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Doctoral Thesis

Using the internal language of toposes in algebraic geometry

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Abstract

Any scheme has its associated little and big Zariski toposes. These toposes support an internal mathematical language which closely resembles the usual formal language of mathematics, but is "local on the base scheme": For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a local triviality condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

We investigate how the internal language of the little Zariski topos can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry. To this end, we build a dictionary relating internal and external notions and demonstrate its utility by giving a simple proof of Grothendieck's generic freeness lemma in full generality. We also employ this framework to state a general transfer principle which relates modules with their induced quasicoherent sheaves, to study the phenomenon that some properties spread from points to open neighborhoods, and to compare general notions of spectra.

We employ the big Zariski topos to set up the foundations of a synthetic account of scheme theory. This account is similar to the synthetic account of differential geometry, but has a distinct algebraic flavor. Central to the theory is the notion of synthetic quasicoherence, which has no analog in synthetic differential geometry. We also discuss how various common subtoposes of the big Zariski topos can be described from the internal point of view and derive explicit descriptions of the geometric theories which are classified by the fppf and by the surjective topology.

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Here in Augsburg, we have a strong bond between undergraduates, PhD students, and members of faculty. I'll try to carry our mathematical pizza seminars and our non-mathematical and slightly nonstandard leisure time activities forward.

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PART I

Basics

1. Introduction

Internal language of toposes. A *topos* is a category which shares certain categorical properties with the category of sets; the archetypical example is the category of sets, and the most important example for the purposes of this thesis is the category of set-valued sheaves on a topological space.

Any topos $\mathcal{E}$ supports an *internal language*. This is a device which allows one to *pretend* that the objects of $\mathcal{E}$ are plain sets and that the morphisms are plain maps between sets, even if in fact they are not. For instance, consider a morphism $\alpha : X \to Y$ in $\mathcal{E}$. From the *internal point of view*, this looks like a map between sets, and we can formulate the condition that this map is surjective; we write this as

$$\mathcal{E} \models \forall y \,{:}\, Y.\ \exists x \,{:}\, X.\ \alpha(x) = y.$$

The appearance of the colons instead of the usual element signs reminds us that this expression is not to be taken literally – X and Y are objects of $\mathcal{E}$ and thus not necessarily sets. The definition of the internal language is made in such a way so that the meaning of this internal statement is that α is an epimorphism. Similarly, the translation of the internal statement that α is injective is that α is a monomorphism.

Furthermore, we can *reason* with the internal language. There is a metatheorem to the effect that if some statement φ holds from the internal point of view of a topos $\mathcal{E}$ and if φ logically implies some further statement ψ, then ψ holds in $\mathcal{E}$ as well. As a simple example, consider the elementary fact that the composition of surjective maps is surjective. Interpreting this statement in the internal language of $\mathcal{E}$, we obtain the more abstract result that the composition of epimorphisms in $\mathcal{E}$ is epic.

There is, however, a slight caveat to this metatheorem. Namely, the internal language of a topos is in general only *intuitionistic*, not *classical*. This means that internally, one cannot use the law of excluded middle ($\varphi \vee \neg\varphi$), the law of double negation elimination ($\neg\neg\varphi \Rightarrow \varphi$), or the axiom of choice. For instance, one rendition of the axiom of choice is that any vector space is free. But it need not be the case that a vector space internal to a topos is free as seen from the internal perspective: By the technique explained in this thesis, this would imply the absurd statement that any sheaf of modules on a reduced scheme is locally free.

The restriction to intuitionistic reasoning is not as confining as it might first appear, in particular because there is a widely applicable metatheorem ensuring that statements of a certain form are provable classically if and only if they are provable intuitionistically. We will discuss practical consequences below (on page 23).

Algebraic geometry. We apply the internal language of toposes to algebraic geometry in two different ways, corresponding to the two different toposes associated to a scheme X: the *little Zariski topos* which is just the topos $\mathrm{Sh}(X)$ of set-valued sheaves on X, and the *big Zariski topos* which we introduce below.

The internal language of the little Zariski topos can be used as follows. The structure sheaf $\mathcal{O}_X$ of a scheme X is a sheaf of rings in that its sets of local sections carry ring structures and these ring structures are compatible with restriction. From the internal point of view of $\mathrm{Sh}(X)$, the structure sheaf $\mathcal{O}_X$ looks much simpler: It looks just like a plain ring (and not a sheaf of rings). Similarly, a sheaf of $\mathcal{O}_X$-modules looks just like a plain module over that ring.

This allows to import notions and facts from basic linear and commutative algebra into the sheaf setting. For instance, it turns out that a sheaf of $\mathcal{O}_X$-modules is of finite type if and only if, from the internal perspective, it is finitely generated as an $\mathcal{O}_X$-module. Now consider the following fact of linear algebra: If in a short exact sequence of modules the two outer ones are finitely generated, then the middle one is too. The usual proof of this fact is intuitionistically valid and can thus be interpreted in the internal language. It then *automatically* yields the following more advanced proposition: If in a short exact sequence of sheaves of $\mathcal{O}_X$-modules the two outer ones are of finite type, then the middle one is too.

This example was not in any way special: *Any (intuitionistically valid) theorem about modules yields a corresponding theorem about sheaves of modules.*

The internal language machinery thus allows us to understand the basic notions and statements of scheme theory as notions and statements of linear and commutative algebra, interpreted in a suitable sheaf topos. This brings conceptual clarity and reduces technical overhead.

In Section 2, we explain how the internal language machinery works, and then develop in Part II a *dictionary* relating common notions of scheme theory and corresponding notions of algebra. Once built, this dictionary can be used arbitrarily often. We stress that no in-depth knowledge of topos theory or categorical logic is necessary to apply this apparatus.

In simple cases, the internal language can be regarded as a tool for ensuring that certain kinds of "fast and loose reasoning" in algebraic geometry can be rigorously justified. For instance, when trying to quickly gauge whether some plausible-looking statement holds for schemes and sheaves, we might content ourselves to check that the statement holds for rings and modules and then trust that it also holds in the general case. Or when trying to construct a certain sheaf of modules, we might content ourselves to construct it over affine open subsets and then appeal to some gluing lemma, without meticulously checking the details.

The internal language apparatus ensures that this kind of reasoning will never result in wrong conclusions, provided that one can formulate the statements and constructions in the internal language and that the correctness proof in the affine setting is intuitionistically valid.

We believe that already this application of the internal language is useful to working algebraic geometers. However, more advanced applications are also possible. They result from considering internal statements whose logical form is more complex, in particular from statements which quantify over subsets or which contain implication and negation signs.

For instance, if X is a reduced scheme, the internal universe of $\mathrm{Sh}(X)$ has the peculiar feature that $\mathcal{O}_X$ is Noetherian and a field, even if X is not locally Noetherian and (as will almost always be the case) the local rings $\mathcal{O}_{X,x}$ are not fields. This fact has no simple external counterpart; it's rather an intricate statement about the interplay between the rings $\Gamma(U, \mathcal{O}_X)$ for varying open subsets $U \subseteq X$.

Thanks to this particular feature, linear and commutative algebra over $\mathcal{O}_X$ are particularly simple from the internal point of view. For instance, Grothendieck's generic freeness lemma, which is usually proved using a somewhat involved series of reduction steps, admits a short, easy, and conceptual proof with this technique.

To briefly indicate a part of this, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules of finite type. A basic version of Grothendieck's generic freeness lemma then states that $\mathcal{F}$ is locally free on some dense open subset of X; this fact is stated in Vakil's lecture notes as an "important hard exercise" [129, Exercise 13.7.K]. In fact, this proposition is just the interpretation of the following basic statement of intuitionistic linear algebra in the sheaf topos: Any finitely generated vector space is *not not* free. The proof of this statement is entirely straightforward.[1]

It is in this way that the internal language unlocks new approaches: by making concepts accessible which would otherwise be too unwieldy to manage and by allowing to import a huge corpus of prior work, namely the entire literature on constructive algebra.

The internal language also sheds light on the phenomenon that sometimes, truth of a property at a point x spreads to some open neighborhood of x; and in particular that sometimes, truth of a property at the generic point spreads to some dense open subset. For instance, if the stalk of a sheaf of finite type is zero at some point, the sheaf is even zero on some open neighborhood; but this spreading does not occur for general sheaves which may fail to be of finite type.

We formalize this by introducing a *modal operator* $\Box$ into the internal language, such that the internal statement $\Box\varphi$ means that φ holds on some open neighborhood of x. Furthermore, we introduce a simple operation on formulas, the $\Box$-*translation* $\varphi \mapsto \varphi^\Box$, such that $\varphi^\Box$ means that φ holds at the point x. This translation is defined on a purely syntactical level. The question whether truth at x spreads to truth on a neighborhood can then be formulated in the following way: Does $\varphi^\Box$ intuitionistically imply $\Box\varphi$?

This allows to deal with the question in a simpler, logical way, with the technicalities of sheaves blinded out. We also give a metatheorem which covers a wide range of cases. Namely, spreading occurs for all those properties which can be formulated in the internal language without using "$\Rightarrow$", "$\forall$", and "$\neg$".

To take up the example above, consider the property of a module $\mathcal{F}$ being the zero module. In the internal language, it can be formulated as $(\forall x : \mathcal{F}.\ x = 0)$. Because of the appearance of "$\forall$", the metatheorem is not applicable to this statement. But if $\mathcal{F}$ is of finite type, there are generators $x_1, \ldots, x_n : \mathcal{F}$ from the internal point of view, and the condition can be reformulated as $x_1 = 0 \wedge \cdots \wedge x_n = 0$; the metatheorem is applicable to this statement.

Synthetic algebraic geometry. All of the applications mentioned above employ the little Zariski topos of the base scheme X, the topos of sheaves on the underlying topological space of X. Its internal language simplifies the treatment of sheaves of rings and modules over X, but the treatment of *schemes* over X is simplified only a little bit: From the internal point of view of $\mathrm{Sh}(X)$, a morphism $T \to X$ of schemes looks like a morphism $T \to \mathrm{pt}$. Therefore relative scheme theory is turned into absolute scheme theory (over the ring $\mathcal{O}_X$), but it still requires the machinery of locally ringed spaces.

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[1]Intuitionistically, the statement that any finitely generated vector space is *free* is stronger than the doubly negated version and cannot be shown. It would imply that any sheaf of finite type is not only locally free on some dense open subset, but locally free on the entire space. We discuss this example in more detail in Section 5 and in particular in Lemma 5.8. A proof of Grothendieck's generic freeness lemma in its full form is given in Section 11.5.

For concreteness, here is the standard intuitionistic proof that any finitely generated vector space V is *not not* free. Let $(x_1, \ldots, x_n)$ be a generating family. If $n = 0$, we are done. Else it's *not not* the case that either some x_i can be expressed as a linear combination of the other vectors, or not. The former implies that $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is a generating family, whereby we can appeal to induction. The latter implies that $(x_1, \ldots, x_n)$ is linearly independent and therefore a basis. In both cases it follows that V is free; thus it's *not not* the case that V is free.

In this argument, we used the intuitionistically valid proof scheme $\neg\neg\varphi \wedge (\varphi \Rightarrow \psi) \Longrightarrow \neg\neg\psi$. We expand on this in Section 2.4.

The internal language of the *big* Zariski topos of X allows for a more far-reaching change of perspective. It incorporates Grothendieck's functor-of-points philosophy in order to cast modern algebraic geometry, relative to the arbitrary base scheme X, in a naive *synthetic* language reminiscent of the classical Italian school.

The synthetic approach is best explained by contrasting it with the usual approach to scheme theory, which is to layer it upon some standard form of set theory: to give a scheme means to firstly give a set of points; then to describe a topology on this set; and finally to equip the resulting space with a local sheaf of rings. Basic objects of study in algebraic geometry, such as closed subschemes of projective spaces, are in this way encoded using a large amount of machinery.

There is also a somewhat lesser used, but philosophically rewarding and more "economical" approach within set theory: Grothendieck's functorial approach. In this account of scheme theory, to give a scheme means to give a functor from the category of commutative rings to the category of sets. For instance, the Fermat scheme is given by the functor

$$A \longmapsto \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\},$$

that is by a *scheme* in the colloquial sense for prescribing a set of solutions for any ring.

This approach requires fewer preparations and involves only objects of intrinsic interest to algebraic geometry: A-valued points, where A ranges over all rings. These tend to be better behaved, for instance in that the set of A-valued points of a product of schemes is isomorphic to the product of the sets of A-valued points, and are more fundamental from a geometric point of view. In contrast, the set-theoretical points of a scheme in the approach using locally ringed spaces actually parameterize irreducible closed subsets, not points in an intuitive sense.

The canonical reference for the functorial approach is the book by Demazure and Gabriel [45]. A summary in English, including a proof of the equivalence with the approach using locally ringed spaces, is contained in the first chapter of [131]. At the Secret Blogging Seminar, there was an insightful long-running discussion on the merits of the functorial approach [112], and further philosophical background is contained in [89]. The thesis [84] contains recent developments on an abstract theory of gluing local models.

The description of basic objects can still be somewhat involved in the functorial approach. For instance, while the functor associated to projective n-space is given on fields by the simple expression

$$\begin{aligned} K \longmapsto &\text{ the set of lines through the origin in } K^{n+1} \\ &\cong \{[x_0 : \cdots : x_n] \mid x_i \neq 0 \text{ for some } i\}, \end{aligned}$$

on general rings it is given by

$$\begin{aligned} A \longmapsto &\text{ the set of quotients } A^n \twoheadrightarrow P, \text{ where } P \text{ is projective,} \\ &\text{modulo isomorphism.} \end{aligned}$$

On the one hand, typically only field-valued points admit a simple description. On the other hand, the A-valued points for more general rings A are crucial in order to impart a meaningful sense of cohesion on the field-valued points. They therefore can't simply be dropped.[2]

[2]For instance, let $\underline{\mathbb{A}}^1 : A \mapsto A$ be the functor associated to the affine line. Yoneda's lemma guarantees that the set of morphisms $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ in the functor category $[\mathrm{Ring}, \mathrm{Set}]$ is in canonical bijection with the set $\mathbb{Z}[U]$, as one would expect: Algebraic functions $\mathbb{A}^1 \to \mathbb{A}^1$ should be given by polynomials. (The discussion could also be relativized so that the answer is the polynomial ring $k[U]$, where k is some base field.) However, if we calculate the set of morphisms in $[\mathrm{Field}, \mathrm{Set}]$ we obtain $\int_{K \in \mathrm{Field}} \mathrm{Hom}(K, K)$, a set which contains pathological functions such as some which permute the elements of the prime fields in arbitrary ways.

We can resolve the tension by incorporating an automatic management of the *stage of definition*, the rings A such that we're considering A-valued points, into our language. Such a language is provided by the internal language of the big Zariski topos. It allows for the Fermat scheme to be given by the naive expression

$$\{(x,y,z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

and for projective n-space to be given by either of the expressions

$$\text{the set of lines through the origin in } (\underline{\mathbb{A}}^1)^{n+1} \quad \text{or} \\ \{[x_0 : \cdots : x_n] \mid x_i \neq 0 \text{ for some } i\}.$$

This is not a specialized trick to give short descriptions of some schemes: Like with the internal universe of any topos, the full power of intuitionistic logic is available to reason about the objects constructed in this way.

We can thus add an approach to the list of ways of giving a rigorous foundation to algebraic geometry, the synthetic approach which layers scheme theory not upon a classical set theory, but rather directly encodes schemes as sets and morphisms of schemes as maps of sets in the nonclassical universe provided by the big Zariski topos of a base scheme. We can therefore use a simple, element-based language to talk about schemes.

This is similar to synthetic approaches to other fields of mathematics, such as differential geometry [76], domain theory [64], computability theory [15], and more recently and very successfully homotopy theory [128] and related subjects [108, 109, 106]. The synthetic approaches allow in each case to encode the objects of study directly as (nonclassical) sets, with geometric, domain-theoretic, computability-theoretic, or homotopic structure being automatically provided for.

The implicit algebro-geometric structure has visible consequences on the internal universe of the big Zariski topos and endows it with a distinctive algebraic flavor. For instance, the statement "*any* map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial function" holds from the internal point of view. This is also a property which sets the internal universe of the big Zariski topos apart from the toposes studied in synthetic differential geometry.

If one is content with building upon classical scheme theory, the big Zariski topos $\mathrm{Zar}(X)$ of a base scheme X can be constructed as the topos of sheaves on the Grothendieck site Sch/X of X-schemes.[3] Explicitly, an object of $\mathrm{Zar}(X)$ is a functor $F : (\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition with respect to Zariski coverings: If $T = \bigcup_i U_i$ is a cover of an X-scheme T by open subsets, the diagram

$$F(T) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

should be an equalizer diagram. A premier example of an object of $\mathrm{Zar}(X)$ is the functor $\underline{Y}$ of points associated to an X-scheme Y, mapping an X-scheme T to $\mathrm{Hom}_X(T,Y)$. It satisfies the gluing condition since one can glue morphisms of schemes in the Zariski topology.

The object $\underline{\mathbb{A}}^1$ which already appeared is the functor of points of the affine line over X, the X-scheme $\mathbb{A}^1_X := X \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{Z}[U]$. Its value on an X-scheme T is

$$\underline{\mathbb{A}}^1_X(T) = \mathrm{Hom}_X(T, \mathbb{A}^1_X) \cong \mathrm{Hom}_{\operatorname{Spec}\mathbb{Z}}(T, \operatorname{Spec}\mathbb{Z}[U]) \cong \Gamma(T, \mathcal{O}_T).$$

This object has a canonical structure as a ring object in $\mathrm{Zar}(X)$. In fact, from the internal point of view of $\mathrm{Zar}(X)$, it is a local ring and even a field in the sense that

[3]Some care is needed in order to avoid set-theoretical issues of size. We discuss this fine point in Section 15. If one is interested in foundational questions and doesn't merely want to use the big Zariski topos in order to employ its convenient internal language, one can rest assured that there's a way to construct it without resorting to classical scheme theory. We sketch this in Section 16.5.

nonzero elements are invertible. In the case $X = \operatorname{Spec}\mathbb{Z}$, this fact was first observed by Kock [77]. At the same time, it is not a reduced ring – a feat possible only in an intuitionistic context. This curious interplay is quite important, since the sets

$$\{x : \underline{\mathbb{A}}^1_X \mid x = 0\} \quad \text{and} \quad \{x : \underline{\mathbb{A}}^1_X \mid x^2 = 0\}$$

should and do describe two different X-schemes: the first is isomorphic to X while the second is an infinitesimal thickening of X, the vanishing scheme of U^2 in $\mathbb{A}^1_X$. In contrast, the sets $\{x : \underline{\mathbb{A}}^1_X \mid x \neq 0\}$ and $\{x : \underline{\mathbb{A}}^1_X \mid x^2 \neq 0\}$ should and do coincide. By the field property, both conditions are equivalent to x being invertible.

The synthetic spectrum of an $\underline{\mathbb{A}}^1_X$-algebra A can be defined as

$$\operatorname{Spec}(A) := \text{the set of } \underline{\mathbb{A}}^1_X\text{-algebra homomorphism } A \to \underline{\mathbb{A}}^1_X.$$

On first sight, this definition seems to overlook potential non-maximal prime ideals of A, since it only gives the $\underline{\mathbb{A}}^1_X$-valued points. But in fact, this description correctly reflects the relative spectrum construction. It yields a simple correspondence between synthetic affine schemes and solution sets of polynomial equations. For instance, it's easy to show that there is a canonical isomorphism

$$\begin{aligned}&\operatorname{Spec}(\underline{\mathbb{A}}^1_X[U_1, \ldots, U_n]/(f_1, \ldots, f_m)) \cong \\ &\quad \{(u_1, \ldots, u_n) : (\underline{\mathbb{A}}^1_X)^n \mid f_1(u_1, \ldots, u_n) = \cdots = f_m(u_1, \ldots, u_n) = 0\}.\end{aligned}$$

We give internal descriptions of further constructions of relative scheme theory in Section 19.

In order to be able to reason internally (in contrast to only using the internal language to describe X-schemes and more general spaces in a simple language), it's crucial to have strong and meaningful axioms available. One such axiom posits that $\underline{\mathbb{A}}^1_X$ is a local and *synthetically quasicoherent* ring and implies all known ring-theoretic properties of $\underline{\mathbb{A}}^1_X$. Synthetic quasicoherence is the internal analog of the usual condition on a sheaf of modules to be quasicoherent. This notion doesn't have a counterpart in synthetic differential geometry and is central to our account of synthetic algebraic geometry, since we derive all of its basic concepts such as open and closed immersions and synthetic schemes from it.

Modal operators are useful in the big topos setting as well. For instance, there is a modal operator $\Box_{\text{ét}}$ in the big Zariski topos such that the internal statement $\Box_{\text{ét}}\varphi$ roughly means that φ holds on an étale covering and such that the translated formula $\varphi^{\Box_{\text{ét}}}$ means that φ holds in the *big étale topos* familiar from étale cohomology. In this way, we can access the internal universe of the big étale topos from within the big Zariski topos. The ring $\underline{\mathbb{A}}^1_X$ enjoys additional properties when studied in the étale topos, where it is separably closed, in the fppf topos, where it is *fppf-local*, and in the ph topos, where it satisfies a strong form of algebraic closure.

Limitations. The internal language is *local*, in the sense that if $X = \bigcup_i U_i$ is an open covering and an internal statement holds in the sheaf toposes $\mathrm{Sh}(U_i)$, it holds in $\mathrm{Sh}(X)$ as well. On the one hand, this property is very useful. But on the other hand, it causes an inherent limitation of the internal language: Global properties of sheaves of modules like "being generated by global sections", "being ample", or "having vanishing sheaf cohomology" and global properties of schemes like "being quasicompact" can *not* be expressed in the internal language.

Thus for global considerations, the internal language of $\mathrm{Sh}(X)$ is only useful in that local subparts can be simplified. Also, some global features reflect themselves in certain metaproperties of the internal language. For instance, a scheme is quasicompact if and only if the internal language has a weak version of the so-called disjunction property of mathematical logic (Section 7).

The locality limitation only refers to locality with respect to the base scheme. For instance, the little and big Zariski toposes of X *can* distinguish between affine and projective n-space over X, even though these are locally isomorphic.

The internal languages of both toposes can be used on a case-by-case basis, employing them as part of longer arguments in the context of ordinary scheme theory where it's useful to do so. However, if one wants to stay solely in one of the provided internal universes and not use ordinary scheme theory at all, then one will of course run into the further limitation that internal scheme theory, as put forward in this thesis, is only developed to a small amount.

Introductory literature. This text is intended to be self-contained, requiring only basic knowledge of scheme theory. In particular, we assume no prior familiarity with topos theory or formal logic.

Nevertheless, a gentle introduction to topos theory is an article by Leinster [81]. Standard references for the internal language of a topos include the book of Mac Lane and Moerdijk [86, Chapter VI], the book of Goldblatt [55, Chapter 14], Caramello's and Streicher's lecture notes [32, 119], the book of Borceux [25, Chapter 6], and Part D of Johnstone's Elephant [67]. Motivation and background on the internal language can also be found in Chapter 0 of Shulman's lecture notes [113].

In the 1970s, there was a flurry of activity on applications of the internal language. An article by Mulvey [94] of this time gives a very accessible introduction to the topic, culminating in an internal proof of the Serre–Swan theorem (with just one external ingredient needed).

Related work. The internal language of toposes was applied to algebraic geometry before. For instance, Wraith used it to construct (and verify the universal property of) the little étale topos of a scheme by internally developing the theory of strict henselization [136]. However, to the best of our knowledge, systematically building a dictionary relating external and internal notions has not been attempted before, and the use of modal operators to study the spreading of properties from points to neighborhoods seems to be new as well.

In particular, Tierney remarked in 1976 that a certain property of the internal universe of the little Zariski topos "is surely important, though its precise significance is still somewhat obscure" [126, p. 209]. This property can now be recognized as a small shadow of the internal characterization of quasicoherence. We expand on this in Section 3.3.

In some regards, this thesis is an extended answer to a MathOverflow question by Gubkin [57].

Brandenburg put forward a related program of internalization in his PhD thesis [27]. However, he internalizes constructions of algebraic geometry not in toposes, but in tensor categories. There is some overlap in working out precise universal properties, particularly when dealing with the big Zariski topos.

In other branches of mathematics, the internal language of toposes is used as well. For instance, there is an ongoing effort in mathematical physics to understand quantum mechanical systems from an internal point of view: To any quantum mechanical system, one can associate a so-called Bohr topos containing an internal mirror image of the system. This mirror image looks like a system of classical mechanics from the internal perspective, and therefore tools like Gelfand duality can be used to construct an internal phase space for the system [28, 63, 60].

In stochastics, the usefulness of an internal language was recently stressed by Tao [121]. Such a language makes the common notational practice of dropping the explicit dependence of the value $X(\omega)$ of a random variable on the sample ω completely rigorous and simplifies the basic theory. Tao also highlighted how a suitable language can be used to simplify "ε/δ management" in analysis [120]. Furthermore, there is a topos-theoretic approach to measure theory, in which the sheaf of measurable real functions on a σ-algebra looks like the ordinary set of real numbers from an internal point of view [65]; this has applications in noncommutative geometry [62].

Intuitionistic methods have found many applications in computer science. Recently, the internal language of a topos of trees and a suitable modal operator was used to study guarded recursion, encompassing, for instance, an internal Banach fixed-point theorem [21].

In constructive mathematics, the internal language of toposes is routinely used to obtain models of intuitionistic theories fulfilling certain anti-classical axioms. For instance, there are toposes in which the axiom "any map $\mathbb{R} \to \mathbb{R}$ is continuous" (appropriately formulated) holds [76, 93] and toposes in which the Church–Turing thesis "any map $\mathbb{N} \to \mathbb{N}$ is computable" holds (certain realizability toposes). The internal language can also be used to extract computational content out of classical constructions. To cite just one recent example, Mannaa and Coquand used it to implement algorithms for working with the algebraic closure of an arbitrary field of characteristic zero [88].

One way this thesis contributes to the program of constructive mathematics is that intuitionistic mathematics gains new areas of application. For instance, the constructive account of the theory of Krull dimension was originally developed to remove Noetherian hypotheses, extract computational meaning, and simplify proofs [37, 40]. It can now also be used to reason about the dimension of schemes, since the topological dimension of a scheme X coincides with the Krull dimension of the structure sheaf $\mathcal{O}_X$ regarded as an ordinary ring from the internal perspective of $\mathrm{Sh}(X)$ (Section 3.4).

We obtained a second contribution to constructive mathematics as a byproduct of deducing transfer principles which relate a module over a ring A with its induced quasicoherent sheaf on $\operatorname{Spec} A$: Using the internal language of the little Zariski topos we can algorithmically turn certain non-constructive arguments concerning prime ideals into constructive ones. We discuss this in Section 11.4; it is related to the *dynamical methods in algebra* explored by Coquand, Coste, Lombardi, Roy, and others [44, 38].

Caramello uses topos theory to build bridges between different mathematical subjects, in a certain precise sense [31, 29]. She exploits that toposes can admit presentations by sites of different character. Our contribution is certainly related to her grand research program in spirit, but since we focus only on specific presentations of a few specific toposes associated to schemes, there is as yet only few direct technical connections.

Notational conventions. To stress that a discussion takes place in an intuitionistic context, we occasionally write "$\forall x : X$" or "$\exists x : X$" instead of "$\forall x \in X$" and "$\exists x \in X$" not only in internal statements, where it's proper to do so, but also when not reasoning internally.

If X and Y are sets, we mean by "$[X, Y]$" the set of all maps from X to Y. This expression will often occur in internal formulas; its external meaning will then be the Hom sheaf. We write pairs $\langle a, b\rangle$ using angle brackets. The preimage of a set M under a map f is written "$f^{-1}[M]$". Similarly, the image is written "$f[N]$".

The constant sheaf with stalks M is written "$\underline{M}$".

2. The internal language of a sheaf topos

At its heart, the internal language of a topos provides a coherent way of translating any mentions of set-theoretical elements to *generalized elements*, carefully keeping track of and adapting the stage of definition. We want to illustrate this with a simple example before giving the formal definition.

A map $f : X \to Y$ of sets is injective if and only if

$$\forall x, x' \in X.\ f(x) = f(x') \Longrightarrow x = x'. \tag{1}$$

This condition can not only be interpreted in Set, but in any category $\mathcal{C}$ whose objects are structured sets and whose morphisms are maps between the underlying sets. If we want to go beyond such kind of categories, we have to restate the condition in purely category-theoretic language:

$$\forall (1 \xrightarrow{x} X), (1 \xrightarrow{x'} X).\ f \circ x = f \circ x' \Longrightarrow x = x'. \tag{2}$$

This condition makes sense in all categories which contain a terminal object 1, and is equivalent to condition (1) in the case $\mathcal{C} = \mathrm{Set}$. This has a deeper reason: The one-element set $1 = \{\star\}$ is a *separator* of Set, that is objects of Set are uniquely determined by their *global elements*, morphisms from the terminal object.

However, in categories in which the terminal object is not a separator, condition (2) is not very meaningful. This is for instance the case if $\mathcal{C}$ is the category $\mathrm{Sh}(X)$ of set-valued sheaves on a topological space X. Global elements of a sheaf $\mathcal{F}$ are in natural one-to-one correspondence with global sections $s \in \mathcal{F}(X)$ (hence the name), whereby condition (2) only states that f is *injective on global sections*. Since many interesting sheaves admit no or only few global sections, this statement is typically not very substantial.

A basic tenet of category theory is therefore to not only refer to global elements $1 \to X$, but also to *generalized elements* $A \to X$, where A ranges over all objects. The domain A is called the *stage of definition* in this context. Bearing this principle in mind, a better translation of the injectivity condition is the statement

$$\forall \text{objects } A \text{ in } \mathcal{C}.\ \forall (A \xrightarrow{x} X), (A \xrightarrow{x'} X) \text{ in } \mathcal{C}.\ \ f \circ x = f \circ x' \implies x = x'. \tag{3}$$

This statement expresses that f is a monomorphism and therefore correctly captures the structural essence of injectivity.

Unlike this manual translation guided by trial and error and categorical philosophy, the internal language provides a purely mechanical translation scheme. It is fully formal, can be analyzed rigorously, works smoothly with arbitrarily convoluted statements, and most importantly can be trusted to support *reasoning*: If a statement formulated in a naive element-based language intuitionistically implies a further such statement, then the translation of the former implies the translation of the latter.

The power of the internal language doesn't unfold in basic situations like with the example above, where one can easily translate statements and even proofs by hand. It unfolds when considering more complex statements. For instance, the short proof of Grothendieck's generic freeness lemma promised in the introduction rests on the internal statement "any ideal of $\mathcal{O}_{\mathrm{Spec}(R)}[U_1, \ldots, U_n]$ is *not not* finitely generated", where R is a reduced ring. For the proof of Grothendieck's generic freeness lemma it's not necessary to actually perform the translation of this statement into external language, but for definiteness we display the translation here nevertheless:

> For any element $f \in R$ and any (not necessarily quasicoherent) sheaf of ideals $\mathcal{J} \hookrightarrow \mathcal{O}_{\mathrm{Spec}(R)}[U_1, \ldots, U_n]|_{D(f)}$: If
> for any element $g \in R$ the condition that
> the sheaf $\mathcal{J}$ is of finite type on $D(g)$
> implies that $g = 0$,
> then $f = 0$.

This statement is obviously quite convoluted, and its proof is even more so; therefore it probably wouldn't occur to one to base a proof of Grothendieck's generic freeness lemma on this statement. The internal language is thus of real use here. We'll expand on this example in Section 3.9 and in Section 11.5.[4]

2.1. Internal statements. Let X be a topological space. Later, X will be the underlying space of a scheme. The meaning of internal statements is given by a set of rules, the *Kripke–Joyal semantics* of the topos of sheaves on X.

Definition 2.1. The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ over U is given by the rules listed in Table 1, recursively in the structure of φ. In a *formula over* U there may appear sheaves defined on U as domains of quantifications, U-sections of sheaves as terms, and morphisms of sheaves on U as function symbols. If $V \subseteq U$ is an open subset, then formulas over U can be pulled back to formulas over V. The symbols "$\top$" and "$\bot$" denote truth and falsehood, respectively. The universal and existential quantifiers come in two flavors: for bounded and unbounded quantification. The translation of $U \models \neg\varphi$ does not have to be separately defined, since negation can be expressed using other symbols: $\neg\varphi :\equiv (\varphi \Rightarrow \bot)$. If we want to emphasize the particular topos, we write

$$\mathrm{Sh}(X) \models \varphi \quad :\Longleftrightarrow \quad X \models \varphi.$$

Remark 2.2. The last two rules in Table 1, concerning *unbounded quantification*, are not part of the classical Kripke–Joyal semantics. They are part of Mike Shulman's stack semantics [114], a slight extension. They are needed so that we can formulate universal properties in the internal language.

Example 2.3. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is a monomorphism of sheaves if and only if, from the internal perspective, α is simply an injective map:

$$\begin{aligned}
& X \models \ulcorner\alpha \text{ is injective}\urcorner \\
\Longleftrightarrow\ & X \models \forall s\,{:}\,\mathcal{F}.\ \forall t\,{:}\,\mathcal{F}.\ \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s \in \Gamma(U, \mathcal{F})\text{:} \\
& \qquad \text{for all open } V \subseteq U, \text{ sections } t \in \Gamma(V, \mathcal{F})\text{:} \\
& \qquad\qquad V \models \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s \in \Gamma(U, \mathcal{F})\text{:} \\
& \qquad \text{for all open } V \subseteq U, \text{ sections } t \in \Gamma(V, \mathcal{F})\text{:} \\
& \qquad\qquad \text{for all open } W \subseteq V\text{:} \\
& \qquad\qquad\qquad \alpha_W(s|_W) = \alpha_W(t|_W) \text{ implies } s|_W = t|_W \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F})\text{:} \\
& \qquad \alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U
\end{aligned}$$

[4] The statement can be proven by hand, but it's much simpler to only verify the case $n = 0$ (and even reduce this case to simple other properties which $\mathcal{O}_{\mathrm{Spec}(R)}$ enjoys from the internal point of view) and then to apply Hilbert's basis theorem. Hilbert's basis theorem is famous for admitting only a nonconstructive proof, and nonconstructive proofs can't be translated by the internal language machinery; but this is only true for the conclusion "any ideal is finitely generated". The intuitionistically weaker conclusion "any ideal is *not not* finitely generated" does admit a constructive proof, and is all what's needed here.

$U \models s = t : \mathcal{F}$	$:\Longleftrightarrow$	$s\|_U = t\|_U \in \Gamma(U, \mathcal{F})$
$U \models s \in \mathcal{G}$	$:\Longleftrightarrow$	$s\|_U \in \Gamma(U, \mathcal{G})$ ($\mathcal{G}$ a subsheaf of $\mathcal{F}$, s a section of $\mathcal{F}$)
$U \models \top$	$:\Longleftrightarrow$	$U = U$ (always fulfilled)
$U \models \bot$	$:\Longleftrightarrow$	$U = \emptyset$
$U \models \varphi \wedge \psi$	$:\Longleftrightarrow$	$U \models \varphi$ and $U \models \psi$
$U \models \bigwedge_{j \in J} \varphi_j$	$:\Longleftrightarrow$	for all $j \in J$: $U \models \varphi_j$ (J an index set)
$U \models \varphi \vee \psi$	$:\Longleftrightarrow$	~~$U \models \varphi$ or $U \models \psi$~~
		there exists a covering $U = \bigcup_i U_i$ such that for all i: $U_i \models \varphi$ or $U_i \models \psi$
$U \models \bigvee_{j \in J} \varphi_j$	$:\Longleftrightarrow$	~~$U \models \varphi_j$ for some $j \in J$~~ (J an index set)
		there exists a covering $U = \bigcup_i U_i$ such that for all i: $U_i \models \varphi_j$ for some $j \in J$
$U \models \varphi \Rightarrow \psi$	$:\Longleftrightarrow$	~~$U \models \varphi$ implies $U \models \varphi$~~
		for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$
$U \models \forall s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	for all sections $s \in \Gamma(V, \mathcal{F})$ on open $V \subseteq U$: $V \models \varphi(s)$
$U \models \exists s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	~~there exists a section $s \in \Gamma(U, \mathcal{F})$ such that $U \models \varphi(s)$~~
		there exists an open covering $U = \bigcup_i U_i$ such that for all i: there exists $s_i \in \Gamma(U_i, \mathcal{F})$ such that $U_i \models \varphi(s_i)$
$U \models \forall \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	for all sheaves $\mathcal{F}$ on open $V \subseteq U$: $V \models \varphi(\mathcal{F})$
$U \models \exists \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	there exists an open covering $U = \bigcup_i U_i$ such that for all i: there exists a sheaf $\mathcal{F}_i$ on U_i such that $U_i \models \varphi(\mathcal{F}_i)$

TABLE 1. The Kripke–Joyal semantics of a sheaf topos.

$\Longleftrightarrow$ α is a monomorphism of sheaves

The corner quotes "$\ulcorner \ldots \urcorner$" indicate that translation into formal language is left to the reader. Similarly, α is an epimorphism of sheaves if and only if, from the internal perspective, α is a surjective map. Notice that injectivity and surjectivity are notions of a simple element-based language. The Kripke–Joyal semantics takes care to properly handle *all* sections, not only global ones.

The rules are not all arbitrary. They are finely concerted to make the following two propositions true, which are crucial for a proper appreciation of the internal language.

Proposition 2.4 (Locality of the internal language)**.** *Let $U = \bigcup_i U_i$ be covered by open subsets. Let φ be a formula over U. Then*

$$U \models \varphi \qquad \textit{iff} \qquad U_i \models \varphi \textit{ for each } i.$$

Proof. Induction on the structure of φ. The canceled rules in Table 1 would make this proposition false. □

As a corollary, one may restrict the open coverings and universal quantifications in the the definition of the Kripke–Joyal semantics (Table 1) to open subsets of some basis of the topology. For instance, if X is a scheme, one may restrict to affine open subsets.

Furthermore, Proposition 2.4 shows that the internal language is monotone in the following sense: If $U \models \varphi$, and V is an open subset of U, then $V \models \varphi$. (This follows by applying the proposition to the trivial covering $U = V \cup U$.)

Proposition 2.5 (Soundness of the internal language)**.** *If a formula φ implies a further formula ψ in intuitionistic logic, then $U \models \varphi$ implies $U \models \psi$.*

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that on each U_i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in Appendix 24. □

In particular, if a formula ψ has an unconditional intuitionistic proof, then $U \models \psi$.

The restriction to intuitionistic logic is really necessary at this point. We will encounter many examples of classically equivalent internal statements whose translations using the Kripke–Joyal semantics are wildly different. To anticipate just one example, the statement

$$X \models \ulcorner \mathcal{F} \text{ is finite free} \urcorner,$$

referring to a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, means that $\mathcal{F}$ is finite locally free. The statement

$$X \models \neg\neg(\ulcorner \mathcal{F} \text{ is finite free} \urcorner)$$

instead means that $\mathcal{F}$ is finite locally free on a dense open subset of X.

In particular, our treatment of modal operators to understand spreading of properties from points to neighborhoods depends on having the ability to make finer distinctions – distinctions which are not visible in classical logic. In Section 2.4 there is a discussion of what the restriction to intuitionistic logic amounts to in practice.

Because of the multitude of quantifiers, literal translations of internal statements can sometimes get slightly unwieldy. There are simplification rules for certain often-occurring special cases:

Proposition 2.6.

$$U \models \forall s\!:\!\mathcal{F}.\ \forall t\!:\!\mathcal{G}.\ \varphi(s,t) \iff \textit{for all open } V \subseteq U, \textit{ sections } s \in \Gamma(V,\mathcal{F}),\ t \in \Gamma(V,\mathcal{G})\textit{: } V \models \varphi(s,t)$$

$$U \models \forall s\!:\!\mathcal{F}.\ \varphi(s) \Rightarrow \psi(s) \iff \textit{for all open } V \subseteq U, \textit{ sections } s \in \Gamma(V,\mathcal{F})\textit{: } V \models \varphi(s) \textit{ implies } V \models \psi(s)$$

$$U \models \exists! s\!:\!\mathcal{F}.\ \varphi(s) \iff \textit{for all open } V \subseteq U, \textit{ there is exactly one section } s \in \Gamma(V,\mathcal{F}) \textit{ with: } V \models \varphi(s)$$

Proof. Straightforward. By way of example, we prove the existence claim in the "only if" direction of the last rule. (This rule formalizes the saying "unique existence implies global existence".) By definition of $\exists!$, it holds that

$$U \models \exists s\!:\!\mathcal{F}.\ \varphi(s) \qquad \text{and} \qquad U \models \forall s,t\!:\!\mathcal{F}.\ \varphi(s) \wedge \varphi(t) \Rightarrow s = t.$$

Let $V \subseteq U$ be an arbitrary open subset. Then there exist local sections $s_i \in \Gamma(V_i, \mathcal{F})$ such that $V_i \models \varphi(s_i)$, where $V = \bigcup_i V_i$ is an open covering. By the locality of the internal language, on intersections it holds that $V_i \cap V_j \models \varphi(s_i)$, so by the uniqueness assumption, it follows that the local sections agree on intersections. They therefore glue to a section $s \in \Gamma(V, \mathcal{F})$. Since $V_i \models \varphi(s)$ for all i, the locality of the internal language allows us to conclude that $V \models \varphi(s)$. □

Remark 2.7. Note that $\mathrm{Sh}(X) \models \neg\varphi$ is in general a much stronger statement than merely saying that $\mathrm{Sh}(X) \models \varphi$ does not hold: The former always implies the latter (unless $X = \emptyset$, in which case *any* internal statement is true), but the converse does not hold: The former statement means that $U = \emptyset$ is the *only* open subset on which φ holds, that is that φ holds *nowhere*. In contrast, the statement $\mathrm{Sh}(X) \not\models \varphi$ only means that φ does *not hold everywhere*.

For instance, let X be a quasicompact scheme and let $f \in \Gamma(X, \mathcal{O}_X)$ be a global function on X. We will see in Section 3 that "$\mathrm{Sh}(X) \not\models \ulcorner f \text{ is invertible}\urcorner$" means that f is not an invertible function in the ring $\Gamma(X, \mathcal{O}_X)$. In contrast, "$\mathrm{Sh}(X) \models \neg(\ulcorner f \text{ is invertible}\urcorner)$" means that f is *nowhere invertible*, not even on smaller non-empty open subsets. This implies that f is nilpotent.

2.2. Internal constructions. The Kripke–Joyal semantics defines the interpretation of internal *statements*. The interpretation of internal *constructions* is given by the following definition.

Definition 2.8. The interpretation of an internal construction T is denoted by $[\![T]\!] \in \mathrm{Sh}(X)$ and given by the following rules.

- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \times \mathcal{G}]\!]$ is the categorical product of $\mathcal{F}$ and $\mathcal{G}$ (i. e. their product as presheaves).
- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \amalg \mathcal{G}]\!]$ is the categorical coproduct of $\mathcal{F}$ and $\mathcal{G}$, i. e. the sheafification of the presheaf $U \mapsto \Gamma(U, \mathcal{F}) \amalg \Gamma(U, \mathcal{G})$.
- If $\mathcal{F}$ is a sheaf, the interpretation $[\![\mathcal{P}(\mathcal{F})]\!]$ of the power set construction is the sheaf given by
$$U \subseteq X \text{ open} \quad \longmapsto \quad \{\mathcal{G} \hookrightarrow \mathcal{F}|_U\},$$
i. e. sections on an open set U are subsheaves of $\mathcal{F}|_U$ (either literally or isomorphism classes of arbitrary monomorphisms into $\mathcal{F}|_U$).
- If $\mathcal{F}$ is a sheaf and $\varphi(s)$ is a formula containing a free variable $s : \mathcal{F}$, the interpretation $[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]$ is given by the subpresheaf of $\mathcal{F}$ defined by
$$U \subseteq X \text{ open} \quad \longmapsto \quad \{s \in \Gamma(U, \mathcal{F}) \mid U \models \varphi(s)\}.$$
By the locality of the internal language, this presheaf is in fact a sheaf.

The definition is made in such a way that, from the internal perspective, the constructions enjoy their expected properties. For instance, it holds that
$$\mathrm{Sh}(X) \models \big(\forall x : [\![\{s : \mathcal{F} \mid \varphi(s)\}]\!].\ \psi(x)\big) \iff \big(\forall x : \mathcal{F}.\ \varphi(x) \Rightarrow \psi(x)\big).$$
We gloss over several details here. See [67, Section D4.1] for a proper treatment.

Morphisms can internally be constructed by appealing to the *principle of unique choice*: Let $\varphi(s, t)$ be a formula with free variables of type $s : \mathcal{F}$, $t : \mathcal{G}$. Assume
$$\mathrm{Sh}(X) \models \forall s : \mathcal{F}.\ \exists! t : \mathcal{G}.\ \varphi(s, t).$$
Then there is one and only one morphism $\alpha : \mathcal{F} \to \mathcal{G}$ of sheaves such that for any local section $s \in \Gamma(U, \mathcal{F})$, $\mathrm{Sh}(X) \models \varphi(s, \alpha(s))$. This follows from the meaning of unique existence with the Kripke–Joyal semantics (Proposition 2.6).

An important application is showing that two sheaves $\mathcal{F}$ and $\mathcal{G}$ are isomorphic (usually as objects with more structure, for instance sheaves of modules). To this end, it suffices to give a formula $\varphi(s, t)$ satisfying, in addition to the condition above, the condition $\mathrm{Sh}(X) \models \forall t\colon \mathcal{G}.\ \exists! s\colon \mathcal{F}.\ \varphi(s,t)$, expressing that the induced morphism α is a bijective map from the internal perspective. This implies the statement

$$\mathrm{Sh}(X) \models \exists \alpha\colon \mathcal{H}\mathrm{om}(\mathcal{F},\mathcal{G}).\ \ulcorner \alpha \text{ is bijective} \urcorner,$$

but this statement is strictly weaker: Its interpretation with the Kripke–Joyal semantics is that the sheaves $\mathcal{F}$ and $\mathcal{G}$ are *locally* isomorphic.

2.3. Geometric formulas and constructions. In formal and categorical logic so-called geometric formulas play a special role. They are named that way because, in a sense which can be made precise, their meaning is preserved under pullback with geometric morphisms.

Definition 2.9. A formula is *geometric* if and only if it consists only of

$$= \quad \in \quad \top \quad \bot \quad \wedge \quad \vee \quad \bigvee \quad \exists,$$

but not "$\bigwedge$" nor "$\Rightarrow$" nor "$\forall$" (and thus not "$\neg$" either, since negation is defined using "$\Rightarrow$"). A *geometric implication* is a formula of the form

$$\forall \cdots \forall.\ (\cdots) \Rightarrow (\cdots)$$

with the bracketed subformulas being geometric.

The *parameters* of a formula φ are the sheaves being quantified over, sections of sheaves appearing as terms, and morphisms of sheaves appearing as function symbols in φ. We say that a formula φ holds *at a point* $x \in X$ if and only if the formula obtained by substituting all parameters in φ with their stalks at x holds in the usual mathematical sense.

Lemma 2.10. *Let $x \in X$ be a point. Let φ be a geometric formula (over some open neighborhood V of x). Then φ holds at x if and only if there exists an open neighborhood $U \subseteq X$ of x (contained in V) such that φ holds on U.*

Proof. This is a very general instance of the phenomenon that sometimes, truth at a point spreads to truth on a neighborhood. It can be proven by induction on the structure of φ, but we will give a more conceptual proof later (Corollary 6.33). $\square$

This lemma is a very useful metatheorem. We will properly discuss its significance in Section 6.7. For now, we just use it to prove a simple criterion for the internal truth of a geometric implication; we will apply this criterion many times.

Corollary 2.11. *A geometric implication holds on X if and only if it holds at every point of X.*

Proof. For notational simplicity, we consider a geometric implication of the form

$$\forall s\colon \mathcal{F}.\ \varphi(s) \Rightarrow \psi(s).$$

For the "only if" direction, assume that this formula holds on X and let $x \in X$ be an arbitrary point. Let $s_x \in \mathcal{F}_x$ be the germ of an arbitrary local section s of $\mathcal{F}$ and assume that $\varphi(s)$ holds at x. By Lemma 2.10, it follows that $\varphi(s)$ holds on some open neighborhood of x. By assumption, $\psi(s)$ holds on this neighborhood as well. Again by the lemma, $\psi(s)$ holds at x.

For the "if" direction, assume that the geometric implication holds at every point. Let $U \subseteq X$ be an arbitrary open subset and let $s \in \Gamma(U,\mathcal{F})$ be a local section such that $\varphi(s)$ holds on U. By the lemma and the locality of the internal language, to show that $\psi(s)$ holds on U, it suffices to show that $\psi(s)$ holds at every point of U. This is clear, since again by the lemma, $\varphi(s)$ holds at every point of U. $\square$

Example 2.12. Injectivity and surjectivity are geometric implications (surjectivity can be spelled $\forall y : \mathcal{G}.\ (\top \Rightarrow \exists x : \mathcal{F}.\ \alpha(x) = y)$). Thus Corollary 2.11 gives a deeper reason for the well-known fact that a morphism of sheaves is a monomorphism resp. an epimorphism if and only if it is stalkwise injective resp. surjective.

A construction is *geometric* if and only if it commutes with pullback under arbitrary geometric morphisms. We do not want to discuss the notion of geometric morphisms here; suffice it to say that calculating the stalk at a point $x \in X$ is an instance of such a pullback. Among others, the following constructions are geometric:

- finite product: $(\mathcal{F} \times \mathcal{G})_x \cong \mathcal{F}_x \times \mathcal{G}_x$
- finite coproduct: $(\mathcal{F} \amalg \mathcal{G})_x \cong \mathcal{F}_x \amalg \mathcal{G}_x$
- arbitrary coproduct: $(\coprod_i \mathcal{F}_i)_x \cong \coprod_i (\mathcal{F}_i)_x$
- set comprehension with respect to a *geometric* formula φ:
$$[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]_x \cong \{[s] \in \mathcal{F}_x \mid \varphi(s) \text{ holds at } x\}$$
- free module: $(\mathcal{R}\langle\mathcal{F}\rangle)_x \cong \mathcal{R}_x\langle\mathcal{F}_x\rangle$ ($\mathcal{R}$ a sheaf of rings, $\mathcal{F}$ a sheaf of sets)
- localization of a module: $\mathcal{F}[\mathcal{S}^{-1}]_x \cong \mathcal{F}_x[\mathcal{S}_x^{-1}]$

Compatibility with taking stalks is not sufficient for geometricity. It is just the most easily visualized requirement. The following constructions are not in general geometric:

- arbitrary product
- set comprehension with respect to a non-geometric formula
- powerset
- internal Hom: $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})_x \not\cong \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$

2.4. Appreciating intuitionistic logic. The principal (and only) difference between classical and intuitionistic logic is that in classical logic, the axioms schemes of *excluded middle* and *double negation elimination* are added.

$$\varphi \vee \neg\varphi \qquad\qquad \neg\neg\varphi \Rightarrow \varphi$$

A classically trained mathematician might legitimately wonder why one should drop these axioms: Are they not obviously true? The pragmatic answer to this question is that the translations of these axioms with the Kripke–Joyal semantics are, except for uninteresting special cases of the base space X, plainly false – irrespective of one's philosophical convictions. Therefore the internal language is in general only sound with respect to intuitionistic logic and not with respect to classical logic. Concretely, there is the following proposition.

Proposition 2.13. *The internal language of a* T_1*-space* X *is* Boolean*, i. e. it verifies the classical axiom schemes displayed above, if and only if* X *is discrete. The internal language of an irreducible or locally Noetherian scheme* X *is Boolean if and only if* X *has dimension* ≤ 0.

Proof. The internal language of $\mathrm{Sh}(X)$ is Boolean if and only if for any open subset $U \subseteq X$ it holds that U is the only dense open subset of U. This can be checked manually, by using the definition of the Kripke–Joyal semantics, but we'll be able to give a more conceptual proof later (Lemma 6.19). The first claim is then an exercise in point-set topology, while the second is more difficult (Corollary 3.15). ☐

However, there is also a more satisfying answer, which furthermore illuminates how to intuitively picture intuitionistic mathematics. Namely, when doing intuitionistic mathematics, we use the same formal symbols as classically, but with *a different intended meaning.* For instance, the classical reading of an existential statement like $\exists x : A.\ \varphi(x)$ is that there exists some element $x : A$ with the property $\varphi(x)$. In contrast, its intuitionistic reading is that such an element can actually be *constructed*, i. e. explicitly given in some form. This is a much stronger statement. Classically, a proof that it is *not* the case that such an element does *not* exist – formally $\neg\neg\exists x : A.\ \varphi(x)$ (or, equivalently even in intuitionistic mathematics, $\neg\forall x : A.\ \neg\varphi(x)$) – suffices to demonstrate the existential statement; this is not so in intuitionistic mathematics.

Similarly, the intuitionistic meaning of a disjunction $\varphi \vee \psi$ is not only that one of the disjuncts is true, but that one can explicitly state which case holds. It is in general not enough to show that it is impossible that both φ and ψ fail.

In this picture, it is obvious that one should not adopt the law of excluded middle or the principle of double negation elimination as axioms. But we don't *reject* those axioms in the sense of postulating their converses, we simply don't use them. Therefore any intuitionistically true result is also true classically. In fact, for some special instances, these two classical axioms do hold intuitionistically. For example, any natural number is zero or is not zero – this is not a triviality, but can be proven by induction.[5]

A consequence of not adopting these axioms is that proofs by contradiction are not generally justified; they are intuitionistically valid only for those statements which can be proven to be true or false. A proof of a *negated formula* is not the same as a proof by contradiction. For instance, the usual proof that $\sqrt{2}$ is not rational is intuitionistically perfectly fine: From the assumption that $\sqrt{2}$ is rational one deduces a contradiction ($\bot$). This is exactly the definition of $\neg(\ulcorner\sqrt{2}\text{ is rational}\urcorner)$.

A more positive consequence of not adopting the law of excluded middle and the principle of double negation elimination is that intuitionistically, we can make *finer distinctions*. For instance, for a formula φ, the doubly negated formula $\neg\neg\varphi$ ("*not not* φ") is a certain kind of weakening of φ: If φ holds, then $\neg\neg\varphi$ does as well, while the converse can not be shown in general.[6] An example from everyday life runs as follows: If in the morning you can't find the key for your apartment, but you know that it must hide somewhere since you used it to open the door in the evening before, you intuitionistically know ($\neg\neg\exists x.\ \ulcorner\text{the key is at position } x\urcorner$), but you cannot claim the unnegated proposition. One cannot model this distinction with pure classical logic.

Double negation also has a concrete geometric meaning with the Kripke–Joyal semantics. Namely, $X \models \neg\neg\varphi$ holds if and only if there is a dense open subset U of X such that $U \models \varphi$. This is of course a weaker statement than $X \models \varphi$. In Section 6, we will discuss this fact and other *modal operators* in more detail. For instance, there is a similarly defined modal operator $\Box$ such that $X \models \Box\varphi$ if and only if there is an open neighborhood U of a given point x such that $U \models \varphi$. Also there is a different operator $\Box$ such that $X \models \Box\varphi$ if and only if φ holds on a scheme-theoretically dense open subset.

For future reference, we remark that if $\varphi \Rightarrow \psi$, then also $\neg\neg\varphi \Rightarrow \neg\neg\psi$; that weakening twice has no further effect, i. e. $\neg\neg\neg\neg\varphi \Leftrightarrow \neg\neg\varphi$;[7] and that the double negation of the law of excluded middle, $\neg\neg(\varphi \vee \neg\varphi)$, holds.

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[5] The analogous statement about real numbers cannot be shown. Intuitively, for a number given by a decimal expansion starting with $0.0000\ldots$ one cannot decide whether the string of zeros will continue indefinitely or whether eventually a non-zero digit will occur. This argument can be made rigorous. The analogous statement about algebraic numbers *can* be proven; the information contained in a witness of algebraicity (a monic polynomial which the given number is a zero of) suffices to make the case distinction [92, Chapter VI.1, p. 140].

[6] A detailed proof of the correct implication goes as follows: Assume φ. We are to show $\neg\neg\varphi$, i. e. $(\neg\varphi \Rightarrow \bot)$. So assume $\neg\varphi$, we are to show $\bot$. Since φ and $\varphi \Rightarrow \bot$, $\bot$ indeed follows.

[7] In fact, negating thrice is the same as negating once: Assume $\neg\neg\neg\varphi$. We are to show $\neg\varphi$. So assume φ, we are to show $\bot$. Since φ, $\neg\neg\varphi$. By $\neg\neg\neg\varphi$, $\bot$ follows.

A classical mathematician might then ask which classical results are valid intuitionistically. Firstly, in linear and commutative algebra, most of the basic theorems stay valid, provided one exercises some caution in formulating them (for instance, one should not arbitrarily weaken assumptions by introducing double negations). This is because the proofs of these statements are usually direct; if intuitionistically unacceptable case distinctions do occur, they can often be eliminated by streamlining the proof.

Consider as a simple example the proposition that the kernel of a linear map is a linear subspace. The case distinction "either the kernel consists just of the zero vector, in which case the claim is trivial, or otherwise ..." is not intuitionistically acceptable, but can be entirely dispensed with: The proof for the general case works in the special case just as well.

Secondly, there is *Barr's theorem*. This metatheorem states that, if a geometric implication has a proof using classical logic and the axiom of choice (from certain axioms which too can be formulated as geometric implications), then it also has a proof using intuitionistic logic. For instance, the Nullstellensatz (any finite system of polynomial equations over an algebraically closed geometric field either possesses a solution or a certificate that there can't be any solutions) is typically proven using maximal ideals and therefore using some forms of the axiom of choice. By Barr's theorem, there is also an intuitionistically valid proof.

Barr's theorem only requires restrictions on the form of the statement and the axioms; the given classical proof can be of any form whatsoever and can freely use statements which are not geometric implications. The proof of Barr's theorem itself is not intuitionistically valid, which puts us in the curious situation that we nonconstructively know that there exists a constructive proof. Because we strive, as far as possible, that these notes can be interpreted in a constructive metatheory, and because we personally prefer direct intuitionistic proofs, we won't use Barr's theorem in what follows. However, it's useful to quickly assess whether a classically known proposition also has an intuitionistic proof.

Finally, we should clarify the status of the axiom of choice. This axiom, which is strictly speaking not part of classical logic, but of a classical set theory, is not accepted in an intuitionistic context: By *Diaconescu's theorem*, it implies the law of excluded middle in presence of the other axioms of set theory.

Standard references for intuitionistic algebra are a textbook by Mines, Richman and Ruitenburg [92] and a textbook by Lombardi [83], the standard reference for intuitionistic analysis is a book by Bishop and Bridges [22]. Further explanations and pointers to relevant literature can be found in an expository article and a recorded lecture by Bauer [17, 16]. A recent survey of intuitionistic logic from a historical and logical point of view is [91].

Remark 2.14. For much of this text, we work in a classical metatheory. This means that we allow ourselves to occasionally use the law of excluded middle and the axiom of choice when reasoning *about* the internal language. In particular, we have the theory of schemes as commonly presented at our disposal. This decision has two reasons.

Firstly, we want to connect the internal world with the usual external framework of algebraic geometry, in order to be directly useful to working algebraic geometers who work in a classical metatheory. We want to prove statements like "a scheme X as classically defined has this-or-that property if and only if, from the internal point of view of $\mathrm{Sh}(X)$, this-or-that holds".

Secondly, as of yet, there is no full constructive account of the theory of schemes with which we could establish a link with the internal language. We sketch how such an account could be developed, and also why one might want to do that, in Section 12.9.

2.5. A fine point on internal natural numbers. The internal world of $\mathrm{Sh}(X)$ contains an object which behaves like the set of natural numbers. For instance, it's possible to prove statements about internal natural numbers by induction and to construct functions on the internal natural numbers by recursion. Externally, this object is the constant sheaf $\underline{\mathbb{N}}$ of locally constant $\mathbb{N}$-valued functions on X.

On the other hand, we can utilize that the internal language supports infinite conjunctions and disjunctions; therefore we may include expressions like "$\bigvee_{n\in\mathbb{N}}$", where n ranges over all external natural numbers, in internal formulas.

The two approaches are related as follows. Any external natural number $n \in \mathbb{N}$ gives rise to a global section $\underline{n}$ of the sheaf $\underline{\mathbb{N}}$, thus to an internal natural number. If $\varphi(n)$ is a formula of the internal language depending on a parameter $n\,{:}\,\underline{\mathbb{N}}$, then

$$\mathrm{Sh}(X) \models \exists n\,{:}\,\underline{\mathbb{N}}.\ \varphi(n) \quad \text{if and only if} \quad \mathrm{Sh}(X) \models \bigvee_{n\in\mathbb{N}} \varphi(\underline{n}),$$

and similarly for "$\forall$" and "$\bigwedge$". In practice we can therefore often ignore the subtle difference. A proof of the equivalence rests on the observation

$$\mathrm{Sh}(X) \models \forall n\,{:}\,\underline{\mathbb{N}}.\ \bigvee_{m\in\mathbb{N}} n = \underline{m},$$

which can be checked by translating this statement using the Kripke–Joyal semantics.

A similar relation holds for internal polynomials. If $\mathcal{R}$ is a sheaf of rings over X, then we can construct, internally in $\mathrm{Sh}(X)$, the ring of polynomials over $\mathcal{R}$. This will yield a certain sheaf $[\![\mathcal{R}[T]]\!]$. If $\varphi(f)$ is a formula of the internal language containing a free variable $f\,{:}\,\mathcal{R}[T]$, then

$$\begin{aligned}&\mathrm{Sh}(X) \models \exists f\,{:}\,\mathcal{R}[T].\ \varphi(f) \quad \text{if and only if} \\ &\qquad\qquad \mathrm{Sh}(X) \models \bigvee_{n\in\mathbb{N}} \exists a_0\,{:}\,\mathcal{R}.\ \ldots \exists a_n\,{:}\,\mathcal{R}.\ \varphi(a_nT^n + \cdots + a_1T + a_0).\end{aligned}$$

PART II

The little Zariski topos

3. Sheaves of rings

Recall that a *sheaf of rings* can be categorically described as a sheaf of sets $\mathcal{R}$ together with maps of sheaves $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$, $- : \mathcal{R} \to \mathcal{R}$, and global elements $0, 1$ such that certain axioms hold. For instance, the axiom on the commutativity of addition is rendered in diagrammatic form as follows:

$$\begin{tikzcd} \mathcal{R} \times \mathcal{R} \arrow[rr, "\text{swap}"] \arrow[dr, "+"'] & & \mathcal{R} \times \mathcal{R} \arrow[dl, "+"] \\ & \mathcal{R} & \end{tikzcd}$$

From the internal perspective, a sheaf of rings looks just like a plain ring. This is the content of the following proposition.

Proposition 3.1. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of sets on X. Let $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $- : \mathcal{R} \to \mathcal{R}$ be maps of sheaves and let $0, 1$ be global elements of $\mathcal{R}$. Then these data define a sheaf of rings if and only if, from the internal perspective, these data fulfill the usual equational ring axioms.*

Proof. We only discuss the commutativity axiom. The internal statement

$$\mathrm{Sh}(X) \models \forall x, y : \mathcal{R}.\ x + y = y + x$$

means that for any open subset $U \subseteq X$ and any local sections $x, y \in \Gamma(U, \mathcal{R})$, it holds that $x + y = y + x \in \Gamma(U, \mathcal{R})$. This is precisely the external commutativity condition. □

Lemma 3.2. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of rings on X. Let f be a global section of $\mathcal{R}$. Then the following statements are equivalent:*

1. *f is invertible from the internal point of view, i. e.* $\mathrm{Sh}(X) \models \exists g : \mathcal{R}.\ fg = 1$.
2. *f is invertible in all stalks $\mathcal{R}_x$.*
3. *f is invertible in* $\Gamma(X, \mathcal{R})$.

Proof. Since invertibility is a geometric implication, the equivalence of the first two statements is clear. Also, it is obvious that the third statement implies the other two. For the remaining direction, note that the uniqueness of inverses in rings can be proven intuitionistically. Therefore, if f is invertible from the internal point of view, it actually holds that

$$\mathrm{Sh}(X) \models \exists! g : \mathcal{R}.\ fg = 1.$$

Since unique internal existence implies global existence (Proposition 2.6), this shows that the first statement implies the third. □

3.1. Reducedness.

Recall that a scheme X is *reduced* if and only if all stalks $\mathcal{O}_{X,x}$ are reduced rings. Since the condition on a ring R to be reduced is a geometric implication,

$$\forall s : R.\ \Big(\bigvee_{n \geq 0} s^n = 0 \Big) \Longrightarrow s = 0,$$

we immediately obtain the following characterization of reducedness in the internal language:

Proposition 3.3. *A scheme X is reduced iff, from the internal point of view, the ring $\mathcal{O}_X$ is reduced.*

3.2. Locality. Recall the usual definition of a local ring: a ring possessing exactly one maximal ideal. This is a so-called *higher-order condition* since it involves quantification over subsets. It is also not of a geometric form. Therefore, for our purposes, it is better to adopt the following elementary definition of a local ring.

Definition 3.4. A *local ring* is a ring R such that $1 \neq 0$ in R and for all $x, y : R$

$$x + y \text{ invertible} \quad \Longrightarrow \quad x \text{ invertible} \ \vee \ y \text{ invertible}.$$

In classical logic, it is an easy exercise to show that this definition is equivalent to the usual one. In intuitionistic logic, we would need to be more precise in order to even state the question of equivalence, since intuitionistically, the notion of a maximal ideal bifurcates into several non-equivalent notions.[8] This is a common phenomenon in intuitionistic mathematics: Classically equivalent notions may bifurcate into related but inequivalent notions intuitionistically, each having a unique character and yielding slightly different theories.

Proposition 3.5. *In the internal language of a scheme X (or a locally ringed space), the ring $\mathcal{O}_X$ is a local ring.*

Proof. The condition in Definition 3.4 is a conjunction of two geometric implications (the first one being $1 = 0 \Rightarrow \bot$, the second being the displayed one) and holds on each stalk. ☐

Remark 3.6. When first exposed to locally ringed spaces, one might ask why the requirement is that the *stalks* $\mathcal{O}_{X,x}$ are local rings, instead of the easier-to-define sets of sections $\mathcal{O}_X(U)$. This question has of course a good geometric answer. Using the internal language, it also has a purely formal answer: The requirement that the stalks are local rings is precisely the requirement that the ring $\mathcal{O}_X$ is a local ring from the perspective of the internal language of X.

3.3. Field properties. From the internal point of view, the structure sheaf $\mathcal{O}_X$ of a scheme X is *almost* a field, in the sense that any element which is not invertible is nilpotent. This is a genuine property of schemes, not shared with arbitrary locally ringed spaces. It is also a specific feature of the internal universe: Neither the local rings $\mathcal{O}_{X,x}$ nor the rings of local sections $\Gamma(U, \mathcal{O}_X)$ have this property in general.

Proposition 3.7. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Rightarrow \ulcorner s \text{ nilpotent} \urcorner.$$

Proof. By the locality of the internal language and since X can be covered by open affine subsets, it is enough to show that for any affine scheme $X = \operatorname{Spec} A$ and any global function $s \in \Gamma(X, \mathcal{O}_X) = A$ it holds that

$$X \models \neg(\ulcorner s \text{ invertible} \urcorner) \quad \text{implies} \quad X \models \ulcorner s \text{ nilpotent} \urcorner.$$

The meaning of the antecedent is that any open subset on which s is invertible is empty. This implies in particular that the standard open subset $D(s)$ is empty. This means that s is an element of any prime ideal of A, thus nilpotent, and therefore implies the a priori weaker statement $X \models \ulcorner s \text{ nilpotent} \urcorner$ (which would allow s to have different indices of nilpotency on an open covering). ☐

[8] For instance, should a maximal ideal $\mathfrak{m}$ be such that if $\mathfrak{n}$ is any ideal with $\mathfrak{m} \subseteq \mathfrak{n} \subsetneq (1)$, then $\mathfrak{m} = \mathfrak{n}$? Or should the condition be that if $\mathfrak{n}$ is any ideal with $\mathfrak{m} \subseteq \mathfrak{n}$, then $\mathfrak{m} = \mathfrak{n}$ or $\mathfrak{n} = (1)$? Intuitionistically, the latter condition is stronger than the former.

Remark 3.8. In classical logic, the statement "not invertible implies nilpotent" is equivalent to "any element is invertible or nilpotent". However, in intuitionistic logic, the latter is strictly stronger than the former. We will see in the next section (Corollary 3.14) that the structure sheaf of a scheme fulfills the latter condition if and only if the scheme is zero-dimensional (or empty). An overview of the basic properties of the intuitionistically different field conditions is contained in [66].

Corollary 3.9. *Let X be a scheme. If X is reduced, the ring $\mathcal{O}_X$ is a field from the internal point of view, in the sense that*

$$\mathrm{Sh}(X) \models \forall s\colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible}\urcorner) \Rightarrow s = 0.$$

Conversely, if $\mathcal{O}_X$ is a field in this internal sense, then X is reduced.

Proof. We can prove this purely in the internal language: It suffices to give an intuitionistic proof of the fact that a local ring which satisfies the condition of the previous proposition fulfills the stated field condition if and only if it is reduced. This is straightforward. ☐

This field property is very useful. We will put it to good use when giving a simple proof of the fact that $\mathcal{O}_X$-modules of finite type on a reduced scheme are locally free on a dense open subset (Lemma 5.8). The field property only holds in the precise form as stated; the classically equivalent condition that any element is invertible or zero is intuitionistically stronger. This is an instance of the already remarked upon phenomenon of intuitionistic bifurcation of notions.

The observation that the structure sheaf is (almost) a field is attributed by Tierney to Mulvey [126, p. 209]. Tierney also states that "[it] is surely important, though its precise significance is still somewhat obscure" (ibid). We think that it's significant as a special case of the following more general proposition, which states that we can deduce a certain unconditional statement from the premise that, under the assumption that some element $f\colon \mathcal{O}_X$ is invertible, an element $s\colon \mathcal{O}_X$ is zero. This is interesting on its own, but will be of particular importance in understanding quasicoherence from the internal point of view (Section 8) and interpreting the relative spectrum as an internal spectrum (Section 12).

Proposition 3.10. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall f\colon \mathcal{O}_X.\ \forall s\colon \mathcal{O}_X.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow \bigvee_{n\geq 0} f^n s = 0.$$

Proof. It is enough to show that for any affine scheme $X = \operatorname{Spec} A$ and any global functions $f, s \in A$ such that

$$X \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0),$$

it holds that $X \models \bigvee_{n\geq 0} f^n s = 0$. This indeed follows, since by assumption such a function s is zero on $D(f)$, i. e. s is zero as an element of $A[f^{-1}]$. ☐

Proposition 3.7 follows from this proposition by setting $s := 1$.

3.4. Krull dimension. Recall that the *Krull dimension* of a ring is usually defined as the supremum of the lengths of strictly ascending chains of prime ideals. As with the classical definition of a local ring, this definition does not lead to a well-behaved notion in an intuitionistic context. Furthermore, it is a higher-order condition, so interpreting it with the Kripke–Joyal semantics is a bit unwieldy.

Luckily, there is an elementary definition of the Krull dimension which works intuitionistically and which is classically equivalent to the usual notion. It was found by Coquand and Lombardi, building upon work by Joyal and Español [37, 40], and can be used to give a short proof that $\dim k[X_1, \ldots, X_n] = n$, where k is a field [39].

Definition 3.11. Let R be a ring. A *complementary sequence* for a sequence $(a_0, \ldots, a_n)$ of elements of R is a sequence $(b_0, \ldots, b_n)$ such that the following inclusions of radical ideals hold:

$$\begin{cases} \sqrt{(1)} & \subseteq & \sqrt{(a_0, b_0)} \\ \sqrt{(a_0 b_0)} & \subseteq & \sqrt{(a_1, b_1)} \\ \sqrt{(a_1 b_1)} & \subseteq & \sqrt{(a_2, b_2)} \\ & \vdots & \\ \sqrt{(a_{n-1} b_{n-1})} & \subseteq & \sqrt{(a_n, b_n)} \\ \sqrt{(a_n b_n)} & \subseteq & \sqrt{(0)} \end{cases}$$

The ring R is *of Krull dimension* $\leq n$ if and only if for any sequence $(a_0, \ldots, a_n)$ there exists a complementary sequence. (The ring R is trivial if and only if it is of Krull dimension ≤ -1.)

Unlike the usual definition, this definition posits only a condition on elements and not on ideals. It is thus of a simpler logical form. (The radical ideals appear only for convenience. We will dispose of them in the proof of Proposition 3.13.) Also note that we do not define the Krull dimension of a ring as some natural number (this is intuitionistically not possible for general rings). Instead, we only define what it means for the Krull dimension to be less than or equal to a given natural number.

For the following, no intuition about the definition is needed; however, we feel that some motivation might be of use. Recall that we can picture inclusions of radical ideals geometrically by considering standard open subsets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}$: The inclusion $\sqrt{(f)} \subseteq \sqrt{(g, h)}$ holds if and only if $D(f) \subseteq D(g) \cup D(h)$, and intersections are calculated by products, i. e. $D(f) \cap D(g) = D(fg)$.

The condition that $(b_0, \ldots, b_n)$ is complementary to $(a_0, \ldots, a_n)$ thus means that $D(a_0)$ and $D(b_0)$ cover all of $\operatorname{Spec} R$; that their intersection is covered by $D(a_1)$ and $D(b_1)$; that in turn their intersection is covered by $D(a_2)$ and $D(b_2)$; $\ldots$; and that finally, the intersection of $D(a_n)$ and $D(b_n)$ is empty.

For the special case $n = 0$, the condition that R is of Krull dimension ≤ 0 means that for any element a_0 there exists an element b_0 such that $D(a_0)$ and $D(b_0)$ cover $\operatorname{Spec} R$ and are disjoint.

The definition of the Krull dimension can be written in such a way as to mimic the definition of the inductive Menger–Urysohn dimension of topological spaces [37, Section 1].

Theorem 3.12. *Let R be a ring.*

(1) *In classical logic, the ring R is of Krull dimension $\leq n$ if and only if its Krull dimension as usually defined using chains of prime ideals is less than or equal to n.*
(2) *If the ring R is of Krull dimension $\leq n$, the radical of any finitely generated ideal is equal to the radical of some ideal which can be generated by $n + 1$ elements. This holds intuitionistically, and there is an explicit algorithm for computing the reduced set of generators from the given ones. (Kronecker's theorem)*

Proof. See [37, Theorem 1.2] for the first statement. The proof relies on the observation that $\dim R \leq n$ if and only if $\dim R[S_x^{-1}] \leq n - 1$ for all $x \in R$, where $S_x = x^{\mathbb{N}}(1 + xR) \subseteq R$. We put the second statement only to demonstrate that the definition of the Krull dimension is constructively sensible. It follows from the identity $\sqrt{(x, a_0, \ldots, a_n)} = \sqrt{(a_0 - xb_0, \ldots, a_n - xb_n)}$, where $(b_0, \ldots, b_n)$ is a complementary sequence for $(a_0, \ldots, a_n)$. $\square$

We can apply the constructive theory of Krull dimension to the structure sheaf $\mathcal{O}_X$ of a scheme X as follows. The condition that a scheme X has dimension exactly n (in the usual sense using ascending chains of closed irreducible subsets) is not local – the dimension may vary on an open cover; hence it isn't possible to characterize this condition in the internal language. However, the condition that the dimension of X is less than or equal to n *is* local, thus there is hope that it can be internalized. And indeed, this is the case.

Proposition 3.13. *Let X be a scheme. Then:*

$$\dim X \leq n \quad \iff \quad \mathrm{Sh}(X) \models \ulcorner \mathcal{O}_X \text{ is of Krull dimension } \leq n \urcorner$$

Proof. A condition of the form "$\sqrt{(f)} \subseteq \sqrt{(g,h)}$" like in the constructive definition of the Krull dimension is not a geometric formula when taken on face value. However, it is equivalent to a geometric condition, namely to

$$\exists a, b : \mathcal{O}_X.\ \bigvee_{m \geq 0} f^m = ag + bh.$$

Therefore the condition $\ulcorner \mathcal{O}_X$ is of Krull dimension $\leq n \urcorner$ is (equivalent to) a geometric implication and thus holds internally if and only if it holds at every point $x \in X$. This in turn means that the Krull dimension of any stalk $\mathcal{O}_{X,x}$ is less than or equal to n. This is equivalent to the (Krull) dimension of X being less than or equal to n. □

We will state and prove a generalization of this lemma about the dimension of closed subschemes later, as Lemma 10.10.

If X is a reduced scheme, we have seen in Corollary 3.9 that $\mathcal{O}_X$ is a field from the internal perspective, in the sense that non-invertible elements are zero. But fields are well-known to be of Krull dimension zero. Why is this not a contradiction to the proposition just proven? Intuitionistically, the notion of a field bifurcates into several non-equivalent notions:

(1) "Any element which is not invertible is zero."
(2) "Any element which is not zero is invertible."
(3) "Any element is either zero or invertible."

Only fields in the sense (3) are automatically of Krull dimension zero. Fields in the weaker senses can have higher Krull dimension, as exhibited by the structure sheaf of reduced schemes with positive dimension.

With out conventions, a scheme X is of dimension ≤ 0 if and only if it is empty or if it's nonempty and of dimension zero.

Corollary 3.14. *Let X be a scheme. Then:*

$$\dim X \leq 0 \quad \iff \quad \mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \ulcorner s \text{ inv.} \urcorner \vee \ulcorner s \text{ nilpotent} \urcorner.$$

If furthermore X is reduced, this is further equivalent to $\mathcal{O}_X$ being a field in the strong sense that any element of $\mathcal{O}_X$ is invertible or zero.

Proof. By Proposition 3.13 and the fact that $\mathcal{O}_X$ is a local ring from the internal perspective, this is an immediate consequence of interpreting the following standard fact of ring theory in the internal language of $\mathrm{Sh}(X)$: A local ring R is of Krull dimension ≤ 0 if and only if any element of R is invertible or nilpotent.

It is well-known that this holds classically; to make sure that it holds intuitionistically as well (so that it can be used in the internal universe), we give a proof of the "only if" direction. Let $a : R$ be arbitrary. By assumption on the Krull dimension, there exists an element $b : R$ such that $\sqrt{(1)} \subseteq \sqrt{(a,b)}$ and $\sqrt{(ab)} = \sqrt{(0)}$. The latter means that ab is nilpotent. Since R is local, the former implies that a is invertible or that b is invertible. In the first case, we are done. In the second case, it follows that a is nilpotent, so we are done as well. $\square$

As a further corollary we note the curious fact that the classicality of the internal language of $\mathrm{Sh}(X)$, where X is a scheme, is tightly coupled with the properties of the ring $\mathcal{O}_X$: Internally, the law of excluded middle and the principle of double negation elimination are "almost equivalent" to the Krull dimension of $\mathcal{O}_X$ being ≤ 0.

Corollary 3.15. *Let X be a scheme. If the internal language of* $\mathrm{Sh}(X)$ *is Boolean, then* $\dim X \leq 0$. *The converse holds if X is irreducible or locally Noetherian.*

Proof. We show that any element of $\mathcal{O}_X$ is invertible or nilpotent, therefore verifying the hypothesis of the previous corollary. Let $s : \mathcal{O}_X$ be given. By assumption, either s is invertible or s is not invertible. In the latter case s is nilpotent by Proposition 3.7.

We defer the converse direction to Proposition 9.19 since we don't want to interrupt the exposition here with a certain necessary technical condition. $\square$

3.5. Integrality. In intuitionistic logic, the notion of an integral domain bifurcates into several inequivalent notions. The following two are important for our purposes:

Definition 3.16. A ring R is an *integral domain in the weak sense* if and only if $1 \neq 0$ in R and

$$\forall x, y : R.\ xy = 0 \Longrightarrow (x = 0) \vee (y = 0).$$

A ring R is an *integral domain in the strong sense* if and only if $1 \neq 0$ in R and

$$\forall x : R.\ x = 0 \vee \ulcorner x \text{ is regular} \urcorner,$$

where $\ulcorner x$ is regular$\urcorner$ means that $xy = 0$ implies $y = 0$ for any $y : R$.

For the following result, recall that a scheme X (or a ringed space) is *integral at a point $x \in X$* if and only if $\mathcal{O}_{X,x}$ is an integral domain (in either sense, since we have adopted a classical metatheory).

Proposition 3.17. *Let X be a ringed space. Then:*

(1) *X is integral at all points if and only if, internally, $\mathcal{O}_X$ is an integral domain in the weak sense.*
(2) *If X is even a locally Noetherian scheme, then $\mathcal{O}_X$ is an integral domain in the weak sense iff it is an integral domain in the strong sense from the internal point of view.*

Proof. The condition on a ring to be an integral domain in the weak sense is a conjunction of two geometric implications, "$1 = 0 \Rightarrow \bot$" and the implication displayed in the definition. Therefore the first statement is obvious.

For the second statement, we observe the condition on a function $f \in \Gamma(U, \mathcal{O}_X)$ to be regular from the internal perspective is open: It holds at a point $x \in U$ if and only if it holds on some open neighborhood of x. We will give a proof of this specific feature of locally Noetherian schemes later on, when we have developed appropriate machinery to do so easily (Proposition 9.4). In any case, this openness property was the essential ingredient for the equivalence between "holding internally" and "holding at every point" (Corollary 2.11). Therefore $\mathcal{O}_X$ is an integral domain in the strong sense from the internal point of view if and only if all local rings $\mathcal{O}_{X,x}$ are integral domains. By the first statement, this is equivalent to $\mathcal{O}_X$ being an integral domain in the weak sense from the internal point of view. $\square$

We record the following lemma for later use. The proof presented here is already simple, but a more conceptual proof is also possible (see Section 11.3).

Lemma 3.18. *Let $X = \operatorname{Spec} A$ be an affine scheme. Let $f \in A$. Then f is a regular element of A if and only if f is a regular element of $\mathcal{O}_X$ from the internal perspective.*

Proof. The Kripke–Joyal translation of internal regularity is:

> For any (without loss of generality: standard) open subset $U \subseteq X$ and any function $g \in \Gamma(U, \mathcal{O}_X)$, $fg = 0$ in $\Gamma(U, \mathcal{O}_X)$ implies $g = 0$ in $\Gamma(U, \mathcal{O}_X)$.

So the "if" direction is clear (use $U := X$). For the "only if" direction, we use that $\Gamma(U, \mathcal{O}_X)$ is a localization of A and that regular elements remain regular in localizations. □

3.6. Bézout property. Recall that a *Bézout ring* is a ring in which any finitely generated ideal is a principal ideal. In intuitionistic mathematics, this is a better notion than that of a principal ideal ring: The requirement that *any* ideal is a principal ideal is far too strong. Intuitively, this is because without any given generators to begin with, one cannot hope to explicitly pinpoint a principal generator. One can (provably) not even verify this property for the ring $\mathbb{Z}$.[9]

Proposition 3.19. *Let X be a scheme (or a ringed space).*

(1) *$\mathcal{O}_X$ is a Bézout ring from the internal perspective if and only if all rings $\mathcal{O}_{X,x}$ are Bézout rings.*
(2) *$\mathcal{O}_X$ is such that, from the internal perspective, of any two elements, one divides the other, if and only if all rings $\mathcal{O}_{X,x}$ are such.*

Proof. Both properties can be formulated as geometric implications:

$$\text{(1)} \quad \forall f, g : \mathcal{O}_X.\ \top \Rightarrow \exists d : \mathcal{O}_X.\ (\exists a, b : \mathcal{O}_X.\ d = af + bg) \wedge (\exists u : \mathcal{O}_X.\ f = ud) \wedge (\exists v : \mathcal{O}_X.\ g = vd)$$

$$\text{(2)} \quad \forall f, g : \mathcal{O}_X.\ \top \Rightarrow (\exists u : \mathcal{O}_X.\ f = ug) \vee (\exists u : \mathcal{O}_X.\ g = uf)$$

□

Corollary 3.20. *Let X be a Dedekind scheme, i. e. a locally Noetherian normal scheme of dimension ≤ 1. Then, from the internal perspective, any matrix over $\mathcal{O}_X$ can be put into Smith canonical form, i. e. is equivalent to a (rectangular) diagonal matrix with diagonal entries $a_1 | a_2 | \cdots | a_n$ successively dividing each other.*

Proof. It is well-known that such a scheme has principal ideal domains as local rings $\mathcal{O}_{X,x}$. For local domains, the Bézout condition is equivalent to the property that of any two elements, one divides the other. Therefore all local rings have this property, and by the previous proposition, the internal ring $\mathcal{O}_X$ has it as well. The statement thus follows from interpreting the following fact of linear algebra in the internal universe: Let R be a ring such that of any two elements, one divides the other. Then any matrix over R can be put into Smith canonical form.

