherent $\mathcal{O}_T$-modules to quasicoherent $\mathcal{O}_S$-modules is fully faithful (the morphism f being affine) and domain and codomain of that morphism are quasicoherent, it suffices to verify that its image under f_* is an isomorphism. This image is the canonical morphism

$$E|_{\mathrm{Sh}(S)} \otimes_{\mathcal{O}_S} \mathcal{A}_0 \longrightarrow f_*(E|_{\mathrm{Sh}(T)}).$$

The assumption of synthetic quasicoherence, applied to the $\underline{\mathbb{A}}^1_S$-algebra $A := (\mathcal{A}_0)^{\mathrm{Zar}}$, shows that this morphism is an isomorphism.

In situation (1), the only step left to do is to generalize the argument is the previous paragraph to morphisms $f : T \to S$ which are locally of finite presentation. This works out because there are open covers of S and T such that the appropriate restrictions of f are of finite presentation and affine. The assumption of synthetic quasicoherence then needs to be applied to to $\underline{\mathbb{A}}^1_S$-algebras in suitable slices of $\mathrm{Zar}(S)$, showing that the canonical morphism (§) is locally an isomorphism and therefore globally as well.

In situation (2), we employ the technique of approximating general S-schemes by S-schemes of finite presentation. Specifically, let $f : T \to S$ be an arbitrary S-scheme contained in the site used to define $\mathrm{Zar}(S)$. Without loss of generality, we may assume that T is an affine scheme. Thus T is quasicompact and quasiseparated, and S is quasiseparated by assumption. We may therefore apply the lemma of relative approximation [69, Tag 09MV] to deduce that T is a directed limit of an inverse system of S-schemes $f_i : T_i \to S$ of finite presentation with affine transition maps. These S-schemes are contained in the site used to define $\mathrm{Zar}(S)$. Furthermore, they inherit quasicompactness and quasiseparatedness from S. Therefore we can apply a comparison result on the categories of quasicoherent modules [69, Tag 01Z0]:

$$E(T) = E(\lim_i T_i) \cong \operatorname{colim}_i E(T_i) \cong \operatorname{colim}_i \Gamma(T_i, f_i^*\mathcal{E}_0) \cong \Gamma(T, f^*\mathcal{E}_0). \qquad \square$$

Scholium 17.3. *Let $E \in \mathrm{Zar}(S)$ be a quasicoherent $\underline{\mathbb{A}}^1_S$-module. Let $A \in \mathrm{Zar}(S)$ be a quasicoherent $\underline{\mathbb{A}}^1_S$-algebra such that $\mathrm{Spec}(A) \in \mathrm{Zar}(S)$ is representable by an object of the site used to define $\mathrm{Zar}(S)$. Then the canonical morphism*

$$E \otimes_{\underline{\mathbb{A}}^1_S} A \longrightarrow [\mathrm{Spec}(A), E]$$

is an isomorphism.

Proof. The second paragraph of the proof of Theorem 17.2 applies. $\square$

Remark 17.4. The condition in Scholium 17.3 that $\mathrm{Spec}(A)$ is representable by an object of the site used to define $\mathrm{Zar}(S)$ is slightly unnatural from a topos-theoretic point of view, since the conclusion of the Scholium depends only on the topos over the site and not the site itself. In fact, the condition can be weakened and made more natural at the same time: It suffices to require that $\mathrm{Spec}(A)$ is *locally* representable by an object of the site.

However, the condition can't be dropped completely. For instance, if we employ the parsimonious sites and consider $S = \operatorname{Spec}\mathbb{Z}$, $E = \underline{\mathbb{A}}^1_S$, and $A = \mathcal{K}_S^{\mathrm{Zar}}$ (where $\mathcal{K}_S$ is the sheaf of rational functions on S, which in this case is the constant sheaf $\underline{\mathbb{Q}}$), then $\mathrm{Spec}(A)$ is the functor of points of the $\mathbb{Z}$-scheme $\mathrm{Spec}(\mathbb{Q})$. This functor coincides with the functor of points of the empty $\mathbb{Z}$-scheme on the parsimonious sites; therefore $\mathrm{Spec}(A) = \emptyset$ from the internal point of view. Thus the codomain of the canonical morphism is the zero algebra, but the domain is not.

Remark 17.5. The quotient $\underline{\mathbb{A}}^1_S/\sqrt{(0)}$ in $\mathrm{Zar}(S)$ is an example for a sheaf of modules which is not quasicoherent even though all of its restrictions to the little Zariski toposes $\mathrm{Sh}(X)$ for morphisms $f : X \to S$ are: Since taking the quotient and taking the radical of an ideal are geometric constructions, we have $(\underline{\mathbb{A}}^1_S/\sqrt{(0)})|_{\mathrm{Sh}(X)} \cong \mathcal{O}_X/\sqrt{(0)}$. These sheaves of modules are quasicoherent (Example 9.7). However, in general, $f^*(\mathcal{O}_S/\sqrt{(0)}) \ncong \mathcal{O}_X/\sqrt{(0)}$. A concrete counterexample is $S = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(k[T]/(T^2))$. In this case $f^*(\mathcal{O}_S/\sqrt{(0)}) \cong f^*(\mathcal{O}_S) \cong \mathcal{O}_X$.

17.2. Special properties of the affine line. The ring object $\underline{\mathbb{A}}^1_S$ in the big Zariski topos enjoys several special properties, some of which are unique in that they're only possible in an intuitionistic context. We compile here a short list of such properties. As was already mentioned, at least one of them, the field property, was already noticed in the 1970s by Kock [50].

The statements and proofs in this subsection are formulated in the internal language. The proofs only use the fact that $\underline{\mathbb{A}}^1_S$ is a synthetically quasicoherent local ring. This illustrates the meta-claim that synthetic quasicoherence is a strong and meaningful condition.

Proposition 17.6. *$\underline{\mathbb{A}}^1_S$ is a field in the sense that any element which is not zero is invertible: $\forall x : \underline{\mathbb{A}}^1_S.\ \neg(x = 0) \Rightarrow \ulcorner x \text{ inv.}\urcorner$. More generally, for any number $n \geq 0$,*

$$\forall x_1, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \neg(x_1 = 0 \wedge \cdots \wedge x_n = 0) \Longrightarrow (\ulcorner x_1 \text{ inv.}\urcorner \vee \cdots \vee \ulcorner x_n \text{ inv.}\urcorner).$$

Proof. Let $x : \underline{\mathbb{A}}^1_S$ be such that $\neg(x = 0)$. We consider the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S/(x)$. Since $\mathrm{Spec}(A) \cong [\![x = 0]\!] = [\![\bot]\!] = \emptyset$, the condition says that the canonical homomorphism

$$\underline{\mathbb{A}}^1_S/(x) \longrightarrow [\emptyset, \underline{\mathbb{A}}^1_S]$$

is an isomorphism. Since its codomain is the zero algebra, so is $\underline{\mathbb{A}}^1_S/(x)$. Therefore $1 \in (x)$, that is, x is invertible.

The more general statement follows in the same way, by using the quasicoherence condition for $A := \underline{\mathbb{A}}^1_S/(x_1, \ldots, x_n)$. This yields $1 \in (x_1, \ldots, x_n)$. Since $\underline{\mathbb{A}}^1_S$ is a local ring, one of the x_i is invertible. $\square$

Proposition 17.7. *$\underline{\mathbb{A}}^1_S$ is not a reduced ring:* $\neg\Big(\forall x : \underline{\mathbb{A}}^1_S.\ (\bigvee_{n\geq 0} x^n = 0) \Rightarrow x = 0\Big)$.

Proof. Assume that $\underline{\mathbb{A}}^1_S$ is reduced. Then the set $\Delta := \{\varepsilon \in \underline{\mathbb{A}}^1_S \mid \varepsilon^2 = 0\}$ is equal to $\{0\}$. By the quasicoherence criterion applied to the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[T]/(T^2)$, the canonical map

$$\underline{\mathbb{A}}^1_S[T]/(T^2) \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S[T]/(T^2)), \underline{\mathbb{A}}^1_S] \cong [\Delta, \underline{\mathbb{A}}^1_S] \cong \underline{\mathbb{A}}^1_S$$

is an isomorphism. It maps $[T]$ to zero (the value of T at $0 \in \Delta$). Thus $T \in (T^2)$ and therefore $1 = 0$ in $\underline{\mathbb{A}}^1_S$. This is a contradiction. $\square$

In classical logic, Proposition 17.6 and Proposition 17.7 would directly contradict each other; only an intuitionistic context allows for fields which are not reduced.

That $\underline{\mathbb{A}}^1_S$ is not reduced, irrespective of the reducedness of the base scheme S, should not come as a surprise: Reducedness is not stable under base change, but all statements of the internal language of $\mathrm{Zar}(S)$ are. If $\underline{\mathbb{A}}^1_S$ was reduced, then all S-schemes (at least those contained in the site used to construct $\mathrm{Zar}(S)$) would be reduced as well. In contrast, the structure sheaf $\mathcal{O}_S$ is reduced from the point of view of the little Zariski topos if and only if S is reduced (Proposition 3.3).

