

USING THE INTERNAL LANGUAGE OF TOPOSES IN ALGEBRAIC GEOMETRY

INGO BLECHSCHMIDT

ABSTRACT. There are several important topoi associated to a scheme, for instance the petit and gros Zariski topoi. These come with an internal mathematical language which closely resembles the usual formal language of mathematics, but is “local on the base scheme”:

For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a certain condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

These notes give an introduction to this topic and show how the internal point of view can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry.

CONTENTS

1. Introduction	1
2. Kripke–Joyal semantics	1
3. Sheaves of rings	1
4. Sheaves of modules	2
5. Rational functions and Cartier divisors	2
6. Relative spectrum	2
7. Modalities	2
8. Unsorted	2

1. INTRODUCTION

2. KRIPKE–JOYAL SEMANTICS

- definition
- fundamental properties
- geometric formulas
- geometric constructions

3. SHEAVES OF RINGS

- reducedness
- field property
- discreteness

4. SHEAVES OF MODULES

- of finite type, of finite presentation, coherent
- basic lemmas
- flatness
- important hard exercise

5. RATIONAL FUNCTIONS AND CARTIER DIVISORS

- internal definition of K_X
- internal definition of Cartier divisors
- correspondence between Cartier divisors and sub- \mathcal{O}_X -modules of K_X

6. RELATIVE SPECTRUM

- ...

7. MODALITIES

- negneg
- spreading of properties from stalk to neighbourhood
- internal sheafification

8. UNSORTED

- Kähler differentials
- completion of the natural numbers, rank function
- meta properties
- locally small categories
- big Zariski topos
- open/closed immersions
- morphisms of schemes...
- proper maps...

E-mail address: iblech@web.de