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[9] Assume that any ideal of $\mathbb{Z}$ is finitely generated. Let φ be an arbitrary statement; we want to intuitionistically deduce $\varphi \vee \neg\varphi$. Consider the ideal $\mathfrak{a} := \{x \in \mathbb{Z} \mid (x = 0) \vee \varphi\} \subseteq \mathbb{Z}$. The definition is such that φ holds if and only if $\mathfrak{a}$ contains an element other than zero; and that $\neg\varphi$ holds if and only if zero is the only element of $\mathfrak{a}$. By assumption, $\mathfrak{a}$ is finitely generated. Since $\mathbb{Z}$ is a Bézout ring, it is therefore even principal: $\mathfrak{a} = (x_0)$ for some $x_0 \in \mathbb{Z}$. Even intuitionistically we have $(x_0 = 0) \vee (x_0 \neq 0)$ (for the natural numbers, this can be proven by induction). In the first case, it follows that $\mathfrak{a}$ contains only zero; in the second case, it follows that $\mathfrak{a}$ contains an element other than zero. Thus $\neg\varphi \vee \varphi$.
This kind of reasoning is called *exhibiting a Brouwerian counterexample*. The definition of $\mathfrak{a}$ may look slightly dubious, considering that φ does not depend on x; but we will see that such definitions actually have a clear geometric meaning – they can be used to define extensions of sheaves by zero in the internal language (Lemma 10.2).

The usual proof of this fact is indeed intuitionistically valid: Let a matrix over R be given. By induction, one can show that for any finite family of ring elements, one divides all the others. Hency some matrix entry is a factor of all the others. We move this entry to the upper left by row and column transformations and then kill the other entries of the first row and the first column. After these operations, it is still the case that the entry in the first row and column is a factor of all other entries. Continuing in this fashion, we obtain a diagonal matrix. Its diagonal entries already fulfill the divisibility condition and thus do not have to be sorted. □

Phrases such as "if by chance the entry in the upper left divides all the others, we can directly proceed with the next step; otherwise, some other entry must be a factor of all entries, so . . . " may not be included in a proof which is intended to be intuitionistically valid. Those phrases assume that one may make the case distinction that for any two ring elements x, y, either x divides y or not. Fortunately, those case distinctions are in fact superfluous.

A consequence of the corollary is that internally to the sheaf topos of a Dedekind scheme, the usual structure theorem on finitely presented $\mathcal{O}_X$-modules is available. We will exploit this in Lemma 4.26, where we give an internal proof of the fact that on Dedekind schemes, torsion-free $\mathcal{O}_X$-modules are locally free.

3.7. Normality. We will discuss the property of a ring to be *normal*, i. e. to be integrally closed in its total field of fractions, in Section 9.3, after giving an internal characterization of the sheaf of rational functions.

3.8. Special properties of constant sheaves of rings. Let R be an ordinary ring and $\underline{R}$ the associated sheaf of locally constant R-valued functions on a topological space. If R is reduced, local, or a field, then $\underline{R}$ is so as well, from the internal point of view.

We will prove this in greater generality: Appropriately formulated, a constant sheaf $\underline{R}$ has some property φ from the internal point of view if and only if R has the property φ externally (Lemma 11.1).

3.9. Noetherian conditions. Recall the usual notion of a Noetherian ring: Any sequence $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots$ of ideals should stabilize, i. e. there should exist a natural number n such that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$.

Intuitionistically, this definition has two problems. Firstly, without the axiom of dependent choice, it is often not possible to construct a *sequence* of ideals: Often, it is only possible to show that there *exists* a suitable ideal $\mathfrak{a}_{n+1}$ depending on $\mathfrak{a}_n$. But since in general there is no canonical choice for this successor ideal, the axiom of dependent choice would be required to collect those into a sequence, i. e. a function from $\mathbb{N}$ to the set of ideals.

Secondly, the conclusion that the sequence stabilizes is too strong. Intuitionistically, one cannot even show that a weakly descending sequence of natural numbers stabilizes in this sense; the statement that one could is equivalent to the *limited principle of omniscience for* $\mathbb{N}$. Intuitionistically, it is only true that a weakly descending sequence $a_0 \geq a_1 \geq \cdots$ of natural numbers eventually *stalls* in the sense that there exists an index n such that $a_n = a_{n+1}$ (but $a_{n+1} > a_{n+2}$ is allowed).[10]

We give two constructively inequivalent notions of Noetherian rings. The first one is of independent constructive interest and enjoys the property that the structure sheaf of a scheme X satisfies the Noetherian condition from the internal point of view of $\mathrm{Sh}(X)$ if and only if X is locally Noetherian.

[10]Classically, the following three statements about a ring are equivalent: (1) Every ascending chain of ideals stabilizes. (2) Every ascending chain of finitely generated ideals stabilizes. (3) Every ascending chain of finitely generated ideals stalls.

The second one is quite weak from a constructive point of view, but still interesting from a geometric point of view and useful enough to derive nontrivial consequences. It is satisfied by the structure sheaf of any (not necessarily locally Noetherian) reduced scheme.

There are several proposals for a constructively sensible definition of Noetherian rings in the literature on constructive algebra, each with unique advantages and disadvantages [105, 92, 100, 99, 101, 123]. Insightful comments on why this is so can be found in the introduction and more specifically on page 27 of the textbook by Lombardi and Quitté [83].

Processly Noetherian rings.

Definition 3.21. Let M be a partially ordered set. An *ascending process with values in* M consists of an initial value $x_0 \in M$ and a function $f : M \to \mathcal{P}(M)$ such that for any $x \in M$ and any $y \in f(x)$, $x \preceq y$, and such that:

- The set $f(x_0)$ is inhabited.
- For any $x_1 \in f(x_0)$, the set $f(x_1)$ is inhabited.
- For any $x_1 \in f(x_0)$ and any $x_2 \in f(x_1)$, the set $f(x_2)$ is inhabited.
- And so on.

Such a process *stalls* if and only if there exists a step n and elements $x_1, \ldots, x_n$ such that $x_{i+1} \in f(x_i)$ for $i = 0, \ldots, n-1$ and such that $x_n \in f(x_n)$. The set M satisfies the *ascending process condition* if and only if every ascending process with values in M stalls.

Intuitively, we picture $f(x)$ as the set of all possible results of running the process for a single step, starting with the value x. This set could be a singleton, in case that the process deterministically produces a single value, but it may also contain more than one element, for instance if the process cannot provide the next value in a canonical way. Instead of arbitrarily choosing a definitive value for its result, the process may instead collect all the possible values in the set $f(x)$.

Remark 3.22. The usual term for what we call "to stall" is "to halt" [105, 101]. However, this choice of wording is slightly unfortunate, since the phrase "the process halts" intuitively suggests that the process stops and won't produce further results in the future, even though this is not what is mathematically meant. We are grateful to Matthias Hutzler for proposing "to stall", which seems quite appropriate.

Definition 3.23. A ring A is *processly Noetherian* if and only if the set of finitely generated ideals in A satisfies the ascending process condition.

An ascending chain of elements $a_0 \preceq a_1 \preceq \cdots$ in a partially ordered set gives rise to an ascending process by setting $x_0 := a_0$ and $f(x) := \{y \mid \exists n.\ x = a_n \wedge y = a_{n+1}\}$. (This process stalls iff there is an index n such that $a_n = a_{n+1}$.) Conversely, the axiom of dependent choice would allow to construct an ascending chain from an ascending process. In the presence of this axiom, for instance in a classical context, a ring is therefore processly Noetherian if and only if it is Noetherian in the usual sense.

The notion of a processly Noetherian ring works well in an intuitionistic context: Important rings such as $\mathbb{Z}$ and more generally $\mathcal{O}_K$ for any algebraic number field K are processly Noetherian, and matrices over Bézout rings which are integral domains in the weak sense and processly Noetherian can be put into Smith canonical form.[11]

[11]In the algorithm to put a matrix into Smith canonical form, one has to repeatedly *choose* generators for principal ideals and associated Bézout representations (see for instance [105, Section 4]). Since these choices are not unique, the algorithm doesn't produce a *sequence* of intermediate ideals, but only a *process*. This example was our main motivation for the notion of processly Noetherian rings.

Richman also studied Noetherian rings in a constructive context without dependent choice [105]. His notion of *ascending tree condition* is equivalent to our ascending process condition. His condition emphasizes the branching nature of a non-deterministic computation, while ours emphasizes the step-for-step picture of computation.

There are three reasons why we did not define a ring to be processly Noetherian if and only if the set of all (not only finitely generated) ideals satisfies the ascending process condition. Firstly, this stricter condition excludes rings as $\mathbb{Z}$.[12] Secondly, restricting to finitely generated ideals in this context is a well-established procedure in constructive mathematics [92, 105] and suffices for the applications of the Noetherian condition one typically expects. Thirdly, our definition provides a link to the external condition on a scheme to be locally Noetherian, as shown by the following proposition.

Proposition 3.24. *Let X be a scheme. The following statements are equivalent:*

(1) *All stalks $\mathcal{O}_{X,x}$ are Noetherian.*
(2) *From the internal point of view of* $\mathrm{Sh}(X)$*, the ring $\mathcal{O}_X$ is processly Noetherian.*

Proof. Statement (1) can be reformulated in a way so it doesn't refer to stalks: For any open affine subscheme $U \subseteq X$ and any ascending chain $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots$ of finitely generated ideals in $\Gamma(U, \mathcal{O}_X)$ there is a partition of unity $1 = \sum_i f_i \in \Gamma(U, \mathcal{O}_X)$ such that for each i there exists an index j such that $\mathfrak{a}_j = \mathfrak{a}_{j+1}$ as ideals of $\Gamma(U, \mathcal{O}_X)[f_i^{-1}]$.

We'll verify the equivalence using this formulation. For proving the direction "(1) $\Rightarrow$ (2)", we may assume that $X = \operatorname{Spec} A$ is affine and that internally, we are given an ascending process on the set of finitely generated ideals of $\mathcal{O}_X$. Externally, this is a finite type sheaf of ideals $\mathcal{I}$ together with a morphism $\mathcal{M} \to \mathcal{P}(\mathcal{M})$ where $\mathcal{M}$ is the sheaf whose U-sections are finite type ideal sheaves of $\mathcal{O}_X|_U$.

Since $X \models \ulcorner f(\mathcal{I})$ is inhabited$\urcorner$, there exists an open covering $X = \bigcup_i U_i$ and finite type sheaves of ideals $\mathcal{I}_i \hookrightarrow \mathcal{O}_X|_{U_i}$ such that $U_i \models \mathcal{I}_i \in f(\mathcal{I})$. Without loss of generality, we may assume that the open sets U_i are standard open sets $D(f_i)$ and that the covering is finite. Since the sheaves $\mathcal{I}_i$ are quasicoherent (being of finite type, they are images of morphisms of the form $\mathcal{O}_X|_{U_i}^n \to \mathcal{O}_X|_{U_i}$), they correspond to ideals $J_i \subseteq A[f_i^{-1}]$. We note for future reference that for $D(g) \subseteq D(f_i)$, the restricted sheaf of ideals $\mathcal{I}_i|_{D(g)}$ corresponds to the extension of J_i in the further localized ring $A[g^{-1}]$.

For each i, $D(f_i) \models \ulcorner f(\mathcal{I}_i)$ is inhabited$\urcorner$. Hence there exists an open covering $D(f_i) = \bigcup_j D(f_{ij})$ and finite type sheaves of ideals $\mathcal{I}_{ij} \hookrightarrow \mathcal{O}_X|_{D(f_{ij})}$; these correspond to ideals $J_{ij} \subseteq A[f_{ij}^{-1}]$ such that $J_i \subseteq J_{ij}$ (where we have suppressed the localization morphism $A[f_i^{-1}] \to A[f_{ij}^{-1}]$ in the notation). Equivalently, writing $J_i' := A \cap J_i$ and $J_{ij}' := A \cap J_{ij}$ for the contractions, we have the inclusions $J_i' \subseteq J_{ij}'$ of ideals of A.

Continuing in this fashion, we obtain an infinite tree of ideals $J'_{i_1 \cdots i_n}$. We now prune this tree in the following fashion: If the node at position $(i_1, \ldots, i_n)$ has the property that $D(f_{i_1 \cdots i_n}) \models \bigvee_{m=0}^{n-1} \mathcal{I}_{i_1 \cdots i_m} = \mathcal{I}_{i_1 \cdots i_{m+1}}$, then we cut of all childs of this node.

The resulting tree doesn't contain an infinite path, since any sequence $J'_{i_1} \subseteq J'_{i_1 i_2} \subseteq \cdots$ locally stalls by assumption on A. Because only finitely many subtrees

[12]The main ingredient in the proof that $\mathbb{Z}$ is Noetherian is that any ideal of $\mathbb{Z}$ is a principal ideal, since (looking at the prime factor decomposition) one can give explicit bounds on the length of strictly ascending chains of principal ideals. However, as detailed in the footnote on page 33, constructively one cannot show that every ideal of $\mathbb{Z}$ is a principal ideal; one can only verify that finitely generated ideals are principal. Geometrically, ideals which are not finitely generated correspond to sheaves of ideals which may fail to be quasicoherent.

branch off at each node, the tree is finite (this is an application of the graph-theoretical *Kőnig's lemma*).

The outermost nodes then yield an open covering of X such that, on each member of the covering, the internal statement $\ulcorner f$ stalls$\urcorner$ holds. By the locality of the internal language, this statement holds on X.

For the converse direction, let an affine open subset $U \subseteq X$ and an ascending sequence $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots$ of finitely generated ideals in $\Gamma(U, \mathcal{O}_X)$ be given. Internally, we construct the process

$$f : \mathcal{M} \longrightarrow \mathcal{P}(\mathcal{M}),\ \mathcal{I} \longmapsto \{\mathcal{J} : \mathcal{M} \mid \bigvee_{n \geq 0} (\mathcal{I} = \mathfrak{a}_n^\sim \wedge \mathcal{J} = \mathfrak{a}_{n+1}^\sim)\}$$

with initial value $\mathfrak{a}_0^\sim$. The assumption that f stalls yields an open covering $U = \bigcup_i D(f_i)$ such that for each i, there is an index n such that $\mathfrak{a}_n^\sim = \mathfrak{a}_{n+1}^\sim$ on $D(f_i)$, that is $\mathfrak{a}_n = \mathfrak{a}_{n+1}$ as ideals of $\Gamma(U, \mathcal{O}_X)[f_i^{-1}]$. □

Remark 3.25. The proof shows that, if the base scheme fulfills the stronger condition that it is locally Noetherian, then internally speaking even the set of all quasicoherent ideals (instead of merely the finitely generated ones) fulfills the ascending process condition. We have not taken this property as the definition of a processly Noetherian ring since it is a notion not usually studied in constructive mathematics (compare Remark 8.9).

There is also an internal characterization of the property that X is locally Noetherian (in contrast to the property that all stalks are Noetherian). However, as described above, the corresponding internal notion is of limited usefulness.

Proposition 3.26. *Let X be a scheme. The following statements are equivalent:*

(1) *The scheme X is locally Noetherian.*
(2) *From the internal point of view of $\mathcal{O}_X$, any ascending chain of finitely generated ideals stabilizes.*

Proof. Similar to, but easier than, the proof of Proposition 3.24. □

Anonymously Noetherian rings. Classically, there is a characterization of Noetherian rings which doesn't involve ascending sequences: A ring is Noetherian if and only if any of its ideals is finitely generated. We mentioned in the footnote on page 33 that this condition is far too strong from a constructive point of view; not even the ring $\mathbb{Z}$ verifies it. However, it can be weakened to yield an interesting notion:

Definition 3.27. A ring A is *anonymously Noetherian* if and only if any ideal of A is *not not* finitely generated. A module M is *anonymously Noetherian* if and only if any submodule of M is *not not* finitely generated.

Example 3.28. There is an intuitionistic proof that the ring $\mathbb{Z}$ is anonymously Noetherian: Let $\mathfrak{a} \subseteq \mathbb{Z}$ be any ideal. Under the assumption that either there exists a nonzero element in $\mathfrak{a}$ or not, the ideal $\mathfrak{a}$ is *not not* finitely generated, even *not not* principal: For in the first case, a minimal element d of $\mathfrak{a} \cap \mathbb{N}^+$ (which *not not* exists) witnesses $\mathfrak{a} = (d)$. In the second case the ideal $\mathfrak{a}$ is the zero ideal. Since the assumption is *not not* satisfied, the ideal $\mathfrak{a}$ is *not not not not* finitely generated, so *not not* finitely generated. (We remark on this proof scheme on page 54.)

It appears that this Noetherian condition has not been studied in the literature on constructive algebra. Indeed, from the philosophical and the computational point of view on constructive mathematics, the notion of an anonymously Noetherian ring is not very useful: From an intuitionistic proof that a given ideal is finitely generated one can mechanically extract explicit generators, thereby satisfying computational or philosophical demands ("the generators are really there"). In contrast, an intuitionistic proof that a given ideal is *not not* finitely generated doesn't contain computational content in general.

Along the same lines, one could dismiss the fact (proved below) that Hilbert's basis theorem, stating that $A[X]$ is Noetherian if A is, holds for the notion of anonymously Noetherian rings. In fact, one could feel mocked by this version of Hilbert's basis theorem: It promises that any ideal of $A[X]$ "has" finitely many generators in some Platonic sense (without providing any clue on how one might go on to find the generators – they remain *anonymous*), provided that any ideal of A "has" finitely many generators in the same sense.

However, applications in the internal universe of toposes provide a further motivation for constructive reasoning, related but distinct from computational or philosophical considerations. A first indication that the notion of anonymously Noetherian rings is useful is that the structure sheaf $\mathcal{O}_X$ of any reduced scheme is anonymously Noetherian from the internal point of view of $\mathrm{Sh}(X)$. We exploit this observation in Section 11.5 to give a short proof of Grothendieck's generic freeness lemma.

Secondly, internal universes of toposes may satisfy certain classicality principles which are not generally satisfied in constructive mathematics. If a topos is set up in an intuitionistically sensible manner, one might then even be able to extract constructive results. For instance, the structure sheaf of a reduced scheme satisfies the principle

$$\mathrm{Sh}(X) \models \forall s\!:\!\mathcal{O}_X.\ \neg\neg(s = 0) \Longrightarrow s = 0.$$

This principle can be put to use as follows. Let's consider the situation that we have an intuitionistic proof that some ring element s is zero under the assumption that some ideal $\mathfrak{a}$ is finitely generated. Hence we also have an intuitionistic proof that s is *not not* zero under the assumption that $\mathfrak{a}$ is *not not* finitely generated. This assumption could be validated by the anonymously Noetherian property, yielding an unconditional proof that s is *not not* zero. Usually in constructive mathematics, we would be stuck at this point; but internally in $\mathrm{Sh}(X)$, we may continue and deduce that s is actually zero.

This observation is the basis for a fully constructive proof of Grothendieck's generic freeness lemma and puts our work in line of the general research program of extracting constructive content from classical proofs [36, 50, 78, 74].

Theorem 3.29. *Let A be an anonymously Noetherian ring. Then the polynomial algebra $A[X]$ is anonymously Noetherian as well, intuitionistically.*

Proof. Classically, this is precisely the statement of Hilbert's basis theorem, whose usual accounts do not care about the sensibilities of constructive mathematics. However, a careful reading of for instance the proof given in [7, Theorem 7.5] shows that the theorem holds intuitionistically as stated. □

Lemma 3.30. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. Intuitionistically, the module M is anonymously Noetherian if and only if M' and M'' are.*

Proof. The usual proof applies. □

Proposition 3.31. *Let X be an arbitrary reduced scheme (not necessarily locally Noetherian). Then $\mathcal{O}_X$ is anonymously Noetherian from the internal point of view of* $\mathrm{Sh}(X)$.

Proof. By Corollary 3.9, the ring $\mathcal{O}_X$ fulfills a suitable field condition from the internal point of view. Therefore it suffices to give an intuitionistic proof of the following statement: Let k be a ring such that any element of k which is not invertible is zero. Then any ideal of k is *not not* finitely generated.

Let $\mathfrak{a} \subseteq k$ be an arbitrary ideal. We have $\neg\neg(1 \in \mathfrak{a} \vee 1 \notin \mathfrak{a})$. Therefore $\neg\neg(\mathfrak{a} = (1) \vee \mathfrak{a} = (0))$. Thus $\mathfrak{a}$ is *not not* finitely generated (even *not not* principal). $\square$

The external translation of the statement that $\mathcal{O}_X[U_1, \ldots, U_n]$ is anonymously Noetherian was displayed on page 17, as an example of a convoluted statement which profits from the simpler internal account.

4. Sheaves of modules

From the internal perspective, a sheaf of $\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of rings, looks just like a plain module over the plain ring $\mathcal{R}$. This is proven just as the correspondence between sheaf of rings and internal rings (Proposition 3.1).

4.1. Finite local freeness. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *finite locally free* if and only if there exists a covering of X by open subsets U such that on each such U, the restricted module $\mathcal{F}|_U$ is isomorphic as an $\mathcal{O}_X|_U$-module to $(\mathcal{O}_X|_U)^n$ for some natural number n (which may depend on U).

Proposition 4.1. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is finite locally free if and only if, from the internal perspective, $\mathcal{F}$ is a finite free module, i. e.*

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \ulcorner \mathcal{F} \cong (\mathcal{O}_X)^n \urcorner,$$

or more elementarily

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_i a_i x_i.$$

Proof. By the expression "$(\mathcal{O}_X)^n$" in the internal language we mean the internally constructed object $\mathcal{O}_X \times \cdots \times \mathcal{O}_X$ with its componentwise $\mathcal{O}_X$-module structure. This coincides with the sheaf $(\mathcal{O}_X)^n$ as usually understood.

It is clear that the two stated internal conditions are equivalent, since the corresponding proof in linear algebra is intuitionistically valid. The equivalence with the external notion of finite local freeness follows because the interpretation of the first condition with the Kripke–Joyal semantics is the following: There exists a covering of X by open subsets U such that for each such U, there exists a natural number n and a morphism of sheaves $\varphi : \mathcal{F}|_U \to (\mathcal{O}_X|_U)^n$ such that

$$U \models \ulcorner \varphi \text{ is } \mathcal{O}_X\text{-linear} \urcorner \quad \text{and} \quad U \models \ulcorner \varphi \text{ is bijective} \urcorner.$$

The first subcondition means that φ is a morphism of sheaves of $\mathcal{O}_X|_U$-modules and the second one means that φ is an isomorphism of sheaves. $\square$

Remark 4.2. There are intuitionistic proofs of the following facts: An R-module is a dualizable object in the monoidal category of all R-modules if and only if it is finitely generated and projective. If R is local, then an R-module is finitely generated and projective if and only if it is finite free. Therefore an $\mathcal{O}_X$-module is internally dualizable if and only if is finite locally free.

4.2. Finite type, finite presentation, coherence. Recall the conditions of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or a ringed space) to be of finite type, of finite presentation, and to be coherent:

(1) $\mathcal{F}$ is *of finite type* if and only if there exists a covering of X by open subsets U such that for each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules.

(2) $\mathcal{F}$ is *of finite presentation* if and only if there exists a covering of X by open subsets U such that for each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^m \longrightarrow (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

(3) $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type and the kernel of any $\mathcal{O}_X|_U$-linear morphism $(\mathcal{O}_X|_U)^n \to \mathcal{F}|_U$, where $U \subseteq X$ is any open subset, is of finite type.

The following proposition gives translations of these definitions into the internal language.

Proposition 4.3. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then:*

(1) *$\mathcal{F}$ is of finite type if and only if $\mathcal{F}$, considered as an ordinary module from the internal perspective, is finitely generated, i. e. if*

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{F}.\ x = \sum_i a_i x_i.$$

(2) *$\mathcal{F}$ is of finite presentation if and only if $\mathcal{F}$ is a finitely presented module from the internal perspective, i. e. if*

$$\mathrm{Sh}(X) \models \bigvee_{n,m \geq 0} \ulcorner\text{there is a short exact sequence } \mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0\urcorner.$$

(3) *$\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is a coherent module from the internal perspective, i. e. if*

$$\begin{aligned} \mathrm{Sh}(X) \models\ & \ulcorner\mathcal{F} \text{ is finitely generated}\urcorner \wedge \\ & \bigwedge_{n \geq 0} \forall \varphi : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \ulcorner\ker\varphi \text{ is finitely generated}\urcorner. \end{aligned}$$

Proof. Straightforward – the translations of the internal statements using the Kripke–Joyal semantics are precisely the corresponding external statements. □

Remark 4.4. We believe that Proposition 4.3 settles a question Lawvere raised on the category theory mailing list [80]: "What concept of finiteness is appropriate for those important mathematical applications in topology for which [Kuratowski-finiteness] doesn't seem right? [...] Especially, a suitably 'finite' module should be a vector bundle or a [coherent sheaf] in the sense of Serre so that our simplified topos theory could apply more directly to those things it should."

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *generated by global sections* if and only if there exist global sections $s_i \in \Gamma(X, \mathcal{F})$ such that for any $x \in X$, the stalk $\mathcal{F}_x$ is generated by the germs of the s_i. This condition is of course not local on the base. Therefore there cannot exist a formula φ such that for any space X and any $\mathcal{O}_X$-module $\mathcal{F}$ it holds that $\mathcal{F}$ is generated by global sections if and only if $\mathrm{Sh}(X) \models \varphi(\mathcal{F})$. But still, global generation can be characterized by a mixed internal/external statement:

Proposition 4.5. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is generated by global sections if and only if there exist global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that*

$$\mathrm{Sh}(X) \models \forall x : \mathcal{F}.\ \bigvee_{J = \{i_1, \ldots, i_n\} \subseteq I \text{ finite}} \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_j a_j x_{i_j}.$$

Furthermore, $\mathcal{F}$ is generated by finitely many global sections if and only if there exist global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{F})$ such that

$$\mathrm{Sh}(X) \models \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_j a_j x_j.$$

Proof. The given internal statements are geometric implications, their validity can thus be checked stalkwise. □

Remark 4.6. The analog of Proposition 4.3 for sheaves of algebras instead of sheaves of modules holds. More precisely, let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras on a scheme X (or a ringed space). Then:

(1) $\mathcal{A}$ is of finite type if and only if $\mathcal{A}$, considered as an ordinary algebra from the internal perspective, is finitely generated, i. e. if

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{A}.\ \forall x : \mathcal{F}.\ \exists p : \mathcal{O}_X[X_1, \ldots, X_n].\ x = p(x_1, \ldots, x_n).$$

(2) $\mathcal{A}$ is of finite presentation if and only if $\mathcal{A}$ is a finitely presented algebra from the internal perspective, i. e. if

$$\mathrm{Sh}(X) \models \bigvee_{n,m \geq 0} \exists f_1, \ldots, f_m : \mathcal{O}_X[X_1, \ldots, X_n].$$
$$\ulcorner \mathcal{A} \cong \mathcal{O}_X[X_1, \ldots, X_n]/(f_1, \ldots, f_m) \urcorner.$$

4.3. Tensor product and flatness. Recall that the tensor product of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X (or a ringed space) is usually constructed as the sheafification of the presheaf

$$U \subseteq X \text{ open} \quad \longmapsto \quad \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$

From the internal point of view, $\mathcal{F}$ and $\mathcal{G}$ look like ordinary modules, so that we can consider their tensor product as usually constructed in commutative algebra, as a certain quotient of the free module on the elements of $\mathcal{F} \times \mathcal{G}$:

$$\mathcal{O}_X\langle x \otimes y \mid x : \mathcal{F}, y : \mathcal{G}\rangle / R,$$

where R is the submodule generated by

$$\begin{gathered} (x + x') \otimes y - x \otimes y - x' \otimes y, \\ x \otimes (y + y') - x \otimes y - x \otimes y', \\ (sx) \otimes y - s(x \otimes y), \\ x \otimes (sy) - s(x \otimes y) \end{gathered}$$

with $x, x' : \mathcal{F}$, $y, y' : \mathcal{G}$, $s : \mathcal{O}_X$. This internal construction gives rise to the same sheaf of modules as the externally defined tensor product:

Proposition 4.7. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the internally constructed tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ coincides with the external one.*

Proof. Since the proof of the corresponding fact of commutative algebra is intuitionistically valid, the internally defined tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ has the following universal property: For any $\mathcal{O}_X$-module H, any $\mathcal{O}_X$-bilinear map $\mathcal{F} \times \mathcal{G} \to H$ uniquely factors over the canonical map $\mathcal{F} \times \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Interpreting this property with the Kripke–Joyal semantics, we see that the internally constructed tensor product has the following external property: For any open subset $U \subseteq X$ and any $\mathcal{O}_X|_U$-module $\mathcal{H}$ on U, any $\mathcal{O}_X|_U$-bilinear morphism $\mathcal{F}|_U \times \mathcal{G}|_U \to \mathcal{H}$ uniquely factors over the canonical morphism $\mathcal{F}|_U \times \mathcal{G}|_U \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U$.

In particular, for $U = X$, this property is well-known to be the universal property of the externally constructed tensor product. Therefore the claim follows. □

A description of the stalks of the tensor product follows purely by considering the logical form of the construction:

Corollary 4.8. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the stalks of the tensor product coincide with the tensor products of the stalks: $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.*

Proof. We constructed the tensor product using the following operations: product of two sets, free module on a set, quotient module with respect to a submodule; submodule generated by a set of elements given by a geometric formula. All of these operations are geometric, so the tensor product construction is geometric as well (see Section 2.3). Hence taking stalks commutes with performing the construction. ☐

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if and only if all stalks $\mathcal{F}_x$ are flat $\mathcal{O}_{X,x}$-modules. We can characterize flatness in the internal language.

Proposition 4.9. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat if and only if, from the internal perspective, $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Recall that flatness of an A-module M can be characterized without reference to tensor products by the following condition (using suggestive vector notation): For any natural number p, any p-tuple $m : M^p$ of elements of M and any p-tuple $a : A^p$ of elements of A, it should hold that

$$a^T m = 0 \implies \bigvee_{q \geq 0} \exists n : M^q, B : A^{p \times q}.\ Bn = m \wedge a^T B = 0.$$

The equivalence of this condition with tensoring being exact holds intuitionistically as well [92, Theorem III.5.3]. This formulation of flatness has the advantage that it is the conjunction of geometric implications (one for each $p \geq 0$); therefore it holds internally if and only if it holds at any point. ☐

4.4. Support. Recall that the *support* of an $\mathcal{O}_X$-module $\mathcal{F}$ is the subset $\operatorname{supp} \mathcal{F} := \{x \in X \mid \mathcal{F}_x \neq 0\} \subseteq X$. If $\mathcal{F}$ is of finite type, this set is closed, since its complement is then open by a standard lemma. (We will give an internal proof of this fact in Lemma 6.40.)

Proposition 4.10. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the interior of the complement of the support of $\mathcal{F}$ can be characterized as the largest open subset of X on which the internal statement $\mathcal{F} = 0$ holds.*

Proof. For any open subset $U \subseteq X$, it holds that:

$$\begin{aligned} & U \subseteq \operatorname{int}(X \setminus \operatorname{supp} \mathcal{F}) \\ \iff & U \subseteq X \setminus \operatorname{supp} \mathcal{F} \\ \iff & U \subseteq \{x \in X \mid \forall s \in \mathcal{F}_x.\ s = 0\} \\ \iff & U \models \forall s : \mathcal{F}.\ s = 0 \\ \iff & U \models \ulcorner \mathcal{F} = 0 \urcorner \end{aligned}$$

The second to last equivalence is because "$\forall s : \mathcal{F}.\ s = 0$" is a geometric implication and can thus be checked stalkwise. ☐

Remark 4.11. The support of a sheaf of *sets* $\mathcal{F}$ is defined as the subset $\{x \in X \mid \mathcal{F}_x \text{ is not a singleton}\}$. A similar proof shows that the interior of its complement can be characterized as the largest open subset of X where the internal statement $\ulcorner\mathcal{F}$ is a singleton$\urcorner$ holds.

4.5. Torsion. Let R be a ring. Recall that the *torsion submodule* M_{tors} of an R-module M is defined as

$$M_{\text{tors}} := \{x : M \mid \exists a : R.\ \ulcorner a \text{ regular} \urcorner \wedge ax = 0\} \subseteq M.$$

This definition is meaningful even if R is not an integral domain. An R-module M is *torsion-free* if and only if M_{tors} is the zero submodule; an R-module M is a *torsion module* if and only if $M_{\text{tors}} = M$.

Recall also that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an integral scheme X, there is a unique subsheaf $\mathcal{F}_{\text{tors}} \subseteq \mathcal{F}$ with the property that $\Gamma(U, \mathcal{F}_{\text{tors}}) = \Gamma(U, \mathcal{F})_{\text{tors}}$ for all affine open subsets $U \subseteq X$. The content of the following proposition is that internally constructing the torsion submodule of $\mathcal{F}$, regarded as a plain module from the internal perspective, gives exactly this subsheaf. There is therefore no harm in using the same notation "$\mathcal{F}_{\text{tors}}$" for the result of the internal construction.

Proposition 4.12. *Let X be an integral scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $U = \operatorname{Spec} A \subseteq X$ be an affine open subset. Let $s \in \Gamma(U, \mathcal{F})$ be a local section. Then*

$$s \in \Gamma(U, \mathcal{F})_{\text{tors}} \quad \textit{if and only if} \quad U \models s \in \mathcal{F}_{\text{tors}}.$$

Proof. The "only if" direction is trivial in view of Lemma 3.18: If s is a torsion element of $\Gamma(U, \mathcal{F})$, there exists a regular element $a \in \Gamma(U, \mathcal{O}_X)$ such that $as = 0$. By the lemma, this element is regular from the internal perspective as well, so $U \models \ulcorner a \text{ regular} \urcorner \wedge as = 0$.

For the "if" direction, we may assume that there exists an open covering $X = \bigcup_i U_i$ by standard open subsets $U_i = D(f_i)$ such that there are sections $a_i \in \Gamma(U_i, \mathcal{O}_X) = A[f_i^{-1}]$ with $U_i \models \ulcorner a_i \text{ regular} \urcorner \wedge a_i s = 0$. Without loss of generality, we may assume that the denominators of the a_i's are ones, that the f_i are finite in number, and that the f_i are regular (i. e. nonzero, since A is an integral domain). By Lemma 3.18, the a_i are regular in $A[f_i^{-1}]$ and by regularity of the f_i also regular in A. Therefore their product $\prod_i a_i \in A$ is regular in A as well and annihilates s. □

Proposition 4.13. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then $(\mathcal{F}_{\text{tors}})_x = (\mathcal{F}_x)_{\text{tors}}$.*

Proof. This would be obvious if the condition on an element $s : \mathcal{F}$ to belong to $\mathcal{F}_{\text{tors}}$ were a geometric formula. Because of the universal quantifier, it is not:

$$s \in \mathcal{F}_{\text{tors}} \quad \Longleftrightarrow \quad \exists a : \mathcal{O}_X.\ (\forall b : \mathcal{O}_X.\ ab = 0 \Rightarrow b = 0) \wedge as = 0.$$

But since X is assumed to be locally Noetherian, regularity is an open property nonetheless (see Proposition 9.4 for an internal proof of this fact). Thus the claim still follows, just like in the proof of Proposition 3.17. □

4.6. Kähler differentials. Let $A \to B$ be a homomorphism of rings. The B-module $\Omega^1_{B|A}$ of Kähler differentials can be constructed as the free B-module on the basis $(db)_{b \in B}$ consisting of formal symbols modulo appropriate relations ensuring that the map $b \mapsto db$ is A-linear and satisfies the Leibniz rule; it verifies the universal property that the map $B \to \Omega^1_{B|A}$ is the initial A-linear derivation of B.

For constructing the sheaf of Kähler differentials for a morphism of schemes, one often resorts to the alternative construction as I/I^2, where $I \subseteq B \otimes_A B$ is the kernel of the multiplication map $B \otimes_A B \to B$. The verification of the universal property is slightly harder for this construction. Gathmann comments on this situation as follows [52, p. 134]:

> Of course, if $f : X \to Y$ is a morphism of general (not necessarily affine) schemes, we want to consider the relative differentials of every restriction of f to affine opens of X and Y, and glue them together to get a quasi-coherent sheaf $\Omega_{X|Y}$. To do this, we have

> to give a different description of the relative differentials, as the construction [via the free module] does not glue very well.

After having constructed a global sheaf of Kähler differentials using the alternative description (only in the case that f is separated, though with a little bit of more work this assumption can be dispensed with), he goes on as follows [52, Remark 7.4.8]:

> It should be stressed that [the definition using the alternative construction] is essentially useless for practical computations. Its only use is to show that a global object $\Omega_{X|Y}$ exists that restricts to the old definition on affine open subsets. For applications, we will always use [the definition using the free module and the calculation of the Kähler differentials of a morphism of the form $k \to k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$] on open subsets.

Vakil chooses a similar route [129, Section 21.2].

Regarding the sheaf of Kähler differentials as the conormal sheaf of the diagonal embedding (and hence using the alternative construction) is of course essential for further developments and hence very useful. However, if the goal is just to construct a global sheaf of Kähler differentials, we can employ the internal language to construct it using only the formal construction via the free module.

Specifically, if $f : X \to S$ is a morphism of schemes, we can construct, in the internal universe of $\mathrm{Sh}(X)$, the module of Kähler differentials of the morphism $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$. A number of basic properties then follow purely formally:

Proposition 4.14. *Let $f : X \to S$ be a morphism of schemes (or of locally ringed spaces). Let $\Omega^1_{X|S}$ be defined as the interpretation of the internal construction $\Omega^1_{\mathcal{O}_X|f^{-1}\mathcal{O}_S}$.*

1. *The sheaf of Kähler differentials has the following universal property: For any open subset $U \subseteq X$, any $(f^{-1}\mathcal{O}_S)|_U$-linear derivation $\mathcal{O}_X|_U \to \mathcal{E}$ over U uniquely factors over $\mathcal{O}_X|_U \to \Omega^1_{X|S}|_U$.*
2. *The stalks $(\Omega^1_{X|S})_x$ are canonically isomorphic to $\Omega^1_{\mathcal{O}_{X,x}|\mathcal{O}_{S,f(x)}}$.*
3. *The sheaf $\Omega^1_{X|S}$ is quasicoherent.*

Proof. The first claim is just the interpretation of the internal universal property using the Kripke–Joyal semantics and using the simplification rule given in Proposition 2.6,

The second claim is immediate because the construction of Kähler differentials via the free module is geometric (see 23). Therefore the operations "take the Kähler differentials" and "take the stalk at x" commute.

For verifying the third claim, it suffices to verify that $\Omega^1_{X|S}$ is quasicoherent in case that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine. Let $\varphi : A \to B$ be homomorphism of rings given by f. We employ the technique and notation of Section 11.2:

$$\begin{aligned}(\Omega^1_{B|A})^\sim &\cong \underline{\Omega^1_{B|A}}[\mathcal{F}^{-1}] \cong \Omega^1_{\underline{B}|\underline{A}}[\mathcal{F}^{-1}] \\ &\cong \Omega^1_{\underline{B}[\mathcal{F}^{-1}]|\underline{A}[(\varphi^{-1}[\mathcal{F}])^{-1}]} \cong \Omega^1_{\mathcal{O}_{\mathrm{Spec}(B)}|f^{-1}\mathcal{O}_{\mathrm{Spec}(A)}}.\end{aligned}$$

The first isomorphism is by Proposition 11.6, the second because the geometric construction "taking Kähler differentials" commutes with "taking constant sheaves" (since it can be expressed as pullback along a geometric morphism), the third by [118, Tag 00RT], and the fourth by Proposition 11.6. □

Incidentally, the Stacks Project too uses the construction via the free module [118, Tag 08RL], however because the Stacks Project doesn't employ the internal language they have to manually sheafify and keep track of open subsets.

4.7. Internal proofs of common lemmas.

Lemma 4.15. *Let X be a scheme (or a ringed space). Let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{H}$ are of finite type, so is $\mathcal{G}$; similarly, if $\mathcal{F}$ and $\mathcal{H}$ are finite locally free, so is $\mathcal{G}$.

Proof. From the internal perspective, we are given a short exact sequence of modules with the outer ones being finitely generated (resp. finite free) and we have to show that the middle one is finitely generated (resp. finite free) as well. It is well-known that this follows; and since the usual proof of this fact is intuitionistically valid, we are done. $\square$

The proof works very generally, in the context of arbitrary ringed spaces, and is still very simple. This is common to proofs using the internal language. Particular features of schemes enter only at clearly recognizable points, for instance when an internal property specific to the structure sheaf of schemes is used (such as in Proposition 3.7).

Lemma 4.16. *Let X be a scheme (or a ringed space).*

(1) *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If two of the three modules are coherent, so is the third.*
(2) *Let $\mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules such that $\mathcal{F}$ is of finite type and $\mathcal{G}$ is coherent. Then its kernel is of finite type as well.*
(3) *If $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module and $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module, the $\mathcal{O}_X$-modules $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ are coherent as well.*

Proof. These statements follow directly from interpreting the corresponding standard proofs of commutative algebra in the internal language. For those standard proofs, see for instance the lecture notes of Ravi Vakil [129, Section 13.8], where they are given as a series of exercises. $\square$

Lemma 4.17. *Let X be a scheme (or a locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of finite locally free $\mathcal{O}_X$-modules. Then the kernel of α is finite locally free as well.*

Proof. It suffices to give an intuitionistic proof of the following statement: The kernel of a matrix over a local ring, which as a linear map is surjective, is finite free.

Let $M : R^{n \times m}$ be such a matrix. Since by the surjectivity assumption some linear combination of the columns is e_1 (the first canonical basis vector), some linear combination of the entries of the first row of M is 1. By locality of R, at least one entry of the first row is invertible. By applying appropriate column and row transformations, we may therefore assume that M is of the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \widetilde{M} & \\ 0 & & & \end{array}\right)$$

with the submatrix $\widetilde{M}$ fulfilling the same condition as M. Continuing in this way, it follows that $m \geq n$ and that we may assume that M is of the form

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right).$$

The kernel of such a matrix is obviously freely generated by the canonical basis vectors corresponding to the zero columns. In particular, the rank of the kernel is $m - n$. ☐

Remark 4.18. The internal language machinery gives no reason to believe that the dual statement is true, i. e. that the cokernel of a monomorphism of finite locally free $\mathcal{O}_X$-modules is finite locally free. This would follow from an intuitionistic proof of the statement that the cokernel of an injective map between finite free modules over a local ring is finite free. But this statement is of course false (consider the exact sequence $0 \longrightarrow \mathbb{Z}_{(2)} \xrightarrow{\cdot 2} \mathbb{Z}_{(2)} \longrightarrow \mathbb{F}_2 \longrightarrow 0$ of $\mathbb{Z}_{(2)}$-modules).

Lemma 4.19. *Let X be a scheme (or a locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of finite locally free $\mathcal{O}_X$-modules of the same rank. Then α is an isomorphism.*

Proof. It suffices to give an intuitionistic proof of the following statement: A square matrix over a local ring, which as a linear map is surjective, is invertible.

This follows from the proof of the previous lemma, since it shows that the kernel of such a matrix is finite free of rank zero. ☐

Remark 4.20. The conclusion of Lemma 4.19 also holds if X is only assumed to be a ringed space. To show this, it suffices to give an intuitionistic proof of the following statement: A square matrix over a (not necessarily local) ring, which as a linear map is surjective, is invertible. Such a matrix A possesses a right inverse. Therefore $\det A$ is invertible. Thus A is invertible with inverse $(\det A)^{-1} \cdot \operatorname{ad} A$.

Lemma 4.21. *Let X be a scheme (or a ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be a monomorphism of finite locally free $\mathcal{O}_X$-modules of the same rank. Then α is an isomorphism.*

Proof. It suffices to give an intuitionistic proof of the following statement: A square matrix over a ring, which as a linear map is injective, is invertible. Such a proof can for instance be found in [104, p. 1013]. ☐

Lemma 4.22. *Let X be a scheme (or a ringed space). Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Then for the closures of the supports there holds the equation* $\operatorname{cl}\operatorname{supp}\mathcal{G} = \operatorname{cl}\operatorname{supp}\mathcal{F} \cup \operatorname{cl}\operatorname{supp}\mathcal{H}$.

Proof. Switching to complements, we have to prove that

$$\operatorname{int}(X \setminus \operatorname{supp}\mathcal{G}) = \operatorname{int}(X \setminus \operatorname{supp}\mathcal{F}) \cap \operatorname{int}(X \setminus \operatorname{supp}\mathcal{H}).$$

By Proposition 4.10, it suffices to prove

$$\mathrm{Sh}(X) \models (\mathcal{G} = 0 \iff \mathcal{F} = 0 \land \mathcal{H} = 0);$$

this is a basic observation in linear algebra, valid intuitionistically. ☐

Of course, a stronger version of this lemma – about the supports themselves instead of their closures – is easily proven without using the internal language. We included this example only for illustrative purposes.

Lemma 4.23. *Let X be a scheme (or a locally ringed space). Let $\mathcal{L}$ be a line bundle on X, i. e. an $\mathcal{O}_X$-module locally free of rank 1. Let $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L})$ be global sections. Then these sections globally generate $\mathcal{F}$ if and only if*

$$\mathrm{Sh}(X) \models \bigvee_i \ulcorner \alpha(s_i) \text{ is invertible for some isomorphism } \alpha : \mathcal{L} \to \mathcal{O}_X \urcorner.$$

Proof. It suffices to give an intuitionistic proof of the following fact: Let R be a local ring. Let L be a free R-module of rank 1. Let $s_1, \ldots, s_n : L$ be given elements. Then L is generated as an R-module by these elements if and only if for some i, the image of s_i under some isomorphism $L \to R$ is invertible.

The choice of such an isomorphism does not matter, since any two such isomorphisms $\alpha, \beta : L \to R$ differ by a unit of R: $\alpha(x) = \alpha(\beta^{-1}(1)) \cdot \beta(x)$ for any $x : L$, and $\alpha(\beta^{-1}(1)) \cdot \beta(\alpha^{-1}(1)) = 1$ in R.

For the "if" direction, we have that some $\alpha(s_i)$ is a generator of R. Since α is an isomorphism, it follows that s_i generates L, and thus that in particular, the family $s_1, \ldots, s_n$ generates L.

For the "only if" direction, we have that the unit of R can be expressed as a linear combination of the $\alpha(s_i)$, where $\alpha : L \to R$ is some isomorphism (whose existence is assured by the assumption on the rank of L). Since R is a local ring, it follows that one of the summands and thus one of the $\alpha(s_i)$ is invertible. □

Remark 4.24. The canonical ring homomorphism $\mathcal{O}_{X,x} \twoheadrightarrow k(x)$ is local. Therefore a germ in $\mathcal{O}_{X,x}$ is invertible if and only if its image in $k(x)$ is not zero. From this one can conclude that global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{F})$ generate $\mathcal{F}$ if and only if, for any point $x \in X$, the images $s_i \in \mathcal{F}|_x$ in the fibers do not vanish simultaneously.

Lemma 4.25. *Let X be a scheme (or a ringed space). Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of rank 1. Then $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.*

Proof. Recall that the dual is defined by $\mathcal{L}^\vee := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. Since "$\mathcal{H}$om" looks like "Hom" from the internal point of view, the dual sheaf $\mathcal{L}^\vee$ looks just like the ordinary dual module. However, to prove the claim, it does *not* suffice to give an intuitionistic proof of the following fact of linear algebra: "Let L be a free R-module of rank 1. Then there exists an isomorphism $L^\vee \otimes_R L \to R$." Since the interpretation of "$\exists$" using the Kripke–Joyal semantics is local existence, this would only show that $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L}$ is *locally* isomorphic to $\mathcal{O}_X$.

Instead, we have to actually *write down* (i. e. explicitly give) a linear map in the internal language – not using the assumption that L is free of rank 1, as this would introduce an existential quantifier again (see Section 2.2). So we have to prove the following fact: Let L be an R-module. Then there explicitly exists a linear map $L^\vee \otimes_R L \to R$ such that this map is an isomorphism if L is free of rank 1.

This is done as usual: Define $\alpha : L^\vee \otimes_R L \to R$ by $\lambda \otimes x \mapsto \lambda(x)$. Since L is free of rank 1, there is an isomorphism $L \cong R$. Precomposing α with the induced isomorphism $R^\vee \otimes_R R \to L^\vee \otimes_R L$, we obtain the linear map $R^\vee \otimes_R R \to R$ given by the same term: $\lambda \otimes x \mapsto \lambda(x)$. One can check that an inverse is given by $x \mapsto \mathrm{id}_R \otimes x$. □

Lemma 4.26. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module.*

1. *Assume X to be a locally Noetherian scheme. Then $\mathcal{F}$ is torsion-free (meaning $\mathcal{F}_{\mathrm{tors}} = 0$) if and only if all stalks $\mathcal{F}_x$ are torsion-free.*
2. *The quotient sheaf $\mathcal{F}/\mathcal{F}_{\mathrm{tors}}$ is torsion-free and the torsion submodule $\mathcal{F}_{\mathrm{tors}}$ is a torsion module.*
3. *The dual sheaf $\mathcal{F}^\vee$ is torsion-free.*
4. *If $\mathcal{F}$ is reflexive (meaning that the canonical morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism), then $\mathcal{F}$ is torsion-free.*
5. *If $\mathcal{F}$ is finite locally free, then $\mathcal{F}$ is reflexive.*
6. *Assume X to be a Dedekind scheme and $\mathcal{F}$ to be of finite presentation. If $\mathcal{F}$ is torsion-free, then $\mathcal{F}$ is finite locally free.*

Proof. The first statement follows from the observation that $(\mathcal{F}_{\mathrm{tors}})_x = (\mathcal{F}_x)_{\mathrm{tors}}$ (Proposition 4.13). All the others follow simply by interpreting the corresponding facts of linear algebra in the internal universe. For concreteness, we give intuitionistic proofs of the last three statements.

So let M be an reflexive R-module. We have to show that M is torsion-free. To this end, let an element $x : M$ and a regular element $a : R$ such that $ax = 0$ be given. For any $\vartheta : M^\vee$, it follows that $\vartheta(x) = 0$, since $a\vartheta(x) = \vartheta(ax) = \vartheta(0) = 0$ and a is regular. Thus the image of x under the canonical map $M \to M^{\vee\vee}$ is zero. By reflexivity, this implies that x is zero.

For statement (5), we have to prove that R-modules of the form R^n are reflexive. This is obvious, the required inverse map is $(R^n)^{\vee\vee} \to R^n,\ \lambda \mapsto \sum_i \lambda(\vartheta_i)$ where $\vartheta_i : R^n \to R,\ (x_j)_j \mapsto x_i$.

In view of Corollary 3.20 we can put matrices over $\mathcal{O}_X$ into Smith canonical form if X is a Dedekind scheme. Therefore it suffices to give an intuitionistic proof of the following fact: Let R be an integral domain in the strong sense such that matrices over R can be put into Smith canonical form. Let M be a finitely presented torsion-free R-module. Then M is finite free.

Such a proof can proceed as follows: Since M is finitely presented, it is the cokernel of some matrix. Without loss of generality, we may assume that the presentation matrix is diagonal, so M is isomorphic to some finite direct sum $\bigoplus_i R/(a_i)$. Since M is torsion-free, all the summands $R/(a_i)$ are torsion-free as well. Since R is an integral domain in the strong sense, the a_i are either zero or invertible. Thus $R/(a_i)$ is isomorphic to R or to the zero module. In any case, $R/(a_i)$ is finite free and therefore M is finite free as well. □

5. Upper semicontinuous functions

5.1. Interlude on natural numbers. In classical logic, the natural numbers are complete in the sense that any inhabited set of natural numbers possesses a minimal element. This statement cannot be proven intuitionistically – intuitively, this is because one cannot explicitly pinpoint the (classically existing) minimal element of an arbitrary inhabited set;[13] see below for a sheaf-theoretic interpretation.

In intuitionistic logic, the completeness principle can be salvaged in two essentially different ways: either by strengthening the premise, or by weakening the conclusion.

Lemma 5.1. *Let $U \subseteq \mathbb{N}$ be an inhabited subset of the natural numbers.*

(1) *Assume U to be* detachable*, i. e. assume that for any natural number n, either $n \in U$ or $n \notin U$. Then U possesses a minimal element.*
(2) *In any case, U does* not not *possess a minimal element.*

Proof. The first statement can be proven by induction on the witness of inhabitation, i. e. the given number n such that $n \in U$. We omit further details, since we will not need this statement in our applications.

For the second statement, we give a careful proof since logical subtleties matter. To simplify the exposition, we assume that U is upward-closed, i. e. that any number larger than some element of U lies in U as well. Any subset can be closed in this way (by considering $\{n \in \mathbb{N} \mid \exists m \in U.\ n \geq m\}$) and a minimal element of the closure will be a minimal element for U as well.

We induct on the number $n \in U$ given by the assumption that U is inhabited. In the case $n = 0$ we are done since 0 is a minimal element of U. For the induction

[13] Let φ be an arbitrary formula. Assuming that any inhabited subset of the natural numbers possesses a minimal element, we want to show that $\varphi \vee \neg\varphi$. Define the subset $U := \{n \in \mathbb{N} \mid (n = 1) \vee \varphi\} \subseteq \mathbb{N}$, which surely is inhabited by $1 \in U$. So by assumption, there exists a number $z \in \mathbb{N}$ which is the minimum of U. We have $z = 0$ or $z > 0$. If $z = 0$, we have $0 \in U$, so $(0 = 1) \vee \varphi$, so φ holds. If $z > 0$, then $\neg\varphi$ holds: If φ were true, zero would be an element of U, contradicting the minimality of z.

step $n \to n+1$, the intuitionistically valid double negation of the law of excluded middle gives

$$\neg\neg(n \in U \vee n \notin U).$$

Because of the tautologies $(\varphi \Rightarrow \psi) \Rightarrow (\neg\neg\varphi \Rightarrow \neg\neg\psi)$ and $\neg\neg\neg\neg\varphi \Rightarrow \neg\neg\varphi$ (see Section 2.4), it suffices to show that $n \in U \vee n \notin U$ implies the conclusion. So assume $n \in U \vee n \notin U$. If $n \in U$, then U does *not not* possess a minimal element by the induction hypothesis. If $n \notin U$, then $n+1$ is a minimal element (and so, in particular, U does *not not* possess a minimal element): If m is any element of U, we have $m \geq n+1$ or $m \leq n$. In the first case, we're done. In the second case, it follows that $n \in U$ because U is upward-closed and so we obtain a contradiction. From this contradiction we can trivially deduce $m \geq n+1$ as well. □

If we want to work with a complete partially ordered set (poset) of natural numbers in intuitionistic logic, we have to construct a suitable completion. The idea of the following definition is to encode numbers as the (not necessarily existing) minimum of inhabited upward-closed subsets.

Definition 5.2. The *completed poset of natural numbers* is the set $\widehat{\mathbb{N}}$ of all inhabited upward-closed subsets of $\mathbb{N}$, ordered by reverse inclusion. The elements of $\widehat{\mathbb{N}}$ are called *generalized natural numbers.*

Lemma 5.3. *The completed poset of natural numbers is the least poset containing* $\mathbb{N}$ *and possessing minima of arbitrary inhabited subsets.*

Proof. The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \quad \longmapsto \quad \uparrow(n) := \{m \in \mathbb{N} \mid m \geq n\}.$$

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

We omit the proof of the universal property. □

Remark 5.4. In classical logic, the map $\widehat{\mathbb{N}} \to \mathbb{N}$, $U \mapsto \min U$ is a well-defined isomorphism of partially ordered sets. In fact, it is the inverse of the canonical embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$. In intuitionistic logic, this embedding is still injective, but it cannot be shown to be surjective: It is only the case that any element of $\widehat{\mathbb{N}}$ does *not not* possess a preimage (by Lemma 5.1).

5.2. A geometric interpretation. We are interested in the completed natural numbers for the following reason: A generalized natural number in the topos of sheaves on a topological space X is the same as an upper semicontinuous function $X \to \mathbb{N}$.

Lemma 5.5. *Let X be a topological space. The sheaf $\widehat{\mathbb{N}}$ of generalized natural numbers on X is canonically isomorphic to the sheaf of upper semicontinuous* $\mathbb{N}$*-valued functions on X.*

Proof. When referring to the natural numbers in the internal language, we actually refer to the constant sheaf $\underline{\mathbb{N}}$ on X. (This is because the sheaf $\underline{\mathbb{N}}$ fulfills the axioms for a natural numbers object, cf. [86, Section VI.1].) Recall that its sections on an open subset $U \subseteq X$ are continuous functions $U \to \mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology.

Therefore, a section of $\widehat{\mathbb{N}}$ on an open subset $U \subseteq X$ is given by a subsheaf $\mathcal{A} \hookrightarrow \underline{\mathbb{N}}|_U$ such that

$$U \models \exists n : \underline{\mathbb{N}}.\ n \in \mathcal{A} \quad \text{and} \quad U \models \forall n, m : \underline{\mathbb{N}}.\ n \geq m \wedge n \in \mathcal{A} \Rightarrow m \in \mathcal{A}.$$

Since these conditions are geometric implications, they are satisfied if and only if any stalk $\mathcal{A}_x$ is an inhabited upward-closed subset of $\underline{\mathbb{N}}_x \cong \mathbb{N}$. The association

$$x \in X \quad \longmapsto \quad \min\{n \in \mathbb{N} \mid n \in \mathcal{A}_x\}$$

thus defines a map $X \to \mathbb{N}$. This map is indeed upper semicontinuous, since if $n \in \mathcal{A}_x$, there exists an open neighborhood V of x such that the constant function with value n is an element of $\Gamma(V, \mathcal{A})$ and therefore $n \in \mathcal{A}_y$ for all $y \in V$.

Conversely, let $\alpha : U \to \mathbb{N}$ be a upper semicontinuous function. Then

$$V \subseteq U \text{ open} \quad \longmapsto \quad \{f : V \to \mathbb{N} \mid f \text{ continuous}, f \geq \alpha \text{ on } V\}$$

is a subobject of $\underline{\mathbb{N}}|_U$ which internally is inhabited and upward-closed. Further details are left to the reader. □

Under the correspondence given by Lemma 5.5, locally *constant* functions map precisely to the (image of the) *ordinary* internal natural numbers (in the completed natural numbers). In a similar vein, the sheaf given by the internal construction of the set of *all* upward-closed subsets of the natural numbers (not only the inhabited ones) is canonically isomorphic to the sheaf of upper semicontinuous functions with values in $\mathbb{N} \cup \{+\infty\}$.

The correspondence can be used to understand classical facts about upper semicontinuous functions as features of intuitionistic number theory. For instance, it is well-known that any upper semicontinuous $\mathbb{N}$-valued function on an arbitrary topological space is locally constant on a dense open subset. This can be explained as follows: The generalized natural number associated to such a function is *not not* an ordinary natural number from the internal point of view. Since "not not" translates to "holding on a dense open subset" (Proposition 6.5), it follows that there is a dense open subset on which the function corresponds to an ordinary internal natural number, i. e. is locally constant.

5.3. The upper semicontinuous rank function. Recall that the rank of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or locally ringed space) at a point $x \in X$ is defined as the $k(x)$-dimension of the vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. If we assume that $\mathcal{F}$ is of finite type around x, this dimension is finite and equals the minimal number of elements needed to generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module by Nakayama's lemma.

In the internal language, we can define an element of $\widehat{\mathbb{N}}$ by

$$\operatorname{rank} \mathcal{F} := \min\{n \in \mathbb{N} \mid \ulcorner\text{there is a gen. family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner\} \in \widehat{\mathbb{N}}.$$

If the module $\mathcal{F}$ is finite locally free, it will be a finite free module from the internal point of view and the rank defined in this way will be an actual natural number (see below); but in general, the rank is really an element of the completion.

Proposition 5.6. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type on a scheme X (or a locally ringed space). Under the correspondence given by Lemma 5.5, the internally defined rank maps to the rank function of $\mathcal{F}$.*

Proof. We have to show that for any point $x \in X$ and natural number n, there exists a generating family for $\mathcal{F}_x$ consisting of n elements if and only if there exists an open neighborhood U of x such that

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

The "if" direction is obvious. For the "only if" direction, consider (liftings to local sections of a) generating family $s_1, \ldots, s_n$ of $\mathcal{F}_x$. Since $\mathcal{F}$ is of finite type, there also exist sections $t_1, \ldots, t_m$ on some neighborhood V of x which generate any stalk $\mathcal{F}_y$, $y \in V$. Since the t_i can be expressed as a linear combination of the s_j in $\mathcal{F}_x$, the same is true on some open neighborhood $U \subseteq V$ of x. On this neighborhood, the s_j generate any stalk $\mathcal{F}_y$, $y \in U$, so by geometricity we have

$$U \models \ulcorner s_1, \ldots, s_n \text{ generate } \mathcal{F}\urcorner. \qquad \square$$

Once we understand when properties spread from points to neighborhoods in logical terms, we will be able to give a simpler proof of Proposition 5.6 (see Lemma 6.42).

Lemma 5.7. *Let X be a scheme (or a locally ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. If $\mathcal{F}$ is finite locally free, its rank function is locally constant. The converse holds if X is a reduced scheme.*

Proof. The rank function is locally constant if and only if internally, the rank of $\mathcal{F}$ is an actual natural number. Also recall that the structure sheaf fulfills a certain field condition if X is a reduced scheme (Corollary 3.9). Therefore it suffices to give a proof of the following fact of intuitionistic linear algebra: Let R be a local ring. Let M be a finitely generated R-module. If M is finite free, its rank is an actual natural number. The converse holds if R fulfills the field condition that any element which is not invertible is zero.

So assume that such a module M is finite free. Then it is isomorphic to R^n for some actual natural number n; by the internal proof in Lemma 4.17, the rank of M is therefore this number n (for any surjection $R^m \twoheadrightarrow R^n$ it holds that $m \geq n$).

Conversely, assume that the rank of M is an actual natural number. Then there exists a minimal generating family $x_1, \ldots, x_n : M$. We can verify that this family is indeed linearly independent (and thus a basis, demonstrating that M is finite free): Let $\sum_i a_i x_i = 0$ with $a_i : R$. If any a_i were invertible, the family $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ would too generate M, contradicting the minimality. So each a_i is not invertible. By the field property of R, each a_i is zero. $\square$

Lemma 5.8. *Let X be a reduced scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then $\mathcal{F}$ is finite locally free on a dense open subset.*

Proof. Since "dense open" translates to "not not" in the internal language (Proposition 6.5), it suffices to give an intuitionistic proof of the following fact: Let R be a local ring which fulfills an appropriate field condition. Let M be a finitely generated R-module. Then M is *not not* finite free.

By Remark 5.4, the rank of such a module M is *not not* an actual natural number. By the last part of the previous proof, it thus follows that M is *not not* finite free. $\square$

Remark 5.9. Besides basics on natural numbers in an intuitionistic setting and some dictionary terms ("reduced", "finite locally free", "finite type", "dense open"), this proof does not depend on any further tools. In particular, it doesn't depend on Nakayama's lemma or on facts about semicontinuous functions. For the (more complex) standard proof of this fact, see for instance [129], where the claim is dubbed an "important hard exercise" (Exercise 13.7.K).

5.4. The upper semicontinuous dimension function. Recall that the dimension of a topological space X at a point $x \in X$ is defined as the infimum

$$\dim_x X := \inf\{\dim U \mid U \text{ open neighborhood of } x\}.$$

The map $X \to \mathbb{N} \cup \{+\infty\}$, $x \mapsto \dim_x X$ is upper semicontinuous and thus corresponds to an internal generalized (possibly unbounded) natural number. The following proposition shows that this number has an explicit description.

Proposition 5.10. *Let X be a scheme. Then the upper semicontinuous function associated to the internal number "Krull dimension of $\mathcal{O}_X$" is the dimension function $x \mapsto \dim_x X$.*

Proof. Internally, we define the Krull dimension of $\mathcal{O}_X$ as the infimum over all natural numbers n such that $\mathcal{O}_X$ is of Krull dimension $\leq n$. This infimum need not exist in the natural numbers, of course; so we really mean the upward-closed set $\mathcal{A}$ of all those numbers. (It is inhabited if and only if, from the external perspective, the dimension of X is locally finite. In this case, it defines a generalized natural number.)

We thus have to show for any point $x \in X$:

$$\inf\{n \in \mathbb{N} \cup \{+\infty\} \mid n \in \mathcal{A}_x\} = \dim_x X.$$

The condition on n can be expressed as follows, where we write "$\underline{n}$" to denote the constant function with value n:

$$\begin{aligned} & n \in \mathcal{A}_x \\ \Longleftrightarrow\ & \text{for some open neighborhood } U \text{ of } x,\ \underline{n} \in \Gamma(U, \mathcal{A}) \\ \Longleftrightarrow\ & \text{for some open neighborhood } U \text{ of } x, \\ & \qquad U \models \ulcorner \mathcal{O}_X \text{ is of Krull dimension } \leq n \urcorner \\ \Longleftrightarrow\ & \text{for some open neighborhood } U \text{ of } x, \\ & \qquad \dim U \leq n \end{aligned}$$

The second equivalence follows from the external description of internally-defined subsheaves given in Section 2.2. We thus have:

$$\inf\{n \mid n \in \mathcal{A}_x\} = \inf\{\dim U \mid U \text{ open neighborhood of } x\} = \dim_x X. \qquad \square$$

6. Modalities

Philosophers and logicians do not only study what is *true*, but also what is *known*, what is *believed*, what is *possible*, and so on. Such *modalities* are absent from the usual mathematical practice. However, it turns out that a specific kind of such modalities plays a role in understanding when properties spread from points to neighborhoods.

Briefly, this is because for any point x of a topological space X, there exists a modal operator $\Box$ such that for any formula φ of the internal language of the sheaf topos $\mathrm{Sh}(X)$, the internal statement $\Box\varphi$ means that φ holds on some open neighborhood of the given point x. In this way, we can reduce sheaf-theoretic questions to questions of modal intuitionistic (non-sheafy) mathematics.

The techniques developed in this section also enable us to use the internal language of $\mathrm{Sh}(X)$ to talk about sheaves on *subspaces* of X (and more general *sublocales* of X).

Topological interpretations of modal logic were studied before, for instance by Awodey and Kishida [11]. However, they study a different kind of modal operators, not corresponding to the Lawvere–Tierney topologies of topos theory, and pursue different goals.

6.1. Basics on truth values and modal operators.

Definition 6.1. The *set of truth values* Ω is the powerset of the singleton set $1 := \{\star\}$, where $\star$ is a formal symbol.

In classical logic, any subset of $\{\star\}$ is either empty or inhabited, so that Ω contains exactly two elements, the empty set ("false") and $\{\star\}$ ("true"). But in intuitionistic logic, this cannot be shown; indeed, if we interpret the definition in the topos of sheaves on a space X, we obtain a (rather large) sheaf Ω with

$$U \subseteq X \text{ open} \quad \longmapsto \quad \Gamma(U, \Omega) = \{V \subseteq U \mid V \text{ open}\}.$$

(This is because by definition of Ω as the power object of the terminal sheaf 1, sections of Ω on an open subset U correspond to subsheaves $\mathcal{F} \hookrightarrow 1|_U$, and those are given by the greatest open subset $V \subseteq U$ such that $\Gamma(V, \mathcal{F})$ is inhabited.) Obviously, in general, this sheaf has many sections, in particular more than the binary coproduct $1 \amalg 1$ (unless any open subset of X is also closed).

The *truth value* of a formula φ is by definition the subset $\{x : 1 \mid \varphi\} \in \Omega$, where "$x$" is a fresh variable not appearing in φ. This subset is inhabited if and only if φ holds and is empty if and only if $\neg\varphi$ holds. Conversely, we can associate to a subset $F \subseteq 1$ the proposition $\ulcorner F$ is inhabited$\urcorner$.

By the above description of Ω in a sheaf topos $\mathrm{Sh}(X)$, the interpretation of the truth value of a formula φ in the internal language of $\mathrm{Sh}(X)$ is a certain open subset of X. Tracing the definitions, we see that this open subset is precisely the largest open subset on which φ holds, i. e. the union of all open subsets $U \subseteq X$ such that $U \models \varphi$.

Under the correspondence of formulas with truth values, logical operations like $\wedge$ and $\vee$ map to set-theoretic operations like $\cap$ and $\cup$ – for instance, we have

$$\{x : 1 \mid \varphi\} \cap \{x : 1 \mid \psi\} = \{x : 1 \mid \varphi \wedge \psi\}.$$

This justifies a certain abuse of notation: We will sometimes treat elements of Ω as propositions and use logical instead of set-theoretic connectives. In particular, if φ and ψ are elements of Ω, we will write "$\varphi \Rightarrow \psi$" to mean $\varphi \subseteq \psi$; "$\perp$" to mean $\emptyset$; and "$\top$" to mean 1.

Definition 6.2. A *modal operator* (or *Lawvere–Tierney topology*) is a map $\Box : \Omega \to \Omega$ such that for all $\varphi, \psi \in \Omega$,

(1) $\varphi \Longrightarrow \Box\varphi$,
(2) $\Box\Box\varphi \Longrightarrow \Box\varphi$,
(3) $\Box(\varphi \wedge \psi) \Longleftrightarrow \Box\varphi \wedge \Box\psi$.

Syntactically, the symbol "$\Box$" binds stronger than any other logical connective. For instance, axiom (2) is shorthand for "$(\Box(\Box(\varphi))) \Rightarrow (\Box(\varphi))$" and axiom (3) is shorthand for "$(\Box(\varphi \wedge \psi)) \Leftrightarrow ((\Box(\varphi)) \wedge (\Box(\psi)))$".

The intuition is that $\Box\varphi$ is a certain weakening of φ, where the precise meaning of "weaker" depends on the modal operator. By the second axiom, weakening twice is the same as weakening once.

In classical logic, where $\Omega = \{\perp, \top\}$, there are only two modal operators: the identity map and the constant map with value $\top$. Both of these are not very interesting: The identity operator does not weaken propositions at all, while the constant operator weakens every proposition to the trivial statement $\top$.

In intuitionistic logic, there can potentially exist further modal operators. For applications to algebraic geometry, the following four operators will have a clear geometric meaning and be of particular importance:

(1) $\Box\varphi :\equiv (\alpha \Rightarrow \varphi)$, where α is a fixed proposition.
(2) $\Box\varphi :\equiv (\varphi \vee \alpha)$, where α is a fixed proposition.
(3) $\Box\varphi :\equiv \neg\neg\varphi$ (the *double negation modality*).
(4) $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition.

Lemma 6.3. *Any modal operator $\Box$ is monotonic, i. e. if $\varphi \Rightarrow \psi$, then $\Box\varphi \Rightarrow \Box\psi$. Furthermore, a modus ponens rule holds: If $\Box\varphi$ holds, and if φ implies $\Box\psi$, then $\Box\psi$ holds as well.*

Proof. Assume $\varphi \Rightarrow \psi$. This is equivalent to supposing $\varphi \wedge \psi \Leftrightarrow \varphi$. We are to show that $\Box\varphi \Rightarrow \Box\psi$, i. e. that $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box\varphi$. This follows since by the third axiom on a modal operator, we have $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box(\varphi \wedge \psi)$, and $\Box$ respects equivalence of propositions.

For the second statement, consider that if $\varphi \Rightarrow \Box\psi$, by monotonicity and the second axiom on a modal operator it follows that $\Box\varphi \Rightarrow \Box\Box\psi \Rightarrow \Box\psi$. □

The modus ponens rule justifies the following proof scheme: When trying to show, given that some boxed statement $\Box\varphi$ holds, that some further boxed statement $\Box\psi$ holds, we may give a proof of $\Box\psi$ under the stronger assumption φ. Symbolically:

$$(\Box\varphi \Rightarrow \Box\psi) \iff (\varphi \Rightarrow \Box\psi).$$

Remark 6.4. There is some contention on what symbol one should use for modal operators in the sense of Definition 6.2. This is because, in the modal logic community, the symbol "$\Box$" usually refers to the modal operator "it is necessary that". For this modal operator, one often imposes the *reflexivity axiom* $\Box\varphi \Rightarrow \varphi$ which we don't impose (and which would trivialize the theory). Conversely, our axiom $\varphi \Rightarrow \Box\varphi$ isn't meaningful in the necessity interpretation. This axiom is meaningful for the modal operator "it is possible that", commonly denoted "$\Diamond$"; but for this modal operator, the axiom $\Diamond(\varphi \wedge \psi) \Leftrightarrow \Diamond\varphi \wedge \Diamond\psi$ isn't meaningful.

A classical modal operator which matches our axioms is "I believe that" under the proviso of ultimate knowledge ("I believe every true statement").

Goldblatt chooses the symbol "∇" for the modal operators in our sense [55, Section 14.5], [54]. The symbol "$\bigcirc$" is also common, particularly in the hardware verification community. A discussion of the relationship between these three kinds of operators is contained in [102]. We are grateful to Tadeusz Litak for valuable comments and references pertaining to this topic.

6.2. Geometric meaning. Let X be a topological space. As discussed above, an open subset $U \subseteq X$ defines an internal truth value (a global section of the sheaf Ω). We also denote it by "U", such that

$$V \models U \quad \iff \quad V \subseteq U$$

for any open subset $V \subseteq X$. (Shortcutting the various intermediate steps, this can also be taken as a definition of "$V \models U$".) If $A \subseteq X$ is a closed subset, there is thus an internal truth value A^c corresponding to the open subset $A^c = X \setminus A$. If $x \in X$ is a point, we define "$!x$" to denote the truth value corresponding to $\operatorname{int}(X \setminus \{x\})$, such that

$$V \models\, !x \quad \iff \quad V \subseteq \operatorname{int}(X \setminus \{x\}) \quad \iff \quad x \notin V.$$

Proposition 6.5. *Let $U \subseteq X$ be a fixed open and $A \subseteq X$ be a fixed closed subset. Let $x \in X$. Then, for any open subset $V \subseteq X$, it holds that:*

$$\begin{array}{lll} V \models (U \Rightarrow \varphi) & \iff & V \cap U \models \varphi. \\ V \models (\varphi \vee A^c) & \iff & \text{there is an open subset } W \subseteq V \\ & & \qquad \text{containing } A \cap V \text{ such that } W \models \varphi. \\ V \models \neg\neg\varphi & \iff & \text{there is a dense open subset } W \subseteq V \text{ s.th. } W \models \varphi. \\ V \models ((\varphi \Rightarrow\, !x) \Rightarrow\, !x) & \iff & x \notin V \text{ or there is an open neighborhood } W \subseteq V \\ & & \qquad \text{of } x \text{ such that } W \models \varphi. \end{array}$$

Proof. (1) Omitted.

(2) Let $V \models \varphi \vee A^c$. Then there exists an open covering $V = \bigcup_i V_i$ such that for each i, $V_i \models \varphi$ or $V_i \subseteq A^c$. Let $W \subseteq V$ be the union of those V_i such that $V_i \models \varphi$. Then $W \models \varphi$ by the locality of the internal language and $A \cap V \subseteq W$.

Conversely, let $W \subseteq V$ be an open subset containing $A \cap V$ such that $W \models \varphi$. Then $V = W \cup (V \cap A^c)$ is an open covering attesting $V \models \varphi \vee A^c$.

Modal operator	associated nucleus	$j(V) = X$ iff ...	subspace
$\Box\varphi \equiv (U \Rightarrow \varphi)$	$j(V) = \operatorname{int}(U^c \cup V)$	$U \subseteq V$	U
$\Box\varphi \equiv (\varphi \vee A^c)$	$j(V) = V \cup A^c$	$A \subseteq V$	A
$\Box\varphi \equiv \neg\neg\varphi$	$j(V) = \operatorname{int}(\operatorname{cl}(V))$	V is dense in X	smallest dense sublo- cale of X
$\Box\varphi \equiv ((\varphi \Rightarrow\, !x) \Rightarrow\, !x)$	$j(V) = X \setminus \operatorname{cl}\{x\}$, if $x \notin V$ $j(V) = X$, if $x \in V$	$x \in V$	$\{x\}$

TABLE 2. List of important modal operators and their associated nuclei (notation as in Proposition 6.5).

(3) For the "only if" direction, let $W \subseteq V$ be the largest open subset on which φ holds, i. e. the union of all open subsets of V on which φ holds. For the "if" direction, we may assume that the given set W is also the largest open subset on which φ holds (by enlarging W if necessary). The claim then follows by the following chain of equivalences:

$$\begin{aligned}
&V \models \neg\neg\varphi \\
\Longleftrightarrow\ & \forall Y \subseteq V \text{ open. } \Big(\forall Z \subseteq Y \text{ open. } (Z \models \varphi) \Rightarrow Z = \emptyset\Big) \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ & \forall Y \subseteq V \text{ open. } \Big(\forall Z \subseteq Y \text{ open. } Z \subseteq W \Rightarrow Z = \emptyset\Big) \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ & \forall Y \subseteq V \text{ open. } Y \cap W = \emptyset \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ & W \text{ is dense in } V.
\end{aligned}$$

(4) Straightforward, since the interpretation of the internal statement with the Kripke–Joyal semantics is

$$\forall Y \subseteq V \text{ open. } \Big(\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow x \notin Z\Big) \Longrightarrow x \notin Y. \qquad \Box$$

6.3. The subspace associated to a modal operator. Any modal operator $\Box : \Omega \to \Omega$ in the sheaf topos of X induces on global sections a map

$$j : \mathcal{T}(X) \to \mathcal{T}(X),$$

where $\mathcal{T}(X) = \Gamma(X, \Omega)$ is the set of open subsets of X. Explicitly, it is given by

$$\begin{aligned}
j(U) &= \text{largest open subset of } X \text{ on which } \Box U \text{ holds} \\
&= \bigcup \{V \subseteq X \mid V \text{ open, } V \models \Box U\}.
\end{aligned}$$

By the axioms for a modal operator, the map j fulfills similar such axioms: For any open subsets $U, V \subseteq X$,

(1) $U \subseteq j(U)$,
(2) $j(j(U)) \subseteq j(U)$,
(3) $j(U \cap V) = j(U) \cap j(V)$.

Such a map is called a *nucleus* on $\mathcal{T}(X)$. Table 2 lists the nuclei associated to the four modal operators of Proposition 6.5.

Any nucleus j defines a subspace X_j of X, to be described below, with a small caveat: In general, the subspace X_j cannot be realized as a topological subspace, but only as a so-called *sublocale*; the notion of a locale is a slight generalization of the notion of a topological space, in which an underlying set of points is not part of the definition. Instead, a locale is simply given by a frame (a partially ordered set with certain properties) of arbitrary *opens* satisfying some axioms – these opens may, but do not necessarily have to, be sets of points. Sheaf theory carries over to locales essentially unchanged, since the notions of presheaves and sheaves only refer to open sets and coverings, but not points.

Accessible introductions to the theory of locales include two notes by Johnstone [68, 69]. Locales are also well-known for a curious application in the theory of randomness [116, 115].

Definition 6.6. Let j be a nucleus on $\mathcal{T}(X)$. Then the sublocale X_j of X is given by the frame of opens $\mathcal{T}(X_j) := \{U \in \mathcal{T}(X) \mid j(U) = U\}$.

If j is induced by a modal operator $\Box$, we also write "$X_\Box$" for X_j. In three of the four cases listed in Table 2, the sublocale $X_\Box$ can indeed be realized as a topological subspace. The only exception is the sublocale $X_{\neg\neg}$ associated to the double negation modality. It can also be described as the *smallest dense sublocale* of X; this is obviously a genuine locale-theoretic notion, since there is (in general) no smallest dense topological subspace (consider $\mathbb{R}$ and its dense subsets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$).

The inclusion $i : X_j \hookrightarrow X$ cannot in general be described on the level of points, since X_j might not be realizable as a topological subspace. But for sheaf-theoretic purposes, it suffices to describe i on the level of opens. This is done as follows:

$$i^{-1} : \mathcal{T}(X) \longrightarrow \mathcal{T}(X_j), \quad U \longmapsto j(U).$$

Thus we can relate the toposes of sheaves on X_j and X by the usual pullback and pushforward functors.

$$\begin{aligned} i^{-1}\mathcal{F} &= \text{sheafification of } (U \mapsto \operatorname*{colim}_{U \preceq i^{-1}V} \Gamma(V, \mathcal{F})) \\ i_*\mathcal{G} &= (U \mapsto \Gamma(i^{-1}U, \mathcal{G})) = (U \mapsto \Gamma(j(U), \mathcal{G})) \end{aligned}$$

As familiar from honest topological subspace inclusions, the pushforward functor $i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X)$ is fully faithful and the composition $i^{-1} \circ i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X_j)$ is (canonically isomorphic to) the identity.

6.4. Internal sheaves and sheafification. It turns out that the image of the pushforward functor $i_* : \mathrm{Sh}(X_\Box) \to \mathrm{Sh}(X)$, where $\Box$ is a modal operator in $\mathrm{Sh}(X)$, can be explicitly described. Namely, it consists exactly of those sheaves which from the internal point of view are so-called $\Box$-*sheaves*, a notion explained below.

Furthermore, if we identify $\mathrm{Sh}(X_\Box)$ with its image in $\mathrm{Sh}(X)$, the pullback functor is given by an internal sheafification process with respect to the modality $\Box$. Thus the external situation of pushforward/pullback translates to forget/sheafify. This broadens the scope of the internal language of $\mathrm{Sh}(X)$: It cannot only be used to talk about sheaves on X in a simple, element-based language, but also to talk about sheaves on arbitrary subspaces of X.

To describe the notion of $\Box$-sheaves and related ones, we switch to the internal perspective and thus forget that we're working over a base space X; we are simply given a modal operator $\Box : \Omega \to \Omega$ and have to take care that our proofs are intuitionistically valid. A reference for the material in this subsection is a preprint by de Vries [135].[14]

[14] On page 5 of that preprint there is a slight typing error: Fact 2.1(i) gives the characterization of j-closedness, not j-denseness. The correct characterization of j-denseness in that context is $\forall b \in B.\ j(b \in A)$.

Recall that a set S is a *subsingleton* if and only if $\forall x, y : S.\ x = y$, and that a set S is a *singleton* if and only if it is a subsingleton and it is inhabited (i. e. $\exists x : S.\ \top$); this amounts to $\exists! x : S.\ \top$.

Definition 6.7. A set F is $\Box$*-separated* if and only if

$$\forall x, y : F.\ \Box(x = y) \Longrightarrow x = y.$$

A set F is a $\Box$*-sheaf* if and only if it is $\Box$-separated and

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton}\urcorner) \Longrightarrow \exists x : F.\ \Box(x \in S).$$

The two conditions can be combined: A set F is a $\Box$-sheaf if and only if

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton}\urcorner) \Longrightarrow \exists! x : F.\ \Box(x \in S).$$

Intuitively, reading "$\Box\varphi$" as "locally φ", a set is $\Box$-separated if elements which are locally equal are in fact equal. A set is a $\Box$-sheaf if furthermore for any set S of elements which locally contains just a single element there is an element which is locally contained in S.

This phrasing is reminiscent of the usual gluing condition, which demands that any family of sections which locally is just a single section (in that the sections of the family agree on their common domain of definition) gives rise to a global section which coincides with the given sections on their respective domain. Remark 6.9 below sketches how to make this relation precise.

Definition 6.8. The *plus construction* of a set F with respect to $\Box$ is the set

$$F^+ := \{S \subseteq F \mid \Box(\ulcorner S \text{ is a singleton}\urcorner)\}/\sim,$$

where the equivalence relation is defined by $S \sim T :\Leftrightarrow \Box(S = T)$. There is a canonical map $F \to F^+$ given by $x \mapsto [\{x\}]$. The $\Box$*-sheafification* of a set F is the set F^{++}.

If F is $\Box$-separated, then for any subset $S \subseteq F$ it holds that

$$\Box(\ulcorner S \text{ is a singleton}\urcorner) \quad\Longleftrightarrow\quad \ulcorner S \text{ is a subsingleton}\urcorner \wedge \Box(\ulcorner S \text{ is inhabited}\urcorner).$$

Remark 6.9. The topos of *pre*sheaves on a topological space X admits an internal language as well [86, Section VI.7, discussion after Theorem 1]. In it, there exists a modal operator $\Box$ reflecting the topology of X. A presheaf on X is separated in the usual sense if, from the internal perspective of $\mathrm{PSh}(X)$, it is $\Box$-separated; and it is a sheaf if, from the internal perspective, it is a $\Box$-sheaf. Furthermore, the $\Box$-sheafification of a presheaf (considered as a set from the internal perspective) coincides with the usual sheafification.

Example 6.10. Any singleton set is a $\Box$-sheaf. The empty set is always $\Box$-separated (trivially) and is a $\Box$-sheaf if and only if $\Box\bot \Rightarrow \bot$.

We will see geometric examples of $\Box$-sheaves in further sections. For instance, on an integral or locally Noetherian scheme X, the structure sheaf $\mathcal{O}_X$ is $\neg\neg$-separated and its $\neg\neg$-sheafification is the sheaf $\mathcal{K}_X$ of rational functions (Proposition 9.9).

Lemma 6.11. *For any set F, it holds that:*

(1) *F^+ is $\Box$-separated.*
(2) *The canonical map $F \to F^+$ is injective if and only if F is $\Box$-separated.*
(3) *If F is $\Box$-separated, then F^+ is a $\Box$-sheaf.*
(4) *If F is a $\Box$-sheaf, then the canonical map $F \to F^+$ is bijective.*

Let $\mathrm{Sh}_\Box(\mathrm{Set})$ be the full subcategory of Set *consisting of the $\Box$-sheaves. Then it holds that:*

(5) *The functor $(\_\_)^+ : \mathrm{Set} \to \mathrm{Set}$ is left exact.*

(6) *The functor* $(\_\_)^{++} : \mathrm{Set} \to \mathrm{Sh}_\Box(\mathrm{Set})$ *is left exact and left adjoint to the forgetful functor* $\mathrm{Sh}_\Box(\mathrm{Set}) \to \mathrm{Set}$, $F \mapsto F$.