Proposition 17.8. *The following statements about an element $x : \underline{\mathbb{A}}^1_S$ are equivalent:*

(1) *x is not invertible.*
(2) *x is nilpotent.*
(3) *x is* not not *zero.*

Proof. Let $x : \underline{\mathbb{A}}^1_S$ be not invertible. We consider the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[x^{-1}]$. Since $\mathrm{Spec}(A) \cong [\![\ulcorner x \text{ inv.}\urcorner]\!] = \emptyset$, it follows that $\underline{\mathbb{A}}^1_S[x^{-1}] = 0$, similarly to the proof of Proposition 17.6. Thus x is nilpotent.

Let $x : \underline{\mathbb{A}}^1_S$ be a nilpotent element. Thus $x^n = 0$ for some number $n \geq 0$. If x was nonzero, then x and therefore x^n would be invertible, in contradiction to $0 \neq 1$ since $\underline{\mathbb{A}}^1_S$ is a local ring.

Let $x : \underline{\mathbb{A}}^1_S$ be *not not* zero. Then x is not invertible, since if x was invertible, then x would be nonzero. $\square$

Summarizing, the following facts about nilpotents hold in the internal universe of the big Zariski topos. Firstly, it's not true that $\underline{\mathbb{A}}^1_S$ is reduced. But this doesn't mean that there actually exists a nilpotent element which is not zero. In fact, any nilpotent is *not not* zero.

Proposition 17.9. *Any function $\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ is given by a unique polynomial in $\underline{\mathbb{A}}^1_S[T]$.*

Proof. Immediate by considering the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[T]$ and noticing that $\mathrm{Spec}(A) \cong \underline{\mathbb{A}}^1_S$. $\square$

This statement too cannot be satisfied in classical logic: for infinite fields the existence part fails and for finite fields the uniqueness part fails.

Proposition 17.10. *$\underline{\mathbb{A}}^1_S$ is* weakly algebraically closed*, in the following sense: Any monic polynomial $p : \underline{\mathbb{A}}^1_S[T]$ of degree at least one does* not not *have a zero.*

Proof. Let $p : \underline{\mathbb{A}}^1_S[T]$ be a monic polynomial of degree at least one. Assume that p doesn't have a zero in $\underline{\mathbb{A}}^1_S$. Then the spectrum of $A := \underline{\mathbb{A}}^1_S[T]/(p)$ is empty. The quasicoherence condition for A therefore implies that $\underline{\mathbb{A}}^1_S[T]/(p)$ is zero. This means that p is invertible in $\underline{\mathbb{A}}^1_S[T]$. A basic lemma in commutative algebra (whose standard proof is constructive) then implies that with the exception of the constant term in p, all coefficients are nilpotent. This contradicts the assumption that p is monic of degree at least one. □

Proposition 17.11. *$\underline{\mathbb{A}}^1_S$ is infinite in the following sense: For any number $n \geq 0$ and any given elements $x_1, \ldots, x_n : \underline{\mathbb{A}}^1_S$, there is* not not *an element y which is distinct from all of the x_i.*

Proof. The polynomial $f(T) := (T - x_1) \cdots (T - x_n) + 1$ does *not not* have a zero $y : \underline{\mathbb{A}}^1_S$, since $\underline{\mathbb{A}}^1_S$ is weakly algebraically closed. This element cannot equal any x_i, since $f(x_i) = 1$ is not zero. □

Proposition 17.12. *$\underline{\mathbb{A}}^1_S$ fulfills a version of the Nullstellensatz: Let $f_1, \ldots, f_m \in \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]$ be polynomials without a common zero in $(\underline{\mathbb{A}}^1_S)^n$. Then there are polynomials $g_1, \ldots, g_m \in \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]$ such that $\sum_i g_i f_i = 1$.*

Proof. We consider the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$. Since $\mathrm{Spec}(A) \cong \{x \in (\underline{\mathbb{A}}^1_S)^n \mid f_1(x) = \ldots = f_m(x) = 0\} = \emptyset$, the condition implies that A is the zero algebra just as in the verification of Proposition 17.6. □

18. Basic constructions of relative scheme theory

With $\underline{\mathbb{A}}^1_S$ at hand, we can perform many of the usual constructions of (relative) scheme theory internally.

Group schemes. The functors associated to the standard group schemes $\mathbb{G}_\mathrm{a}$, $\mathbb{G}_\mathrm{m}$, GL_n, and μ_n are given by the internal expressions

$$\begin{aligned}
\mathbb{G}_\mathrm{a} &:= \underline{\mathbb{A}}^1_S \text{ (as an additive group)}, \\
\mathbb{G}_\mathrm{m} &:= \{x : \underline{\mathbb{A}}^1_S \mid \ulcorner x \text{ inv.} \urcorner\}, \\
\mathrm{GL}_n &:= \{M : (\underline{\mathbb{A}}^1_S)^{n \times n} \mid \ulcorner M \text{ inv.} \urcorner\}, \\
\mu_n &:= \{x : \underline{\mathbb{A}}^1_S \mid x^n = 1\}.
\end{aligned}$$

Affine and projective space. Affine n-space over S is given by $(\underline{\mathbb{A}}^1_S)^n$, i. e. internally the set of n-tuples of elements of $\underline{\mathbb{A}}^1_S$. The functor of points of projective n-space over X, with all its nontrivial topological and ring-theoretical structure, is described by the astoundingly naive expression

$$\mathbb{P}^n := \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1_S)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/\sim,$$

where the equivalence relation is the usual rescaling relation from the internal point of view. This example was suggested by Zhen Lin Low (private communication).

More generally, for an S-scheme X, affine and projective n-space over X are given by $\underline{X} \times (\underline{\mathbb{A}}^1_S)^n$ and $\underline{X} \times \mathbb{P}^n$, respectively.

18.1. Tangent bundle. For an S-scheme X, the internal Hom $[\Delta, \underline{X}] \in \mathrm{Zar}(S)$ describes the tangent bundle of X, i. e. the S-scheme $\underline{\mathrm{Spec}}_X \, \mathrm{Sym}(\Omega^1_{X/S}) \to X \to S$, as can be seen by chasing the definitions [21, Lemma 5.12.1]. Intuitively, a map $f : \Delta \to \underline{X}$ from the internal point of view is given by slightly more data than merely the point $f(0)$; one also has to specify first-order information.

This description of the (not necessarily locally trivial) tangent bundle fits nicely with the intuition of tangent vectors as infinitesimal curves, and in fact is precisely the definition of the tangent bundle in synthetic differential geometry [49, Def. 7.1].

18.2. Relative spectrum.

Definition 18.1. The *synthetic spectrum* of an R-algebra A is

$$\mathrm{Spec}(A) := [A, R]_{\mathrm{Alg}(R)},$$

the set of R-algebra homomorphisms from A to R.

Example 18.2. The synthetic spectrum of R is the one-element set. More generally, the synthetic spectrum of the algebra $R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ is the solution set $\{x : R^n \mid f_1(x) = \cdots = f_n(x) = 0\}$.

Example 18.3. The synthetic spectrum of $R/(f)$ is $[\![f = 0]\!]$, the truth value of the formula "$f = 0$", the subsingleton set $\{\star \mid f = 0\}$. If classical logic is available, then this set contains $\star$ or is empty, depending on whether f is zero or not. Similarly, the synthetic spectrum of $R[f^{-1}]$ is $[\![\ulcorner f \text{ inv.}\urcorner]\!]$.

Proposition 18.4. *Let $\mathcal{A}_0$ be an $\mathcal{O}_S$-algebra (not necessarily quasicoherent). Then the synthetic spectrum of the $\underline{\mathbb{A}}^1_S$-algebra $(\mathcal{A}_0)^{\mathrm{Zar}}$, as constructed in the internal language of* $\mathrm{Zar}(S)$*, is the functor of points of* $\underline{\mathrm{Spec}}_S\, \mathcal{A}_0$.

Proof. The Hom set occuring in the definition of the synthetic spectrum is interpreted by the internal Hom when using the internal language. For any S-scheme $f : T \to S$ contained in the site used to define $\mathrm{Zar}(S)$, we have the following chain of isomorphisms.

$$\begin{aligned}
(\mathrm{Spec}(A))(T) &= [(\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(T) \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T}, [(\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}) \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times (\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S)_{\ldots} \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)/\underline{T}}(\underline{T} \times (\mathcal{A}_0)^{\mathrm{Zar}}, \underline{T} \times \underline{\mathbb{A}}^1_S)_{\ldots} \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Zar}(T)}(\underline{\mathbb{A}}^1_T)}((f^*\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_T) \\
&= \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Zar}(T)}(\underline{\mathbb{A}}^1_T)}(\pi^{-1}(f^*\mathcal{A}_0) \otimes_{\pi^{-1}\mathcal{O}_T} \underline{\mathbb{A}}^1_T, \underline{\mathbb{A}}^1_T) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Zar}(T)}(\pi^{-1}\mathcal{O}_T)}(\pi^{-1}(f^*\mathcal{A}_0), \underline{\mathbb{A}}^1_T) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Sh}(T)}(\mathcal{O}_T)}(f^*\mathcal{A}_0, \mathcal{O}_T) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Sh}(S)}(\mathcal{O}_S)}(\mathcal{A}_0, f_*\mathcal{O}_T) \\
&\cong \mathrm{Hom}_S(T, \underline{\mathrm{Spec}}_S\, \mathcal{A}_0).
\end{aligned}$$

The omitted subscripts "..." should denote that we're only taking the subset of the Hom set where, for each fixed first argument, the morphisms are morphisms of $\underline{\mathbb{A}}^1_S$-algebras. ☐

If $X \in \mathrm{Zar}(S)$ is an arbitrary object, there is a canonical morphism $X \to \mathrm{Spec}([X, \underline{\mathbb{A}}^1_S])$. In the internal language of $\mathrm{Zar}(X)$ it looks like the "inclusion into the double dual":

$$x \longmapsto \_\_(x), \quad \text{where } \_\_(x) : [X, \underline{\mathbb{A}}^1_S] \to \underline{\mathbb{A}}^1_S,\ \varphi \mapsto \varphi(x).$$

The following proposition shows that bijectivity of this map is related to X being the functor of points of an affine S-scheme (an S-scheme whose structure morphism to S is affine).