Proof. These are all straightforward, and in fact simpler than their classical counterparts, since there are no colimit formulas which would have to be dealt with. □

Remark 6.12. As is to be expected from the familiar inclusion of sheaves in presheaves on topological spaces, the forgetful functor $\mathrm{Sh}_\Box(\mathrm{Set}) \to \mathrm{Set}$ does not in general preserve colimits. It is instructive to see why epimorphisms in $\mathrm{Sh}_\Box(\mathrm{Set})$ need not be epimorphisms in Set: A map $f : A \to B$ between $\Box$-sheaves is an epimorphism in $\mathrm{Sh}_\Box(\mathrm{Set})$ if and only if

$$\forall y : B.\ \Box(\exists x : X.\ f(x) = y),$$

that is preimages do not need to exist, it suffices for them to "$\Box$-exist". (Using results about the $\Box$-translation, to be introduced below, this characterization will be obvious.) This condition is intuitionistically weaker than the condition that f is an epimorphism in Set, i. e. that f is surjective. This should be compared to the failure of the forgetful functor $\mathrm{Sh}(X) \to \mathrm{PSh}(X)$ to preserve epimorphisms: A morphism of sheaves does not need to have preimages for any local section in order to be an epimorphism. Instead, it suffices for any local section to *locally* have preimages.

Proposition 6.13. *Let X be a topological space. Let $\Box$ be a modal operator in* $\mathrm{Sh}(X)$. *Let $i : X_\Box \hookrightarrow X$ be the inclusion of the associated sublocale. Corestricting the pushforward functor $i_* : \mathrm{Sh}(X_\Box) \to \mathrm{Sh}(X)$ to its essential image, it induces an equivalence $\mathrm{Sh}(X_\Box) \simeq \mathrm{Sh}_\Box(\mathrm{Sh}(X))$ between the category of sheaves on $X_\Box$ and the category of $\Box$-sheaves in* $\mathrm{Sh}(X)$.

Proof. For the further development of the theory, we need the statement of this proposition, but not the proof, which really is routine in dealing with subtoposes and modal operators. Nevertheless, a proof can proceed by combining Example A4.6.2(a) and Theorem C1.4.7 of [67], observing that for a topos of sheaves on a locale Y it holds that $\mathcal{T}(Y) = \Gamma(Y, \Omega_{\mathrm{Sh}(Y)})$, and that the subobject classifier of $\mathrm{Sh}_\Box(\mathrm{Sh}(X))$ is $\{\varphi : \Omega_{\mathrm{Sh}(X)} \mid \Box\varphi \Leftrightarrow \varphi\}$. □

Remark 6.14. It's possible to rewrite the sheaf condition in the following form. A set F is $\Box$-separated if and only if, for any truth value $\varphi : \Omega$ such that $\Box\varphi$, the canonical map

$$F \longrightarrow F^\varphi,$$

which maps an element $x : F$ to the constant map $\varphi \to X$ with value x (where φ is considered as a subset of the terminal set 1), is injective. The set F is a $\Box$-sheaf if and only if furthermore this map is surjective for all such truth values.

6.5. Sheaves for the double negation modality. Recall that if $\Box$ is the modal operator associated to a sub*space* Y of a topological space X, then the sheaves on X which are $\Box$-sheaves are easy to describe: These are precisely the sheaves in the essential image of the pushforward functor $\mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. For the double negation modality, the same is true, only that Y is then the perhaps unfamiliar *smallest dense sublocale* of X.

The following proposition gives a characterization of $\neg\neg$-separated presheaves and $\neg\neg$-sheaves in explicit terms.

Proposition 6.15. *Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X. Then:*

(1) *$\mathcal{F}$ is $\neg\neg$-separated if and only if any two local sections of $\mathcal{F}$, which are defined on a common domain and which agree on a dense open subset of their domain, are already equal.*

(2) *$\mathcal{F}$ is a $\neg\neg$-sheaf if and only if it is $\neg\neg$-separated and for any open $U \subseteq X$ and any open $V \subseteq U$ dense in U, any V-section of $\mathcal{F}$ extends to an U-section of $\mathcal{F}$.*

(3) *If $\mathcal{F}$ is $\neg\neg$-separated, the sections of $\mathcal{F}^+$ on an open subset $U \subseteq X$ can be described by pairs $\langle V, s\rangle$, where V is a dense open subset of U and s is a section of $\mathcal{F}$ on V. Two such pairs $\langle V, s\rangle, \langle V', s'\rangle$ determine the same element in $\Gamma(U, \mathcal{F}^+)$ if and only if s and s' agree on $V \cap V'$.*

Proof. The first statement is obvious from the definition of $\neg\neg$-separatedness (Definition 6.7 for $\Box = \neg\neg$) and the geometric interpretation of double negation (Proposition 6.5).

For the second statement, we need to show that, assuming that $\mathcal{F}$ is $\neg\neg$-separated, the sheaf $\mathcal{F}$ has the extension property if and only if

$$\mathrm{Sh}(X) \models \forall \mathcal{S} : \mathcal{P}(\mathcal{F}).\ \ulcorner \mathcal{S} \text{ is a subsingleton} \urcorner \wedge \neg\neg(\ulcorner \mathcal{S} \text{ is inhabited} \urcorner) \Longrightarrow \exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S}).$$

A section $\mathcal{S} \in \Gamma(U, \mathcal{P}(\mathcal{F}))$ which internally is a subsingleton and *not not* inhabited is precisely a subsheaf $\mathcal{S} \hookrightarrow \mathcal{F}|_U$ such that all stalks $\mathcal{S}_x$, $x \in U$ are subsingletons and such that for some dense open subset $V \subseteq U$, the stalks $\mathcal{S}_x$, $x \in V$ are inhabited. This is precisely the datum of a section of $\mathcal{F}$ defined on some dense open subset of U, considering the gluing of the unique germs in $\mathcal{S}_x$ for those points x such that $\mathcal{S}_x$ is inhabited. (Conversely, a section $s \in \Gamma(V, \mathcal{F})$ defines a subsheaf $\mathcal{S}$ by setting $\Gamma(W, \mathcal{S}) := \{s|_W \mid W \subseteq V\}$.)

In view of this explicit description and the observation that the asserted existence ("$\exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S})$") is actually a question of unique existence, the second statement follows.

For the third statement, one can check that the presheaf on X defined by

$$U \subseteq X \text{ open} \quad \longmapsto \quad \{\langle V, s\rangle \mid V \subseteq U \text{ dense open},\ s \in \Gamma(V, \mathcal{F})\}/\!\sim$$

is in fact a sheaf (with respect to the topology of X), internally a $\neg\neg$-sheaf, and that it has the universal property of the $\neg\neg$-sheafification of $\mathcal{F}$. $\Box$

The conditions (1) and (2) of Proposition 6.15 can be summarized as follows: A sheaf $\mathcal{F}$ on a topological space is a $\neg\neg$-sheaf if and only if, for any open subset $U \subseteq X$, the restriction map $\Gamma(\operatorname{int}\operatorname{cl} U, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is bijective [65, Lemma 36].

In the case that X contains a *generic point*, that is a point $\xi \in X$ such that $\operatorname{cl}\{\xi\} = X$, we can describe the sublocale $X_{\neg\neg}$ in simple terms: In this case, it coincides with the subspace $\{\xi\}$. For instance, such a generic point exists and is unique if X is an irreducible scheme.

Lemma 6.16. *Let X be a topological space and $\xi \in X$ be a point such that $\operatorname{cl}\{\xi\} = X$. Then the modal operator $\Box :\equiv ((\_\_ \Rightarrow\, !\xi) \Rightarrow\, !\xi)$ coincides with the double negation modality and $X_{\neg\neg} = \{\xi\}$ as sublocales of X.*

Proof. The semantics of the formula $!\xi$ was defined by the equivalence

$$U \models\, !\xi \quad \Longleftrightarrow \quad \xi \notin U.$$

By the assumption on ξ, this is equivalent to requiring $U = \emptyset$. Thus for any open subset U the formulas $!\xi$ and $\bot$ have the same meaning; they are therefore logically equivalent from the internal point of view. The given modal operator thus simplifies:

$$\Box\varphi \quad \equiv \quad ((\varphi \Rightarrow\, !\xi) \Rightarrow\, !\xi) \quad \Leftrightarrow \quad ((\varphi \Rightarrow \bot) \Rightarrow \bot) \quad \Leftrightarrow \quad \neg\neg\varphi.$$

The second claim follows from Table 2. $\Box$

Corollary 6.17. *Let X be a topological space and let $\xi \in X$ be a point such that $\mathrm{cl}\{\xi\} = X$. Since $X_{\neg\neg} = \{\xi\}$, the category of $\neg\neg$-sheaves in $\mathrm{Sh}(X)$ coincides with the category of sheaves on $\{\xi\}$ and can therefore be identified with the category of sets. Under this identification,*

(1) *sheafifying an object $\mathcal{F} \in \mathrm{Sh}(X)$ with respect to the double negation modality (i. e. pulling back to $X_{\neg\neg}$) is the same as calculating its generic stalk $\mathcal{F}_\xi$ and*
(2) *pushing forward a set M along $X_{\neg\neg} \hookrightarrow X$ is the same as calculating the constant sheaf associated to M.*

Proof. The first statement follows because pulling back to $X_{\neg\neg}$ is the same as pulling back to $\{\xi\}$. The pushforward of a set M, considered as a sheaf on $X_{\neg\neg}$, to X is explicitly given by

$$U \quad \longmapsto \quad \begin{cases} M, & \text{if } U \neq \emptyset, \\ \{\star\}, & \text{else.} \end{cases}$$

We omit the routine verification that this sheaf coincides with the constant sheaf $\underline{M}$ associated to M. □

The following technical lemma will occasionally be handy. It is an internal reflection of the fact that an open subset of an affine scheme can always be written as the union of standard open subsets. We will generalize it to schemes which are not necessarily integral in Section 9 (see Lemma 9.18).

Lemma 6.18. *Let X be an integral scheme. Let φ be any formula over X. Then*

$$\mathrm{Sh}(X) \models \neg\neg\varphi \Longrightarrow \exists f : \mathcal{O}_X.\ \neg\neg(\ulcorner f \text{ inv.}\urcorner) \wedge (\ulcorner f \text{ inv.}\urcorner \Rightarrow \varphi).$$

Proof. We may assume that X is the spectrum of an integral domain A and that there is a dense open subset $U \subseteq X$ on which φ holds. The open set U may be covered by standard open subsets $D(f_i)$; since X is irreducible, at least one of these is itself dense. We may take this f_i as the sought f. □

We can now also follow up on a promise made in Section 2.4 and prove the following somewhat tangential lemma.

Lemma 6.19. *Let X be a topological space. The internal language of $\mathrm{Sh}(X)$ is Boolean if and only if for any open subset $U \subseteq X$ it holds that U is the only dense open subset of U.*

Proof. That the internal language of $\mathrm{Sh}(X)$ is Boolean amounts to

$$\mathrm{Sh}(X) \models \forall \varphi : \Omega.\ \neg\neg\varphi \Rightarrow \varphi.$$

This is equivalent to the external statement that for any open subset $U \subseteq X$ and for any open subset $V \subseteq U$ it holds that: If V is dense in U, then V is equal to U. □

6.6. The $\Box$-translation. In logic, there is certain well-known transformation $\varphi \mapsto \varphi^{\neg\neg}$ on formulas, the *double negation translation*, with the following curious property: A formula φ is derivable in classical logic if and only if its translation $\varphi^{\neg\neg}$ is derivable in intuitionistic logic. The translation $\varphi^{\neg\neg}$ is obtained from φ by putting "$\neg\neg$" before any subformula, i. e. before any "$\exists$" and "$\forall$", around any logical connective, and around any atomic statement ("$x = y$", "$x \in A$"). For instance, the double negation translation of "f is surjective" is

$$\neg\neg\forall y : Y.\ \neg\neg\exists x : X.\ \neg\neg f(x) = y.$$

We will describe a slight generalization of the double negation translation, the $\Box$-translation for any modal operator $\Box$. It will be pivotal for using the internal language of a space X to express internal statements about sheaves defined on subspaces of X. The $\Box$-translation has been studied in other contexts before [1, 48]. To the best of our knowledge, this application – expressing the internal language of subtoposes in the internal language of the ambient topos – is new.

Definition 6.20. The $\Box$-*translation* is recursively defined as follows.

$$\begin{aligned}
(f = g)^\Box &:\equiv \Box(f = g) \\
(x \in A)^\Box &:\equiv \Box(x \in A) \\
\top^\Box &:\equiv \Box\top \quad (\Leftrightarrow \top) \\
\bot^\Box &:\equiv \Box\bot \\
(\varphi \wedge \psi)^\Box &:\equiv \Box(\varphi^\Box \wedge \psi^\Box) & (\textstyle\bigwedge_i \varphi_i)^\Box &:\equiv \Box(\textstyle\bigwedge_i \varphi_i^\Box) \\
(\varphi \vee \psi)^\Box &:\equiv \Box(\varphi^\Box \vee \psi^\Box) & (\textstyle\bigvee_i \varphi_i)^\Box &:\equiv \Box(\textstyle\bigvee_i \varphi_i^\Box) \\
(\varphi \Rightarrow \psi)^\Box &:\equiv \Box(\varphi^\Box \Rightarrow \psi^\Box) \\
(\forall x : X.\ \varphi)^\Box &:\equiv \Box(\forall x : X.\ \varphi^\Box) & (\forall X.\ \varphi)^\Box &:\equiv \Box(\forall X.\ \varphi^\Box) \\
(\exists x : X.\ \varphi)^\Box &:\equiv \Box(\exists x : X.\ \varphi^\Box) & (\exists X.\ \varphi)^\Box &:\equiv \Box(\exists X.\ \varphi^\Box)
\end{aligned}$$

Definition 6.21. A formula φ is $\Box$-*stable* if and only if $\Box\varphi$ implies φ.

Lemma 6.22.
(1) *Formulas in the image of the $\Box$-translation are $\Box$-stable, i. e. for any formula φ it holds that $\Box(\varphi^\Box) \Longrightarrow \varphi^\Box$.*
(2) *In the definition of the $\Box$-translation, one may omit the boxes printed in gray.*

Proof. The first statement is obvious, since one of the axioms for a modal operator demands that $\Box\Box\varphi \Rightarrow \Box\varphi$ for any formula φ. The second statement follows by an induction on the formula structure. By way of example, we prove the case for "$\Rightarrow$":

$$\begin{aligned}
&(\varphi \Rightarrow \psi)^{\Box\text{ with the gray parts}} \\
\Longleftrightarrow\ &\Box(\varphi^{\Box\text{ with the gray parts}} \Rightarrow \psi^{\Box\text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box\text{ with the gray parts}} \Rightarrow \psi^{\Box\text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box\text{ without the gray parts}} \Rightarrow \psi^{\Box\text{ without the gray parts}}) \\
\Longleftrightarrow\ &(\varphi \Rightarrow \psi)^{\Box\text{ without the gray parts}}
\end{aligned}$$

The first step is by definition; the second by $\Box$-stability of $\psi^{\Box\text{ with the gray parts}}$ and the intuitionistic tautology $\Box(\alpha \Rightarrow \beta) \Leftrightarrow (\alpha \Rightarrow \beta)$ for $\Box$-stable formulas β; the third by the induction hypothesis; and the fourth by definition. $\Box$

Lemma 6.23. *The $\Box$-translation is sound with respect to intuitionistic logic: Assume that there exists an intuitionistic proof of an implication $\varphi \Rightarrow \psi$. Then there is also an intuitionistic proof of the translated implication $\varphi^\Box \Rightarrow \psi^\Box$.*

Proof. This follows by an induction on the structure of intuitionistic proofs. We have to verify that we can mirror any inference rule of intuitionistic logic in the translation. For instance, one of the disjunction rules justifies the following proof scheme: In order to prove $\varphi \vee \psi \Rightarrow \chi$, it suffices to give proofs of $\varphi \Rightarrow \chi$ and $\psi \Rightarrow \chi$. We have to justify the translated proof scheme: In order to prove $(\varphi \vee \psi)^\Box \Rightarrow \chi^\Box$, it suffices to give proofs of $\varphi^\Box \Rightarrow \chi^\Box$ and $\psi^\Box \Rightarrow \chi^\Box$.

So assume that proofs of the two implications are given. Further assume $(\varphi \vee \psi)^\Box$, i. e. $\Box(\varphi^\Box \vee \psi^\Box)$. We want to show $\chi^\Box$. Since this is a $\Box$-stable statement, we may assume that in fact $\varphi^\Box \vee \psi^\Box$ holds. Then the claim is obvious by the two given proofs.

The cases for the other rules (see Appendix 24 for a list) are similar and left to the reader. $\square$

Remark 6.24. The reader well-versed in formal logic will have noticed that we are mixing syntax and semantics here. The proper way to state Lemma 6.23 would be to formally adjoin a box operator to the language of intuitionistic logic, governed by three inference rules which are modeled on the three axioms for a modal operator. This formal box operator could then be instantiated by any concrete modal operator $\square : \Omega \to \Omega$.

Soundness of the $\square$-translation is important for the following reason. If φ and φ' are equivalent formulas, we are accustomed to be able to freely substitute φ by φ' anywhere we want. Since a modal operator $\square$ is semantically defined as a map $\Omega \to \Omega$, it is trivially justified that $\square\varphi$ and $\square\varphi'$ are equivalent: The formulas φ and φ' give rise to the *same* element $\{x : 1 \mid \varphi\} = \{x : 1 \mid \varphi'\}$ of Ω, and therefore their images under $\square$ are equal as well.

However, it is *not* clear and in fact wrong in general that the translated formulas $\varphi^\square$ and $(\varphi')^\square$ are equivalent. This follows only if the soundness lemma can be applied (two times, once for each direction). We should stress that to apply this lemma, it is not enough to merely *know* that φ and φ' are equivalent; instead, there has to be an intuitionistic proof of this equivalence. This is really a stronger requirement, since an equivalence $\varphi \Leftrightarrow \varphi'$ might hold in a particular model, i. e. in the internal language of some particular topos, without possessing an intuitionistic proof, i. e. holding in any topos. We give an explicit example of this situation below (Example 6.39).

Lemma 6.25. *Let φ be a formula such that for any subformulas ψ appearing as antecedents of implications, it holds that $\psi^\square \Rightarrow \square\psi$. (In particular, this condition is satisfied if there are no "$\Rightarrow$" signs in φ or if φ is a geometric implication.) Then $\square\varphi \Rightarrow \varphi^\square$.*

Proof. We prove this by an induction on the formula structure. All cases except for "$\Rightarrow$" are obvious. For this case, assume $\square(\psi \Rightarrow \chi)$; we are to show that $(\psi^\square \Rightarrow \chi^\square)$. Since this is a $\square$-stable statement, we can in fact assume that $(\psi \Rightarrow \chi)$. We then have

$$\psi^\square \Longrightarrow \square\psi \Longrightarrow \square\chi \Longrightarrow \chi^\square,$$

with the first step being by the requirement on antecedents, the second by the monotonicity of $\square$, and the third by the induction hypothesis. $\square$

Lemma 6.26. *Let φ be a geometric formula. Then $\varphi^\square \Leftrightarrow \square\varphi$.*

Proof. The "$\Leftarrow$" direction is by Lemma 6.25. The "$\Rightarrow$" direction is an induction on the formula structure. By way of example, we verify the case of "$\bigvee$". So assume $\square(\bigvee_i \varphi_i^\square)$; we are to show that $\square(\bigvee_i \varphi_i)$. Since this is a boxed statement, we may in fact assume $\bigvee_i \varphi_i^\square$, so for some index j, it holds that $\varphi_j^\square$. By the induction hypothesis, it follows that $\square\varphi_j$. By $\varphi_j \Rightarrow \bigvee_i \varphi_i$ and the monotonicity of $\square$, it follows that $\square(\bigvee_i \varphi_i)$. $\square$

An analogous argument for infinite conjunctions is not valid: Assume $(\bigwedge_i \varphi_i)^\square$. So for all j, $\varphi_j^\square$ holds. By the induction hypothesis, $\square\varphi_j$ holds for any j. But from this we may not deduce $\square \bigwedge_i \varphi_i$, since the axioms for a modal operator only require commutativity with finite conjunctions. This failure also has a geometric interpretation, for instance in the special case $\square = \neg\neg$: Given dense open subsets U_i on which formulas φ_i hold, we may not conclude that there exists a single dense open subset U on which all the formulas φ_i hold.

Remark 6.27. In the special case that $\Box$ is the double negation modality, Lemma 6.26 holds with slightly weaker hypotheses: Namely, implications may occur in φ, provided that for their antecedents ψ it holds that $\psi \Rightarrow \psi^\Box$. This is because for the double negation modality, the formula $\Box(\psi \Rightarrow \chi)$ is equivalent to $\psi \Rightarrow \Box\chi$. (In general, for an arbitrary modality, only the former implies the latter, but not vice versa.) The case for "$\Rightarrow$" in the inductive proof then goes as follows: Assume $(\psi \Rightarrow \chi)^\Box$. Then $\psi \Rightarrow \psi^\Box \Rightarrow \chi^\Box \Rightarrow \Box\chi$, so $\Box(\psi \Rightarrow \chi)$.

Lemma 6.28. *Let φ, φ', ψ be formulas. Assume that:*

(1) *The formula φ' is geometric. (More generally, it suffices for $(\varphi')^\Box$ to imply $\Box\varphi'$.)*
(2) *There is an intuitionistic proof that φ and φ' are equivalent under the (only) hypothesis ψ.*
(3) *Both $\Box\psi$ and $\psi^\Box$ hold.*

Then $\varphi^\Box \Rightarrow \Box\varphi$.

Proof. Assume $\varphi^\Box$. Since $\psi^\Box$, $(\varphi \wedge \psi)^\Box$. Because the $\Box$-translation is sound with respect to intuitionistic logic (Lemma 6.23) it follows that $(\varphi')^\Box$. As φ' is geometric, it follows that $\Box\varphi'$. Since $\Box\psi$ holds, it follows that $\Box\varphi$. □

Example 6.29. Let M be an R-module. The statement that M is zero is not geometric: $\varphi :\equiv (\forall x : M.\ x = 0)$. But if M is generated by some finite family $x_1, \ldots, x_n : M$, then φ is equivalent to the statement $\varphi' :\equiv (x_1 = 0 \wedge \cdots \wedge x_n = 0)$ which is geometric; and there is an intuitionistic proof of this equivalence. Since no implication signs occur in $\psi :\equiv \ulcorner M$ is generated by $x_1, \ldots, x_n \urcorner$, Lemma 6.28 is applicable and shows that $\varphi^\Box$ implies $\Box\varphi$. This example will gain geometric meaning in Lemma 6.40.

Lemma 6.30. *For the modality $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition, the $\Box$-translation of the law of excluded middle holds. In particular, this applies to the double negation modality $\Box = \neg\neg$, where $\alpha = \bot$.*

Proof. We are to show that $(\varphi \vee \neg\varphi)^\Box$, i. e. that

$$((\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)) \Longrightarrow \alpha) \Longrightarrow \alpha.$$

So assume that the antecedent holds. If $\varphi^\Box$ holds, then in particular $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α hold. Therefore it follows that $(\varphi^\Box \Rightarrow \alpha)$. This implies $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α. □

6.7. Truth at stalks vs. truth on neighborhoods. We now state the crucial property of the $\Box$-translation. Recall that "$X_\Box$" denotes the sublocale of X induced by $\Box$ (Definition 6.6).

Theorem 6.31. *Let X be a topological space. Let $\Box$ be a modal operator in $\mathrm{Sh}(X)$. Let φ be a formula over X. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(X_\Box) \models \varphi,$$

where on the right hand side, all parameters occurring in φ were pulled back to $X_\Box$ along the inclusion $X_\Box \hookrightarrow X$.

We have not yet explicitly stated the Kripke–Joyal semantics for a sheaf topos over a locale, which $X_\Box$ is in general. The definition is exactly the same as in the case for sheaf toposes over a topological space, only that any mention of "open sets" has to be substituted by the more general "opens" and any mention of the union operator "$\bigcup$" has to be interpreted by the supremum operator in the frame of opens of the locale. For $X_\Box$, this is $\sup U_i = j(\bigcup_i U_i)$. Before giving a proof of Theorem 6.31, we want to discuss some of its consequences.

Corollary 6.32. *Let X be a topological space.*

(1) *Let $U \subseteq X$ be an open subset and let $\Box\varphi :\equiv (U \Rightarrow \varphi)$. Then*
$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(U) \models \varphi.$$

(2) *Let $A \subseteq X$ be a closed subset and let $\Box\varphi :\equiv (\varphi \vee A^c)$. Then*
$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(A) \models \varphi.$$

(3) *Let $\Box\varphi :\equiv \neg\neg\varphi$. Then*
$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(X_{\neg\neg}) \models \varphi.$$

(4) *Let $x \in X$ be a point and let $\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$. Then*
$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \varphi \textit{ holds at } x.$$

Proof. Combine Theorem 6.31 and Table 2. $\Box$

We want to discuss the last case of Corollary 6.32 in more detail. Let x be a point of a topological space X and let φ be a formula. Let $\Box$ be the modal operator given in the corollary. Then φ *holds at* x if and only if, from the internal perspective of $\mathrm{Sh}(X)$, the translated formula $\varphi^\Box$ holds; and φ *holds on some open neighborhood of* x if and only if, from the internal perspective, the formula $\Box\varphi$ holds.

Thus the question whether the truth of φ at the point x spreads to some open neighborhood can be formulated in the following way:

Does $\varphi^\Box$ imply $\Box\varphi$ in the internal language of $\mathrm{Sh}(X)$*?*

Phrased this way, technicalities like appropriately shrinking open neighborhoods are blinded out. A purposefully trivial example to illustrate this is the following. Let X be a scheme (or a ringed space). Let $f, g \in \Gamma(X, \mathcal{O}_X)$ be global functions. Suppose that the germs of f and g are zero in some stalk $\mathcal{O}_{X,x}$; we want to show that they are zero on a common open neighborhood of x.

Usual proof. Since the germ of f vanishes in $\mathcal{O}_{X,x}$, there is an open neighborhood U_1 of x such that $f|_{U_1} = 0$ in $\Gamma(U_1, \mathcal{O}_X)$. Since furthermore the germ of g vanishes in the same stalk, there exists an open neighborhood U_2 of x such that $g|_{U_2} = 0$. The intersection of both neighborhoods is still an open neighborhood of x; on this neighborhood both f and g vanish. $\Box$

Proof in the internal language. We may suppose that $(f = 0 \wedge g = 0)^\Box$, that is $\Box(f = 0) \wedge \Box(g = 0)$, and have to prove that $\Box(f = 0 \wedge g = 0)$. (To this end, we could simply invoke the third axiom on a modal operator, but we want to stay close to the given external proof.) So by assumption, both $\Box(f = 0)$ and $\Box(g = 0)$ hold. Since our goal is to prove a boxed statement, we may in fact assume that $f = 0$ and $g = 0$. Thus $f = 0 \wedge g = 0$. $\Box$

By using the internal language with its modal operators, we can thus reduce basic facts of scheme theory which deal with stalks and neighborhoods to facts of algebra in a *modal intuitionistic context*. As with using the internal language in its basic form without modalities, this brings conceptual clarity and reduced technical overhead. There are, however, two more distinctive advantages. Firstly, many internal proofs do not require specific properties of the modal operator and thus work with any modal operator. By interpreting such a proof using different operators, one obtains an entire family of external statements without any additional work (see Lemma 6.40 for an example).

Secondly, the following corollary gives a general metatheorem which is applicable to a wide range of cases. It allows to decide whether spreading will occur (or is likely not to occur) simply by looking at the *logical form* of the statement in question.

Corollary 6.33. *Let X be a topological space. Let φ be a formula. If φ is geometric, truth of φ at a point $x \in X$ implies truth of φ on some open neighborhood of x, and vice versa.*

Proof. By the purely logical lemmas of Section 6.6, it holds that $\varphi^\Box \Leftrightarrow \Box\varphi$. $\square$

Corollary 6.34. *Let X be a topological space. Let φ be a formula. If φ is geometric, the property "φ holds at a point $x \in X$" is open.*

Proof. This is just a reformulation of the previous corollary: If φ holds at a point $x \in X$, it holds on some open neighborhood U of x as well. Going back to stalks, it follows that φ holds at every point of U. $\square$

Example 6.35. Let X be a scheme (or a ringed space). Since the condition for a function $f : \mathcal{O}_X$ to be nilpotent is geometric (it is $\bigvee_{n \geq 0} f^n = 0$), nilpotency of f at a point is equivalent to nilpotency on some open neighborhood.

Combined with Lemma 6.28, this metatheorem is quite useful. We will illustrate it with several examples in the next subsection.

An important special case of spreading from stalks to neighborhoods is the case of spreading from the generic point (should it exist) to a dense open subset. Whether this occurs can be phrased by Lemma 6.16 as follows:

Does $\varphi^{\neg\neg}$ imply $\neg\neg\varphi$ in the internal language of $\mathrm{Sh}(X)$*?*

This question is a question of ordinary (non-modal) intuitionistic algebra.

Example 6.36. We have seen in Remark 6.12 that a morphism $f : A \to B$ in $\mathrm{Sh}(X_\Box) \simeq \mathrm{Sh}_\Box(\mathrm{Sh}(X))$ is an epimorphism if and only if

$$\mathrm{Sh}(X) \models \forall y : B.\ \Box(\exists x : X.\ f(x) = y).$$

We can now understand a simple proof of this fact:

$$\begin{aligned} & f \text{ is an epimorphism in } \mathrm{Sh}_\Box(\mathrm{Sh}(X)) \\ \Longleftrightarrow\ & \mathrm{Sh}_\Box(\mathrm{Sh}(X)) \models \ulcorner f \text{ is surjective} \urcorner \\ \Longleftrightarrow\ & \mathrm{Sh}(X) \models (\ulcorner f \text{ is surjective} \urcorner)^\Box \\ \Longleftrightarrow\ & \mathrm{Sh}(X) \models \forall y : B.\ \Box(\exists x : X.\ \Box(f(x) = y)) \\ \Longleftrightarrow\ & \mathrm{Sh}(X) \models \forall y : B.\ \Box(\exists x : X.\ f(x) = y). \end{aligned}$$

The ultimate equivalence is by Lemma 6.26, applied to the geometric subformula "$\exists x : X.\ f(x) = y$".

Remark 6.37. Theorem 6.31 can also be motivated by purely logical considerations. Namely, one can check that interpreting a formula φ by $\mathrm{Sh}(X) \models \varphi^\Box$ gives rise to a model of intuitionistic logic – if φ intuitionistically implies ψ, then $\mathrm{Sh}(X) \models \varphi^\Box$ implies $\mathrm{Sh}(X) \models \psi^\Box$. It is therefore a natural question whether there exists a topos $\mathcal{E}$ such that $\mathcal{E} \models \varphi$ if and only if $\mathrm{Sh}(X) \models \varphi^\Box$. Theorem 6.31 gives an affirmative answer to this question, explicitly stating that $\mathcal{E} := \mathrm{Sh}(X_\Box)$ is such a topos.

Proof of Theorem 6.31. A fancy proof goes as follows. First, one shows intuitionistically that for a modal operator $\Box$ in Set, it holds that

$$\mathrm{Set} \models \varphi^\Box \quad \Longleftrightarrow \quad \mathrm{Sh}_\Box(\mathrm{Set}) \models \varphi.$$

This can be done by an easy and nontechnical induction on the structure of formulas φ. Then one interprets this result in the sheaf topos $\mathrm{Sh}(X)$:

$$\begin{aligned}
& \mathrm{Sh}(X) \models \varphi^{\Box} & \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \ulcorner \mathrm{Set} \models \varphi^{\Box} \urcorner & \text{by idempotency} \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \ulcorner \mathrm{Sh}_{\Box}(\mathrm{Set}) \models \varphi \urcorner & \text{by the first step} \\
\Longleftrightarrow\ & \mathrm{Sh}_{\Box}(\mathrm{Sh}(X)) \models \varphi & \text{by idempotency} \\
\Longleftrightarrow\ & \mathrm{Sh}(X_{\Box}) \models \varphi & \text{since } \mathrm{Sh}_{\Box}(\mathrm{Sh}(X)) \simeq \mathrm{Sh}(X_{\Box})
\end{aligned}$$

By *idempotency*, we mean that internally employing the Kripke–Joyal semantics to interpret doubly-internal statements is the same as using the Kripke–Joyal semantics once. However, we do not want to discuss this here any further; some details can be found in the original article on the stack semantics [114, Lemma 7.20], but the statement given there is not general enough to justify the second use of idempotency above. For this, one would have to extend the stack semantics to support internal statements about locally internal categories like $\mathrm{Sh}(X_{\Box}) \hookrightarrow \mathrm{Sh}(X)$ (which then look like locally small categories from the internal point of view). This is worthwhile for other reasons too, but shall not be pursued here.

Therefore, we give a more explicit proof. By induction, we are going to prove that for any open subset $U \subseteq X$ and any formula φ over U, it holds that

$$U \models_X \varphi^{\Box} \quad \Longleftrightarrow \quad j(U) \models_{X_{\Box}} \varphi,$$

where the internal statements are to be interpreted by the Kripke–Joyal semantics of X and $X_{\Box}$ respectively and j is the nucleus associated to $\Box$. We may assume that any sheaves occurring in φ as domains of quantifications are in fact $\Box$-sheaves; we justify this with a separate lemma below.

The cases $\varphi \equiv \top$, $\varphi \equiv (\psi \wedge \chi)$, and $\varphi \equiv \bigwedge_i \psi_i$ are trivial. For $\varphi \equiv \bot$, the claim is that $U \models_X \Box\bot$ if and only if $j(U) \models_{X_{\Box}} \bot$. The former means $U \subseteq j(\emptyset)$ and the latter means $j(U) \leq \sup \emptyset = j(\emptyset)$, so the claim follows from the first two axioms for a nucleus.

We omit the verification of the remaining cases. $\Box$

Lemma 6.38. *Let $\Box$ be a modal operator. Let φ be a formula. Let $\psi :\equiv \varphi^{\Box}$ be the $\Box$-translation of φ. Let ψ' be the formula obtained from ψ by substituting any occurring domain of quantification by its $\Box$-sheafification, as syntactically defined in Definition 6.8. Then ψ and ψ' are intuitionistically equivalent.*

Proof. For any formula φ, we denote by "$\varphi^{\boxplus}$" the result of first applying the $\Box$-translation to φ and then substituting any set F occurring in φ as a domain of quantification by the plus construction F^+. Recall that for any such F there is a canonical map $F \to F^+,\ x \mapsto [\{x\}]$. We are going to show by induction that for any formula $\varphi(x_1, \ldots, x_n)$ in which elements $x_i : F_i$ may occur as terms, it holds that $\varphi^{\Box}(x_1, \ldots, x_n)$ is equivalent to $\varphi^{\boxplus}([\{x_1\}], \ldots, [\{x_n\}])$. This suffices to prove the lemma.

The cases for

$$\top \quad \bot \quad \wedge \quad \bigwedge \quad \vee \quad \bigvee \quad \Longrightarrow$$

are trivial. The cases for unbounded "$\forall$" and "$\exists$" are trivial as well. The case for "$=$" is slightly more interesting; let $\varphi(x, y) \equiv (x = y)$. Then we are to show that $\varphi^{\Box}(x, y) \equiv \Box(x = y)$ (equality in some set F) is equivalent to $\varphi^{\boxplus}([\{x\}], [\{y\}]) \equiv \Box([\{x\}] = [\{y\}])$ (equality in F^+). This follows by the definition of the plus construction. The case for "$\in$" is similar.

Let $\varphi \equiv (\exists x : F.\ \psi(x))$, where we have dropped further variables occurring in ψ for simplicity. Then we are to show that $\varphi^{\Box} \equiv \Box(\exists x : F.\ \psi^{\Box}(x))$ is equivalent to $\varphi^{\boxplus} \equiv \Box(\exists \bar{x} : F^+.\ \psi^{\boxplus}(\bar{x}))$. The "only if" direction is trivial (set $\bar{x} := [\{x\}]$). For the "if" direction, we may assume that there exists $\bar{x} : F^+$ such that $\psi^{\boxplus}(\bar{x})$, since we want to prove a boxed statement. By definition of the plus construction, it holds that $\Box(\ulcorner \bar{x}$ is a singleton$\urcorner)$. So, again since we want to prove a boxed statement, we may assume that $\bar{x}$ is actually a singleton. Therefore there exists $x : F$ such that $\bar{x} = [\{x\}]$ and that $\psi^{\boxplus}([\{x\}])$ holds. By the induction hypothesis, it follows that $\psi^{\Box}(x)$. From this the claim follows.

The case for "$\forall$" is similar. $\square$

Example 6.39. Let X be a scheme. Let f be a global function on X. Let $\varphi :\equiv \neg(\ulcorner f \text{ inv.}\urcorner)$ and $\varphi' :\equiv \ulcorner f \text{ nilpotent}\urcorner$. Then, by Proposition 3.10, we have $\mathrm{Sh}(X) \models (\varphi \Leftrightarrow \varphi')$. But in general, this does not imply that $\mathrm{Sh}(X) \models (\varphi^{\Box} \Leftrightarrow (\varphi')^{\Box})$. Consider for instance the modal operator given by $\Box\alpha :\equiv ((\alpha \Rightarrow\ !x) \Rightarrow\ !x)$ associated to a point $x \in X$. Then $\mathrm{Sh}(X) \models (\varphi^{\Box} \Leftrightarrow (\varphi')^{\Box})$ means that the equivalence $\varphi \Leftrightarrow \varphi'$ holds at the point x. This is false for $X = \operatorname{Spec} \mathbb{Z}$, $f = 2$, and $x = (2)$, since in the local ring $\mathcal{O}_{X,x} = \mathbb{Z}_{(2)}$, the element f is not invertible while also not being nilpotent.

6.8. Internal proofs of common lemmas.

Lemma 6.40. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.*

1. *Let $x \in X$ be a point. Then the stalk $\mathcal{F}_x$ is zero if and only if $\mathcal{F}$ is zero on some open neighborhood of x.*
2. *Let $A \subseteq X$ be a closed subset. Then the restriction $\mathcal{F}|_A$ (i. e. the pullback of $\mathcal{F}$ to A) is zero if and only if $\mathcal{F}$ is zero on some open subset of X containing A.*

Proof. Both statements are simply internalizations of Example 6.29, using the modal operators $\Box = (\_\_ \vee A^c)$ and $\Box = ((\_\_ \Rightarrow\ !x) \Rightarrow\ !x)$. $\square$

Remark 6.41. Lemma 6.40 fails if one drops the hypothesis that $\mathcal{F}$ is of finite type. Indeed, in this case one cannot reformulate the condition that $\mathcal{F}$ is zero in a geometric way.

In a remark after the proof of Proposition 5.6, we promised to present a simpler proof of it once we would have developed the theory for doing so. We can now follow up on this promise.

Lemma 6.42. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point. Let n be a natural number. Then the following statements are equivalent:*

1. *There exists a generating family for $\mathcal{F}_x$ consisting of n elements.*
2. *There exists an open neighborhood U of x such that*

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

Proof. Using the modal operator $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow\ !x) \Rightarrow\ !x)$, we have to show that the following statements in the internal language are equivalent:

1. $\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner^{\Box}$.
2. $\Box(\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner)$.

By Lemma 6.25, the second statement implies the first, since in a formal spelling of the statement in quotes,

$$\exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i, \qquad (\star)$$

no implication signs occur. To show the converse direction, we may assume that there is a generating family $y_1, \ldots, y_m : \mathcal{F}$ for $\mathcal{F}$ (since $\mathcal{F}$ is, externally speaking, of finite type). Then the $\Box$-translation of the statement that the y_i generate $\mathcal{F}$ holds as well (again by Lemma 6.25). Since there is an intuitionistic proof of

$$\ulcorner y_1, \ldots, y_m \text{ generate } \mathcal{F}\urcorner \Longrightarrow \Big(\ulcorner\text{there exist } x_1, \ldots, x_n : \mathcal{F} \text{ which generate } \mathcal{F}\urcorner \Longleftrightarrow \exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m \times n}.\ \ulcorner\vec{y} = A\vec{x}\urcorner\Big),$$

Lemma 6.28 can substitute the non-geometric formula ($\star$) by the geometric formula

$$\exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m \times n}.\ \ulcorner\vec{y} = A\vec{x}\urcorner$$

(Lemma 6.28). Thus the claim follows. $\Box$

Lemma 6.43. *Let X be a scheme (or a ringed space). Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be of finite type and assume that $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for some point $x \in X$. Then α is an epimorphism on some open neighborhood of x.*

Proof. In the presence of generators $y_1, \ldots, y_n : \mathcal{G}$, the non-geometric surjectivity condition ($\forall y : \mathcal{G}.\ \exists x : \mathcal{F}.\ \alpha(x) = y$) can be reformulated in a geometric way: $\bigwedge_{i=1}^n \exists x : \mathcal{F}.\ \alpha(x) = y_i$. Thus the claim follows by Lemma 6.28. $\Box$

Lemma 6.44. *Let $i : A \hookrightarrow X$ be a closed immersion of schemes (or ringed spaces). Let $\mathcal{F}$ be an $\mathcal{O}_A$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. Let $\Box$ be the modal operator defined by $\Box\varphi :\equiv (\varphi \vee A^c)$. From the internal perspective, we have a surjective ring homomorphism $i^\sharp : \mathcal{O}_X \to \mathcal{O}_A$, where we omit the forgetful functor i_* from $\Box$-sheaves to arbitrary sets in the notation, and an $\mathcal{O}_A$-module $\mathcal{F}$. Furthermore, we may assume that $\mathcal{F}$ is a $\Box$-sheaf. We can regard $\mathcal{F}$ as an $\mathcal{O}_X$-module by $i^\sharp$.

Note that $A^c \Rightarrow (\mathcal{F} = 0)$, by $\Box$-separatedness of $\mathcal{F}$.

We are to show that $\mathcal{F}$ is a finitely generated $\mathcal{O}_X$-module if and only if the $\Box$-translation of "$\mathcal{F}$ is a finitely generated $\mathcal{O}_A$-module" holds. In explicit terms, we have to show the equivalence of the following statements:

(1) $\bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_i i^\sharp(a_i) x_i$.
(2) $\Box(\bigvee_{n \geq 0} \Box(\exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \Box(\exists b_1, \ldots, b_n : \mathcal{O}_A.\ \Box(x = \sum_i b_i x_i))))$.

It is clear that the first statement implies the second. For the converse direction, we just have to repeatedly use the observation that $\Box\varphi$ implies $\varphi \vee (\mathcal{F} = 0)$ (once for each occurrence of $\Box$). So in each step, we either obtain the statement we want or may assume that $\mathcal{F}$ is the trivial module, in which case any subclaim trivially follows. By surjectivity of $i^\sharp$, we may write any $b : \mathcal{O}_A$ as $b = i^\sharp(a)$ for some $a : \mathcal{O}_X$. $\Box$

Lemma 6.45. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. Then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.*

Proof. It suffices to give an intuitionistic proof of the following fact: The construction $\mathrm{Hom}_R(M, \_\_)$ is geometric if M is a finitely presented R-module. So assume that M is the cokernel of a presentation matrix $(a_{ij}) : R^{n \times m}$. Then we can calculate the Hom with any R-module N as

$$\mathrm{Hom}_R(M, N) \cong \Big\{x : N^n \ \Big|\ \bigwedge_{j=1}^m \sum_{i=1}^n a_{ij} x_i = 0 : N\Big\},$$

and this construction is patently geometric, as a set comprehension with respect to a geometric formula. $\Box$

Lemma 6.46. *Let X be a scheme (or a ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $x \in X$. Then the stalk $\mathcal{F}_x$ is a finite free $\mathcal{O}_{X,x}$-module if and only if $\mathcal{F}$ is finite locally free on some open neighborhood of x.*

Proof. The internal statement that $\mathcal{F}$ is a finite free module is not geometric:

$$\bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i.$$

But it can equivalently be reformulated as

$$\bigvee_{n \geq 0} \exists \alpha : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n).\ \exists \beta : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \alpha \circ \beta = \mathrm{id} \wedge \beta \circ \alpha = \mathrm{id}.$$

This reformulation is geometric, therefore it holds at x if and only if it holds on some open neighborhood of x. The claim follows since, by the previous proposition, taking stalks commutes with calculating $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \underline{\ \ })$ resp. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \underline{\ \ })$; thus the pulled back formula indeed expresses that $\mathcal{F}_x$ is finite free as an $\mathcal{O}_{X,x}$-module. $\square$

Lemma 6.47. *Let X be an integral scheme with generic point ξ. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a torsion module if and only if its generic stalk $\mathcal{F}_\xi$ vanishes.*

Proof. The generic stalk vanishes if and only if the internal statement "$(\mathcal{F} = 0)^{\neg\neg}$" holds. Therefore it suffices to give an intuitionistic proof of the following internal statement: The module $\mathcal{F}$ is torsion if and only if any element of $\mathcal{F}$ is *not not* zero.

For the "only if" direction, let $x : \mathcal{F}$ be an arbitrary element. Since $\mathcal{F}$ is a torsion module, there exists a regular element $a : \mathcal{O}_X$ such that $ax = 0$. Since X is reduced, regularity is equivalent to not-not-invertibility. Since we want to verify the $\neg\neg$-stable statement "$\neg\neg(x = 0)$", we may in fact assume that a is invertible. Then $x = 0$ obviously follows.

For the "if" direction, let $x : \mathcal{F}$ be an arbitrary element; by assumption, x is *not not* zero. Since X is integral, Lemma 6.18 is applicable. Therefore there exists an element $a : \mathcal{O}_X$ such that a is *not not* invertible and such that invertibility of a implies $x = 0$. Since $\mathcal{F}$ is quasicoherent, for some natural number n it holds that $a^n x = 0$ (Theorem 8.3 below). Since a is *not not* invertible, it is regular (see Lemma 9.7 below for a short and self-contained proof), and therefore a^n is regular. So $x \in \mathcal{F}_{\mathrm{tors}}$. $\square$

By simply using a different modal operator than "*not not*", we will – without any additional work – obtain a more general form of this lemma, applicable to non-integral schemes (see Lemma 9.20).

7. Compactness and metaproperties

7.1. Quasicompactness. As stated in the introduction, quasicompactness of a space cannot be detected by the internal language: There cannot exist a formula φ such that a topological space is quasicompact if and only if $\mathrm{Sh}(X) \models \varphi$, since the latter is always a local property on X while quasicompactness is not. However, quasicompactness can be characterized by a *metaproperty* of the internal language.

This result is best stated in a way which does not explicitly refer to a notion of finiteness. So recall that quasicompactness of a topological space X can be phrased in the following way: For any directed set I and any monotone family $(U_i)_{i \in I}$ of open subsets, if $X = \bigcup_i U_i$ then $X = U_i$ for some $i \in I$. As usual, a *directed set* is an inhabited partially ordered set such that for any two elements there exists a common upper bound. A family $(U_i)_{i \in I}$ is *monotone* if and only if $i \preceq j$ implies $U_i \subseteq U_j$.

Proposition 7.1. *Let X be a topological space. Then X is quasicompact if and only if the internal language of* $\mathrm{Sh}(X)$ *has the following metaproperty: For any directed set I and any monotone family $(\varphi_i)_{i\in I}$ of formulas over X,*

$$\mathrm{Sh}(X) \models \bigvee_{i\in I} \varphi_i \quad \textit{implies} \quad \textit{for some } i \in I,\ \mathrm{Sh}(X) \models \varphi_i.$$

The monotonicity condition means that $\mathrm{Sh}(X) \models (\varphi_i \Rightarrow \varphi_j)$ *for any $i \preceq j$ in I.*

Stated more succinctly, a topological space X is quasicompact if and only if "$\mathrm{Sh}(X) \models$" commutes with directed "$\bigvee_{i\in I}$"'s.

Proof. For the "only if" direction, let such a family of formulas be given. Declare U_i to be the largest open subset of X where φ_i holds. Then by assumption, the sets U_i form a monotone family and cover X. By quasicompactness of X, some single member U_i covers X as well, whereby the corresponding formula φ_i holds on X.

For the "if" direction, we observe that a monotone family (U_i) of open subsets induces a monotone family of formulas by defining $\varphi_i :\equiv U_i$, employing the notational convention set out in Section 6.2. This correspondence is such that $\mathrm{Sh}(X) \models \bigvee_i \varphi_i$ holds if and only if $X = \bigcup_i U_i$ and such that $\mathrm{Sh}(X) \models \varphi_i$ if and only if $X = U_i$. With these observations the claim is obvious. □

Example 7.2. Let X be a quasicompact scheme (or quasicompact ringed space). Let $f \in \Gamma(X, \mathcal{O}_X)$ be a global function. Let the set of natural numbers be endowed with the usual ordering. Then the family of formulas given by $(f^n = 0)_{n\in\mathbb{N}}$ is monotone. Thus, if it internally holds that f is nilpotent, then f is nilpotent as an element of $\Gamma(X, \mathcal{O}_X)$ as well.

Proposition 7.3. *Let X be a topological space. Let $K \subseteq X$ be an open subset which is* locally quasicompact *in the sense that there exists an open covering $X = \bigcup_j U_j$ such that each $K \cap U_j$ is quasicompact. Then the internal language of* $\mathrm{Sh}(X)$ *has the following metaproperty: For any directed set I and any monotone family $(\varphi_i)_{i\in I}$ of formulas over X it holds that*

$$\mathrm{Sh}(X) \models \big(K \Rightarrow \bigvee_i \varphi_i\big) \quad \textit{implies} \quad \mathrm{Sh}(X) \models \bigvee_i (K \Rightarrow \varphi_i).$$

If additionally for any open subset $V \subseteq X$ the set $K \cap V$ is locally quasicompact in V, the following stronger and purely internal statement holds:

$$\mathrm{Sh}(X) \models \big(K \Rightarrow \bigvee_i \varphi_i\big) \Longrightarrow \bigvee_i (K \Rightarrow \varphi_i).$$

Proof. Assume that $\mathrm{Sh}(X) \models (K \Rightarrow \bigvee_i \varphi_i)$. This is equivalent to $K \models \bigvee_i \varphi_i$. By the locality of the internal language, it follows that $K \cap U_j \models \bigvee_i \varphi_i$ for each j. Since $K \cap U_j$ is quasicompact, it follows by Proposition 7.1 that there exists an index $i_j \in I$ such that $K \cap U_j \models \varphi_{i_j}$. This is equivalent to $U_j \models (K \Rightarrow \varphi_{i_j})$. In particular, it holds that $U_j \models \bigvee_i (K \Rightarrow \varphi_i)$. Since this is true for any j, it follows that $X \models \bigvee_i (K \Rightarrow \varphi_i)$, again by the locality of the internal language.

The second statement is a corollary of the first one. □

Example 7.4. Any retrocompact subset of a scheme is locally quasicompact in the sense of Proposition 7.3.

Example 7.5. Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ be a global function. Then the open set $D(f) = \{x \in X \mid f_x \text{ is invertible in } \mathcal{O}_{X,x}\}$ is locally quasicompact in the sense of Proposition 7.3, even in the stronger sense: Let $V \subseteq X$ be any open set. Consider a covering $V = \bigcup_i U_i$ by open affine subsets $U_i = \operatorname{Spec} A_i$. Then $D(f) \cap U_i \cong \operatorname{Spec} A_i[f^{-1}]$ is quasicompact.

From this example it will trivially follow that the nilradical $\sqrt{(0)} \subseteq \mathcal{O}_X$ of a scheme and indeed the radical of any quasicoherent sheaf of ideals is quasicoherent (Example 8.7). This example is also pivotal for giving a simple description of the quasicoherator (Proposition 8.13), which in turn is needed for an internal understanding of the relative spectrum (Section 12).

Remark 7.6. In applications, the open set K of Proposition 7.3 is often given as the largest open subset on which some formula ψ holds. (For instance, in Example 7.5, K was given by the formula $\ulcorner f$ is invertible in $\mathcal{O}_X\urcorner$.) Then the conclusion of the proposition is that *assuming that ψ holds commutes with directed disjunctions.*

7.2. Locality. A stronger condition on a topological space X than quasicompactness is locality: A topological space is *local* if and only if for any open covering $X = \bigcup_i U_i$ (not necessarily directed) a certain single subset U_i covers X as well. For instance, the spectrum of a ring A is local if and only if A is a local ring. Locality has the following characterization as a metaproperty of $\mathrm{Sh}(X)$.

Proposition 7.7. *Let X be a topological space. Then X is local if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any set I and any family $(\varphi_i)_{i\in I}$ of formulas over X, it holds that*

$$\mathrm{Sh}(X) \models \bigvee_{i\in I} \varphi_i \quad \textit{implies} \quad \textit{for some } i \in I,\ \mathrm{Sh}(X) \models \varphi_i.$$

In this case, the internal language has additionally the following (weaker) metaproperty: For any sheaf $\mathcal{F}$ on X and any formula $\varphi(s)$ containing a variable $s : \mathcal{F}$, it holds that

$$\mathrm{Sh}(X) \models \exists s : \mathcal{F}.\ \varphi(s) \quad \textit{implies} \quad \textit{for some } s \in \Gamma(X, \mathcal{F}),\ \mathrm{Sh}(X) \models \varphi(s).$$

Proof. The proof of the first part is very similar to the proof of Proposition 7.3. For the "only if" direction of the second part, note that the antecedent implies that there exist local sections $s_i \in \Gamma(U_i, \mathcal{F})$ such that $U_i \models \varphi(s_i)$ for some open covering $X = \bigcup_i U_i$. By locality of X, one such U_i suffices to cover X; so the corresponding section s_i is actually a global section and verifies $X \models \varphi(s_i)$. ☐

Remark 7.8. The second metaproperty stated in the proposition is indeed weaker than the condition that X is local. For instance, let X be a space consisting of two discrete points. Then $\mathrm{Sh}(X)$ has the second metaproperty, but X is not local.

7.3. Irreducibility. In intuitionistic logic, De Morgan's law $\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$ is not generally justified; therefore we can't use it when working internally to the topos of sheaves on a general scheme X. The following proposition demonstrates that if X is irreducible, the law does hold.

Proposition 7.9. *A topological space X is irreducible if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any formulas φ and ψ*

$$\mathrm{Sh}(X) \models \neg(\varphi \wedge \psi) \quad \textit{implies} \quad \mathrm{Sh}(X) \models \neg\varphi \textit{ or } \mathrm{Sh}(X) \models \neg\psi,$$

and not $\mathrm{Sh}(X) \models \bot$. Furthermore, in this case the following internal logical principle holds:

$$\mathrm{Sh}(X) \models \forall \alpha, \beta : \Omega.\ \neg(\alpha \wedge \beta) \Rightarrow (\neg\alpha \vee \neg\beta).$$

Proof. The statement "$\mathrm{Sh}(X) \models \neg(\varphi \wedge \psi)$" means that $U \cap V = \emptyset$, where U and V are the largest open subsets on which φ respectively ψ hold. The disjunction "$\mathrm{Sh}(X) \models \neg\varphi$ or $\mathrm{Sh}(X) \models \neg\psi$" means that $U = \emptyset$ or $V = \emptyset$. And "$\mathrm{Sh}(X) \models \bot$" is equivalent to $X = \emptyset$.

Therefore, if X is irreducible, then the internal language has the claimed metaproperty. The converse can be seen by instantiating φ and ψ with the formulas associated to given open subsets having empty intersection. It then follows that one of these formulas is false in the internal language; thus the associated subset is empty.

The stated internal logical principle holds since nonempty open subsets of irreducible spaces are irreducible. □

7.4. Internal proofs of common lemmas.

Lemma 7.10. *Let X be an irreducible reduced scheme. Then all local rings $\mathcal{O}_{X,x}$ are integral domains.*

Proof. It suffices to give a proof of the following statement: Let R be a local ring such that elements which are not invertible are nilpotent. Furthermore assume that R is reduced. Then R is an integral domain in the weak sense of Definition 3.16.

This proof may, additionally to the rules of intuitionistic logic, use the classical axiom stated in Proposition 7.9.

So let arbitrary elements $x, y : R$ with $xy = 0$ be given. Then it is not the case that x and y are both invertible: If they were, their product xy would be invertible as well, contradicting $1 \neq 0$. By the classicality principle, it follows that x is not invertible or that y is not invertible. Thus x or y is nilpotent and therefore zero. □

Lemma 7.11. *Let X be a scheme (or a ringed space). Let $(\mathcal{E}_i)_i$ be a directed system of $\mathcal{O}_X$-modules such that $\mathcal{E} := \operatorname{colim}_i \mathcal{E}_i$ is of finite type. If X is quasicompact, there is an index i such that $\mathcal{E}_i$ is of finite type and such that the coprojection $\mathcal{E}_i \to \mathcal{E}$ is an epimorphism of sheaves of modules.*

Proof. Since the usual proof of the statement "if a directed colimit of modules $(M_i)_i$ is finitely generated, then so is one of the modules and its coprojection into $\operatorname{colim}_i M_i$ is surjective" is intuitionistic, it can be applied in the internal universe of $\mathrm{Sh}(X)$. Hence we have

$$\mathrm{Sh}(X) \models \bigvee_i \ulcorner \mathcal{E}_i \text{ is finitely generated and } \mathcal{E}_i \to \mathcal{E} \text{ is surjective} \urcorner.$$

Therefore we can conclude by Proposition 4.3 and by Proposition 7.1. □

Caveat 7.12. There's a lemma stating that on a quasicompact and quasiseparated scheme, every quasicoherent sheaf of modules is a filtered colimit of finitely presented sheaves of modules [118, Tag 07V9]. There's also the corresponding statement for modules, whose standard proof is intuitionistic: Any module M is a filtered colimit of finitely presented modules (namely the finitely presented modules mapping to M).

However, the stated lemma does not immediately follow by applying the statement for modules in the internal universe. This only yields that any sheaf of modules is an *internal* filtered colimit; those are more general.[15]

8. Quasicoherent sheaves of modules

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space X is *quasicoherent* if and only if there exists a covering of X by open subsets U such that on each such set U, there

[15] Any small category $\mathcal{I}$ induces a small category $\underline{\mathcal{I}}$ internal to $\mathrm{Sh}(X)$ in such a way that the category of diagrams over $\mathcal{I}$ coincides with the category of internal diagrams over $\underline{\mathcal{I}}$ and the corresponding notions of limit and colimit agree. However, not every internal small category is of the form $\underline{\mathcal{I}}$. Therefore internal limits and colimits are more flexible than external ones.
For instance, it's not true that any sheaf of sets is a colimit of a suitable system of copies of the terminal sheaf. In contrast, internally, any set is a colimit of a suitable system of copies of the singleton set.

exists an exact sequence

$$(\mathcal{O}_X|_U)^J \longrightarrow (\mathcal{O}_X|_U)^I \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules, where I and J are arbitrary sets (which may depend on U).

If X is a scheme, quasicoherence can also be characterized in terms of inclusions of distinguished open subsets of affines: An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any open affine subscheme $U = \operatorname{Spec} A$ of X and any function $f \in A$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}),\ \tfrac{s}{f^n} \longmapsto f^{-n} s|_{D(f)}$$

is an isomorphism of $A[f^{-1}]$-modules. Here $D(f) \subseteq U$ denotes the standard open subset $\{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$. Both conditions can be internalized.

Proposition 8.1. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if*

$$\operatorname{Sh}(X) \models \exists I, J \text{ lc. } \ulcorner\text{there exists an exact sequence } \mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner.$$

*The “*lc*” indicates that when interpreting this internal statement with the Kripke–Joyal semantics, I and J should only be instantiated with* locally constant *sheaves.*

Proof. We only sketch the proof. The translation of the internal statement is that there exists a covering of X by open subsets U such that for each such U, there exist sets I, J and an exact sequence

$$(\mathcal{O}_X|_U)^{\underline{J}} \longrightarrow (\mathcal{O}_X|_U)^{\underline{I}} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

where $\underline{I}$ and $\underline{J}$ are the constant sheaves associated to I respectively J. The term “$(\mathcal{O}_X|_U)^{\underline{I}}$” refers to the internally defined free $\mathcal{O}_X$-module with basis the elements of $\underline{I}$. By exploiting that $\underline{I}$ is a discrete set from the internal point of view (i. e. any two elements are either equal or not), one can show that this is the same as $(\mathcal{O}_X|_U)^I$; similarly for J. With this observation, the statement follows. □

Remark 8.2. The restriction to locally constant sheaves is really necessary: The internal statement $\operatorname{Sh}(X) \models \exists I, J. \ulcorner\text{there exists an exact sequence } \mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner$ is true for *any* $\mathcal{O}_X$-module $\mathcal{F}$. This is because the usual proof of the fact that any module admits a resolution by (not necessarily finite) free modules is intuitionistically valid and thus also valid in the internal universe.

We don't know a useful internal characterization of locally constant sheaves (but see Section 8.2). The alternative internal condition given by the following theorem does not need such a characterization.

Theorem 8.3. *Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if, from the internal perspective, for any $f : \mathcal{O}_X$, the localized module $\mathcal{F}[f^{-1}]$ is a sheaf for the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \underline{\quad})$.*

In detail, the internal condition is that for any $f : \mathcal{O}_X$, it holds that

$$\forall s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow s = 0$$

and for any subsingleton $\mathcal{S} \subseteq \mathcal{F}[f^{-1}]$ it holds that

$$(\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner \mathcal{S} \text{ inhabited}\urcorner) \Longrightarrow \exists s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{S}).$$

Unlike with the internalizations of finite type, finite presentation and coherence, this condition is *not* a standard condition of commutative algebra. In fact, in classical logic, this condition is always satisfied – for trivial logical reasons if f is invertible, and because $\mathcal{F}[f^{-1}]$ is the zero module if f is not invertible (since f is nilpotent then, by Proposition 3.7).

That this condition in not known in commutative algebra is to be expected: Quasicoherence is a condition on sheaves of modules, ensuring that they are locally isomorphic to sheaves of the form $M^{\sim}$, where M is a plain module. But in commutative algebra, one *only* studies plain modules (and not sheaves of modules). The quasicoherence condition is imported into the realm of commutative algebra only by the internal language.

We give the proof of Theorem 8.3 below, after first discussing some examples and consequences. The proof will explain the origin of this condition. The localized module $\mathcal{F}[f^{-1}]$ appearing in the theorem is externally a certain sheaf. If $f \in \Gamma(U, \mathcal{O}_X)$, then it is the sheafification of the presheaf on U given by $V \mapsto \Gamma(V, \mathcal{F})[f^{-1}]$.

Example 8.4. The zero $\mathcal{O}_X$-module is quasicoherent, since (it and) all localizations of it are singleton sets from the internal perspective and thus $\Box$-sheaves for any modal operator $\Box$ by Example 6.10.

Corollary 8.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subseteq \mathcal{F}$ be a submodule. Then $\mathcal{G}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \forall f \colon \mathcal{O}_X.\ \forall s \colon \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{G}) \Longrightarrow \bigvee_{n \geq 0} f^n s \in \mathcal{G}.$$

Proof. We can give a purely internal proof. Let $f \colon \mathcal{O}_X$. Since subpresheaves of separated sheaves are separated, the module $\mathcal{G}[f^{-1}]$ is in any case separated with respect to the modal operator $\Box$ with $\Box\varphi :\equiv (\ulcorner f \text{ inv.}\urcorner \Rightarrow \varphi)$.

Now suppose that $\mathcal{G}$ is quasicoherent. Let $f \colon \mathcal{O}_X$. Let $s \colon \mathcal{F}$ and assume that if f were invertible, s would be an element of $\mathcal{G}$. Define the subsingleton $S := \{t \colon \mathcal{G}[f^{-1}] \mid \ulcorner f \text{ inv.}\urcorner \wedge t = s/1\}$. Then S would be inhabited by $s/1$ if f were invertible. Since $\mathcal{G}[f^{-1}]$ is a $\Box$-sheaf, it follows that there exists an element u/f^n of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be the case that $u/f^n = s/1 \in \mathcal{G}[f^{-1}] \subseteq \mathcal{F}[f^{-1}]$. Since $\mathcal{F}[f^{-1}]$ is $\Box$-separated, it follows that it actually holds that $u/f^n = s/1 \in \mathcal{F}[f^{-1}]$. Therefore there exists $m \colon \mathbb{N}$ such that $f^m f^n s = f^m u \in \mathcal{F}$. Thus $f^{m+n}s$ is an element of $\mathcal{G}$.

For the converse direction, assume that $\mathcal{G}$ fulfills the stated condition. Let$f \colon \mathcal{O}_X$. Let $S \subseteq \mathcal{G}[f^{-1}]$ be a subsingleton which would be inhabited if f were invertible. By regarding S as a subset of $\mathcal{F}[f^{-1}]$, it follows that there exists an element $u/f^n \in \mathcal{F}[f^{-1}]$ such that, if f were invertible, u/f^n would be an element of S. In particular, u would be an element of $\mathcal{G}$. By assumption it follows that there exists $m \colon \mathbb{N}$ such that $f^m u \in G$. Thus $(f^m u)/(f^m f^n)$ is an element of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be an element of S. $\Box$

Example 8.6. Let X be a scheme and s be a global section of $\mathcal{O}_X$. Then the annihilator of s, i. e. the sheaf of ideals internally defined by the formula

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t \colon \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X$$

is quasicoherent. To prove this in the internal language it suffices to verify the condition of Corollary 8.5. So let $f \colon \mathcal{O}_X$ and $t \colon \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.}\urcorner \Rightarrow t \in I$, i. e. assume that if f were invertible, then st would be zero. By Proposition 3.10 it follows that $f^n st = 0$ for some $n \colon \mathbb{N}$, i. e. that $f^n t \in I$.

Example 8.7. Let X be a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasicoherent sheaf of ideals. Then the radical of $\mathcal{I}$, internally definable as

$$\sqrt{\mathcal{I}} := \Big\{ s \colon \mathcal{O}_X \,\Big|\, \bigvee_{n \geq 0} s^n \in \mathcal{I} \Big\},$$

is quasicoherent as well: Let $f \colon \mathcal{O}_X$ and $s \colon \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \sqrt{\mathcal{I}}$, i. e. assume that if f were invertible, some power s^n would be an element of $\mathcal{I}$. Since *assuming that f is invertible commutes with directed disjunctions* (Example 7.5), it follows that for some natural number n, it holds that $\ulcorner f \text{ inv.}\urcorner \Rightarrow s^n \in \mathcal{I}$. By quasicoherence of $\mathcal{I}$, we may deduce that for some natural number m, it holds that $f^m s^n \in \mathcal{I}$. Thus $fs \in \sqrt{\mathcal{I}}$.

Proposition 8.8. *Let X be a scheme of dimension ≤ 0. Then any $\mathcal{O}_X$-module is quasicoherent.*

Proof. By Corollary 3.14, any element $f : \mathcal{O}_X$ is invertible or nilpotent. Therefore the quasicoherence condition of Theorem 8.3 is trivially satisfied for any $\mathcal{O}_X$-module. □

Remark 8.9. In general intuitionistic mathematics – not inside the internal universe of a scheme – the notion of quasicoherence as given by the internal condition of Theorem 8.3 does not seem to be very interesting: For many important rings, there are few quasicoherent modules in this sense. For instance, let M be a module over a ring R in which every element is invertible or not invertible. (The ring $\mathbb{Z}$ is such a ring.) Then M is quasicoherent if and only if for any $f : R$ which is not invertible, the localized module $M[f^{-1}]$ is the zero module, i. e. any element of M is annihilated by some power f^n. As a concrete example, any $\mathbb{Z}$-submodule of $\mathbb{Z}$ which contains a nonzero element fails to be quasicoherent.

Proof of Theorem 8.3. By the well-known characterization of quasicoherence in terms of inclusions of distinguished open subsets, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}), \ s/f^n \longmapsto f^{-n} s|_{D(f)} \tag{4}$$

is bijective. We will see that this map is injective for all such U and f if and only if from the internal perspective, for any $f : \mathcal{O}_X$, the set $\mathcal{F}[f^{-1}]$ is a separated presheaf with respect to the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$; and we will see that in this case, the map is additionally surjective for all such U and f if the full sheaf condition is fulfilled.

Since the sheaf $\mathcal{F}[f^{-1}]$ does not appear in the stated characterization, we will first reformulate the separatedness and the sheaf condition in terms of $\mathcal{F}$ instead of $\mathcal{F}[f^{-1}]$. To this end, we observe that the separatedness condition is equivalent to

$$\forall f : \mathcal{O}_X.\ \forall s : \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0 : \mathcal{F}) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0 : \mathcal{F}. \tag{5}$$

The equivalence can easily be proven in the internal language. The sheaf condition is equivalent to the conjunction of the separatedness condition and

$$\forall f : \mathcal{O}_X.\ \forall \mathcal{K} \subseteq \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner K \text{ is a singleton}\urcorner) \Longrightarrow \bigvee_{n \geq 0} \exists s : \mathcal{F}.\ \ulcorner f \text{ inv.}\urcorner \Rightarrow f^{-n} s \in \mathcal{K}. \tag{6}$$

In one direction, a set $\mathcal{S} \subseteq \mathcal{F}[f^{-1}]$ is given; construct $K := \{s : \mathcal{F} \mid s/1 \in \mathcal{S}\} \subseteq \mathcal{F}$. In the other direction, a set $\mathcal{K} \subseteq \mathcal{F}$ is given; construct $S := \{s : \mathcal{F}[f^{-1}] \mid \exists s' : \mathcal{F}.\ s' \in \mathcal{K} \wedge s = s'/1\} \subseteq \mathcal{F}[f^{-1}]$. The remaining details can easily be filled in.

We now interpret the internal statement (5) with the Kripke–Joyal semantics. Using the simplification rules, the external meaning is that for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$ the following condition is satisfied: For any section $s \in \Gamma(U, \mathcal{F})$ it should hold that

$$U \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \quad \text{implies} \quad U \models \bigvee_{n \geq 0} f^n s = 0.$$

The antecedent is equivalent to saying that s is zero in $\Gamma(D(f), \mathcal{F})$. The consequent is (by quasicompactness of U, see Example 7.2) equivalent to saying that for some $n \geq 0$, the section $f^n s$ is zero in $\Gamma(U, \mathcal{F})$, i. e. that s is zero in $\Gamma(U, \mathcal{F})[f^{-1}]$. So this condition is precisely the injectivity of the canonical map (4).

The external meaning of statement (6) is that for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$ the following condition is satisfied: For any subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ it should hold that

$$U \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner \mathcal{K} \text{ is a singleton}\urcorner) \quad \text{implies}$$
$$U \models \bigvee_{n \geq 0} \exists s : \mathcal{F}.\ \ulcorner f \text{ inv.}\urcorner \Rightarrow f^{-n} s \in \mathcal{K}.$$

Given the injectivity of the canonical map (4) (for any affine open subset, not only U), this condition is equivalent to its surjectivity: To see that surjectivity is sufficient, let a subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ verifying the antecedent be given. Since $\mathcal{K}|_{D(f)}$ is a singleton sheaf, we can consider its unique section $u \in \Gamma(D(f), \mathcal{K}) \subseteq \Gamma(D(f), \mathcal{F})$. By surjectivity, there exists a preimage, i. e. a fraction $s/f^n \in \Gamma(U, \mathcal{F})[f^{-1}]$ such that $u = f^{-n} s|_{D(f)}$ in $\Gamma(D(f), \mathcal{F})$. Thus $U \models f^{-n} s \in \mathcal{K}$ holds and the consequent is verified.

To see that surjectivity is necessary, let a section $u \in \Gamma(D(f), \mathcal{F})$ be given. Define a subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ by setting $\Gamma(V, \mathcal{K}) := \{u|_V \mid V \subseteq D(f)\}$. Then $\mathcal{K}$ verifies the antecedent. Thus the consequent holds: There exists an open covering $U = \bigcup_i U_i$ such that for each i, there exists a natural number n_i and a section $s_i \in \Gamma(U_i, \mathcal{F})$ such that $f^{-n_i} s_i = u$ on $U_i \cap D(f)$. Without loss of generality, we may assume that the U_i are distinguished open subsets $D(g_i) \subseteq U$; that they are finite in number; and that the natural numbers n_i agree with each other and thus equal some number n. Since $s_i = s_j$ in $\Gamma(U_i \cap U_j \cap D(f), \mathcal{F})$, injectivity of the canonical map (4) (on the affine set $U_i \cap U_j = D(g_i g_j)$) implies that $s_i = s_j$ in $\Gamma(U_i \cap U_j, \mathcal{F})[f^{-1}]$. Thus for any indices i, j there exists a natural number m_{ij} such that $f^{m_{ij}} s_i = f^{m_{ij}} s_j$ in $\Gamma(U_i \cap U_j, \mathcal{F})$. We may assume that the numbers m_{ij} equal some common number m; thus the local sections $f^m s_i$ glue to a section $s \in \Gamma(U, \mathcal{F})$. The sought preimage of u is the fraction s/f^{n+m}, since $f^{-(n+m)} s|_{D(f)}$ equals u in $\Gamma(D(f), \mathcal{F})$ (as this is true on the covering $D(f) = \bigcup_i (D(f) \cap U_i)$). □

8.1. The quasicoherator for radical ideals. For applications in Section 12 about interpreting the relative spectrum as an internal spectrum, we want to specialize to radical sheaves of ideals. In particular, we want to describe the *quasicoherator* – the left adjoint to the inclusion of the quasicoherent radical ideals in the poset of all radical ideals – in simple terms.

Caveat 8.10. The quasicoherator we refer to does *not* coincide with the quasicoherator of $\mathcal{O}_X$-modules [118, Tag 077P], [125], which is the *right* adjoint to the inclusion of category of quasicoherent $\mathcal{O}_X$-modules in the category of all $\mathcal{O}_X$-modules. We discuss this in more detail in Example 8.12 below.

Proposition 8.11. *Let X be a scheme. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a radical ideal.*

(1) *The ideal $\mathcal{I}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ (\ulcorner s \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I}) \Rightarrow s \in \mathcal{I}.$$

(2) *The reflection of $\mathcal{I}$ in the poset of quasicoherent radical ideals is the sheaf $\overline{\mathcal{I}}$ given by the internal expression*

$$\overline{\mathcal{I}} := \{s : \mathcal{O}_X \mid \ulcorner s \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I}\}.$$

Proof. Both claims can be verified by purely internal reasoning. The first claim is a straightforward calculation using the characterization given in Corollary 8.5. We discuss the second one in more detail.

Firstly, it's obvious that $\overline{\mathcal{I}}$ contains $\mathcal{I}$ and that $\overline{\mathcal{I}}$ is a radical ideal. To verify that $\overline{\mathcal{I}}$ is quasicoherent, let $s : \mathcal{O}_X$ be given such that, if s were invertible, then s would be an element of $\overline{\mathcal{I}}$. Symbolically, we have

$$\ulcorner s \text{ inv.} \urcorner \Longrightarrow (\ulcorner s \text{ inv.} \urcorner \Rightarrow s \in \mathcal{I}),$$

which of course implies

$$\ulcorner s \text{ inv.} \urcorner \Longrightarrow s \in \mathcal{I}.$$

This is precisely the condition for s to be an element of $\overline{I}$.

To verify that the construction $\mathcal{I} \mapsto \overline{\mathcal{I}}$ is really left adjoint to the inclusion, let a quasicoherent radical ideal $\mathcal{J}$ be given such that $\mathcal{I} \subseteq \mathcal{J}$. We have to show that $\overline{\mathcal{I}} \subseteq \mathcal{J}$. This is straightforward. □

Example 8.12. Let $X := \mathbb{A}^1_k = \operatorname{Spec} k[T]$ be the affine line over a field k. Let $j : U := \mathbb{A}^1_k \setminus \{0\} \hookrightarrow X$ be the open inclusion of the punctured line. Then $\mathcal{I} := j_! \mathcal{O}_U \hookrightarrow \mathcal{O}_X$ is the standard example of a radical sheaf of ideals which is not quasicoherent. The quasicoherator of modules maps $\mathcal{I}$ to $(\Gamma(X, \mathcal{I}))^\sim$, so to the zero module. In contrast, the reflection of $\mathcal{I}$ in the poset of quasicoherent radical ideals is (T).

Generally, the reflection $\overline{\mathcal{I}}$ of a radical ideal $\mathcal{I}$ is the unique radical ideal such that $\overline{\mathcal{I}}$ is quasicoherent and such that $D(\mathcal{I}) = D(\overline{\mathcal{I}})$. Explicitly, it is the subsheaf of $\mathcal{O}_X$ given by

$$U \longmapsto \{f \in \mathcal{O}_X(U) \mid 1 \in \mathcal{I}(D(f))\}.$$

For arbitrary $\mathcal{O}_X$-algebras $\mathcal{A}$, the description of the quasicoherator for radical ideals of $\mathcal{A}$ is more involved than the description given in Proposition 8.11(b), but still sufficiently explicit for the applications in Section 12.

Proposition 8.13. *Let X be a scheme. Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Then the reflection of a radical ideal $\mathcal{I} \subseteq \mathcal{A}$ in the poset of quasicoherent radical ideals of $\mathcal{A}$ is given by the internal expression*

$$\overline{\mathcal{I}} := \bigcup_{n \geq 0} \mathcal{I}_n,$$

where $(\mathcal{I}_n)$ is the family of radical ideals defined recursively by

$$\mathcal{I}_0 := \mathcal{I},$$
$$\mathcal{I}_{n+1} := \text{the radical ideal generated by } \{fs \mid f : \mathcal{O}_X, s : \mathcal{A}, (\ulcorner f \text{ inv.} \urcorner \Rightarrow s \in \mathcal{I}_n)\}.$$

Proof. We argue internally. The set $\overline{\mathcal{I}}$ contains $\mathcal{I}$ and is a radical ideal, as an ascending union of radical ideals. To verify that $\overline{\mathcal{I}}$ is quasicoherent, let $f : \mathcal{O}_X$ and $s : \mathcal{A}$ be given such that, if f were invertible, then s would be an element of $\overline{\mathcal{I}}$. This means that we have

$$\ulcorner f \text{ inv.} \urcorner \Longrightarrow \bigvee_{n \geq 0} s \in \mathcal{I}_n.$$

Since assuming that f is invertible commutes with directed disjunctions (Example 7.5), there is a natural number n such that

$$\ulcorner f \text{ inv.} \urcorner \Longrightarrow s \in \mathcal{I}_n.$$

Therefore $fs \in \mathcal{I}_{n+1} \subseteq \overline{\mathcal{I}}$.

Finally, to verify that the construction $\mathcal{I} \mapsto \overline{\mathcal{I}}$ is indeed left adjoint to the inclusion of the quasicoherent radical ideals in all radical ideals, let a quasicoherent radical ideal $\mathcal{J}$ be given such that $\mathcal{I} \subseteq \mathcal{J}$. By induction we can show that $\mathcal{I}_n \subseteq \mathcal{J}$ for all natural numbers n. Therefore $\overline{\mathcal{I}} \subseteq \mathcal{J}$. □

Remark 8.14. If the goal was to close a given radical ideal under the condition

$$\forall s : \mathcal{A}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I}) \Longrightarrow fs \in \mathcal{I},$$

where $f : \mathcal{O}_X$ is a fixed element, no infinite iteration would be necessary. The closure would in this case simply be given by

$$\overline{\mathcal{I}}^f := \text{the radical ideal generated by the set } \{fs \mid s : \mathcal{A}, (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I})\}.$$

There is also a purely formal description of the reflector, given by

$$\mathcal{I} \longmapsto \bigcap\{\mathcal{J} \subseteq \mathcal{A} \mid \mathcal{J} \text{ is a quasicoherent radical ideal such that } \mathcal{I} \subseteq \mathcal{J}\}.$$

Verifying that this construction has the universal property of the reflector is straightforward. However, it is not sufficiently concrete for calculations. In particular, we don't see a way to prove the following corollary without the explicit description given by Proposition 8.13.

Corollary 8.15. *Let X be a scheme. Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Let $\mathcal{I}$ and $\mathcal{J}$ be radical ideals of $\mathcal{A}$. Then $\overline{\mathcal{I} \cap \mathcal{J}} = \overline{\mathcal{I}} \cap \overline{\mathcal{J}}$.*

Proof. The claim is not purely formal. As a left adjoint, the reflector preserves arbitrary suprema (as a map from the poset of all radical ideals into the poset of all quasicoherent radical ideals); but the claim is that it preserves (finite) intersections.

Since the reflector is monotone, it is clear that $\overline{\mathcal{I} \cap \mathcal{J}} \subseteq \overline{\mathcal{I}} \cap \overline{\mathcal{J}}$.

To verify the converse direction, we show by induction that $\mathcal{I}_n \cap \mathcal{J}_m \subseteq \overline{\mathcal{I} \cap \mathcal{J}}$ for all natural numbers n and m. The base case is trivial, since $\mathcal{I}_0 \cap \mathcal{J}_0 = \mathcal{I} \cap \mathcal{J}$. For the induction step let $x \in \mathcal{I}_{n+1} \cap \mathcal{J}_m$. Then $x^\ell = \sum_i f_i s_i$ for some natural number ℓ and elements $f_i : \mathcal{O}_X$, $s_i : \mathcal{A}$ such that $\ulcorner f_i \text{ inv.}\urcorner \Rightarrow s_i \in \mathcal{I}_n$. In particular we have $\ulcorner f_i \text{ inv.}\urcorner \Rightarrow s_i x \in \mathcal{I}_n \cap \mathcal{J}_m$, so by the induction hypothesis $\ulcorner f_i \text{ inv.}\urcorner \Rightarrow s_i x \in \overline{\mathcal{I} \cap \mathcal{J}}$. This implies $f_i s_i x \in \overline{\mathcal{I} \cap \mathcal{J}}$, since $\overline{\mathcal{I} \cap \mathcal{J}}$ is quasicoherent. Therefore $x^{\ell+1} \in \overline{\mathcal{I} \cap \mathcal{J}}$ and thus $x \in \overline{\mathcal{I} \cap \mathcal{J}}$. □

Remark 8.16. If in the situation of Proposition 8.13 the algebra $\mathcal{A}$ is not quasicoherent, the construction $\mathcal{I} \mapsto \overline{\mathcal{I}}$ is still left adjoint to the inclusion of the radical sheaves of ideals which satisfy the (then somewhat unmotivated) internal condition given in Corollary 8.5 in the poset of all radical sheaves of ideals. Also Corollary 8.15 remains valid. This is even the case if X is an arbitrary ringed space; in this case, the proofs of Proposition 8.13 and Corollary 8.5 have to be modified, since then we may not suppose that assuming that an element of $\mathcal{O}_X$ is invertible commutes with directed disjunctions.

Instead, the reflector $\mathcal{I} \mapsto \overline{\mathcal{I}}$ has to be characterized by

$$\overline{\mathcal{I}} := \text{least fixed point of } P \text{ above } \mathcal{I},$$

where P is the monotone operator on the set of radical ideals which takes a radical ideal $\mathcal{I}$ to the radical ideal generated by $\{fs \mid f : \mathcal{O}_X, s : \mathcal{A}, (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I})\}$. The existence of these fixed points is guaranteed by the Knaster–Tarski theorem, which is intuitionistically valid in the version we need [18].

The following proof scheme is useful for verifying properties of the least fixed point. Let $\varphi(\mathcal{J})$ be a statement on radical ideals $\mathcal{J}$ such that $\varphi(\sup_i \mathcal{J}_i) \Leftrightarrow \bigvee_i \varphi(\mathcal{J}_i)$ for every family $(\mathcal{J}_i)_i$ of radical ideals. If

$$\varphi(P(\mathcal{J})) \Longrightarrow \varphi(\mathcal{J})$$

for all radical ideals $\mathcal{J}$ containing $\mathcal{I}$, then $\varphi(\overline{\mathcal{I}}) \Rightarrow \varphi(\mathcal{I})$. This proof scheme is a special case of the following more general scheme, which is also sometimes needed for reasoning about the least fixed point.

Let L be a complete partial order. Let α be a map from the set of radical ideals to L such that $\alpha(\sup_i \mathcal{J}_i) = \sup_i \alpha(\mathcal{J}_i)$ for every family $(\mathcal{J}_i)_i$ of radical ideals. If

$$\alpha(P(\mathcal{J})) \preceq \alpha(\mathcal{J})$$

for all radical ideals $\mathcal{J}$ containing $\mathcal{I}$, then $\alpha(\overline{\mathcal{I}}) \preceq \alpha(\mathcal{I})$.

Remark 8.17. The reflector can also be given by the formula

$$\overline{\mathcal{I}} = \bigcap_{\mathcal{J}} \Big(\mathcal{J} : \bigcap_{f : \mathcal{O}_X} (\mathcal{J} : \overline{\mathcal{I}}^f) \Big),$$

where $\overline{\mathcal{I}}^f$ is as in Remark 8.14 and the first intersection is indexed by all radical ideals $\mathcal{J} \subseteq \mathcal{A}$. This identity follows by the description of $\overline{\mathcal{I}}$ as a least fixed point and the explicit formula for the least fixed point from the proof of its existence [18]. It also follows from the observation that the operation $\mathcal{J} \mapsto \overline{\mathcal{J}}$ is the nucleus associated to the intersection of the sublocales given by the nuclei $\mathcal{J} \mapsto \overline{\mathcal{J}}^f$, which in turn is evident from the description of the relative spectrum as a classifying locale given in Proposition 12.14.

8.2. Characterizing locally constant sheaves. We don't think that there is a characterization of locally constant sheaves in the internal language of an arbitrary topos of sheaves, other than the following trivial one: A sheaf $\mathcal{E}$ on a topological space X (or a locale, or a site) is locally constant if and only if

$$\mathrm{Sh}(X) \models \bigvee_M \ulcorner \mathcal{E} \cong \underline{M} \urcorner,$$

where the disjunction is over *all sets* and $\underline{M}$ is the constant sheaf associated to the set M. Strictly speaking, because of the class-sized disjunction, this statement is not even well-formed; however one can still make sense of its Kripke–Joyal translation.

In the special case that X is a scheme, however, there might be an internal characterization. We failed to disprove the following speculation:

Speculation 8.18. Let X be a scheme. Let $\mathcal{E}$ be a sheaf of sets on X. Then $\mathcal{E}$ is locally constant if and only if $\mathcal{O}_X\langle\mathcal{E}\rangle$, the free $\mathcal{O}_X$-module on $\mathcal{E}$ (constructed internally), is quasicoherent.

The free module occurring in this speculation is the sheafification of the presheaf

$$U \longmapsto \Gamma(U, \mathcal{O}_X)\langle\Gamma(U, \mathcal{E})\rangle$$

and can also be described as $f_! f^{-1}\mathcal{O}_X$, where $f : \mathrm{\acute{E}t}(X) \to X$ is the projection of the étalé space associated to $\mathcal{E}$ (and $\mathrm{\acute{E}t}(X)$ is equipped with a scheme structure by exploiting that $\mathrm{\acute{E}t}(X)$ is locally homeomorphic to X).

The "only if" direction of Speculation 8.18 certainly holds; in fact, if $\mathcal{E}$ is locally constant, then $\mathcal{O}_X\langle\mathcal{E}\rangle$ is even locally free. There are the following indications that the converse might hold.

If X happens to be local as a topological space, then the converse holds: Exploiting that in this case $\Gamma(X, \mathcal{O}_X\langle\mathcal{E}\rangle) \cong \Gamma(X, \mathcal{O}_X)\langle\Gamma(X, \mathcal{E})\rangle$ one can show that the canonical morphism $\underline{\Gamma(X, \mathcal{E})} \to \mathcal{E}$ is an isomorphism. Returning to the general situation, we see that the pullback of $\mathcal{E}$ to any of the $\mathrm{Spec}(\mathcal{O}_{X,x})$ is constant if $\mathcal{O}_X\langle\mathcal{E}\rangle$ is quasicoherent. Thus $\mathcal{E}$ is "constant on all infinitesimal neighborhoods".

If $\mathcal{O}_X\langle\mathcal{E}\rangle$ is not only quasicoherent, but even locally free (locally isomorphic to a module of the form $\mathcal{O}_X^{\oplus M}$), then locally we have $\mathcal{O}_{X,x}\langle\mathcal{E}_x\rangle \cong \mathcal{O}_{X,x}\langle M\rangle$, so $\mathcal{E}_x \cong M$, so at least the stalks are locally constant. Similarly, if $\mathcal{O}_X\langle\mathcal{E}\rangle$ is of finite presentation, then $\mathcal{E}$ is locally constant (with finite stalks).

Finally, let $j : V \hookrightarrow X$ be the inclusion of an open subset. Let $\mathcal{E}$ be $j_!(1)$, the extension of the terminal sheaf on V by the empty set. This sheaf is locally constant iff V is a clopen subset. Now furthermore assume that X is integral. In this case one can check that $\mathcal{O}_X\langle\mathcal{E}\rangle = j_!(\mathcal{O}_V)$ (extension by zero) is quasicoherent iff V is a clopen subset. Thus the converse holds in this case.

9. Rational functions and Cartier divisors

9.1. The sheaf of rational functions. Recall that the sheaf $\mathcal{K}_X$ of rational functions on a scheme X (or a ringed space) can be defined as the sheafification of the presheaf

$$U \subseteq X \text{ open} \quad \longmapsto \quad \Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S})^{-1}],$$

where $\Gamma(U, \mathcal{S})$ is the multiplicative set of those sections of $\mathcal{O}_X$ on U which are regular in each stalk $\mathcal{O}_{X,x}$, $x \in U$. Recall also that there are some wrong definitions in the literature [75].

Using the internal language, we can give a simpler definition of $\mathcal{K}_X$. Recall that we can associate to any ring R its total quotient ring, i. e. its localization at the multiplicative subset of regular elements. Since from the internal perspective $\mathcal{O}_X$ is an ordinary ring, we can associate to it its total quotient ring $\mathcal{O}_X[\mathcal{S}^{-1}]$, where $\mathcal{S}$ is internally defined by the formula

$$\mathcal{S} := \{s : \mathcal{O}_X \mid \ulcorner s \text{ is regular}\urcorner\} \subseteq \mathcal{O}_X.$$

Externally, this ring is the sheaf $\mathcal{K}_X$.

Proposition 9.1. *Let X be a scheme (or a ringed space). The sheaf of rings defined in the internal language by localizing $\mathcal{O}_X$ at its set of regular elements is (canonically isomorphic to) the sheaf $\mathcal{K}_X$ of rational functions.*

Proof. Internally, the ring $\mathcal{O}_X[\mathcal{S}^{-1}]$ has the following universal property: For any ring R and any homomorphism $\mathcal{O}_X \to R$ which maps the elements of $\mathcal{S}$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}] \to R$ which renders the evident diagram commutative.

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr] \arrow[dr] & & R \\
& \mathcal{O}_X[\mathcal{S}^{-1}] \arrow[ur, dashed] &
\end{tikzcd}$$

The translation using the Kripke–Joyal semantics gives the following universal property: For any open subset $U \subseteq X$, any sheaf of rings $\mathcal{R}$ on U and any homomorphism $\mathcal{O}_X|_U \to \mathcal{R}$ which maps all elements of $\Gamma(V, \mathcal{S})$ for open subsets $V \subseteq U$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}]|_U \to \mathcal{R}$ which renders the evident diagram commutative. It is well-known that the sheaf $\mathcal{K}_X$ as usually defined has this universal property as well. $\square$

Proposition 9.2. *Let X be a scheme (or a ringed space). Then the stalks of $\mathcal{K}_X$ are given by*

$$\mathcal{K}_{X,x} = \mathcal{O}_{X,x}[\mathcal{S}_x^{-1}].$$

The elements of $\mathcal{S}_x$ are exactly the germs of those local sections which are regular not only in $\mathcal{O}_{X,x}$, but in all rings $\mathcal{O}_{X,y}$ where y ranges over some open neighborhood of x (depending on the section).

Proof. Since localization is a geometric construction, the first statement is made entirely trivial by our framework. The second statement follows since

$$\Gamma(U, \mathcal{S}) = \{s \in \Gamma(U, \mathcal{O}_X) \mid U \models \ulcorner s \text{ is regular}\urcorner\}$$

and since regularity is a geometric implication, so that $U \models \ulcorner s \text{ is regular}\urcorner$ if and only if the germ s_y is regular in $\mathcal{O}_{X,y}$ for all $y \in U$. $\square$

Remark 9.3. Speaking internally, the multiplicative set $\mathcal{S}$ is saturated. Therefore an element $s/t : \mathcal{K}_X$ is invertible in $\mathcal{K}_X$ if and only if the numerator s belongs to $\mathcal{S}$, that is if s is an regular element of $\mathcal{O}_X$.

9.2. Regularity of local functions. It is well-known that on a locally Noetherian scheme, regularity spreads from stalks to neighborhoods, that is that a section of $\mathcal{O}_X$ is regular in $\mathcal{O}_{X,x}$ if and only if it is regular on some open neighborhood of x. This fact has a simple proof in the internal language.

Proposition 9.4. *Let X be a locally Noetherian scheme. Let $s \in \Gamma(U, \mathcal{O}_X)$ be a local function on X. Let $x \in U$. Then the following statements are equivalent:*

1. *The section s is regular in $\mathcal{O}_{X,x}$.*
2. *The section s is regular in all local rings $\mathcal{O}_{X,y}$ where y ranges over some open neighborhood of x.*

Proof. Let $\Box$ be the modal operator defined by $\Box\varphi :\equiv ((\varphi \Rightarrow {!x}) \Rightarrow {!x})$. By Corollary 6.32, we are to show that the following statements of the internal language are equivalent:

1. $(\ulcorner s \text{ is regular}\urcorner)^{\Box}$, i. e. $\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow \Box(t = 0)$.
2. $\Box(\ulcorner s \text{ is regular}\urcorner)$, i. e. $\Box(\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow t = 0)$.

It is clear that the second statement implies the first – in fact, this is true without any assumptions on X: Let $t : \mathcal{O}_X$ be such that $st = 0$. Since we want to prove the boxed statement $\Box(t = 0)$, we may assume that s is regular and prove $t = 0$. This is immediate. (This direction also follows simply by examining the logical form and applying Lemma 6.25.)

For the converse direction, consider the annihilator of s, i. e. the ideal

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X.$$

This ideal satisfies the quasicoherence condition (we will explain how to prove this internally in 8.6), thus I is a quasicoherent submodule of a finitely generated module. Since X is locally Noetherian, it follows that I is finitely generated as well, say by $x_1, \ldots, x_n : I$. By assumption, each generator $x_i : I$ fulfills $\Box(x_i = 0)$. Since we want to prove a boxed statement, we may in fact assume $x_i = 0$. Thus $I = (0)$ and the assertion that s is regular follows. ◻

The proof critically depends on the ideal I being finitely generated, since a modal operator need only commute with finite conjunctions. Intuitively, each time we use the modus ponens rule $(\Box\varphi \wedge (\varphi \Rightarrow \psi)) \Rightarrow \Box\psi$, we restrict to a smaller open neighborhood of x. Since infinite intersections of open sets need not be open, we cannot expect an infinitary modus ponens rule to hold.

Corollary 9.5. *Let X be a locally Noetherian scheme. Then the stalks $\mathcal{K}_{X,x}$ of the sheaf of rational functions are given by the total quotient rings of the local rings $\mathcal{O}_{X,x}$.*

Proof. Follows by combining Proposition 9.2 and Proposition 9.4. ◻

9.3. Normality. Recall that a ring R is *normal* if and only if it is integrally closed in its total quotient ring. Recall also that a scheme X (or a ringed space) is *normal* if and only if all rings $\mathcal{O}_{X,x}$ are normal.

Proposition 9.6. *A locally Noetherian scheme is normal if and only if the ring $\mathcal{O}_X$ is normal from the internal perspective.*

Proof. The condition of normality can be put into a form which is almost a geometric implication:

$$\begin{aligned}&\forall s, t : \mathcal{O}_X.\ \big(\ulcorner t \text{ regular}\urcorner \wedge \\ &\qquad \exists a_0, \ldots, a_{n-1} : \mathcal{O}_X.\ s^n + a_{n-1} t s^{n-1} + \cdots + a_1 t^{n-1} s + a_0 t^n = 0\big) \Longrightarrow \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \exists u : \mathcal{O}_X.\ s = ut.\end{aligned}$$

The only non-geometric subpart is the condition on t to be regular. However, by Proposition 9.4, for the purposes of comparing its truth at points vs. on neighborhoods, it behaves just like a geometric formula. Therefore the claim follows. □

9.4. Geometric interpretation of rational functions. Recall that on integral schemes, rational functions (i. e. sections of $\mathcal{K}_X$) are the same thing as regular functions defined on dense open subsets. This amounts to saying that $\mathcal{K}_X$ *is the* $\neg\neg$*-sheafification of* $\mathcal{O}_X$ (see Proposition 6.15). We want to rederive this result, as far as possible in the internal language, and generalize it to arbitrary (not necessarily locally Noetherian) schemes.

Lemma 9.7. *Let X be a reduced scheme. Then:*

(1) *$\mathcal{O}_X$ is $\neg\neg$-separated.*
(2) *Internally, an element $s : \mathcal{O}_X$ is regular if and only if it is* not not *invertible.*

Proof. Recall from Corollary 3.9 that

$$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible}\urcorner) \Leftrightarrow s = 0. \tag{⋈}$$

From this we can deduce that $\mathcal{O}_X$ is $\neg\neg$-separated: Assume $\neg\neg(s = 0)$ for $s : \mathcal{O}_X$. If s were invertible, we would have $\neg\neg(1 = 0)$ and thus $\perp$. Therefore s is not invertible and thus zero.

For the "only if" direction of the second statement, note that a regular element is not zero (if it were, then the true statement $0 \cdot 0 = 0 \cdot 1$ would imply the false statement $0 = 1$) and thus *not not* invertible (by the contrapositive of equivalence (⋈)). For the "if" direction, let $st = 0$ in $\mathcal{O}_X$. Since s is *not not* invertible, it follows that t is *not not* zero. Since $\mathcal{O}_X$ is $\neg\neg$-separated, this implies that t really is zero. □

For the following, we need two technical conditions. Say that an affine scheme $\operatorname{Spec} A$ has property ($\star$) if and only if:

> Every open dense subset $U \subseteq \operatorname{Spec} A$ contains a *standard open* dense subset.

Say that $\operatorname{Spec} A$ has property ($\star\star$) if and only if:

> Every open scheme-theoretically dense subset $U \subseteq \operatorname{Spec} A$ contains a *standard open* scheme-theoretically dense subset.

The first condition is satisfied if A is an irreducible ring (i. e. if $\operatorname{Spec} A$ is irreducible) or more generally if A contains only finitely many minimal prime ideals. Both conditions are satisfied if A is integral or if A is Noetherian; for convenience, we give a proof in the latter case.

Proposition 9.8. *Let A be a Noetherian ring. Then* $\operatorname{Spec} A$ *has properties* ($\star$) *and* ($\star\star$).

Proof. Recall that, under the Noetherian hypothesis, an open subset of $\operatorname{Spec} A$ is dense if and only if it contains all minimal prime ideals (this fact holds more generally if there are only finitely many minimal prime ideals) and that it is scheme-theoretically dense if and only if it contains all associated prime ideals. There are only a finite number of these prime ideals. Therefore the claim is reduced to the following statement:

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be a finite number of points of an open subset $U \subseteq \operatorname{Spec} A$. Then there exists a standard open subset $D(f) \subseteq U$ which also contains these points.

The proof of this statement is a direct application of the prime avoidance lemma. □

Proposition 9.9. *Let X be a reduced scheme. Assume that every open affine subscheme has property ($\star$). (For instance, this condition is satisfied if X is integral, the set of irreducible components is locally finite, or if X is locally Noetherian.) Then $\mathcal{K}_X$ is the $\neg\neg$-sheafification of $\mathcal{O}_X$.*

Proof. We first show that $\mathcal{K}_X$ is $\neg\neg$-separated, so assume $\neg\neg(a/s = 0)$ for $a/s : \mathcal{K}_X$. Since $\mathcal{K}_X$ is obtained from $\mathcal{O}_X$ by localizing at regular elements, the fraction a/s vanishes in $\mathcal{K}_X$ if and only if $a = 0$ in $\mathcal{O}_X$. Thus it follows that $\neg\neg(a = 0)$ in $\mathcal{O}_X$ and therefore $a = 0$ in $\mathcal{O}_X$; in particular, $a/s = 0$ in $\mathcal{K}_X$.

We defer the proof that $\mathcal{K}_X$ is a $\neg\neg$-sheaf to the end and first verify the universal property of $\neg\neg$-sheafification. So let G be a $\neg\neg$-sheaf and let $\alpha : \mathcal{O}_X \to G$ be a map. We define an extension $\bar{\alpha} : \mathcal{K}_X \to G$ in the following way: Let $f : \mathcal{K}_X$. Define the subsingleton $S := \{x : G \mid \exists b : \mathcal{O}_X.\ f = b/1 \wedge x = \alpha(b)\} \subseteq G$. Since f can be written in the form a/s with s *not not* invertible, it follows that S is *not not* inhabited. Since G is a $\neg\neg$-sheaf, there exists a unique $x : G$ such that $\neg\neg(x \in S)$. We declare $\bar{\alpha}(f)$ to be this x. It is straightforward to check that the composition $\mathcal{O}_X \to \mathcal{K}_X \to G$ equals α and that $\bar{\alpha}$ is unique with this property.

Up to this point, the proof did not need that X is a scheme – it was enough for X to be a ringed space such that equivalence ($\bowtie$) holds and such that $\neg(0 = 1)$ in $\mathcal{O}_X$. Only now, in showing that $\mathcal{K}_X$ is a $\neg\neg$-sheaf, the scheme condition enters. To this end, we first reformulate the sheaf condition in a way such that it only refers to $\mathcal{O}_X$, not $\mathcal{K}_X$: The quotient ring $\mathcal{K}_X$ is a $\neg\neg$-sheaf if and only if

$$\begin{aligned}\mathrm{Sh}(X) \models \forall T \subseteq \mathcal{O}_X.\ \ulcorner T \text{ is a subsingleton} \urcorner \wedge \neg\neg(\ulcorner T \text{ is inhabited} \urcorner) \Longrightarrow \\ \exists a, b : \mathcal{O}_X.\ \ulcorner b \text{ is regular} \urcorner \wedge \neg\neg(b^{-1} a \in T).\end{aligned}$$

This is done just as in the proof of Theorem 8.3. The expression "b^{-1}" refers to the inverse of b which indeed exists in a doubly negated context, since b is assumed regular. More explicitly, we should write

$$\neg\neg(\exists c : \mathcal{O}_X.\ bc = 1 \wedge ca \in T) \quad \text{instead of} \quad \neg\neg(b^{-1} a \in T).$$

To verify the Kripke–Joyal interpretation of the rewritten sheaf condition, let an affine open subset $U = \operatorname{Spec} A \subseteq X$ having property ($\star$) and a subsheaf $T \hookrightarrow \mathcal{O}_X|_U$ be given such that T is internally a subsingleton and *not not* inhabited. We may glue the unique germs in the inhabited stalks of T to obtain a section $s \in \Gamma(V, \mathcal{O}_X)$ where $V \subseteq U$ is a dense open subset. Since U has property ($\star$), we may assume that $V = D(f)$ is a standard open subset. Because V is dense and A is reduced, the function f is a regular element of A. Since $\Gamma(V, \mathcal{O}_X) = A[f^{-1}]$, we can write $s = a/f^n$ with $a \in A$ and $n \geq 0$.

By Lemma 3.18, the function $b := f^n$ is also regular as an element of $\mathcal{O}_U$ from the internal point of view. The function b is invertible on V, since $V = D(f) = D(b)$. It follows that on the dense open subset $V \subseteq U$, the sections s and $b^{-1}a$ agree. This observation concludes the proof. □

Corollary 9.10. *Let X be a reduced scheme such that any open affine subscheme has property ($\star$). Then $\mathcal{K}_X$ is the result of pulling back $\mathcal{O}_X$ to the sublocale $X_{\neg\neg}$ and then pushing forward again. If X is irreducible with generic point ξ, then $\mathcal{K}_X$ is the constant sheaf associated to the set $\mathcal{O}_{X,\xi}$.*

Proof. Recall from Section 6.4 that pulling back to $X_{\neg\neg}$ is equivalent to sheafifying with respect to the double negation modality; and that pushing forward is equivalent to forgetting the sheaf property. Therefore the first statement holds.

For the second statement, recall from Lemma 6.16 that the sublocale $X_{\neg\neg}$ is given by the subspace $\{\xi\}$; that the sheafification functor $\mathrm{Sh}(X) \to \mathrm{Sh}(\{\xi\}) \simeq \mathrm{Set}$ is given by calculating the stalk at ξ; and that the inclusion functor $\mathrm{Set} \simeq \mathrm{Sh}(\{\xi\}) \hookrightarrow \mathrm{Sh}(X)$ is given by the constant sheaf construction. □

If X is a general scheme (not necessarily reduced), we can describe $\mathcal{K}_X$ in a similar way as a sheafification of $\mathcal{O}_X$; specifically, it is the sheafification with respect to the modal operator defined by

$$\widehat{\Box}\varphi :\equiv \ulcorner \mathcal{O}_X \text{ is } (\varphi \Rightarrow \_\_)\text{-separated}\urcorner$$

in the internal language of $\mathrm{Sh}(X)$, i. e.

$$\widehat{\Box}\varphi :\equiv (\forall s : \mathcal{O}_X.\ (\varphi \Rightarrow s = 0) \Rightarrow s = 0).$$

This modal operator has an explicit scheme-theoretic description.

Lemma 9.11. *Let U be an open subset of a scheme X. Then $\mathrm{Sh}(X) \models \widehat{\Box}U$ if and only if U is scheme-theoretically dense in X.*

Proof. We have the following chain of equivalences.

$$\begin{aligned}
& X \models \widehat{\Box}U \\
\Longleftrightarrow\ & \ulcorner \mathcal{O}_X \text{ is } (U \Rightarrow __)\text{-separated}\urcorner \\
\Longleftrightarrow\ & X \models \ulcorner \mathcal{O}_X \to \mathcal{O}_X^{+} \text{ is injective}\urcorner \\
& \qquad\qquad (\text{where the plus construction is wrt. the modality } (U \Rightarrow __)) \\
\Longleftrightarrow\ & X \models \ulcorner \mathcal{O}_X \to \mathcal{O}_X^{++} \text{ is injective}\urcorner \\
& \qquad\qquad (\text{by the factorization } \mathcal{O}_X \to \mathcal{O}_X^{+} \to \mathcal{O}_X^{++}) \\
\Longleftrightarrow\ & \text{the canonical morphism } \mathcal{O}_X \to j_*\mathcal{O}_U \text{ (with } j : U \hookrightarrow X) \text{ is injective} \\
\Longleftrightarrow\ & U \text{ is scheme-theoretically dense in } X.
\end{aligned}$$
□

Using the internal language of a scheme, talking about scheme-theoretically dense open subsets is therefore just as easy as talking about ordinary topologically dense open subsets; the difference simply amounts to using the modal operator $\widehat{\Box}$ instead of "*not not*".

Proposition 9.12. *Let X be a ringed space. Then:*

(1) *The operator $\widehat{\Box}$ fulfills the axioms for a modal operator.*
(2) *$\mathcal{O}_X$ is $\widehat{\Box}$-separated.*
(3) *$\mathcal{K}_X$ is $\widehat{\Box}$-separated.*
(4) *Internally, it holds that $\widehat{\Box}(\ulcorner f \text{ inv.}\urcorner)$ implies that f is regular for any $f : \mathcal{O}_X$.*

Suppose furthermore that X is a scheme. Then:

(5) *The converse in (4) holds.*
(6) *If every open affine subscheme of X has property $(\star\star)$, then $\mathcal{K}_X$ is the $\widehat{\Box}$-sheafification of $\mathcal{O}_X$.*

Proof. The first four properties are entirely formal; we thus skip over some details. For the first property, we verify the second axiom on a modal operator. So we assume $\widehat{\Box}\widehat{\Box}\varphi$ and have to show $\widehat{\Box}\varphi$. To this end, let $s : \mathcal{O}_X$ be arbitrary such that $\varphi \Rightarrow (s = 0)$; we have to prove that $s = 0$. If $\mathcal{O}_X$ were separated with respect to the modal operator $(\varphi \Rightarrow \_\_)$, it would follow that $s = 0$. So unconditionally it holds that $\widehat{\square}\varphi \Rightarrow (s = 0)$. Since by assumption $\mathcal{O}_X$ is $(\widehat{\square}\varphi \Rightarrow \_\_)$-separated, the claim follows.

For the second property, let $s : \mathcal{O}_X$ be arbitrary such that $\widehat{\square}(s = 0)$. Obviously it holds that $(s = 0) \Rightarrow (s = 0)$. Thus, since $\mathcal{O}_X$ is separated with respect to $((s = 0) \Rightarrow \_\_)$, it follows that $s = 0$. The proof of the third property is similar.

For the fourth property, assume $\widehat{\square}(\ulcorner f \text{ inv.}\urcorner)$ and let $h : \mathcal{O}_X$ be arbitrary such that $fh = 0$. Then, trivially, it holds that $\ulcorner f \text{ inv.}\urcorner \Rightarrow h = 0$. Since $\mathcal{O}_X$ is separated with respect to $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$, it follows that $h = 0$.

We now suppose that X is a scheme. To verify the fifth property, let a regular element $f : \mathcal{O}_X$ be given. We have to show that $\mathcal{O}_X$ is separated with respect to the modality $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$. So assume that $\ulcorner f \text{ inv.}\urcorner \Rightarrow (s = 0)$ for some $s : \mathcal{O}_X$. By Proposition 3.10 it follows that $f^n s = 0$ for some natural number n. Since f is regular, we may conclude that $s = 0$.

The verification of the universal property of $\mathcal{K}_X$ is done analogously as in the case that X is reduced: For the proof of Proposition 9.9, it was critical that regular elements of $\mathcal{O}_X$ are *not not* invertible. We now need (and have) that regular elements of $\mathcal{O}_X$ are $\widehat{\square}(\ulcorner\text{invertible}\urcorner)$.

Thus it only remains to verify that $\mathcal{K}_X$ is a $\widehat{\square}$-sheaf. We may again imitate the proof of Proposition 9.9; using the same notation, we may now suppose that V is a standard open subset such that $U \models \widehat{\square}V$ (previously, we supposed that $U \models \neg\neg V$). The proof that the denominator b is regular (as seen from the internal perspective, as an element of $\mathcal{O}_U$) now goes as follows: We have $V \subseteq D(b)$. Therefore $U \models \widehat{\square}V$ implies $U \models \widehat{\square}(\ulcorner b \text{ inv.}\urcorner)$. By the fourth property, it follows that $U \models \ulcorner b \text{ is regular}\urcorner$. $\square$

Remark 9.13. The modal operator $\widehat{\square}$ is the largest (weakest) operator such that $\mathcal{O}_X$ is $\widehat{\square}$-separated, i. e. if $\square$ is any modal operator such that $\mathcal{O}_X$ is $\square$-separated, then $\square\varphi \Rightarrow \widehat{\square}\varphi$ for any proposition φ.

In the special case that X is a reduced scheme, Proposition 9.12 recovers the result of Proposition 9.9:

Proposition 9.14. *Let X be a scheme. Then* $\mathrm{Sh}(X) \models \forall \varphi : \Omega.\ \widehat{\square}\varphi \Rightarrow \neg\neg\varphi$. *The converse holds if X is reduced, so that in this case the modal operator $\widehat{\square}$ coincides with the double negation modality.*

Proof. We argue internally. Let φ be an arbitrary truth value and assume that $\widehat{\square}\varphi$. The negation $\neg\varphi$ (which is defined as $\varphi \Rightarrow \bot$) is equivalent to $\varphi \Rightarrow (1 = 0)$. Since by assumption $\mathcal{O}_X$ is separated with respect to the $(\varphi \Rightarrow \_\_)$-modality, this in turn is equivalent to $1 = 0 : \mathcal{O}_X$, i. e. to $\bot$. Thus $\neg\neg\varphi$.

For the converse direction, let $\varphi \Rightarrow (s = 0)$ for some $s : \mathcal{O}_X$; we have to show that in fact $s = 0$. Since by assumption $\neg\neg\varphi$, it follows that s is *not not* zero. Since X is reduced, $\mathcal{O}_X$ is $\neg\neg$-separated, so this implies that s is really zero. $\square$

As a corollary, we can reprove the following basic lemma about scheme-theoretical denseness.

Lemma 9.15. *Let U be an open subset of a scheme X. If U is scheme-theoretically dense, then U is also dense in the plain topological sense. The converse holds if X is reduced.*

Proof. The set U is scheme-theoretically dense if and only if $\mathrm{Sh}(X) \models \widehat{\square}U$ and is dense if and only if $\mathrm{Sh}(X) \models \neg\neg U$. Therefore the claim follows from Proposition 9.14. $\square$

Proposition 9.16. *Let X be a scheme such that any open affine subscheme has property $(\star\star)$. Then $\mathcal{K}_X$ is the result of pulling back $\mathcal{O}_X$ to the sublocale $X_{\widehat{\Box}}$ associated to the modal operator $\widehat{\Box}$ and then pushing forward again. If X is locally Noetherian, this sublocale is the subspace of associated points in X.*

In formulas, the proposition states that the canonical map

$$\mathcal{K}_X \longrightarrow i_* i^{-1} \mathcal{O}_X$$

is an isomorphism, where $i : X_{\widehat{\Box}} \hookrightarrow X$ is the inclusion of the sublocale $X_{\widehat{\Box}}$. This result requires a cover with property $(\star\star)$, but no Noetherian hypothesis.

Proof. The first statement follows trivially by the results of Section 6.4 and the fact that $\mathcal{K}_X$ is the $\widehat{\Box}$-sheafification of $\mathcal{O}_X$.

For the second statement, we need to verify that the nucleus $j_{\mathrm{Ass}(\mathcal{O}_X)}$ associated to the subspace of associated points coincides with the nucleus $j_{\widehat{\Box}}$ associated to the modal operator $\widehat{\Box}$. Recall from Subsection 6.3 that the latter is given by

$$\begin{aligned} j_{\widehat{\Box}}(U) &= \text{largest open subset of } X \text{ on which } \widehat{\Box} U \text{ holds} \\ &= \bigcup \{ V \subseteq X \mid V \text{ open},\ V \models \widehat{\Box} U \} \end{aligned}$$

and note that the former is given by

$$j_{\mathrm{Ass}(\mathcal{O}_X)}(U) = \bigcup \{ V \subseteq X \mid V \text{ open},\ V \cap \mathrm{Ass}(\mathcal{O}_X) \subseteq U \}.$$

This is a general fact of locale theory, not depending on particular properties of $\mathrm{Ass}(\mathcal{O}_X)$. To verify this, one needs to check that $j_{\mathrm{Ass}(\mathcal{O}_X)}$ is indeed a nucleus and that the canonical map

$$\{ U \in \mathcal{T}(X) \,|\, j_{\mathrm{Ass}(\mathcal{O}_X)}(U) = U \} \longrightarrow \mathcal{T}(\mathrm{Ass}(\mathcal{O}_X)),\ U \longmapsto \mathrm{Ass}(\mathcal{O}_X) \cap U$$

is an isomorphism of frames with inverse given by $\mathrm{Ass}(\mathcal{O}_X) \cap U \mapsto j_{\mathrm{Ass}(\mathcal{O}_X)}(U)$.

The equivalence thus follows from a standard result on the set of associated points on locally Noetherian schemes:

$$\begin{aligned} & V \cap \mathrm{Ass}(\mathcal{O}_X) \subseteq U \\ \Longleftrightarrow\ & \mathrm{Ass}(\mathcal{O}_V) \subseteq U \\ \Longleftrightarrow\ & U \cap V \text{ is scheme-theoretically dense in } U \\ & \qquad\qquad \text{(this step requires the Noetherian assumption)} \\ \Longleftrightarrow\ & V \models \widehat{\Box} U. \end{aligned}$$

□

Lemma 9.17. *Let X be a scheme such that any open affine subscheme has property $(\star\star)$. Let $j : U \hookrightarrow X$ be the inclusion of an open subset containing the sublocale $X_{\widehat{\Box}}$. (If X is locally Noetherian, this is equivalent to requiring that U contains $\mathrm{Ass}(\mathcal{O}_X)$.) Then the canonical morphism $\mathcal{K}_X \to j_* \mathcal{K}_U$ is an isomorphism.*

Proof. Write $i : X_{\widehat{\Box}} \hookrightarrow X$ and $i' : X_{\widehat{\Box}} \hookrightarrow U$ for the inclusions. By Proposition 9.16, the sheaf $\mathcal{K}_X$ is given by $i_* i^{-1} \mathcal{O}_X$. Similarly, the sheaf $j_* \mathcal{K}_U$ is given by $j_* i'_* i'^{-1} j^{-1} \mathcal{O}_X$. The claim follows since $j \circ i' = i$. □

Lemma 9.18. *Let X be a scheme such that any open affine subscheme has property $(\star)$ respectively $(\star\star)$. Then*

$$\mathrm{Sh}(X) \models \forall \varphi : \Omega.\ \neg\neg\varphi \Longrightarrow \exists f : \mathcal{O}_X.\ \neg\neg(\ulcorner f \text{ inv.} \urcorner) \wedge (\ulcorner f \text{ inv.} \urcorner \Rightarrow \varphi)$$

respectively

$$\mathrm{Sh}(X) \models \forall \varphi : \Omega.\ \widehat{\Box}\varphi \Longrightarrow \exists f : \mathcal{O}_X.\ \widehat{\Box}(\ulcorner f \text{ inv.} \urcorner) \wedge (\ulcorner f \text{ inv.} \urcorner \Rightarrow \varphi).$$

Proof. The proof of Lemma 6.18 carries over, *mutatis mutandis.* □

Proposition 9.19. *Let X be a scheme of dimension ≤ 0 such that the set of irreducible components is locally finite or such that X is locally Noetherian. Then the internal language of* $\mathrm{Sh}(X)$ *is Boolean. (The converse holds as well and was already stated as Corollary 3.15.)*

Proof. It suffices to verify the principle of double negation elimination, since the law of excluded middle is equivalent to it.[16] So let φ be an arbitrary formula and assume $\neg\neg\varphi$. By the previous lemma there exists an element $f : \mathcal{O}_X$ such that f is *not not* invertible and such that ($\ulcorner f$ inv.$\urcorner \Rightarrow \varphi$). Since $\dim X \leq 0$, this element is invertible or nilpotent (Corollary 3.14). In the first case, we are done. In the second case, some power f^n is zero and therefore in particular *not not* zero. Since f is *not not* invertible, this implies that *not not* $1 = 0$. On the other hand $1 \neq 0$, so we obtain a contradiction; from this contradiction φ trivially follows. $\square$

Lemma 9.20. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a torsion module if and only if the restriction of $\mathcal{F}$ to* $\mathrm{Ass}(\mathcal{O}_X)$ *vanishes.*

Proof. By Proposition 9.16 and Lemma 9.18 it suffices to repeat the proof of Lemma 6.47 with "*not not*" substituted by "$\widehat{\Box}$". $\square$

9.5. Cartier divisors. Let X be a scheme (or a ringed space). Recall that a *Cartier divisor* on X is a global section of the sheaf of groups $\mathcal{K}_X^\times/\mathcal{O}_X^\times$. This sheaf can be constructed internally, with the same notation: It is the quotient of the group of invertible elements of the ring $\mathcal{K}_X$ by the subgroup of invertible elements of the ring $\mathcal{O}_X$. So an arbitrary section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ is internally of the form $[s/t]$ with $s, t : \mathcal{O}_X$ being regular elements; this is a simpler description than the usual external one as a family $(f_i)_i$ of functions $f_i \in \Gamma(U_i, \mathcal{K}_X^\times)$ such that $f_i^{-1}|_{U_i \cap U_j} \cdot f_j|_{U_i \cap U_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$ for all i, j.

We can sketch the basic theory of Cartier divisors completely from the internal perspective. In accordance with common practice, we write the group operation of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$ (which is induced by multiplication of elements in $\mathcal{K}_X^\times$) additively.

Definition 9.21. A Cartier divisor is *effective* if and only if, from the internal perspective, it can be written in the form $[s/1]$ with $s : \mathcal{O}_X$ being a regular element.

Thus a Cartier divisor $[s/t]$ is effective if and only if s is an $\mathcal{O}_X$-multiple of t.

Definition 9.22. A Cartier divisor D is *principal* if and only if there exists a global section $f \in \Gamma(X, \mathcal{K}_X^\times)$ such that internally, $D = [f]$. Two Cartier divisors are *linearly equivalent* if and only if their difference is a principal divisor.

Decidedly, principality is a global notion: For *any* divisor D it is true that locally there exists sections f of $\mathcal{K}_X^\times$ such that $D = [f]$.

Definition 9.23. The *line bundle associated to a Cartier divisor D* is the $\mathcal{O}_X$-submodule

$$\mathcal{O}_X(D) := \{g : \mathcal{K}_X \mid gD \in \mathcal{O}_X\} = D^{-1}\mathcal{O}_X \subseteq \mathcal{K}_X$$

of $\mathcal{K}_X$. Here we are abusing language for "$gD \in \mathcal{O}_X$" to mean that $gf \in \mathcal{O}_X$ if $D = [f]$ with $f : \mathcal{K}_X$; and for "$D^{-1}\mathcal{O}_X$" to mean $f^{-1}\mathcal{O}_X$. This condition respectively submodule does not depend on the representative f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^\times$.

[16] This is a standard fact of intuitionistic logic. Assume that the principle of double negation elimination holds. We want to verify the law of excluded middle, so let an arbitrary formula φ be given. Even intuitionistically it holds that $\neg\neg(\varphi \vee \neg\varphi)$. By double negation elimination it follows that $\varphi \vee \neg\varphi$.

The submodule $\mathcal{O}_X(D)$ is indeed locally free of rank 1, since internally f^{-1} gives a one-element basis. The divisor D is effective if and only if $\mathcal{O}_X(-D)$ is a subset of $\mathcal{O}_X$ from the internal perspective (this comparison makes sense, since $\mathcal{O}_X(-D)$ and $\mathcal{O}_X$ are both canonically embedded in $\mathcal{K}_X$). In this case, we can define the *support* of D to be the closed subscheme of X associated to the sheaf of ideals $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$.

The line bundle $\mathcal{O}_X(D)$ can also be written in the familiar form

$$\mathcal{O}_X(D) = \{g : \mathcal{K}_X \mid \operatorname{div}(g) + D \geq 0\},$$

if we define "$\operatorname{div}(g)$" as the equivalence class $[g] : \mathcal{K}_X/\mathcal{O}_X^\times$, interpret the left-hand side of the inequality as an element of $\mathcal{K}_X/\mathcal{O}_X^\times$, and declare that $[s/t] \geq 0$ if and only if s is an $\mathcal{O}_X$-multiple of t.

On the other hand, a description like

$$\text{“}\mathcal{O}_X(D) = \{0\} \cup \{g : \mathcal{K}_X^\times \mid \operatorname{div}(g) + D \geq 0\}\text{”}$$

is not possible, since the case distinction necessary for a verification of the inclusion "$\subseteq$" is not possible intuitionistically.

Definition 9.24. The *Cartier divisor associated to a free $\mathcal{O}_X$-submodule $\mathcal{L} \subseteq \mathcal{K}_X$ of rank 1* is $D := [f^{-1}]$, where $f : \mathcal{K}_X$ is the unique element of some one-element basis of $\mathcal{L}$.

The basis element $f : \mathcal{K}_X$ does indeed lie in $\mathcal{K}_X^\times$: Write $f = s/t$ with $s, t : \mathcal{O}_X$. It suffices to show that s is a regular element of $\mathcal{O}_X$. So let $h : \mathcal{O}_X$ such that $sh = 0$ in $\mathcal{O}_X$. Then in particular $hf = 0$ in $\mathcal{K}_X$. By linear independence, it follows that $h = 0$ in $\mathcal{K}_X$ and thus $h = 0$ in $\mathcal{O}_X$.

Furthermore, the associated divisor does not depend on the choice of f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^\times$: If $f\mathcal{O}_X = g\mathcal{O}_X \subseteq \mathcal{K}_X$, then there exist elements $u, v : \mathcal{O}_X$ such that $fu = g$ and $gv = f$ in $\mathcal{K}_X$. It follows that $uv = fuvf^{-1} = gvf^{-1} = ff^{-1} = 1$ in $\mathcal{K}_X$ and thus in $\mathcal{O}_X$, by injectivity of the localization morphism $\mathcal{O}_X \to \mathcal{K}_X$. Therefore u and v are elements of $\mathcal{O}_X^\times$.

Lemma 9.25. *Let D and D' be divisors on X. Then $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \cong \mathcal{O}_X(D + D')$.*

Proof. The wanted morphism of sheaves $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D') \to \mathcal{O}_X(D + D')$ is given by multiplication. That this is well-defined and an isomorphism can be checked from the internal point of view, where the claims are obvious. □

Proposition 9.26. *The association $D \mapsto \mathcal{O}_X(D)$ defines a one-to-one correspondence between Cartier divisors on X and rank-one submodules of $\mathcal{K}_X$. This correspondence descends to a one-to-one correspondence between Cartier divisors up to linear equivalence and rank-one submodules of $\mathcal{K}_X$ up to isomorphism (as abstract $\mathcal{O}_X$-modules, ignoring their embedding into $\mathcal{K}_X$).*

Proof. The first statement is obvious from the definitions. For the second statement, it suffices to show that $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$ if and only if D is principal. An isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ gives a global section $f \in \Gamma(X, \mathcal{K}_X^\times)$ (by considering the image of the unit element) such that internally, $D = [f^{-1}]$; this shows that D is principal. The converse is similar. □

For the following definition, recall that we can localize an $\mathcal{O}_X$-module $\mathcal{L}$ at the set $\mathcal{S} \subseteq \mathcal{O}_X$ of regular elements to obtain a $\mathcal{K}_X$-module $\mathcal{L}[S^{-1}]$.

Definition 9.27. Let $f : \mathcal{L}[\mathcal{S}^{-1}]$ be a rational section of a line bundle $\mathcal{L}$ on X. Assume that "f is nontrivial", that is multiplication by f is an injective map $\mathcal{O}_X \to \mathcal{L}[\mathcal{S}^{-1}]$. Then the *associated divisor* of f is $\operatorname{div}(f) := [\psi(s)/t]$, where $f = s/t$ with $s : \mathcal{L}$ and $t : \mathcal{O}_X$ and $\psi : \mathcal{L} \to \mathcal{O}_X$ is an isomorphism.

One can check that $\psi(s)$ is a regular element of $\mathcal{O}_X$; this statement is equivalent to the multiplication map $\mathcal{O}_X \to \mathcal{L}[\mathcal{S}^{-1}]$ being injective. Furthermore one can check that $[\psi(s)/t]$ does not depend on the choice of s, t, and ψ.

Proposition 9.28. *Let $f : \mathcal{L}[\mathcal{S}^{-1}]$ be a nontrivial rational section of a line bundle $\mathcal{L}$ on X. Then multiplication by f induces an isomorphism $\mathcal{O}_X(\operatorname{div}(f)) \to \mathcal{L}$.*

Proof. The isomorphism should map a rational function g to gf. This is a priori an element of $\mathcal{L}[\mathcal{S}^{-1}]$; we have to check that it can be regarded as an element of $\mathcal{L}$. Just as in the definition of $\operatorname{div}(f)$, write $f = s/t$ and fix an isomorphism $\psi : \mathcal{L} \to \mathcal{O}_X$. Write $g = (t/\psi(s)) \cdot h$ for some function $h : \mathcal{O}_X$. Then $gf = sh/\psi(s) = h\psi^{-1}(1)$, since $s = \psi^{-1}(\psi(s)) = \psi(s) \cdot \psi^{-1}(1)$. The element $h\psi^{-1}(1)$ can indeed be considered as an element of $\mathcal{L}$.

Injectivity of the map $\mathcal{O}_X(\operatorname{div}(f)) \to \mathcal{L}$ is by nontriviality of f. For surjectivity, we observe that $(t/\psi(s)) \cdot \psi(v)$ is a preimage to $v : \mathcal{L}$, since $(t/\psi(s)) \cdot \psi(v) \cdot f = \psi(v)\psi(s)\psi^{-1}(1)/\psi(s) = v$. □

Proposition 9.29. *Let $\mathcal{L}$ be a line bundle on X. Assume that $\mathcal{L}$ can be embedded into $\mathcal{K}_X$. Then $\mathcal{L}$ possesses a nontrivial rational section.*

Proof. Let $i : \mathcal{L} \to \mathcal{K}_X$ be the given injection. Let (v) be an one-element basis for $\mathcal{L}$. Write $i(v) = s/t$. Then s is regular, since $hs = 0$ implies $i(hv) = 0$ and thus $h = 0$, for any $h : \mathcal{O}_X$. Therefore $f := tv/s$ is a well-defined element of $\mathcal{L}[\mathcal{S}^{-1}]$. Furthermore it is nontrivial in the desired sense: If $h \cdot (tv/s) = 0$, then $htv = 0$, thus $ht = 0$ and $h = 0$.

It remains to check that f is independent of the choice of v and of the representation $i(v) = s/t$; else we defined only local sections which might not glue to a single nontrivial rational section (externally speaking). This verification is trivial. □

Proposition 9.30. *Let D be an effective divisor on X. Then the complement of its support is scheme-theoretically dense.*

Proof. The complement of the support of D, that is the open subset $D(\mathcal{O}_X(-D))$ (where we consider $\mathcal{O}_X(-D)$ as an ideal of $\mathcal{O}_X$), is the truth value of the statement "$1 \in \mathcal{O}_X(-D)$". By Lemma 9.11, we therefore have to verify that $\mathcal{O}_X$ is separated with respect to the modal operator $(1 \in \mathcal{O}_X(-D) \Rightarrow \_\_)$.

Let $s : \mathcal{O}_X$ be given such that $1 \in \mathcal{O}_X(-D) \Rightarrow s = 0$; we have to show that $s = 0$. Writing $D = [f/1]$ where $f : \mathcal{O}_X$ is a regular element, this condition is equivalent to $\ulcorner f \text{ inv.} \urcorner \Rightarrow s = 0$. By Proposition 3.10 it follows that $f^n s = 0$ for some $n \geq 0$. Since f is regular, we may cancel f^n in this equation. □

Proposition 9.31. *Assume that X is an integral scheme. Then any line bundle on X is (uncanonically) a submodule of $\mathcal{K}_X$.*

Proof. Let ξ be the generic point of X and let $\Box := \neg\neg$ denote the modal operator such that internal sheafification with respect to $\Box$ is the same as pulling back to $\{\xi\}$ and then pushing forward to X again (see Section 6.5). Let $\mathcal{L}$ be a line bundle on X. Since $\mathcal{L}_\xi \cong \mathcal{O}_{X,\xi}$ (uncanonically), there is some injection $\mathcal{L}_\xi \to \mathcal{K}_{X,\xi}$; this corresponds internally to an injection $\mathcal{L}^{++} \to \mathcal{K}_X^{++}$. Since $\mathcal{K}_X$ is already a $\Box$-sheaf (Proposition 9.9) and $\mathcal{L}$ is $\Box$-separated (being isomorphic to $\mathcal{O}_X$), we have the global injection

$$\mathcal{L} \hookrightarrow \mathcal{L}^{++} \hookrightarrow \mathcal{K}_X^{++} \xrightarrow{(\cong)^{-1}} \mathcal{K}_X. \qquad \Box$$

10. Subschemes

10.1. Sheaves on open and closed subspaces. It is well-known that sheaves defined on open or closed subspaces of a topological space X can be related with certain sheaves on X, by using appropriate extension functors. We can define these functors and show their basic properties in the internal language. Recall from Section 6.2 that we have defined a formula "U" for any open subset $U \subseteq X$ such that $V \models U$ if and only if $V \subseteq U$.

Lemma 10.1. *Let X be a topological space. Let $j : U \hookrightarrow X$ be the inclusion of an open subspace. Then there is a canonical functor $j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ called* extension by the empty set *with the following properties:*

(1) *The functor $j_!$ is left adjoint to the restriction functor $j^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$.*
(2) *The composition $j^{-1} \circ j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(U)$ is (canonically isomorphic to) the identity.*
(3) *The essential image of $j_!$ consists of exactly those sheaves on X whose stalks are empty at all points of U^c. For those sheaves $\mathcal{F}$ it holds that $j_! j^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Internally, for a set $\mathcal{F}$, we can define $j_!(\mathcal{F})$ simply to be the set comprehension

$$j_!(\mathcal{F}) := \{x : \mathcal{F} \mid U\}.$$

Externally, the sections of the thus defined sheaf on an open subset $V \subseteq X$ are given by $\{x \in \Gamma(V, \mathcal{F}) \mid V \subseteq U\}$, i.e. all of $\Gamma(V, \mathcal{F})$ if $V \subseteq U$ and the empty set otherwise. With this short internal description, all of the stated properties can be easily verified in the internal language.

For instance, recall that internally the functor j^{-1} is given by sheafifying with respect to the modal operator $\Box :\equiv (U \Rightarrow \_\_)$. Thus, to show the second statement, we have to give a bijection $(j_!(\mathcal{F}))^{++} \to \mathcal{F}$ for any $\Box$-sheaf $\mathcal{F}$. (This map has to be given explicitly, to not only show a weaker statement about a local isomorphism – see Section 2.2). To this end, we can use the composition

$$(j_!(\mathcal{F}))^{++} \hookrightarrow \mathcal{F}^{++} \xrightarrow{(\cong)^{-1}} \mathcal{F},$$

where the first map is injective since sheafifying is exact. It is also surjective, since the $\Box$-translation of the statement $\ulcorner j_!(\mathcal{F}) \to \mathcal{F}$ is surjective$\urcorner$ holds: For any element $x : \mathcal{F}$, it holds that $\Box(\ulcorner x$ possesses a preimage$\urcorner)$.

For the third property, we observe that a sheaf $\mathcal{F}$ on X fulfills the stated condition on stalks if and only if, from the internal perspective, it holds that $U \Rightarrow \ulcorner \mathcal{F}$ is inhabited$\urcorner$. We omit further details. $\Box$

Lemma 10.2. *Let X be a ringed space. Let $j : U \hookrightarrow X$ be the inclusion of an open subspace. Then there is a canonical functor $j_! : \mathrm{Mod}_U(\mathcal{O}_U) \to \mathrm{Mod}_X(\mathcal{O}_X)$ called* extension by zero *with the following properties:*

(1) *The functor $j_!$ is left adjoint to the restriction functor $j^{-1} : \mathrm{Mod}_X(\mathcal{O}_X) \to \mathrm{Mod}_U(\mathcal{O}_U)$.*
(2) *The composition $j^{-1} \circ j_! : \mathrm{Mod}_U(\mathcal{O}_U) \to \mathrm{Mod}_U(\mathcal{O}_U)$ is (canonically isomorphic to) the identity.*
(3) *The essential image of $j_!$ consists of exactly those $\mathcal{O}_X$-modules whose stalks are zero at all points of U^c. For those sheaves $\mathcal{F}$ it holds that $j_! j^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Internally, a sheaf of modules on $\mathcal{O}_U$ is simply a module on $\mathcal{O}_X^{++}$ which is a $\Box$-sheaf, where $\Box :\equiv (U \Rightarrow \_\_)$. The suitable internal definition for the extension by zero of such a module $\mathcal{F}$ is

$$j_!(\mathcal{F}) := \{x : \mathcal{F} \mid (x = 0) \vee U\}.$$

With this description, all necessary verifications are easy. Note that an $\mathcal{O}_X$-module $\mathcal{F}$ fulfills the stated condition on stalks if and only if internally, it holds that $\forall x : \mathcal{F}.\ ((x = 0) \vee U)$. $\Box$

Lemma 10.3. *Let X be a topological space. Let $i : A \hookrightarrow X$ be the inclusion of a closed subspace. The essential image of the inclusion $i_* : \mathrm{Sh}(A) \to \mathrm{Sh}(X)$ consists of exactly those sheaves whose support is a subset of A. For those sheaves $\mathcal{F}$ it holds that $i_* i^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Recall that the modal operator associated to A is $\Box\varphi :\equiv (\varphi \vee A^c)$, and that by Section 6.4 the essential image of i_* consists of exactly those sheaves which are $\Box$-sheaves from the internal perspective. Let $\mathcal{F}$ be a sheaf on X. Then it holds that

$$\operatorname{supp}\mathcal{F} \subseteq A \quad\Longleftrightarrow\quad A^c \subseteq X \setminus \operatorname{supp}\mathcal{F} \quad\Longleftrightarrow\quad A^c \subseteq \operatorname{int}(X \setminus \operatorname{supp}\mathcal{F}).$$

Since the interior of the complement of $\operatorname{supp}\mathcal{F}$ can be characterized as the largest open subset of X on which the internal statement "$\mathcal{F}$ is a singleton" holds (Remark 4.11), the condition on the support is fulfilled if and only if

$$\mathrm{Sh}(X) \models (A^c \Rightarrow \ulcorner\mathcal{F} \text{ is a singleton}\urcorner).$$

We thus have to show that this internal condition is equivalent to $\mathcal{F}$ being a $\Box$-sheaf. For the "if" direction, assume A^c. Then the empty subset $S \subseteq \mathcal{F}$ trivially verifies the condition that $\Box(\ulcorner S \text{ is a singleton}\urcorner)$. There thus exists an element $x : \mathcal{F}$ (such that $\Box(x \in S)$). If we're given a further element $y : \mathcal{F}$, it trivially holds that $\Box(x = y)$. By $\Box$-separatedness, it thus follows that $x = y$. Thus $\mathcal{F}$ is the singleton $\{x\}$. The proof of the "only if" direction is similar.

The second statement claims that internally, sheafifying a $\Box$-sheaf with respect to the modal operator $\Box$ and then forgetting that the result is a $\Box$-sheaf amounts to doing nothing. This is obvious. $\Box$

10.2. Closed subschemes. Let X be a ringed space. Recall that a sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ defines a closed subset $V(\mathcal{I}) = \{x \in X \mid \mathcal{I}_x \neq (1) \subseteq \mathcal{O}_{X,x}\}$, a sheaf of rings $\mathcal{O}_X/\mathcal{I}$, and a ringed space $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ where $\mathcal{O}_{V(\mathcal{I})}$ is the pullback of $\mathcal{O}_X/\mathcal{I}$ to $V(\mathcal{I})$. In the internal universe, we can reify $V(\mathcal{I})$ by giving a modal operator $\Box$ such that externally, the subspace $X_\Box$ coincides with $V(\mathcal{I})$.

Proposition 10.4. *Let X be a ringed space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals. Then:*

1. *The subspace of X associated to the modal operator $\Box$ defined by $\Box\varphi :\equiv (\varphi \vee (1 \in \mathcal{I}))$ is $V(\mathcal{I})$.*
2. *The support of $\mathcal{O}_X/\mathcal{I}$ is exactly $V(\mathcal{I})$.*
3. *The canonical morphism $i : V(\mathcal{I}) \to X$ is a closed immersion of ringed spaces.*

Proof. For any open subset $U \subseteq X$, it holds that $U \models 1 \in \mathcal{I}$ if and only if $U \subseteq D(\mathcal{I}) = X \setminus V(\mathcal{I})$. Thus $D(\mathcal{I})$ can be characterized as the largest open subset on which "$1 \in \mathcal{I}$" holds. According to Table 2 on page 55, the stated modal operator thus defines the subspace $D(\mathcal{I})^c$, i. e. $V(\mathcal{I})$.

For the second statement, we observe that since $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings, its support is closed. Therefore the largest open subset of X where the internal statement "$\mathcal{O}_X/\mathcal{I} = 0$" holds is the complement of the support (Proposition 4.10). Since $D(\mathcal{I})$ is the largest open subset where the internal statement "$\mathcal{I} = (1)$" holds, it suffices to show that internally, $\mathcal{O}_X/\mathcal{I} = 0$ if and only if $\mathcal{I} = (1)$. This is obvious.

The topological part of the third statement is clear. For the ring-theoretic part, we have to show that the canonical ring homomorphism $\mathcal{O}_X \to i_*\mathcal{O}_{V(\mathcal{I})}$, that is the canonical projection $\mathcal{O}_X \to \mathcal{O}_X/(\mathcal{I})$, is an epimorphism of sheaves. This is obvious. $\Box$

By Lemma 10.3, the sheaf $\mathcal{O}_X/\mathcal{I}$ is thus a $\Box$-sheaf from the internal perspective.

Proposition 10.5. *Let X be a locally ringed space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals. Then the ringed space $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ is locally ringed as well.*

Proof. We have to show that

$$\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is a local ring} \urcorner.$$

By Theorem 6.31, this is equivalent to

$$\mathrm{Sh}(X) \models (\ulcorner \mathcal{O}_X/\mathcal{I} \text{ is a local ring} \urcorner)^{\Box},$$

where $\Box$ is the modal operator given by $\Box\varphi :\equiv (\varphi \vee (1 \in \mathcal{I}))$. We therefore have to give an intuitionistic proof of the fact

$$\forall x, y : \mathcal{O}_X/\mathcal{I}.\ \ulcorner x + y \text{ inv.} \urcorner \Longrightarrow \Box(\ulcorner x \text{ inv.} \urcorner \vee \ulcorner y \text{ inv.} \urcorner).$$

So let $x = [s], y = [t] : \mathcal{O}_X/\mathcal{I}$ such that $x + y$ is invertible in $\mathcal{O}_X/\mathcal{I}$. This means that there exists $u : \mathcal{O}_X$ and $v : \mathcal{I}$ such that $us + ut + v = 1$ in $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a local ring, it follows that us, ut, or v is invertible. In the first two cases, it follows that x respectively y are invertible in $\mathcal{O}_X/\mathcal{I}$. In the third case, it follows that $1 \in \mathcal{I}$ and thus any boxed statement is trivially true. $\Box$

If X is a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ is a sheaf of ideals, it is well-known that the locally ringed space $V(\mathcal{I})$ is a scheme if and only if $\mathcal{I}$ is quasicoherent. We cannot give an internal proof of this fact since we lack an internal characterization of being a scheme.

Lemma 10.6. *Let X be a scheme (or a ringed space). Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals. The ringed space $V(\mathcal{I})$ is reduced if and only if, from the internal perspective of* $\mathrm{Sh}(X)$*, the ideal $\mathcal{I}$ is a radical ideal.*

Proof. The following chain of equivalences holds:

$$\begin{aligned}
& \mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is a reduced ring} \urcorner \\
\Longleftrightarrow\ & \mathrm{Sh}(V(\mathcal{I})) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_{V(\mathcal{I})}.\ s^n = 0 \Longrightarrow s = 0 \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \Big(\bigwedge_{n \geq 0} \forall s : \mathcal{O}_X/\mathcal{I}.\ s^n = 0 \Rightarrow s = 0\Big)^{\Box} \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X/\mathcal{I}.\ s^n = 0 \Rightarrow \Box(s = 0) \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X.\ s^n \in \mathcal{I} \Rightarrow \Box(s \in \mathcal{I}) \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X.\ s^n \in \mathcal{I} \Rightarrow s \in \mathcal{I} \\
\Longleftrightarrow\ & \mathrm{Sh}(X) \models \ulcorner \mathcal{I} \text{ is a radical ideal} \urcorner
\end{aligned}$$

In the second-to-last step, we used that $\Box(s \in \mathcal{I}) \equiv ((s \in \mathcal{I}) \vee (1 \in \mathcal{I}))$ implies $s \in \mathcal{I}$. This is trivial in both cases of the disjunction. $\Box$

Lemma 10.7. *Let X be a scheme (or a ringed space).*

(1) *There exists a reduced closed sub-ringed space $X_{\mathrm{red}} \hookrightarrow X$ having the same underlying topological space as X with the following universal property: Any morphism $Y \to X$ of (ringed or locally ringed) spaces such that Y is reduced factors uniquely over the closed immersion $X_{\mathrm{red}} \hookrightarrow X$.*
(2) *Let $A \subseteq X$ be a closed subset. Then there exists a structure of a reduced closed ringed subspace on A with a similar universal property.*

Proof. For the first statement, let $\mathcal{N} \subseteq \mathcal{O}_X$ be the nilradical of $\mathcal{O}_X$. This can internally be simply defined by $\mathcal{N} := \sqrt{(0)} = \{s : \mathcal{O}_X \mid \bigvee_{n \geq 0} s^n = 0\}$. Define X_{red} as the closed subspace associated to this sheaf of ideals. This ringed space is reduced by the previous lemma. If X is a scheme, then quasicoherence of $\mathcal{N}$ (which is necessary and sufficient for X_{red} to be a scheme) can be shown internally (Example 8.7). The proof of the universal property can also be done in the internal language, by using the basic fact of locale theory that the category of locales over X is equivalent to internal locales in $\mathrm{Sh}(X)$; but we do not want to discuss this further.

For the second statement, internally define the ideal $\mathcal{I} := \{s : \mathcal{O}_X \mid \ulcorner s \text{ inv.}\urcorner \Rightarrow A^c\} \subseteq \mathcal{O}_X$. Then $1 \in \mathcal{I}$ if and only if A^c, thus by Proposition 10.4 the closed ringed subspace defined by $\mathcal{I}$ has A as underlying topological space. It is reduced since $\mathcal{I}$ is a radical ideal. $\square$

Remark 10.8. By Proposition 8.11, the ideal $\mathcal{I}$ defined in the proof of Lemma 10.7 is internally quasicoherent. Therefore the closed ringed subspace defined by $\mathcal{I}$ is a scheme if X is.

Lemma 10.9. *Let X be a scheme of dimension $\leq n$. Let $V(\mathcal{I}) \hookrightarrow X$ be a closed subscheme which is locally cut out by a regular equation. Then* $\dim V(\mathcal{I}) \leq n - 1$.

Proof. By Proposition 3.13, it suffices to give an intuitionistic proof of the following fact of dimension theory: Let A be an arbitrary ring of dimension $\leq n$. Let $I = (s) \subseteq A$ be an ideal which is generated by a regular element $s : A$. Then the $\Box$-translation of "A/I is of dimension $\leq n - 1$" holds. In fact, we can show that A/I really is of dimension $\leq n - 1$; since no implication signs occur in a formal rendering of "being of dimension $\leq n - 1$", Lemma 6.25 is applicable and implies that this a stronger statement.

For this, let a sequence $([a_0], \ldots, [a_{n-1}])$ of elements in A/I be given. We can lift and extend this sequence to the sequence $(a_0, \ldots, a_{n-1}, s)$ of elements of A. Since $\dim A \leq n$, there exists a complementary sequence $(b_0, \ldots, b_{n-1}, b_n)$. Since s is regular, the inclusion $\sqrt{(sb_n)} \subseteq \sqrt{(0)}$ given by the definition of complementarity implies that b_n is nilpotent. Thus we have that $\sqrt{(a_{n-1}b_{n-1})} \subseteq \sqrt{(s, b_n)} = \sqrt{(s)}$ in A, which translates to $\sqrt{([a_{n-1}][b_{n-1}])} \subseteq \sqrt{(0)}$ in A/I. Therefore $([b_0], \ldots, [b_{n-1}])$ is a complementary sequence to $([a_0], \ldots, [a_{n-1}])$ in A/I. $\square$

Lemma 10.10. *Let X be a scheme. Let $\mathcal{I}$ be a sheaf of $\mathcal{O}_X$-modules. Then:*

$$\dim V(\mathcal{I}) \leq n \quad \Longleftrightarrow \quad \mathrm{Sh}(X) \models \ulcorner \mathcal{O}_X/\mathcal{I} \text{ is of Krull dimension} \leq n \urcorner.$$

Proof. By Proposition 3.13, the condition $\dim V(\mathcal{I}) \leq n$ is equivalent to

$$\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is of Krull dimension} \leq n \urcorner.$$

By Theorem 6.31 this is equivalent to

$$\mathrm{Sh}(X) \models (\ulcorner \mathcal{O}_X/\mathcal{I} \text{ is of Krull dimension} \leq n \urcorner)^{\Box},$$

where $\Box$ is the modal operator given by $\Box\varphi := (\varphi \vee (1 \in \mathcal{I}))$. The claimed equivalence then follows by Lemma 6.25 (for "$\Leftarrow$") and by direct inspection similar to the proof of Lemma 6.44 (for "$\Rightarrow$"). $\square$

11. Transfer principles

Let M be an A-module. A natural question is how properties of M relate to properties of the induced quasicoherent sheaf $M^\sim$ on $\operatorname{Spec} A$. For instance it is well-known that

- M is finitely generated iff $M^\sim$ is of finite type,
- M is flat over A iff $M^\sim$ is flat over $\mathcal{O}_{\operatorname{Spec} A}$, and

- M is torsion iff $M^\sim$ is a torsion sheaf.

Using the internal language of the little Zariski topos of $\operatorname{Spec} A$, we can give a simple, conceptual, and uniform explanation of these equivalences. Namely, from the internal point of view, the module $M^\sim$ is obtained from the constant sheaf $\underline{M}$ by localizing at the *generic filter*, a particular multiplicative subset to be introduced below, and the set M and the sheaf $\underline{M}$ share the same properties (by Lemma 11.1 below).

This makes it obvious that, for instance, properties which are stable under localization pass from M to $M^\sim$.

11.1. Internal properties of constant sheaves.

Lemma 11.1. *Let φ be a formula in which arbitrary sets and elements may occur as parameters. Let X be a topological space and let $U \subseteq X$ be an open subset. Then*

$$U \models \varphi \quad \textit{iff} \quad (U \textit{ inhabited} \Rightarrow \varphi).$$

We are abusing notation on the left hand side: The parameters of φ, which are sets and elements, must be read as the induced constant sheaves and constant functions (sections of that sheaves). Unbounded quantifiers have to be read as ranging only over locally constant sheaves, not all sheaves.

Proof. By induction on the structure of φ. By way of example, we give the argument in the case that $\varphi \equiv (a = b)$, where a and b are elements of some set M. Then $U \models \varphi$ means by definition that the constant functions $U \to M$ with value a respectively b coincide. This is equivalent to saying that a and b coincide if U is inhabited. □

The lemma in particular implies that constant sheaves enjoy several classical properties from the internal point of view (if they are present in the metatheory), even though the internal language only supports intuitionistic reasoning in general. For instance, for a constant sheaf $\underline{M}$ it holds that

$$\mathrm{Sh}(X) \models \forall x, y : \underline{M}.\ \neg\neg(x = y) \Rightarrow x = y$$

and even

$$\mathrm{Sh}(X) \models \forall x, y : \underline{M}.\ x = y \vee x \neq y.$$

Remark 11.2. Lemma 11.1 is also valid for locales instead of topological spaces. If one works in an intuitionistic metatheory, one has to add the additional requirement that the locale is *overt*; classically, every locale is overt, and intuitionistically, at least locales arising from topological spaces are overt. We'll revisit this subtle point in Section 12.9, where we sketch how scheme theory can be developed in an intuitionistic context, and more specifically in Lemma 12.48.

11.2. The generic filter.

Let A be a ring.

Definition 11.3. A *filter* of A is a subset $F \subseteq A$ such that

- $0 \notin F$,
- $x + y \in F \Longrightarrow (x \in F) \vee (y \in F)$, and
- $1 \in F$,
- $xy \in F \Longleftrightarrow (x \in F) \wedge (y \in F)$

for all $x, y : A$.

In classical logic, the complement of a prime ideal is a filter and furthermore every filter is of such a form. In constructive mathematics however, it is useful to axiomatize complements of prime ideals directly, avoiding negations. Intuitionistically, since De Morgan's law $\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$ is not available, one can neither show that the complement of a prime ideal is a filter nor that the complement of a filter is a prime ideal.

A filter is in particular a multiplicative subset. Inverting the elements of a filter results in a local ring, while intuitionistically the localization of a ring at a prime ideal cannot in general be verified to be local.

Definition 11.4. The *generic filter* $\mathcal{F}$ is the subsheaf of $\underline{A}$ on $\operatorname{Spec} A$ given by

$$\Gamma(U, \mathcal{F}) := \{f : U \to A \mid f(\mathfrak{p}) \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in U\}.$$

Proposition 11.5.

(1) *Let $f \in A$ and $x \in A$. Then $D(f) \models x \in \mathcal{F}$ if and only if $f \in \sqrt{(x)}$.*
(2) *The stalk $\mathcal{F}_\mathfrak{p}$ at a point $\mathfrak{p} \in \operatorname{Spec} A$ is in canonical bijection with $A \setminus \mathfrak{p}$.*
(3) *From the internal point of view of* $\mathrm{Sh}(\operatorname{Spec} A)$*, the generic filter is indeed a filter of $\underline{A}$.*

Proof. By definition $D(f) \models x \in \mathcal{F}$ means that $x \notin \mathfrak{p}$ for all prime ideals $\mathfrak{p}$ with $f \notin \mathfrak{p}$. This is well-known to be equivalent to $f \in \sqrt{(x)}$.

For the claim about stalks, we observe that the canonical map $\mathcal{F}_\mathfrak{p} \to A \setminus \mathfrak{p}$ sending a germ $[f]$ to $f(\mathfrak{p})$ is invertible with inverse being the map which sends an element $x \notin \mathfrak{p}$ to the germ of the constant function with value x (defined on $D(x)$).

Regarding the third statement we only verify the axiom regarding sums, the other verifications being easier. Interpreting this axiom with the Kripke–Joyal semantics and restricting without loss of generality to open subsets where given locally constant functions are constant, let elements $x, y \in A$ be given such that $D(f) \models x + y \in \mathcal{F}$. By the first statement $f \in \sqrt{(x+y)}$. Therefore $D(f) \subseteq D(x) \cup D(y)$, and on $D(x)$ it holds that $x \in \mathcal{F}$ and on $D(y)$ it holds that $y \in \mathcal{F}$. □

The significance of the generic filter is given by the following proposition.

Proposition 11.6. *From the internal point of view of* $\mathrm{Sh}(\operatorname{Spec} A)$,

(1) *the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is the localization of the constant sheaf $\underline{A}$ at the generic filter: $\mathcal{O}_{\operatorname{Spec} A} = \underline{A}[\mathcal{F}^{-1}]$, and*
(2) *the quasicoherent sheaf of modules $M^\sim$ associated to an A-module M is the localization of the constant sheaf $\underline{M}$ at the generic filter.*

Proof. Ignoring the ring respectively module structure, the second statement is more general; therefore we prove this one. We didn't discuss the case of quotients in Section 2.2. However it should be perspicuous that the interpretation of $\underline{M}[\mathcal{F}^{-1}]$ is defined as the colimit of $\mathcal{E} \twoheadrightarrow \underline{M} \times \mathcal{F}$, taken in the category of sheaves on $\operatorname{Spec} A$, where $\mathcal{E}$ is the subsheaf of $\mathcal{F} \times (\underline{M} \times \mathcal{F}) \times (\underline{M} \times \mathcal{F})$ given by $\mathcal{E}(U) := \{(s, (x, t), (y, u)) \mid sux = sty\}$.

This colimit can be obtained as the sheafification of the similarly defined presheaf colimit $\mathcal{E}' \twoheadrightarrow \underline{M}_{\mathrm{pre}} \times \mathcal{F}$, where $\underline{M}_{\mathrm{pre}}$ is the constant *presheaf* associated to M. On an open subset U this presheaf colimit is simply the localization $\Gamma(U, \underline{M}_{\mathrm{pre}})[\Gamma(U, \mathcal{F})^{-1}] = M[\Gamma(U, \mathcal{F})^{-1}]$. In the special case that $U = D(f)$ is a standard open subset, Proposition 11.5(a) shows that this module is canonically isomorphic to $M[f^{-1}]$. The quasicoherent sheaf $M^\sim$ of modules admits the same description. □

Recognizing $\mathcal{O}_{\operatorname{Spec} A}$ as a localization of $\underline{A}$ fits nicely into the following abstract algebraic motivation for schemes: Does the ring A admit a *universal localization*, i. e. a homomorphism $A \to A'$ into a local ring such that every homomorphism $A \to B$ into a local ring factors via a local map over $A \to A'$? Intuitively speaking, can we localize a ring at all prime ideals at once, or equivalently at all filters at once? The answer is *no* in general,[17] but always *yes* if we are willing to change the topos

[17]Assume that the universal localization A' of a ring A exists as an ordinary ring in Set. Then any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A are equal: Let $s \notin \mathfrak{p}$. Since s is invertible in the local ring $A_\mathfrak{p}$

in which we look for a solution: The universal localization of A is given by the ring $\mathcal{O}_{\operatorname{Spec} A}$ in the topos $\mathrm{Sh}(\operatorname{Spec} A)$; this ring is constructed by localizing $\underline{A}$ at the generic filter, a filter which exists in $\mathrm{Sh}(\operatorname{Spec} A)$ but not in Set.

We expand on this point of view in Section 12 on the relative spectrum.

For transferring properties of $M^\sim$ to M, the following metatheorem is crucial.

Proposition 11.7. *Let $\mathcal{I}$ be an ideal in $\underline{A}$ such that, for all inhabited open subsets $U \subseteq \operatorname{Spec} A$ and elements $x \in A$, the set $\Gamma(X, \mathcal{I})$ contains the constant function with value x if $\Gamma(U, \mathcal{I})$ does. Then*

$$D(f) \models \ulcorner \mathcal{I} \cap \mathcal{F} \text{ is inhabited} \urcorner \quad \textit{implies} \quad \textit{for some } n \geq 0,\ D(f) \models f^n \in \mathcal{I}.$$

Lemma 11.1 gives a simple and purely syntactical criterion for the hypothesis on $\mathcal{I}$: It suffices for $\mathcal{I}$ to be internally defined by an expression of the form $\{a : \underline{A} \mid \varphi(a)\}$, where φ is a formula which refers only to constant sheaves.

The metatheorem reflects the following well-known fact of classical ring theory: If an ideal meets every filter (that is, the complement of every prime ideal), it is the unit ideal. In this particular formulation the statement can't be proven intuitionistically; the occurrence of "every filter" has to be replaced by "generic filter". Intuitively, the generic filter is a reification of the abstract idea of an "arbitrary filter", a filter about which nothing is known except that it satisfies the filter axioms.

Proof. Let $D(f) \models \ulcorner \mathcal{I} \cap \mathcal{F} \text{ is inhabited} \urcorner$. Then there exists an open cover $D(f) = \bigcup_i D(f_i)$ and elements $x_i \in A$ such that $D(f_i) \models x_i \in \mathcal{F}$ and $D(f_i) \models x_i \in \mathcal{I}$. By Proposition 11.5 we have that $f_i \in \sqrt{(x_i)}$ and therefore $D(f_i) \models f_i^{m_i} \in \mathcal{I}$ for some $m_i \geq 0$. We may assume that all the $D(f_i)$ are inhabited and that the exponents m_i are all equal to some number m. The assumption on $\mathcal{I}$ implies $D(f) \models f_i^m \in \mathcal{I}$ for all i. By a standard argument we can write $f^n = \sum_i a_i f_i^m$ for some coefficients a_i; thus $D(f) \models f^n \in \mathcal{I}$. □

Remark 11.8. The stronger statement

$$D(f) \models (\ulcorner \mathcal{I} \cap \mathcal{F} \text{ is inhabited} \urcorner \Rightarrow \bigvee_{n \geq 0} (f^n \in \mathcal{I}))$$

does not hold in general. Indeed, consider the example $f := 1$ and $\mathcal{I} := [\![(g)]\!] := [\![\{a : \underline{A} \mid \exists b : \underline{A}.\ a = bg\}]\!]$, where g is a fixed element of A which is not nilpotent and not invertible. Since $D(g) \models g \in \mathcal{I} \cap \mathcal{F}$, the stronger statement would imply $D(g) \models 1 \in \mathcal{I}$. By Lemma 11.1, this is equivalent to g being invertible in A.

Remark 11.9. Recall from Proposition 9.1 that the sheaf $\mathcal{K}_{\operatorname{Spec} A}$ of rational functions can internally by obtained by localizing $\mathcal{O}_{\operatorname{Spec} A}$ at the set of regular elements. Since $\mathcal{O}_{\operatorname{Spec} A}$ is itself a localization, the sheaf $\mathcal{K}_{\operatorname{Spec} A}$ is therefore obtained by a two-step process. It can also be obtained in a single step by localizing $\underline{A}$ at $\mathcal{T}$, where $\mathcal{T}$ is the subsheaf of $\underline{A}$ defined by

$$\Gamma(U, \mathcal{T}) = \{f : U \to A \mid f(\mathfrak{p}) \text{ is regular in } A_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in U\}.$$

This subsheaf is characterized by the property that, for all $f \in A$ and $x \in A$, $D(f) \models x \in \mathcal{T}$ if and only if x is regular in $A[f^{-1}]$.

11.3. Internal proofs of common lemmas.

Lemma 11.10. *Let A be a ring. Then A is reduced if and only if the scheme* $\operatorname{Spec} A$ *is reduced.*

---

and the map $A' \to A_{\mathfrak{p}}$ induced by $A \to A_{\mathfrak{p}}$ is local, it is also invertible in A'. Therefore the image of s in $A_{\mathfrak{q}}$ is invertible as well. Thus $s \notin \mathfrak{q}$.

Proof. By Proposition 3.3 the scheme $\operatorname{Spec} A$ is reduced if and only if $\mathcal{O}_{\operatorname{Spec} A}$ is a reduced ring from the internal point of view of $\mathrm{Sh}(\operatorname{Spec} A)$.

For the "only if" direction assume that A is reduced. Then $\underline{A}$ is reduced as well, by Lemma 11.1. Since localizations of reduced rings are reduced (and this fact has an intuitionistic proof), in particular $\mathcal{O}_{\operatorname{Spec} A} = \underline{A}[\mathcal{F}^{-1}]$ is reduced.

For the "if" direction let $x \in A$ be an element such that $x^n = 0$. Since $\mathcal{O}_{\operatorname{Spec} A} = \underline{A}[\mathcal{F}^{-1}]$ is reduced from the internal point of view, the element x is zero in that ring, that is

$$\mathrm{Sh}(\operatorname{Spec} A) \models \exists s : \mathcal{F}.\ sx = 0.$$

Therefore the ideal internally defined by

$$\mathcal{I} := \{a : \underline{A} \mid ax = 0\}$$

meets the generic filter. By Proposition 11.7 it follows that $\mathrm{Sh}(\operatorname{Spec} A) \models 1 \in \mathcal{I}$. By Lemma 11.1 this is equivalent to $1 \cdot x = 0$ as elements of A. $\square$

The "if" direction also admits a shorter proof, by simply considering the Kripke–Joyal interpretation of $\mathrm{Sh}(\operatorname{Spec} A) \models \ulcorner \mathcal{O}_{\operatorname{Spec} A}$ is reduced$\urcorner$ and using the canonical isomorphism $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \cong A$. We included the given proof to give a simple example of the mixed internal/external reasoning with the generic filter. In a similar way we could reprove Lemma 3.18, the statement that a ring element $f \in A$ is regular in A if and only if, from the internal point of view, it is regular in $\mathcal{O}_{\operatorname{Spec} A}$.

Lemma 11.11. *Let M be an A-module. Then $M^\sim$ is of finite type if and only if M is finitely generated.*

Proof. First assume that M is finitely generated over A. Then $\underline{M}$ is finitely generated over $\underline{A}$, by Lemma 11.1. Since localizations of finitely generated modules are finitely generated (over the localized ring), the module $M^\sim = \underline{M}[\mathcal{F}^{-1}]$ is finitely generated from the internal point of view. By Proposition 4.3 this means that $M^\sim$ is of finite type from the external point of view.

For the "only if" direction, we assume that $M^\sim$ is finitely generated over $\mathcal{O}_{\operatorname{Spec} A}$ from the internal point of view and have to verify that M is finitely generated over A. So it holds that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \underline{M}[\mathcal{F}^{-1}].\ \ulcorner\text{the } x_i \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner.$$

Since multiplying a generating family by an unit results again in a generating family, we have in fact that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \underline{M}.\ \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner$$

or equivalently

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{n \geq 0, x_1, \ldots, x_n \in M} \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner.$$

Since this is a directed disjunction and $\operatorname{Spec} A$ is quasicompact, Proposition 7.1 is applicable and shows that there exists a natural number $n \geq 0$ and elements $x_1, \ldots, x_n \in M$ such that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner.$$

We claim that these x_i also span M as an A-module. So let $x \in M$ be arbitrary. By elementary linear algebra we can deduce that

$$\mathrm{Sh}(\operatorname{Spec} A) \models \exists s \in \mathcal{F}.\ \exists a_1, \ldots, a_n : \underline{A}.\ sx = \sum_i a_i x_i.$$

Therefore the ideal internally defined by

$$\mathcal{I} := \{s : \underline{A} \mid \exists a_1, \ldots, a_n : \underline{A}.\ sx = \textstyle\sum_i a_i x_i\}$$

meets the generic filter. Proposition 11.7 shows that $\mathrm{Sh}(\operatorname{Spec} A) \models 1 \in \mathcal{I}$, that is x is an element of the A-span of the x_i. □

Remark 11.12. If $M^\sim$ can be generated by $\leq n$ elements over $\mathcal{O}_{\operatorname{Spec} A}$ from the internal point of view, it needn't be the case that M can be generated by $\leq n$ elements over A. It is instructive to see where the appropriately modified version of the above proof fails: In this case we still have

$$\mathrm{Sh}(\operatorname{Spec} A) \models \bigvee_{x_1,\ldots,x_n \in M} \ulcorner\text{the } x_i/1 \text{ span } \underline{M}[\mathcal{F}^{-1}] \text{ over } \underline{A}[\mathcal{F}^{-1}]\urcorner,$$

but this disjunction is no longer directed.

Lemma 11.13. *Let X be a scheme. Then kernels and cokernels of morphisms between quasicoherent $\mathcal{O}_X$-modules are quasicoherent.*

Proof. We may assume that $X = \operatorname{Spec} A$ is affine. A morphism between quasicoherent $\mathcal{O}_X$-modules is of the form $\underline{\varphi}[\mathcal{F}^{-1}] : \underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$, where $\varphi : M \to N$ is a linear map between A-modules. Since taking constant shaves and localization are exact, we have the chain of isomorphisms

$$(\underline{\ker(\varphi)})[\mathcal{F}^{-1}] \cong (\ker(\underline{\varphi}))[\mathcal{F}^{-1}] \cong \ker(\underline{\varphi}[\mathcal{F}^{-1}]),$$

and similarly for the cokernel. □

11.4. An application to constructive mathematics. The generic filter has a practical application in constructive mathematics. Recall that intuitionistically prime and maximal ideals don't work very well, since one often needs the axiom of choice or related set-theoretical principles in dealing with them. This is unfortunate, since calculations with prime and maximal ideals are often quite useful. For example:

- To verify that a ring element is nilpotent, it suffices to verify that it is an element of every prime ideal. For instance, this is calculationally simpler when proving that the coefficients of a nilpotent polynomial are themselves nilpotent.
- To verify that there is an relation of the form $1 = p_1 f_1 + \cdots + p_m f_m$ among polynomials $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ where K is an algebraically closed field, it suffices to show that the f_i don't have a common zero.

One could of course blithely switch to classical logic in this case. However this might not be desirable, as a constructive proof would contain more information: For instance, if we have classically proven that an element x is an element of every prime ideal, then we know that some power x^n is zero. But from such a proof we can't directly read off any upper bound on n. From a constructive proof of nilpotency, we could.

There is a way to combine some of the powerful tools of classical ring theory with the advantages that constructive reasoning provides. Namely we can devise a language in which we can usefully talk about prime ideals, but which substitutes all non-constructive arguments by constructive arguments "behind the scenes". The key idea is to substitute the phrase "for all prime ideals" (or equivalently "for all filters") by "for the generic filter".

This was already explored by Coquand, Coste, Lombardi, Roy, and others under the theme of *dynamical methods in algebra* [44, 38]. Here we show how one can use the generic filter, as reified by a sheaf in the little Zariski topos, to achieve similar effects.

Statement	constructive substitution	meaning
$x \in \mathfrak{p}$ for all $\mathfrak{p}$.	$x \notin \mathcal{F}$.	x is nilpotent.
$x \in \mathfrak{p}$ for all $\mathfrak{p}$ such that $y \in \mathfrak{p}$.	$x \in \mathcal{F} \Rightarrow y \in \mathcal{F}$.	$x \in \sqrt{(y)}$.
x is regular in all stalks $A_\mathfrak{p}$.	x is regular in $\underline{A}[\mathcal{F}^{-1}]$.	x is regular in A.
The stalks $A_\mathfrak{p}$ are reduced.	$\underline{A}[\mathcal{F}^{-1}]$ is reduced.	A is reduced.
The stalks $M_\mathfrak{p}$ vanish.	$\underline{M}[\mathcal{F}^{-1}] = 0$.	$M = 0$.
The stalks $M_\mathfrak{p}$ are fin. gen. over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is fin. gen. over $\underline{A}[\mathcal{F}^{-1}]$.	M is fin. gen. over A.
The stalks $M_\mathfrak{p}$ are flat over $A_\mathfrak{p}$.	$\underline{M}[\mathcal{F}^{-1}]$ is flat over $\underline{A}[\mathcal{F}^{-1}]$.	M is flat over A.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are injective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is injective.	$M \to N$ is injective.
The maps $M_\mathfrak{p} \to N_\mathfrak{p}$ are surjective.	$\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.	$M \to N$ is surjective.

TABLE 3. Substituting the use of prime ideals by the generic filter.

Proposition 11.14. *Let M and N be A-modules. Let $\alpha : M \to N$ be a linear map. The interpretations of the statements in the second column of Table 3 in the internal language of* $\mathrm{Sh}(\operatorname{Spec} A)$ *are intuitionistically equivalent to the statements given in the third column.*

Proof. To demonstrate the technique we verify the first and the last claim. To make the following proofs constructive we have to define $\operatorname{Spec} A$, its sheaf topos, and the generic filter in a constructive fashion, not using prime ideals. This can be done, by constructing $\operatorname{Spec} A$ as a locale instead of a topological space. We expand on this in Section 12.2 and in Section 12.9.

The interpretation of $\mathrm{Sh}(\operatorname{Spec} A) \models x \notin \mathcal{F}$ by the Kripke–Joyal semantics is that $D(f) \models x \in \mathcal{F}$ implies $D(f) = \emptyset$ for all $f \in A$. By Proposition 11.5(a) this is equivalent to

$$\forall f \in A.\ f \in \sqrt{(x)} \Rightarrow f \in \sqrt{(0)},$$

that is the statement that x is nilpotent in A.