Proposition 18.5. *A sheaf $X \in \mathrm{Zar}(S)$ is isomorphic to the functor of points of an affine S-scheme if, in the internal language of* $\mathrm{Zar}(S)$*, the $\underline{\mathbb{A}}^1_S$-algebra $[X, \underline{\mathbb{A}}^1_S]$ is quasicoherent and the canonical map $X \to \mathrm{Spec}([X, \underline{\mathbb{A}}^1_S])$ is bijective. The converse holds in any of the following situations:*

(1) *The affine S-scheme which X represents is of finite presentation over S.*

(2) *The site used to define* $\mathrm{Zar}(S)$ *is defined using a partial universe and the affine* S*-scheme which* X *represents if of finite type over* S*.*
(3) *The affine* S*-scheme which* X *represents is contained in the site used to define* $\mathrm{Zar}(S)$*. (This situation subsumes the previous ones.)*

Proof. The "if" direction is straightforward, since the assumption expresses X as the functor of points of the relative spectrum of a quasicoherent $\mathcal{O}_S$-algebra.

For the "only if" direction, we abuse notation and denote the given affine S-scheme whose functor of points is X by "$f : X \to S$". Then $f_*\mathcal{O}_X$ is quasicoherent and the canonical morphism $X \to \underline{\mathrm{Spec}}_S\, f_*\mathcal{O}_X$ is an isomorphism. In any of the listed situations, the internal Hom $[X, \underline{\mathbb{A}}^1_S]$ is canonically isomorphic to $(f_*\mathcal{O}_X)^{\mathrm{Zar}}$, since for any object $T \xrightarrow{g} S$ of the site used to define $\mathrm{Zar}(S)$ we have that

$$\begin{aligned}[X, \underline{\mathbb{A}}^1_S](T) &\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T}, [X, \underline{\mathbb{A}}^1_S]) \cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times X, \underline{\mathbb{A}}^1_S) \\ &\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times \underline{X}, \underline{\mathbb{A}}^1_S) \cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T \times_S X}, \underline{\mathbb{A}}^1_S) \\ &\cong \underline{\mathbb{A}}^1_S(T \times_S X) \cong (g^* f_*\mathcal{O}_X)(T) = (f_*\mathcal{O}_X)^{\mathrm{Zar}}(T).\end{aligned}$$

Therefore $[X, \underline{\mathbb{A}}^1_S]$ is quasicoherent. The map induced by the isomorphism $X \to \underline{\mathrm{Spec}}_S\, f_*\mathcal{O}_X$ on the level of functors of points is precisely the canonical map $X \to \mathrm{Spec}([X, \underline{\mathbb{A}}^1_S])$ as defined in the internal language; therefore this map is bijective from the internal point of view. $\square$

Remark 18.6. As noted in Remark 17.4 in a slightly different context, Condition (3) in the previous proposition is unnatural from a topos-theoretic point of view and should be weakened to require only local representability.

Remark 18.7. Let $\mathcal{A}_0$ be an $\mathcal{O}_S$-algebra. Then one can form, internally to $\mathrm{Zar}(S)$, two locales related to $\mathcal{A}_0$: the discrete locale on the synthetic spectrum of $(\mathcal{A}_0)^{\mathrm{Zar}}$, and the local spectrum of $(\mathcal{A}_0)^{\mathrm{Zar}}$ over $\underline{\mathbb{A}}^1_S$ as described in Definition 12.8. These locales don't coincide. In fact, the pullback of a discrete locale is discrete, whereas the pullback of the local spectrum to any of the little Zariski toposes $\mathrm{Sh}(X)$, where $f : X \to S$ is an S-scheme contained in the site used to define $\mathrm{Zar}(S)$, is the relative spectrum $\underline{\mathrm{Spec}}_X(f^*\mathcal{A}_0)$, which is typically not discrete as an X-locale. (This is because the locale spectrum construction is geometric, by Proposition 12.22.)

There is, however, a comparison morphism from the discrete locale on the synthetic spectrum to the local spectrum. On points, it sends an $\underline{\mathbb{A}}^1_S$-algebra homomorphism $\varphi : (\mathcal{A}_0)^{\mathrm{Zar}} \to \underline{\mathbb{A}}^1_S$ to the filter $\varphi^{-1}[(\underline{\mathbb{A}}^1_S)^\times]$.

One can also form, internally to $\mathrm{Zar}(S)$, the classifying topos of $(\mathcal{A}_0)^{\mathrm{Zar}}$-algebras which are local over $\underline{\mathbb{A}}^1_S$. This topos doesn't coincide with the (toposes of sheaves over) the mentioned two locales, either. The pullback of that classifying topos to any of the $\mathrm{Sh}(X)$ is the big Zariski topos of $\underline{\mathrm{Spec}}_X(f^*\mathcal{A}_0)$ (built using one of the parsimonious sites).

18.3. Relative Proj construction.

Definition 18.8. The *synthetic Proj* of a graded R-algebra A is the set

$$\mathrm{Proj}(A) := (\text{set of all surj. graded } R\text{-algebra homomorphisms } A \to R[T])/R^\times.$$

Example 18.9. The synthetic Proj of $R[X_0, \ldots, X_n]$ is canonically isomorphic to the set of n-tuples of homogeneous coordinates $[x_0 : \cdots : x_n]$ with at least one coordinate being invertible.

Proposition 18.10. *Let* $\mathcal{A}_0$ *be a graded* $\mathcal{O}_S$*-algebra (not necessarily quasicoherent). Then the synthetic Proj of the* $\underline{\mathbb{A}}^1_S$*-algebra* $(\mathcal{A}_0)^{\mathrm{Zar}}$*, as constructed in the internal language of* $\mathrm{Zar}(S)$*, is the functor of points of* $\underline{\mathrm{Proj}}_S\, \mathcal{A}_0$*.*

XXX: proof

18.4. Open immersions. A basic concept in the functor-of-points approach to algebraic geometry is the concept of an *open subfunctor*. It is used to delimit schemes from more general kinds of spaces: A functor is deemed to be a scheme if and only if it admits a covering by open subfunctors which are representable.

The following definition is phrased in such a way as to apply to any of the several ways to define the big Zariski topos $\mathrm{Zar}(S)$. In particular, it applies to the definition using the site consisting of affine schemes which are locally of finite presentation over S. If S is affine, the definition only refers to affine schemes and open subschemes of affine schemes and is therefore suitable if one wants to found the theory of schemes using the functorial approach.

Definition 18.11 ([34, Définition I.1.3.6 on page 10], [69, Tag 01JI])**.** A subfunctor $U \hookrightarrow X$ in $\mathrm{Zar}(S)$ is an *open subfunctor* if and only if for any object $(T \to S)$ of the site used to define $\mathrm{Zar}(S)$ and any $x \in X(T)$ there exists an open subscheme $T_0 \subseteq T$ such that for any object $(T' \xrightarrow{f} T \to S)$ of the site used to define $\mathrm{Zar}(S)$ the map $T' \to T$ factors over T_0 if and only if $X(f)(x) \in U(T')$.

The open subschemes $T_0 \subseteq T$ appearing in this definition are uniquely determined by their universal property. The relation of open subfunctors to open immersions is as follows.

Proposition 18.12. *Let X be an S-scheme.*

1. *Let $U \subseteq X$ be an open subscheme. Then the subfunctor $\underline{U} \hookrightarrow \underline{X}$ is open.*
2. *If $\underline{X}$ is locally representable by an object of the site used to define* $\mathrm{Zar}(S)$*, any open subfunctor $U \hookrightarrow \underline{X}$ is isomorphic to the open subfunctor associated to an open subscheme of X.*

Proof. For the first claim, let $(T \to S)$ be an object of the site used to define $\mathrm{Zar}(S)$ and let $x \in \underline{X}(T)$. The open subscheme $T_0 \subseteq T$ required by the definition of an open subfunctor can then be chosen as $x^{-1}[U]$.

For the second claim, assuming for notational simplicity that $\underline{X}$ is directly representable without having to pass to a cover, the desired open subscheme of X can be obtained as the witnessing subscheme "T_0" as it appears in the definition of an open subfunctor in the special case $(T \to S) := (X \to S)$. □

From the point of view of the internal language of $\mathrm{Zar}(S)$, a subfunctor $U \hookrightarrow X$ looks like the inclusion of a subset. The natural question how one can characterize those inclusions which externally correspond to open subfunctors is answered as follows.

Definition 18.13. In the context of a specified local ring, as for instance $\underline{\mathbb{A}}^1_S$ of the big Zariski topos of a scheme, a truth value $U \subseteq 1$ is *open* if and only if there exists an ideal $J \subseteq \underline{\mathbb{A}}^1_S$ such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent (Definition 17.1) and such that U holds if and only if $1 \in J$. (Section 6.1 contains generalities on truth values.)

Example 18.14. Let $f : \underline{\mathbb{A}}^1_S$. Then "$f \neq 0$" is an open truth value with witnessing ideal $J = (f)$. The quotient $\underline{\mathbb{A}}^1_S/J$ is indeed synthetically quasicoherent, since it is finitely presented. More generally, let $f_1, \ldots, f_n : \underline{\mathbb{A}}^1_S$. Then "$f_1 \neq 0 \vee \cdots \vee f_n \neq 0$" is an open truth value with witnessing ideal $J = (f_1, \ldots, f_n)$.

Definition 18.15. In the context of a specified local ring, a map $U \to X$ of sets is a *synthetic open immersion* if and only if it is injective and for any $x : X$ the truth value of "the fiber of x is inhabited" is open.

Example 18.16. The inclusion $(\underline{\mathbb{A}}^1_S)^\times \hookrightarrow \underline{\mathbb{A}}^1_S$ of the invertible elements is a synthetic open immersion, since for $x : \underline{\mathbb{A}}^1_S$ the truth value of "the fiber of x is inhabited" equals the truth value of "$x \neq 0$".

Example 18.17. Let $X \in \mathrm{Zar}(S)$. Let $f : [X, \underline{\mathbb{A}}^1_S]$ from the internal point of view. The inclusion $\{x : X \mid f(x) \neq 0\} \hookrightarrow X$ is a synthetic open immersion.

Proposition 18.18. *Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. A subfunctor $U \hookrightarrow X$ is open if and only if, from the internal point of view of $\mathrm{Zar}(S)$, the map $U \hookrightarrow X$ is a synthetic open immersion. In formal language:*

$$\begin{aligned}&\mathrm{Zar}(S) \models \forall x : X.\ \exists J \subseteq \underline{\mathbb{A}}^1_S. \\ &\qquad \ulcorner J \text{ is an ideal}\urcorner \wedge \ulcorner \underline{\mathbb{A}}^1_S / J \text{ is synth. quasicoherent}\urcorner \wedge (x \in U \Leftrightarrow 1 \in J).\end{aligned}$$

We postpone the proof of this proposition in order to give a bit of context first.

Firstly, the displayed condition is only meaningful in an intuitionistic context as provided by the big Zariski topos. In classical logic, the condition would be trivially satisfied for any subfunctor $U \hookrightarrow X$: Classically, we have $(x \in U) \vee (x \notin U)$. If $x \in U$, we can pick $J = (1)$, and if $x \notin U$, we can pick $J = (0)$ (whereby the quotient $\underline{\mathbb{A}}^1_S / J$ is isomorphic to $\underline{\mathbb{A}}^1_S$, thus finitely presented and therefore in particular synthetically quasicoherent).[29]

The proposition is often used in the following weakened form.

Corollary 18.19. *Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. Let $U \hookrightarrow X$ be a subfunctor. If*

$$\mathrm{Zar}(S) \models \forall x : X.\ \bigvee_{n \geq 0} \exists f_1, \ldots, f_n : \underline{\mathbb{A}}^1_S.\ (x \in U \Leftrightarrow \bigvee_i \ulcorner f_i \text{ inv.}\urcorner),$$

then the subfunctor is open.

Proof. We show that the assumption implies the displayed condition of Proposition 18.18 in the internal language. Given elements $f_1, \ldots, f_n$ as in the assumption, we construct the ideal $J := (f_1, \ldots, f_n) \subseteq \underline{\mathbb{A}}^1_S$. The quotient $\underline{\mathbb{A}}^1_S / J$ is indeed synthetically quasicoherent, since it is finitely presented, and the statement that $1 \in J$ is equivalent to one of the f_i being invertible by locality of $\underline{\mathbb{A}}^1_S$. □

The internal condition appearing in the corollary reflects basic intuition about openness in algebraic geometry: Intuitively, a subset is open if it is given by inequations, so that to decide whether a point belongs to the subset one has to check that at least one of some numbers is not zero.

Of course, in classical scheme theory, one would put some condition on these numbers in order not to trivialize the notion. For instance, one would require that these numbers depend continuously on the point in some sense or, more specifically, that these numbers are given by evaluating certain locally defined regular functions at the point.

On first sight, such a condition seems to be lacking in Corollary 18.19. However, it's implicitly built into the language, since by the Kripke–Joyal semantics the external meaning of "$\exists f : \underline{\mathbb{A}}^1_S$" is that there exist, locally on an open cover, suitable elements of $\underline{\mathbb{A}}^1_S(T)$, that is regular functions on T.

The notion of open truth values is not unique to our account of synthetic algebraic geometry. Rather, it's a concept in the established and more general framework of synthetic topology [35, 53] which aims to do topology in a synthetic fashion: Any set should have an intrinsic topology and any map should be automatically continuous with respect to this intrinsic topology.

This automatic continuity reflects as stability of open subfunctors under pullbacks:

[29] Strictly speaking, incompability with classical logical surfaces even earlier: in our synthetic quasicoherence condition. The map $E \otimes_{\underline{\mathbb{A}}^1_S} A \to [\mathrm{Spec}(A), E]$ which the condition demands to be bijective has hardly any chance to be surjective if the law of excluded middle is available to define maps $\mathrm{Spec}(A) \to E$ by case distinction.

Lemma 18.20. *Let $f : X \to Y$ be a morphism in* $\mathrm{Zar}(S)$*. Let $U \hookrightarrow Y$ be an open subfunctor. Then its pullback along f, denoted "$f^{-1}U \hookrightarrow X$", is too an open subfunctor.*

Proof. From the internal point of view of $\mathrm{Zar}(S)$, the subfunctor $f^{-1}U \hookrightarrow X$ looks like the inclusion of the preimage $f^{-1}[U] \subseteq X$.

So, to verify the claim, let internally an element $x : X$ be given. We are to show that the truth value of "$x \in f^{-1}[U]$" is open. This truth value equals the truth value of "$f(x) \in U$" which is open by assumption, and is therefore open. □

Remark 18.21. In the internal language of toposes used to carry out synthetic differential geometry, there is the concept of an *Penon-open* subset [64, Chapitre III]: A subset $U \subseteq X$ is Penon-open if and only if

$$\forall x \in U.\ \forall y : X.\ (x \neq y) \vee (y \in U).$$

This notion is not useful in synthetic algebraic geometry, since it is much too weak: Any subset of the one-element set 1 is Penon-open. However, not every subfunctor of the terminal functor in $\mathrm{Zar}(S)$ is an open subfunctor.

In many flavours of synthetic topology, open truth values φ are $\neg\neg$-stable in that $\neg\neg\varphi$ implies φ. With a small caveat, this is true for open truth values in the big Zariski topos as well.

Proposition 18.22. *Let $U \hookrightarrow 1$ be a subfunctor in* $\mathrm{Zar}(S)$ *such that* $\mathrm{Zar}(S) \models \neg\neg U$*. Then in any of the following situations it follows that* $\mathrm{Zar}(S) \models U$*:*

(1) *U is a quasicompact open truth value.*
(2) *U is an arbitrary open truth value and the site used to define* $\mathrm{Zar}(S)$ *is closed under domains of closed immersions.*

Proof. We give two proofs, an internal one and an external one, since they employ different ideas.

Internal proof. Since U is an open truth value, there exists an ideal $J \subseteq \underline{\mathbb{A}}^1_S$ such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent and such that U holds if and only if $1 \in J$. By assumption, the element 1 is *not not* an element of J; we want to verify that it's actually an element of J.

By Scholium 17.3, the canonical homomorphism

$$\underline{\mathbb{A}}^1_S/J \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S/J), \underline{\mathbb{A}}^1_S]$$

is bijective; the assumptions of that scholium are satisfied in either of the two situations. The set $\mathrm{Spec}(\underline{\mathbb{A}}^1_S/J)$ is isomorphic to $[\![J = (0)]\!]$. Since $\neg\neg(1 \in J)$, we also have $\neg(J = (0))$. Therefore $\mathrm{Spec}(\underline{\mathbb{A}}^1_S/J)$ is empty and the codomain of the displayed isomorphism is the zero algebra. Thus $\underline{\mathbb{A}}^1_S/J$ is trivial as well, showing $1 \in J$.