Assume that $\alpha : M \to N$ is surjective. By Lemma 11.1 the induced map $\underline{M} \to \underline{N}$ is surjective from the internal point of view. Since localization preserves surjectivity, also the map $\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective.

Conversely, assume that $\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ is surjective from the internal point of view. To verify that $\alpha : M \to N$ is surjective, let $y \in N$. The assumption implies that the ideal internally defined by

$$\mathcal{I} := \{s : \underline{A} \mid \exists x : \underline{A}.\ sy = \underline{\alpha}(x)\}$$

meets the generic filter. By Proposition 11.7 this implies that $\mathrm{Sh}(\operatorname{Spec} A) \models 1 \in \mathcal{I}$, that is there exists an element $x \in A$ such that $\alpha(x) = y$. □

Remark 11.15. As is apparent from Table 3, there is a slight mismatch between the external "for any prime ideal" and the internal "for the generic filter". It's not true that a module is finitely generated if and only if all its stalks are finitely generated (a counterexample is the $\mathbb{Z}$-module $\bigoplus_p \mathbb{Z}/(p)$). But it is true that an A-module M is finitely generated if and only if, internally to $\mathrm{Sh}(\operatorname{Spec}(A))$, the generic stalk $\underline{M}[\mathcal{F}^{-1}]$ is finitely generated.

Intuitively, verifying a statement about the generic stalk doesn't only mean that it holds for all (ordinary) stalks; it means that it holds for the ordinary stalks in a *uniform manner*. This extra bit of rigidity is what allows to draw slightly stronger conclusions.

The other entries in Table 3 don't show this slight difference in semantics.

The sheaf-theoretical approach using the generic filter is different from the dynamical methods in the following aspect. We have to reword classical arguments using (the generic) filter instead of (the generic) prime ideal. Depending on the situation this might be a nuisance. One might be tempted to employ the complement of the generic filter, but this is only an ideal, not a prime ideal from the internal point of view.[18]

11.5. An internal proof of Grothendieck's generic freeness lemma. The goal of this subsection is to give a simple proof of Grothendieck's generic freeness lemma in the following general form.

Theorem 11.16. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. Then there is a dense open subset $U \subseteq \operatorname{Spec}(A)$ such that over U,*

(1) *$B^\sim$ is finitely presented as an $\mathcal{O}_{\operatorname{Spec}(A)}$-algebra,*
(2) *$M^\sim$ is of finite presentation over $B^\sim$, and*
(3) *$M^\sim$ is (not necessarily finite) locally free as an $\mathcal{O}_{\operatorname{Spec}(A)}$-module.*

The usual proofs of Grothendieck's generic freeness lemma proceed using a series of reduction steps which are arguably not very memorable or straightforward, see for instance [118, Tag 051Q] or [117]. In particular, there doesn't seem to be a published proof which tackles the Noetherian and non-Noetherian cases in one go. Employing the internal language, Grothendieck's generic freeness lemma can be proved in a simple, conceptual, and constructive way without any reduction steps.

This section was prompted by a MathOverflow thread [34] and greatly benefited from discussions with Brandenburg.

Proof of Theorem 11.16. Since "dense open" translates to "not not" in the internal language (Proposition 6.5), it suffices to prove that, from the internal point of view of $\operatorname{Sh}(\operatorname{Spec}(A))$, it's *not not* the case that

(1) $B^\sim$ is of finite presentation over $\mathcal{O}_{\operatorname{Spec}(A)}$,
(2) $M^\sim$ is finitely presented as a $B^\sim$-module, and
(3) $M^\sim$ is (not necessarily finite) free over $\mathcal{O}_{\operatorname{Spec}(A)}$.

Since $B^\sim$ is finitely generated as an $\mathcal{O}_{\operatorname{Spec}(A)}$-algebra, it is isomorphic to an algebra of the form $\mathcal{O}_{\operatorname{Spec}(A)}[X_1, \ldots, X_n]/\mathfrak{a}$ for some number $n \geq 0$ and some ideal $\mathfrak{a}$. By Proposition 3.31 and Theorem 3.29, the ring $\mathcal{O}_{\operatorname{Spec}(A)}[X_1, \ldots, X_n]$ is anonymously Noetherian. Therefore $\mathfrak{a}$ is *not not* finitely generated, showing that $B^\sim$ is *not not* of finite presentation over $\mathcal{O}_{\operatorname{Spec}(A)}$.

Similarly, the module $M^\sim$ is of the form $(B^\sim)^m/U$ for some number $m \geq 0$ and some submodule U. Since $(B^\sim)^m$ is anonymously Noetherian as a direct sum of anonymously Noetherian modules, the submodule U is *not not* finitely generated. Thus $M^\sim$ is *not not* a finitely presented $B^\sim$-module.

The basic idea to show that $M^\sim$ is *not not* free over $\mathcal{O}_{\operatorname{Spec}(A)}$ is as follows. Since $\mathcal{O}_{\operatorname{Spec}(A)}$ is a field in the sense that noninvertible elements are zero, minimal generating families are already linearly independent; we observed this in the proof of Lemma 5.7. By the finiteness hypotheses, the module $M^\sim$ admits a countable generating family. It's *not not* the case that either one of these vectors can be expressed as a linear combination of the others, or not. In the second case we're

[18]One can check that the complement of $\mathcal{F}$ in $\underline{A}$ is the subsheaf $\mathcal{P}$ defined by $\Gamma(U, \mathcal{P}) := \{f : U \to A \mid f(\mathfrak{p}) \in \mathfrak{p} \text{ for all } \mathfrak{p} \in U\}$ and that $D(f) \models x \in \mathcal{P}$ if and only if fx is nilpotent. This can be used to show that the statement $\operatorname{Sh}(\operatorname{Spec} A) \models \forall x, y : \underline{A}.\ xy \in \mathcal{P} \Rightarrow x \in \mathcal{P} \vee y \in \mathcal{P}$ is false in general. A concrete counterexample is given by $A = \mathbb{Z}[U, V]/(UV)$. Then $\operatorname{Sh}(\operatorname{Spec} A) \models [U] \cdot [V] \in \mathcal{P}$, but $\operatorname{Sh}(\operatorname{Spec} A) \not\models [U] \in \mathcal{P} \vee [V] \in \mathcal{P}$.

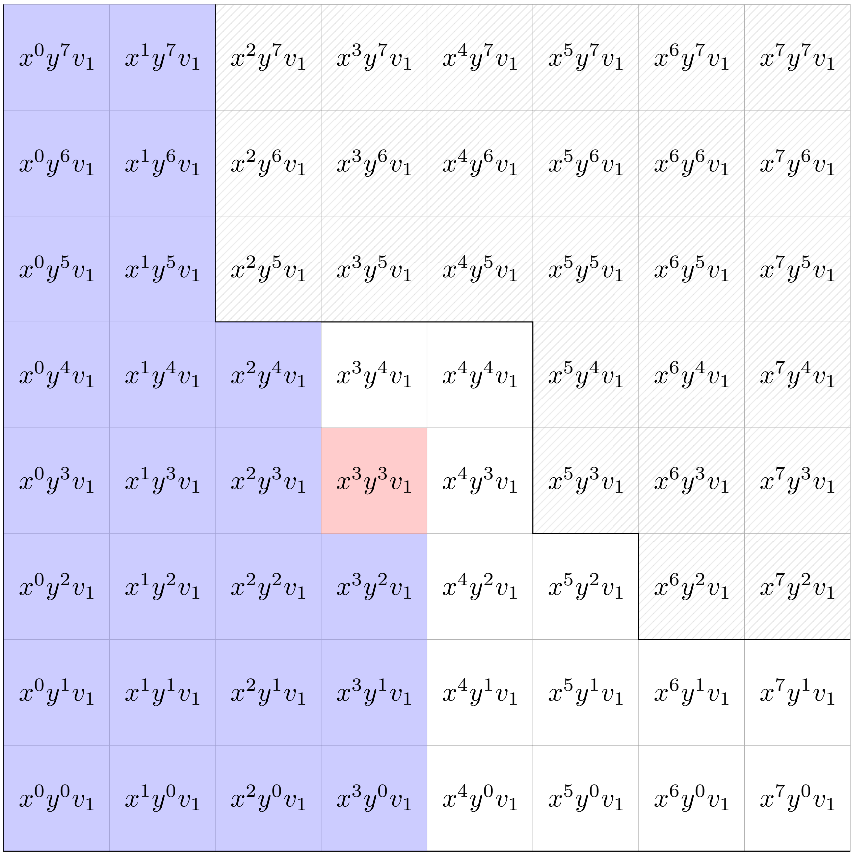

FIGURE 1. A single step in the iterative process used in the proof of Theorem 11.16, in the special case $n = 2, m = 1$. The hatched cells indicate vectors which have already been removed from the generating family. The vector in the red cell was found to be expressible as a linear combination of vectors with smaller index (blue cells). It is therefore about to be removed, along with the vectors in all cells to the top and to the right of the red cell.

done; in the first case, we remove the redundant vector and continue in the same fashion.

However, if we shrink the given generating family in this naive fashion, the process may not terminate in finitely many steps. In a classical context, Zorn's lemma could be used to iterate the process transfinitely and eventually obtain a minimal generating family, but Zorn's lemma is not available in the internal universe of the little Zariski topos. We therefore have to pick the vectors we'll remove in a more systematic fashion.

Let $(x_1, \ldots, x_n)$ be a generating family for $B^\sim$ as an $\mathcal{O}_{\mathrm{Spec}(A)}$-algebra and let $(v_1, \ldots, v_m)$ be a generating family for $M^\sim$ as a $B^\sim$-module. We endow the set

$$I := \{(j, i_1, \ldots, i_n) \mid j \in \{1, \ldots, m\}, i_1, \ldots, i_n \in \{0, 1, \ldots\}\}$$

with the lexicographic order. We choose the family $(x_1^{i_1} \cdots x_n^{i_n} v_j)_{j,i_1,\ldots,i_n}$ as the starting point of the shrinking process. In each step, we use that it's *not not* the case that

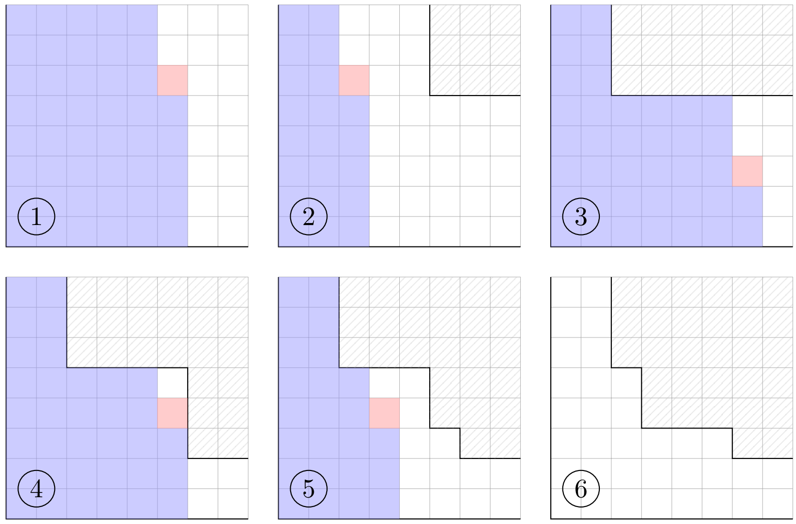

FIGURE 2. The iterative process used in the proof of Theorem 11.16, in the special case $n = 2, m = 1$. The process terminates after reducing the generating family a finite number of times.

- either one of the vectors of the generating family can be expressed as a linear combination of vectors in the family with a smaller index,
- or not.

In the second case, the generating family is linearly independent: For any linear combination summing to zero, we can show that all coefficients are zero, beginning with the coefficient which is paired with the vector of greatest index.

Figure 1 illustrates our action in the first case. We remove the redundant vector $x_1^{i_1} \cdots x_n^{i_n} v_j$ *and also any vector with greater powers of the* $x_1, \ldots, x_n$ from the generating family. The resulting family will still be a generating family, since the linear combination witnessing the redundancy of $x_1^{i_1} \cdots x_n^{i_n} v_j$ successively gives rise to linear combinations witnessing the redundancy of the vectors $x_1^{\geq i_1} \cdots x_n^{\geq i_n} v_j$; we maintain the invariant that any member of the starting generating family can be expressed as a linear combination of vectors of the current generating family with smaller or equal index.

As indicated in Figure 2, this process terminates after finitely many steps. This fact is related to the fact that the ordinal ω^n is well-founded. □

Since the given internal proof was (necessarily) intuitionistically valid, the internal language machinery is intuitionistically valid, and the construction of the spectrum can be set up in an intuitionistically sensible way (Section 12), an intuitionistic external proof not employing the topos machinery can be extracted from the given argument. The resulting proof will verify Grothendieck's generic freeness lemma in the following form.

Theorem 11.17. *Let A be a reduced ring. Let B be an A-algebra of finite type. Let M be a finitely generated B-module. Assume that the only element $f \in A$ such that*

1. *$B[f^{-1}]$ is of finite presentation over $A[f^{-1}]$,*
2. *$M[f^{-1}]$ is finitely presented as a $B[f^{-1}]$-module, and*
3. *$M[f^{-1}]$ is free over $A[f^{-1}]$*

is $f = 0$. Then $A = 0$.

In classical logic, this form implies Grothendieck's generic freeness lemma in its more abstract formulation by a routine argument: Let $U \subseteq \mathrm{Spec}(A)$ be the union over all standard open subsets $D(f)$ such that the statements (1), (2), and (3) in Theorem 11.17 hold. The statements (1), (2), and (3) of Theorem 11.16 hold on this open subset, therefore it remains to show that U is dense.

So let a nonempty open subset V of $\mathrm{Spec}(A)$ be given. This contains a standard open subset $D(g) \subseteq V$ such that g is not nilpotent. Therefore the localized ring $A[g^{-1}]$ is not zero. Thus the conclusion of Theorem 11.17 is not satisfied. Since we assume classical logic, there is a nonzero element $f \in A[g^{-1}]$ such that statements (1), (2), and (3) in Theorem 11.17 hold for $A[g^{-1}][f^{-1}]$, $B[g^{-1}][f^{-1}]$, and $M[g^{-1}][f^{-1}]$. Writing $f = h/g^n$, we see that $U \cap V$ contains the nonzero open subset $D(gh)$.

We refrain from giving the resulting explicit proof of Theorem 11.17 here, but will report on it in the future [24]. A part of the proof was included by Brandenburg in a paper of his [26].

Remark 11.18. There is no hope that there is an intuitionistic proof of Grothendieck's generic freeness lemma in the form of Theorem 11.16 even if the spectrum is constructed in an intuitionistically sensible way, since there is the following Brouwerian counterexample. Let φ be an arbitrary statement. Then the $\mathbb{Z}$-module $M := \mathbb{Z}/\mathfrak{a}$, where $\mathfrak{a} := \{x \in \mathbb{Z} \,|\, (x = 0) \vee \varphi\}$ as in Footnote 9 on page 33, is finitely generated. By assumption, there exists a nonzero element $f \in \mathbb{Z}$ such that $M[f^{-1}]$ is a finite free module over $A[f^{-1}]$ of some rank n. If $n = 0$, then $f^m \in \mathfrak{a}$ for some $m \geq 0$, therefore φ holds. If $n \geq 1$, then $\neg\varphi$ holds, since φ would imply $\mathfrak{a} = \mathbb{Z}$ and therefore $M[f^{-1}] = 0$. Since $n = 0 \vee n \geq 1$, it follows that $\varphi \vee \neg\varphi$.

11.6. A note on $\mathbb{Q}$-algebras which are finitely generated over $\mathbb{Z}$. In this section, we want to show how the internal language of $\mathrm{Sh}(\mathrm{Spec}(\mathbb{Z}))$ can be used to give a proof of the following fact.

Proposition 11.19. *Let A be a finitely presented $\mathbb{Z}$-algebra such that any positive natural number is invertible in A. Then $1 = 0$ in A.*

A slick classical proof runs as follows: Assume that $1 \neq 0$ in A. Then there exists a maximal ideal $\mathfrak{m} \subseteq A$. The preimage of $\mathfrak{m}$ in $\mathbb{Z}$ is maximal since $\mathbb{Z}$ is a Jacobson ring [118, Tag 00GB] and therefore of the form (p) for a prime number p. Thus $p = 0$ in A. Since p is also invertible in A, it follows that $1 = 0$ in A.

We intend the following proof as an example of how one can extend, in some cases, the applicability of theorems about fields to rings using the internal language. If one sets up the spectrum in an intuitionistically sensible way, as described in Section 12, the proof avoids the axiom of choice.

Proof of Proposition 11.19. Noether's normalization lemma is intuitionistically valid in the following form: "Let K be a ring such that $1 \neq 0$ and such that $\neg(\ulcorner s \text{ inv.}\urcorner) \Rightarrow s = 0$ for all $s : K$. Let $\mathfrak{a} \subseteq K[X_1, \ldots, X_n]$ be an ideal. Then it's *not not* the case that either $\mathfrak{a} = (1)$ or that there exists a number $r \geq 0$ and a finite injective homomorphism $K[Y_1, \ldots, Y_r] \to K[X_1, \ldots, X_n]/\mathfrak{a}$ of K-algebras."

In the internal universe of $\mathrm{Sh}(\mathrm{Spec}(\mathbb{Z}))$, the structure sheaf $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}$ satisfies the assumption on K by Corollary 3.9. We can therefore apply the Noether normalization lemma to the $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}$-algebra $A^\sim$.

Writing $A = \mathbb{Z}[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$, we thus obtain that, internally, it's *not not* the case that $1 \in (f_1, \ldots, f_m)$ or that there is a finite injective morphism $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}[Y_1, \ldots, Y_r] \to \mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}[X_1, \ldots, X_n]/(f_1, \ldots, f_m) = A^\sim$ for some number $r \geq 0$. Since any positive natural number is invertible in $A^\sim$ and finite injective homomorphisms of rings reflect invertibility, any positive natural number is also invertible in $\mathcal{O}_{\operatorname{Spec}(\mathbb{Z})}[Y_1, \ldots, Y_r]$ and therefore in $\mathcal{O}_{\operatorname{Spec}(\mathbb{Z})}$. This, however, is false.

We therefore have $\operatorname{Sh}(\operatorname{Spec}(\mathbb{Z})) \models \neg\neg(1 \in (f_1, \ldots, f_m))$. Thus $D(h) \models 1 \in (f_1, \ldots, f_m)$ for some dense open subset $D(h) \subseteq \operatorname{Spec}(\mathbb{Z})$. This implies that $h^l \in (f_1, \ldots, f_m)$ for some $l \geq 0$. Therefore $h^l = 0$ in A; on the other hand, h^l is invertible in A. Thus $1 = 0$ in A. □

12. Relative spectrum

Recall that if $\mathcal{A}$ is a quasicoherent $\mathcal{O}_X$-algebra on a scheme X, one can construct the *relative spectrum* $\underline{\operatorname{Spec}}_X \mathcal{A}$ by appropriately gluing the spectra $\operatorname{Spec} \Gamma(U, \mathcal{A})$ where U ranges over the affine opens of X. This relative spectrum comes equipped with a canonical morphism $\underline{\operatorname{Spec}}_X \mathcal{A} \to X$.

From the internal point of view of $\operatorname{Sh}(X)$, the sheaf $\mathcal{A}$ looks just like a plain algebra, to which therefore the usual (absolute) spectrum construction can be applied. One could hope that this construction yields the relative spectrum.

In this section, we discuss generalities on how to make sense of this internal construction; we show that this proposed construction is too naive and doesn't yield the relative spectrum; we give a refined internal construction which does yield the relative spectrum, discuss its relation to the naive construction, and phrase it in topos-theoretic terms; and we deduce, as an application, a description of limits in the category of locally ringed spaces. We also cover the relative Proj construction.

In much of the following, it's not actually necessary that X is a scheme and $\mathcal{A}$ is a quasicoherent algebra. If X is not a scheme or $\mathcal{A}$ is not quasicoherent, then $\underline{\operatorname{Spec}}_X(\mathcal{A})$ might fail to be a scheme and can of course not be constructed by gluing usual spectra, but it still exists as a more general kind of space and still verifies a meaningful universal property. We give details on this generalization below.

12.1. Internal locales. Let X be a topological space (or a locale). A fundamental fact in the theory of locales is that there is a canonical equivalence between the category of *locales over* X – that is locales Y equipped with a morphism $Y \to X$ – and *internal locales in* $\operatorname{Sh}(X)$ [69, p. 49]. An internal locale in a topos $\mathcal{E}$ is given by an object L of $\mathcal{E}$ (the internal frame of opens of the locale) together with a binary relation $(\preceq) \hookrightarrow L \times L$ such that the axioms for a locale hold from the internal point of view. (For our purposes, we do not need a precise wording of these axioms.)

The equivalence is described as follows: A locale $f : Y \to X$ over X induces an internal locale $I(Y)$ with object of opens given by $\mathcal{T}(I(Y)) := f_*\Omega_{\operatorname{Sh}(Y)} \in \operatorname{Sh}(X)$, where f_* is the pushforward functor and $\Omega_{\operatorname{Sh}(Y)}$ is the object of truth values in the topos of sheaves on Y. Conversely, an internal locale given by an internal frame $\mathcal{L} \in \operatorname{Sh}(X)$ induces an (external) locale $E(\mathcal{L})$ with frame of opens given by $\mathcal{T}(E(\mathcal{L})) := \Gamma(X, \mathcal{L})$. This comes equipped with a canonical morphism $Y \to X$ of locales which we do not need to describe explicitly [67, Section C1.6].

As a special case, the internalization of the trivial locale $\mathrm{id} : X \to X$ over X has as frame of opens the object $\mathrm{id}_*\Omega_{\operatorname{Sh}(X)} = \Omega_{\operatorname{Sh}(X)} = \mathcal{P}(1)$. This is precisely the frame of opens of the one-point space. Thus $I(X) \cong \mathrm{pt}$. This illustrates the intuition behind working internally in $\operatorname{Sh}(X)$: From the perspective of $\operatorname{Sh}(X)$, the space X looks like the one-point space (even if in fact it is not).

One can associate to an internal locale T in a topos $\mathcal{E}$ a topos of internal sheaves on it: $\operatorname{Sh}_{\mathcal{E}}(T)$. The correspondence is made in such a way that the topos of sheaves on a locale Y over X is equivalent to the topos of sheaves on the internal locale $I(Y)$: $\operatorname{Sh}(Y) \simeq \operatorname{Sh}_{\operatorname{Sh}(X)}(I(Y))$.

There is no similarly nice correspondence between topological spaces over X and internal topological spaces in $\operatorname{Sh}(X)$ [67, Corollary C1.6.7]. This is one of the reasons why locales are better suited for working internally and for switching between internal and external perspectives.

For verification of properties of such sheaves, the *idempotency* of the internal language is useful: If φ is a formula over Y, then

$$\mathrm{Sh}(Y) \models \varphi \qquad \text{if and only if} \qquad \mathrm{Sh}(X) \models \ulcorner \mathrm{Sh}(I(Y)) \models \varphi \urcorner.$$

Here we're abusing notation in two ways. Firstly, the formula φ has to be appropriately interpreted in the expression "$\mathrm{Sh}(I(Y)) \models \varphi$". Secondly, the expression "$\mathrm{Sh}(I(Y))$" doesn't actually refer to the category $\mathrm{Sh}_{\mathrm{Sh}(X)}(I(Y))$, but to the *locally internal* category induced by the canonical geometric morphism $\mathrm{Sh}_{\mathrm{Sh}(X)}(I(Y)) \to \mathrm{Sh}(X)$. We give some details on this point in Section 16. However, in the situations encountered in this section, the meaning will always be reasonably clear.

12.2. The spectrum of a ring as a locale. Recall that the spectrum of a ring A is usually constructed as the set

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

endowed with a certain topology and a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$. From an intuitionistic (and thus internal) point of view, this construction does not work well: Prime ideals are intuitionistically much more elusive than classically, where one can appeal to Zorn's lemma to obtain maximal (and thus prime) ideals. More to the point, one cannot show that this construction of the spectrum as a topological space verifies the expected universal property, namely

$$\mathrm{Hom}_{\mathrm{LRS}}(X, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X))$$

for all locally ringed spaces X (or some variant of this property involving more general kinds of spaces).

On the other hand, the frame of opens of $\operatorname{Spec} A$ admits a simple description not requiring the notion of prime ideals:

$$\mathcal{T}(\operatorname{Spec} A) \cong \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \text{ is a radical ideal}\}.$$

An open subset $U \subseteq \operatorname{Spec} A$ corresponds to the radical ideal $\{h \in A \mid D(h) \subseteq U\}$ (so in particular, the open subset $D(f)$ corresponds to the radical ideal $\sqrt{(f)}$); conversely, a radical ideal $\mathfrak{a}$ corresponds to the open subset $\bigcup_{h \in \mathfrak{a}} D(h)$.

Thus, in an intuitionistic context, we will construct the spectrum of a ring A as a locale, not as a topological space, and adopt the following definition.

Definition 12.1. The *spectrum* $\operatorname{Spec}(A)$ of a ring A is the locale whose frame of opens is the frame of radical ideals of A. We endow it with the structure sheaf $\mathcal{O}_{\operatorname{Spec}(A)} := \underline{A}[\mathcal{F}^{-1}]$, where $\mathcal{F}$ is the generic filter as described in Section 11.2.

This construction has the expected universal property, namely that it is adjoint to the global functions functor:

$$\mathrm{Hom}_{\mathrm{LRL}}(X, \operatorname{Spec} A) \cong \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Here, "LRL" refers to the category of *locally ringed locales*, i. e. locales X equipped with a sheaf of rings $\mathcal{O}_X$ such that from the internal point of view of $\mathrm{Sh}(X)$, the ring $\mathcal{O}_X$ is a local ring. A morphism $Y \to X$ of locally ringed locales consists of a locale morphism $f : Y \to X$ and a morphism $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ of sheaves of rings on Y such that, from the internal point of view of $\mathrm{Sh}(Y)$, the ring homomorphism $f^\sharp$ is a local homomorphism. The notion of a locally ringed locale is thus a straightforward generalization of that of a locally ringed space.

Schemes are usually regarded as locally ringed spaces, not as locally ringed locales. However, in a classical context where the axiom of choice is available, schemes are *sober* topological spaces [118, Tag 01IS]. For sober topological spaces, the passage from the space to its induced locale (forgetting the set of points and only keeping the frame of open subsets) doesn't lose information: The category of sober topological spaces with arbitrary continuous maps embeds into the category of locales as a full subcategory. Therefore the category of schemes can just as well be viewed as a full subcategory of the category of locally ringed locales.

Describing morphisms between locally ringed locales is just as simple as describing morphisms between locally ringed spaces; using the viewpoint of classifying locales, one may even pretend that it suffices to give a map of points. We expand on this in Section 12.9.

The importance of a locale-theoretic approach to spectra of rings, especially in relative situations, has also been stressed by Lurie [85, p. 37].

Remark 12.2. In contrast to the prime spectrum, the spectrum of maximal ideals can't in general be realized as a locale. This is because the maximal spectrum is in general not sober; its soberification is the subspace of the prime spectrum consisting of those prime ideals which are intersections of maximal ideals. (For Jacobson rings, every prime ideal is of this form.)

Points of the locale-theoretic spectrum. Constructing the spectrum as a locale instead of a topological space sidesteps any issues with prime ideals, since points are not a defining ingredient of a locale. However, points are still meaningful as a *derived concept*: A point of locale X is a morphism $1 \to X$, where 1 is the terminal locale, the locale corresponding to the one-point topological space with frame of opens $\mathcal{P}(1) = \Omega$. Therefore it's still an interesting question what the points of the locale $\mathrm{Spec}(A)$ look like.

Proposition 12.3. *Let A be a ring. Then the points of the locale* $\mathrm{Spec}(A)$ *are in canonical one-to-one correspondence with the filters of A (as in Definition 11.3), even intuitionistically.*

Proof. The points of a locale X are in canonical one-to-one correspondence with the *completely prime filters* of $\mathcal{T}(X)$, subsets $K \subseteq \mathcal{T}(X)$ which are upward-closed, downward-directed, and have the property that, whenever a supremum of a set $M \subseteq \mathcal{T}(X)$ is contained in K, then so is some element of M.

Such a completely prime filter $K \subseteq \mathcal{T}(\mathrm{Spec}(A))$ corresponds to the ring-theoretic filter

$$F := \{s : A \mid \sqrt{(s)} \in K\} \subseteq A,$$

and a ring-theoretic filter $F \subseteq A$ corresponds to the completely prime filter

$$K := \{\mathfrak{a} : \mathcal{T}(\mathrm{Spec}(A)) \mid \mathfrak{a} \cap F \text{ is inhabited}\}.$$

We omit the required routine verifications. □

In classical logic, where complementation yields a one-to-one correspondence between filters and prime ideals, the points of $\mathrm{Spec}(A)$ are therefore in canonical bijection with the prime ideals of A, just as one would expect.

Observing that intuitionistically the points of the locale $\mathrm{Spec}(A)$ are filters, not prime ideals, one might wonder: Is the locale-theoretic approach really necessary? Wouldn't it suffice to define $\mathrm{Spec}(A)$ as the topological space of filters of A? Indeed, for some time this was believed [79, Section 3]; however, this hope turned out to be too naive: Joyal gave an explicit example of a nontrivial ring in a certain topos without any filters [126, pp. 200f.], thus showing that the construction can't have the expected universal property and that therefore a true pointfree approach as provided by lattice theory/locale theory [41], topos theory, or formal topology [111] is necessary to construct the spectrum in an intuitionistic context.[19]

The spectrum as a classifying locale. The fact that the points of $\mathrm{Spec}(A)$ are in canonical one-to-one correspondence with the filters of A is a shadow of a more general fact. Namely, for any locale X (and in fact any topos), maps $X \to \mathrm{Spec}(A)$ are in canonical one-to-one correspondence with the internal filters of A in $\mathrm{Sh}(X)$, that is subsheaves of the constant sheaf $\underline{A}$ satisfying the filter axioms from the point of view of the internal language of $\mathrm{Sh}(X)$: The locale $\mathrm{Spec}(A)$ is the *classifying locale of the theory of filters of A*.

The fact about the points of $\mathrm{Spec}(A)$ can be recovered from this observation as follows. A point of $\mathrm{Spec}(A)$ is a map $1 \to \mathrm{Spec}(A)$ and therefore corresponds to a subsheaf of the constant sheaf $\underline{A}$ satisfying the filter axioms from the point of view of $\mathrm{Sh}(1)$. Since $\mathrm{Sh}(1) \simeq \mathrm{Set}$, such a subsheaf amounts to a subset of A satisfying the filter axioms.

The notion of classifying locales provides a pleasant way of approaching the problem of constructing a space of models of a *propositional geometric theory* (in the case of the spectrum the theory of filters), simultaneously streamlining the usual topological approach and generalizing it to work in an intuitionistic context: Instead of first constructing the *set* of models (filters of A) and then manually endowing this set with a suitable topology (the Zariski topology), one can simply consider the *locale* of models, that is the classifying locale of the theory. Its sets of points coincides with the set of models of the topological approach, but the locale is not determined by its sets of points, facilitating a better behavior in contexts where the points might be elusive.

Put more concisely, the topological space of filters doesn't work well in an intuitionistic context, but the locale of filters does. This phenomenon is an instance of Vickers's motto "if you define points by a geometric theory, then the topology is implicit". A lucid expository account of the theory of classifying locales can be found in a survey article by him [133].

Remark 12.4. For comparison with a refined geometric theory discussed below, we describe the geometric theory of filters of A here explicitly. It has one atomic proposition "$s \in F$" for each element $s : A$, and its axioms are given by the following axiom schemes:

(1) $\top \vdash 1 \in F$
(2) $st \in F \dashv\vdash s \in F \wedge t \in F$ (two axioms for each $s, t : A$)
(3) $0 \in F \vdash \bot$
(4) $s + t \in F \vdash s \in F \vee t \in F$ (one axiom for each $s, t : A$)

A trivial case. For later use, we study the question when the spectrum is the one-point space. The answer is well-known classically, but since we want to use this result in an internal context, we have to give an intuitionistic proof.

Lemma 12.5. *Let A be a ring. Its spectrum is a one-point space (as a locale) if and only if $1 \neq 0$ in A any element of A is nilpotent or invertible.*

Proof. The locale $\mathrm{Spec}\, A$ is a one-point space if and only if the unique continuous map $\mathrm{Spec}(A) \to \mathrm{pt}$ of locales is an isomorphism. This is the case if and only if the

[19] When following reference [126], note that Tierney calls "primes" what we call "filters". Joyal's example was none other than the ring $\underline{\mathbb{A}}^1_S$ in the functor category $[\mathrm{Ring}_{\mathrm{fp}}, \mathrm{Set}]$. The big Zariski topos of $\mathrm{Spec}(\mathbb{Z})$, when defined using the parsimonious sites, is a subtopos of that topos; in it, the ring $\underline{\mathbb{A}}^1_S$ does have filters, for instance the filter of units. These two facts are not contradictory, since not having any filters is not a geometric implication and is therefore not preserved by inverse image parts of geometric morphisms.

canonical frame homomorphism

$$\begin{array}{rcl} \Omega = \mathcal{P}(1) & \longrightarrow & \mathcal{T}(\operatorname{Spec} A) \\ \varphi & \longmapsto & \mathfrak{a}_\varphi := \sup\{\sqrt{(1)} \,|\, \varphi\} = \{x : A \,|\, \ulcorner x \text{ nilpotent} \urcorner \vee \varphi\} \end{array}$$

is surjective and reflects the ordering (and is therefore automatically injective). If $1 = 0$ in A, this homomorphism is not injective, since $\bot$ and $\top$ get both mapped to $\sqrt{(0)}$. For the rest of the proof, we'll therefore assume that $1 \neq 0$ in A.

Under this assumption, the homomorphism reflects the ordering: If $\mathfrak{a}_\varphi \subseteq \mathfrak{a}_\psi$, then $(1 \in \mathfrak{a}_\varphi) \Rightarrow (1 \in \mathfrak{a}_\psi)$. Since the unit of A is not nilpotent, this amounts to $\varphi \Rightarrow \psi$.

The homomorphism is surjective if and only if for any radical ideal $\mathfrak{a} \subseteq A$, it holds that $\mathfrak{a} = \{x : A \,|\, \ulcorner x \text{ nilpotent} \urcorner \vee \varphi\}$ for some proposition φ. By considering the condition "$1 \in \mathfrak{a}$", it follows that this proposition φ must be equivalent to the proposition "$1 \in \mathfrak{a}$" (if it is at all possible to write $\mathfrak{a}$ in such a way).

So the map is surjective if and only if for any radical ideal $\mathfrak{a} \subseteq A$ and any element x of A it holds that

$$x \in \mathfrak{a} \quad \Longleftrightarrow \quad \ulcorner x \text{ nilpotent} \urcorner \vee (1 \in \mathfrak{a}).$$

The "if" direction always holds. If any element of A is nilpotent or invertible, the "only if" direction holds as well (for any $\mathfrak{a}$ and any x). Conversely, if the "only if" direction holds, then any element of A is nilpotent or invertible. This follows by considering the radical ideal $\sqrt{(f)}$ for an element $f : A$. ☐

Remark 12.6. The structure sheaf $\mathcal{O}_X$ of a scheme fulfills almost, but not quite, the condition given in Lemma 12.5: By Proposition 3.7, it has the property that any element which is not invertible is nilpotent. In classical logic, this statement is equivalent to the statement that every element is nilpotent or invertible. However, intuitionistically the former is a weaker statement than the latter. This observation entails that the internally constructed spectrum does *not* coincide with the relative spectrum, and that instead a refined approach is necessary. Section 12.4 is devoted to studying this difference.

12.3. Digression: Further topologies on the set of prime ideals. The Zariski topology is not the only interesting topology on the set of prime ideals. For instance, the constructible topology and the flat topology studied by Tarizadeh [122] (also called "co-Zariski topology") too have their uses.

The universal properties given in the following two propositions should be compared with the following way of phrasing the universal property of the ordinary locale-theoretic spectrum. The usual phrasing employs the categories RL and LRL of (locally) ringed locales, therefore emphasizing the spatial character. But the dual categories $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$ can be used just as well; since the morphisms in $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$ go in the direction of the ring-theoretic parts, they can be thought of as the category of *all* rings respectively *all* local rings, where "all" refers to the fact that these categories don't only include the (local) rings in Set, but the (local) rings in arbitrary localic sheaf toposes.

Formulated using $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$, and adopting the notation to suppress mention of the involved spaces (instead of the involved sheaves of rings), the universal property of $\mathrm{Spec}(A)$ reads as follows: For any local ring $\mathcal{O}_Y$ over any locale Y,

$$\mathrm{Hom}_{\mathrm{LRL}^{\mathrm{op}}}(\mathcal{O}_{\mathrm{Spec}(A)}, \mathcal{O}_Y) \cong \mathrm{Hom}_{\mathrm{RL}^{\mathrm{op}}}(A, \mathcal{O}_Y).$$

The morphism $A \to \mathcal{O}_{\mathrm{Spec}(A)}$ in $\mathrm{RL}^{\mathrm{op}}$ is therefore the *universal localization* of A.

Proposition 12.7. *Let A be a ring. The locale given by the space of prime ideals of A with the flat topology is the classifying locale of prime ideals of A. Equipped* *with $\underline{A}/\mathcal{P}$ as structure sheaf, where $\mathcal{P}$ is the generic prime ideal, it is the universal way of mapping A to an integral domain in the weak sense (as defined in Section 3.5).*

Proof. See [66, Proposition 4.5]. □

Proposition 12.8. *Let A be a ring. The locale given by the space of prime ideals of A with the constructible topology is the classifying locale of detachable prime ideals (or equivalently detachable filters) of A. Equipped with $\underline{A}/\mathcal{P}$ as structure sheaf, where $\mathcal{P}$ is the generic prime ideal, it is the universal way of mapping A to an integral domain in the strong sense. Equipped with $\underline{A}[\mathcal{F}^{-1}]$, where $\mathcal{F}$ is the generic filter, it is the universal way of mapping A to a local ring in which invertibility is decidable.*

Proof. This is mostly covered in [66, p. 253]. □

In constructive mathematics, a subset $U \subseteq A$ is *detachable* if and only if for every element $a : A$, either $a \in U$ or $a \notin U$. While intuitionistically the complement of a filter might fail to be a prime ideal and the complement of a prime ideal might fail to be a filter, the complement of a detachable filter is a detachable prime ideal, and vice versa.

12.4. The relative spectrum as an ordinary spectrum from the internal point of view. Let X be a scheme and $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Since $\mathcal{A}$ looks like a plain algebra from the internal perspective of $\mathrm{Sh}(X)$, we can consider its internally defined spectrum. This is a locale internal to $\mathrm{Sh}(X)$; we might hope that its externalization is precisely the relative spectrum of $\mathcal{A}$ (considered as a locale):

$$E(\operatorname{Spec}\mathcal{A}) \stackrel{?}{\cong} \underline{\operatorname{Spec}}_X \mathcal{A}.$$

However, this turns out to be too naive. The locale $E(\operatorname{Spec}(\mathcal{A}))$ is equipped with a map to X, being an externalization of a locale internal to $\mathrm{Sh}(X)$, and it is equipped with a sheaf of rings (because we can transport the internally defined structure sheaf along the equivalence $\mathrm{Sh}_{\mathrm{Sh}(X)}(\operatorname{Spec}(\mathcal{A})) \simeq \mathrm{Sh}(E(\operatorname{Spec}(A)))$. Furthermore, this sheaf of rings is local, since we know

$$\mathrm{Sh}(X) \models \ulcorner \mathrm{Sh}(\operatorname{Spec}(\mathcal{A})) \models \ulcorner \mathcal{O}_{\operatorname{Spec}(\mathcal{A})} \text{ is a local ring} \urcorner\urcorner$$

which by idempotency of the internal language is equivalent to

$$\mathrm{Sh}(E(\operatorname{Spec}(\mathcal{A}))) \models \ulcorner \mathcal{O}_{\operatorname{Spec}(\mathcal{A})} \text{ is a local ring} \urcorner.$$

However, the map $E(\operatorname{Spec}(\mathcal{A})) \to X$ is only part of a morphism of ringed locales, not of locally ringed locales (even though domain and codomain happen to be locally ringed): Internally, the morphism $(\operatorname{Spec}(\mathcal{A}), \mathcal{O}_{\operatorname{Spec}(\mathcal{A})}) \to (\mathrm{pt}, \mathcal{O}_X)$ of ringed locales, which is defined using the $\mathcal{O}_X$-algebra structure of $\mathcal{A}$, is not a morphism of locally ringed locales (even though domain and codomain happen to be locally ringed).

In contrast, the true relative spectrum $\underline{\operatorname{Spec}}_X(\mathcal{A})$ is equipped with a morphism of locally ringed locales to X.

It's illuminating to compare the different universal properties of $E(\operatorname{Spec}(\mathcal{A}))$ and $\underline{\operatorname{Spec}}_X(\mathcal{A})$. There is a canonical morphism $E(\operatorname{Spec}(\mathcal{A})) \to E(\operatorname{Spec}(\mathcal{O}_X))$ of locally ringed locales (the externalization of the canonical morphism $\operatorname{Spec}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{O}_X)$ given by the $\mathcal{O}_X$-algebra structure of $\mathcal{A}$), but in general, the locales $E(\operatorname{Spec}(\mathcal{O}_X))$ and X are not isomorphic.

As we justify below, the externalization of the internally defined spectrum has the universal property

$$\operatorname{Hom}_{\mathrm{LRL}/E(\operatorname{Spec}\mathcal{O}_X)}(Y, E(\operatorname{Spec}\mathcal{A})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mu_*\mathcal{O}_Y)$$

for all locally ringed locales Y over $E(\operatorname{Spec}\mathcal{O}_X)$. Here, μ is the structure morphism $Y \to \operatorname{Spec}\mathcal{O}_X$, $E(\operatorname{Spec}\mathcal{O}_X)$ is the locally ringed locale associated to the internally defined spectrum of $\mathcal{O}_X$, and $\mathrm{LRL}_{\mathrm{Sh}(X)}$ is the category of locally ringed locales internal to $\mathrm{Sh}(X)$. In contrast, the relative spectrum has the different universal property

$$\mathrm{Hom}_{\mathrm{LRL}/X}(Y, \underline{\mathrm{Spec}}_X\, \mathcal{A}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mu_* \mathcal{O}_Y)$$

for all locally ringed locales Y over X.[20] The crucial difference is that in general, the internally defined locally ringed locale $\mathrm{Spec}\,\mathcal{O}_X$ does *not* coincide with the internal locally ringed locale $(\mathrm{pt}, \mathcal{O}_X)$ (which is simply $(X, \mathcal{O}_X)$ from the external point of view). More succinctly, the functor $E \circ \mathrm{Spec}$ is an adjoint to the pushforward-of-sheaf-of-functions functor $\mathrm{LRL}/E(\mathrm{Spec}\,\mathcal{O}_X) \to \mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}}$, while the relative spectrum functor is an adjoint to the analogous functor $\mathrm{LRL}/X \to \mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}}$.

The universal property of $E(\mathrm{Spec}(\mathcal{A}))$ can be determined as follows. From the internal point of view of $\mathrm{Sh}(X)$, the locally ringed locale $E(\mathrm{Spec}(\mathcal{A}))$ looks like the ordinary locale-theoretic spectrum $\mathrm{Spec}(\mathcal{A})$ and therefore has the universal property

$$\mathrm{Hom}_{\mathrm{LRL}}(Y, \mathrm{Spec}(\mathcal{A})) \cong \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{A}, \Gamma(Y, \mathcal{O}_Y))$$

for any locally ringed locale Y.[21] If we restrict the right-hand side to the set of $\mathcal{O}_X$-algebra homomorphisms, the left-hand side restricts to the set of morphisms $Y \to \mathrm{Spec}(\mathcal{A})$ of locally ringed locales over the locally ringed locale $\mathrm{Spec}(\mathcal{O}_X)$. So we have

$$\mathrm{Hom}_{\mathrm{LRL}/\,\mathrm{Spec}(\mathcal{O}_X)}(Y, \mathrm{Spec}(\mathcal{A})) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \Gamma(Y, \mathcal{O}_Y)).$$

This discussion took place in the internal universe of $\mathrm{Sh}(X)$. Externally, the displayed universal property implies that for any locally ringed locale $\mu : Y \to X$ over $E(\mathrm{Spec}(\mathcal{O}_X))$,

$$\mathrm{Hom}_{\mathrm{LRL}/E(\mathrm{Spec}(\mathcal{O}_X))}(Y, E(\mathrm{Spec}(\mathcal{A}))) \cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_* \mathcal{O}_Y),$$

as claimed above.

Definition 12.9. Let R be a ring. Let A be an R-algebra. The *local spectrum* of A over R is the locale $\mathrm{Spec}(A|R)$ with frame of opens given by

$$\begin{aligned}\mathcal{T}(\mathrm{Spec}(A|R)) := \{\mathfrak{a} \subseteq A \,|\, &\mathfrak{a} \text{ is a radical ideal such that} \\ &\forall f : R.\ \forall s : A.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathfrak{a}) \Rightarrow fs \in \mathfrak{a}\}.\end{aligned}$$

We'll equip the local spectrum with the structure of a locally ringed locale below. It is this refined construction which correctly internalizes the relative spectrum:

Theorem 12.10. *Let X be a scheme (or a locally ringed locale). Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. Then the externalization $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))$ coincides with $\underline{\mathrm{Spec}}_X(\mathcal{A})$ as locally ringed locales over X.*

Before giving the proof, we want to clarify some details of the construction.

Firstly, the base ring R directly enters the construction. This is in contrast to the usual spectrum: If A is an R-algebra, the construction of $\mathrm{Spec}(A)$ does not depend on the R-algebra structure of A. The algebra structure only enters in the construction of a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$.

Secondly, in the case that X is a scheme and $\mathcal{A}$ is a quasicoherent $\mathcal{O}_X$-algebra, we can compare the externalization of $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$ with the result of the construction of $\underline{\mathrm{Spec}}_X(\mathcal{A})$ by gluing spectra:

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[20] If X is a scheme and $\mathcal{A}$ is quasicoherent, this universal property is well-known, even though it's usually only stated for schemes Y over X instead of general locally ringed locales over X. In any case, we take this universal property as the definition of what the relative spectrum should be.

[21] Externally, this implies that for any locally ringed locale over the underlying locale of X (that is, for any locale Y equipped with a morphism $\mu : Y \to X$ and a local sheaf of rings), we have

$$\mathrm{Hom}_{\mathrm{LR(L}/X)}(Y, E(\mathrm{Spec}(\mathcal{A}))) \cong \mathrm{Hom}_{\mathrm{Ring}_{\mathrm{Sh}(X)}}(\mathcal{A}, \mu_* \mathcal{O}_Y).$$

Proposition 12.11. *Let X be a scheme. Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Then $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))$ coincides with $\underline{\mathrm{Spec}}_X(\mathcal{A})$ as locales over X.*

Proof. The condition

$$\forall f : \mathcal{O}_X.\ \forall s : \mathcal{A}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathfrak{a}) \Longrightarrow fs \in \mathfrak{a}$$

appearing in Definition 12.9 is precisely the internal quasicoherence condition of Corollary 8.5 (slightly simplified in view that $\mathfrak{a}$ is a radical ideal). The sections of the sheaf $[\![\mathcal{T}(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))]\!]$ on an open subset $U \subseteq X$ are therefore precisely the quasicoherent sheaves of radical ideals $\mathfrak{a} \hookrightarrow \mathcal{A}|_U$. Let $\pi : \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ be the canonical morphism. If U is affine, then

$$\pi^{-1}U \cong \underline{\mathrm{Spec}}_X(\mathcal{A}) \times_X U \cong \underline{\mathrm{Spec}}_U(\mathcal{A}|_U) \cong \mathrm{Spec}(\Gamma(U, \mathcal{A}))$$

is affine as well and

$$\begin{aligned} \Gamma(U, \mathcal{T}(I(\underline{\mathrm{Spec}}_X(\mathcal{A})))) &= \Gamma(U, \pi_*\Omega_{\underline{\mathrm{Spec}}_X(\mathcal{A})}) = \Omega_{\underline{\mathrm{Spec}}_X(\mathcal{A})}(\pi^{-1}U) \\ &\cong \text{set of open subsets of } \pi^{-1}U \\ &\cong \text{set of open subsets of } \mathrm{Spec}(\Gamma(U, \mathcal{A})) \\ &\cong \text{set of radical ideals of } \Gamma(U, \mathcal{A}) \\ &\cong \text{set of quasicoherent sheaves of radical ideals of } \mathcal{A}|_U \\ &\cong \Gamma(U, [\![\mathcal{T}(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))]\!]). \end{aligned}$$

Therefore $I(\underline{\mathrm{Spec}}_X(\mathcal{A}))$ is canonically isomorphic to $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$ as locales internal to $\mathrm{Sh}(X)$. Expressed externally: The relative spectrum $\underline{\mathrm{Spec}}_X(\mathcal{A})$ coincides with the externalization of $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$ as locales over X, as claimed. ☐

Thirdly, the partial order $\mathcal{T}(\mathrm{Spec}(A|R))$ is indeed a frame. A quick way to verify this is to recognize that it is related to the frame of opens of $\mathrm{Spec}(A)$ by the formula

$$\mathcal{T}(\mathrm{Spec}(A|R)) = \{\mathfrak{a} : \mathcal{T}(\mathrm{Spec}(A)) \mid \mathfrak{a} = \overline{\mathfrak{a}}\},$$

where $(\mathfrak{a} \mapsto \overline{\mathfrak{a}})$ is the quasicoherator described in Remark 8.16. Since the quasicoherator satisfies the axioms for a nucleus, this formula exhibits $\mathrm{Spec}(A|R)$ as a sublocale of $\mathrm{Spec}(A)$. In particular, suprema are calculated in $\mathcal{T}(\mathrm{Spec}(A|R))$ by applying the quasicoherator to the suprema calculated in $\mathcal{T}(\mathrm{Spec}(A))$. We denote the inclusion $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$ by "i".

Lastly, it's interesting to know what the points of $\mathrm{Spec}(A|R)$ are, even though these don't determine $\mathrm{Spec}(A)$.

Definition 12.12. Let R be a ring. Let $\varphi : R \to A$ be an algebra. A filter $F \subseteq A$ *lies over the filter of units* if and only if $\varphi^{-1}F \subseteq R^\times$, that is if

$$\varphi(r) \in F \Longrightarrow r \text{ is invertible in } R$$

for all $r : R$. (The reverse inclusion "$\varphi^{-1}F \supseteq R^\times$" holds automatically.)

This definition will mostly be used in situations where the ring R is local, in which case the subset $R^\times$ is actually a filter and the phrase "filter of units" is therefore justified.

It's illuminating to consider Definition 12.12 in a classical context, even though the use case we have in mind is to apply it in the internal language of the little Zariski topos of a base scheme. Classically, a filter F lies over the filter of units if and only if $\varphi^{-1}\mathfrak{p} \supseteq R \setminus R^\times$, where $\mathfrak{p} := F^c = A \setminus F$ is the prime ideal associated to F. If R is local, the set $R \setminus R^\times$ is the unique maximal ideal $\mathfrak{m}$ of R. Thus F lies over the filter of units if and only if $\mathfrak{p}$ lies over the maximal ideal.

Proposition 12.13. *Let R be a ring. Let $\varphi : R \to A$ be an R-algebra. Then the points of* $\operatorname{Spec}(A|R)$ *are intuitionistically in canonical one-to-one correspondence with those filters of A which lie over the filter of units.*

Proof. The correspondence outlined in Proposition 12.3 can be adapted to the situation at hand. A completely prime filter $K \subseteq \mathcal{T}(\operatorname{Spec}(A|R))$ corresponds to the ring-theoretic filter

$$F := \{s : A \mid \sqrt{(s)} \in K\}$$

and a ring-theoretic filter F corresponds to the completely prime filter

$$K := \{\mathfrak{a} : \mathcal{T}(\operatorname{Spec}(A|R)) \mid \mathfrak{a} \cap F \text{ is inhabited}\}.$$

It's instructive to perform some of the necessary verifications, to see how the quasicoherator is used, even though Proposition 12.14 will subsume this correspondence.

The filter F corresponding to K has the displayed property for the following reason. Let $\varphi(r) \in F$. We want to verify that r is invertible in R. Under the assumption that r is invertible in R, it's trivial that 1 is an element of

$$\begin{aligned}\mathfrak{a} &:= \sup\{\sqrt{(1)} \mid r \text{ is invertible in } R\} \\ &= \{s : A \mid s \text{ is nilpotent or } r \text{ is invertible in } R\} \in \mathcal{T}(\operatorname{Spec}(A)).\end{aligned}$$

Therefore, without any assumption on r, we have that $r \cdot 1 = \varphi(r)$ is an element of $\overline{\mathfrak{a}}$ and therefore $\sqrt{(\varphi(r))} \subseteq \overline{\mathfrak{a}}$. Since K is upward-closed, it follows that $\overline{\mathfrak{a}} \in K$. Since $\overline{\mathfrak{a}}$ is the supremum of the set $\{\sqrt{(1)} \mid r \text{ is invertible}\}$ in $\mathcal{T}(\operatorname{Spec}(A|R))$ and K is completely prime, it follows that this set is inhabited. Thus r is invertible in R.

The set K corresponding to a ring-theoretic filter F is completely prime for the following reason. Let $\sup_i \mathfrak{a}_i = \overline{\sqrt{\sum_i \mathfrak{a}_i}} \in K$. Then $\overline{\sqrt{\sum_i \mathfrak{a}_i}} \cap F$ is inhabited. By the special assumption on F, the intersection $\sqrt{\sum_i \mathfrak{a}_i} \cap F$ is inhabited as well: In the case that X is a scheme, this follows easily using the description of the quasicoherator given in Proposition 8.13. In the general case, we use the proof scheme outlined in Remark 8.16 – using the notation of that remark, if $P(\mathfrak{b}) \cap F$ is inhabited, then $\mathfrak{b} \cap F$ is as well.

A short calculation using the filter axioms then shows that there exists an index i such that $\mathfrak{a}_i \cap F$ is inhabited. ☐

Proposition 12.14. *Let R be a ring. Let $\varphi : R \to A$ be an algebra. Then* $\operatorname{Spec}(A|R)$ *is the classifying locale of the theory of filters of A which lie over the filter of units, that is of the geometric theory with atomic propositions "$s \in F$" for $s : A$ and axioms given by the following axiom schemes:*

(1) $\top \vdash 1 \in F$
(2) $st \in F \dashv\vdash s \in F \wedge t \in F$ *(two axioms for each $s, t : A$)*
(3) $0 \in F \vdash \bot$
(4) $s + t \in F \vdash s \in F \vee t \in F$ *(one axiom for each $s, t : A$)*
(5) $\varphi(r) \in F \vdash \bigvee\{\top \mid r \text{ is invertible in } R\}$ *(one axiom for each $r : R$)*

Proof. The frame of the classifying locale of the given theory T is the free frame on generators "$s \in F$" for $s : A$ subject to the relations given by the axioms of the theory. More explicitly, it's the Lindenbaum algebra $L(T)$ of the theory, so its elements are the formulas of the theory up to provable equivalence and the ordering is defined by $[\varphi] \preceq [\psi] :\Leftrightarrow (\varphi \vdash \psi)$. We want to verify that this frame is isomorphic to $\mathcal{T}(\operatorname{Spec}(A|R))$.

We define a frame homomorphism $L(T) \to \mathcal{T}(\operatorname{Spec}(A|R))$ by sending the generators $[s \in F]$ to the radical ideal $\overline{\sqrt{(s)}}$. This respects the relations and therefore gives a well-defined map. The map is surjective, since a preimage to $\mathfrak{a} : \mathcal{T}(\operatorname{Spec}(A|R))$

is $[\bigvee_{s\in\mathfrak{a}}(s \in F)]$. To verify that it is an isomorphism of frames, we therefore only have to verify that it reflects the ordering.

By the axiom schemes (1) and (2), any formula of T is provably equivalent to a formula of the form $\bigvee_i(s_i \in F)$. It therefore suffices to verify that, for any families $(s_i)_i$ and $(t_j)_j$ such that $\overline{\sqrt{(s_i)_i}} \subseteq \overline{\sqrt{(t_j)_j}}$, the sequent $\bigvee_i(s_i \in F) \vdash \bigvee_j(t_j \in F)$ is derivable. We'll show more generally: If $\mathfrak{a}$ and $\mathfrak{b}$ are radical ideals such that $\overline{\mathfrak{a}} \subseteq \overline{\mathfrak{b}}$, then $\bigvee_{s\in\mathfrak{a}}(s \in F) \vdash \bigvee_{t\in\mathfrak{b}}(t \in F)$. This follows from the following chain of deductions:

$$\bigvee_{s\in\mathfrak{a}}(s \in F) \vdash \bigvee_{s\in\overline{\mathfrak{a}}}(s \in F) \vdash \bigvee_{s\in\overline{\mathfrak{b}}}(s \in F) \vdash \bigvee_{s\in\mathfrak{b}}(s \in F).$$

All but the final step are trivial. The final step is an application of the general proof scheme outlined in Remark 8.16. In the notation of that remark, we set $\alpha(\mathcal{J}) := [\bigvee_{s\in\mathcal{J}}(s \in F)]$ and exploit that, if $s \in P(\mathcal{J})$, then $s \in F \vdash \bigvee_{t\in\mathcal{J}}(t \in F)$. This is because s can be written as $s^n = \sum_j a_j f_j u_j$ such that, for each j, if f_j is invertible in R then $u_j \in \mathcal{J}$, and we have the following chain of deductions.

$$\begin{aligned} s \in F \vdash{} & s^n \in F \\ \vdash{} & \bigvee_j (t_j f_j u_j \in F) \\ \vdash{} & \bigvee_j (\varphi(f_j) \in F \wedge u_j \in F) \\ \vdash{} & \bigvee_j \left(\bigvee\{\top \mid f_j \text{ invertible in } R\} \wedge u_j \in F\right) \\ \vdash{} & \bigvee_j \bigvee\{(u_j \in F) \mid f_j \text{ invertible in } R\} \\ \vdash{} & \bigvee_{t\in\mathcal{J}} (t \in F). \end{aligned}$$

□

Lemma 12.15. *Let R be a local ring. Let $\varphi : R \to A$ be an R-algebra. Then, intuitionistically, the locale* $\mathrm{Spec}(A|R)$ *carries a canonical structure as a locally ringed locale over* (pt, R) *and has the following universal property: For any locally ringed locale* $(Y, \mathcal{O}_Y)$ *over* (pt, R)*,*

$$\mathrm{Hom}_{\mathrm{LRL}/(\mathrm{pt},R)}(Y, \mathrm{Spec}(A|R)) \cong \mathrm{Hom}_{\mathrm{Alg}(R)}(A, \Gamma(Y, \mathcal{O}_Y)).$$

Proof. Since $\mathrm{Spec}(A|R)$ is a sublocale of $\mathrm{Spec}(A)$, we can equip $\mathrm{Spec}(A|R)$ with the restriction of $\mathcal{O}_{\mathrm{Spec}(A)}$ to $\mathrm{Spec}(A|R)$ as the structure sheaf:

$$\mathcal{O}_{\mathrm{Spec}(A|R)} := i^{-1}\mathcal{O}_{\mathrm{Spec}(A)} = i^{-1}(\underline{A}[\mathcal{F}^{-1}]) \cong (i^{-1}\underline{A})[(i^{-1}\mathcal{F})^{-1}] \cong \underline{A}[(i^{-1}\mathcal{F})^{-1}].$$

The generic filter $\mathcal{F}$ was described in Section 11.2. The penultimate isomorphism is because localizing is a geometric construction. Since locality of a ring is a geometric implication, this structure sheaf is indeed a local sheaf of rings. Thus $\mathrm{Spec}(A|R)$ is a locally ringed locale.

Next, we have to describe a morphism $(\mathrm{Spec}(A|R), \mathcal{O}_{\mathrm{Spec}(A|R)}) \to (\mathrm{pt}, R)$. Locale-theoretically, this morphism is given by the unique map $! : \mathrm{Spec}(A|R) \to \mathrm{pt}$. The ring-theoretic part is given by the composition

$$!^{-1}R = \underline{R} \longrightarrow \underline{A} \longrightarrow \underline{A}[(i^{-1}\mathcal{F})^{-1}] = \mathcal{O}_{\mathrm{Spec}(A|R)}.$$

This homomorphism of rings which happen to be local is indeed a local homomorphism, that is, it reflects invertibility. More precisely,

$$\mathrm{Spec}(A|R) \models \forall f : \underline{R}.\ \ulcorner \varphi(f) \text{ is inv. in } \mathcal{O}_{\mathrm{Spec}(A|R)} \urcorner \Rightarrow \ulcorner f \text{ is inv. in } \underline{R} \urcorner.$$

Denoting the modal operator associated to the sublocale inclusion $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$ by "$\Box$", this statement is equivalent to

$$\mathrm{Spec}(A) \models (\forall f : \underline{R}.\ \underline{\varphi}(f) \in \mathcal{F} \Rightarrow \ulcorner f \text{ is inv. in } \underline{R}\urcorner)^{\Box}$$

by Theorem 6.31 and Lemma 6.23. To verify this, let $s : A$ and $f : R$ be given such that $\sqrt{(s)} \models \varphi(f) \in \mathcal{F}$, that is, $s \in \sqrt{(\varphi(f))}$. We are to show that $\sqrt{(s)} \models \Box(\ulcorner f \text{ is invertible in } \underline{R}\urcorner)$.

The largest open in $\mathrm{Spec}(A)$ on which $\ulcorner f$ is invertible in $\underline{R}\urcorner$ holds is

$$\begin{aligned}\mathfrak{a} &:= \sup\{\sqrt{(1)} \mid f \text{ is invertible in } R\} \\ &= \{t : A \mid t \text{ is nilpotent or } f \text{ is invertible in } R\} \in \mathcal{T}(\mathrm{Spec}(A)),\end{aligned}$$

by Lemma 11.1. Under the assumption that f is invertible in R, trivially $1 \in \mathfrak{a}$. Therefore, without any assumptions on f, we have that $\varphi(f) \in \overline{\mathfrak{a}}$. Thus $\sqrt{(\varphi(f))} \subseteq \overline{\mathfrak{a}}$ and therefore $\sqrt{(\varphi(f))} \models \Box(\ulcorner f \text{ is invertible in } \underline{R}\urcorner)$. Since $\sqrt{(s)} \subseteq \sqrt{(\varphi(f))}$, the monotonicity of the internal language implies $\sqrt{(s)} \models \Box(\ulcorner f \text{ is invertible in } \underline{R}\urcorner)$.

Finally, we verify the universal property. Let Y be a locally ringed locale over (pt, R) and let a morphism $A \to \Gamma(Y, \mathcal{O}_Y)$ of R-algebras be given. We like this data to uniquely induce a morphism $Y \to \mathrm{Spec}(A|R)$ of locally ringed locales over (pt, R).

To obtain a locale-theoretic map $f : Y \to \mathrm{Spec}(A|R)$, by Proposition 12.14 we need to specify a filter of $\underline{A}$ in $\mathrm{Sh}(Y)$ which lies over the filter of units. The given morphism $A \to \Gamma(Y, \mathcal{O}_Y)$ induces a morphism $\alpha : \underline{A} \to \mathcal{O}_Y$ in $\mathrm{Sh}(Y)$. Since $\mathcal{O}_Y$ is a local ring, the subsheaf $\mathcal{O}_Y^\times$ is a filter. Its preimage $F := \alpha^{-1}\mathcal{O}_Y^\times$ is the sought filter of $\underline{A}$. It lies over the filter of units because the composition $\underline{R} \to \underline{A} \to \mathcal{O}_Y$ is local. By the general theory, the pullback of the generic filter in $\mathrm{Sh}(\mathrm{Spec}(A|R))$ to $\mathrm{Sh}(Y)$ along f is F.

The ring-theoretic part of the sought morphism $Y \to \mathrm{Spec}(A|R)$ of locally ringed locales over (pt, R) is the canonical homomorphism

$$f^{-1}\mathcal{O}_{\mathrm{Spec}(A|R)} = f^{-1}(\underline{A}[(i^{-1}\mathcal{F})^{-1}]) = \underline{A}[F^{-1}] \longrightarrow \mathcal{O}_Y$$

of local rings.

This finishes the description of the construction. We omit further verifications that the construction works as claimed. $\Box$

Remark 12.16. The modal operator $\Box$ associated to the inclusion $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$ can be defined in the internal language of $\mathrm{Sh}(\mathrm{Spec}(A))$. Namely, it's the smallest operator such that the $\Box$-translated statement

$$(\ulcorner \text{the morphism } \underline{R} \to \mathcal{O}_{\mathrm{Spec}(A)} \text{ is local}\urcorner)^{\Box}$$

holds. It is thus the smallest operator such that for any $f : \underline{R}$ with $\underline{\varphi}(f) \in \mathcal{F}$, $\Box(\ulcorner f \text{ is invertible in } \underline{R}\urcorner)$. The sublocale $\mathrm{Spec}(A|R)$ is therefore the largest sublocale of $\mathrm{Spec}(A)$ on which the morphism $\underline{R} \to \mathcal{O}_{\mathrm{Spec}(A)}$ is local.

Remark 12.17. In classical logic, the sublocale $\mathrm{Spec}(A|R)$ is closed in $\mathrm{Spec}(A)$, coinciding with $V(\mathfrak{m}_R A)$ (see Remark 12.24). But we don't think that this property can be verified intuitionistically.[22] For technical reasons, it would be nice to know that $\mathrm{Spec}(A|R)$ is an essential sublocale, since the pullback functor admits a simpler description for essential sublocales. It is an intersection of open, hence essential, sublocales, but the intersection of essential sublocales needn't be essential [72]. We

[22] The nucleus corresponding to the sublocale $\mathrm{Spec}(A|R)$ is the quasicoherator $\mathfrak{a} \mapsto \overline{\mathfrak{a}}$. Assume that the sublocale is closed. Then there is a radial ideal $\mathfrak{b}$ such that $\overline{\mathfrak{a}} = \sqrt{\mathfrak{a} + \mathfrak{b}}$ for all radical ideals $\mathfrak{a} \subseteq A$. This radical ideal $\mathfrak{b}$ is uniquely determined by $\mathfrak{b} = \overline{\sqrt{(0)}}$. Thus we obtain the simple description $\overline{\mathfrak{a}} = \sqrt{\mathfrak{a} + \overline{\sqrt{(0)}}}$ for the quasicoherator. We don't believe that this is right in general.

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Proof of Theorem 12.10. Follows immediately by interpreting the intuitionistic proof of Lemma 12.15 in the internal language of $\mathrm{Sh}(X)$, applied to $R := \mathcal{O}_X$ and $A := \mathcal{A}$. Then "$(\mathrm{pt}, \mathcal{O}_X)$" actually refers to the locally ringed locale $(X, \mathcal{O}_X)$ and "$\Gamma(Y, \mathcal{O}_Y)$" refers to $\mu_* \mathcal{O}_Y$, where $\mu : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a locally ringed locale over $(X, \mathcal{O}_X)$. □

Theorem 12.10 settles the question how the the little Zariski topos of $\underline{\mathrm{Spec}}_X(\mathcal{A})$ looks like from the internal point of view of $\mathrm{Sh}(X)$. A related question is how the big Zariski topos looks like. We give the answer in Theorem 16.9.

A basic fact about the ordinary spectrum is that the ring of global sections of $\mathcal{O}_{\mathrm{Spec}(A)}$ is canonically isomorphic to A. This is not true for the local spectrum: A trivial example is given by any nonzero algebra over a ring R which is not local in the sense that there are some nonunits which sum to a unit. In this case the theory which $\mathrm{Spec}(A|R)$ classifies is inconsistent. Thus $\mathrm{Spec}(A|R)$ is the empty locale and the ring of global sections of $\mathcal{O}_{\mathrm{Spec}(A|R)}$ is the zero ring.[23]

Proposition 12.18. *Let R be a local ring. Let A be an R-algebra. The canonical homomorphism $A \to \Gamma(\mathrm{Spec}(A|R), \mathcal{O}_{\mathrm{Spec}(A|R)})$ is an isomorphism in the following situations:*

1. *The algebra A satisfies the quasicoherence condition given in Theorem 8.3.*
2. *The algebra A is local and the structure morphism $R \to A$ is local.*

Furthermore, the sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$ is a sheaf for the modal operator associated to the sublocale $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$ *if and only if A satisfies the quasicoherence condition.*

Proof. We only verify the claim in the second situation. In this case $1 \in \overline{\mathfrak{a}}$ implies $1 \in \mathfrak{a}$ for any radical ideal of A, as can be checked using the proof scheme given in Remark 8.16. Hence $\mathrm{Spec}(A|R)$ is a local locale, meaning that for any covering $\sqrt{(1)} = \bigvee_i \mathfrak{a}_i = \sqrt{\sum_i \mathfrak{a}_i}$ of the top element of $\mathcal{T}(\mathrm{Spec}(A|R))$, there is an index i such that $\mathfrak{a}_i = \sqrt{(1)}$.

The locale $\mathrm{Spec}(A|R)$ thus has an initial (locale-theoretic) point. This focal point can be explicitly described: it is the filter $A^\times$ (which lies over the filter of units because $R \to A$ is local). As generally the case for local locales, taking global sections is the same as taking the stalk at the focal point. Therefore we can conclude by the following string of isomorphisms.

$$\Gamma(\mathrm{Spec}(A|R), \mathcal{O}_{\mathrm{Spec}(A|R)}) \cong \mathcal{O}_{\mathrm{Spec}(A|R), A^\times} \cong A[(A^\times)^{-1}] \cong A. \qquad \square$$

Corollary 12.19. *Let X be a scheme. Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. Let $f : \underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ be the canonical projection morphism. The canonical morphism $\mathcal{A} \to f_* \mathcal{O}_{\underline{\mathrm{Spec}}_X(\mathcal{A})}$ of $\mathcal{O}_X$-algebras is an isomorphism in the following situations:*

1. *The algebra $\mathcal{A}$ is quasicoherent.*
2. *From the point of view of* $\mathrm{Sh}(X)$*, the algebra $\mathcal{A}$ is local and the homomorphism $\mathcal{O}_X \to \mathcal{A}$ is local. (This means that, for every point $x \in X$, the stalk $\mathcal{A}_x$ is local and the homomorphism $\mathcal{O}_{X,x} \to \mathcal{A}_x$ is local.)*

Proof. This is just the interpretation of Proposition 12.18 internal to $\mathrm{Sh}(X)$. □

[23] This failure is not entirely unexpected, since Coste's general result on sheaf representations [43, Theorem 5.1.1], which would immediately guarantee that the global sections of $\mathcal{O}_{\mathrm{Spec}(A|R)}$ are in canonical one-to-one correspondence with the elements of A, is not applicable: The theory which $\mathrm{Spec}(A|R)$ classifies is not a coherent theory. The set-indexed disjunction appearing in axiom scheme (5) of the description given in Proposition 12.14 can't be rewritten as a finite disjunction.

Remark 12.20. Naively one might think that the canonical morphism $\mathcal{A} \to f_*\mathcal{O}_{\underline{\mathrm{Spec}}_X(\mathcal{A})}$ of Corollary 12.19 is the canonical morphism from $\mathcal{A}$ to the quasicoherization of $\mathcal{A}$. This is not the case. Firstly, the canonical morphism obtained by quasicoherization goes in the other direction. Secondly, as stated in Corollary 12.19, the canonical morphism can be an isomorphism even if $\mathcal{A}$ is not quasicoherent.

12.5. Comparing the different spectrum constructions. For rings and algebras, there are at least the following spectrum constructions.

- The ordinary spectrum of a ring, possibly realized as a locale instead of a topological space in order to work in an intuitionistic setting: $\mathrm{Ring}^{\mathrm{op}} \to \mathrm{LRS}$ or $\mathrm{Ring}^{\mathrm{op}} \to \mathrm{LRL}$
- The local spectrum of an algebra: $\mathrm{Alg}(R)^{\mathrm{op}} \to \mathrm{LRL}/(\mathrm{pt}, R)$
- Gillam's spectrum of a sheaf of algebras [53]: $\mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}} \to \mathrm{LRS}/(X, \mathcal{O}_X)$
- Hakim's spectrum of a ringed topos [58], yielding a locally ringed topos: $\mathrm{RT} \to \mathrm{LRT}$.
- Cole's general framework for spectrum constructions [33] (also reported on at [70, Theorem 6.58])

These are related as follows.

As described in Section 12.4, the ordinary spectrum construction cannot only be applied to rings, but also to sheaves of rings and indeed ring objects internal to arbitrary elementary toposes equipped with a natural numbers object, by employing the internal language. Applied to a ring $\mathcal{O}$ internal to such a topos $\mathcal{E}$, it yields a locally ringed locale internal to $\mathcal{E}$, or equivalently a locally ringed localic topos internal to $\mathcal{E}$. Externally, this corresponds to a locally ringed topos which is equipped with a localic geometric morphism to $\mathcal{E}$.

The ordinary spectrum construction can therefore be used to turn a ringed topos $(\mathcal{E}, \mathcal{O})$ (with a natural numbers object) into a locally ringed topos (which will be equipped with a morphism of ringed toposes to $(\mathcal{E}, \mathcal{O})$, but which will, even if $\mathcal{O}$ happens to be a local ring, not be equipped with a morphism of locally ringed toposes to $(\mathcal{E}, \mathcal{O})$).

By comparing the universal properties one sees that this kind of internal application of the ordinary spectrum construction coincides with the result of Hakim's spectrum construction. In fact, it can be interpreted as a simultaneous simplification and generalization of Hakim's construction: It's simpler, since it's just the familiar spectrum construction and no explicit site calculations are required; and it's more general, since Hakim's construction only applies to ringed Grothendieck toposes whereas the internally-performed construction of the ordinary spectrum applies to ringed elementary toposes with natural numbers object.

Gillam's spectrum coincides with internally performing the construction of the local spectrum, with the caveat that Gillam's construction starts with and yields a locally ringed space, whereas ours starts with and yields a locally ringed locale.[24] More precisely:

For a locale Y, let Y_P be the topological space of points of Y, and for a topological space T, let T_L be the induced locale. Let $(X, \mathcal{O}_X)$ be a sober locally ringed topological space. Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. Then we have a morphism $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)) \to X_L$ of locally ringed locales. Since $X \cong (X_L)_P$, there is an induced morphism $E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))_P \to X$ of locally ringed spaces. The adjunction $(\_\_)_L \dashv (\_\_)_P$ relating locales and topological spaces then yields, for any

[24] More generally, the local spectrum construction can be applied to any algebra over a local ring $\mathcal{O}$ internal to an elementary topos $\mathcal{E}$ with a natural numbers objects and yields a locally ringed topos equipped with a morphism of locally ringed toposes to $(\mathcal{E}, \mathcal{O})$.

locally ringed space $\mu : Y \to X$ over X,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{LRS}/X}(Y, E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))_P) &\cong \mathrm{Hom}_{\mathrm{LRL}/X_L}(Y_L, E(\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X))) \\ &\cong \mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_X)}(\mathcal{A}, \mu_*\mathcal{O}_Y).\end{aligned}$$

This is precisely the universal property which Gillam's spectrum enjoys.

Cole's framework for spectrum constructions is sufficiently general to encompass both the ordinary spectrum and the local spectrum, and by extension Hakim's spectrum and Gillam's spectrum. As is well-known, the ordinary spectrum can be obtained from Cole's framework by applying it to the geometric theory $\mathbb{S}$ of rings, its quotient theory $\mathbb{T}$ of local rings, and the admissible class $\mathbb{A}$ of local homomorphisms (notation as in [70, Theorem 6.58]). The local spectrum can be obtained by applying it to the geometric theory $\mathbb{S}$ of $\mathcal{O}_X$-algebras, its quotient theory $\mathbb{T}$ of local $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$, and the admissible class of local homomorphisms. For this to make sense, one has to interpret Cole's framework in the internal language of $\mathrm{Sh}(X)$, since there are no external geometric theories of (local) $\mathcal{O}_X$-algebras.

In general, the local spectrum doesn't coincide with the usual spectrum and Gillam's spectrum doesn't coincide with Hakim's spectrum. However, if the base space is a scheme of dimension ≤ 0, they do coincide.

Proposition 12.21. *Let X be a scheme. Then $E(\mathrm{Spec}(\mathcal{O}_X)) \cong X$ as locales over X if and only if $\dim X \leq 0$.*

Proof. The externalization of $\mathrm{Spec}\,\mathcal{O}_X$ coincides with X if and only if from the internal point of view, the locale $\mathrm{Spec}\,\mathcal{O}_X$ coincides with the one-point locale. By interpreting Lemma 12.5 in the internal language of $\mathrm{Sh}(X)$, it follows that this is the case if and only if

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \ulcorner f \text{ nilpotent}\urcorner \vee \ulcorner f \text{ invertible}\urcorner.$$

(Internally, it always holds that $\neg(1 = 0)$ in $\mathcal{O}_X$, even if X happens to be the empty scheme. Therefore the lemma is indeed applicable.) By Corollary 3.14, this condition is equivalent to the dimension of X being less than or equal to zero (i. e. to X being empty or having dimension exactly zero). ☐

Corollary 12.22. *Let X be a scheme. Then the relative spectrum of $\mathcal{O}_X$-algebras can be calculated by the internal spectrum (instead of the internal local spectrum) if and only if $\dim X \leq 0$.*

Proof. The externalization of the internal spectrum of arbitrary $\mathcal{O}_X$-algebras $\mathcal{A}$ coincides with the relative spectrum if and only if it coincides in the special case $\mathcal{A} = \mathcal{O}_X$. This is apparent by the universal properties of both constructions. Thus the claim follows from Proposition 12.21. ☐

Which construction is more fundamental, the ordinary spectrum of a ring or the local spectrum of an algebra? The ordinary spectrum $\mathrm{Spec}(A)$ can be expressed as the local spectrum $\mathrm{Spec}(A^\sim|\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})})$, where $A^\sim$ is the induced quasicoherent algebra on $\mathrm{Sh}(\mathrm{Spec}(\mathbb{Z}))$. This fact is well-known in the alternate form "$\underline{\mathrm{Spec}}_{\mathrm{Spec}(\mathbb{Z})}(A^\sim) \cong \mathrm{Spec}(A)$".

Fast and loose reasoning as follows could lead one to believe that it's similarly possible to express the local spectrum as an ordinary spectrum. Let R be a local ring. Let $\varphi : R \to A$ be an algebra. The points of $\mathrm{Spec}(A|R)$ are those filters $F \subseteq A$ such that $\varphi^{-1}F = R^\times$. Illicitly assuming classical logic, the points of $\mathrm{Spec}(A|R)$ are in canonical one-to-one correspondence with those prime ideals $\mathfrak{p} \subseteq A$ such that $\varphi^{-1}\mathfrak{p} = \mathfrak{m}_R$. The points of $\mathrm{Spec}(A|R)$ are therefore in canonical one-to-one correspondence with the points of $\mathrm{Spec}(A \otimes_R k)$, where $k = R/\mathfrak{m}_R$ is the residue field of R. Therefore $\mathrm{Spec}(A|R)$ and $\mathrm{Spec}(A \otimes_R k)$ might coincide.

However, we have the following negative result.[25]

Proposition 12.23. *In general, the local spectrum of an algebra can't be expressed as an ordinary spectrum.*

Proof. It is well-known that the ordinary spectrum is always quasicompact. The local spectrum, however, can fail to be quasicompact. A quick way to see this is to notice that, if that was the case, the locale-theoretic part of the projection morphism $\underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ would always be a proper map of locales [130].

There's also a more direct way of seeing this, which in fact proves a slightly stronger statement. Let X be a scheme. Let $f \in \Gamma(X, \mathcal{O}_X)$. From the internal point of view of $\mathrm{Sh}(X)$, the local spectrum $\mathrm{Spec}(\mathcal{O}_X[f^{-1}]|\mathcal{O}_X) \hookrightarrow \mathrm{Spec}(\mathcal{O}_X|\mathcal{O}_X) \cong \mathrm{pt}$ is the open sublocale of pt corresponding to the truth value of "f is invertible". Explicitly, the frame of opens of $\mathrm{Spec}(\mathcal{O}_X[f^{-1}]|\mathcal{O}_X)$ is isomorphic to $\{\psi : \Omega \mid \psi \Rightarrow f \text{ is invertible}\}$.

The ordinary spectrum always has the Frobenius reciprocity property, being quasicompact. In contrast, the locale $\mathrm{Spec}(\mathcal{O}_X[f^{-1}]|\mathcal{O}_X)$ has this property if and only if f is nilpotent or invertible. ☐

Remark 12.24. Even in classical logic, where the local spectrum $\mathrm{Spec}(A|R)$ and the ordinary spectrum $\mathrm{Spec}(A \otimes_R R/\mathfrak{m}_R)$ coincide as locales, they do not coincide as locally ringed locales. The structure sheaf of $\mathrm{Spec}(A|R)$, regarded as a sheaf on $\mathrm{Spec}(A \otimes_R R/\mathfrak{m}_R)$, is $i^{-1}\mathcal{O}_{\mathrm{Spec}(A)}$, where $i : \mathrm{Spec}(A \otimes_R R/\mathfrak{m}_R) \hookrightarrow \mathrm{Spec}(A)$ is the closed immersion corresponding to the inclusion $\mathrm{Spec}(A|R) \hookrightarrow \mathrm{Spec}(A)$. It's in general not $\mathcal{O}_{\mathrm{Spec}(A \otimes_R R/\mathfrak{m}_R)}$.

Finally, we want to restate the universal properties of the ordinary spectrum and the local spectrum in ring-theoretic language, employing the dual categories $\mathrm{RL}^{\mathrm{op}}$ and $\mathrm{LRL}^{\mathrm{op}}$, as in Section 12.3.

Let A be a ring. The morphism $A \to \mathcal{O}_{\mathrm{Spec}(A)}$ in $\mathrm{RL}^{\mathrm{op}}$ (the ring-theoretic part of the canonical morphism $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \to (\mathrm{Set}, A)$) is the *universal localization* of A: The ring $\mathcal{O}_{\mathrm{Spec}(A)}$ is local, and for any morphism $A \to \mathcal{B}$ into a local ring $\mathcal{B}$ (over any locale), there is a unique local morphism $\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{B}$ rendering the diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{\mathcal{B}} \\
& \underset{\text{local}}{\mathcal{O}_{\mathrm{Spec}(A)}} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

commutative. In contrast, the universal property of the local spectrum is as follows. Let R be a ring. Let A be an R-algebra. The morphism $A \to \mathcal{O}_{\mathrm{Spec}(A|R)}$ is the

[25]Intuitionistically, it's still true that the prime ideals of a quotient ring $A/\mathfrak{p}$ are in one-to-one correspondence with those prime ideals of A which contain $\mathfrak{p}$. However, the analogous statement "filters of A/F correspond to those filters of A which are contained in F" can't be shown intuitionistically, if A/F is defined as A/F^c. However, informally speaking, this failure is not the fault of the statement, but of the definition of A/F. The definition raises red flags from an intuitionistic point of view, since not F, but only its complement F^c enters the construction.
The statement can be salvaged by defining "A/F" to mean the set A equipped with a new *apartness relation* defined by $a \# b :\Leftrightarrow a - b \in F$. (A basic example for a ring-with-apartness-relation is the field of real numbers equipped with $x \# y :\Leftrightarrow \exists q \in \mathbb{Q}.\ |x - y| \geq q > 0$.) A filter G of this ring-with-apartness-relation A is by definition a subset $G \subseteq A$ which verifies the filter axioms and which is *open with respect to the apartness relation* in that for any elements $a, b : A$, the implication $a \in G \Rightarrow (b \in G) \vee (a \# b)$ holds.
This construction provides one of several motivations for developing the theory of rings using apartness relations and anti-ideals; one can even define the spectrum of a ring-with-apartness-relation. However, we'll not pursue these ideas further here.

universal way of turning A into a local ring *which is local over* R: The ring $\mathcal{O}_{\mathrm{Spec}(A|R)}$ is local, the composition $R \to A \to \mathcal{O}_{\mathrm{Spec}(A|R)}$ is local, and for any morphism $A \to \mathcal{B}$ into a local ring (over any locale) such that the composition $R \to A \to \mathcal{B}$ is local, there is a unique local morphism $\mathcal{O}_{\mathrm{Spec}(A|R)} \to \mathcal{B}$ such that the diagram

$$\begin{tikzcd}
R \arrow[r] \arrow[rrr, bend left, "\text{local}"] \arrow[drr, bend right, "\text{local}"'] & A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{\mathcal{B}} \\
& & \underset{\text{local}}{\mathcal{O}_{\mathrm{Spec}(A|R)}} \arrow[ur, dashed, "\text{local}"'] &
\end{tikzcd}$$

commutes.