External proof. Since $U \hookrightarrow 1$ is an open subfunctor, there is an open subscheme $S_0 \subseteq S$ such that a morphism $f : T \to S$ factors over S_0 if and only if $U(T)$ is inhabited. In both situations it's possible to endow $X := S \subseteq S_0$ with the structure of a closed subscheme such that X is contained in the site used to define $\mathrm{Zar}(S)$. By the universal property of S_0, we have $X \models \neg U$. Since $\mathrm{Zar}(S) \models \neg\neg U$, it follows that X is empty. Therefore $S_0 = S$ and U is globally inhabited. □

Corollary 18.23. *Let $\gamma : \Delta \to X$ be a morphism in* $\mathrm{Zar}(S)$*. Let $U \hookrightarrow X$ be an open subfunctor such that* $\mathrm{Zar}(S) \models \gamma(0) \in U$*. Then, in any of the situations in Proposition 18.22, the morphism γ factors over U.*

Proof. We give an internal proof. Let $\varepsilon \in \Delta$. Then $\neg\neg(\varepsilon = 0)$. Therefore $\neg\neg(\gamma(\varepsilon) \in U)$. Since being an element of U is $\neg\neg$-stable, it follows that $\gamma(\varepsilon) \in U$. □

In ordinary scheme theory, an inclusion of a standard open subset $D(f) \hookrightarrow X$ is isomorphic to the structure morphism of the relative spectrum $\underline{\mathrm{Spec}}_X\, \mathcal{O}_X[f^{-1}]$. Inclusions of more general open subsets can typically not be described using the relative spectrum construction, the standard example being the inclusion $\mathbb{A}^2_k \setminus \{0\} \hookrightarrow \mathbb{A}^2_k$ whose domain is not affine.

An interesting feature of the internal universe of the big Zariski topos is that it's flexible enough to express *any* open subset as a spectrum. The contradiction is only apparent since the algebra used for constructing such a spectrum is not in general quasicoherent.

Proposition 18.24. *Let $U \hookrightarrow 1$ be an open truth value. In any of the situations of Proposition 18.22, there is a (not necessarily quasicoherent) $\underline{\mathbb{A}}^1_S$-algebra A such that the inclusion is isomorphic to the morphism* $\mathrm{Spec}(A) \to 1$.

Proof. The open truth value U is given by an ideal $J \subseteq \underline{\mathbb{A}}^1_S$ such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent and such that U holds if and only if $1 \in J$. We set $A := \underline{\mathbb{A}}^1_S[M^{-1}]$, where M is the multiplicatively closed subset

$$M := \{f : \underline{\mathbb{A}}^1_S \mid 1 \in J \Rightarrow \ulcorner f \text{ inv.}\urcorner\} \subseteq \underline{\mathbb{A}}^1_S.$$

The spectrum of A is inhabited if and only if $M \subseteq (\underline{\mathbb{A}}^1_S)^\times$, in which case the unique element of $\mathrm{Spec}(A)$ is the inverse of the localization morphism $\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S[M^{-1}]$. Thus $\mathrm{Spec}(A)$ is isomorphic to $[\![M \subseteq (\underline{\mathbb{A}}^1_S)^\times]\!]$. Therefore we have to verify that U holds if and only if $M \subseteq (\underline{\mathbb{A}}^1_S)^\times$.

The "only if" direction is trivial.

For the "if" direction, we exploit the $\neg\neg$-stability of U. If $\neg U$, then $\neg(1 \in J)$, so $M = \underline{\mathbb{A}}^1_S$, and since $M \subseteq (\underline{\mathbb{A}}^1_S)^\times$ by assumption, it follows that zero is invertible. This is a contradiction. Thus $\neg\neg U$. ☐

Proof of Proposition 18.18. **XXX: fill in proof** ☐

Remark 18.25. The radical $\sqrt{J}$ of the ideal J appearing in Proposition 18.18 is unique: It is equal to the radical ideal

$$K := \{f : \underline{\mathbb{A}}^1_S \mid \ulcorner f \text{ inv.}\urcorner \Rightarrow (x \in U)\} \subseteq \underline{\mathbb{A}}^1_S.$$

It's obvious that $J \subseteq K$ and therefore $\sqrt{J} \subseteq K$. For the converse direction, let $f \in K$ be given. Since $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent, the canonical map

$$(\underline{\mathbb{A}}^1_S/J)[f^{-1}] \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S[f^{-1}]), \underline{\mathbb{A}}^1_S/J]$$

is bijective. Since $\mathrm{Spec}(\underline{\mathbb{A}}^1_S[f^{-1}]) \cong [\![\ulcorner f \text{ inv.}\urcorner]\!]$, the image of 1 is zero: If $\mathrm{Spec}(\underline{\mathbb{A}}^1_S[f^{-1}])$ is inhabited, the element f is invertible and therefore x is an element of U. This implies that $1 \in J$. Thereby $\underline{\mathbb{A}}^1_S/J = 0$. By injectivity of the canonical map, the algebra $(\underline{\mathbb{A}}^1_S/J)[f^{-1}]$ is zero. Therefore $f^n \in J$ for some natural number n.

Remark 18.26. In view of the previous remark, one might hope to be able to simplify the condition in Proposition 18.18 as follows: "For any $x : X$, the quotient $\underline{\mathbb{A}}^1_S/K$ modulo the ideal $K = \{f : \underline{\mathbb{A}}^1_S \mid \ulcorner f \text{ inv.}\urcorner \Rightarrow (x \in U)\}$ is synthetically quasicoherent." However, this doesn't work out. This statement implies the condition in the proposition, but the converse direction does not hold, since $\underline{\mathbb{A}}^1_S/K \cong \underline{\mathbb{A}}^1_S/\sqrt{J}$ might fail to be synthetically quasicoherent. For instance that's the case if $U = \emptyset$; then $K = \sqrt{(0)}$ by Proposition 17.8. The quotient $\underline{\mathbb{A}}^1_S/\sqrt{(0)}$ is not synthetically quasicoherent by Remark 17.5.

18.5. Closed immersions.

Definition 18.27. In the context of a specified local ring, as for instance $\underline{\mathbb{A}}^1_S$ of the big Zariski topos of a scheme, a truth value $Z \subseteq 1$ is *closed* if and only if there exists an ideal $J \subseteq \underline{\mathbb{A}}^1_S$ such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent (Definition 17.1) and such that Z holds if and only if $J = (0)$.

In other words, a truth value $Z \subseteq 1$ is closed if and only if Z is isomorphic to the spectrum of a synthetically quasicoherent quotient algebra of $\underline{\mathbb{A}}^1_S$.

Example 18.28. Let $f : \underline{\mathbb{A}}^1_S$. Then "$f = 0$" is a closed truth value with witnessing ideal $J = (f)$. More generally, if $f_1, \ldots, f_n : \underline{\mathbb{A}}^1_S$, the truth value "$f_1 = \cdots = f_n = 0$" is closed.

Definition 18.29. In the context of a specified local ring, a map $Z \to X$ of sets is a *synthetic closed immersion* if and only if it is injective and for any $x : X$ the truth value of "the fiber of x is inhabited" is closed.

Example 18.30. The inclusion $\{0\} \hookrightarrow \underline{\mathbb{A}}^1_S$ is a synthetic closed immersion. More generally, for any functions $f_1, \ldots, f_m : (\underline{\mathbb{A}}^1_S)^n \to \underline{\mathbb{A}}^1_S$, the inclusion of the set of their common zeros in $(\underline{\mathbb{A}}^1_S)^n$ is a synthetic closed immersion.

Example 18.31. Let $X \in \mathrm{Zar}(S)$. Let $f : [X, \underline{\mathbb{A}}^1_S]$ from the internal point of view. The inclusion $\{x : X \mid f(x) = 0\} \hookrightarrow X$ is a synthetic closed immersion.

Proposition 18.32. *Let X be an S-scheme.*

1. *Let $Z \subseteq X$ be a closed subscheme. Then the subfunctor $\underline{Z} \hookrightarrow \underline{X}$ is a synthetic closed immersion from the internal point of view of* $\mathrm{Zar}(S)$.
2. *If $\underline{X}$ is locally representable by an object of the site used to define* $\mathrm{Zar}(S)$, *any synthetic closed immersion $Z \hookrightarrow \underline{X}$ is isomorphic to the subfunctor associated to a closed subscheme of X.*

Proof. To verify the first claim, let a quasicoherent $\mathcal{O}_X$-algebra $\mathcal{J}_0$ be given such that the closed subscheme $Z \subseteq X$ is the vanishing scheme of $\mathcal{J}_0$. Following the translation with the Kripke–Joyal semantics, let $f : T \to S$ be an object of the site used to define $\mathrm{Zar}(S)$ and let $x \in \underline{X}(T)$. We define $J := (f^*\mathcal{J}_0)^{\mathrm{Zar}} \in \mathrm{Zar}(S)/\underline{T}$. Then $T \models \ulcorner \underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent$\urcorner$ and $T \models (x \in \underline{Z} \Leftrightarrow J = (0))$, therefore "$x \in \underline{Z}$" is a closed truth value. **XXX: For the second claim, ...** ☐

18.6. Surjective morphisms.

Proposition 18.33. *Let φ be a formula over S.*

1. *Assume that for all points $s \in S$ there is a field extension K of the residue field $k(s)$ such that the S-scheme* $\mathrm{Spec}(K)$ *is contained in the site used to define* $\mathrm{Zar}(S)$ *and such that* $\mathrm{Spec}(K) \models \varphi$. *Then* $\mathrm{Zar}(S) \models \neg\neg\varphi$.
2. *The converse holds if the site used to define* $\mathrm{Zar}(S)$ *contains the spectra of residue fields. (This is not the case for the parsimonious sites.)*

Proposition 18.34. *Let $f : X \to Y$ be a morphism of S-schemes. Then f is set-theoretically surjective if and only if*

$$\mathrm{Zar}(S) \models \forall y : \underline{Y}.\ \neg\neg\exists x : \underline{X}.\ \underline{f}(x) = y.$$

18.7. Quasicompact morphisms.

18.8. Quasiseparated morphisms.

18.9. Proper morphisms.

19. Case studies

19.1. Punctured plane.

Definition 19.1. The *synthetic punctured plane* is the set $P := (\underline{\mathbb{A}}^1_S)^2 \setminus \{0\}$.