Remark 12.25. It's possible to state the universal property of the structure sheaf of the big Zariski topos of a ring A, more precisely of the canonical morphism $(\mathrm{Zar}(A), \underline{\mathbb{A}}^1_{\mathrm{Spec}\,A}) \to (\mathrm{Set}, A)$ of ringed toposes, in a similar manner, employing the dual categories $\mathrm{RT}^{\mathrm{op}}$ and $\mathrm{LRT}^{\mathrm{op}}$ of the categories of (locally) ringed toposes. However, unlike the universal property of the spectrum, this universal property looks slightly odd from an algebraic point of view: For any morphism $A \to \mathcal{B}$ into a local ring (over any topos $\mathcal{E}$), there is a unique bijective homomorphism $\underline{\mathbb{A}}^1_{\mathrm{Spec}\,A} \to \mathcal{B}$ rendering the diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \overset{\text{local}}{\mathcal{B}} \\
& \underset{\text{local}}{\underline{\mathbb{A}}^1_{\mathrm{Spec}\,A}} \arrow[ur, dashed, "\text{bijective}"'] &
\end{tikzcd}$$

commutative. By "bijective" we mean that the ring-theoretic part $f^\sharp : f^{-1}\underline{\mathbb{A}}^1_{\mathrm{Spec}\,A} \to \mathcal{B}$ of the morphism $f : (\mathcal{E}, \mathcal{B}) \to (\mathrm{Zar}(A), \underline{\mathbb{A}}^1_{\mathrm{Spec}\,A})$ is bijective as seen from the internal point of view of $\mathcal{E}$.

12.6. The spectrum of the generic ring. Let Set[Ring] be the classifying topos of the theory of rings; explicitly, it's the topos of presheaves on $\mathrm{Ring}_{\mathrm{fp}}^{\mathrm{op}}$, the dual of the category of finitely presented rings. This topos contains the *generic ring* U (explicitly the presheaf $R \mapsto R$): any ring in any topos is the pullback of U along a suitable geometric morphism.

Let Set[LocRing] be the classifying topos of the theory of local rings. Explicitly, it's the big Zariski topos $\mathrm{Zar}(\mathrm{Spec}(\mathbb{Z}))$ (built using one of the *parsimonious sites*, as described in Section 15). This topos contains the *generic local ring* U': any local ring in any topos is the pullback of U' along a suitable geometric morphism.

Let A be a ring. By the universal property of Set[Ring], there is a geometric morphism $g : \mathrm{Set} \to \mathrm{Set[Ring]}$ such that $g^{-1}U \cong A$. Since U' is in particular a ring, again by the universal property of Set[Ring], there is a geometric morphism $f : \mathrm{Set[LocRing]} \to \mathrm{Set[Ring]}$ such that $f^{-1}U \cong U'$. By the universal property of Set[LocRing], the topos of sheaves over the spectrum of A admits a geometric morphism g' to Set[LocRing] such that $(g')^{-1}U' \cong \mathcal{O}_{\mathrm{Spec}(A)}$.

The resulting solid diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[dr, dotted, "h"] \arrow[drr, dotted, bend left, "\tilde{g}"] \arrow[ddr, dotted, bend right, "\tilde{f}"'] & & \\
& \mathrm{Sh}(\mathrm{Spec}(A)) \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Nearrow \eta"] & \mathrm{Set}[\mathrm{LocRing}] \arrow[d, "f"] \\
& \mathrm{Set} \arrow[r, "g"'] & \mathrm{Set}[\mathrm{Ring}]
\end{tikzcd}$$

commutes up to a non-invertible natural transformation η; under the equivalence

$$\begin{aligned}
&\text{category of geometric morphisms } \mathrm{Sh}(\mathrm{Spec}(A)) \to \mathrm{Set}[\mathrm{Ring}] \simeq \\
&\qquad\qquad\qquad\qquad\text{category of ring objects in } \mathrm{Sh}(\mathrm{Spec}(A))
\end{aligned}$$

this transformation corresponds to the non-invertible localization homomorphism $\underline{A} \to \underline{A}[\mathcal{F}^{-1}] = \mathcal{O}_{\mathrm{Spec}(A)}$. It is folklore that this square is a lax pullback square in the 2-category of Grothendieck toposes (for instance, this is reported on at [6]); however, this is not true.

Given a topos $\mathcal{E}$ together with geometric morphisms $\tilde{f} : \mathcal{E} \to \mathrm{Set}$ and $\tilde{g} : \mathcal{E} \to \mathrm{Set}[\mathrm{LocRing}]$ and a natural transformation $\tilde{\eta} : \tilde{f}^{-1} \circ g^{-1} \Rightarrow \tilde{g}^{-1} \circ f^{-1}$ (these data correspond to a local ring $\mathcal{O}_{\mathcal{E}}$ in $\mathcal{E}$ together with a ring homomorphism $\varphi : \underline{A} \to \mathcal{O}_{\mathcal{E}}$), there is a canonical geometric morphism $h : \mathcal{E} \to \mathrm{Sh}(\mathrm{Spec}(A))$ (determined by requiring that $h^{-1}\mathcal{F} \cong \mathcal{F}_0 := \varphi^{-1}[\mathcal{O}_{\mathcal{E}}^{\times}]$), and this morphism renders the lower left triangle commutative up to a natural isomorphism, but it renders the upper right triangle commutative only up to a non-invertible natural transformation (corresponding to the non-invertible ring homomorphism $\underline{A}[\mathcal{F}_0^{-1}] \to \mathcal{O}_E$).

The observation that the square is not a lax pullback is joint with Peter Arndt and Matthias Hutzler. The observation raises two questions: What is the lax pullback (which exists by general theory), if it's not $\mathrm{Sh}(\mathrm{Spec}(A))$? And how can $\mathrm{Sh}(\mathrm{Spec}(A))$ be described as a pullback? The following two propositions answer these questions. The geometric morphism $\mathrm{Set} \to \mathrm{Set}[\mathrm{Ring}]$ which they implicitly refer to is the morphism g mentioned above.

Proposition 12.26. *Let A be a ring. The lax pullback* $(\mathrm{Set} \Rightarrow_{\mathrm{Set}[\mathrm{Ring}]} \mathrm{Set}[\mathrm{LocRing}])$ *is the big Zariski topos of* $\mathrm{Spec}(A)$ *(built using one of the parsimonious sites, as described in Section 15).*

Proof. The claim can be checked by hand, but it's more instructive to employ the general theory of classifying toposes. In the situation

$$\begin{tikzcd}
(\mathrm{Set}[T] \Rightarrow_{\mathrm{Set}[T_0]} \mathrm{Set}[T']) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \eta"] & \mathrm{Set}[T'] \arrow[d, "f"] \\
\mathrm{Set}[T] \arrow[r, "g"'] & \mathrm{Set}[T_0],
\end{tikzcd}$$

where T_0, T, and T' are arbitrary geometric theories, the lax pullback classifies the geometric theory whose models consist of a model M of T, a model N of T', and a homomorphism $G(M) \to F(N)$ of T_0-models. The constructions G and F are given by the geometric morphisms g and f as follows:

Any object of $\mathrm{Set}[T]$ can be obtained by geometric constructions from U_T, the universal model of T in $\mathrm{Set}[T]$. In particular, the pullback $g^{-1}U_{T_0}$, which is a model of T_0, can be obtained by geometric constructions from U_T. Therefore the geometric morphism g displays a way to turn the generic model of T into a model of T_0 using only geometric constructions. The same constructions can be applied to any model M of T, yielding a model $G(M)$ of T_0.

In the concrete situation at hand, the theory T is the empty theory (admitting in any topos a unique model M), the theory T' is the theory of local rings, and T_0 is the theory of rings. The T_0-model $G(M)$ is the ring A. The T_0-model $F(N)$ of a local ring N is the underlying ring of N.

Therefore the lax pullback $(\mathrm{Set} \Rightarrow_{\mathrm{Set[Ring]}} \mathrm{Set[LocRing]})$ classifies ring homomorphisms $A \to R$ where R is a local ring, that is, local A-algebras. It's well-known that $\mathrm{Zar}(\mathrm{Spec}(A))$ classifies these as well. □

Proposition 12.27. *Let A be a ring. The pullback of the spectrum of the generic ring along* $\mathrm{Set} \to \mathrm{Set[Ring]}$ *is the spectrum of A.*

Proof. There are two related ways of making the statement precise. Firstly, the spectrum of the generic ring U can be interpreted as a (locally ringed) locale internal to Set[Ring]. Locales can be pulled back along geometric morphisms (even though the pullback of a frame along a geometric morphism typically fails to be a frame) [134]. In this way $\mathrm{Spec}(U)$ pulls back to a locale internal to Set, that is an ordinary external locale. The claim is that this locale is canonically isomorphic to $\mathrm{Spec}(A)$.

A second way to interpret the statement of the proposition is to regard the spectrum of the generic ring as a localic geometric morphism with codomain Set[Ring]. The claim is then that the diagram

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{Spec}(A)) \arrow[r] \arrow[d] & \mathrm{Sh}_{\mathrm{Set[Ring]}}(\mathrm{Spec}(U)) \arrow[d] \\
\mathrm{Set} \arrow[r, "g"'] & \mathrm{Set[Ring]}
\end{tikzcd}$$

is a pullback diagram in the 2-category of toposes.

Using the language of classifying locales and classifying toposes, both claims are easy to establish. The pulled-back locale (or topos) classifies the pulled-back geometric theory [134, Corollary 5.4]. Since the description of the theory which $\mathrm{Spec}(U)$ classifies – the theory of filters of U – is itself geometric, the pulled-back theory is the theory of filters of $g^{-1}U \cong A$.[26] □

Proposition 12.28. (1) *Let A be an R-algebra. The local spectrum* $\mathrm{Spec}(A|R)$ *is the pullback of* $\mathrm{Spec}(U''|R)$ *along the geometric morphism* $\mathrm{Set} \to \mathcal{E}$ *given by A, where U'' is the* generic R-algebra *contained in the classifying topos $\mathcal{E}$ of R-algebras.*

(2) *Let X be a scheme (or a locally ringed locale). Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra. The relative spectrum* $\underline{\mathrm{Spec}}_X(\mathcal{A})$ *is the pullback of* $\mathrm{Spec}(U''|\mathcal{O}_X)$ *along the geometric morphism* $\mathrm{Sh}(X) \to \mathcal{E}$ *given by $\mathcal{A}$, where U'' is the generic $\mathcal{O}_X$-algebra contained in the classifying* $\mathrm{Sh}(X)$*-topos $\mathcal{E}$ of $\mathcal{O}_X$-algebras.*

Proof. Straightforward modification of the proof of Proposition 12.27. □

Remark 12.29. The big Zariski topos $\mathrm{Zar}(\mathrm{Spec}(A))$ can be obtained as the pullback of the big Zariski topos of the generic ring U, if both toposes are understood to be defined using the parsimonious sites as described in Section 15.

[26] In the notation of [134, Section 5], the theory of filters of U is represented by a GRD-system with $G = U$ and $R = 1 \amalg U^2 \amalg U^2 \amalg 1 \amalg U^2$ (one summand for each axiom scheme).

12.7. Limits in the category of locally ringed locales. The category of ringed locales has small limits, by the naive construction. For instance, the fiber product $X \times_Z Y$ of ringed locales is given by the fiber product of the underlying locales and the structure sheaf $\pi_X^{-1}\mathcal{O}_X \otimes_{\pi_Z^{-1}\mathcal{O}_Z} \pi_Y^{-1}\mathcal{O}_Y$. More generally, the limit of a small diagram of ringed locales is given by the limit L of the underlying locales and the colimit of the pulled-back structure sheaves (calculated in the category of sheaves of rings on L).

However, when applied to a diagram of locally ringed locales, the ringed locale which this simple construction yields is in general not locally ringed. This can be nicely understood from the internal point of view: Let R be a local ring. Let $R \to A$ and $R \to B$ be local R-algebras which are furthermore local over R. Then the tensor product $A \otimes_R B$ is in general not a local ring. Indeed, this fails even in the easiest case, where all rings involved are fields: The rings $\mathbb{R}$ and $\mathbb{C}$ are local, and the inclusion $\mathbb{R} \to \mathbb{C}$ is local, but $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}[X]/(X^2+1) \cong \mathbb{C}[X]/(X^2+1) \cong \mathbb{C}\times\mathbb{C}$ is not.

The following proposition explains that the true limit in the category of locally ringed locales is obtained by *relocalizing* the limit in the category of ringed locales.

Proposition 12.30. *The category of locally ringed locales has all small limits.*

Proof. For notational simplicity, we describe how products in the category of locally ringed locales can be constructed. The general case is entirely analogous.

Let X and Y be locally ringed locales. Their product P as ringed locales has two defects: Firstly, it's not locally ringed. Secondly, the ring-theoretic parts of the projection morphisms $\pi_X : P \to X$ and $\pi_Y : P \to Y$ aren't local, that is, don't reflect invertibility.

The first issue could be solved by constructing, internally to $\mathrm{Sh}(P)$, the ordinary spectrum of $\mathcal{O}_P$. From the external point of view, this would yield a locally ringed locale equipped with morphisms of ringed, but not of locally ringed, locales to X and Y.

To solve both issues, we need to employ a refined spectrum construction, similar to the modification required by the internal account of the relative spectrum: Internally to $\mathrm{Sh}(P)$, we construct the classifying locale of the theory of those filters of $\mathcal{O}_P$ which simultaneously lie over the filter of units of $\pi_X^{-1}\mathcal{O}_X$ and which lie over the filter of units of $\pi_Y^{-1}\mathcal{O}_Y$. This locale is a sublocale of $\mathrm{Spec}(\mathcal{O}_P)$, the largest such that the morphisms to $(\mathrm{pt}, \pi_X^{-1}\mathcal{O}_X)$ and to $(\mathrm{pt}, \pi_Y^{-1}\mathcal{O}_Y)$ are morphisms of locally ringed locales.

The externalization of the internal locally ringed locale obtained in this way is the sought product of X and Y in the category of locally ringed locales. □

Remark 12.31. The category of locally ringed locales embeds as a (non-full) coreflective subcategory into the category of ringed locales; the coreflector maps a ringed locale $(X, \mathcal{O}_X)$ to the externalization of $\mathrm{Spec}(\mathcal{O}_X)$ (constructed internally to $\mathrm{Sh}(X)$). However, as is familiar in situations where the embedding is not full [4], it's in general not the case that limits in LRL are calculated by applying the coreflector to the limit calculated in RL. Employing the language of the proof of Proposition 12.30, applying the coreflector only solves the first issue, but not the second.

It's instructive to determine the points of limits in LRL, even though a locale is of course not determined by its points. For instance, the construction in Proposition 12.30 shows that the points of the product $X \times Y$ of locally ringed locales in LRL are in canonical one-to-one correspondence with tuples (x, y, F), where x is a point of X, y is a point of y, and F is a filter of $\mathcal{O}_{X,x} \otimes_{\mathbb{Z}} \mathcal{O}_{Y,y}$ which lies over the filter of units of $\mathcal{O}_{X,x}$ and of $\mathcal{O}_{Y,y}$. In classical logic, those tuples are in canonical one-to-one correspondence with tuples $(x, y, \mathfrak{p})$, where x and y are as before and $\mathfrak{p}$ is a prime ideal of $k(x) \otimes_{\mathbb{Z}} k(y)$.

Similarly, points of the fiber product $X \times_Z Y$ are in canonical one-to-one correspondence with tuples (x, y, F), where x is a point of X and y is a point of y such that both map to the same point z of Z, and F is a filter of $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Z,z}} \mathcal{O}_{Y,y}$ lying over the filter of units of $\mathcal{O}_{X,x}$ and of $\mathcal{O}_{Y,y}$ (and therefore automatically of $\mathcal{O}_{Z,z}$). In classical logic, those tuples are in canonical one-to-one correspondence with tuples $(x, y, \mathfrak{p})$, where x and y are as before and $\mathfrak{p}$ is a prime ideal of $k(x) \otimes_{k(z)} k(y)$.

Remark 12.32. By the adjunction $(\underline{\ \ })_L \dashv (\underline{\ \ })_P$ relating locales and topological spaces, limits of locally ringed spaces which happen to be sober can be calculated by regarding them as locally ringed locales by $(\underline{\ \ })_L$, calculating their limit in LRL, and taking the associated topological space of the limit by $(\underline{\ \ })_P$.

Small diagrams of arbitrary locally ringed spaces admit limits as well. Indeed, the proof of Proposition 12.30 was adapted from Gillam's proof of this fact [53, Corollary 5].

12.8. Relative Proj construction. Similar issues as with the relative spectrum arise with the Proj construction: The standard definition of the Proj construction as a topological space of homogeneous prime ideals gives rise to a space which can't intuitionistically be shown to satisfy the expected universal property. The construction has to be reimagined as a locale instead of a topological space. A certain sublocale of this locale then yields the relative Proj construction when interpreted in the internal language of the little Zariski topos of a base scheme (or a locally ringed locale).

Definition 12.33. The *Proj construction* of an $\mathbb{N}$-graded ring S is the locale with frame of opens given by

$$\begin{aligned}\mathcal{T}(\mathrm{Proj}(S)) := \{\mathfrak{a} \subseteq S \mid \mathfrak{a} \text{ is a homogeneous radical ideal such that} \\ \forall x : S.\ xS_+ \subseteq \mathfrak{a} \Rightarrow x \in \mathfrak{a}\},\end{aligned}$$

where $S_+ = \bigoplus_{i>0} S_i$ is the irrelevant ideal.

A quick way to see that the partial order $\mathcal{T}(\mathrm{Proj}(S))$ is a frame is to recognize that it's the frame of opens of a sublocale of $\mathrm{Spec}(S)$. The associated nucleus $j : \mathcal{T}(\mathrm{Spec}(S)) \to \mathcal{T}(\mathrm{Spec}(S))$ is given by

$$j(\mathfrak{a}) := (\sqrt{\mathfrak{a}^h} : S_+),$$

where $\mathfrak{a}^h$ is the homogenization of $\mathfrak{a}$, the ideal of S generated by all homogeneous components of the elements of $\mathfrak{a}$. Since $\mathfrak{a} \subseteq \mathfrak{a}^h \subseteq \sqrt{\mathfrak{a}^h} \subseteq j(\mathfrak{a})$, a radical ideal $\mathfrak{a}$ is an element of $\mathcal{T}(\mathrm{Proj}(S))$ if and only if $\mathfrak{a} = j(\mathfrak{a})$.

One way to derive this definition is to start, within a classical context, with the general expression for the nucleus associated to the subspace of $\mathrm{Spec}(S)$ consisting of those prime ideals which are homogeneous and don't contain S_+, and then rewrite this expression to not refer to prime ideals.

Definition 12.34. A filter $F \subseteq S$ in an $\mathbb{N}$-graded ring S is *homogeneous* if and only if, for any element $a : S$, the filter F contains a if it contains at least one of the homogeneous components of a. It *meets the irrelevant ideal* if and only if $F \cap S_+$ is inhabited.

In classical logic, a subset is a homogeneous filter meeting the irrelevant ideal if and only if its complement is a homogeneous prime ideal not containing the irrelevant ideal. Intuitionistically, neither direction can be shown.

Proposition 12.35. *Let S be an $\mathbb{N}$-graded ring. Then* $\mathrm{Proj}(S)$ *is the classifying locale of any of the following geometric theories.*

(1) *The theory of homogeneous filters of S meeting the irrelevant ideal, that is the theory of Remark 12.4 supplemented by the following two axiom schemes:*
- $\bigvee_i (a_i \in F) \vdash a \in F$ *(one axiom for each decomposition $a = \sum_i a_i$ of an element of S into homogeneous components)*
- $\top \vdash \bigvee_{a \in S_+} (a \in F)$ *(one axiom)*

(2) *The theory given by one atomic proposition "$a \in F_i$" for each homogeneous element a of degree i in S and axioms given by the following axiom schemes:*
- $\top \vdash 1 \in F_0$ *(one axiom)*
- $st \in F_{i+j} \dashv\vdash s \in F_i \wedge t \in F_j$ *(two axioms for each $i, j \geq 0$, $s \in S_i$, $t \in S_j$)*
- $0 \in F_i \vdash \bot$ *(one axiom for each $i \geq 0$)*
- $s + t \in F_i \vdash s \in F_i \vee t \in F_i$ *(one axiom for each $i \geq 0$, $s, t \in A_i$)*
- $\top \vdash \bigvee_{i \geq 1} \bigvee_{a \in S_i} (a \in F_i)$ *(one axiom)*

(3) *The same theory as in (2), but with atomic propositions only for homogeneous elements of degree ≥ 1 and without the first axiom "$\top \vdash 1 \in F_0$".*

Proof. That $\mathrm{Proj}(S)$ coincides with the classifying locale of the theory given in (1), can be verified by a direct calculation. By the general theory, the nucleus associated to the quotient theory given in (1) maps a radical ideal $\mathfrak{a} : \mathcal{T}(\mathrm{Spec}(S))$ to the least fixed point above of $\mathfrak{a}$ of the map

$$\mathfrak{b} \longmapsto \mathfrak{b} \vee \bigvee_{a : S} \Big(\sqrt{(a_i)_i} \cap \big(\sqrt{(a)} \to \mathfrak{b} \big) \Big) \vee \Big(\sqrt{(a)_{a \in S_+}} \to \mathfrak{b} \Big),$$

where $(\mathfrak{c} \to \mathfrak{b}) = (\mathfrak{b} : \mathfrak{c})$ is the Heyting implication and "$\vee$" is the join in $\mathcal{T}(\mathrm{Spec}(S))$. We omit the intermediate steps of the calculation.

The theories given in (1) and in (2) are bi-interpretable. The interpretation of the atomic propositions "$a \in F_i$" of theory (2) using the signature of theory (1) is "$a \in F$". Verifying the axioms is straightforward. Conversely, the interpretation of "$a \in F$" in the signature of theory (2) is "$\bigvee_i (a_i \in F_i)$", where $a = \sum_i a_i$ is the decomposition into homogeneous components. For verifying the axioms, one needs the lemma that

$$\bigvee_i (s_i \in F_i) \wedge \bigvee_j (t_j \in F_j) \;\dashv\vdash\; \bigvee_n \Big(\sum_{i+j=n} s_i t_j \in F_n \Big)$$

is derivable in theory (2), for any decompositions $s = \sum_i s_i$ and $t = \sum_j t_j$ of elements of S into homogeneous components. In the guise "$\sqrt{(s_i)_i} \cap \sqrt{(t_j)_j} = \sqrt{(\sum_{i+j=n} s_i t_j)_n}$" this is a familiar fact on the content of polynomials [12, Proposition 1].

Also theories (2) and (3) are bi-interpretable. The interpretation of "$a \in F_0$" in the signature of theory (3) is "$\bigvee_{i \geq 1} \bigvee_{h \in S_i} (ha \in F_i)$". □

Corollary 12.36. *Let S be an $\mathbb{N}$-graded ring. The points of* $\mathrm{Proj}(S)$ *are in canonical one-to-one correspondence with the homogeneous filters of S meeting the irrelevant ideal.*

Proof. Points of $\mathrm{Proj}(S)$ are given by models of the theory of homogeneous filters of S meeting the irrelevant ideal in Set. □

Remark 12.37. The same presentation as in Proposition 12.35(3) has been used to construct $\mathrm{Proj}(S)$ not as a locale, but as a distributive lattice [42].

Definition 12.38. Let S be an $\mathbb{N}$-graded ring. The *generic homogeneous filter meeting the irrelevant ideal* is the subsheaf $\mathcal{F} \hookrightarrow \underline{S}$ over $\mathrm{Proj}(S)$ generated by the sections a over $D_+(a) := j(\sqrt{(a)})$.

Equivalently, the generic homogeneous filter meeting the irrelevant ideal is the pullback of the generic filter in $\mathrm{Sh}(\mathrm{Spec}(S))$ to $\mathrm{Sh}(\mathrm{Proj}(S))$.

Definition 12.39. Let S be an $\mathbb{N}$-graded ring. The structure sheaf of $\mathrm{Proj}(S)$ is the homogeneous localization $\underline{S}[\mathcal{F}^{-1}]_0$ of the ring $\underline{S}$ at the generic homogeneous filter meeting the irrelevant ideal, that is the degree-zero part of $\underline{S}[\mathcal{F}^{-1}]$. The *tilde construction* of a graded S-module M is $M^\sim := \underline{M}[\mathcal{F}^{-1}]_0$.

The locally ringed locale $\mathrm{Proj}(S)$ and the tilde construction defined in this way enjoy their familiar properties. For instance, we have the following lemma.

Lemma 12.40. *Let S be an $\mathbb{N}$-graded ring.*

1. *Let $f : S$ be homogeneous of degree $d \geq 1$. Then $D_+(h) \cong \mathrm{Spec}(S[f^{-1}]_0)$.*
2. *Assume that S is generated as an S_0-algebra by S_1. Let M and N be graded S-modules. Then $M^\sim \otimes_{\mathcal{O}_{\mathrm{Proj}(S)}} N^\sim \cong (M \otimes_S N)^\sim$.*
3. *Under the same assumption as in (2), the twisting sheaves $\mathcal{O}(m) := (S(m))^\sim$ are finite locally free of rank 1.*

Proof. For the first statement, it suffices to verify that the theories of homogeneous filters of S meeting the irrelevant ideal and containing h and of filters of $S[f^{-1}]_0$ are bi-interpretable. It's slightly more convenient to use the presentation given by Proposition 12.35(2) for the former theory.

The interpretation of "$q \in F$" for $q : S[f^{-1}]_0$ in the signature of the theory given by Proposition 12.35(2) is

$$\bigvee \{ (x \in F_{di}) \mid q = x/f^i \text{ for some } x : S,\ i \geq 0 \}.$$

Conversely, the interpretation of "$a \in F_i$" in the signature of the theory of filters of $S[f^{-1}]_0$ is "$x^d/h^i \in F$".

The second statement follows from the calculation

$$\begin{aligned} M^\sim \otimes_{\mathcal{O}_{\mathrm{Proj}(S)}} N^\sim &= \underline{M}[\mathcal{F}^{-1}]_0 \otimes_{\underline{S}[\mathcal{F}^{-1}]_0} \underline{N}[\mathcal{F}^{-1}]_0 \\ &\cong (\underline{M} \otimes_{\underline{S}} \underline{N})[\mathcal{F}^{-1}]_0 \cong (\underline{M \otimes_S N})[\mathcal{F}^{-1}]_0 = (M \otimes_S N)^\sim. \end{aligned}$$

The first isomorphism maps $x/s \otimes y/t$ to $(x \otimes y)/(st)$. By the assumption that S is generated as an S_0-algebra by S_1, the generic filter contains a homogeneous element h of degree 1 from the internal point of view of $\mathrm{Sh}(\mathrm{Proj}(S))$. Therefore the map has an inverse sending $(a \otimes b)/u$, where a and b are homogeneous of degrees i and j, to $(h^j a)/u \otimes b/h^j$. The second isomorphism is because the tensor product is a geometric construction and therefore commutes with constructing the constant sheaf.

For the proof of the third statement, we show that $(S(m))^\sim$ is a finite free module of rank 1 from the internal point of view. We again use that the generic filter contains a homogeneous element $h : \underline{S}$ of degree 1 from the internal point of view. Such an element allows to define an isomorphism $\mathcal{O}_{\mathrm{Proj}(S)} = \underline{S}[\mathcal{F}^{-1}]_0 \to \underline{S(m)}[\mathcal{F}^{-1}]_0 = \mathcal{O}(m)$ by mapping x/s to $(h^m x)/s$ if $m \geq 0$ and to $x/(h^{-m}s)$ otherwise. □

Definition 12.41. Let R be a ring. Let S be an $\mathbb{N}$-graded R-algebra. The *local Proj construction* of S over R is the sublocale $\mathrm{Proj}(S|R)$ of $\mathrm{Proj}(S)$ with frame of opens given by

$$\mathcal{T}(\mathrm{Proj}(S|R)) := \{ \mathfrak{a} : \mathcal{T}(\mathrm{Proj}(S)) \mid \forall f : R.\ \forall s : S.\ (\ulcorner f \text{ inv.} \urcorner \Rightarrow s \in \mathfrak{a}) \Rightarrow fs \in \mathfrak{a} \}$$

and with the pullback of $\mathcal{O}_{\mathrm{Proj}(S)}$ as the structure sheaf.

Proposition 12.42. *Let R be a ring. Let S be an $\mathbb{N}$-graded R-algebra. Then the local Proj construction* $\mathrm{Proj}(S|R)$ *is the classifying locale of the theory of homogeneous filters of S meeting the irrelevant ideal and lying over the filter of units.*

Proof. Direct calculation similar to the proof of Proposition 12.35. □

Since pullback and localization commute, the structure sheaf of $\mathrm{Proj}(S|R)$ can also be described as $\underline{S}[\mathcal{F}^{-1}]_0$, where by abuse of notation we mean by "$\mathcal{F}$" the pullback of the generic filter on $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S|R)$. This filter has the special property

$$\mathrm{Sh}(\mathrm{Proj}(S|R)) \models \forall r : \underline{R}.\ r \in \mathcal{F} \Rightarrow \ulcorner r \text{ inv. in } \underline{R}\urcorner.$$

Theorem 12.43. *Let X be a scheme (or a locally ringed locale). Let $\mathcal{S}$ be an $\mathbb{N}$-graded $\mathcal{O}_X$-algebra. Then the externalization $E(\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X))$ coincides with the relative Proj construction $\underline{\mathrm{Proj}}_X(\mathcal{S})$ as locally ringed locales over X.*

Proof. For simplicity, we assume that $\mathcal{S}$ is generated as an $\mathcal{S}_0$-algebra by $\mathcal{S}_1$. In this case, the expected universal property of the relative Proj construction is that it's a locally ringed locale over X such that, for all locally ringed locales $\mu : Y \to X$ over X, the set $\mathrm{Hom}_{\mathrm{LRL}/X}(Y, \underline{\mathrm{Proj}}_X(\mathcal{S}))$ is canonically isomorphic (by pullback of the standard such datum on $\underline{\mathrm{Proj}}_X(\mathcal{S})$) to the set of pairs $(\mathcal{L}, \psi)$ such that

- $\mathcal{L}$ is a line bundle on Y and
- $\psi : \mu^*\mathcal{S} \to \bigoplus_{n\geq 0} \mathcal{L}^{\otimes n}$ is a graded morphism of $\mathcal{O}_Y$-algebras such that the degree-1 part of ψ is a surjective morphism $\mu^*\mathcal{S}_1 \to \mathcal{L}$

modulo equivalence. For instance, it is known that this property is satisfied in the case that X is a scheme and $\mathcal{S}$ is quasicoherent [118, Tag 01O4].

We verify that $E(\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X))$ enjoys the same property, even if X is not a scheme or $\mathcal{S}$ is not quasicoherent. For the rest of the proof, we switch to the internal universe of $\mathrm{Sh}(X)$.

The local Proj construction is a locally ringed locale over $(\mathrm{pt}, \mathcal{O}_X)$ by the unique morphism $! : \mathrm{Proj}(\mathcal{S}|\mathcal{O}_X) \to \mathrm{pt}$ of locales and by the canonical morphism $!^\sharp : \underline{\mathcal{O}_X} \to \underline{\mathcal{S}}_0 \to \underline{S}[\mathcal{F}^{-1}]_0 = \mathcal{O}_{\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)}$ of local rings.

As the standard datum on $\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$, we choose the line bundle $\mathcal{O}(1)$ (pulled back to $\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$) together with the canonical morphism $!^*\mathcal{S} \to \oplus_{n\geq 0}\mathcal{O}(1)^{\otimes n}$.

Let Y be a locally ringed locale over $(\mathrm{pt}, \mathcal{O}_X)$. Let a pair $(\mathcal{L}, \psi)$ be given. In the internal language of $\mathrm{Sh}(Y)$, we define a filter by the formula

$$\mathcal{F}' := \{s : \underline{S} \,|\, \ulcorner\text{there exists } i \text{ such that } (\psi(s_i \otimes 1)) \text{ is a basis of } \mathcal{L}^{\otimes i}\urcorner\} \subseteq \underline{S},$$

where s_i refers to the homogeneous component of s of degree i. Since $\mathcal{L}^{\otimes i}$ is finite free of rank 1, a one-element family in $\mathcal{L}^{\otimes i}$ is a basis if and only if it's a generating family. This observation can be repeatedly used to verify that $\mathcal{F}'$ is homogeneous, meets the irrelevant ideal, and lies over the filter of units. Since $\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$ is the classifying locale of such filters (Proposition 12.42), we obtain a morphism $f : Y \to \mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$ of locales which is unique with the property that $f^{-1}\mathcal{F} = \mathcal{F}'$.

To obtain a morphism $Y \to \mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)$ of locally ringed locales, it remains to define a morphism $f^\sharp : f^{-1}\mathcal{O}_{\mathrm{Proj}(\mathcal{S}|\mathcal{O}_X)} = \underline{S}[\mathcal{F}'^{-1}]_0 \to \mathcal{O}_Y$. A canonical choice is

$$x/s \mapsto \ulcorner \text{ the coefficient of } \psi(x \otimes 1) \text{ with respect to the basis } (\psi(s \otimes 1))\urcorner.$$

We omit further verifications. □

12.9. A constructive account of scheme theory. Scheme theory as classically set up heavily relies on prime ideals and therefore only works well in a classical context, where the law of excluded middle and (at least some forms of) the axiom of choice are available. However, the actual mathematical ideas often do not fundamentally require classical logic; we don't begin the proof that the kernel of a morphism between quasicoherent sheaves of modules is quasicoherent by supposing that it's not. Instead, classical logic is only needed because the usual foundations of scheme theory involving locally ringed spaces require it.

In this section, we sketch how scheme theory can be developed in an intuitionistic metatheory; there are several reasons why it's desirable to have such an account. Firstly, as is familiar from constructive treatments of other subjects, the constraint to set up all definitions in an intuitionistically sensible way is a useful guiding principle which can increase the perceived elegance of the theory and result in more direct proofs.

It would be interesting to know which advanced results in algebraic geometry *actually* require classical logic (or at least classicality hypotheses on the ground ring); to study this question, one has to use a foundation which doesn't itself require classical logic just for organizational purposes.[27] McLarty and other researchers study a similar question: Which axioms of set theory are actually needed for algebraic geometry, in particular for proving Fermat's Last Theorem? [90]

Secondly, one might be interested in concrete computations and might therefore leverage the fact that one can mechanically extract algorithms from constructive proofs. For instance, an intuitionistic proof that some cohomology is finite dimensional yields an algorithm for computing the dimension and even a basis.

Finally, one might want to apply scheme theory in the intuitionistic internal universe of the little Zariski topos of a base scheme, in order to generalize results of absolute scheme theory to relative scheme theory with little effort and no duplication of proofs. The starting point for such a transfer is that locally ringed locales over a locally ringed locale X look like locally ringed locales over the point from the internal point of view of $\mathrm{Sh}(X)$, as discussed in Section 12.1.

The internal language of the big Zariski topos, presented in Part III, is too a vehicle for relative scheme theory; however, its language looks quite different from what one is accustomed to.

In this section, we only sketch how the basics of constructive scheme theory could look like. Some parts are certainly folklore among constructive mathematicians, but to the best of our knowledge no coherent summary appeared in print before.

There is a vast literature on algorithmic computations in algebraic geometry (to exemplarily cite just two references, Eisenbud's textbook on syzygies [46] and the GAP project [124] are well-known). However, these results are often still set in a classical context, relying on classical logic for termination or correctness proofs. They therefore don't contain an intuitionistic development of scheme theory.

Constructive algebra. Any constructive development of scheme theory needs to rest on a constructive development of commutative algebra. Such an account is readily available [92, 83].

Local models. As discussed in Section 12.2, defining the spectrum of a ring as a topological space isn't sensible from a constructive point of view. A working alternative is defining the spectrum as a locally ringed locale, employing the frame of radical ideals. By considering sheaves over it, this yields a locally ringed topos; this topos can also be presented by a more parsimonious site, namely the site whose objects are the elements of the ring and whose coverings are those finite families $(g_i \to f)_i$ such that $\sqrt{(f)} = \sqrt{(g_i)_i}$.

[27] For instance, some results in linear algebra can intuitionistically only be shown for *discrete* fields – fields such that any element is zero or not zero. Such hypotheses are computationally meaningful and will entail similar hypotheses for some results in algebraic geometry.

This construction is due to Joyal [71, 49, 126] and was further explored by several researchers [41, 42]. It's also possible to employ the framework of formal topology [111].

The universal property of the localic spectrum ensures that morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of locally ringed locales (or locally ringed toposes) are in canonical one-to-one correspondence with ring homomorphisms $A \to B$, as it should be. There are two ways for explicitly constructing a morphism between spectra. One is to specify a morphisms of frames going in the other direction. For instance, given a ring homomorphism $\varphi : A \to B$, one can map a radical ideal $\mathfrak{a} \subseteq A$ to $\sqrt{\mathfrak{a}B} \subseteq B$; this yields a morphism $\mathcal{T}(\mathrm{Spec}(A)) \to \mathcal{T}(\mathrm{Spec}(B))$.

Using the device of classifying locales, there is also another way which more closely mimics the classical approach of taking preimages of prime ideals. To give a morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of locales amounts to give a model of the theory of filters of A in $\mathrm{Sh}(\mathrm{Spec}(B))$. The sheaf topos over $\mathrm{Spec}(B)$ contains the generic filter $\mathcal{F}$ of B; given a ring homomorphism $\varphi : A \to B$, this filter can be turned into a filter of A by taking the preimage $\varphi^{-1}[\mathcal{F}]$.

Classically, the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ would be described by $\mathfrak{p} \mapsto \varphi^{-1}[\mathfrak{p}]$, where $\mathfrak{p}$ ranges over all prime ideals of B, and after defining the map in this way, one would have to verify its continuity; constructively, we can describe it as $\mathcal{F} \mapsto \varphi^{-1}[\mathcal{F}]$, where $\mathcal{F}$ is just a single special filter, and get continuity for free. For more on this way of pretending that morphisms between locales are just maps between points, we highly recommend an expository survey by Vickers on this topic [132].

As we have seen in Section 11, for deriving transfer principles it's useful to be able to quickly gauge properties of constant sheaves over $\mathrm{Spec}(A)$. For topological spaces, Lemma 11.1 could be used to this effect. For locales in an intuitionistic metatheory, this lemma has to be modified slightly.

Definition 12.44. (1) A locale X is *overt* if and only if the unique morphism $X \to \mathrm{pt}$ of locales is open.
(2) A *positivity predicate* on a frame P is a predicate on the set of elements of P, written "$U > 0$" for $U : P$, such that for any element $U : P$ and any subset $M \subseteq P$,
- if $U > 0$ and $U \preceq \bigvee M$, then there exists an element $V \in M$ such that $V > 0$, and
- if $U > 0 \Longrightarrow U \preceq \bigvee M$, then $U \preceq \bigvee M$.

Example 12.45. The frame of open subsets of a topological space X has a positivity predicate, namely declaring $U > 0$ if and only if U is inhabited.

Example 12.46. Assuming classical logic, any frame admits the positivity predicate defined by declaring $U > 0$ if and only if $U \neq \bot$.

Proposition 12.47. *A locale X is overt if and only if its frame of opens admits a* positivity predicate*.*

Proof. Instructive unraveling of the definitions. ☐

Lemma 12.48. *Let φ be a first-order formula in which arbitrary sets and elements may occur as parameters. Let X be a locale and let U be an open of X. Consider the following statements:*

(1) *$U \models \varphi$ (with the same abuse of notation as in Lemma 11.1*
(2) *$U \preceq \bigvee\{\top \mid \varphi\}$.*
(3) *(If X has a positivity predicate.) $U > 0 \Longrightarrow \varphi$.*
(4) *φ.*

Then:

- *"(4) $\Rightarrow$ (2)".*
- *If φ is a geometric formula, then "(1) $\Leftrightarrow$ (2)".*
- *If all subformulas of φ appearing as antecedents of implications satisfy "(1) $\Rightarrow$ (2)" (for instance, because they are geometric formulas or because φ doesn't contain any "$\Rightarrow$" signs), then "(2) $\Rightarrow$ (1)".*
- *If X is overt, then "(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)".*

Proof. The implication "(4) $\Rightarrow$ (2)" is trivial. The other claims can be checked by induction on the structure of φ. $\square$

Remark 12.49. It is no coincidence that the conditions in Lemma 12.48 are reminiscent of the conditions in Lemma 6.25. In fact, associated to any locale X is a modal operator $\Box_X$ on Set, which implicitly appeared in Lemma 12.48. It maps a truth value φ to the truth value $[\![X \preceq \bigvee\{\top \mid \varphi\}]\!]$. The associated sublocale $\mathrm{pt}_{\Box_X}$ of the one-point locale is the image of the unique locale morphism $X \to \mathrm{pt}$.

Proposition 12.50. *The localic spectrum of a ring A is overt if and only if any element of A is nilpotent or not nilpotent.*

Proof. For the "if" direction, we can define a positivity predicate by declaring $\mathfrak{a} > 0$ if and only if $\mathfrak{a}$ contains an element which is not nilpotent.

For the "only if" direction, let $f : A$ be an arbitrary element. Then

$$\begin{aligned}\sqrt{(f)} &\subseteq \bigvee\{\sqrt{(1)} \mid \sqrt{(f)} > 0\} \\ &\subseteq \bigvee\{\sqrt{(1)} \mid f \text{ is not nilpotent}\} \\ &= \{s : A \mid s \text{ is nilpotent or } f \text{ is not nilpotent}\}.\end{aligned}$$

Considering that $f \in \sqrt{(f)}$, it follows that f is nilpotent or f is not nilpotent. $\square$

Remark 12.51. We don't know when the local spectrum $\mathrm{Spec}(A|R)$ is overt. This question is related to openness of morphisms between schemes as follows. Let X be a scheme (in a classical context). Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Then the relative spectrum $\underline{\mathrm{Spec}}_X(\mathcal{A})$ exists as a topological space, and is given by the externalization of the local spectrum $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$. If the canonical morphism $\underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$ is open, then the induced morphism of locales is open as well (the converse doesn't hold in general [86, Proposition IX.7.5]). This is the case if and only if $\mathrm{Spec}(\mathcal{A}|\mathcal{O}_X)$ is an overt locale from the internal point of view of $\mathrm{Sh}(X)$.

Since Proposition 12.50 shows that the spectrum of a ring is in general not overt, Lemma 12.48 is not applicable to the spectrum in its full power. However, there is a substitute which is often sufficient: For a ring element $f : A$, it holds that

$$\sqrt{(f)} \subseteq \bigvee\{\top \mid \varphi\} \qquad \text{if and only if} \qquad \ulcorner f \text{ is nilpotent}\urcorner \vee \varphi,$$

The case that f is nilpotent often trivializes the situation, allowing to extend Lemma 12.48, at least morally. For instance, it still holds that an A-module M is finitely generated if and only if $\underline{M}$ is finitely generated as an $\underline{A}$-module from the internal point of view of $\mathrm{Spec}(A)$. (This then implies that M is finitely generated if and only if $M^\sim$ is of finite type, as in the proof of Lemma 11.11.) The "only if" direction is straightforward. For the "if" direction, we may assume that we're given a covering $\sqrt{(1)} = \bigvee_i \sqrt{(f_i)}$ such that, for each i, there are elements $x_{i1}, \ldots, x_{i,n_i} : M$ satisfying

$$\sqrt{(f_i)} \models \forall x : \underline{M}.\ \exists a_1, \ldots, a_{n_i} : \underline{A}.\ x = \textstyle\sum_j a_j x_{ij}.$$

Without loss of generality, we may assume that the covering is finite. We can then verify that the joint system $(x_{ij})_{ij}$ generates M. Let $x : M$. For each index i, there exists a finite covering $\sqrt{(f_i)} = \bigvee_k \sqrt{(g_{ik})}$ such that, for each index k, there exist elements $a_1, \ldots, a_{n_i} : A$ such that

$$g_{ik} \text{ is nilpotent} \qquad \text{or} \qquad x = \textstyle\sum_j a_j x_{ij}.$$

If the second case occurs for at least one pair (i, k) of indices, we are done. Else all the g_{ik} are nilpotent. This implies that all the f_i are nilpotent, which in turn implies that the unit of A is nilpotent. Thus A is the zero ring. In this case $x = 0$; thus we are done as well.

Gluing. The following definition is intuitionistically sensible:

Definition 12.52. An *affine scheme* is a locally ringed locale which is isomorphic to the spectrum of a ring. A *scheme* is a locally ringed locale which is locally (on an open cover) isomorphic to the spectrum of a ring.

It's crucial that we're able to verify the affine communication lemma [129, Lemma 5.3.2] in this setting; this is the lemma which ensures that for many properties, there is no difference between mandating that they hold for the members of some open affine cover or that they hold on any affine open. Its validity rests solely on the following technical statement.

Proposition 12.53. *Let $(X, \mathcal{O}_X)$ be a locally ringed locale. Let U and V be opens of X such that $(U, \mathcal{O}_X|_U)$ and $(V, \mathcal{O}_X|_V)$ are affine. Then the meet $U \wedge V$ admits a covering by opens which are simultaneously standard opens of $(U, \mathcal{O}_X|_U)$ and of $(V, \mathcal{O}_X|_V)$.*

Proof. Since U is affine,

$$U \wedge V = \bigvee \{W \preceq U \wedge V \mid W \hookrightarrow U \text{ is a standard open}\}.$$

For any such open W,

$$W = \bigvee \{W' \preceq W \mid W' \hookrightarrow W \hookrightarrow V \text{ is a standard open}\}$$

since V is affine. We show that any such open W' is also standard open in U; this suffices to establish the claim.

Since W is standard open in U, there is a function $f : \Gamma(U, \mathcal{O}_X)$ such that $W = D(f)$, where

$$D(f) = \bigvee \{A \preceq X \mid A \models \ulcorner f \text{ inv.} \urcorner\}.$$

Since W' is standard open in V, there is a function $g : \Gamma(V, \mathcal{O}_X)$ such that $W' = D(g)$. The restriction $g|_W$ can be regarded as an element of $\Gamma(U, \mathcal{O}_X)[f^{-1}]$; as such, it is of the form h/f^n. Then $W' = D(f) \wedge D(h) = D(fh)$. The open W' therefore coincides, as an open of U, with $\sqrt{(fh)}$ and is thus standard open in U. $\square$

Properties of sheaves. Schemes in the sense of Definition 12.52 can't intuitionistically be shown to have enough points [126]. Classically, they can; this ensures that classically there is no difference between the category of schemes as usually defined and the category of schemes in the sense of Definition 12.52.

As a consequence, properties of morphisms of sheaves can't be checked on stalks. For instance, for a morphism $\alpha : \mathcal{G} \to \mathcal{H}$ of sheaves on a locale X to be an epimorphism it's not enough that $\alpha_x : \mathcal{G}_x \to \mathcal{H}_x$ is surjective for all locale-theoretic points of X. Instead, for every local section $s : \mathcal{H}(U)$ there has to be a covering $U = \bigvee_i U_i$ such that, for each i, there is a preimage of $s|_{U_i}$.

Many of the results in Section 3 and Section 4 thus have to be made into definitions. For instance, a sheaf of modules should be declared *flat* if and only if it is flat as an ordinary module from the internal point of view.

The results in Section 11.4 can be used to keep up the appearance that testing on stalks suffices. For instance, let $\alpha : M^\sim \to N^\sim$ be a morphism of quasicoherent sheaves on $\mathrm{Spec}(A)$. The points of $\mathrm{Spec}(A)$ are the filters of A; but as remarked it doesn't suffice to test the stalks $\alpha_F : M[F^{-1}] \to N[F^{-1}]$. However, it does suffice to test, internally to $\mathrm{Sh}(\mathrm{Spec}(A))$, the map $\underline{M}[\mathcal{F}^{-1}] \to \underline{N}[\mathcal{F}^{-1}]$ (which is just α), where $\mathcal{F}$ is the generic filter.

Big toposes. Just as classically, the big Zariski topos of a scheme S can be defined as the topos of sheaves over the parsimonious sites $(\mathrm{Aff}/S)_{\mathrm{lfp}}$ or $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ (details about the possible choices for the site are in Section 15). The proof that this topos classifies local rings over S is intuitionistically valid.

We don't know whether all of the common subtoposes of the big Zariski topos corresponding to finer topologies like the étale or fppf topology, have all the properties which are classically expected of them. In any case, if it's classically known that the subtopos of the big Zariski topos corresponding to a finer topology classifies a certain explicitly presented geometry theory, one could adopt such a result as a definition in an intuitionistic context. For instance, the big étale topos of a scheme S can be defined as the classifying topos of separably closed local rings over S and the big fppf topos can be defined as the classifying topos of fppf-local rings over S (Section 21).

Cohomology. We don't know how a general constructive framework for cohomology might look like (besides Čech cohomology, which has its well-known shortcomings) and can only remark that Grothendieck's approach using injective resolutions can't work, since it's consistent with Zermelo–Fraenkel set theory that no nontrivial injective abelian groups exist [23], and that the account of Kempf [73] looks promising, since he employs flabby resolutions instead of injective ones.

However, Barakat and Lange-Hegermann pioneered constructive approaches to cohomology of certain base schemes, which are not only mathematically elegant but also work very well in practice (much more efficiently than Čech methods). We refer to their articles for details [13, 14].

13. Higher direct images and other derived functors

13.1. Flabby sheaves. Recall that a sheaf $\mathcal{F}$ of sets on a topological space (or a locale) X is *flabby* if and only if, for any open subset $U \subseteq X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Flabbiness is a local property, even though it doesn't seem like that at first sight: If the restrictions $\mathcal{F}|_{U_i}$ of $\mathcal{F}$ to the members of an open covering $X = \bigcup_i U_i$ are flabby, then the verification that $\mathcal{F}$ is flabby can't proceed as follows. "Let $s \in \mathcal{F}(U)$ be an arbitrary section. Since each $\mathcal{F}|_{U_i}$ is flabby, the section $s|_{U \cap U_i}$ extends to a section on U_i." The reason is that the individual extensions obtained in this way might not glue.

A correct proof employs Zorn's lemma in a typical way, considering a maximal extension and then verifying that the subset this maximal extension is defined on is all of X.

Since flabbiness is a local property, it's not unreasonable to expect that flabbiness can be characterized in the internal language. The following proposition shows that this is indeed the case.

Proposition 13.1. *Let $\mathcal{F}$ be a sheaf of sets on a topological space X (or a locale). Then the following statements are equivalent:*

(1) *$\mathcal{F}$ is flabby.*

(2) *"Any section of $\mathcal{F}$ can be locally extended": For any open $U \subseteq X$ and any section $s \in \mathcal{F}(U)$ there is an open covering $X = \bigcup_i V_i$ such that, for each i, there is an extension of s to $U \cup V_i$ (that is, a section $s' \in \mathcal{F}(U \cup V_i)$ such that $s'|_U = s$).*

(If X is a space instead of a locale, this can be equivalently formulated as follows: For any open subset $U \subseteq X$, any section $s \in \mathcal{F}(U)$, and any point $x \in X$, there is an open neighborhood V of x and an extension of s to $U \cup V$.)

(3) *From the point of view of the internal language of* $\mathrm{Sh}(X)$*, for any subsingleton $K \subseteq \mathcal{F}$ there exists an element $s : \mathcal{F}$ such that $s \in K$ if K is inhabited. More precisely,*

$$\begin{aligned} \mathrm{Sh}(X) \models \forall K \subseteq \mathcal{F}.\ (\forall s, s' : K.\ s = s') \Longrightarrow \\ \exists s : \mathcal{F}.\ (K \textit{ is inhabited} \Rightarrow s \in K). \end{aligned}$$

(4) *The canonical map $\mathcal{F} \to \mathcal{P}_{\leq 1}(\mathcal{F}), s \mapsto \{s\}$ is final from the internal point of view, that is*

$$\mathrm{Sh}(X) \models \forall K : \mathcal{P}_{\leq 1}(\mathcal{F}).\ \exists s : \mathcal{F}.\ K \subseteq \{s\},$$

where $\mathcal{P}_{\leq 1}(\mathcal{F})$ is the object of subsingletons of $\mathcal{F}$.

Proof. The implication "(1) $\Rightarrow$ (2)" is trivial. The converse direction uses a typical argument with Zorn's lemma, considering a maximal extension. The equivalence "(2) $\Leftrightarrow$ (3)" is routine, using the Kripke–Joyal semantics to interpret the internal statement. Condition (4) is a straightforward reformulation of Condition (3). $\square$

Condition (2) of the proposition is, unlike the standard definition of flabbiness, manifestly local. Also its equivalence with Condition (3) and Condition (4) is intuitionistically valid; therefore one might consider to adopt Condition (2) as the definition of flabbiness.

The object $\mathcal{P}_{\leq 1}(\mathcal{F})$ of subsingletons of $\mathcal{F}$ can be interpreted as the object of *partially-defined elements* of $\mathcal{F}$. In this view, the empty subset is the maximally undefined element and a singleton is a maximally defined element. In classical logic, there are no further examples of partially-defined elements, but intuitionistically, there might; and indeed, in the model of intuitionistic logic provided by $\mathrm{Sh}(X)$, there are many more. An explicit description of the sheaf $\mathcal{P}_{\leq 1}(\mathcal{F})$ is given in Remark 13.10.

The proposition shows that a sheaf $\mathcal{F}$ is flabby if and only if any partially-defined element of $\mathcal{F}$ can be refined to an honest element of $\mathcal{F}$.

13.2. Injective sheaves. Recall that an object I of a category $\mathcal{C}$ is *injective* if and only if, for any monomorphism $X \hookrightarrow Y$ in $\mathcal{C}$ and any morphism $X \to I$, there is an extension such that the diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & Y \arrow[dl, dashed] \\ I & \end{tikzcd}$$

commutes. Equivalently, an object I is injective if and only if the Hom functor $\mathrm{Hom}_{\mathcal{C}}(\_\_, I) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ maps monomorphisms in $\mathcal{C}$ to surjective maps. This general definition is often specialized to one of these cases: to the category of modules over a ring, to the category of set-valued sheaves on a topological space, and to the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.

The definition is seldom applied in the category of sets, since in a classical context it's easy to show that a set is injective if and only if it's inhabited, thereby completely settling the question which objects are injective in a trivial manner.

The question is more interesting in an intuitionistic setting, since intuitionistically one cannot prove that inhabited sets are injective [2]; but one can still verify that any set embeds into an injective set: The powerset $\mathcal{P}(X)$ and even the smaller set $\mathcal{P}_{\leq 1}(X)$ of subsingletons of a given set X are injective. This fact is well-known in the constructive mathematics community, but for convenience we spell out the proof as Lemma 13.9.

For a cartesian or monoidal closed category $\mathcal{C}$, there is also the notion of an *internally injective* object. This is an object I such that the internal Hom functor $[\_\_, I] : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ maps monomorphisms in $\mathcal{C}$ to epimorphisms. In the special case that $\mathcal{C}$ is a elementary topos with a natural numbers object, such as the topos of set-valued sheaves on a space, this condition can be rephrased in several ways. The following proposition lists five of these conditions. The equivalence of the first four is due to Harting [59].

Proposition 13.2. *Let $\mathcal{E}$ be an elementary topos. Then the following statements about an object $I \in \mathcal{E}$ are equivalent.*

(1) *I is internally injective.*
(2) *The functor $[\_\_, I] : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ maps monomorphisms in $\mathcal{E}$ to morphisms for which any global element of the target locally (after change of base along an epimorphism) possesses a preimage.*
(3) *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object p^*I has property (1) as an object of $\mathcal{E}/A$.*
(4) *For any morphism $p : A \to 1$ in $\mathcal{E}$, the object p^*I has property (2) as an object of $\mathcal{E}/A$.*
(5) *From the point of view of the internal language of $\mathcal{E}$, the object I is injective.*[28]

Proof. The implications "(1) $\Rightarrow$ (2)", "(3) $\Rightarrow$ (4)", "(3) $\Rightarrow$ (1)", and "(4) $\Rightarrow$ (2)" are trivial.

The equivalence "(3) $\Leftrightarrow$ (5)" follows directly from the interpretation rules of the stack semantics.

The implication "(2) $\Rightarrow$ (4)" employs the extra left adjoint $p_! : \mathcal{E}/A \to \mathcal{E}$ of $p^* : \mathcal{E} \to \mathcal{E}/A$ (which maps an object $(X \to A)$ to X), as in the usual proof that injective sheaves remain injective when restricted to smaller open subsets: We have that $p_* \circ [\_\_, p^*I]_{\mathcal{E}/A} \cong [\_\_, I]_{\mathcal{E}} \circ p_!$, the functor $p_!$ preserves monomorphisms, and one can check that p_* reflects the property that global elements locally possess preimages. Details are in [59, Thm. 1.1].[29]

The implication "(4) $\Rightarrow$ (3)" follows by performing an extra change of base, since any non-global element becomes a global element after a suitable change of base. $\square$

Somewhat surprisingly, and in stark contrast with the situation for internally projective objects (which are defined dually), internal injectivity coincides with external injectivity for sheaf toposes over spaces.

Theorem 13.3. *Let X be a topological space (or a locale). An object $\mathcal{I} \in \mathrm{Sh}(X)$ is injective if and only if it is internally injective.*

Proof. For the "only if" direction, let $\mathcal{I}$ be an injective sheaf of sets. Then $\mathcal{I}$ satisfies Condition (2) in Proposition 13.2, even without having to pass to covers.

[28] In Section 2, we have only introduced the internal language for sheaf toposes. The general definition is in [114, Section 7].

[29] Harting formulates the statement for abelian group objects, and has to assume that $\mathcal{E}$ contains a natural numbers object to ensure the existence of an abelian version of $p_!$.

For the "if" direction, let $\mathcal{I}$ be an internally injective object. Let $m : \mathcal{E} \hookrightarrow \mathcal{F}$ be a monomorphism in $\mathrm{Sh}(X)$ and let $k : \mathcal{E} \to \mathcal{I}$ be an arbitrary morphism. We want to show that there exists an extension $\mathcal{F} \to \mathcal{I}$ of k along m. To this end, we consider the sheaf defined by the internal expression

$$\mathcal{G} := [\![\{k' : [\mathcal{F}, \mathcal{I}] \mid k' \circ m = k\}]\!].$$

Global sections of $\mathcal{G}$ are extensions of the kind we're looking for. Therefore it suffices to show that $\mathcal{G}$ is flabby. We do this by verifying Condition (3) of Proposition 13.1 in the internal language of $\mathrm{Sh}(X)$.

Let $K \subseteq \mathcal{G}$ be a subsingleton. We consider the injectivity diagram

$$\begin{tikzcd} m[\mathcal{E}] \cup \mathcal{F}' \arrow[r, hook] \arrow[d] & \mathcal{F} \arrow[dl, dashed] \\ \mathcal{I} & \end{tikzcd}$$

where $\mathcal{F}'$ is the set $\{s : \mathcal{F} \mid K \text{ is inhabited}\}$ and the solid vertical arrow is defined in the following way: It should map an element $s \in \mathcal{F}'$ to $k'(s)$, where k' is any element of K; and it should map an element $m(u) \in m[\mathcal{E}]$ to $k(u)$. These prescriptions determine a well-defined map.

Since $\mathcal{I}$ is injective from the internal point of view we're taking up here, there exists a dotted map rendering the diagram commutative. This map is an element of $\mathcal{G}$. Furthermore, this map is an element of K, if K is inhabited. □

Theorem 13.4. *Let $(X, \mathcal{O}_X)$ be a ringed topological space (or a ringed locale). An $\mathcal{O}_X$-module $\mathcal{I}$ is injective if and only if it is internally injective.*

Proof. Proposition 13.2 can be adapted from sheaves to sets to sheaves of modules, with the same proof. The extra left adjoint $p_! : \mathrm{Mod}_{\mathrm{Sh}(X)/A}(\mathcal{O}_X \times A) \to \mathrm{Mod}_{\mathrm{Sh}(X)}(\mathcal{O}_X)$ required by the proof maps a module $M \to A$ to the internal direct sum $\bigoplus_{a : A} M(a)$.

The proof of Theorem 13.3 can be adopted as well. It suffices to change "$m[\mathcal{E}] \cup \mathcal{F}'$" to "$m[\mathcal{E}] + \mathcal{F}''$", where $\mathcal{F}'' := \{s : \mathcal{F} \mid s = 0 \text{ or } K \text{ is inhabited}\}$. □

Remark 13.5. The proof of Theorem 13.3 crucially rests on Proposition 13.1 and therefore on Zorn's lemma, in ensure that the sheaf $\mathcal{G}$ defined in the proof which has the property that any of its sections can be locally extended admits a global section. The proof is therefore not intuitionistically valid.

On a related note, we don't think that the statement of Theorem 13.3 can be generalized to arbitrary (Grothendieck) toposes. The proof gradually refines the trivial generalized element of $\mathcal{G}$ (defined on the empty stage) to a global element. Such a procedure is not really meaningful for sheaf toposes over sites for which not any object is a subobject of the terminal object.

13.3. Internal proofs of common lemmas.

Lemma 13.6. *A sheaf of sets or a sheaf of modules is injective if and only if it is locally injective.*

Proof. By Theorem 13.3 respectively Theorem 13.4, injectivity can be characterized in the internal language. Any such property is local. □

Lemma 13.7. *Let X be a topological space (or a locale).*

1. *Let $\mathcal{I}$ be an injective sheaf of sets over X. Let $\mathcal{F}$ be an arbitrary sheaf of sets. Then $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{I})$ is flabby.*
2. *Let $\mathcal{I}$ be an injective sheaf of modules over some sheaf $\mathcal{O}_X$ of rings over X. Let $\mathcal{F}$ be an arbitrary sheaf of modules. Then $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})$ is flabby.*

Proof. We first cover the case of sheaves of sets. By Theorem 13.3 and Proposition 13.1, it suffices to give an intuitionistic proof of the following statement: If I is an injective set and F is an arbitrary set, then partially defined elements of the set $[F, I]$ of all maps $F \to I$ can be refined to honest elements.

Thus let a subsingleton $K \subseteq [F, I]$ be given. We consider the injectivity diagram

$$\begin{tikzcd} F' \arrow[r] \arrow[d] & F \arrow[dl, dashed] \\ I & \end{tikzcd}$$

where F' is the subset $\{s : F \mid K \text{ is inhabited}\} \subseteq F$ and the solid vertical map sends $s \in F'$ to $f(s)$, where f is an arbitrary element of K. This association is well-defined. Since I is injective, a dotted lift as indicated exists. If K is inhabited, this lift is an element of K.

The same kind of argument applies to the case of sheaves of modules, relying on Theorem 13.4 and defining F' as the submodule "$\{s : F \mid s = 0 \text{ or } K \text{ is inhabited}\}$". □

Corollary 13.8. *Injective sheaves of sets and injective sheaves of modules are flabby.*

Proof. Follows from the previous lemma by considering the special cases $\mathcal{F} := 1$ respectively $\mathcal{F} := \mathcal{O}_X$. □

Lemma 13.9. *Let X be a topological space (or a locale). Any sheaf of sets over X can be embedded into an injective (therefore flabby) sheaf of sets.*

Proof. By Proposition 13.1, it suffices to give an intuitionistic proof of the following statement: Any set F can be embedded into an injective set.

As already indicated, there at least two simple ways how F can be embedded into an injective set: by embedding F in its powerset $\mathcal{P}(F)$ or by embedding F in $\mathcal{P}_{\leq 1}(F)$, the set of subsingletons of F. For conciseness, we only verify that $\mathcal{P}_{\leq 1}(F)$ is injective.

So let $m : A \hookrightarrow B$ be an injective map and let $k : A \to \mathcal{P}_{\leq 1}(F)$ be an arbitrary map. Then we can extend k to a map $k' : B \to \mathcal{P}_{\leq 1}(F)$ by defining for $y : B$

$$\begin{aligned} k'(y) &:= \bigcup k[m^{-1}[\{y\}]] \\ &= \{s : F \mid s \in k(x) \text{ for some } x \in A \text{ such that } m(x) = y\}. \end{aligned}$$ □

Remark 13.10. The *Godement construction* provides a well-known way of embedding an inhabited sheaf of sets $\mathcal{F}$ into an injective sheaf, namely the sheaf of not necessarily continuous sections of the étale space of $\mathcal{F}$:

$$U \subseteq X \quad \longmapsto \quad \prod_{x \in U} \mathcal{F}_x.$$

The sheaf $\mathcal{P}_{\leq 1}(\mathcal{F})$ does not coincide with this construction. Instead by Definition 2.8, it is the sheaf with

$$U \subseteq X \quad \longmapsto \quad \{\langle V, s\rangle \mid V \subseteq U \text{ open}, s \in \mathcal{F}(V)\}.$$

It's not possible to describe the Godement construction in the internal language of $\mathrm{Sh}(X)$, since the Godement construction depends on the underlying set of X. But the sheaf topos of X doesn't remember this set. For instance, if X is an inhabited indiscrete topological space, then $\mathrm{Sh}(X)$ is equivalent to Set.

Remark 13.11. It's not known to us whether it's possible to intuitionistically prove that any module can be embedded into a module which satisfies the internal flabbiness criterion of Proposition 13.1. This would give an internal proof of the well-known fact that any sheaf of modules can be embedded into a flabby sheaf of modules. The naive candidates don't work: The set $\mathcal{P}_{\leq 1}(F)$ doesn't admit a canonical module structure (though it does admit the structure of a commutative monoid), and the free module over that set is not flabby in general.

Since by the Godement construction the statement that any sheaf of modules can be embedded into a flabby sheaf of modules is true in many models of intuitionistic logic, the sheaf toposes over topological spaces, and furthermore the proof that the Godement construction yields a flabby sheaf is intuitionistically valid,[30] it's not entirely unreasonable to believe that such an intuitionistic proof is possible.[31]

On the other hand, it's certainly not possible to intuitionistically prove that any module can be embedded into an injective module, since it's consistent with Zermelo–Fraenkel set theory that no nontrivial injective abelian groups exist [23].

Lemma 13.12. *Let X be a ringed space (or a ringed locale). Let $0 \to \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{E}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{E}'$ is flabby, then the induced sequence*

$$0 \longrightarrow \Gamma(X, \mathcal{E}') \longrightarrow \Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{E}'') \longrightarrow 0$$

is exact.

Proof. Since taking global sections is left exact (being a right adjoint functor), it suffices to verify that the map $\Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{E}'')$ is surjective. We'll do this by showing, in the internal language of $\mathrm{Sh}(X)$, that the sheaf of preimages of a given global section $s \in \Gamma(X, \mathcal{E}'')$ is flabby and therefore has a global section.

In the internal language, this sheaf has the description $F := \{u : \mathcal{E} \mid \beta(u) = s\}$. To verify the internal condition of Proposition 13.1, let a subsingleton $K \subseteq F$ be given. Since β is surjective, there is a preimage $u_0 \in F$. The translated set $K - u_0 \subseteq \mathcal{E}$ is still a subsingleton, and its preimage under α is as well. By the assumption on $\mathcal{E}'$, there is an element $v : \mathcal{E}'$ such that $v \in \alpha^{-1}[K - u_0]$ if $\alpha^{-1}[K - u_0]$ is inhabited. We'll now verify that $u_0 + \alpha(v) \in K$ if K is inhabited.

So assume that K is inhabited. Then $K - u_0$ is as well. Since the image of its unique element under β is zero and the given sequence is exact, the set $\alpha^{-1}[K - u_0]$ is inhabited as well. Therefore $v \in \alpha^{-1}[K - u_0]$. Thus $u_0 + \alpha(v) \in K$. ☐

Lemma 13.13. *Let X be a ringed space (or a ringed locale). Let $0 \to \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{E}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{E}'$ and $\mathcal{E}''$ are flabby, then $\mathcal{E}$ is flabby as well.*

Proof. We verify the condition of Proposition 13.1 in the internal language of $\mathrm{Sh}(X)$.

Let $K \subseteq \mathcal{E}$ be a subsingleton. Then its image $\beta[K] \subseteq \mathcal{E}''$ is a subsingleton as well. Since partial elements of $\mathcal{E}''$ can be refined to honest elements, there is an element $s : \mathcal{E}''$ such that $\beta[K] \subseteq \{s\}$.

Since β is surjective, there is an element $t_0 : \mathcal{E}$ such that $\beta(t_0) = s$.

[30]In order for the Godement construction to work in a intuitionistic metatheory, one has to tweak its definition a little bit. Instead of mapping an open subset U to $\prod_{x \in U} \mathcal{F}_x$, one has to map U to $\prod_{x \in U} \mathcal{P}_{\leq 1}(\mathcal{F}_x)$. This has the added advantage that it works even if $\mathcal{F}$ is not inhabited.

[31]There is a metatheorem guaranteeing that a statement is intuitionistically provable if and only if it holds in the sheaf topos over any topological space [10, Theorem B]. However, this metatheorem requires the considered statements to be of a certain form, which in particular forbids them from mentioning the object of truth values. The internal statements given in Proposition 13.1 depend on this object in a crucial way.

The preimage $\alpha^{-1}[K - t_0] \subseteq \mathcal{E}'$ is a subsingleton. This partial element can be refined to an honest element, so there exists an element $u : \mathcal{E}'$ such that $\alpha^{-1}[K - t_0] \subseteq \{u\}$.

The partial element K can thereby refined to the honest element $t := t_0 + \alpha(u)$. $\square$

13.4. Tor and sheaf Ext. The following lemma expresses a prototype result for constructing sheaves in the internal language. We'll use it to internally define derived functors.

Lemma 13.14. *Let X be a ringed topological space (or a ringed locale). Let $\varphi(\mathcal{E})$ be a property of sheaves of $\mathcal{O}_X$-modules, formulated in the internal language of* $\mathrm{Sh}(X)$*. Let $\psi(f)$ be a property of morphisms of sheaves of $\mathcal{O}_X$-modules, formulated in the internal language of* $\mathrm{Sh}(X)$ *and stable under composition. Assume that*

(1) $\mathrm{Sh}(X) \models \exists \mathcal{E}\ \mathcal{O}_X\text{-module}.\ \varphi(\mathcal{E})$ *and*
(2) $\mathrm{Sh}(X) \models \forall \mathcal{E}, \mathcal{E}'\ \mathcal{O}_X\text{-modules}.\ \varphi(\mathcal{E}) \wedge \varphi(\mathcal{E}') \Longrightarrow \exists! f : \mathcal{E} \to \mathcal{E}' \text{ linear}.\ \psi(f)$.

Then there exists a sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules such that $\mathrm{Sh}(X) \models \varphi(\mathcal{E})$, and any two such sheaves are isomorphic via a unique isomorphism which satisfies ψ from the internal point of view.

Proof. This is a reformulation of the well-known fact that we have descent for sheaves of $\mathcal{O}_X$-modules. By the first assumption, there is an open covering $X = \bigcup_i U_i$ such that for each i, there is an $\mathcal{O}_X|_{U_i}$-module $\mathcal{E}_i$ with $U_i \models \varphi(\mathcal{E}_i)$. By the second assumption and by Proposition 2.6, for each pair (i, j) of indices there is a unique morphism $f_{ij} : \mathcal{E}_i|_{U_i \cap U_j} \to \mathcal{E}_j|_{U_i \cap U_j}$ such that $U_i \cap U_j \models \psi(f_{ij})$. Since the property ψ is stable under composition, these morphisms are isomorphisms which satisfy the cocycle condition. Thus the $\mathcal{O}_X$-modules $\mathcal{E}_i$ glue to a global $\mathcal{O}_X$-module $\mathcal{E}$, which satisfies property φ because it does so locally.

The uniqueness claim is immediate by Proposition 2.6 and by the assumption that property ψ is stable under composition. $\square$

Lemma 13.14 can be generalized in two ways: from sheaves of modules to other kinds of algebraic structures, for instance complexes of sheaves of modules; and from sheaf toposes over locales to more general Grothendieck toposes, by the descent theorem for Grothendieck toposes. We will use the former, but not the latter generalization.

Lemma 13.15. *Let X be a ringed space (or a ringed locale). From the internal point of view of* $\mathrm{Sh}(X)$*, any $\mathcal{O}_X$-module admits a resolution by injective $\mathcal{O}_X$-modules, and any two such are related by a morphism of complexes which is unique up to homotopy with the property that it induces the identity on the resolved module.*

Proof. There can't be an intuitionistic proof of this fact, since it's consistent with Zermelo–Fraenkel set theory that no nontrivial injective abelian groups exist [23]. But working in a classical metatheory, it's well-known that, for any open subset $U \subseteq X$, the category of sheaves of $\mathcal{O}_X|_U$-modules has enough injectives. Since externally injective sheaves of modules look like injective modules from the internal point of view, by (the easy part of) Theorem 13.4, the internal statement "any $\mathcal{O}_X$-module can be embedded into an injective $\mathcal{O}_X$-module" holds.

Under the assumption of the existence of enough injectives, the usual proof that any object admits a resolution by injective objects is intuitionistically valid. We can therefore interpret this proof in the internal language of $\mathrm{Sh}(X)$ and conclude – were it not for a subtle fine point regarding the failure of the axiom of countable choice in $\mathrm{Sh}(X)$.

A resolution is an infinite complex of modules. The assumption of the existence of enough injectives allows us to extend any finite partially-constructed injective resolution to a longer one; but collecting all of the resulting injective objects into a complete resolution requires some form of choice.

There are two ways to counter this problem. If one wants to prove the lemma exactly as stated, one has to construct the injective resolution externally (and appeal to the axiom of choice in the metatheory) instead of internally constructing it step by step. But for many purposes, there's also an alternative: Often, a full injective resolution isn't actually needed. For instance, for evaluating an n-th derived functor on an object, it suffices to have a finite partial injective resolution. If one adopts this stance, then it's enough to adopt the statement "any $\mathcal{O}_X$-module can be embedded into an injective $\mathcal{O}_X$-module" as an axiom in the internal language. □

Remark 13.16. It's known that the axiom of choice suffices to construct injective resolutions of abelian groups,[32] and also that the axiom of choice implies the law of excluded middle in the presence of the other axioms of set theory (Diaconescu's theorem). Lemma 13.15 shows that the axiom "any abelian group can be embedded into an injective abelian group" does not imply the law of excluded middle, since (assuming the axiom of choice in the metatheory) this statement is true in the internal language of the sheaf topos over any topological space and such toposes typically do not satisfy the law of excluded middle.

We use Lemma 13.15 as follows to construct the sheaf Ext in the internal language. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{O}_X$-modules over a ringed space (or a ringed locale). Internally, we define $\mathcal{E}\mathrm{xt}^n(\mathcal{E},\mathcal{F}) := H^n([\mathcal{E},\mathcal{I}^\bullet]_{\mathrm{Mod}(\mathcal{O}_X)})$ where $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution and $[\mathcal{E},\mathcal{I}^k]_{\mathrm{Mod}(\mathcal{O}_X)}$ is the set of $\mathcal{O}_X$-linear maps $\mathcal{E} \to \mathcal{I}^k$. The module constructed in this way depends on the chosen injective resolution, but for any two such resolutions, there is a unique isomorphism in cohomology which is induced by a morphism of resolutions.

Externally, this definition gives rise to a well-defined sheaf on X, by arguing similarly as in the proof of Lemma 13.14: We obtain $\mathcal{O}_X$-modules on an open cover; on intersections, we find coherent isomorphisms by the uniqueness statement; therefore we can glue. The $\mathcal{O}_X$-module constructed in this way coincides with the sheaf Ext as usually conceived.

Along the same lines, we can construct Tor sheaves internally. Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{O}_X$-modules. Assume that $\mathcal{F}$ is of finite type. Internally, we define $\mathcal{T}\mathrm{or}_n(\mathcal{E},\mathcal{F}) := H_n(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}_\bullet)$, where $\mathcal{P}_\bullet \to \mathcal{F} \to 0$ is a projective resolution. Since $\mathcal{F}$ is finitely generated, such a resolution exists; in fact, we can resolve $\mathcal{F}$ by finite free modules (which are projective even without the axiom of choice). As with $\mathcal{E}\mathrm{xt}^n$, the module constructed in this way is unique up to a unique isomorphism induced by a morphism of resolutions.

13.5. Higher direct images. Higher direct images are thought of as a relative "fiberwise" version of cohomology. One way to make this precise is to show that higher direct images can be made sense of internally to the topos of sheaves over the base, where they then look like ordinary cohomology.

Let $f : Y \to X$ be a morphism of ringed locales. As discussed in Section 12.1, there is a locale $I(Y)$ internal to $\mathrm{Sh}(X)$ mirroring Y; from the point of view of $\mathrm{Sh}(X)$, the given morphism f looks like the unique morphism $I(Y) \to \mathrm{pt}$.

Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_Y$-modules on Y. This sheaf corresponds to a sheaf on $I(Y)$. Internally to $\mathrm{Sh}(X)$, we can take an injective resolution $\mathcal{J}^\bullet$ of this sheaf and define $H^n(I(Y),\mathcal{E}) := H^n(\Gamma(I(Y),\mathcal{J}^\bullet))$. Just like with sheaf Ext and Tor presented in Section 13.4, this internal description gives rise to a sheaf of $\mathcal{O}_X$-modules on X;

[32]More precisely, it is intuitionistically provable that any abelian group admits a resolution by divisible groups, even a canonical such. Some form of choice is needed to verify that such a resolution is actually a resolution by injective abelian groups.

the sheaf constructed in this way coincides with the higher direct image $R^n f_*(\mathcal{E})$ as usually defined.

The conception of higher direct images as internal cohomology entails that basic statements about cohomology yield corresponding statements about higher direct images. For instance:

- That $R^{\geq 1}\mathrm{id}_*(\mathcal{E})$ vanishes is a reflection of the fact that $H^{\geq 1}(\mathrm{pt}, \mathcal{E})$ vanishes.
- Čech methods to compute cohomology entail Čech methods to compute higher direct images.
- The computation of the cohomology of projective n-space over a ring immediately yields the higher direct images of the canonical morphism $\mathbb{P}^n_S \to S$ for any base scheme S.

Also the failure of higher direct images to commute with arbitrary base change gains a logical interpretation. Let

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

be a pullback diagram of ringed locales. Let $\mathcal{E}$ be a sheaf of modules on Y. If taking cohomology was a geometric construction, then it would be preserved by pullback along arbitrary geometric morphisms. Since higher direct images are just cohomology from the internal point of view, we would therefore have a canonical isomorphism

$$g^*(R^n f_*(\mathcal{E})) \cong R^n f'_*((g')^*\mathcal{E}).$$

However, taking cohomology is not a geometric construction, and indeed in general there is no such isomorphism. It's an open question whether the well-known cases where there is such an isomorphism can be treated by a purely or mostly logical framework.

PART III

The big Zariski topos

The preceding part demonstrated that working in the internal universe of the little Zariski topos of a scheme S, the topos of sheaves on S, is useful for simplifying local work on S. The basic tenet was that sheaves of modules look just like plain modules and that theorems of intuitionistic algebra yield theorems about sheaves.

But the little Zariski topos is not particularly suited for dealing with *schemes* over S. For this, we need a related topos. For the scope of this introduction only, we blithely employ the following slightly problematic definition which we'll correct in Section 15. We'll keep the base scheme S fixed throughout this part.

For some material in this part, we assume basic familiarity with classifying toposes.

14. Basics

Definition 14.1 (provisional)**.** The *big Zariski topos* $\mathrm{Zar}(S)$ of a scheme S is the topos of sheaves on the Grothendieck site Sch/S of schemes over S.

Explicitly, an object of $\mathrm{Zar}(S)$ is a functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition with respect to ordinary Zariski coverings: If $X = \bigcup_i U_i$ is a cover of an S-scheme X by open subsets, the canonical diagram

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

should be an equalizer diagram.

Internal language. Just like the topos of sheaves on a topological space or on a locale admits an internal language, so does the big Zariski topos. The necessary modifications of the Kripke–Joyal semantics (Definition 2.1) are straightforward. Instead of defining recursively the meaning of "$U \models \varphi$" for open subsets $U \subseteq S$, we define the meaning of "$T \models \varphi$" for S-schemes T and slightly rewrite the rules for implication and universal quantification. Instead of

$$\begin{aligned} U \models \varphi \Rightarrow \psi \quad &:\Longleftrightarrow \quad \text{for all open } V \subseteq U\text{:} \\ &\qquad\qquad V \models \varphi \text{ implies } V \models \psi \\ U \models \forall s\,{:}\,\mathcal{F}.\ \varphi(s) \quad &:\Longleftrightarrow \quad \text{for all sections } s \in \Gamma(V, \mathcal{F}) \text{ on open } V \subseteq U\text{:} \\ &\qquad\qquad V \models \varphi(s) \end{aligned}$$

they have to read as follows.

$$\begin{aligned} T \models \varphi \Rightarrow \psi \quad &:\Longleftrightarrow \quad \text{for all morphisms } T' \to T \text{ in } \mathrm{Sch}/S\text{:} \\ &\qquad\qquad T' \models \varphi \text{ implies } T' \models \psi \\ T \models \forall s\,{:}\,F.\ \varphi(s) \quad &:\Longleftrightarrow \quad \text{for all morphisms } T' \to T \text{ in } \mathrm{Sch}/S \text{ and} \\ &\qquad\qquad \text{all sections } s \in \Gamma(T', F)\text{:} \\ &\qquad\qquad T' \models \varphi(s) \end{aligned}$$

The analogs of Proposition 2.4 and Proposition 2.5 are true for the internal language of the big Zariski topos:

Proposition 14.2. *Let T be an S-scheme and φ be a formula over T.*

(1) *If $T \models \varphi$ and if there is an intuitionistic proof that φ implies a further formula ψ, then $T \models \psi$.*
(2) *Let $T' \to T$ be a morphism of S-schemes. If $T \models \varphi$, then $T' \models \varphi$.*
(3) *If $T = \bigcup_i T_i$ is an open covering and if $T_i \models \varphi$ for all i, then $T \models \varphi$.*

Proof. The proofs of Proposition 2.4 and Proposition 2.5 carry over. □

When working with the internal language of the little Zariski topos, we often used the fact that if a formula holds on some open subset U, then it also holds on all open subsets contained in U. Proposition 14.2(2) states a stronger version of this: All properties which can be expressed using the internal language of the big Zariski topos are automatically *stable under base change.*

Important objects in the big Zariski topos. It's convenient to introduce notation for objects which often appear when working with the big Zariski topos.

Let X be an S-scheme. Its functor of points, which maps an S-scheme T to $\mathrm{Hom}_S(T, X)$, is an object of $\mathrm{Zar}(S)$. We denote it by "$\underline{X}$".

From the internal point of view of $\mathrm{Zar}(S)$, such a functor $\underline{X}$ looks like a single set. It can be pictured as the "set of points of X", where "point" doesn't mean "point of the underlying topological space of X", but rather "T-point of X", where T varies over all S-schemes. The internal language of the big Zariski topos hides any explicit mentions of the stage T; it is therefore a device for reifying the multitude of points of X, defined on varying stages, as a single entity.

Particularly important is $\underline{\mathbb{A}}^1_S$, the functor of points of the affine line over S. The object $\underline{S}$ is the terminal object in $\mathrm{Zar}(S)$. This fits into the philosophy: From the point of view of the big Zariski topos, the base scheme should simply look like a point. The functor of points of $S \amalg S$ looks like a two-element set from the internal point of view.

Let $\mathcal{F}$ be a sheaf of sets on S. For reasons explained in Section 16, we denote by "$\pi^{-1}\mathcal{F}$" the induced sheaf on Sch/S mapping an S-scheme $(f : T \to S)$ to $\Gamma(T, f^{-1}\mathcal{F})$.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules. We denote by "$\mathcal{F}^{\mathrm{Zar}}$" the induced sheaf on Sch/S mapping an S-scheme $(f : T \to S)$ to $\Gamma(T, f^*\mathcal{F})$.

A first example illustrating the Kripke–Joyal translation rules. Since all the sets $\underline{\mathbb{A}}^1_S(T) \cong \Gamma(T, \mathcal{O}_T)$ carry ring structures and do so in a compatible way, the object $\underline{\mathbb{A}}^1_S$ can be endowed with a canonical structure as a ring object in $\mathrm{Zar}(S)$. For a particular S-scheme T, the ring $\underline{\mathbb{A}}^1_S(T)$ will almost never be a field, but the system of these rings, conceptualized as a single entity from the internal point of view, does satisfy a field axiom. In the case $S = \operatorname{Spec} \mathbb{Z}$, this was first observed by Kock [77].

Proposition 14.3. *The ring $\underline{\mathbb{A}}^1_S$ is a field from the internal point of view of* $\mathrm{Zar}(S)$*, in the sense that*

$$\mathrm{Zar}(S) \models \forall f : \underline{\mathbb{A}}^1_S.\ \neg(f = 0) \Rightarrow \ulcorner f \text{ inv.} \urcorner.$$

Proof. According to the Kripke–Joyal semantics of $\mathrm{Zar}(S)$, we have to show that for any S-scheme T and any function $f \in \Gamma(T, \mathcal{O}_T)$ the statement $T \models \neg(f = 0)$ implies $T \models \ulcorner f \text{ inv.} \urcorner$. The antecedent states that, for any T-scheme T', if the pullback of f to T' vanishes, then T' is the empty scheme.

As with the analogous statement about the little Zariski topos (Lemma 3.2), the consequent means that f is invertible in $\Gamma(T, \mathcal{O}_T)$.

The claim follows by considering the particular T-scheme $T' := V(f)$. Since f vanishes on $V(f)$, this subscheme is empty and therefore its complement $D(f)$ is all of T. $\square$

The field property can be interpreted as follows. A function f not being the zero function does not imply that it's invertible. But if f is *universally nonzero* in that the only scheme such that pullback of f to that scheme vanishes is the empty scheme, then f is indeed invertible.

We'll revisit the field property in Section 18.4; it turns out that it has a deeper reason than the manual proof given here showed.

15. On the proper choice of a big Zariski site

Unlike with the construction of the little Zariski topos, set-theoretical issues of size arise when constructing the big Zariski topos. These can be solved in several different manners, yielding toposes which are not equivalent, and actually differ in some important aspects, but otherwise enjoy very similar properties.

Naive approach. Some authors construct the big Zariski topos of S as the topos of sheaves over the site Sch/S of all schemes over S. This option is quite attractive since the Yoneda embedding $\mathrm{Sch}/S \to \mathrm{Sh}(\mathrm{Sch}/S)$, which sends an S-scheme to its functor of points, is fully faithful, therefore the internal language of $\mathrm{Sh}(\mathrm{Sch}/S)$ can distinguish arbitrary schemes.

However, because Sch/S is not essentially small, forming the sheaf topos is not possible in plain Zermelo–Fraenkel set theory.

Since it's still possible to meaningfully speak of individual functors $(\mathrm{Sch}/S)^{\mathrm{op}} \to S$, we can attach a Kripke–Joyal semantics to $\mathrm{Sh}(\mathrm{Sch}/S)$, as long as we keep in mind that $\mathrm{Sh}(\mathrm{Sch}/S)$ might not contain a subobject classifier and might not be cartesian closed. From the internal point of view, powersets and function sets might therefore not exist.

Using Grothendieck universes. We could also assume the existence of a Grothendieck universe $\mathcal{U}$ containing S and construct $\mathrm{Zar}(S)$ as the topos of sheaves over the small site $\mathrm{Sch}_{\mathcal{U}}/S$, the category of S-schemes contained in U.

By the *comparison lemma* (see, for instance, [32, Theorem 3.7]), we could also construct $\mathrm{Zar}(S)$ as the topos of sheaves over $\mathrm{Aff}_{\mathcal{U}}/S$, the category of S-schemes in U which are affine (as absolute schemes), and obtain an equivalent topos.

In this case, the Yoneda functor $\mathrm{Sch}/S \to \mathrm{Zar}(S)$ might not be faithful, but the restricted Yoneda functor $\mathrm{Sch}_{\mathcal{U}}/S \to \mathrm{Zar}(S)$ will.

Approach of the Stacks Project. The Stacks Project proposes a more nuanced approach, namely expanding a given set M_0 of schemes containing S to a superset M which is closed (up to isomorphism) under several constructions [118, Tag 000H]: fiber products, countable coproducts, domains of open and closed immersions and of morphisms of finite type, spectra of local rings $\mathcal{O}_{X,x}$, spectra of residue fields, and others.

The Stacks Project then defines $\mathrm{Zar}(S)$ as $\mathrm{Sh}(\mathrm{Sch}_M/S)$, where Sch_M/S is the small category of S-schemes in M, or equivalently as $\mathrm{Sh}(\mathrm{Aff}_M/S)$. This approach has the advantage that one doesn't have to assume the existence of a Grothendieck universe; the *partial universe* M can be constructed entirely within ZFC set theory using transfinite recursion.

Employing parsimonious sites. From a topos-theoretical point of view, it's natural to settle for an even more parsimonious site: the site $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ consisting of the S-schemes which are locally of finite presentation over S, or equivalently the essentially small site $(\mathrm{Aff}/S)_{\mathrm{lfp}}$ of the S-schemes which are locally of finite presentation over S and affine (as absolute schemes).[33]

In the special case that $S = \mathrm{Spec}(A)$ is affine, this site is the dual of the category of finitely presented A-algebras; in this case the topos-theoretic points of the resulting topos are precisely the local A-algebras, and moreover, the resulting topos is the classifying topos of the theory of local A-algebras, such that for any Grothendieck topos $\mathcal{E}$, geometric morphisms $\mathcal{E} \to \mathrm{Sh}((\mathrm{Aff}/S)_{\mathrm{lfp}})$ correspond to local A-algebras internal to $\mathcal{E}$. A textbook reference for these facts is [86, Section VIII.6].

In contrast, the toposes arising when using the larger sites have categories of points which contain further objects in addition to all local A-algebras; and no simple description of the theory they classify is known.[34]

A further advantage of these parsimonious sites is that they don't require arbitrary choices of a starting set M_0 or of a way of expanding M_0 to a sufficiently ample set M of schemes.

However, the parsimonious sites also have a serious disadvantage, namely that with them, the Yoneda functor is only fully faithful when restricted to $(\mathrm{Sch}/S)_{\mathrm{lfp}}$. For instance, in the case $S = \mathrm{Spec}(\mathbb{Z})$, the schemes $\mathrm{Spec}(\mathbb{Q})$ and the empty scheme have isomorphic functors of points by Proposition 11.19. Therefore $\mathrm{Spec}(\mathbb{Q})$ looks like the empty set from the internal point of view.

In the following, we do not commit to a single one of these options for resolving the set-theoretical size issues, but rather keep any of them in mind. This approach will sometimes necessitate phrases such as "for any S-scheme T contained in the site used to define $\mathrm{Zar}(S)$", which might seem awkward to a topos-theorist when taken out of context, since the site used to construct a Grothendieck topos is not at all uniquely determined by the resulting topos.

We will indicate the few places where the choice of site makes a difference. When the definition of the Kripke–Joyal semantics for $\mathrm{Zar}(S)$ refers to S-schemes, it actually refers only to the S-schemes contained in the site. Similarly, one has to restrict oneself to such schemes in the statement of Proposition 14.2. Proposition 14.3 holds for any choice of site.

It's possible to define the big Zariski topos of a scheme without recourse to classical scheme theory; we discuss this in Section 16.5.

Remark 15.1. Some authors define the big Zariski topos of S as the topos of sheaves over the category of affine S-schemes (that is, S-schemes $f : X \to S$ where f is affine) contained in some universe. In case that the diagonal morphism $S \to S \times S$ is affine, the resulting topos is equivalent to what we regard as the big Zariski topos of S, when defined using the same universe. This is because in this case morphisms

[33] It's not reasonable to restrict to the even smaller site consisting of the finitely presented S-schemes, since open immersions can fail to be finitely presented. We want the site used to construct $\mathrm{Zar}(S)$ to be closed under domains of open immersions, for instance to facilitate a comparison with the little Zariski topos $\mathrm{Sh}(S)$, whose site does contain all open subsets of S. Furthermore, since a finitely presented S-scheme might not admit an open covering by finitely presented S-schemes which are affine (as absolute schemes), the toposes $\mathrm{Sh}((\mathrm{Sch}/S)_{\mathrm{fp}})$ and $\mathrm{Sh}((\mathrm{Aff}/S)_{\mathrm{fp}})$ can differ.

[34] The category of points of a presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ coincides with $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$, the ind-completion of $\mathcal{C}^{\mathrm{op}}$. This general fact explains that in the case that $\mathcal{C}$ is the category of finitely presented A-algebras, the category of points coincides with the category of A-algebras, since $\mathrm{Ind}(\mathrm{Alg}(A)_{\mathrm{fp}}) \simeq \mathrm{Alg}(A)$. For the larger sites, understanding the structure of their points is therefore tantamount to understanding the structure of the ind-completion of their dual category (and understanding which points of the presheaf topos are also points of the sheaf topos).

of the form $\mathrm{Spec}(A) \to S$ are affine and affine morphisms with codomain S can be refined by such morphisms.

16. Relation between the big and little Zariski toposes

The big Zariski topos $\mathrm{Zar}(S)$ is a topos over the little Zariski topos $\mathrm{Sh}(S)$ in that there is a canonical geometric morphism

$$\pi : \mathrm{Zar}(S) \longrightarrow \mathrm{Sh}(S)$$

with direct and inverse image parts given by

$$\pi_* E = E|_{\mathrm{Sh}(S)} \qquad \text{and} \qquad \pi^{-1}\mathcal{F} = ((T \xrightarrow{f} S) \mapsto \Gamma(T, f^{-1}\mathcal{F})).$$

Since π^{-1} is fully faithful, this geometric morphism is connected; and furthermore, it is a local geometric morphism (a further right adjoint $\pi^!$ which is fully faithful exists).

By general results on local geometric morphisms, the adjoint pair $(\pi_* \dashv \pi^!)$ is a geometric morphism which is right inverse to π and which exhibits $\mathrm{Sh}(S)$ as a subtopos of $\mathrm{Zar}(S)$, similarly to how Set is a subtopos of a sheaf topos over a local topological space. In this context, it's customary to introduce notation for the idempotent monad $\sharp$ and the idempotent comonad $\flat$ arising from the adjoint triple $\pi^{-1} \dashv \pi_* \dashv \pi^!$:

$$\sharp E = \pi^!(E|_{\mathrm{Sh}(S)}) \qquad \text{and} \qquad \flat E = \pi^{-1}(E|_{\mathrm{Sh}(S)}).$$

In the case that $S = \mathrm{Spec}(A)$ is an affine scheme and we employ one of the parsimonious sites to construct $\mathrm{Zar}(S)$, it's well-known that $\mathrm{Sh}(S)$ classifies local localizations of A and that $\mathrm{Zar}(S)$ classifies arbitrary local A-algebras. On points, the morphism π sends a local A-algebra $\varphi : A \to R$ to the local localization $A \to A[(\varphi^{-1}[R^\times])^{-1}]$, and its right inverse sends a local localization $A \to A[F^{-1}]$ to itself.

16.1. Recovering the big Zariski topos from the little Zariski topos. What does $\mathrm{Zar}(S)$ classify in the case that S is an arbitrary scheme? We don't know a nontautologous answer to this question, but we can answer a related one: What does $\mathrm{Zar}(S)$ classify as seen from the internal point of view of $\mathrm{Sh}(S)$?

To make sense of this question, we employ a slight extension of Shulman's stacks semantics which allows to refer to locally internal categories [97] over a base topos $\mathcal{E}$ from the internal language. Using this extension, a locally internal category over $\mathcal{E}$ looks like a locally small category from the internal point of view of $\mathcal{E}$. In particular, a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ gives rise to a locally internal category (which over an object $A \in \mathcal{E}$ is given by the $\mathcal{E}/A$-enriched category $\mathcal{F}/f^{-1}A$) which will look like an ordinary topos from the internal point of view of $\mathcal{E}$.

For instance, the trivial $\mathcal{E}$-topos $\mathcal{E}$ will look like Set and the slice topos $\mathcal{E}/X$ will look like Set/X from the internal point of view of $\mathcal{E}$.

Theorem 16.1. *In the situation that the site used to construct* $\mathrm{Zar}(S)$ *is one of the parsimonious sites, the big Zariski topos* $\mathrm{Zar}(S)$ *is, from the internal point of view of* $\mathrm{Sh}(S)$*, the classifying topos of the theory of local* $\mathcal{O}_S$*-algebras which are local over* $\mathcal{O}_S$*.*

For an arbitrary topos $\mathcal{F}$ over Set, the concept of an "$\mathcal{O}_S$-algebra in $\mathcal{F}$" doesn't make any sense – in contrast to the concept of an A-algebra in $\mathcal{F}$, which can either be defined as a ring homomorphism $\underline{A} \to R$ in $\mathcal{F}$ (where $\underline{A}$ is the pullback of $A \in \mathrm{Set}$ to $\mathcal{F}$) or as a ring object which is equipped with an A-indexed family of endomorphisms satisfying suitable axioms. However, for a $\mathrm{Sh}(S)$-topos $f : \mathcal{F} \to \mathrm{Sh}(S)$, the concept of an $\mathcal{O}_S$-algebra in $\mathcal{F}$ is meaningful: It's a ring homomorphism $f^{-1}\mathcal{O}_S \to R$ in $\mathcal{F}$.

Similarly, there is no absolute "geometric theory of $\mathcal{O}_S$-algebras". However, there is a geometric theory of $\mathcal{O}_S$-algebras *internal to* $\mathrm{Sh}(S)$. Theorem 16.1 should be viewed in this light. A detailed account of internal geometric theories and internal classifying toposes is given in [61, Chapter II].

The proviso "local over $\mathcal{O}_S$" is as in the discussion of the relative spectrum from the internal point of view (Section 12).

Proof of Theorem 16.1. We have to verify that, from the point of view of $\mathrm{Sh}(S)$, the topos $\mathrm{Zar}(S)$ contains a canonical local and local-over-$\mathcal{O}_S$ $\mathcal{O}_S$-algebra and that for any Grothendieck topos $\mathcal{F}$, pulling back this canonical algebra yields an equivalence between the category of geometric morphisms $\mathcal{F} \to \mathrm{Zar}(S)$ and the category of local and local-over-$\mathcal{O}_S$ $\mathcal{O}_S$-algebras in $\mathcal{F}$.

The canonical local and local-over-$\mathcal{O}_S$ $\mathcal{O}_S$-algebra in $\mathrm{Zar}(S)$ is the algebra $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$. Indeed, the ring $\underline{\mathbb{A}}^1_S$ is local and the homomorphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ is local, since its restriction to any sheaf topos $\mathrm{Sh}(X)$, where $f : X \to S$ is an S-scheme contained in the site used to define $\mathrm{Zar}(S)$, is local: It's the morphism $f^\sharp : f^{-1}\mathcal{O}_S \to \mathcal{O}_X$.

We now want to verify the universal property, which expressed internally to $\mathrm{Sh}(S)$ reads as

$$\mathrm{Hom}(\mathcal{E}, \mathrm{Zar}(S)) \simeq \text{category of local and local-over-}\mathcal{O}_S\ \mathcal{O}_S\text{-algebras in } \mathcal{E}.$$

Externally, this means that for any open subset $U \subseteq S$ and any topos $\mathcal{E}$ over $\mathrm{Sh}(S)/\underline{U}$,

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{Sh}(S)/\underline{U}}(\mathcal{E}, \mathrm{Zar}(S)/\pi^{-1}\underline{U}) \simeq \\ &\qquad \text{category of local and local-over-}\pi^{-1}\mathcal{O}_S\ \pi^{-1}\mathcal{O}_S\text{-algebras in } \mathcal{E}.\end{aligned}$$

We will verify this equivalence in the case that $S = \mathrm{Spec}(A)$ is affine and that $S = U$. This suffices to establish the theorem, since $\mathrm{Sh}(S)/\underline{U} \simeq \mathrm{Sh}(U)$, $\mathrm{Zar}(S)/\pi^{-1}\underline{U} \simeq \mathrm{Zar}(U)$, and since the internal language is local.

So let $f : \mathcal{E} \to \mathrm{Sh}(\mathrm{Spec}(A))$ be a $\mathrm{Sh}(\mathrm{Spec}(A))$-topos. By the universal property of $\mathrm{Zar}(\mathrm{Spec}(A))$ as the classifying topos of local A-algebras, a geometric morphism $g : \mathcal{E} \to \mathrm{Zar}(\mathrm{Spec}(A))$ is uniquely determined by a local A-algebra $\varphi : \underline{A} \to \mathcal{B}$ in $\mathcal{E}$. By the universal property of $\mathrm{Sh}(\mathrm{Spec}(A))$ as the classifying topos of local localizations of A, the composition $\pi \circ g : \mathcal{E} \to \mathrm{Sh}(\mathrm{Spec}(A))$ is uniquely determined by the local localization $\underline{A} \to g^{-1}\pi^{-1}\mathcal{O}_{\mathrm{Spec}(A)} = g^{-1}(\flat\underline{\mathbb{A}}^1_S)$ in $\mathcal{E}$.

In the composition

$$\underline{A} \longrightarrow \flat\underline{\mathbb{A}}^1_S \longrightarrow \underline{\mathbb{A}}^1_S,$$

the first morphism is a local localization and the second morphism is local. Since these properties can be formulated as geometric implications,[35] they are preserved by the functor g^{-1}. Since furthermore such a factorization is unique, the localization $\underline{A} \to g^{-1}(\flat\underline{\mathbb{A}}^1_S)$ which determines $\pi \circ g$ coincides with the localization $\underline{A}[(\varphi^{-1}((\underline{\mathbb{A}}^1_S)^\times))^{-1}]$. Referring directly to the involved filters, the filter $g^{-1}\mathcal{F}$ which determines $\pi \circ g$ (where $\mathcal{F}$ is the generic filter of $\underline{A}$ in $\mathrm{Sh}(\mathrm{Spec}(A))$) coincides

[35] A ring homomorphism $\alpha : R \to T$ is a localization (that is, isomorphic to the canonical localization morphism $R \to R[S^{-1}]$ for some multiplicative subset S) if and only if the canonical comparison morphism $R[(\alpha^{-1}T^\times)^{-1}] \to T$ is bijective. This is the case if and only if

$$\begin{aligned}&\forall y : T.\ \exists x : R.\ \exists s : R.\ \ulcorner\alpha(s)\ \text{inv.}\urcorner \wedge \alpha(s)y = x \qquad \text{and} \\ &\qquad\qquad \forall x : R.\ \alpha(x) = 0 \Rightarrow \exists s : R.\ \ulcorner\alpha(s)\ \text{inv.}\urcorner \wedge sx = 0.\end{aligned}$$

with the filter $\varphi^{-1}((\underline{\mathbb{A}}^1_S)^\times)$. This explains the first equivalence in the chain

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Sh}(\mathrm{Spec}(A))}(\mathcal{E}, \mathrm{Zar}(\mathrm{Spec}(A))) \\
\simeq\ &\text{category of local algebras } \varphi : \underline{A} \to \mathcal{B} \text{ in } \mathcal{E} \text{ such that } \varphi^{-1}\mathcal{B}^\times = f^{-1}\mathcal{F} \\
\simeq\ &\text{category of local algebras } \psi : f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{B} \text{ in } \mathcal{E} \text{ such that } \psi \text{ is local.}
\end{aligned}$$

The second equivalence maps an algebra φ to $\underline{A}[(\varphi^{-1}\mathcal{B}^\times)^{-1}] \to \mathcal{B}$; conversely, an algebra ψ is mapped to the composition $\underline{A} \to f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \xrightarrow{\psi} \mathcal{B}$. □

Similarly to how Theorem 16.1 shows how the big Zariski topos of S looks like from the point of view of $\mathrm{Sh}(S)$, it's possible to give an internal description of what the big Zariski topos of an arbitrary relative spectrum over S looks like. We state and verify such a description in Theorem 16.9.

It is well-known that the points of $\mathrm{Zar}(\mathrm{Spec}(R))$, when constructed using one of the parsimonious sites, are in canonical bijection with the local R-algebras; for instance, this follows from the description of $\mathrm{Zar}(\mathrm{Spec}(R))$ as the classifying topos of the theory of local R-algebras. For the case of a general base scheme, we introduce the following definition.

Definition 16.2. A *ring over* S is a ring A together with a morphism $\mathrm{Spec}(A) \to S$ of locally ringed locales. A *morphism of rings over* S is a ring homomorphism which is compatible with the structure morphisms to S.

Proposition 16.3. *In the situation that one of the parsimonious sites is used to define* $\mathrm{Zar}(S)$*, the category of points of* $\mathrm{Zar}(S)$ *is canonically equivalent to the full subcategory of the rings over* S *whose underlying ring is local.*

Proof. By Theorem 16.1, a point of $\mathrm{Zar}(S)$ is given by a point of $\mathrm{Sh}(S)$, that is by a point s of S,[36] together with a local $\mathcal{O}_{S,s}$-algebra A which is local over $\mathcal{O}_{S,s}$. These data define a ring over S, namely the ring A together with the composite $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathcal{O}_{S,s}) \to S$. Since the structure morphism $\mathcal{O}_{S,s} \to A$ is local, this composite maps the focal point of $\mathrm{Spec}(A)$ to the given point $s \in S$.

Conversely, let a local ring A together with a morphism $f : \mathrm{Spec}(A) \to S$ of locally ringed locales be given. Let $x \in \mathrm{Spec}(A)$ be the focal point of $\mathrm{Spec}(A)$. We set $s := f(x)$; then A is an $\mathcal{O}_{S,s}$-algebra by $(f^\sharp)_x$. It is local over $\mathcal{O}_{S,s}$ since $f^\sharp$ is a local homomorphism.

These constructions are mutually inverse since the morphisms $\mathrm{Spec}(\mathcal{O}_{S,s}) \to S$ are monomorphisms in the category of locally ringed locales. □

Remark 16.4. In the situation that one of the parsimonious sites is used to define the big Zariski topos of S, it classifies the theory of local rings over S. This is a restatement of Theorem 16.1. Explicitly, the theory of local rings over R is given by:

(1) A theory which $\mathrm{Sh}(S)$ classifies.
(2) Structure and axioms for a ring R.
(3) Structure and axioms which guarantee that the interpretation of R in any cocomplete topos coincides with the pullback of $\mathcal{O}_S$.
(4) Structure and axioms for a local ring A and a local homomorphism $R \to A$.

The fourth item can be substituted by:

[36] The topos-theoretic points of the topos of sheaves over a topological space T are in canonical bijection with the locale-theoretic points of T, that is with locale morphisms $1 \to T$. If T is sober, such points are in canonical bijection with the elements of the underlying set of T. In a classical metatheory, schemes are sober [118, Tag 01IS]. If one wants the proof to work intuitionistically, the base scheme S has to be defined in a intuitionistically sensible way, for instance as a locally ringed locale. Correspondingly, the point s of S has to be interpreted in the locale-theoretic sense.

(4') Structure and axioms for a local ring A and a morphism $\mathrm{Spec}(A) \to (\mathrm{pt}, R)$ of locally ringed locales.

This is because such a morphism is given by a local homomorphism $\underline{R} \to \mathcal{O}_{\mathrm{Spec}(A)}$ of sheaves of rings which in turn is given by a local ring homomorphism $R \to \Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) = A$. (Taking global sections of a local homomorphism of sheaves of rings yields a homomorphism of rings which will typically fail to be local. However, here taking global sections coincides with calculating the stalk at the focal point of $\mathrm{Spec}(A)$, and pullback preserves locality of ring homomorphisms.)

Corollary 18.16 gives a description of the theory which the big Zariski topos of $\mathbb{P}^1_{\mathbb{Z}}$ classifies, building upon Remark 16.4.

16.2. Recovering the little Zariski topos from the big Zariski topos. Theorem 16.1 shows that $\mathrm{Zar}(S)$ can be reconstructed from $\mathrm{Sh}(S)$ (and its structure sheaf $\mathcal{O}_S$). Similarly, it's possible to reconstruct $\mathrm{Sh}(S)$ from $\mathrm{Zar}(S)$ (and the canonical morphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$).

Theorem 16.5. *In the situation that the site used to construct* $\mathrm{Zar}(S)$ *is one of the parsimonious sites, the little Zariski topos* $\mathrm{Sh}(S)$ *is the largest subtopos of* $\mathrm{Zar}(S)$ *where the canonical morphism* $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ *is an isomorphism.*

In other words, the little Zariski topos is the largest subtopos $\mathcal{E} \hookrightarrow \mathrm{Zar}(S)$ such that $\mathrm{Zar}(S) \models (\ulcorner \flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S \text{ is bijective}\urcorner)^\Box$ (where $\Box$ is the modal operator corresponding to the subtopos), that is that the pullback of the canonical morphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ to $\mathcal{E}$ is an isomorphism.

In the case that $S = \mathrm{Spec}(A)$ is affine, we also have the ring $\underline{A}$ in $\mathrm{Zar}(S)$ available. In this case the condition is equivalent to

$$\mathrm{Zar}(S) \models \ulcorner \underline{A} \to \underline{\mathbb{A}}^1_S \text{ is a localization}\urcorner^\Box,$$

since in the composition $\underline{A} \to \flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ the first morphism is a localization.

Proof of Theorem 16.5. The little Zariski topos is a subtopos of the big Zariski topos via the right inverse s of $\pi : \mathrm{Zar}(S) \to \mathrm{Sh}(S)$, the geometric morphism $(\pi_* \dashv \pi^!)$. The pullback of $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ to $\mathrm{Sh}(S)$ is therefore the morphism $(\flat\underline{\mathbb{A}}^1_S)|_{\mathrm{Sh}(S)} \to \underline{\mathbb{A}}^1_S|_{\mathrm{Sh}(S)}$, that is $\mathcal{O}_S \to \mathcal{O}_S$, which is an isomorphism.

Let $f : \mathcal{E} \hookrightarrow \mathrm{Zar}(S)$ be any subtopos such that the pullback of $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ to $\mathcal{E}$ is an isomorphism. We want to verify that f factors over the inclusion $s : \mathrm{Sh}(S) \hookrightarrow \mathrm{Zar}(S)$.

$$\begin{tikzcd} \mathcal{E} \arrow[rr, hook, "f"] \arrow[dr, dashed] & & \mathrm{Zar}(S) \\ & \mathrm{Sh}(S) \arrow[ur, hook, "s"'] & \end{tikzcd}$$

A candidate for a morphism $\mathcal{E} \to \mathrm{Sh}(S)$ witnessing this factorization is the composite $\pi \circ f$. It remains to show that $s \circ (\pi \circ f) = f$. Both $s \circ (\pi \circ f)$ and f are morphisms of $\mathrm{Sh}(S)$-toposes, where $\mathcal{E}$ is regarded as a $\mathrm{Sh}(S)$-topos by the composition $\pi \circ f$. By the universal property of the big Zariski topos given in Theorem 16.1, they are therefore uniquely determined by the $\mathcal{O}_S$-algebra they classify.

The morphism $s \circ (\pi \circ f)$ classifies the $\mathcal{O}_S$-algebra $f^{-1}\pi^{-1}s^{-1}\underline{\mathbb{A}}^1_S = f^{-1}(\flat\underline{\mathbb{A}}^1_S)$. The morphism f classifies the $\mathcal{O}_S$-algebra $f^{-1}\underline{\mathbb{A}}^1_S$. Since $f^{-1}(\flat\underline{\mathbb{A}}^1_S) \to f^{-1}\underline{\mathbb{A}}^1_S$ is an isomorphism, these algebras coincide. $\Box$

16.3. Change of base. Let $f : X \to S$ be a morphism of schemes. In any of the situations that

(1) the parsimonious sites are used to construct the big Zariski toposes and f is locally of finite presentation, or
(2) the same (Grothendieck or partial) universe is used for constructing both Zariski toposes and both X and S are contained in the universe,

the morphism f induces an essential geometric morphism $\mathrm{Zar}(X) \to \mathrm{Zar}(S)$ which we again denote by "f". Explicitly, the big Zariski toposes are related by the adjoint triple $f_! \dashv f^{-1} \dashv f_*$ with

$$\begin{aligned} f_* : \mathrm{Zar}(X) &\longrightarrow \mathrm{Zar}(S),\ F \longmapsto ((T \xrightarrow{g} S) \mapsto F(T \times_S X)), \\ f^{-1} : \mathrm{Zar}(S) &\longrightarrow \mathrm{Zar}(X),\ E \longmapsto ((T \xrightarrow{g} X) \mapsto E(T \xrightarrow{g} X \xrightarrow{f} S)), \\ f_! : \mathrm{Zar}(X) &\longrightarrow \mathrm{Zar}(S),\ F \longmapsto ((T \xrightarrow{g} S) \mapsto \coprod_{h:T\to X} F(T \xrightarrow{h} X)). \end{aligned}$$

In situation (2), the well-definedness of these functors is trivial. In situation (1), the well-definedness rests on the lemma that an S-morphism $h : T \to X$ is locally of finite presentation if T and X are locally of finite presentation over S [118, Tag 02FV].

The objects of $\mathrm{Zar}(S)$ listed on page 142 pull back as expected:

- Let Y be an S-scheme. Then $f^{-1}\underline{Y} = \underline{Y \times_S X}$, by the universal property of the fiber product.
- In particular, $f^{-1}\underline{\mathbb{A}}^1_S = \underline{\mathbb{A}}^1_X$, since $\mathbb{A}^1_S \times_S X = \mathbb{A}^1_X$.
- Let $\mathcal{F}$ be a sheaf of sets on S. Then $f^{-1}\pi_S^{-1}\mathcal{F} = \pi_X^{-1} f^{-1}\mathcal{F}$.
- Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules. Then $f^{-1}\mathcal{F}^{\mathrm{Zar}} = (f^*\mathcal{F})^{\mathrm{Zar}}$.

The functors $f_! \dashv f^{-1}$ induce an equivalence

$$\mathrm{Zar}(X) \simeq \mathrm{Zar}(S)/\underline{X},$$

explicitly described by

$$\begin{aligned} F &\longmapsto (f_!F \to f_!1), \\ ((T \xrightarrow{g} X) \mapsto \{s \in (f^{-1}E)(T) \mid \alpha(s) = g\}) &\longleftarrow (E \xrightarrow{\alpha} \underline{X}). \end{aligned}$$

From the internal point of view of $\mathrm{Zar}(S)$, the big Zariski topos of X is therefore simply $\mathrm{Set}/\underline{X}$, the category of $\underline{X}$-indexed families of sets or equivalently the category of sheaves on $\underline{X}$ considered as a *discrete* locale. This fits nicely with the philosophy that "S-schemes are plain unstructured sets from the internal point of view of $\mathrm{Zar}(S)$".[37]

In contrast, for the little Zariski toposes, there is no similarly simple description of the little Zariski topos of X as a slice of the little Zariski topos of S. From the internal point of view of $\mathrm{Sh}(S)$, the topos $\mathrm{Sh}(X)$ looks like the topos of sheaves over a locale which is not discrete, and the topos $\mathrm{Zar}(X)$ doesn't even look like a topos of sheaves over an arbitrary locale (discrete or not).

The internal language of a slice topos $\mathcal{E}/I$ admits a simple description from the point of view of $\mathcal{E}$. Namely, for any formula φ over $\mathcal{E}/I$,

$$\mathcal{E}/I \models \varphi \qquad \text{iff} \qquad \mathcal{E} \models \forall i\,{:}\,I.\ \varphi(i).$$

For the right hand side to make sense, it has to be interpreted in the following way. Any object $(p : M \to I)$ of $\mathcal{E}/I$ which appears in φ, for instance as a domain of

[37]The requirement on f mentioned at the beginning of this subsection is really necessary. For instance, let f be the unique morphism $\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$. This morphism is not locally of finite presentation. By Proposition 11.19, the functor of points $\underline{\mathbb{Q}} \in \mathrm{Zar}(\mathrm{Spec}(\mathbb{Z}))$ coincides with the functor of points of the empty scheme if the parsimonious sites are used and is thus the initial object in $\mathrm{Zar}(S)$. Therefore $\mathrm{Zar}(\mathrm{Spec}(\mathbb{Z}))/\underline{\mathbb{Q}}$ is the trivial topos. In contrast, $\mathrm{Zar}(\mathrm{Spec}(\mathbb{Q}))$ is not.

quantification, has to be substituted by the internal expression "$p^{-1}[\{i\}]$" denoting the fiber of p over $i : I$.[38] For example, if $(p : M \to I)$ is such an object of $\mathcal{E}/I$,

$$\begin{aligned} & \mathcal{E}/I \models \ulcorner M \text{ is inhabited}\urcorner \\ \text{iff} \quad & \mathcal{E} \models \forall i : I.\ \ulcorner\text{the fiber of } p \text{ over } i \text{ is inhabited}\urcorner \\ \text{iff} \quad & \mathcal{E} \models \forall i : I.\ \exists m : M.\ p(m) = i. \end{aligned}$$

Thanks to this description of the internal language of a slice topos, the equivalence $\mathrm{Zar}(X) \simeq \mathrm{Zar}(S)/\underline{X}$ is useful for lifting internal characterizations concerning properties of S-schemes to properties of morphisms of S-schemes. For instance, we will see in Proposition 19.37 that the structure morphism of an S-scheme $f : Y \to S$ is surjective if and only if $\mathrm{Zar}(S) \models \neg\neg(\ulcorner\underline{Y} \text{ is inhabited}\urcorner)$. This automatically implies (Corollary 19.38) that a morphism $p : Y \to X$ of S-schemes is surjective if and only if

$$\mathrm{Zar}(S) \models \forall x : \underline{X}.\ \neg\neg(\ulcorner\text{the fiber of } \underline{p} \text{ over } x \text{ is inhabited}\urcorner).$$

Many properties of morphisms in algebraic geometry, and any properties which can be characterized using the internal language of the big Zariski topos, are stable under base change. For those kinds of properties P, if a morphism $Y \to X$ is P, then for any point $x \in X$ the base change $Y_x \to \mathrm{Spec}(k(x))$ along $\mathrm{Spec}(k(x)) \to X$ is P as well. The converse is usually false, but the motto "a morphism is P if all its fibers are P in a continuous fashion" is still useful for intuition. The equivalence $\mathrm{Zar}(X) \simeq \mathrm{Zar}(S)/\underline{X}$ makes this motto precise: For any morphism $p : Y \to X$ of S-schemes and any formula $\varphi(M)$ of $\mathrm{Zar}(S)$ containing a free variable M,

$$\mathrm{Zar}(X) \models \varphi(\underline{Y}) \qquad \text{iff} \qquad \mathrm{Zar}(S) \models \forall x : \underline{X}.\ \varphi(\underline{p}^{-1}[\{x\}]),$$

that is Y has property φ when regarded as an X-scheme if and only if all the fibers of $Y \to X$ have property φ when regarded as S-schemes.

The family of geometric morphisms

$$\mathrm{Sh}(X) \to \mathrm{Zar}(X) \to \mathrm{Zar}(S)$$

with inverse image $E \mapsto E|_{\mathrm{Sh}(X)}$, where X ranges over all S-schemes contained in the site used to define $\mathrm{Zar}(S)$, is jointly surjective. That is, the restriction functors $\mathrm{Zar}(S) \to \mathrm{Sh}(X)$ are jointly conservative. This fact, together with the fact that these functors commute with geometric constructions, is frequently useful to relate truth in $\mathrm{Zar}(S)$ with truth in all of the $\mathrm{Sh}(X)$.

Lemma 16.6. (1) *The functor π^{-1} from $\mathcal{O}_S$-modules to π^{-1}-modules commutes with tensor product.*
(2) *The functor π_* from $\underline{\mathbb{A}}^1_S$-modules to $\mathcal{O}_S$-modules commutes with tensor product.*
(3) *The $\underline{\mathbb{A}}^1_S$-module $\mathcal{E}^{\mathrm{Zar}}$ associated to an $\mathcal{O}_S$-module $\mathcal{E}$ (defined on page 142) is canonically isomorphic to $\pi^{-1}\mathcal{E} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S$.*
(4) *The functor $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{Zar}}$ from $\mathcal{O}_S$-modules to $\underline{\mathbb{A}}^1_S$-modules commutes with tensor product.*

[38]This substitution is less ad hoc as it might at first appear. The internal language of a topos $\mathcal{E}$ is *dependently typed*, meaning that the types one can quantify over may depend on previously introduced values. Types in the empty context, depending on no values, correspond to objects of $\mathcal{E}$. Types in the context of a variable $i : I$ correspond to objects $(p : M \to I)$ of $\mathcal{E}/I$. For instance, in this case one can form formulas of the form "$\forall i : I.\ \forall m : M(i).\ \psi(i, m)$". If in the translation process using the Kripke–Joyal semantics a formal variable i was substituted by a generalized element $i_0 : A \to I$, the expression "$M(i_0)$" has to be interpreted as the pullback $i_0^* M$.

Proof. Forming the tensor product is a geometric construction. It is therefore preserved by π^{-1} and by π_*, since both functors are inverse-image-parts of geometric morphisms. Claim (3) follows from the calculation

$$\begin{aligned}(\pi^{-1}\mathcal{E} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S)|_{\mathrm{Sh}(X)} &\cong (\pi^{-1}\mathcal{E})|_{\mathrm{Sh}(X)} \otimes_{(\pi^{-1}\mathcal{O}_S)|_{\mathrm{Sh}(X)}} \underline{\mathbb{A}}^1_S|_{\mathrm{Sh}(X)} \\ &\cong (f^{-1}\mathcal{E}) \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X \\ &\cong f^*\mathcal{E}\end{aligned}$$

for S-schemes $(f : X \to S)$ contained in the site used to define $\mathrm{Zar}(S)$. Finally, claim (4) follows by using a standard property for tensor products (whose proof is intuitionistic and therefore valid in $\mathrm{Zar}(S)$):

$$\begin{aligned}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{F})^{\mathrm{Zar}} &\cong \pi^{-1}(\mathcal{E} \otimes_{\mathcal{S}} \mathcal{F}) \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S \\ &\cong (\pi^{-1}\mathcal{E} \otimes_{\pi^{-1}\mathcal{O}_S} \pi^{-1}\mathcal{F}) \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S \\ &\cong (\pi^{-1}\mathcal{E} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S) \otimes_{\underline{\mathbb{A}}^1_S} (\pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S) \\ &\cong \mathcal{E}^{\mathrm{Zar}} \otimes_{\underline{\mathbb{A}}^1_S} \mathcal{F}^{\mathrm{Zar}}.\end{aligned}$$

□

Example 16.7. Constructing, internally in $\mathrm{Zar}(S)$, the module $\Omega^1_{\underline{\mathbb{A}}^1_S|\flat\underline{\mathbb{A}}^1_S}$ of Kähler differentials of the ring morphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ yields the "universal cotangent sheaf"

$$X \longmapsto \Gamma(X, \Omega^1_{X|S}).$$

This is because constructing the module of Kähler differentials is a geometric construction and the restriction functors $\mathrm{Zar}(S) \to \mathrm{Zar}(X)$ commute with geometric constructions, so

$$[\![\Omega^1_{\underline{\mathbb{A}}^1_S|\flat\underline{\mathbb{A}}^1_S}]\!]|_{\mathrm{Sh}(X)} \cong \Omega^1_{(\underline{\mathbb{A}}^1_S|_{\mathrm{Sh}(X)})|((\flat\underline{\mathbb{A}}^1_S)|_{\mathrm{Sh}(X)})} \cong \Omega^1_{\mathcal{O}_X|f^{-1}\mathcal{O}_S} \cong \Omega^1_{X|S}$$

for any S-scheme $(f : X \to S)$ contained in the site used to define $\mathrm{Zar}(S)$.

This universal cotangent sheaf doesn't have any finiteness properties from the internal point of view. For instance, it isn't finitely generated, because else all the individual cotangent sheaves $\Omega^1_{X|S}$ would be of finite type (with a uniform bound on the number of generators) by Proposition 18.1.

We feel that the significance of the canonical morphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ hasn't been adequately explored yet.

Remark 16.8. Some care is needed when dealing with the modalities $\flat$ and $\sharp$, since they are not compatible with change of base. If $f : X \to S$ is a morphism of schemes, then in general $f^{-1}(\flat E) \not\cong \flat(f^{-1}E)$, since

$$\begin{aligned}f^{-1}(\flat E) &= ((T \xrightarrow{g} X) \mapsto \Gamma(T, g^{-1}f^{-1}(E|_{\mathrm{Sh}(S)}))), \quad \text{but} \\ \flat(f^{-1}E) &= ((T \xrightarrow{g} X) \mapsto \Gamma(T, g^{-1}(E|_{\mathrm{Sh}(X)}))).\end{aligned}$$

A special case in which the canonical morphism $f^{-1}(\flat E) \to \flat(f^{-1}E)$ is an isomorphism is when f is an open immersion.

A consequence of the fact that $\flat$ and $\sharp$ aren't compatible with change of base is that these modalities can't be defined in the internal language of $\mathrm{Zar}(S)$, since any construction which can be described in the internal language is automatically compatible with change of base. However, the modalities can still be used and their general properties can even be elementarily axiomatized [9].

16.4. The big Zariski topos of a relative spectrum.

Theorem 16.9. *Let $\mathcal{A}$ be a quasicoherent $\mathcal{O}_S$-algebra. In the situation that the parsimonious sites are used for constructing big Zariski toposes, the big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is, from the internal point of view of $\mathrm{Sh}(S)$, the classifying topos of the theory of local $\mathcal{A}$-algebras which are local over $\mathcal{O}_S$.*

Proof. The proof is similar to the proof of Theorem 16.1. Let $X = \underline{\mathrm{Spec}}_S(\mathcal{A})$ and $f : X \to S$ be the canonical morphism. The big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ is a $\mathrm{Sh}(S)$-topos by the composition $\mathrm{Zar}(\underline{\mathrm{Spec}}_S(\mathcal{A})) \to \mathrm{Zar}(S) \to \mathrm{Sh}(S)$. The pullback of $\mathcal{O}_S$ along this geometric morphism is $f^{-1}(\flat\underline{\mathbb{A}}^1_S)$. A canonical $\mathcal{O}_S$-algebra in $\mathrm{Zar}(\underline{\mathrm{Spec}}_S(\mathcal{A}))$ is therefore

$$f^{-1}(\flat\underline{\mathbb{A}}^1_S) \longrightarrow \flat\underline{\mathbb{A}}^1_X \longrightarrow \underline{\mathbb{A}}^1_X.$$

This algebra is indeed local and local over $f^{-1}(\flat\underline{\mathbb{A}}^1_S)$.

For verifying the universal property, it suffices to restrict to the case that $S = \mathrm{Spec}(R)$ is affine, as in the proof of Theorem 16.1, and consider a geometric morphism $f : \mathcal{E} \to \mathrm{Sh}(S)$. In this case $\mathcal{A} = A^\sim$ and $X = \underline{\mathrm{Spec}}_S(\mathcal{A}) = \mathrm{Spec}(A)$. Let $\alpha : R \to A$ be the structure morphism of A. We then have the chain of equivalences

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Sh}(S)}(\mathcal{E}, \mathrm{Zar}(X)) \\
\simeq\ &\text{cat. of local algebras } \varphi : \underline{A} \to \mathcal{B} \text{ in } \mathrm{Zar}(X) \text{ such that } \underline{\alpha}^{-1}\varphi^{-1}\mathcal{B}^\times = f^{-1}\mathcal{F} \\
\simeq\ &\text{cat. of local algebras } \psi : f^{-1}\mathcal{A} \to \mathcal{B} \text{ such that } f^{-1}\mathcal{O}_S \to f^{-1}\mathcal{A} \to \mathcal{B} \text{ is local.}
\end{aligned}$$

The first equivalence maps a geometric morphism $g : \mathcal{E} \to \mathrm{Zar}(X)$ to $\underline{A} \to g^{-1}\underline{\mathbb{A}}^1_X$. The second equivalence acts as follows. Given an algebra $\varphi : \underline{A} \to \mathcal{B}$ such that $\underline{\alpha}^{-1}\varphi^{-1}\mathcal{B}^\times = f^{-1}\mathcal{F}$, we can factor $\underline{R} \to \underline{A} \to \mathcal{B}$ uniquely as a localization $\underline{R} \to C$ followed by a local homomorphism $C \to \mathcal{B}$. By the condition on filters, the localization C is isomorphic to $f^{-1}\mathcal{O}_S$. From the description $\mathcal{A} = \underline{A}[\mathcal{F}^{-1}]$ it is apparent that $\underline{A} \to \mathcal{B}$ factors over $\underline{A} \to f^{-1}\mathcal{A}$. In this way, we obtain morphisms $f^{-1}\mathcal{O}_S \to f^{-1}\mathcal{A} \to \mathcal{B}$. □

The only reason why we have supposed that $\mathcal{A}$ is quasicoherent in the statement of Theorem 16.9 is because else $\underline{\mathrm{Spec}}_S(\mathcal{A})$ might fail to be a scheme, whereby the notion "big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$" is not defined.

In fact, we propose the following definition: If $(X, \mathcal{O}_X)$ is an arbitrary locally ringed locale (or even a locally ringed topos), then the big Zariski topos of X should be the classifying $\mathrm{Sh}(X)$-topos of the theory (internal to $\mathrm{Sh}(X)$) of local $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$. The following proposition shows that this definition is consistent with Theorem 16.1 and with Theorem 16.9.

Proposition 16.10. *Let $\mathcal{A}$ be an $\mathcal{O}_S$-algebra. The following constructions, performed internally to $\mathrm{Sh}(S)$, yield canonically equivalent toposes:*

(1) *Constructing first the local spectrum $X := \mathrm{Spec}(\mathcal{A}|\mathcal{O}_S)$ and then, internally to $\mathrm{Sh}_{\mathrm{Sh}(S)}(X)$, the classifying topos of the theory of $\mathcal{O}_X$-algebras which are local over $\mathcal{O}_X$.*
(2) *Constructing the classifying topos of the theory of $\mathcal{A}$-algebras which are local over $\mathcal{O}_S$.*

If furthermore $\mathcal{A}$ is finitely presented as an $\mathcal{O}_S$-algebra from the internal point of view of $\mathrm{Sh}(S)$, then the following construction yields the same result as well:

(3) *Constructing first the big Zariski topos of S as the classifying topos of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$ and then constructing, internally to that topos, the slice topos over $[\mathcal{A}^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}$.*

Proof. If S is indeed a scheme, as is supposed throughout this part, and $\mathcal{A}$ is quasicoherent, then all three constructions yield the big Zariski topos of $\underline{\mathrm{Spec}}_S(\mathcal{A})$ (defined using one of the parsimonious sites). For the first construction, this is by Theorem 12.10 and Theorem 16.1; for the second construction, this is by Theorem 16.9; and for the third construction, this is by Theorem 16.1, Proposition 18.8, and the description of the slice topos in Section 16.3. However, the claim also holds if $\mathcal{A}$ is not quasicoherent or if S is an arbitrary locally ringed locale, and it's instructive to see the proof in this more general situation.

We work in the internal universe of $\mathrm{Sh}(S)$. Let $\mathcal{E}$ be an arbitrary (Grothendieck) topos. Then $\mathcal{E}$-valued points of the three toposes are given by:

(1) a filter $F \subseteq \mathcal{A}$ lying over the filter of units of $\mathcal{O}_S$ together with a local $\mathcal{A}_F$-algebra R which is local over $\mathcal{A}_F$
(2) a local $\mathcal{A}$-algebra which is local over $\mathcal{O}_S$
(3) a local $\mathcal{O}_S$-algebra R which is local over $\mathcal{O}_S$ together with an element of the stalk of $[\mathcal{A}^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}$ at the point corresponding to R

In the case that $\mathcal{A}$ is finitely presented, the stalk appearing in description (3) is canonically isomorphic to the set of R-algebra homomorphisms $\mathcal{A} \otimes_{\mathcal{O}_S} R \to R$, as discussed in Lemma 6.45.

With these descriptions, the equivalence is immediate. For instance, a datum $(F \subseteq \mathcal{A}, \mathcal{A}_F \to R)$ as in description (1) gives rise to the datum $(\mathcal{O}_S \to \mathcal{A}_F \to R)$ as in description (2). Conversely, the structure morphism of a datum as in description (2) can be factored as a localization followed by a local homomorphism to yield a datum as in (1). ☐

16.5. Constructing the big Zariski topos without recourse to classical scheme theory. Given a scheme S, is it possible to construct the big Zariski topos of S without recourse to classical scheme theory? Without employing a site which refers to schemes as classically conceived?

Taken literally, this question is ill-posed, since the datum S is given as a classical scheme. A better question is: Is it possible to setup the basics of the theory of schemes using only big Zariski toposes, preferably even in an intuitionistic fashion?

This is indeed possible, and we wish to sketch how this can be done. Given a base ring A, the big Zariski topos of $\mathrm{Spec}(A)$ can be defined as the topos of sheaves over the parsimonious site $\mathrm{Alg}(A)^{\mathrm{op}}_{\mathrm{fp}}$ consisting of (formal duals) of finitely presented A-algebras. We can then declare an A-scheme to be an object of $\mathrm{Zar}(\mathrm{Spec}(A))$ having certain properties, for instance being a *finitely presented synthetic scheme* (Definition 19.49).

The big Zariski topos of such an A-scheme X can then simply be defined as the slice topos $\mathrm{Zar}(\mathrm{Spec}(A))/X$, in accordance with the equivalence noted in Section 16.3. This slice topos can serve as the base over which further schemes and their big Zariski toposes can be constructed.

Inaccessible to this approach to scheme theory are schemes which are not locally of finite presentation over the base ring. If one wants to account for such schemes, one has to substitute the parsimonious site for a larger one; however, some problems remain, as indicated in Section 19.10.

17. The double negation modality

Proposition 17.1. *Let φ be a formula over S. Consider the following statements:*

(1) $\mathrm{Zar}(S) \models \neg\neg\varphi$.
(2) *For all points $s \in S$, there is a field extension $K \mid k(s)$ such that* $\mathrm{Spec}(K) \to \mathrm{Spec}(k(s)) \to S$ *is contained in the site used to define* $\mathrm{Zar}(S)$ *and such that* $\mathrm{Spec}(K) \models \varphi$.
(3) *For all closed points $s \in S$, there is a finite field extension $K \mid k(s)$ such that* $\mathrm{Spec}(K) \models \varphi$.

Then:

- *Condition (2) implies condition (1). The converse holds if the site used to define* $\mathrm{Zar}(S)$ *is closed under taking spectra of residue fields (this is satisfied for all sites listed in Section 15 except for the parsimonious sites).*

- *If one of the parsimonious sites is used to define* $\mathrm{Zar}(S)$ *and* S *is locally Noetherian, condition (1) implies condition (3). The converse holds if additionally* S *is locally of finite type over a field.*

Proof. We begin with showing that condition (2) implies condition (1). By the Kripke–Joyal translation, we need to verify that

$$\forall(X \to S).\ \Big(\forall(T \to X).\ (T \models \varphi) \Rightarrow T = \emptyset\Big) \Longrightarrow X = \emptyset,$$

where the universal quantifiers range over all schemes contained in the site used to define $\mathrm{Zar}(S)$. So let such an S-scheme $f : X \to S$ be given. We show that the fiber over any point $s \in S$ is empty. By assumption, there is a field extension $K \mid k(s)$ such that $\mathrm{Spec}(K) \to \mathrm{Spec}(k(s)) \to S$ is contained in the site used to define $\mathrm{Zar}(S)$ and such that $\mathrm{Spec}(K) \models \varphi$. The base change T of the fiber X_s to $\mathrm{Spec}(K)$ as indicated in the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d] & X_s \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(k(s)) \arrow[r] & S \end{tikzcd}$$

is contained in the site used to define $\mathrm{Zar}(S)$ as well, therefore saying "$T \models \varphi$" is meaningful. And indeed $T \models \varphi$, since $\mathrm{Spec}(K) \models \varphi$. Therefore $T = \emptyset$. Thus $X_s = \emptyset$ as claimed.

For the direction "(1) $\Rightarrow$ (2)", let a point $s \in S$ be given. Since we assume that the site used to define $\mathrm{Zar}(S)$ contains the S-scheme $X := \mathrm{Spec}(k(s))$ and since $X \neq \emptyset$, the assumption implies that there exists an nonempty X-scheme T such that $T \models \varphi$. Since T is nonempty, there exists a point $t \in T$. By the morphism $\mathrm{Spec}(k(t)) \to T \to X$, the field $K := k(t)$ is an extension of $k(s)$, and since $\mathrm{Spec}(K) \to T \to X \to S$ is contained in the site, we have $\mathrm{Spec}(K) \models \varphi$.

The proof that condition (1) implies condition (3) in the case that one of the parsimonious sites is used to define $\mathrm{Zar}(S)$ and that S is locally Noetherian is similar. For a closed point $s \in S$, the residue field $k(s)$ can be calculated as $A/\mathfrak{m}$, where A is the ring of functions of an open affine neighborhood of s and $\mathfrak{m}$ is a maximal ideal in A. Since A is Noetherian, the ideal $\mathfrak{m}$ is finitely generated and therefore $A/\mathfrak{m}$ is finitely presented as an A-algebra. Thus the canonical morphism $\mathrm{Spec}(k(s)) \to \mathrm{Spec}(A) \to S$ is locally of finite presentation and thereby contained in the parsimonious site. The hypothesis is therefore applicable to $X := \mathrm{Spec}(k(s))$ and yields a nonempty X-scheme T which is locally of finite presentation over X such that $T \models \varphi$.

Since the structure morphism $T \to X$ is locally of finite presentation, the scheme T inherits the property to be locally Noetherian from X. Let $U \subseteq T$ be a nonempty open affine subset and let $t \in U$ be a point which is closed in U. With the same reasoning as above, the canonical morphism $\mathrm{Spec}(k(t)) \to U \to T$ is therefore contained in the parsimonious site. Thus $\mathrm{Spec}(k(t)) \models \varphi$. The field $K := k(t)$ is finitely presented as an $k(s)$-algebra. By Noether normalization, it is also of finite dimension as an $k(s)$-vector space.

Finally, we verify that condition (3) implies condition (1) if one of the parsimonious sites is used to define $\mathrm{Zar}(S)$ and if S is locally of finite type over a field (and therefore in particular Noetherian). We adopt the notation of the proof of "(2) $\Rightarrow$ (1)". The argument there shows that all fibers of f over closed points are empty. If X is not empty, it contains a closed point x (since X is locally of finite type over a field, any point which is closed in an open affine neighborhood will do). Since X is locally of finite type over a field, the point $f(x)$ is closed in S. Therefore x is contained in the fiber over a closed point; a contradiction. $\square$

Remark 17.2. The proof of Proposition 17.1 uses classical logic in a substantial way, since repeatedly the lemma that a scheme is trivial if it doesn't have any points was used. Even if scheme theory is set up in an intuitionistic sensible way (for instance defining a scheme to be a locally ringed locale which is locally isomorphic to the locale-theoretic spectra of rings as discussed in Section 12.2), one should therefore not expect the proposition to admit an intuitionistic proof without additional hypotheses.

Lemma 17.3. *Let $f : X \to S$ and $g : Y \to S$ be S-schemes which are locally contained in the site used to define* $\mathrm{Zar}(S)$*. In the case that the site is one of the parsimonious sites, further assume that f and g are quasicompact and quasiseparated. The following statements are equivalent:*

(1) *The image of f coincides with the image of g topologically.*
(2) $\mathrm{Zar}(S) \models \neg\neg(\ulcorner \underline{X}$ inhabited$\urcorner) \Leftrightarrow \neg\neg(\ulcorner \underline{Y}$ inhabited$\urcorner)$.

Proof. By Proposition 19.37, which we'll prove below, statement (2) is equivalent to:

> For any S-scheme $h : T \to S$ contained in the site used to define $\mathrm{Zar}(S)$, the morphism $X \times_S T \to T$ is surjective if and only if $Y \times_T T \to T$ is.

We verify that this statement implies statement (1). Let $s \in \mathrm{im}(f)$. Then the canonical morphism $X_s \to \mathrm{Spec}(k(s))$ is surjective. Therefore there exists an S-scheme $h : T \to S$ which is contained in the site used to define $\mathrm{Zar}(S)$ such that $X \times_S T \to T$ is surjective and such that $s \in \mathrm{im}(h)$: If $\mathrm{Zar}(S)$ is defined using a Grothendieck or partial universe, this claim is trivial, since we can take $T := \mathrm{Spec}(k(s))$. If $\mathrm{Zar}(S)$ is defined using one of the parsimonious sites, we employ the technique of relative approximation.[39]

The assumption yields that the induced morphism $Y \times_T T \to T$ is surjective. Since $s \in \mathrm{im}(h)$, also $s \in \mathrm{im}(g)$.

The proof of the converse containment relation is analogous.

The direction "(1) $\Rightarrow$ (2)" is immediate, since

$$\mathrm{im}(X \times_S T \to T) = h^{-1}\,\mathrm{im}(f) = h^{-1}\,\mathrm{im}(g) = \mathrm{im}(Y \times_S T \to T). \qquad \square$$

Remark 17.4. Let $f : X \to S$ be contained in the site used to define $\mathrm{Zar}(S)$. In the case that the site is one of the parsimonious sites, further assume that f is quasicompact and quasiseparated. The expression "$\neg\neg(\ulcorner \underline{X}$ is inhabited$\urcorner)$" of the internal language of $\mathrm{Zar}(S)$ denotes the subfunctor of the terminal functor $\underline{S} = 1 \in \mathrm{Zar}(S)$ given by

$$(h : T \to S) \longmapsto \{\star \mid \mathrm{im}(h) \subseteq \mathrm{im}(f)\}.$$

If f is an open immersion, then this functor coincides with the functor of points of X, since the set-theoretic image of a morphism of schemes is contained in an open subset $U \subseteq S$ if and only if it factors over the open immersion $U \hookrightarrow S$.

If f is a closed immersion, this functor is the functor of points of the formal completion of S along X. More generally, for an arbitrary S-scheme X and a closed subscheme $Z \hookrightarrow X$ (such that both X and S are locally contained in the site used

[39] More specifically, we may assume that S is affine. Then the lemma on relative approximation [118, Tag 09MV] can be applied to write $\mathrm{Spec}(k(s))$ as a directed limit of an inverse system of finitely presented S-schemes T_i with affine transition maps. Let $U \subseteq X$ be an open affine subset containing a preimage of s. The property that $U \hookrightarrow X \to \mathrm{Spec}(k(s))$ is surjective descends to one of the morphisms $U \times_S T_i \to T_i$ [118, Tag 07RR]. In particular, the morphism $X \times_S T_i \to T_i$ is surjective. We can therefore take $T := T_i$. The image of $T_i \to S$ contains s since $\mathrm{Spec}(k(s)) \to S$ factors over $T_i \to S$.

to define $\mathrm{Zar}(S)$), the internal expression "$\{x : \underline{X} \mid \neg\neg(x \in \underline{Z})\}$" denotes the functor of points of the formal completion of X along Z. For instance, the expression

$$\{f : \underline{\mathbb{A}}^1_S \mid \neg\neg(f = 0)\} = \{f : \underline{\mathbb{A}}^1_S \mid \ulcorner f \text{ is nilpotent}\urcorner\}$$

denotes the formal neighborhood of the origin in the affine line $\mathbb{A}^1_S$. (The equivalence $\neg\neg(f = 0) \Leftrightarrow \ulcorner f$ is nilpotent$\urcorner$ is by Proposition 18.29.)

18. Sheaves of rings, algebras, and modules

Proposition 18.1. *Let $\mathcal{E}$ be an $\mathcal{O}_S$-module. Properties of $\mathcal{E}$ and of the induced $\underline{\mathbb{A}}^1_S$-module $\mathcal{E}^{\mathrm{Zar}}$ are related as follows:*

- *$\mathcal{E}$ is finite locally free if and only if $\mathcal{E}^{\mathrm{Zar}}$ is finite free as an $\underline{\mathbb{A}}^1_S$-module from the internal point of view of* $\mathrm{Zar}(S)$*.*
- *$\mathcal{E}$ is of finite type if and only if $\mathcal{E}^{\mathrm{Zar}}$ is finitely generated as an $\underline{\mathbb{A}}^1_S$-module from the internal point of view of* $\mathrm{Zar}(S)$*.*
- *$\mathcal{E}$ is of finite presentation if and only if $\mathcal{E}^{\mathrm{Zar}}$ is finitely presented as an $\underline{\mathbb{A}}^1_S$-module from the internal point of view of* $\mathrm{Zar}(S)$*.*

Proof. The "if" directions follow just as in Proposition 4.3. The proofs of the "only if" directions can too proceed by hand, further exploiting only that generators and relations are stable under base change. But there's also a more conceptual proof: From the point of view of $\mathrm{Sh}(S)$, the $\underline{\mathbb{A}}^1_S$-module $\mathcal{E}^{\mathrm{Zar}}$ admits the description $\mathcal{E}^{\mathrm{Zar}} = \underline{\mathcal{E}} \otimes_{\underline{\mathcal{O}_S}} \underline{\mathbb{A}}^1_S$, where the underline denotes the constant sheaf construction (which externally is interpreted by the functor π^{-1} described in Section 16). By Lemma 12.48 (generalized from locales to toposes), the $\underline{\mathcal{O}_S}$-module $\underline{\mathcal{E}}$ inherits any property from $\mathcal{E}$ which is of a certain logical form, and the properties under discussion are stable under tensoring. ∎

An analogous result holds for sheaves of algebras and their finiteness conditions.

Coherence is missing from the list in Proposition 18.1 since coherence is not stable under pullback (for instance, the pullback of $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}$ is not stable under any morphism $X \to \mathrm{Spec}(\mathbb{Z})$ for which $\mathcal{O}_X$ is not coherent) and can therefore not be characterized by any formula in the internal language of a big Zariski topos whose site is sufficiently encompassing.

However, coherence is stable under pullback along locally finitely presented morphisms. Therefore we do have the following result.

Proposition 18.2. *Let $\mathcal{E}$ be an $\mathcal{O}_S$-module. If the induced $\underline{\mathbb{A}}^1_S$-module $\mathcal{E}^{\mathrm{Zar}}$ is coherent from the internal point of view of* $\mathrm{Zar}(S)$*, then $\mathcal{E}$ is coherent. The converse holds if* $\mathrm{Zar}(S)$ *is defined using one of the parsimonious sites.*

Proposition 18.3. *The inclusion functor*

$$\mathrm{Mod}_{\mathrm{Sh}(S)}(\mathcal{O}_S) \longrightarrow \mathrm{Mod}_{\mathrm{Zar}(S)}(\underline{\mathbb{A}}^1_S),\ \mathcal{E} \longmapsto \mathcal{E}^{\mathrm{Zar}}$$

is fully faithful.

Proof. This follows from the following string of isomorphisms, employing the adjunction $\pi^{-1} \dashv \pi_*$ and the fact that π_* commutes with tensor product by Lemma 16.6.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(\mathcal{E}^{\mathrm{Zar}}, \mathcal{F}^{\mathrm{Zar}}) &= \mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(\pi^{-1}\mathcal{E} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S, \pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S) \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\pi^{-1}\mathcal{O}_S)}(\pi^{-1}\mathcal{E}, \pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S) \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_S)}(\mathcal{E}, \pi_*(\pi^{-1}\mathcal{F} \otimes_{\pi^{-1}\mathcal{O}_S} \underline{\mathbb{A}}^1_S)) \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_S)}(\mathcal{E}, \pi_*\pi^{-1}\mathcal{F} \otimes_{\pi_*\pi^{-1}\mathcal{O}_S} \pi_*\underline{\mathbb{A}}^1_S)) \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_S)}(\mathcal{E}, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_S) \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{O}_S)}(\mathcal{E}, \mathcal{F}).
\end{aligned}$$

∎

Caveat 18.4. The inclusion functor from the category of $\mathcal{O}_S$-modules to the category of $\underline{\mathbb{A}}^1_S$-modules is right exact, and preserves even arbitrary colimits, but is not left exact.

18.1. Relative spectrum.

Definition 18.5. In the context of a specified local ring R, the *synthetic spectrum* of an R-algebra A is

$$\mathrm{Spec}(A) := [A, R]_{\mathrm{Alg}(R)},$$

the set of R-algebra homomorphisms from A to R.

Example 18.6. The synthetic spectrum of R is the one-element set. More generally, the synthetic spectrum of the algebra $R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ is the solution set $\{x : R^n \mid f_1(x) = \cdots = f_n(x) = 0\}$.

Example 18.7. The synthetic spectrum of $R/(f)$ is $[\![f = 0]\!]$, the truth value of the formula "$f = 0$", the subsingleton set $\{\star \mid f = 0\}$. If classical logic is available, then this set contains $\star$ or is empty, depending on whether f is zero or not. Similarly, the synthetic spectrum of $R[f^{-1}]$ is $[\![\ulcorner f \text{ inv.} \urcorner]\!]$.

Proposition 18.8. *Let $\mathcal{A}_0$ be an $\mathcal{O}_S$-algebra (not necessarily quasicoherent). Then the synthetic spectrum of the $\underline{\mathbb{A}}^1_S$-algebra $(\mathcal{A}_0)^{\mathrm{Zar}}$, as constructed in the internal language of* $\mathrm{Zar}(S)$*, is the functor of points of* $\underline{\mathrm{Spec}}_S\, \mathcal{A}_0$.

Proof. The Hom set occurring in the definition of the synthetic spectrum is interpreted by the internal Hom when using the internal language. For any S-scheme $f : T \to S$ contained in the site used to define $\mathrm{Zar}(S)$, we have the following chain of isomorphisms.

$$\begin{aligned}
[\![\mathrm{Spec}(A)]\!](T) &= [(\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(T) \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T}, [(\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}) \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times (\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_S)_{\ldots} \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)/\underline{T}}(\underline{T} \times (\mathcal{A}_0)^{\mathrm{Zar}}, \underline{T} \times \underline{\mathbb{A}}^1_S)_{\ldots} \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Zar}(T)}(\underline{\mathbb{A}}^1_T)}((f^*\mathcal{A}_0)^{\mathrm{Zar}}, \underline{\mathbb{A}}^1_T) \\
&= \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Zar}(T)}(\underline{\mathbb{A}}^1_T)}(\pi^{-1}(f^*\mathcal{A}_0) \otimes_{\pi^{-1}\mathcal{O}_T} \underline{\mathbb{A}}^1_T, \underline{\mathbb{A}}^1_T) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Zar}(T)}(\pi^{-1}\mathcal{O}_T)}(\pi^{-1}(f^*\mathcal{A}_0), \underline{\mathbb{A}}^1_T) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Sh}(T)}(\mathcal{O}_T)}(f^*\mathcal{A}_0, \mathcal{O}_T) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}_{\mathrm{Sh}(S)}(\mathcal{O}_S)}(\mathcal{A}_0, f_*\mathcal{O}_T) \\
&\cong \mathrm{Hom}_S(T, \underline{\mathrm{Spec}}_S\, \mathcal{A}_0).
\end{aligned}$$

The omitted subscripts "…" should denote that we're only taking the subset of the Hom set where, for each fixed first argument, the morphisms are morphisms of $\underline{\mathbb{A}}^1_S$-algebras. $\square$

If $X \in \mathrm{Zar}(S)$ is an arbitrary object, there is a canonical morphism $X \to \mathrm{Spec}([X, \underline{\mathbb{A}}^1_S])$. In the internal language of $\mathrm{Zar}(X)$ it looks like the "inclusion into the double dual":

$$x \longmapsto \_\_(x), \quad \text{where } \_\_(x) : [X, \underline{\mathbb{A}}^1_S] \to \underline{\mathbb{A}}^1_S,\ \varphi \mapsto \varphi(x).$$

The following proposition shows that bijectivity of this map is related to X being the functor of points of an affine S-scheme (an S-scheme whose structure morphism to S is affine).

Proposition 18.9. *Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. Consider the following statements:*

(1) *The sheaf X is isomorphic to the functor of points of an affine S-scheme.*
(2) *In the internal language of $\mathrm{Zar}(S)$, the $\underline{\mathbb{A}}^1_S$-algebra $[X, \underline{\mathbb{A}}^1_S]$ is synthetically quasicoherent (Definition 18.18) and the canonical map $X \to \mathrm{Spec}([X, \underline{\mathbb{A}}^1_S])$ is bijective.*

If one of the parsimonious sites is used to define $\mathrm{Zar}(S)$ or if, internally, the algebra $[X, \underline{\mathbb{A}}^1_S]$ is finitely presented, then "(2) $\Rightarrow$ (1)". The converse holds in any of the following situations:

- *The affine S-scheme which X represents is of finite presentation over S.*
- *The site used to define $\mathrm{Zar}(S)$ is defined using a partial universe and the affine S-scheme which X represents if of finite type over S.*
- *The affine S-scheme which X represents is contained in the site used to define $\mathrm{Zar}(S)$. (This situation subsumes the previous ones.)*

Proof. The direction "(2) $\Rightarrow$ (1)" is straightforward, since the assumption expresses X as the functor of points of the relative spectrum of a quasicoherent $\mathcal{O}_S$-algebra. Theorem 18.19, which is needed to obtain the sought quasicoherent $\mathcal{O}_S$-algebra, is applicable.

For the converse direction, we abuse notation and denote the given affine S-scheme whose functor of points is X by "$f : X \to S$". Then $f_*\mathcal{O}_X$ is quasicoherent and the canonical morphism $X \to \underline{\mathrm{Spec}}_S\, f_*\mathcal{O}_X$ is an isomorphism. In any of the listed situations, the internal Hom $[X, \underline{\mathbb{A}}^1_S]$ is canonically isomorphic to $(f_*\mathcal{O}_X)^{\mathrm{Zar}}$, since for any object $T \xrightarrow{g} S$ of the site used to define $\mathrm{Zar}(S)$ we have that

$$\begin{aligned}
[X, \underline{\mathbb{A}}^1_S](T) &\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T}, [X, \underline{\mathbb{A}}^1_S]) \cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times X, \underline{\mathbb{A}}^1_S) \\
&\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times \underline{X}, \underline{\mathbb{A}}^1_S) \cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T \times_S X}, \underline{\mathbb{A}}^1_S) \\
&\cong \mathbb{A}^1_S(T \times_S X) \cong (g^* f_*\mathcal{O}_X)(T) = (f_*\mathcal{O}_X)^{\mathrm{Zar}}(T).
\end{aligned}$$

Therefore $[X, \underline{\mathbb{A}}^1_S]$ is quasicoherent. The map induced by the isomorphism $X \to \underline{\mathrm{Spec}}_S\, f_*\mathcal{O}_X$ on the level of functors of points is precisely the canonical map $X \to \mathrm{Spec}([X, \underline{\mathbb{A}}^1_S])$ as defined in the internal language; therefore this map is bijective from the internal point of view. ☐

Remark 18.10. The condition in Proposition 18.9 that $\mathrm{Spec}(A)$ is representable by an object of the site used to define $\mathrm{Zar}(S)$ is slightly unnatural from a topos-theoretic point of view, since the conclusion of the Scholium depends only on the topos over the site and not the site itself. In fact, the condition can be weakened and made more natural at the same time: It suffices to require that $\mathrm{Spec}(A)$ is *locally* representable by an object of the site.

The following corollary answers a question by Madore [87, entry 2002-07-07:044].

Corollary 18.11. *A morphism $f : X \to S$ of schemes if finite if, from the internal point of view of $\mathrm{Zar}(S)$, the canonical map $\underline{X} \to \mathrm{Spec}([\underline{X}, \underline{\mathbb{A}}^1_S])$ is bijective and the $\underline{\mathbb{A}}^1_S$-algebra $[\underline{X}, \underline{\mathbb{A}}^1_S]$ is synthetically quasicoherent and finitely generated. The converse holds if X is locally contained in the site used to define $\mathrm{Zar}(S)$.*

Proof. Immediate using Proposition 18.9 and Proposition 18.1. ☐

Remark 18.12. Let $\mathcal{A}_0$ be an $\mathcal{O}_S$-algebra. Then one can form, internally to $\mathrm{Zar}(S)$, two locales related to $\mathcal{A}_0$: the discrete locale on the synthetic spectrum of $(\mathcal{A}_0)^{\mathrm{Zar}}$, and the local spectrum of $(\mathcal{A}_0)^{\mathrm{Zar}}$ over $\underline{\mathbb{A}}^1_S$ as described in Definition 12.4. These locales don't coincide. In fact, the pullback of a discrete locale is discrete, whereas the pullback of the local spectrum to any of the little Zariski toposes $\mathrm{Sh}(X)$, where $f : X \to S$ is an S-scheme contained in the site used to define $\mathrm{Zar}(S)$, is the relative spectrum $\underline{\mathrm{Spec}}_X(f^*\mathcal{A}_0)$, which is typically not discrete as an X-locale. (This is because the locale spectrum construction is geometric, by Proposition 12.28.)

There is, however, a comparison morphism from the discrete locale on the synthetic spectrum to the local spectrum. On points, it sends an $\underline{\mathbb{A}}^1_S$-algebra homomorphism $\varphi : (\mathcal{A}_0)^{\mathrm{Zar}} \to \underline{\mathbb{A}}^1_S$ to the filter $\varphi^{-1}[(\underline{\mathbb{A}}^1_S)^\times]$.

One can also form, internally to $\mathrm{Zar}(S)$, the classifying topos of $(\mathcal{A}_0)^{\mathrm{Zar}}$-algebras which are local over $\underline{\mathbb{A}}^1_S$. This topos doesn't coincide with the (toposes of sheaves over) the mentioned two locales, either. The pullback of that classifying topos to any of the $\mathrm{Sh}(X)$ is the big Zariski topos of $\underline{\mathrm{Spec}}_X(f^*\mathcal{A}_0)$ (built using one of the parsimonious sites).

18.2. Relative Proj construction.

Definition 18.13. In the context of a specified local ring R, the *synthetic Proj* of an $\mathbb{N}$-graded R-algebra A which is generated as an A_0-algebra by A_1 is the set

$$\mathrm{Proj}(A) := (\text{set of all surj. graded } R\text{-algebra homomorphisms } A \to R[T])/R^\times.$$

Example 18.14. The synthetic Proj of $R[X_0, \ldots, X_n]$ is canonically isomorphic to the set of points $[x_0 : \cdots : x_n]$ with at least one invertible coordinate.

Proposition 18.15. *Let $\mathcal{A}$ be an $\mathbb{N}$-graded $\mathcal{O}_S$-algebra (not necessarily quasicoherent). Assume that $\mathcal{A}$ is generated as an $\mathcal{A}_0$-algebra by $\mathcal{A}_1$. Then the synthetic Proj of the $\underline{\mathbb{A}}^1_S$-algebra $\mathcal{A}^{\mathrm{Zar}}$, as constructed in the internal language of $\mathrm{Zar}(S)$, is the functor of points of $\underline{\mathrm{Proj}}_S\,\mathcal{A}$.*

Proof. We omit the somewhat tedious verification. □

The following corollary was prompted by a question on MathOverflow [35]. We are grateful to Yuhao Huang for the impulse.

Corollary 18.16. *The big Zariski topos of the projective line $\mathbb{P}^1_{\mathbb{Z}}$ classifies the theory of "a local ring together with a point $[a : b]$" (that is a pair (a, b) of ring elements, where at least one coordinate is invertible, up to multiplication by units). Explicitly, this theory is given by:*

1. *A sort A together with function symbols, constants, and axioms expressing that A is a local ring.*
2. *A sort P (to be thought of as the set of $[a : b]$ with $a, b : A$ where at least one coordinate is invertible) together with a relation $\langle\cdot,\cdot,\cdot\rangle$ on $A \times A \times P$ and the following axioms:*
   - $\ulcorner a \text{ inv.}\urcorner \vee \ulcorner b \text{ inv.}\urcorner \dashv\vdash_{a,b:A} \exists p : P.\ \langle a, b, p\rangle$
   - $\langle a, b, p\rangle \wedge \langle a, b, p'\rangle \vdash_{a,b:A,\,p,p':P} p = p'$
   - $\top \vdash_{p:P} \exists a, b : A.\ \langle a, b, p\rangle$
   - $\langle a, b, p\rangle \wedge \langle a', b', p\rangle \dashv\vdash_{a,a',b,b':A,\,p:P} \exists s : A.\ \ulcorner s \text{ inv.}\urcorner \wedge a' = sa \wedge b' = sb$
3. *A constant of sort P.*

Proof. The big Zariski topos of $\mathbb{P}^1_{\mathbb{Z}}$ is a topos over the big Zariski topos of $\mathrm{Spec}(\mathbb{Z})$; from the point of view of $\mathrm{Zar}(\mathrm{Spec}(\mathbb{Z}))$, it is the classifying topos of a point $[a : b]$ where $a, b : \underline{\mathbb{A}}^1_{\mathrm{Spec}\,\mathbb{Z}}$, since $\mathrm{Zar}(\mathbb{P}^1_{\mathbb{Z}}) \simeq \mathrm{Zar}(\mathrm{Spec}(\mathbb{Z}))/\underline{\mathbb{P}^1_{\mathbb{Z}}}$ as discussed in Section 16.3 and $\underline{\mathbb{P}^1_{\mathbb{Z}}} \cong \{[a : b] \mid a, b : \underline{\mathbb{A}}^1_{\mathrm{Spec}\,\mathbb{Z}}\}$ by Proposition 18.15 and Example 18.14. The big Zariski topos of $\mathrm{Spec}(\mathbb{Z})$ classifies local rings. Therefore the claim follows by considering the combined geometric theory.

An alternative proof builds upon Remark 16.4 and the description of the theory which the little Zariski topos of $\mathbb{P}^1_{\mathbb{Z}}$ classifies (Proposition 12.35). Combining these, we see that $\mathrm{Zar}(\mathbb{P}^1_{\mathbb{Z}})$ classifies the theory of a homogeneous filter F of $\mathbb{Z}[X, Y]$ meeting the irrelevant ideal together with a local homomorphism $\alpha : \mathbb{Z}[X, Y][F^{-1}]_0 \to A$

into a local ring A. Such data gives rise to a point $[\alpha(X/u) : \alpha(Y/u)]$, where u is a homogeneous element of degree 1 contained in F; and conversely any point $[a : b]$ gives rise to a filter

$$F := \{f \in \mathbb{Z}[X,Y] \mid f_n(a,b) \text{ is invertible in } A \text{ for some } n \geq 0\},$$

where f_n is the homogeneous component of degree n of f, and a local homomorphism $\alpha : \mathbb{Z}[X,Y][F^{-1}]_0 \to A$ mapping f/g to $f(a,b)/g(a,b)$. □

18.3. Quasicoherence. The goal of this section is to give an internal characterization of quasicoherence. We'll build several notions of synthetic algebraic geometry on quasicoherence; it is therefore central to the theory.

Lemma 18.17. *Let E be an $\underline{\mathbb{A}}^1_S$-module. Let $S = \bigcup_i U_i$ be an open covering such that the restrictions $E|_{\mathrm{Zar}(U_i)}$ are quasicoherent, that is of the form $(\mathcal{E}_i)^{\mathrm{Zar}}$ for quasicoherent $\mathcal{O}_{U_i}$-modules $\mathcal{E}_i$. Then E is quasicoherent, that is of the form $(\mathcal{E}_0)^{\mathrm{Zar}}$ for a quasicoherent $\mathcal{O}_S$-module $\mathcal{E}_0$.*

Proof. The given modules $\mathcal{E}_i$ glue to a quasicoherent $\mathcal{O}_S$-module $\mathcal{E}_0$, and the sheaf condition ensures that E is isomorphic to $(\mathcal{E}_0)^{\mathrm{Zar}}$. The details are given in [118, Tag 03DN]. □

Definition 18.18. An R-module E is *synthetically quasicoherent* if and only if, for any finitely presented R-algebra A, the canonical R-algebra homomorphism

$$E \otimes_R A \longrightarrow [\mathrm{Spec}(A), E] = [[A, R]_{\mathrm{Alg}(R)}, E]$$

which maps a pure tensor $x \otimes f$ to the function $(\varphi \mapsto \varphi(f)x)$ is bijective. Here and in the following, the set $[\mathrm{Spec}(A), E]$ is the set of all maps $\mathrm{Spec}(A) \to E$, and $[A, R]_{\mathrm{Alg}(R)}$ is the set of all R-algebra homomorphisms $A \to R$.

This definition has the following interpretation. The codomain of the displayed canonical map is the set of all E-valued functions on $\mathrm{Spec}(A)$. Elements of $E \otimes_R A$ induce such functions; these induced functions can reasonably be called "algebraic". In a synthetic context, there should be no other E-valued functions as these algebraic ones, and different algebraic expressions should yield different functions. This is precisely what the postulated bijectivity expresses.

The notion of synthetic quasicoherence is only meaningful in an intuitionistic context. For instance, even R itself can't be synthetically quasicoherent in the presence of the law of excluded middle, since it forces the canonical evaluation morphism $R[T] \to [R, R]$ (obtained by setting $A := R[T]$ in the definition of synthetic quasicoherence) to never be bijective: If R is finite, then the evaluation morphism isn't injective, since $\prod_{x \in R}(T - x)$ is mapped to the same function as the zero polynomial is. If R is infinite, then $R[T]$ has cardinality $|R|$ while the set of functions $R \to R$ has strictly greater cardinality.

Theorem 18.19. *Let $E \in \mathrm{Zar}(S)$ be an $\underline{\mathbb{A}}^1_S$-module. If E is quasicoherent, that is of the form $(\mathcal{E}_0)^{\mathrm{Zar}}$ for some quasicoherent $\mathcal{O}_S$-module $\mathcal{E}_0$, then E is synthetically quasicoherent from the internal point of view of* $\mathrm{Zar}(S)$*. The converse holds in any of the following situations:*

1. *The site used to construct* $\mathrm{Zar}(S)$ *is one of the parsimonious sites.*
2. *The functor E maps directed limits of inverse systems of S-schemes with affine transition morphisms to colimits in* Set.
3. *From the internal point of view of* $\mathrm{Zar}(S)$*, the module E is even finitely presented.*

Proof. Let $E = (\mathcal{E}_0)^{\mathrm{Zar}}$ for a quasicoherent $\mathcal{O}_S$-module $\mathcal{E}_0$. To verify that E is synthetically quasicoherent, we have to verify a condition for $\underline{\mathbb{A}}^1_S$-algebras A in any slice $\mathrm{Zar}(S)/\underline{T}$. If such an algebra is finitely presented from the internal point of view, then there is a covering $T = \bigcup_i T_i$ such that each of the restrictions of the algebra to the schemes T_i is of the form $(\mathcal{A}_0)^{\mathrm{Zar}}$ for some finitely presented $\mathcal{O}_{T_i}$-algebra $\mathcal{A}_0$. Without loss of generality, we will just assume that A itself is of the form $(\mathcal{A}_0)^{\mathrm{Zar}}$ for a finitely presented $\mathcal{O}_S$-algebra $\mathcal{A}_0$.

By Proposition 18.8, the interpretation $[\![\mathrm{Spec}(A)]\!]$ of the internal spectrum is the functor of points of $\underline{\mathrm{Spec}}_S\,\mathcal{A}_0$. For any S-scheme $f : T \to S$ contained in the site used to define $\mathrm{Zar}(S)$, we consider the fiber product

$$\begin{tikzcd} \underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0) \arrow[r, "f'"] \arrow[d, "p'"'] & \underline{\mathrm{Spec}}_S\,\mathcal{A}_0 \arrow[d, "p"] \\ T \arrow[r, "f"'] & S. \end{tikzcd}$$

Since $\underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0) \to S$ is contained in the site (for any of our admissible sites), we may conclude using the following chain of isomorphisms:

$$\begin{aligned} [\mathrm{Spec}(A), E](T) &\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T}, [\mathrm{Spec}(A), E]) \cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T} \times \mathrm{Spec}(A), E) \\ &\cong \mathrm{Hom}_{\mathrm{Zar}(S)}(\underline{T \times_S \underline{\mathrm{Spec}}_S\,\mathcal{A}_0}, E) \cong E(\underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0)) \\ &\cong \Gamma(\underline{\mathrm{Spec}}_T(f^*\mathcal{A}_0), (p')^* f^*\mathcal{E}_0) \cong \Gamma(T, (p')_*(p')^* f^*\mathcal{E}_0) \\ &\cong \Gamma(T, f^*\mathcal{E}_0 \otimes_{\mathcal{O}_T} f^*\mathcal{A}_0) \cong (\mathcal{E}_0 \otimes_{\mathcal{O}_S} \mathcal{A}_0)^{\mathrm{Zar}})(T) \\ &\cong ((\mathcal{E}_0)^{\mathrm{Zar}} \otimes_{\underline{\mathbb{A}}^1_S} (\mathcal{A}_0)^{\mathrm{Zar}})(T) \cong (E \otimes_{\underline{\mathbb{A}}^1_S} A)(T). \end{aligned}$$

The antepenultimate isomorphism is because pullback of modules in $\mathrm{Sh}(S)$ to modules in $\mathrm{Sh}(T)$ commutes with tensor product. The penultimate isomorphism is because pullback of a sheaf in $\mathrm{Sh}(S)$ to a sheaf in $\mathrm{Zar}(S)$ commutes with tensor product (Lemma 16.6).

For the converse direction, we first verify that the restriction $E|_{\mathrm{Sh}(T)}$ to the little Zariski topos of any S-scheme T contained in the site used to define $\mathrm{Zar}(S)$ is a quasicoherent $\mathcal{O}_T$-module. For this, we employ the quasicoherence criterion of Theorem 8.3: For any open affine subset $T' \subseteq T$ and any function $h \in \Gamma(T', \mathcal{O}_T)$ we verify that the canonical morphism

$$E|_{\mathrm{Sh}(T)}[h^{-1}] \longrightarrow j_*(E|_{\mathrm{Sh}(D(h))}) \qquad (\dagger)$$

is an isomorphism, where $j : D(h) \hookrightarrow T'$ denotes the inclusion. This follows from the assumption of synthetic quasicoherence by considering the $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[h^{-1}]$ (in the slice $\mathrm{Zar}(S)/\underline{T'}$): This expresses that the canonical morphism

$$E \otimes_{\underline{\mathbb{A}}^1_S} \underline{\mathbb{A}}^1_S[h^{-1}] \longrightarrow [\mathrm{Spec}(A), E] \qquad (\ddagger)$$

is an isomorphism (of $\underline{\mathbb{A}}^1_S$-modules in $\mathrm{Zar}(S)/\underline{T'}$). Restricting the domain to $\mathrm{Sh}(T')$ yields the sheaf $E|_{\mathrm{Sh}(T')} \otimes_{\mathcal{O}_{T'}} \mathcal{O}_{T'}[h^{-1}]$, since restricting commutes with the geometric constructions "forming the tensor product" and "localizing away from h". Since $\mathrm{Spec}(A)$ is the functor of points of $D(h)$, restricting the codomain to $\mathrm{Sh}(T')$ yields the sheaf $j_*(E|_{\mathrm{Sh}(D(h))})$. The canonical morphism (†) which we want to recognize as an isomorphism is therefore the restriction of the canonical morphism (‡) which we know to be an isomorphism.