Proposition 19.2. *The synthetic punctured plane, as constructed by the internal language of* $\mathrm{Zar}(S)$*, is the functor of points of the ordinary punctured plane over* S*, that is the open subscheme* $D(X) \cup D(Y) \hookrightarrow \mathbb{A}^1_S$.

Proposition 19.3. *The evaluation morphism* $\underline{\mathbb{A}}^1_S[X,Y] \to [P, \underline{\mathbb{A}}^1_S]$ *is bijective.*

Proof. The synthetic punctured plane can be expressed as the pushout

$$P \cong D(X) \amalg_{D(X) \cap D(Y)} D(Y).$$

Therefore we have the chain of isomorphisms

$$\begin{aligned}
[P, \underline{\mathbb{A}}^1_S] &\cong [D(X) \amalg_{D(X)\cap D(Y)} D(Y), \underline{\mathbb{A}}^1_S] \\
&\cong [D(X), \underline{\mathbb{A}}^1_S] \times_{[D(X)\cap D(Y), \underline{\mathbb{A}}^1_S]} [D(Y), \underline{\mathbb{A}}^1_S] \\
&\cong \underline{\mathbb{A}}^1_S[X, X^{-1}] \times_{\underline{\mathbb{A}}^1_S[XY,(XY)^{-1}]} \underline{\mathbb{A}}^1_S[Y, Y^{-1}] \\
&\cong \underline{\mathbb{A}}^1_S[X,Y].
\end{aligned}$$

The penultimate isomorphism exploits the synthetic quasicoherence of $\underline{\mathbb{A}}^1_S$, which guarantees that the canonical map

$$\underline{\mathbb{A}}^1_S[X, X^{-1}] \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S[X, X^{-1}]), \underline{\mathbb{A}}^1_S] \cong [D(X), \underline{\mathbb{A}}^1_S]$$

is bijective. The ultimate isomorphism rests on the purely algebraic argument that elements of $\underline{\mathbb{A}}^1_S[X, X^{-1}]$ and $\underline{\mathbb{A}}^1_S[Y, Y^{-1}]$ which agree as elements of $\underline{\mathbb{A}}^1_S[(XY), (XY)^{-1}]$ are both given by an element of $\underline{\mathbb{A}}^1_S[X,Y]$ and in fact by the same element. □

Corollary 19.4. *The punctured plane is not affine.*

Proof. The canonical map $P \to \mathrm{Spec}([P, \underline{\mathbb{A}}^1_S])$ is isomorphic to the inclusion $P \hookrightarrow (\underline{\mathbb{A}}^1_S)^2$ and therefore not bijective. □

19.2. Cohomology of projective space.

19.3. Categorical group quotients.

19.4. Grassmannian. Let $\mathcal{V}$ be a finite locally free $\mathcal{O}_S$-module. We want to illustrate the synthetic approach by verifying the basic fact that the Grassmannian $\mathrm{Gr}(\mathcal{V}, r)$ of rank-r locally free quotients of $\mathcal{V}$, defined as a certain functor of points, is representable by an S-scheme of finite presentation using the internal language of $\mathrm{Zar}(S)$.

Definition 19.5. The *Grassmannian* $\mathrm{Gr}(\mathcal{V}, r)$ is the functor which associates to an S-scheme $f : T \to S$ the set

$$\mathrm{Gr}(\mathcal{V}, r)(T) := \{U \subseteq f^*\mathcal{V} \text{ sub-}\mathcal{O}_T\text{-module} \,|\, (f^*\mathcal{V})/U \text{ is locally free of rank } r\}.$$

Definition 19.6. The *synthetic Grassmannian* of rank-r quotients of a module V is the set

$$\mathrm{Gr}(V, r) := \{U \subseteq V \text{ submodule} \,|\, V/U \text{ is free of rank } r\}.$$

We could just as well define the synthetic Grassmannian somewhat more directly as the set of free rank r-quotients (up to isomorphism). This set is canonically isomorphic to the Grassmannian as we chose to define it, by mapping a quotient $\pi : V \twoheadrightarrow Q$ to the kernel of π.

Proposition 19.7. *The synthetic Grassmannian of* $\mathcal{V}$*, as constructed by the internal language of* $\mathrm{Zar}(S)$ *where* $\mathcal{V}$ *looks like an ordinary free module, coincides with the functorially defined Grassmannian.*

Proof. Immediate from Definition 2.8 and Proposition **??**. □

Having established that the internally constructed synthetic Grassmannian actually describes the external Grassmannian which we're interested in, we can switch to a fully internal perspective. We'll reflect this switch notationally by referring to the $\underline{\mathbb{A}}^1_S$-module $V := \mathcal{V}^{\mathrm{Zar}}$ instead of $\mathcal{V}$.

We define for any free submodule $W \subseteq V$ of rank r the subset

$$G_W := \{U \in \mathrm{Gr}(V, r) \mid W \to V \to V/U \text{ is bijective}\}.$$

This sets admits a more concrete description, since it is in canonical bijection to the set

$$G'_W := \{\pi : V \to W \mid \pi \circ \iota = \mathrm{id}\}$$

of all splittings of the inclusion $\iota : W \hookrightarrow V$: An element $U \in G_W$ corresponds to the splitting $V \twoheadrightarrow V/U \xrightarrow{(\cong)^{-1}} W$. Conversely, a splitting π corresponds to $U := \ker(\pi) \in G_W$.

Proposition 19.8. *The union of the subsets* G_W *is* $\mathrm{Gr}(V, r)$*.*

Proof. Let $U \in \mathrm{Gr}(V, r)$. Then there exists a basis $([v_1], \ldots, [v_r])$ of V/U. The family $(v_1, \ldots, v_r)$ is linearly independent in V, therefore the submodule $W := \mathrm{span}(v_1, \ldots, v_r) \subseteq V$ is free of rank r. The canonical linear map $W \hookrightarrow V \twoheadrightarrow V/U$ maps the basis $(v_i)_i$ to the basis $([v_i])_i$ and is therefore bijective. Thus $U \in G_W$. □

Proposition 19.9. *The sets* G_W *are (quasicompact-)open subsets of* $\mathrm{Gr}(V, r)$*.*

Proof. Let $U \in \mathrm{Gr}(V, r)$. Then $U \in G_W$ if and only if the canonical linear map $W \hookrightarrow V \twoheadrightarrow V/U$ is bijective. Since W and V/U are both free modules of rank r, this map is given by an $(r \times r)$-matrix M over $\underline{\mathbb{A}}^1_S$; therefore it's bijective if and only if the determinant of M is invertible.

Thus we've found a number which is invertible if and only if $U \in G_W$. By Corollary 18.19, the truth value of "$U \in G_W$" is open. □

Proposition 19.10. *The sets* G_W *are affine. Moreover, the algebras which the* G_W *are spectra of are finitely presented.*

Proof. The set of all linear maps $V \to W$ is the spectrum of the $\underline{\mathbb{A}}^1_S$-algebra $A := \mathrm{Sym}(\mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(V, W)^\vee)$, since

$$\begin{aligned}
\mathrm{Spec}(A) &= \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(\mathrm{Sym}(\mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V, W)^\vee), \underline{\mathbb{A}}^1_S) \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(\mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V, W)^\vee, \underline{\mathbb{A}}^1_S) \\
&= \mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V, W)^{\vee\vee} \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V, W).
\end{aligned}$$

In the last step the assumption that not only W, but also V is a free module of finite rank enters. (This is the first time in this development that we need this assumption.)

The set G'_W is a closed subset of this spectrum, namely the locus where the generic linear map $V \to W$ is a splitting of the inclusion $\iota : W \hookrightarrow V$. If we choose bases of V and W, whereby $\mathrm{Sym}(\mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(V, W)^\vee)$ is isomorphic to $\underline{\mathbb{A}}^1_S[M_{11}, \ldots, M_{rn}]$, we can be more explicit: The set G'_W is isomorphic to

$$\mathrm{Spec}(k[M_{11}, \ldots, M_{rn}]/(MN - I)),$$

where I is the $(r \times r)$ identity matrix, M is the generic matrix $M = (M_{ij})_{ij}$, and N is the matrix of ι with respect to the chosen bases. The notation "$(MN - I)$" denotes the ideal generated by the entries of $MN - I$. $\square$

Corollary 19.11. *The Grassmannian* $\mathrm{Gr}(V, r)$ *is a finitely presented scheme.*

Proof. We need to verify that $\mathrm{Gr}(V, r)$ admits a finite covering by spectra of finitely presented $\underline{\mathbb{A}}^1_S$-algebras. We already know that $\mathrm{Gr}(V, r)$ can be covered by the open subsets G_W and that these sets are spectra of finitely presented algebras. Therefore it remains to prove that finitely many of these subsets suffice to cover $\mathrm{Gr}(V, r)$.

In fact, if we choose an isomorphism $V \cong (\underline{\mathbb{A}}^1_S)^n$, we see that $\binom{n}{r}$ of these subsets suffice: namely those where W is one of the standard submodules of $(\underline{\mathbb{A}}^1_S)^n$ (generated by the standard basis vectors). For if $U \in \mathrm{Gr}((\underline{\mathbb{A}}^1_S)^n, r)$, the surjection $V \to V/U$ maps the basis of at least one of these standard submodules to a basis and is therefore bijective. This is because from any surjective $(r \times n)$-matrix over a local ring one can select r columns which form an linearly independent family. $\square$

Proposition 19.12. *Let* $U \in \mathrm{Gr}(V, r)$*. Then the tangent space at* U *is given by* $T_U \mathrm{Gr}(V, r) \cong \mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(U, V/U)$.

Proof. **XXX: fill in proof** $\square$

20. Beyond the Zariski topology

The Zariski topology is of course not the only interesting topology on Sch/S. For any finer topology τ, such as the étale, smooth, or fppf topology, the big τ-topos of S, that is the topos of sheaves on Sch/S with respect to τ, is a subtopos of the big Zariski topos. Therefore there is a modal operator $\Box_\tau$ in $\mathrm{Zar}(S)$ reflecting the topology τ. Explicitly, for an S-scheme T and a formula φ over T, the meaning of

$$T \models \Box_\tau \varphi$$

is that there exists a τ-covering $(T_i \to T)_i$ of T such that $T_i \models \varphi$ for all i (where parameters appearing in φ have to be pulled back along $T_i \to T$). Succinctly, the formula "$\Box_\tau \varphi$" means that φ holds τ-locally. Generalizing Theorem 6.30 from sheaves on locales to sheaves on arbitrary Grothendieck sites we also have

$$\mathrm{Zar}(S) \models \varphi^{\Box_\tau} \qquad \text{iff} \qquad \mathrm{Sh}((\mathrm{Sch}/S)_\tau) \models \varphi.$$

A basic illustration of these modal operators is provided by the Kummer sequence, that is the short sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_{\mathrm{m}} \xrightarrow{(\_)^n} \mathbb{G}_{\mathrm{m}} \longrightarrow 1$$

of multiplicatively-written commutative group objects in $\mathrm{Zar}(S)$. With the internal description of μ_n and $\mathbb{G}_{\mathrm{m}}$, there is a purely internal and straightforward proof that this sequence is exact at the first two terms. But except for trivial cases, the n-th power map $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ will fail to be an epimorphism; internally speaking, the statement

$$\forall f : (\underline{\mathbb{A}}^1_S)^\times. \qquad \exists g : (\underline{\mathbb{A}}^1_S)^\times.\ f = g^n$$

is false in general. However, if n is invertible in $\Gamma(S, \mathcal{O}_S)$, the internal statement

$$\forall f : (\underline{\mathbb{A}}^1_S)^\times.\ \Box_{\text{ét}}(\exists g : (\underline{\mathbb{A}}^1_S)^\times.\ f = g^n)$$

is true. In fact, the more general statement

$$\forall p : \underline{\mathbb{A}}^1_S[X].\ \ulcorner p \text{ is monic, of positive degree, and separable} \urcorner \Longrightarrow$$
$$\Box_{\text{ét}}(\exists x : \underline{\mathbb{A}}^1_S.\ p(x) = 0 \wedge \ulcorner p'(x) \text{ inv.} \urcorner)$$

is true from the internal point of view, where a polynomial p is called *separable* if and only if there exists a Bézout representation $ap + bp' = 1$. After simplifying, the intepretation of that statement with the Kripke–Joyal semantics is that for any S-scheme T and any monic separable polynomial $p \in \Gamma(T, \mathcal{O}_T)[X]$ of positive degree there exists an étale covering $(T_i \to T)_i$ of T such that the pullbacks of p to each of the T_i possess a simple zero. The required covering is given by the canonical surjective étale map $\underline{\mathrm{Spec}}_T\, \mathcal{O}_T[X]/(p) \to T$.

The following theorem shows that the modal operator $\Box_{\text{ét}}$ corresponding to the étale topology admits a purely internal characterization in $\mathrm{Zar}(S)$, which furthermore resonates well with the intuition about the étale topology.

Theorem 20.1. *Let S be a scheme. The modal operator $\Box_{\text{ét}}$ in $\mathrm{Zar}(S)$ corresponding to the étale topology is the smallest operator $\Box$ such that the $\Box$-translation of the statement "$\underline{\mathbb{A}}^1_S$ is separably closed" holds.*

Here, a ring A is *separably closed* if and only if

$$\forall p : A[X].\ \ulcorner p \text{ is monic, of positive degree, and unramifiable} \urcorner \Longrightarrow \\ \exists x : A.\ p(x) = 0 \wedge \ulcorner p'(x) \text{ inv.} \urcorner.$$

We call a polynomial p over a ring A *unramifiable* if and only if it admits at least one simple root in every algebraically closed field over A. Since quantifying over algebraically closed fields raises red flags from an intuitionistic point of view, just as quantifying over maximal ideals does, this condition has to be formulated in a sensible way. One possibility is to use the *hyperdiscriminants* of p, i. e. the elementary symmetric polynomials in the values of p' at the roots of p, resulting in a simple existential statement involving only the coefficients of p; in particular, the condition for a polynomial to be unramifiable is a geometric formula. See [82, p. 751] for the precise formulation.

In more detail, the claim is that firstly $\Box_{\text{ét}}$ is a modal operator such that the displayed formula holds and that secondly, if $\Box$ is any modal operator verifying the formula, internally it holds that $\Box_{\text{ét}}\varphi \Rightarrow \Box\varphi$ for any truth value $\varphi : \Omega$.

Proof. For the proof we require some familiarity with the concept of classifying toposes. We are grateful to Felix Geißler for contributing a key step of the argument.

To verify the first statement, note that the displayed formula is a geometric implication and that the big étale topos $\text{Ét}(S)$ has *enough points*. Therefore it suffices to show that for any S-scheme T and any geometric point $\bar{t}$ of T, the stalk $\mathcal{O}_{T,\bar{t}}$ is separably closed. It is well-known that this is true.

For the second statement we may assume without loss of generality that $S = \mathrm{Spec}\, A$ is affine. It is well-known that, for any cocomplete topos $\mathcal{E}$, geometric morphisms $\mathcal{E} \to \mathrm{Zar}(\mathrm{Spec}\, A)$ are in canonical one-to-one correspondence with local algebras over $\underline{A}$ in $\mathcal{E}$ (where $\underline{A}$ denotes the pullback of A along the unique geometric morphism $\mathcal{E} \to \mathrm{Set}$) and that geometric morphisms $\mathcal{E} \to \text{Ét}(\mathrm{Spec}\, A)$ are in canonical one-to-one correspondence with algebras over $\underline{A}$ which are local and separably closed from the internal point of view of $\mathcal{E}$; see [56, Section VIII.6] and [4].

Therefore a geometric morphism $\mathcal{E} \to \mathrm{Zar}(\mathrm{Spec}\, A)$ factors over the geometric embedding $\text{Ét}(\mathrm{Spec}\, A) \hookrightarrow \mathrm{Zar}(\mathrm{Spec}\, A)$ if and only if the pullback of $\underline{\mathbb{A}}^1_{\mathrm{Spec}\, A}$ along $\mathcal{E} \to \mathrm{Zar}(\mathrm{Spec}\, A)$ is separably closed.

Let $\Box$ be a modal operator in $\mathrm{Zar}(\mathrm{Spec}\, A)$ such that the $\Box$-translation of "$\underline{\mathbb{A}}^1_{\mathrm{Spec}\, A}$ is separably closed" holds. Then the pullback of $\underline{\mathbb{A}}^1_{\mathrm{Spec}\, A}$ along $\mathrm{Zar}(\mathrm{Spec}\, A)_\Box \hookrightarrow \mathrm{Zar}(\mathrm{Spec}\, A)$ is separably closed and therefore this geometric embedding factors over $\text{Ét}(\mathrm{Spec}\, A) \hookrightarrow \mathrm{Zar}(\mathrm{Spec}\, A)$. This shows that any $\Box$-sheaf is also a $\Box_{\text{ét}}$-sheaf.

The claim that $\Box_{\text{ét}}\varphi \Rightarrow \Box\varphi$ for any truth value $\varphi : \Omega$ then follows by combining the following two basic observations of the theory of modal operators, valid for any modal operator $\Box$:

(1) $\Box\varphi \iff \forall \psi : \Omega.\ ((\Box\psi \Rightarrow \psi) \wedge (\varphi \Rightarrow \psi)) \Rightarrow \psi$.
(2) $(\Box\psi \Rightarrow \psi) \iff \ulcorner\{x \in 1 \mid \psi\}$ is a $\Box$-sheaf$\urcorner$. $\Box$

XXX: other topologies?

- synthetic topology?

21. Unsorted

- "functoriality"
- Kähler differentials
- closed and open subschemes
- $j_!\mathcal{O}_U$ flat over $\mathcal{O}_X$, ...
- Koszul resolution; Beilinson resolution?
- meta properties: some lemmas about limits of modules
- locally small categories
- open/closed immersions
- morphisms of schemes...
- proper maps...
- limits and colimits...
- Kähler differentials; clear myth that the definition via free modules "does not glue very well" (`http://www.mathematik.uni-kl.de/~gathmann/class/algeom-2002/chapter-7.pdf`)

Appendix

22. Dictionary between internal and external notions

External	Internal	Reference
Sheaves of sets		
sheaf of sets	set	
$\alpha : \mathcal{F} \to \mathcal{G}$ monomorphism	α injective	Ex. 2.3
$\alpha : \mathcal{F} \to \mathcal{G}$ epimorphism	α surjective	Ex. 2.3
$\mathrm{int}(X \setminus \mathrm{supp}\,\mathcal{F})$	truth value of "$\mathcal{F}$ is a singleton"	Rem. 4.8
$f : X \to \mathbb{N}$ upper semicont.	element of $\widehat{\mathbb{N}}$	Lemma 5.5
$f : X \to \mathbb{N}$ locally constant	element of $\mathbb{N}$	Lemma 5.5
Sheaves of rings		
sheaf of rings	ring	Prop. 3.1
local sheaf of rings	local ring	Prop. 3.5
X is reduced	$\mathcal{O}_X$ is reduced (and $\neg$invertible $\Rightarrow$ zero)	Prop. 3.3
$\dim X \leq n$	Krull dimension of $\mathcal{O}_X$ is $\leq n$	Prop. 3.13
X is integral at all points	$\mathcal{O}_X$ is a integral domain	Prop. 3.17
X is locally Noetherian	$\mathcal{O}_X$ is processly Noetherian	Prop. 3.23
X is normal	$\mathcal{O}_X$ is normal (assuming that X is locally Noetherian)	Prop. 7.6
Sheaves of modules		
sheaf of modules	module	
$\mathcal{F}$ is finite locally free	$\mathcal{F}$ is finite free	Prop. 4.1
$\mathcal{F}$ is of finite type	$\mathcal{F}$ is finitely generated	Prop. 4.2
$\mathcal{F}$ is of finite presentation	$\mathcal{F}$ is finitely presented	Prop. 4.2
$\mathcal{F}$ is coherent	$\mathcal{F}$ is coherent	Prop. 4.2
$\mathcal{F}$ is quasicoherent	$\mathcal{F}[f^{-1}]$ is a sheaf wrt. $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$ for $f : \mathcal{O}_X$	Thm. 9.3
$\mathcal{F}$ is flat	$\mathcal{F}$ is flat	Prop. 4.6
$\mathcal{F}$ is torsion	$\mathcal{F}$ is torsion	Prop. 4.9
$M^\sim$	$\underline{M}[\mathcal{F}^{-1}]$ (localization at generic filter)	Prop. 11.5
tensor product $\mathcal{F} \otimes \mathcal{G}$	tensor product $\mathcal{F} \otimes \mathcal{G}$	Prop. 4.4
dual $\mathcal{F}^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$	dual $\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$	
$\mathrm{int}(X \setminus \mathrm{supp}\,\mathcal{F})$	truth value of "$\mathcal{F} = 0$"	Prop. 4.7
quasicoherator of $\mathcal{I}$	$\{s : \mathcal{O}_X \mid \ulcorner s \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I}\}$ ($\mathcal{I}$ a radical ideal)	Prop. 9.10
rank function of $\mathcal{F}$	minimal number of generators for $\mathcal{F}$	Prop. 5.6

Subspaces ($i : A \hookrightarrow X$ closed immersion, $j : U \hookrightarrow X$ open immersion)

sheaf supported on A	$\Box$-sheaf, where $\Box = (\_\_ \vee A^c)$	Lemma 10.3
sheaf of the form $j_*(\mathcal{F})$	$\Box$-sheaf, where $\Box = (U \Rightarrow \_\_)$	
extension of $\mathcal{F}$ by the empty set	$j_!(\mathcal{F}) = \{x : \mathcal{F} \mid U\}$	Lemma 10.1
extension of $\mathcal{F}$ by zero	$j_!(\mathcal{F}) = \{x : \mathcal{F} \mid (x = 0) \vee U\}$	Lemma 10.2
sheaf with empty/zero stalks on U^c	sheaf of the form $j_!(\mathcal{F})$	
sections of $\mathcal{F}$ are equal if they agree on dense open	$\mathcal{F}$ is $\neg\neg$-separated	Prop. 6.14
sheaf of sections of $\mathcal{F}$ defined on dense open subsets	$\mathcal{F}^{++}$ with respect to $\Box = \neg\neg$ (assuming that $\mathcal{F}$ is $\neg\neg$-separated)	Prop. 6.14
U is dense	$\neg\neg U$	Prop. 6.4
U is scheme-theoretically dense	$\widehat{\Box} U$, i. e. $\mathcal{O}_X$ is separated wrt. $(U \Rightarrow \_\_)$	Lemma 7.11
$V(\mathcal{I})$ is reduced	$\mathcal{I}$ is a radical ideal	Lemma 10.6
$\mathcal{O}_{X_{\text{red}}}$	$\mathcal{O}_X / \sqrt{(0)}$	Lemma 10.7

Rational functions and Cartier divisors

$\mathcal{K}_X$	total quotient ring of $\mathcal{O}_X$	Prop. 7.1
Cartier divisor	element of $\mathcal{K}_X^\times / \mathcal{O}_X^\times$	
effective Cartier divisor	$[s/1]$ with $s : \mathcal{O}_X$ regular	Def. 7.21
line bundle $\mathcal{O}_X(D)$	$D^{-1}\mathcal{O}_X \subseteq \mathcal{K}_X$	Def. 7.23

Topological properties

X is quasicompact	"$\mathrm{Sh}(X) \models$" commutes with directed disjunctions	Prop. 8.1
X is local	"$\mathrm{Sh}(X) \models$" commutes with arbitrary disjunctions	Prop. 8.7
X is irreducible	if $\neg(\varphi \wedge \psi)$, then $\neg\varphi$ or $\neg\psi$	Prop. 8.9

23. The inference rules of intuitionistic logic

XXX: cite [44, Section D1.3.1], talk about $\in$, and explain contexts

Structural rules

$$\frac{}{\varphi \vdash_{\vec{x}} \varphi} \qquad \frac{\varphi \vdash_{\vec{x}} \psi}{\varphi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]} \qquad \frac{\varphi \vdash_{\vec{x}} \psi \qquad \psi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \chi}$$

Rules for nullary and binary conjunction

$$\frac{}{\varphi \vdash_{\vec{x}} \top} \qquad \frac{}{\varphi \wedge \psi \vdash_{\vec{x}} \varphi} \qquad \frac{}{\varphi \wedge \psi \vdash_{\vec{x}} \psi} \qquad \frac{\varphi \vdash_{\vec{x}} \psi \qquad \varphi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \psi \wedge \chi}$$

Rules for nullary and binary disjunction

$$\frac{}{\bot \vdash_{\vec{x}} \varphi} \qquad \frac{}{\varphi \vdash_{\vec{x}} \varphi \vee \psi} \qquad \frac{}{\psi \vdash_{\vec{x}} \varphi \vee \psi} \qquad \frac{\varphi \vdash_{\vec{x}} \chi \qquad \psi \vdash_{\vec{x}} \chi}{\varphi \vee \psi \vdash_{\vec{x}} \chi}$$

Rules for arbitrary set-indexed conjunction and disjunction

$$\frac{}{\bigwedge_{i\in I}\varphi_i \vdash_{\vec{x}} \varphi_j \text{ for all } j \in I} \qquad \frac{\varphi \vdash_{\vec{x}} \psi_j \text{ for all } j \in I}{\varphi \vdash_{\vec{x}} \bigwedge_{i\in I}\psi_i}$$

$$\frac{}{\varphi_j \vdash_{\vec{x}} \bigvee_{i\in I}\varphi_i \text{ for all } j \in I} \qquad \frac{\varphi_j \vdash_{\vec{x}} \psi \text{ for all } j \in I}{\bigvee_{i\in I}\varphi_i \vdash_{\vec{x}} \psi}$$

Double rule for implication

$$\frac{\varphi \wedge \psi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \psi \Rightarrow \chi}$$

Double rules for bounded and unbounded quantification

$$\frac{\varphi \vdash_{\vec{x},y} \psi}{\exists y : Y.\, \varphi \vdash_{\vec{x}} \psi}\ (y \text{ not occuring in } \psi) \qquad \frac{\varphi \vdash_{\vec{x},y} \psi}{\varphi \vdash_{\vec{x}} \forall y : Y.\, \psi}\ (y \text{ not occuring in } \varphi)$$

$$\frac{\varphi \vdash_{\vec{x},Y} \psi}{\exists Y.\, \varphi \vdash_{\vec{x}} \psi}\ (Y \text{ not occuring in } \psi) \qquad \frac{\varphi \vdash_{\vec{x},Y} \psi}{\varphi \vdash_{\vec{x}} \forall Y.\, \psi}\ (Y \text{ not occuring in } \varphi)$$

Rules for equality

$$\frac{}{\top \vdash_x x = x} \qquad \frac{}{(\vec{x} = \vec{y}) \wedge \varphi \vdash_{\vec{z}} \varphi[\vec{y}/\vec{x}]}$$

("$\vec{x} = \vec{y}$" is short for "$x_1 = y_1 \wedge \cdots \wedge x_n = y_n$".)

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XXX: remark on possible pitfalls