A natural candidate for a quasicoherent $\mathcal{O}_S$-module $\mathcal{E}_0$ with $E \cong (\mathcal{E}_0)^{\mathrm{Zar}}$ is $\mathcal{E}_0 := E|_{\mathrm{Sh}(S)}$. We'll show that this is indeed true. Let $f : T \to S$ be any S-scheme contained in the site used to define $\mathrm{Zar}(S)$. We assume for the time being that f is of finite presentation and affine, so $T \cong \underline{\mathrm{Spec}}_S\,\mathcal{A}_0$ for some finitely presented $\mathcal{O}_S$-algebra $\mathcal{A}_0$. We want to verify that the canonical morphism

$$f^*(E|_{\mathrm{Sh}(S)}) \longrightarrow E|_{\mathrm{Sh}(T)} \qquad (\S)$$

is an isomorphism. Since the functor f_* from quasicoherent $\mathcal{O}_T$-modules to quasicoherent $\mathcal{O}_S$-modules reflects isomorphisms (the morphism f being affine) and the domain and codomain of morphism (§) are quasicoherent, it suffices to verify that its image under f_* is an isomorphism. This image is the canonical morphism

$$E|_{\mathrm{Sh}(S)} \otimes_{\mathcal{O}_S} \mathcal{A}_0 \longrightarrow f_*(E|_{\mathrm{Sh}(T)}).$$

The assumption of synthetic quasicoherence, applied to the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := (\mathcal{A}_0)^{\mathrm{Zar}}$, shows that this morphism is an isomorphism.

In situation (1), the only step left to do is to generalize the argument in the previous paragraph to morphisms $f : T \to S$ which are locally of finite presentation. This works out because there are open covers of S and T such that the appropriate restrictions of f are of finite presentation and affine. The assumption of synthetic quasicoherence then needs to be applied to $\underline{\mathbb{A}}^1_S$-algebras in suitable slices of $\mathrm{Zar}(S)$, showing that the canonical morphism (§) is locally an isomorphism and therefore globally as well.

In situation (2), we may by Lemma 18.17 assume without loss of generality that S is affine. We then employ the technique of approximating general S-schemes by S-schemes of finite presentation. Specifically, let $f : T \to S$ be an arbitrary S-scheme contained in the site used to define $\mathrm{Zar}(S)$. Without loss of generality, we may assume that T is an affine scheme. Thus T is quasicompact and quasiseparated, and S is quasiseparated since it is affine. We may therefore apply the lemma of relative approximation [118, Tag 09MV] to deduce that T is a directed limit of an inverse system of S-schemes $f_i : T_i \to S$ of finite presentation with affine transition maps. These S-schemes are contained in the site used to define $\mathrm{Zar}(S)$. Furthermore, they inherit quasicompactness and quasiseparatedness from S. Therefore we can apply a comparison result on the categories of quasicoherent modules [118, Tag 01Z0]:

$$E(T) = E(\lim_i T_i) \cong \operatorname{colim}_i E(T_i) \cong \operatorname{colim}_i \Gamma(T_i, f_i^* \mathcal{E}_0) \cong \Gamma(T, f^* \mathcal{E}_0).$$

In situation (3), we may assume by Lemma 18.17 that E is the cokernel of a morphism $\alpha : (\underline{\mathbb{A}}^1_S)^m \to (\underline{\mathbb{A}}^1_S)^n$ of $\underline{\mathbb{A}}^1_S$-modules. This morphism induces a morphism $\alpha|_{\mathrm{Sh}(S)} : \mathcal{O}_S^m \to \mathcal{O}_S^n$ of $\mathcal{O}_S$-modules. One can then check that E is canonically isomorphic to $(\operatorname{cok}(\alpha|_{\mathrm{Sh}(S)}))^{\mathrm{Zar}}$, by using that the restriction functors $\mathrm{Mod}_{\mathrm{Zar}(S)}(\underline{\mathbb{A}}^1_S) \to \mathrm{Mod}_{\mathrm{Sh}(X)}(\mathcal{O}_X)$ are jointly conservative and right exact. □

Scholium 18.20. *Let $E \in \mathrm{Zar}(S)$ be a quasicoherent $\underline{\mathbb{A}}^1_S$-module. Let $A \in \mathrm{Zar}(S)$ be a quasicoherent $\underline{\mathbb{A}}^1_S$-algebra such that $[\![\mathrm{Spec}(A)]\!] \in \mathrm{Zar}(S)$ is representable by an object of the site used to define $\mathrm{Zar}(S)$. Then the canonical morphism*

$$E \otimes_{\underline{\mathbb{A}}^1_S} A \longrightarrow [\mathrm{Spec}(A), E]$$

is an isomorphism.

Proof. The second paragraph of the proof of Theorem 18.19 applies. □

Remark 18.21. As noted in Remark 18.10 in a slightly different context, the condition in Scholium 18.20 that $\mathrm{Spec}(A)$ is representable by an object of the site is unnatural from a topos-theoretic point of view and should be weakened to require only local representability.

However, the condition can't be dropped completely. For instance, if we employ the parsimonious sites and consider $S = \mathrm{Spec}\,\mathbb{Z}$, $E = \underline{\mathbb{A}}^1_S$, and $A = (\mathcal{K}_S)^{\mathrm{Zar}}$ (where $\mathcal{K}_S$ is the sheaf of rational functions on S, which in this case is the constant sheaf $\underline{\mathbb{Q}}$), then $[\![\mathrm{Spec}(A)]\!]$ is the functor of points of the $\mathbb{Z}$-scheme $\mathrm{Spec}(\mathbb{Q})$. By Proposition 11.19, this functor coincides with the functor of points of the empty $\mathbb{Z}$-scheme on the parsimonious sites; therefore $\mathrm{Spec}(A) = \emptyset$ from the internal point of view. Thus the codomain of the canonical morphism is the zero algebra, but the domain is not.

The internal quasicoherence condition for the little Zariski topos (Theorem 8.3) is related to the notion of synthetic quasicoherence as follows. Recall that an $\mathcal{O}_S$-module $\mathcal{E}_0$ is quasicoherent if and only if, from the internal point of view of $\mathrm{Sh}(S)$, the localized module $\mathcal{E}_0[f^{-1}]$ is a sheaf with respect to the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \underline{\quad})$ for any function $f : \mathcal{O}_S$. The sublocale associated with this modal operator is the open sublocale $j : \mathrm{Spec}(\mathcal{O}_S[f^{-1}] | \mathcal{O}_S) \hookrightarrow \mathrm{pt}$. The condition can therefore also be put in the form

$$\mathrm{Sh}(S) \models \forall f : \mathcal{O}_S.\ \ulcorner \mathcal{E}_0[f^{-1}] \longrightarrow j_* j^{-1}(\mathcal{E}_0[f^{-1}]) \text{ is bijective}\urcorner.$$

One can verify that the functor $j_* \circ j^{-1}$ is canonically isomorphic to the functor $[[\![\ulcorner f \text{ inv.}\urcorner]\!], \underline{\quad}]$, hence to the functor $[\mathrm{Spec}(\mathcal{O}_S[f^{-1}]), \underline{\quad}]$. Since $\mathcal{E}_0[f^{-1}] \cong \mathcal{E}_0 \otimes_{\mathcal{O}_S} \mathcal{O}_S[f^{-1}]$, the condition can therefore also be put in the form

$$\mathrm{Sh}(S) \models \forall f : \mathcal{O}_S.\ \mathcal{E}_0 \otimes_{\mathcal{O}_S} \mathcal{O}_S[f^{-1}] \longrightarrow [\mathrm{Spec}(\mathcal{O}_S[f^{-1}]), \mathcal{E}_0].$$

The synthetic quasicoherence condition therefore implies the condition which characterizes quasicoherence in the little Zariski topos as a special case.

Lemma 18.22. *Let $J \hookrightarrow \underline{\mathbb{A}}^1_S$ be an ideal such that $\underline{\mathbb{A}}^1_S/J$ is of the form $(\mathcal{E}_0)^{\mathrm{Zar}}$ for an $\mathcal{O}_S$-module $\mathcal{E}_0$. Let $\mathcal{I}$ be the kernel of the epimorphism $\mathcal{O}_S \twoheadrightarrow \mathcal{E}_0$ induced by the quotient morphism $\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S/J$. Then, for any S-scheme $(f : X \to S)$ contained in the site used to define* $\mathrm{Zar}(S)$*, there is a canonical isomorphism*

$$\mathrm{im}(f^*\mathcal{I} \to \mathcal{O}_X) \cong J|_{\mathrm{Sh}(X)}.$$

Proof. The short exact sequence $0 \to J \to \underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S/J \to 0$ of $\underline{\mathbb{A}}^1_S$-modules in $\mathrm{Zar}(S)$ remains exact restricting to $\mathrm{Sh}(X)$, since restricting to $\mathrm{Sh}(X)$ is taking the inverse image along a geometric morphism. Hence the sequence $0 \to J|_{\mathrm{Sh}(X)} \to \mathcal{O}_X \to f^*\mathcal{E}_0 \to 0$ is exact. On the other hand, the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_S \to \mathcal{E}_0 \to 0$ yields the short exact sequence $0 \to \mathrm{im}(f^*\mathcal{I} \to \mathcal{O}_X) \to \mathcal{O}_X \to f^*\mathcal{E}_0 \to 0$. □

Remark 18.23. The quotient $\underline{\mathbb{A}}^1_S/\sqrt{(0)}$ in $\mathrm{Zar}(S)$ is an example for a sheaf of modules which is not quasicoherent even though all of its restrictions to the little Zariski toposes $\mathrm{Sh}(X)$ for morphisms $f : X \to S$ are: Since taking the quotient and taking the radical of an ideal are geometric constructions, we have $(\underline{\mathbb{A}}^1_S/\sqrt{(0)})|_{\mathrm{Sh}(X)} \cong \mathcal{O}_X/\sqrt{(0)}$. These sheaves of modules are quasicoherent (Example 8.7). However, in general, $f^*(\mathcal{O}_S/\sqrt{(0)}) \not\cong \mathcal{O}_X/\sqrt{(0)}$. A specific counterexample is $S = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(k[T]/(T^2))$. In this case $f^*(\mathcal{O}_S/\sqrt{(0)}) \cong f^*(\mathcal{O}_S) \cong \mathcal{O}_X \not\cong \mathcal{O}_X/\sqrt{(0)}$.

Caveat 18.24. The kernel of a morphism of quasicoherent $\underline{\mathbb{A}}^1_S$-modules, calculated in the category of all $\underline{\mathbb{A}}^1_S$-modules, is in general not quasicoherent. This fact is evident from Lemma 18.22. In the notation of that lemma, the $\underline{\mathbb{A}}^1_S$-module J is in general not quasicoherent, since if it was, the restriction $J|_{\mathrm{Sh}(X)}$ would be canonically isomorphic to $f^*\mathcal{I}$.

The category of quasicoherent $\underline{\mathbb{A}}^1_S$-modules possesses kernels, since it is equivalent to the category of quasicoherent $\mathcal{O}_S$-modules by Proposition 18.3, but the inclusion from quasicoherent $\underline{\mathbb{A}}^1_S$-modules to arbitrary $\underline{\mathbb{A}}^1_S$-modules does not preserve them.

Lemma 18.25. *Let* $\mathrm{Zar}(S)$ *be defined using one of the parsimonious sites. Then tensor products of synthetically quasicoherent modules and cokernels of morphisms between synthetically quasicoherent modules are synthetically quasicoherent from the internal point of view of* $\mathrm{Zar}(S)$.

Proof. By Theorem 18.19 and the description of the slice toposes of $\mathrm{Zar}(S)$ given in Section 16.3, the first claim reduces to the fact that the tensor product of quasicoherent $\underline{\mathbb{A}}^1_S$-modules is quasicoherent. Indeed, the tensor product of $(\mathcal{E}_0)^{\mathrm{Zar}}$ with $(\mathcal{F}_0)^{\mathrm{Zar}}$ is $(\mathcal{E}_0 \otimes_{\mathcal{O}_S} \mathcal{F}_0)^{\mathrm{Zar}}$ by Lemma 16.6 and the tensor product of quasicoherent $\mathcal{O}_S$-modules is quasicoherent.

The second claim reduces to the statement that the cokernel of a morphism $\alpha : (\mathcal{E}_0)^{\mathrm{Zar}} \to (\mathcal{F}_0)^{\mathrm{Zar}}$, calculated in the category of $\underline{\mathbb{A}}^1_S$-modules, coincides with $(\mathrm{cok}(\alpha|_{\mathrm{Sh}(S)}))^\sim$ and that the cokernel of a morphism of quasicoherent $\mathcal{O}_S$-modules is quasicoherent. $\square$

It's somewhat embarassing that we didn't give an internal proof of Lemma 18.25. Also we don't know whether the result holds when using one of the larger sites (the given proof doesn't generalize to this situation since Theorem 18.19 can't be applied). We expand on this in Section 22.

Lemma 18.26. *Let A and B be $\underline{\mathbb{A}}^1_S$-algebras. The canonical map*

$$\mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(A, B) \longrightarrow \mathrm{Hom}_{\mathrm{Zar}(S)}([\![\mathrm{Spec}(B)]\!], [\![\mathrm{Spec}(A)]\!])$$

is bijective in the following situations:

(1) *The algebra B is finitely presented.*
(2) *The algebra B is quasicoherent and the functor $[\![\mathrm{Spec}(B)]\!] \in \mathrm{Zar}(S)$ is locally representable by an object of the site used to define $\mathrm{Zar}(S)$. (This situation subsumes the previous one.)*

Proof. By Scholium 18.20, the canonical morphism $B \to [\mathrm{Spec}(B), \underline{\mathbb{A}}^1_S]$ is an isomorphism. Hence the claim follows by the following entirely formal calculation:

$$\begin{aligned}
&\mathrm{Hom}([\![\mathrm{Spec}(B)]\!], [\![\mathrm{Spec}(A)]\!]) \\
&= \mathrm{Hom}([B, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}, [A, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}) \\
&\cong \mathrm{Hom}([B, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)} \times A, \underline{\mathbb{A}}^1_S)_{\underline{\mathbb{A}}^1_S\text{-homomorphism in the second argument}} \\
&\cong \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(A, [[B, \underline{\mathbb{A}}^1_S]_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}, \underline{\mathbb{A}}^1_S]) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(A, [[\![\mathrm{Spec}(B)]\!], \underline{\mathbb{A}}^1_S]) \\
&\cong \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(A, B).
\end{aligned}$$

$\square$

It's a basic fact that for an $\mathcal{O}_S$-algebra $\mathcal{A}$, the canonical map

$$\mathrm{Hom}_{\mathrm{Alg}(\mathcal{O}_S)}(\mathcal{A}_0, f_*\mathcal{O}_X) \longrightarrow \mathrm{Hom}_{\mathrm{LRL}/S}(X, \underline{\mathrm{Spec}}_S(\mathcal{A}_0))$$

is bijective for any S-scheme $(f : X \to S)$. One should not expect that the similar statement that for an $\underline{\mathbb{A}}^1_S$-algebra A, the canonical map

$$\mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(A, [X, \underline{\mathbb{A}}^1_S]) \longrightarrow \mathrm{Hom}_{\mathrm{Zar}(S)}(X, [\![\mathrm{Spec}(A)]\!])$$

is bijective for arbitrary functors $X \in \mathrm{Zar}(S)$ is bijective, holds. This is because the objects of $\mathrm{Zar}(S)$ are quite a bit more general than locally ringed spaces (or locales) over S.

18.4. Properties of the affine line. The ring object $\underline{\mathbb{A}}^1_S$ in the big Zariski topos enjoys several special properties, some of which are unique in that they're only possible in an intuitionistic context. We compile here a short list of such properties. As was already mentioned, at least one of them, the field property, was already noticed in the 1970s by Kock [77].

The statements and proofs in this subsection are formulated in the internal language. The proofs only use the fact that $\underline{\mathbb{A}}^1_S$ is a synthetically quasicoherent local ring. This supports the meta-claim that synthetic quasicoherence is a strong and meaningful condition.

Proposition 18.27. *$\underline{\mathbb{A}}^1_S$ is a field in the sense that any element which is not zero is invertible: $\forall x : \underline{\mathbb{A}}^1_S.\ \neg(x = 0) \Rightarrow \ulcorner x \text{ inv.}\urcorner$. More generally, for any number $n \geq 0$,*

$$\forall x_1, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \neg(x_1 = 0 \wedge \cdots \wedge x_n = 0) \Longrightarrow (\ulcorner x_1 \text{ inv.}\urcorner \vee \cdots \vee \ulcorner x_n \text{ inv.}\urcorner).$$

Proof. Let $x : \underline{\mathbb{A}}^1_S$ be such that $\neg(x = 0)$. We consider the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S/(x)$. Since $\mathrm{Spec}(A) \cong [\![x = 0]\!] = [\![\bot]\!] = \emptyset$, the condition posits that the canonical homomorphism

$$\underline{\mathbb{A}}^1_S/(x) \longrightarrow [\emptyset, \underline{\mathbb{A}}^1_S]$$

is an isomorphism. Since its codomain is the zero algebra, so is $\underline{\mathbb{A}}^1_S/(x)$. Therefore $1 \in (x)$, that is, x is invertible.

The more general statement follows in the same way, by using the quasicoherence condition for $A := \underline{\mathbb{A}}^1_S/(x_1, \ldots, x_n)$. This yields $1 \in (x_1, \ldots, x_n)$. Since $\underline{\mathbb{A}}^1_S$ is a local ring, one of the x_i is invertible. □

Proposition 18.28. *$\underline{\mathbb{A}}^1_S$ is not a reduced ring:*

$$\neg\big(\forall x : \underline{\mathbb{A}}^1_S.\ \big(\bigvee_{n \geq 0} x^n = 0\big) \Rightarrow x = 0\big).$$

Proof. Assume that $\underline{\mathbb{A}}^1_S$ is reduced. Then the set $\Delta := \{\varepsilon \in \underline{\mathbb{A}}^1_S \mid \varepsilon^2 = 0\}$ is equal to $\{0\}$. By the quasicoherence criterion applied to the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[T]/(T^2)$, the canonical map

$$\underline{\mathbb{A}}^1_S[T]/(T^2) \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S[T]/(T^2)), \underline{\mathbb{A}}^1_S] \cong [\Delta, \underline{\mathbb{A}}^1_S]$$

is an isomorphism. It maps $[T]$ to zero (the value of T at $0 \in \Delta$). Thus $T \in (T^2)$ and therefore $1 = 0$ in $\underline{\mathbb{A}}^1_S$. This is a contradiction. □

In classical logic, Proposition 18.27 and Proposition 18.28 would directly contradict each other; only an intuitionistic context allows for fields which are not reduced.

That $\underline{\mathbb{A}}^1_S$ is not reduced, irrespective of the reducedness of the base scheme S, should not come as a surprise: Reducedness is not stable under base change, but all statements of the internal language of $\mathrm{Zar}(S)$ are. If $\underline{\mathbb{A}}^1_S$ was reduced, then all S-schemes (at least those contained in the site used to construct $\mathrm{Zar}(S)$) would be reduced as well. In contrast, the structure sheaf $\mathcal{O}_S$ is reduced from the point of view of the little Zariski topos if and only if S is reduced (Proposition 3.3).

Proposition 18.29. *The following statements about an element $x : \underline{\mathbb{A}}^1_S$ are equivalent:*

(1) *x is not invertible.*
(2) *x is nilpotent.*
(3) *x is* not not *zero.*

Proof. Let $x : \underline{\mathbb{A}}^1_S$ be not invertible. We consider the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[x^{-1}]$. Since $\mathrm{Spec}(A) \cong [\![\ulcorner x \text{ inv.}\urcorner]\!] = \emptyset$, it follows that $\underline{\mathbb{A}}^1_S[x^{-1}] = 0$, similarly to the proof of Proposition 18.27. Thus x is nilpotent.

Let $x : \underline{\mathbb{A}}^1_S$ be a nilpotent element. Thus $x^n = 0$ for some number $n \geq 0$. If x was nonzero, then x and therefore x^n would be invertible, in contradiction to $0 \neq 1$ since $\underline{\mathbb{A}}^1_S$ is a local ring.

Let $x : \underline{\mathbb{A}}^1_S$ be *not not* zero. Then x is not invertible, since if x was invertible, then x would be nonzero. □

Summarizing, the following facts about nilpotents hold in the internal universe of the big Zariski topos. Firstly, it's not true that $\underline{\mathbb{A}}^1_S$ is reduced. But this doesn't mean that there actually exist nilpotent elements which are not zero. In fact, any nilpotent is *not not* zero.

Proposition 18.30. *Any function $\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$ is given by a unique polynomial in $\underline{\mathbb{A}}^1_S[T]$.*

Proof. Immediate by considering the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[T]$ and noticing that $\mathrm{Spec}(A) \cong \underline{\mathbb{A}}^1_S$. ☐

This statement too cannot be satisfied in classical logic: for infinite fields the existence part fails and for finite fields the uniqueness part fails.

In synthetic differential geometry, the *axiom of microaffinity* is central to the theory. It is fulfilled by the image of $\mathbb{R}^1$ in any well-adapted model of synthetic differential geometry, and also by $\underline{\mathbb{A}}^1_S \in \mathrm{Zar}(S)$. This fact is well-known; we include the proof only to show that it is a consequence of synthetic quasicoherence.

Proposition 18.31. *$\underline{\mathbb{A}}^1_S$ fulfills the axiom of microaffinity: Let $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1_S \mid \varepsilon^2 = 0\}$. Let $f : \Delta \to \underline{\mathbb{A}}^1_S$ be an arbitrary function. Then there are unique elements $a, b : \underline{\mathbb{A}}^1_S$ such that $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon : \Delta$.*

Proof. Immediate from the definition of synthetic quasicoherence, considering the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $\underline{\mathbb{A}}^1_S[T]/(T^2)$. ☐

Proposition 18.32. *$\underline{\mathbb{A}}^1_S$ is* anonymously algebraically closed*, in the following sense: Any monic polynomial $p : \underline{\mathbb{A}}^1_S[T]$ of degree at least one does* not not *have a zero.*

Proof. Let $p : \underline{\mathbb{A}}^1_S[T]$ be a monic polynomial of degree at least one. Assume that p doesn't have a zero in $\underline{\mathbb{A}}^1_S$. Then the spectrum of $A := \underline{\mathbb{A}}^1_S[T]/(p)$ is empty. The quasicoherence condition for A therefore implies that $\underline{\mathbb{A}}^1_S[T]/(p)$ is zero. This means that p is invertible in $\underline{\mathbb{A}}^1_S[T]$. A basic lemma in commutative algebra (whose standard proof is constructive) then implies that with the exception of the constant term in p, all coefficients are nilpotent. This contradicts the assumption that p is monic of degree at least one. ☐

Proposition 18.33. *$\underline{\mathbb{A}}^1_S$ is infinite in the following sense: For any number $n \geq 0$ and any given elements $x_1, \ldots, x_n : \underline{\mathbb{A}}^1_S$, there is* not not *an element y which is distinct from all of the x_i.*

Proof. The polynomial $f(T) := (T - x_1) \cdots (T - x_n) + 1$ does *not not* have a zero $y : \underline{\mathbb{A}}^1_S$, since $\underline{\mathbb{A}}^1_S$ is anonymously algebraically closed. This element cannot equal any of the elements x_i, since $f(x_i) = 1$ is not zero. ☐

Proposition 18.34. *$\underline{\mathbb{A}}^1_S$ fulfills the following version of the Nullstellensatz: Let $f_1, \ldots, f_m \in \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]$ be polynomials without a common zero in $(\underline{\mathbb{A}}^1_S)^n$. Then there are polynomials $g_1, \ldots, g_m \in \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]$ such that $\sum_i g_i f_i = 1$.*

Proof. We consider the quasicoherence condition for the finitely presented $\underline{\mathbb{A}}^1_S$-algebra $A := \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$. Since $\mathrm{Spec}(A) \cong \{x \in (\underline{\mathbb{A}}^1_S)^n \mid f_1(x) = \ldots = f_m(x) = 0\} = \emptyset$, the condition implies that A is the zero algebra just as in the verification of Proposition 18.27. ☐

Remark 18.35. The Krull dimension of the ring $\mathcal{O}_S$ of the little Zariski topos coincides with the dimension of S (Proposition 3.13). The analogous statement for $\underline{\mathbb{A}}^1_S$ in the big Zariski topos is false. Unless S is the empty scheme, the internal statement

$$\mathrm{Zar}(S) \models \ulcorner \underline{\mathbb{A}}^1_S \text{ is of Krull dimension} \leq n \urcorner$$

is false for any natural number $n \geq 0$: Since the property of having Krull dimension $\leq n$ is a geometric implication, this statement would imply that for any S-scheme X (contained in the site used to define $\mathrm{Zar}(S)$) the ring $\mathcal{O}_X$ in $\mathrm{Sh}(X)$ is of Krull dimension $\leq n$.

The ring $\underline{\mathbb{A}}^1_S$ in $\mathrm{Zar}(S)$ is therefore an example for a ring of infinite Krull dimension which nevertheless fulfills a field condition. A ring in the big Zariski topos which does reflect the dimension of S is $\flat\underline{\mathbb{A}}^1_S$. The scheme S is of dimension $\leq n$ if and only if $\flat\underline{\mathbb{A}}^1_S$ is of Krull dimension $\leq n$ from the internal point of view of $\mathrm{Zar}(S)$.

19. Basic constructions of relative scheme theory

With $\underline{\mathbb{A}}^1_S$ at hand, we can perform many of the usual constructions of (relative) scheme theory internally.

19.1. Group schemes. The functors associated to the standard group schemes $\mathbb{G}_\mathrm{a}$, $\mathbb{G}_\mathrm{m}$, GL_n, and μ_n are given by the internal expressions

$$\begin{aligned} \mathbb{G}_\mathrm{a} &:= \underline{\mathbb{A}}^1_S \text{ (as an additive group)}, \\ \mathbb{G}_\mathrm{m} &:= \{x : \underline{\mathbb{A}}^1_S \mid \ulcorner x \text{ inv.} \urcorner\}, \\ \mathrm{GL}_n &:= \{M : (\underline{\mathbb{A}}^1_S)^{n\times n} \mid \ulcorner M \text{ inv.} \urcorner\}, \\ \mu_n &:= \{x : \underline{\mathbb{A}}^1_S \mid x^n = 1\}. \end{aligned}$$

19.2. Affine and projective space. Affine n-space over S is given by $(\underline{\mathbb{A}}^1_S)^n$, i. e. internally the set of n-tuples of elements of $\underline{\mathbb{A}}^1_S$. The functor of points of projective n-space over X, with all its nontrivial topological and ring-theoretical structure, is described by the naive expression

$$\mathbb{P}^n := \{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1_S)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/\sim,$$

where the equivalence relation is the usual rescaling relation from the internal point of view. This example was suggested by Zhen Lin Low (private communication).

More generally, for an S-scheme X, affine and projective n-space over X are given by $\underline{X} \times (\underline{\mathbb{A}}^1_S)^n$ and $\underline{X} \times \mathbb{P}^n$, respectively.

19.3. Tangent bundle. For an S-scheme X, the internal Hom $[\Delta, \underline{X}] \in \mathrm{Zar}(S)$ describes the tangent bundle of X, i. e. the S-scheme $\underline{\mathrm{Spec}}_X \operatorname{Sym}(\Omega^1_{X/S}) \to X \to S$, as can be seen by chasing the definitions [27, Lemma 5.12.1]. Intuitively, a map $f : \Delta \to \underline{X}$ from the internal point of view is given by slightly more data than merely the point $f(0)$; one also has to specify first-order information.

This description of the (not necessarily locally trivial) tangent bundle fits nicely with the intuition of tangent vectors as infinitesimal curves, and in fact is precisely the definition of the tangent bundle in synthetic differential geometry [76, Def. 7.1].

19.4. The tilde construction. Let A be an $\underline{\mathbb{A}}^1_S$-algebra. If the functor $[\![\mathrm{Spec}(A)]\!]$ is representable by a scheme contained in the site used to define $\mathrm{Zar}(S)$, then Section 16.3 described the big Zariski topos of that scheme as the slice topos $\mathrm{Zar}(S)/\operatorname{Spec}(A)$. From the point of view of $\mathrm{Zar}(S)$, this topos looks like “$\mathrm{Set}/\operatorname{Spec}(A)$”, the topos of $\operatorname{Spec}(A)$-indexed families of sets.

In this picture, the three functors $f_! \dashv f^{-1} \dashv f_*$ relating $\mathrm{Zar}(S)$ and $\mathrm{Zar}(S)/\operatorname{Spec}(A)$ look as follows. We set $X := \operatorname{Spec}(A)$.

1. The functor $f_!$ maps a family $(M_x)_{x:X}$ to its *dependent sum* $\coprod_{x:X} M_x$, the “total space” of the family.
2. The functor f^{-1} maps a set N to the constant family $(N)_{x:X}$.
3. The functor f_* maps a family $(M_x)_{x:X}$ to its *dependent product*, the set $\prod_{x:X} M_x$ of sections of the projection $(\coprod_{x:X} M_x) \to X$.

The affine line over X is the constant family $\underline{\mathbb{A}}^1_X = (\underline{\mathbb{A}}^1_S)_{x:X}$.

We can canonically associate an $\underline{\mathbb{A}}^1_X$-module $E^\sim$ to a given A-module E. This module lives in the topos Set/X, so is an X-indexed family of $\underline{\mathbb{A}}^1_S$-modules. Explicitly, it is the family

$$E^\sim := (E \otimes_A \underline{\mathbb{A}}^1_S)_{x:X},$$

where for the tensor product in the component $x : X$ the ring $\underline{\mathbb{A}}^1_S$ is regarded as an A-algebra by the $\underline{\mathbb{A}}^1_S$-homomorphism $x : A \to \underline{\mathbb{A}}^1_S$. To render the dependence of the tensor product on x explicit, we use the familiar notation $E(x) := E \otimes_A \underline{\mathbb{A}}^1_S$ to denote the fiber of E over x.

Conversely, given an $\underline{\mathbb{A}}^1_X$-module $F = (F_x)_{x:X}$, there is a canonically associated A-module given by calculating the dependent product. In analogy with the classical situation, we write

$$\Gamma(X, F) := \prod_{x:X} F_x$$

for this module.

A morphism $F \to F'$ of $\underline{\mathbb{A}}^1_X$-modules is an X-indexed family $(F_x \to F'_x)_{x:X}$ of morphism of $\underline{\mathbb{A}}^1_S$-modules.

Proposition 19.1. *The functor* $(\_\_)^\sim : \mathrm{Mod}(A) \to \mathrm{Mod}(\underline{\mathbb{A}}^1_X)$ *is left adjoint to the functor* $\Gamma(X, \_\_) : \mathrm{Mod}(\underline{\mathbb{A}}^1_X) \to \mathrm{Mod}(A)$.

Proof. For an A-module E and an $\underline{\mathbb{A}}^1_X$-module F, the required bijection $\mathrm{Hom}_{\underline{\mathbb{A}}^1_X}(E^\sim, F) \cong \mathrm{Hom}_A(E, \Gamma(X, \mathcal{F}))$ is given by

$$(\alpha_x : E(x) \to F_x)_{x:X} \longmapsto (v \mapsto (\alpha_x([v \otimes 1]))_{x:X})$$

with inverse given by

$$(v \otimes f \mapsto f \cdot \beta(v)_x) \longleftarrow \beta. \qquad \square$$

It's fruitful to study the unit of the adjunction $(\_\_)^\sim \dashv \Gamma(X, \_\_)$ in more detail.

Definition 19.2. An A-module E has *cohesive fibers* if and only if the canonical linear map $E \to \prod_{x:X} E(x)$ is a bijection.

Ordinary modules in classical logic almost never have cohesive fibers. Intuitively, the law of excluded middle allows to define "discontinuous" elements of $\prod_{x:X} E(x)$. These can't be induced from elements $v : E$, whose values $[v \otimes 1] : E(x)$ always "vary continuously" in $x : X$. In particular, the trivial A-module A won't have cohesive fibers in a classical context, since A might not be a product of copies of the base ring.

However, in the intuitionistic context of the internal language of the big Zariski topos, the law of excluded middle is not available to define discontinuous families of values. And indeed, we have the following proposition.

Proposition 19.3. *Assume that there is a quasicoherent* $\mathcal{O}_S$*-algebra* $\mathcal{A}_0$ *such that* $A = \mathcal{A}_0^{\mathrm{Zar}}$. *Further assume that* $[\![\mathrm{Spec}(A)]\!]$ *is locally representable by an object of the site used to define* $\mathrm{Zar}(S)$. *Let* $\mathcal{E}_0$ *be a quasicoherent* $\mathcal{A}_0$*-module. Then* $(\mathcal{E}_0)^{\mathrm{Zar}}$ *has cohesive fibers from the internal point of view of* $\mathrm{Zar}(S)$.

Proof. This just reflects the fact that the canonical morphism $\mathcal{E}_0 \to f_*((\mathcal{E}_0)^\sim)$, where $f : \underline{\mathrm{Spec}}_S(\mathcal{A}_0) \to S$ is the structure morphism of the relative spectrum and $(\mathcal{E}_0)^\sim$ is the result of applying the (ordinary) relative tilde construction to $\mathcal{E}_0$, is an isomorphism [118, Tag 01SB]. $\square$

Corollary 19.4. *Let* $\mathrm{Zar}(S)$ *be defined using one of the parsimonious sites. Assume that* A *is a finitely presented* $\underline{\mathbb{A}}^1_S$*-algebra. Then the following statements hold from the internal point of view of* $\mathrm{Zar}(S)$.

(1) *The algebra A satisfies the following "higher-typed" version of the synthetic quasicoherence condition: Any A-module which is synthetically quasicoherent as an $\underline{\mathbb{A}}^1_S$-module (for instance because it is finitely presented) has cohesive fibers.*
(2) *For any $\underline{\mathbb{A}}^1_X$-module $F = (F_x)_{x:X}$ such that F_x is a synthetically quasicoherent $\underline{\mathbb{A}}^1_S$-module for all $x : X$, the canonical map $\Gamma(X, F)^\sim \to F$ is an isomorphism.*

Proof. The first claim is immediate from Proposition 19.3 and Theorem 18.19. The second claim is an internal rendition of [118, Tag 01SB]. □

Remark 19.5. Since the internal language of Set/ Spec(A) is just ordinary language applied to all fibers, the following properties of an $\underline{\mathbb{A}}^1_X$-module $F = (F_x)_{x:X}$ are equivalent:

(1) For any $x : X$, the $\underline{\mathbb{A}}^1_S$-module F_x is synthetically quasicoherent.
(2) From the point of view of Set/ Spec(A), the $\underline{\mathbb{A}}^1_X$-module F is synthetically quasicoherent.

We believe that the condition that A is synthetically quasicoherent is not strong enough to allow an internal proof of Corollary 19.4. The conclusion of Corollary 19.4 seems quite natural from a synthetic point of view; we therefore propose to adopt it as an axiom for synthetic algebraic geometry.

19.5. Open immersions. A basic concept in the functor-of-points approach to algebraic geometry is the concept of an *open subfunctor*. It is used to delimit schemes from more general kinds of spaces: A functor is deemed to be a scheme if and only if it admits a covering by open subfunctors which are representable.

The following definition is phrased in such a way as to apply to any of the several ways to define the big Zariski topos Zar(S). In particular, it applies to the definition using the site consisting of affine schemes which are locally of finite presentation over S. If S is affine, the definition only refers to affine schemes and open subschemes of affine schemes and is therefore suitable if one wants to found the theory of schemes using the functorial approach.

Definition 19.6 ([45, Définition I.1.3.6 on page 10], [118, Tag 01JI])**.** A subfunctor $U \hookrightarrow X$ in Zar(S) is an *open subfunctor* if and only if for any object $(T \to S)$ of the site used to define Zar(S) and any $x \in X(T)$ there exists an open subscheme $T_0 \subseteq T$ such that for any object $(T' \xrightarrow{f} T \to S)$ of the site used to define Zar(S) the map $T' \to T$ factors over T_0 if and only if $X(f)(x) \in U(T')$.

The open subschemes $T_0 \subseteq T$ appearing in this definition are uniquely determined by their universal property. The relation of open subfunctors to open immersions is as follows.

Proposition 19.7. *Let X be an S-scheme.*

(1) *Let $U \subseteq X$ be an open subscheme. Then the subfunctor $\underline{U} \hookrightarrow \underline{X}$ is open.*
(2) *If $\underline{X}$ is locally representable by an object of the site used to define* Zar(S), *any open subfunctor $U \hookrightarrow \underline{X}$ is isomorphic to the open subfunctor associated to an open subscheme of X.*

Proof. For the first claim, let $(T \to S)$ be an object of the site used to define Zar(S) and let $x \in \underline{X}(T)$. The open subscheme $T_0 \subseteq T$ required by the definition of an open subfunctor can then be chosen as $x^{-1}[U]$.

For the second claim, assuming for notational simplicity that $\underline{X}$ is directly representable without having to pass to a cover, the desired open subscheme of X can be obtained as the witnessing subscheme "T_0" as it appears in the definition of an open subfunctor in the special case $(T \to S) := (X \to S)$. □

From the point of view of the internal language of $\mathrm{Zar}(S)$, a subfunctor $U \hookrightarrow X$ looks like the inclusion of a subset. The natural question how one can characterize those inclusions which externally correspond to open subfunctors is answered as follows.

Definition 19.8. In the context of a specified local ring R, as for instance $\underline{\mathbb{A}}^1_S$ of the big Zariski topos of a scheme, a truth value φ is *open* if and only if there exists an ideal $J \subseteq R$ such that R/J is synthetically quasicoherent (Definition 18.18) and such that φ holds if and only if $1 \in J$. (Section 6.1 contains generalities on truth values.)

Example 19.9. Let $f : R$. Then "f is invertible" is an open truth value with witnessing ideal $J = (f)$. The quotient R/J is indeed synthetically quasicoherent, since it is finitely presented. More generally, let $f_1, \ldots, f_n : R$. Then "one of the f_i is invertible" is an open truth value with witnessing ideal $J = (f_1, \ldots, f_n)$.

In case that R fulfills the same field condition as $\underline{\mathbb{A}}^1_S$ does, one can write this truth value also as "$f_1 \neq 0 \vee \cdots \vee f_n \neq 0$".

Definition 19.10. In the context of a specified local ring, a map $U \to X$ of sets is a *synthetic open immersion* if and only if it is injective and for any $x : X$ the truth value of "the fiber of x is inhabited" is open.

Example 19.11. The inclusion $R^\times \hookrightarrow R$ of the invertible elements is a synthetic open immersion, since for $x : R$ the truth value of "the fiber of x is inhabited" equals the truth value of "x is invertible".

Example 19.12. Let X be a set. Let $f : X \to R$ be a function. The inclusion $\{x : X \mid f(x) \text{ is invertible}\} \hookrightarrow X$ is a synthetic open immersion.

Proposition 19.13. *Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. If a subfunctor $U \hookrightarrow X$ is open, then the map $U \hookrightarrow X$ is a synthetic open immersion from the internal point of view of $\mathrm{Zar}(S)$, that is*

$$\begin{aligned}\mathrm{Zar}(S) \models \forall x : X.\ \exists J \subseteq \underline{\mathbb{A}}^1_S.&\\ &\ulcorner J \text{ is an ideal}\urcorner \wedge \ulcorner \underline{\mathbb{A}}^1_S/J \text{ is synth. quasicoherent}\urcorner \wedge (x \in U \Leftrightarrow 1 \in J).\end{aligned}$$

The converse holds if one of the parsimonious sites is used to define $\mathrm{Zar}(S)$ or if the ideals J are required to be finitely generated from the internal point of view.

We postpone the proof of this proposition in order to give a bit of context first.

Firstly, the displayed condition is only meaningful in an intuitionistic context as provided by the big Zariski topos. In classical logic, the condition would be trivially satisfied for any subfunctor $U \hookrightarrow X$: Classically, we have $(x \in U) \vee (x \notin U)$. If $x \in U$, we can pick $J = (1)$, and if $x \notin U$, we can pick $J = (0)$ (whereby the quotient $\underline{\mathbb{A}}^1_S/J$ is isomorphic to $\underline{\mathbb{A}}^1_S$, thus finitely presented and therefore in particular synthetically quasicoherent).[40]

Proposition 19.13 is often used in the following form, which is weaker because it only gives one direction, but which is applicable for any of our choices for the site used to define $\mathrm{Zar}(S)$.

Corollary 19.14. *Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. Let $U \hookrightarrow X$ be a subfunctor. If*

$$\mathrm{Zar}(S) \models \forall x : X.\ \bigvee_{n \geq 0} \exists f_1, \ldots, f_n : \underline{\mathbb{A}}^1_S.\ (x \in U \Leftrightarrow \bigvee_i \ulcorner f_i \text{ inv.}\urcorner),$$

[40]Strictly speaking, incompatibility with classical logical surfaces even earlier: in our synthetic quasicoherence condition. The map $E \otimes_{\underline{\mathbb{A}}^1_S} A \to [\mathrm{Spec}(A), E]$ which the condition demands to be bijective has hardly any chance to be surjective if the law of excluded middle is available to define maps $\mathrm{Spec}(A) \to E$ by case distinction.

then the subfunctor is open.

Proof. We show that the assumption implies the displayed condition of Proposition 19.13 in the internal language. Given elements $f_1, \ldots, f_n$ as in the assumption, we construct the ideal $J := (f_1, \ldots, f_n) \subseteq \underline{\mathbb{A}}^1_S$. The quotient $\underline{\mathbb{A}}^1_S / J$ is finitely presented, hence synthetically quasicoherent, and the statement that $1 \in J$ is equivalent to one of the f_i being invertible by locality of $\underline{\mathbb{A}}^1_S$. $\square$

The internal condition appearing in Corollary 19.14 reflects basic intuition about openness in algebraic geometry: Intuitively, a subset is open if it is given by inequations, so that to decide whether a point belongs to the subset one has to check that at least one of some numbers is not zero.

Of course, in classical scheme theory, one would put some condition on these numbers in order not to trivialize the notion. For instance, one would require that they depend continuously on the point in some sense or, more specifically, that these numbers are given by evaluating certain locally defined regular functions at the point.

On first sight, such a condition seems to be lacking in Corollary 19.14. However, it's implicitly built into the language, since by the Kripke–Joyal semantics the external meaning of "$\exists f : \underline{\mathbb{A}}^1_S$" is that there exist, locally on an open cover, suitable elements of $\underline{\mathbb{A}}^1_S(T)$, that is regular functions on T.

It's useful to give a name for the kind of the subfunctors appearing in Corollary 19.14.

Definition 19.15.
- A subfunctor $U \hookrightarrow X$ in $\mathrm{Zar}(S)$ is a *quasicompact open subfunctor* if and only if for any object $(T \to S)$ of the site used to define $\mathrm{Zar}(S)$ and any $x \in X(T)$ there exists an open subscheme $T_0 \subseteq T$ such that the open immersion $T_0 \hookrightarrow T$ is quasicompact and such that for any object $(T' \xrightarrow{f} T \to S)$ of the site used to define $\mathrm{Zar}(S)$ the map $T' \to T$ factors over T_0 if and only if $X(f)(x) \in U(T')$.
- In the context of a specified local ring, a truth value φ is *quasicompact open* if and only if there exists a finitely generated ideal $J \subseteq R$ such that φ holds if and only if $1 \in J$.
- In the context of a specified local ring, a map $U \to X$ of sets is a *synthetic quasicompact open immersion* if and only if it is injective and for any $x : X$ the truth value of "the fiber of x is inhabited" is quasicompact open.

Proof of Proposition 19.13. We begin with the "only if" direction. Let T be an S-scheme contained in the site used to define $\mathrm{Zar}(S)$ and let $x \in X(T)$. By assumption there is an open subscheme $T_0 \subseteq T$ such that, for any object $(T' \xrightarrow{f} T \to S)$ of the site the morphism f factors over T_0 if and only if $X(f)(x) \in U(T')$.

There is a quasicoherent sheaf $\mathcal{I}$ of ideals on T such that $T_0 = D(\mathcal{I})$. We set $J := \mathrm{im}(\mathcal{I}^{\mathrm{Zar}} \to \underline{\mathbb{A}}^1_T)$. Then the quotient module $\underline{\mathbb{A}}^1_T / J$ coincides with $(\mathcal{O}_T/\mathcal{I})^{\mathrm{Zar}}$ and is therefore quasicoherent, hence synthetically quasicoherent from the internal point of view of $\mathrm{Zar}(T)$ by Theorem 18.19.

To verify that $T \models (x \in U \Leftrightarrow 1 \in J)$, let an object $(T' \xrightarrow{f} T \to S)$ of the site be given. Then we have the chain of equivalences

$$\begin{aligned} & X(f)(x) \in U(T') \\ \iff & T' \to T \text{ factors over } T_0 \\ \iff & T' \models 1 \in \mathrm{im}(f^*\mathcal{I} \to \mathcal{O}_{T'}) \\ \iff & T' \models 1 \in J, \end{aligned}$$

where the last step follows by Lemma 18.22.

For the converse direction, let T be an S-scheme contained in the site and let $x \in X(T)$. By assumption there is an open covering $T = \bigcup_i V_i$ and ideals $J_i \in \mathrm{Zar}(V_i)$ such that $\underline{\mathbb{A}}^1_{V_i}/J_i$ is synthetically quasicoherent and such that $V_i \models (x \in U \Leftrightarrow 1 \in J_i)$.

By Theorem 18.19, there are quasicoherent modules $\mathcal{E}_i$ over V_i such that $\underline{\mathbb{A}}^1_{V_i}/J_i \cong (\mathcal{E}_i)^{\mathrm{Zar}}$. We set $\mathcal{I}_i := \ker(\mathcal{O}_{V_i} \to \mathcal{E}_i)$. One can check that $D(\mathcal{I}_i) \cap V_j = D(\mathcal{I}_j) \cap V_i$ for all i and j. We set $T_0 := \bigcup_i D(\mathcal{I}_i)$.

To verify that this construction satisfies what is expected of it, let $(T' \xrightarrow{f} T \to S)$ be an object of the site. We then have the following chain of equivalences:

$$\begin{aligned}
&T' \to T \text{ factors over } T_0 \\
\iff &\text{for all } i,\ f_i : f^{-1}V_i \to V_i \text{ factors over } D(\mathcal{I}_i) \\
\iff &\text{for all } i,\ f^{-1}V_i \models 1 \in \mathrm{im}(f_i^*\mathcal{I}_i \to \mathcal{O}_{f^{-1}V_i}) \\
\overset{\star}{\iff} &\text{for all } i,\ f^{-1}V_i \models 1 \in J_i \\
\iff &\text{for all } i,\ f^{-1}V_i \models x \in U \\
\iff &T' \models x \in U \\
\iff &X(f)(x) \in U(T')
\end{aligned}$$

For this calculation, it's not important that the ideals $\mathcal{I}_i$ are quasicoherent. The assumption that the quotient modules $\underline{\mathbb{A}}^1_{V_i}/J_i$ are quasicoherent is only needed to ensure that Lemma 18.22 can be applied in the marked step. □

Proposition 19.16. *Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. A subfunctor $U \hookrightarrow X$ is a quasicompact open subfunctor if and only if $U \hookrightarrow X$ is a synthetic quasicompact open immersion from the internal point of view of* $\mathrm{Zar}(S)$.

Proof. The proof of Proposition 19.13 can be adapted. □

Lemma 19.17. *Finite conjunctions and finite disjunctions of quasicompact open truth values are quasicompact open. If* $\mathrm{Zar}(S)$ *is defined using one of the parsimonious sites, then furthermore finite conjunctions and finite disjunctions of open truth values are open from the internal point of view of* $\mathrm{Zar}(S)$.

Proof. Let φ and ψ be quasicompact open truth values with witnessing finitely generated ideals I and J. Because the ring is local, $\varphi \wedge \psi$ is equivalent to $1 \in I \cdot J$, and $\varphi \vee \psi$ is equivalent to $1 \in I + J$. The ideals $I \cdot J$ and $I + J$ are finitely generated. Therefore $\varphi \wedge \psi$ and $\varphi \vee \psi$ are quasicompact open truth values.

For the case of arbitrary open truth values, we need to verify that $\underline{\mathbb{A}}^1_S/(I \cdot J)$ and $\underline{\mathbb{A}}^1_S/(I + J)$ are synthetically quasicoherent if $\underline{\mathbb{A}}^1_S/I$ and $\underline{\mathbb{A}}^1_S/J$ are. This second claim from Lemma 18.25, since $\underline{\mathbb{A}}^1_S/(I + J) \cong \underline{\mathbb{A}}^1_S/I \otimes_{\underline{\mathbb{A}}^1_S} \underline{\mathbb{A}}^1_S/J$. The first claim follows from a similar calculation. □

The notion of open truth values is not unique to our account of synthetic algebraic geometry. Rather, it's a concept in the established and more general framework of synthetic topology [47, 82] which aims to do topology in a synthetic fashion: Any set should have an intrinsic topology and any map should be automatically continuous with respect to this intrinsic topology.

This automatic continuity reflects as stability of open subfunctors under pullbacks:

Lemma 19.18. *Let $f : X \to Y$ be a morphism in $\mathrm{Zar}(S)$. Let $U \hookrightarrow Y$ be an open subfunctor. Then its pullback along f, denoted "$f^{-1}U \hookrightarrow X$", is too an open subfunctor.*

Proof. From the internal point of view of $\mathrm{Zar}(S)$, the subfunctor $f^{-1}U \hookrightarrow X$ looks like the inclusion of the preimage $f^{-1}[U] \subseteq X$.

So, to verify the claim, let internally an element $x : X$ be given. We are to show that the truth value of "$x \in f^{-1}[U]$" is open. This truth value equals the truth value of "$f(x) \in U$" which is open by assumption, and is therefore open. $\square$

Remark 19.19. In the internal language of toposes used to carry out synthetic differential geometry, there is the concept of an *Penon-open* subset [98, Chapitre III]: A subset $U \subseteq X$ is Penon-open if and only if

$$\forall x \in U.\ \forall y : X.\ (x \neq y) \vee (y \in U).$$

This notion is not useful in synthetic algebraic geometry, since it is much too weak: Any subset of the one-element set 1 is Penon-open. However, not every subfunctor of the terminal functor in $\mathrm{Zar}(S)$ is an open subfunctor.

In many flavors of synthetic topology, open truth values φ are $\neg\neg$-stable in that $\neg\neg\varphi$ implies φ. With a small caveat, this is true for open truth values in the big Zariski topos as well.

Proposition 19.20. *Let $U \hookrightarrow 1$ be a subfunctor in $\mathrm{Zar}(S)$ such that $\mathrm{Zar}(S) \models \neg\neg U$. Then in any of the following situations it follows that $\mathrm{Zar}(S) \models U$:*

1. *U is a quasicompact open truth value.*
2. *U is an arbitrary open truth value and the site used to define $\mathrm{Zar}(S)$ is closed under domains of closed immersions. (This is for instance satisfied for the sites built using a Grothendieck or a partial universe. It is satisfied for the parsimonious sites if S is locally Noetherian.)*

Proof. We give two proofs, an internal one and an external one, since they employ different ideas.

Internal proof. Since U is an open truth value, there exists an ideal $J \subseteq \underline{\mathbb{A}}^1_S$ such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent and such that U holds if and only if $1 \in J$. By assumption, the element 1 is *not not* an element of J; we want to verify that it's actually an element of J.

By Scholium 18.20, the canonical homomorphism

$$\underline{\mathbb{A}}^1_S/J \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S/J), \underline{\mathbb{A}}^1_S]$$

is bijective; the assumptions of that scholium are satisfied in either of the two situations. The set $\mathrm{Spec}(\underline{\mathbb{A}}^1_S/J)$ is isomorphic to $[\![J = (0)]\!]$. Since $\neg\neg(1 \in J)$, we also have $\neg(J = (0))$. Therefore $\mathrm{Spec}(\underline{\mathbb{A}}^1_S/J)$ is empty and the codomain of the displayed isomorphism is the zero algebra. Thus $\underline{\mathbb{A}}^1_S/J$ is trivial as well, showing $1 \in J$.

External proof. Since $U \hookrightarrow 1$ is an open subfunctor, there is an open subscheme $S_0 \subseteq S$ such that a morphism $f : T \to S$ factors over S_0 if and only if $U(T)$ is inhabited. In both situations it's possible to endow $X := S \subseteq S_0$ with the structure of a closed subscheme such that X is contained in the site used to define $\mathrm{Zar}(S)$. By the universal property of S_0, we have $X \models \neg U$. Since $\mathrm{Zar}(S) \models \neg\neg U$, it follows that X is empty. Therefore $S_0 = S$ and U is globally inhabited. $\square$

Corollary 19.21. *Let $\gamma : \Delta \to X$ be a morphism in $\mathrm{Zar}(S)$. Let $U \hookrightarrow X$ be an open subfunctor such that $\mathrm{Zar}(S) \models \gamma(0) \in U$. Then, in any of the situations in Proposition 19.20, the morphism γ factors over U.*

Proof. We give an internal proof. Let $\varepsilon \in \Delta$. Then $\neg\neg(\varepsilon = 0)$. Therefore $\neg\neg(\gamma(\varepsilon) \in U)$. Since being an element of U is $\neg\neg$-stable, it follows that $\gamma(\varepsilon) \in U$. $\square$

Remark 19.22. Subobjects $U \hookrightarrow X$ for which any morphism $\gamma : \Delta \to X$ with $\gamma(0) \in U$ factors over U are called "D_1-open" in the literature on synthetic differential geometry [103, p. 60]. Corollary 19.21 shows that open subfunctors are D_1-open.

In ordinary scheme theory, an inclusion of a standard open subset $D(f) \hookrightarrow X$ is isomorphic to the structure morphism of the relative spectrum $\underline{\mathrm{Spec}}_X\, \mathcal{O}_X[f^{-1}]$. Inclusions of more general open subsets can typically not be described using the relative spectrum construction, the standard example being the inclusion $\mathbb{A}^2_k \setminus \{0\} \hookrightarrow \mathbb{A}^2_k$ whose domain is not affine.

An interesting feature of the internal universe of the big Zariski topos is that it's flexible enough to express *any* open subset as a spectrum. The contradiction is only apparent since the algebra used for constructing such a spectrum is not in general quasicoherent.

Proposition 19.23. *Let $U \hookrightarrow 1$ be an open truth value. In any of the situations of Proposition 19.20, there is a (not necessarily quasicoherent) $\underline{\mathbb{A}}^1_S$-algebra A such that the inclusion is isomorphic to the morphism* $\mathrm{Spec}(A) \to 1$.

Proof. The open truth value U is given by an ideal $J \subseteq \underline{\mathbb{A}}^1_S$ such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent and such that U holds if and only if $1 \in J$. We set $A := \underline{\mathbb{A}}^1_S[M^{-1}]$, where M is the multiplicatively closed subset

$$M := \{f : \underline{\mathbb{A}}^1_S \mid 1 \in J \Rightarrow \ulcorner f \text{ inv.}\urcorner\} \subseteq \underline{\mathbb{A}}^1_S.$$

The spectrum of A is inhabited if and only if $M \subseteq (\underline{\mathbb{A}}^1_S)^\times$, in which case the unique element of $\mathrm{Spec}(A)$ is the inverse of the localization morphism $\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S[M^{-1}]$. Thus $\mathrm{Spec}(A)$ is isomorphic to $[\![M \subseteq (\underline{\mathbb{A}}^1_S)^\times]\!]$. Therefore we have to verify that U holds if and only if $M \subseteq (\underline{\mathbb{A}}^1_S)^\times$.

The "only if" direction is trivial.

For the "if" direction, we exploit the $\neg\neg$-stability of U. If $\neg U$, then $\neg(1 \in J)$, so $M = \underline{\mathbb{A}}^1_S$, and since $M \subseteq (\underline{\mathbb{A}}^1_S)^\times$ by assumption, it follows that zero is invertible. This is a contradiction. Thus $\neg\neg U$. ☐

Remark 19.24. The radical $\sqrt{J}$ of the ideal J appearing in Proposition 19.13 is unique: It is equal to the radical ideal

$$K := \{f : \underline{\mathbb{A}}^1_S \mid \ulcorner f \text{ inv.}\urcorner \Rightarrow (x \in U)\} \subseteq \underline{\mathbb{A}}^1_S.$$

It's obvious that $J \subseteq K$ and therefore $\sqrt{J} \subseteq K$. For the converse direction, let $f \in K$ be given. Since $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent, the canonical map

$$(\underline{\mathbb{A}}^1_S/J)[f^{-1}] \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S[f^{-1}]), \underline{\mathbb{A}}^1_S/J]$$

is bijective. Since $\mathrm{Spec}(\underline{\mathbb{A}}^1_S[f^{-1}]) \cong [\![\ulcorner f \text{ inv.}\urcorner]\!]$, the image of 1 is zero: If $\mathrm{Spec}(\underline{\mathbb{A}}^1_S[f^{-1}])$ is inhabited, the element f is invertible and therefore x is an element of U. This implies that $1 \in J$. Thereby $\underline{\mathbb{A}}^1_S/J = 0$. By injectivity of the canonical map, the algebra $(\underline{\mathbb{A}}^1_S/J)[f^{-1}]$ is zero. Therefore $f^n \in J$ for some natural number n.

Remark 19.25. In view of the previous remark, one might hope to be able to simplify the condition in Proposition 19.13 as follows: "For any $x : X$, the quotient $\underline{\mathbb{A}}^1_S/K$ modulo the ideal $K = \{f : \underline{\mathbb{A}}^1_S \mid \ulcorner f \text{ inv.}\urcorner \Rightarrow (x \in U)\}$ is synthetically quasicoherent." However, this doesn't work out. This statement implies the condition in the proposition, but the converse direction does not hold, since $\underline{\mathbb{A}}^1_S/K \cong \underline{\mathbb{A}}^1_S/\sqrt{J}$ might fail to be synthetically quasicoherent. For instance that's the case if $U = \emptyset$; then $K = \sqrt{(0)}$ by Proposition 18.29. The quotient $\underline{\mathbb{A}}^1_S/\sqrt{(0)}$ is not synthetically quasicoherent by Remark 18.23.

Remark 19.26. There is the notion of an open geometric morphism of toposes. For the big Zariski toposes, this notion is not related to open morphisms or open immersions between schemes: If $X \to S$ is any morphism of schemes, the induced geometric morphism $\mathrm{Zar}(X) \to \mathrm{Zar}(S)$ is isomorphic to the canonical morphism $\mathrm{Zar}(S)/\underline{X} \to \mathrm{Zar}(S)$, as detailed in Section 16.3. Geometric morphisms of the form $\mathcal{E}/A \to \mathcal{E}$ are always open.

19.6. Closed immersions.

Definition 19.27. In the context of a specified local ring R, as for instance $\underline{\mathbb{A}}^1_S$ of the big Zariski topos of a scheme, a truth value φ is *closed* if and only if there exists an ideal $J \subseteq R$ such that R/J is synthetically quasicoherent (Definition 18.18) and such that φ holds if and only if $J = (0)$.

In other words, a truth value $Z \subseteq 1$ is closed if and only if Z is isomorphic to the spectrum of a synthetically quasicoherent quotient algebra of R.

Example 19.28. Let $f : R$. Then "$f = 0$" is a closed truth value with witnessing ideal $J = (f)$. More generally, if $f_1, \ldots, f_n : R$, the truth value "$f_1 = \cdots = f_n = 0$" is closed with witnessing ideal $(f_1, \ldots, f_n)$.

Definition 19.29. In the context of a specified local ring, a map $Z \to X$ of sets is a *synthetic closed immersion* if and only if it is injective and for any $x : X$ the truth value of "the fiber of x is inhabited" is closed.

Example 19.30. The inclusion $\{0\} \hookrightarrow R$ is a synthetic closed immersion. More generally, for any functions $f_1, \ldots, f_m : R^n \to R$, the inclusion of the set of their common zeros in R^n is a synthetic closed immersion.

Example 19.31. Let X be a set. Let $f : X \to R$ be a function. The inclusion $\{x : X \mid f(x) = 0\} \hookrightarrow X$ is a synthetic closed immersion.

Proposition 19.32. *Let X be an S-scheme.*

1. *Let $Z \hookrightarrow X$ be a closed subscheme. Then the subfunctor $\underline{Z} \hookrightarrow \underline{X}$ is a synthetic closed immersion from the internal point of view of* $\mathrm{Zar}(S)$.
2. *Assume that $\underline{X}$ is locally representable by an object of the site used to define* $\mathrm{Zar}(S)$. *Let $Z \hookrightarrow \underline{X}$ be a synthetic closed immersion. If the site is one of the parsimonious sites or if the witnessing ideals for the immersion are finitely generated from the internal point of view, then $Z \hookrightarrow \underline{X}$ is isomorphic to the subfunctor associated to a closed subscheme of X.*

Proof. To verify the first claim, let a quasicoherent $\mathcal{O}_X$-algebra $\mathcal{J}_0$ be given such that the closed subscheme $Z \hookrightarrow X$ is the vanishing scheme of $\mathcal{J}_0$. Following the translation with the Kripke–Joyal semantics, let $f : T \to S$ be an object of the site used to define $\mathrm{Zar}(S)$ and let $x \in \underline{X}(T)$. We define $J := (f^*\mathcal{J}_0)^{\mathrm{Zar}} \in \mathrm{Zar}(S)/\underline{T}$. Then $T \models \ulcorner \underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent$\urcorner$ and $T \models (x \in \underline{Z} \Leftrightarrow J = (0))$, therefore "$x \in \underline{Z}$" is a closed truth value.

For the converse direction, we may assume that $X = S$ since $\mathrm{Zar}(S)/\underline{X} \simeq \mathrm{Zar}(X)$, as discussed in Section 16.3. We then observe that the problem is local on S, since we can glue matching schemes defined over the members of an open covering of S [118, Tag 01JJ]. We may therefore assume that we are given an $\underline{\mathbb{A}}^1_S$-algebra J such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent from the internal point of view and such that $Z = [\![J = (0)]\!]$. The assumptions ensure that Theorem 18.19 is applicable. Therefore there is a quasicoherent $\mathcal{O}_S$-module $\mathcal{E}$ such that $\underline{\mathbb{A}}^1_S/J \cong \mathcal{E}^{\mathrm{Zar}}$. We set $\mathcal{J} := \ker(\mathcal{O}_S \twoheadrightarrow \mathcal{E})$. Lemma 18.22 then implies that Z is the functor of points of the S-scheme $V(\mathcal{J})$. ☐

Proposition 19.33. *Let* $\mathrm{Zar}(S)$ *be defined using a parsimonious site and assume that S is locally Noetherian. Then the witnessing ideals of closed truth values are uniquely determined by the truth value.*

Proof. We argue internally. Let φ be a closed truth value. Let $J \subseteq \underline{\mathbb{A}}^1_S$ be an ideal such that $\underline{\mathbb{A}}^1_S/J$ is synthetically quasicoherent and such that φ holds if and only if $J = (0)$.

We set $K := \{f : \underline{\mathbb{A}}^1_S \mid \varphi \Rightarrow f = 0\}$. Then, trivially, $J \subseteq K$. For the converse containment relation, let $f \in K$. The canonical homomorphism $\underline{\mathbb{A}}^1_S/J \to [[\![\varphi]\!], \underline{\mathbb{A}}^1_S]$ is bijective by synthetic quasicoherence of $\underline{\mathbb{A}}^1_S/J$ and by Scholium 18.20 (it is here that we need that S is locally Noetherian – else we can't ensure that the S-scheme $\underline{\mathrm{Spec}}_S((\underline{\mathbb{A}}^1_S/J)|_{\mathrm{Sh}(S)})$ is locally finitely presented). The image of $[f]$ is zero, hence f is an element of J. $\square$

Remark 19.34. Without any assumptions on S or on the site used to define $\mathrm{Zar}(S)$, Proposition 19.33 holds for those closed truth values which admit a finitely generated witnessing ideal. For those closed truth values, any given finitely generated witnessing ideals are equal.

Lemma 19.35. *If* $\mathrm{Zar}(S)$ *is defined using one of the parsimonious sites, then finite conjunctions of closed truth values are closed from the internal point of view of* $\mathrm{Zar}(S)$.

Proof. We argue internally. Let φ and ψ be closed truth values with witnessing ideals I and J. Then $\varphi \wedge \psi$ is equivalent to $I + J = (0)$ and the module $\underline{\mathbb{A}}^1_S/(I+J)$ is synthetically quasicoherent (as in the proof of Lemma 19.17). Hence $\varphi \wedge \psi$ is a closed truth value. $\square$

It's in general not the case that finite disjunctions of closed truth values are closed, and it's instructive to see why. The external interpretation of this failure is the following: Let $A \hookrightarrow S$ and $B \hookrightarrow S$ be closed subschemes. Then the functor of points of the union $A \cup B$ does *not* coincide with the union of the subfunctors $\underline{A} \hookrightarrow \underline{S}$ and $\underline{B} \hookrightarrow \underline{S}$. An explicit description of the former functor is

$$(X \xrightarrow{f} S) \longmapsto \{\star \mid f \text{ factors over } A \cup B\}$$

and of the latter is

$$(X \xrightarrow{f} S) \longmapsto \{\star \mid \text{locally on the target, } f \text{ factors over } A \text{ or over } B\}.$$

The inclusion $A \cup B \hookrightarrow S$ trivially factors over $A \cup B$, but in general there isn't an open covering $S = \bigcup_i U_i$ such that for each i, the restriction $(A \cup B) \cap U_i \hookrightarrow U_i$ factors over $A \cap U_i$ or over $B \cap U_i$. For instance, this isn't the case if S is the affine plane over a field and A and B are the two axes.

This phenomenon doesn't happen for open subschemes, which explains why finite disjunctions of open truth values are open.

Remark 19.36. There is the notion of a closed geometric morphism of toposes. For an arbitrary topos $\mathcal{E}$ and an object $A \in \mathcal{E}$, the canonical geometric morphism $\mathcal{E}/A \to E$ is closed if and only if

$$\forall U \subseteq A.\ \forall \varphi : \Omega. \quad A \subseteq (U \cup \{x \in A \mid \varphi\}) \quad \Longrightarrow \quad (A \subseteq U) \vee \varphi$$

from the internal point of view of $\mathcal{E}$. If $X \to S$ is a closed morphism of schemes, then the induced geometric morphism $\mathrm{Sh}(X) \to \mathrm{Sh}(S)$ between the little Zariski toposes is closed in this sense.

However, the induced geometric morphism $\mathrm{Zar}(X) \to \mathrm{Zar}(S)$ is typically not closed. For instance, if $X \to S$ is the embedding of a closed subset $V(f)$ with $f \in \Gamma(S, \mathcal{O}_S)$, then the morphism $\mathrm{Zar}(X) \to \mathrm{Zar}(S)$ is isomorphic to $\mathrm{Zar}(S)/\underline{V(f)} \to \mathrm{Zar}(S)$, as discussed in Section 16.3. In the special case $U := \emptyset$ and $\varphi := [\![f = 0]\!]$, the displayed closedness condition simplifies to $\ulcorner f \text{ inv.}\urcorner \vee (f = 0)$. This is typically not true in the internal language of $\mathrm{Zar}(S)$. A specific counterexample is given in Example 6.39.

19.7. Surjective morphisms.

Proposition 19.37. *Let $f : X \to S$ be an arbitrary S-scheme. Consider the following statements:*

1. *The morphism f is surjective.*
2. *From the internal point of view of* $\mathrm{Zar}(S)$*, it's not the case that $\underline{X}$ is empty, that is*
$$\mathrm{Zar}(S) \models \neg\neg(\ulcorner \underline{X} \text{ is inhabited} \urcorner).$$

If X is locally contained in the site used to define $\mathrm{Zar}(S)$ *(for instance, if X is contained in the universe used to define* $\mathrm{Zar}(S)$ *or if one of the parsimonious sites is used and X is locally of finite presentation over S), then (1) implies (2). The converse holds if the site is closed under taking spectra of residue fields or if one of the parsimonious sites is used and f is quasicompact and quasiseparated.*

Proof. The translation of the internal statement using the Kripke–Joyal semantics is:

> For any S-scheme T of the site used to define $\mathrm{Zar}(S)$, if $\underline{X \times_S T} = \underline{\emptyset}$ (as functors of points of T-schemes), then $T = \emptyset$.

In the case that the site used to define $\mathrm{Zar}(S)$ is closed under taking spectra of residue fields, this implies that f is surjective as follows. Let $s \in S$ be an arbitrary point. The S-scheme $T := \mathrm{Spec}(k(s))$ is not empty. Therefore the fiber $X_s = X \times_S T$ of f over s is not empty.

If one of the parsimonious sites is used to define $\mathrm{Zar}(S)$, we can't apply the assumption to the S-scheme $T = \mathrm{Spec}(k(s))$ since it might not be locally of finite presentation over S. We therefore argue as follows. Without loss of generality, we may assume that S is affine. Writing $k(s)$ as the canonical filtered colimit of all finitely presented $\Gamma(S, \mathcal{O}_S)$-algebras mapping to $k(s)$ (and then rewriting this filtered colimit as a directed colimit [3, Theorem 1.5]), we see that $\mathrm{Spec}(k(s))$ is the directed limit of an inverse system of finitely presented affine S-schemes T_i with affine transition maps. In particular, the structure morphisms $T_i \to S$ are quasicompact and quasiseparated. By the assumption that the morphism $X \to S$ is quasicompact and quasiseparated as well, the schemes $X \times_S T_i$ are quasicompact and quasiseparated (as absolute schemes). Therefore, if $X_s = X \times_S T = \lim_i (X \times_S T_i)$ is empty, then so is $X \times_S T_i$ for some i [118, Tag 01ZC]. Thus $T_i = \emptyset$ and hence $\mathrm{Spec}(k(s)) = \emptyset$; this is a contradiction.

For the converse direction, let an S-scheme T contained in the site used to define $\mathrm{Zar}(S)$ be given such that $\underline{X \times_S T} = \underline{\emptyset}$ as functors of points of T-schemes. The assumption that X is locally contained in the site used to define $\mathrm{Zar}(S)$ implies that $X \times_S T = \emptyset$ as schemes. Since the base change $X \times_S T \to T$ of f is surjective, this implies that T is empty. □

Corollary 19.38. *Let $p : X \to Y$ be a morphism of S-schemes. Assume that Y is is locally contained in the site used to define* $\mathrm{Zar}(S)$*. Further assume that the site used to define the big Zariski toposes are closed under taking spectra of residue fields or that the parsimonious sites are used and that p is quasicompact and quasiseparated. Consider the following statements:*

1. *The morphism p is surjective.*
2. *From the internal point of view of* $\mathrm{Zar}(S)$ *all fibers of $\underline{p}$ are nonempty, that is*
$$\mathrm{Zar}(S) \models \forall y : \underline{Y}.\ \neg\neg\exists x : \underline{X}.\ \underline{p}(x) = y.$$

If X is locally contained in the site used to define $\mathrm{Zar}(Y)$*, then (1) implies (2). The converse holds if the sites used to define the big Zariski toposes are closed under* *taking spectra of residue fields or that the parsimonious sites are used and that p is quasicompact and quasiseparated.*

Proof. Immediate using Proposition 19.37 and the equivalence $\mathrm{Zar}(Y) \simeq \mathrm{Zar}(S)/\underline{Y}$, as explained in Section 16.3. □

Remark 19.39. Combining Proposition 17.1 and Proposition 19.37 yields a proof of the fact that a quasicompact, quasiseparated, and locally finitely presented morphism $X \to S$, where S is locally of finite type over a field, is surjective if it is surjective on closed points.

Remark 19.40. In the case that the parsimonious sites are used, the assumption in Proposition 19.37 that the morphism f is quasicompact can't be dropped. For instance, let k be an algebraically closed field. Then the canonical morphism

$$X := \coprod_{a \in k} \mathrm{Spec}(k[X]/(X-a)) \longrightarrow \mathrm{Spec}(k[X]) =: S$$

is surjective on closed points. By Proposition 17.1, it's not the case that $\underline{X}$ is empty from the internal point of view of $\mathrm{Zar}(S)$. However, the morphism is not surjective.

19.8. Universally injective morphisms.

Proposition 19.41. *Let $f : X \to S$ be an S-scheme which is locally contained in the site used to define* $\mathrm{Zar}(S)$*. In the case that the parsimonious sites are used to define* $\mathrm{Zar}(S)$*, further assume that f is quasicompact and quasiseparated. Then the following statements are equivalent:*

1. *The morphism f is universally injective.*
2. *The diagonal morphism $X \to X \times_S X$ is surjective.*
3. *From the internal point of view of* $\mathrm{Zar}(S)$*, any given elements of $\underline{X}$ are* not not *equal, that is*

$$\mathrm{Zar}(S) \models \forall x : \underline{X}.\ \forall x' : \underline{X}.\ \neg\neg(x = x').$$

Proof. The equivalence "(1) ⇔ (2)" is well-known [118, Tag 01S4]. The equivalence "(2) ⇔ (3)" is by Corollary 19.38 and the fact that, internally, there is *not not* a preimage for any element of $\underline{X} \times \underline{X}$ under the diagonal map $\underline{X} \to \underline{X} \times \underline{X}$ if and only if any given elements of X are *not not* equal. □

Corollary 19.42. *Let $p : X \to Y$ be a morphisms of S-schemes which are locally contained in the site used to define* $\mathrm{Zar}(S)$*. In the case that the parsimonious sites are used to define* $\mathrm{Zar}(S)$*, further assume that f is quasicompact and quasiseparated. Then the following statements are equivalent:*

1. *The morphism p is universally injective.*
2. *From the internal point of view of* $\mathrm{Zar}(S)$*, any given elements of any fiber of p are* not not *equal.*

Proof. Immediate using Proposition 19.41 and the equivalence $\mathrm{Zar}(Y) \simeq \mathrm{Zar}(S)/\underline{Y}$, as explained in Section 16.3. □

19.9. Universally closed morphisms.

Definition 19.43. In the context of a specified local ring, a set X is *synthetically closed* if and only if, for any synthetic closed immersion $Z \hookrightarrow X$, there is a closed truth value φ such that Z is *not not* inhabited if and only if $\neg\neg\varphi$.

Example 19.44. Any singleton set is synthetically closed.

Example 19.45. The specified local ring R is typically not synthetically closed. For let $f : R$ be an element. Then the inclusion $Z := \{g : R \mid fg - 1 = 0\} \hookrightarrow R$ is a synthetic closed immersion. The set Z is *not not* inhabited if and only if f is *not not* invertible if and only if f is invertible (by Proposition 18.27); typically, there is no closed truth value φ such that $\neg\neg\varphi$ is equivalent to the open truth value "f is invertible".

Proposition 19.46. *Assume that S is locally Noetherian. Let $f : X \to S$ be a finitely presented morphism. In the situation that one the parsimonious sites is used to define* $\mathrm{Zar}(S)$*, the following statements are equivalent:*

1. *The morphism f has closed image.*
2. *The morphism f has universally closed image, that is for any S-scheme T the image of the induced morphism $X \times_S T \to T$ is closed.*
3. $\mathrm{Zar}(S) \models \exists \varphi : \Omega.\ \ulcorner\varphi$ is a closed truth value$\urcorner \wedge (\neg\neg(\ulcorner\underline{X}$ inhabited$\urcorner) \Leftrightarrow \neg\neg\varphi)$.

Proof. The direction "(2) ⇒ (1)" is trivial, and the direction "(1) ⇒ (2)" is immediate, since the image of $X \times_T T \to T$ is the preimage of the image of f under the structure morphism $T \to S$.

For the "(1) ⇒ (3)" direction, we may pick the subfunctor of $\underline{S}$ induced by the closed immersion $\mathrm{im}(f) \hookrightarrow S$ as the sought truth value. This truth value is closed by Proposition 19.32 and its double negation is equivalent to $\ulcorner\underline{X}$ is inhabited$\urcorner$ by Lemma 17.3.

For the converse direction, we see that, after passing to an open covering of S which we won't reflect in the notation, there is a closed subfunctor $Z \hookrightarrow 1$ such that $\mathrm{Zar}(S) \models \neg\neg(\ulcorner\underline{X}$ inhabited$\urcorner) \Leftrightarrow \neg\neg(\ulcorner Z$ is inhabited$\urcorner)$. By Proposition 19.32, this subfunctor is the functor of points of a closed subscheme of S. Since S is locally Noetherian, this subscheme is locally of finite presentation over S. Therefore Lemma 17.3 is applicable and yields that the image of $X \to S$ coincides with the found closed subscheme. This concludes the proof. □

Corollary 19.47. *Assume that S is locally Noetherian. Let $f : X \to S$ be a finitely presented morphism. In the situation that one the parsimonious sites is used to define* $\mathrm{Zar}(S)$*, the following statements are equivalent:*

1. *The morphism f is universally closed.*
2. $\mathrm{Zar}(S) \models \ulcorner\underline{X}$ is synthetically closed$\urcorner$.

Proof. Immediate using Proposition 19.32 and Proposition 19.46. □

In view of Corollary 19.47, Example 19.44 and Example 19.45 have a geometric interpretation. The first example reflects the fact that the idenity morphism $S \to S$ is universally closed. The second example reflects the fact that the projection morphism $\underline{\mathbb{A}^1_S} \to S$ is typically not universally closed.

19.10. Quasicompact and quasiseparated morphisms.

Definition 19.48. In the context of a specified local ring R, a *synthetic affine scheme* is a set which is isomorphic (as a set) to the synthetic spectrum of a synthetically quasicoherent R-algebra.

Definition 19.49. In the context of a specified local ring R:

1. A *quasicompact synthetic scheme* is a set X which admits a finite open covering $X = \bigcup_{i=1}^n U_i$ by synthetic affine schemes U_i.
2. A *locally finitely presented quasicompact synthetic scheme* is a set X which admits a finite open covering $X = \bigcup_{i=1}^n U_i$ such that the sets U_i are isomorphic to spectra of finitely presented R-algebras.

(3) A *finitely presented synthetic scheme* is a set X which admits a finite open covering $X = \bigcup_{i=1}^n U_i$ such that the sets U_i are isomorphic to spectra of finitely presented R-algebras and such that the intersections $U_i \cap U_j$ can be covered by finitely many open subsets which are isomorphic to spectra of finitely presented R-algebras.

Proposition 19.50. *Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. Consider the following statements:*

(1) *X is the functor of points of a quasicompact S-scheme.*
(2) *X is the functor of points of a locally finitely presented quasicompact S-scheme.*
(3) *X is the functor of points of a finitely presented S-scheme (locally finitely presented, quasicompact, and quasiseparated over S).*
(1') $\mathrm{Zar}(S) \models \ulcorner X$ is a quasicompact synthetic scheme$\urcorner$.
(2') $\mathrm{Zar}(S) \models \ulcorner X$ is a locally finitely presented quasicompact synthetic scheme$\urcorner$.
(3') $\mathrm{Zar}(S) \models \ulcorner X$ is a finitely presented synthetic scheme$\urcorner$.

Then:

- *(1) ⇒ (1'), (2) ⇒ (2'), and (3) ⇒ (3').*
- *If the parsimonious sites are used to define* $\mathrm{Zar}(S)$*, then all three converses hold.*
- *If the given synthetic open immersions are even quasicompact open immersions, then (2') ⇒ (2) and (3') ⇒ (3).*

Proof. For proving (1) ⇒ (1'), (2) ⇒ (2'), and (3) ⇒ (3'), we may assume that S is affine, since the internal language is local. Let X_0 be an S-scheme representing the functor X. Since the structure morphism $X_0 \to S$ is quasicompact and S is quasicompact, there exist finitely many open affine subschemes $U_i \subseteq X_0$ which cover X_0. By Proposition 19.7, the subfunctors $\underline{U_i} \hookrightarrow X$ are synthetic open immersions from the internal point of view of $\mathrm{Zar}(S)$. The internal union $\bigcup_i \underline{U_i} \hookrightarrow X$ is the functor

$$T/S \longmapsto \{f : T \to X_0 \mid \text{locally, the morphism } f \text{ factors over one of the opens } U_i\}$$

and therefore coincides with X.

Since each scheme U_i can be realized as a relative spectrum of a quasicoherent $\mathcal{O}_S$-algebra, both U_i and S being affine, the sets U_i are synthetic affine schemes from the internal point of view. Hence (1) ⇒ (1'). If (2) holds, then the U_i are spectra of finitely presented $\mathcal{O}_S$-algebras, so (2') holds. If (3) holds, then the intersections $U_i \cap U_j$ can be covered by finitely many open subschemes which are spectra of finitely presented $\mathcal{O}_S$-algebras, so (3') holds.

For the converse directions, we first note that the problem is local on S, since we can glue matching schemes defined over the members of an open covering of S [118, Tag 01JJ]. We may therefore assume that we are given subfunctors $U_1, \ldots, U_n \hookrightarrow X$ such that, from the internal point of view of $\mathrm{Zar}(S)$, the subsets $U_i \subseteq X$ are synthetic affine schemes and the inclusions $U_i \hookrightarrow X$ are synthetic open immersions. The extra assumptions ensure that Proposition 19.13 and Proposition 18.9 are applicable. These imply that the functors U_i are representable by affine S-schemes. These can be glued to yield an S-scheme which represents the functor X [118, Tag 01JJ].

In case that internally further finiteness conditions are satisfied, the resulting scheme X will satisfy the corresponding external finiteness conditions. □

One can reasonably wonder why we didn't include the following notion in Definition 19.49: A *synthetic scheme* is a set X which admits an arbitrary open covering by synthetic affine schemes. The reason is that, with this definition, any subset X of the singleton set $1 = \{\star\}$ is a synthetic scheme, since it admits the open affine covering $X = \bigcup\{1 \mid \star \in X\}$. But not any subfunctor of the terminal functor is representable by a scheme.

This phenomenon is well-known in synthetic topology; one has to put some restrictions on the kind of allowed open coverings. Being finite is the simplest such condition.

Proposition 19.51. *Let* $\mathrm{Zar}(S)$ *be defined using the parsimonious sites. Let* $f : X \to S$ *be a locally finitely presented morphism. Then:*

1. *The morphism f is quasicompact if and only if $\underline{X}$ is a quasicompact synthetic scheme from the internal point of view of* $\mathrm{Zar}(S)$.
2. *The morphism f is quasiseparated if and only if, internally, for any elements $x, y : \underline{X}$ the set $[\![x = y]\!]$ is a quasicompact synthetic scheme.*
3. *The morphism f is separated if and only if, internally, for any elements $x, y : \underline{X}$ the truth value $[\![x = y]\!]$ is closed. If this is the case, the witnessing ideal can be chosen to be finitely generated.*

Proof. The first statement is by Proposition 19.50 and by the fact that the scheme representing a representable functor is unique.

The second statement follows from the first by applying it to the diagonal morphism $\Delta : X \to X \times_S X$. More precisely, we have the following chain of equivalences:

$$\begin{aligned} &\mathrm{Zar}(S) \models \forall x, y : \underline{X}.\ \ulcorner [\![x = y]\!] \text{ is a quasicompact synthetic scheme}\urcorner \\ \iff &\mathrm{Zar}(X \times_S X) \models \ulcorner \underline{X} \text{ is a quasicompact synthetic scheme}\urcorner \\ \iff &\Delta : X \to X \times_S X \text{ is quasicompact} \end{aligned}$$

The first step is by the discussion in Section 16.3 and the second by applying the first statement; the diagonal morphism is locally of finite presentation [118, Tag 0818], as required.

The third statement follows in a similar way, but employing Proposition 19.32 instead of the first statement. □

Lemma 19.52. *Internally in* $\mathrm{Zar}(S)$, *a set X is separated if and only if all of its points are closed. More formally, the following two statements are equivalent:*

1. *For all $x : X$, the inclusion $\{x\} \hookrightarrow X$ is a synthetic closed immersion.*
2. *The diagonal morphism $X \hookrightarrow X \times X$ is a synthetic closed immersion.*

Proof. We argue internally. Let $x : X$ be fixed. The inclusion $\{x\} \hookrightarrow X$ is a synthetic closed immersion if and only if, for all $y : X$, the truth value "the fiber over y is inhabited" is closed. This truth value is equal to the truth value of "$x = y$".

With these observations, the equivalence of the two statements is immediate. □

19.11. Proper morphisms.

Proposition 19.53. *Let* $\mathrm{Zar}(S)$ *be defined using the parsimonious sites. Assume that S is locally Noetherian. Let $X \in \mathrm{Zar}(S)$ be a Zariski sheaf. Then the following statements are equivalent:*

1. *The sheaf X is the functor of points of a proper S-scheme.*
2. $\mathrm{Zar}(S) \models \ulcorner X$ is a finitely presented synthetic scheme and synthetically closed$\urcorner$.

Proof. Follows immediately from Proposition 19.50, Proposition 19.51, and Corollary 19.47. □

20. Case studies

20.1. Punctured plane.

Definition 20.1. The *synthetic punctured plane* is the set $P := (\underline{\mathbb{A}}^1_S)^2 \setminus \{0\}$.

Proposition 20.2. *The synthetic punctured plane, as constructed by the internal language of* $\mathrm{Zar}(S)$*, is the functor of points of the ordinary punctured plane over* S*, that is the open subscheme* $D(X) \cup D(Y) \hookrightarrow \mathbb{A}^2_S$*.*

Proof. The set P can be written as $\{(x,y) : (\underline{\mathbb{A}}^1_S)^2 \mid x \text{ inv.} \vee y \text{ inv.}\}$. □

Proposition 20.3. *The evaluation morphism* $\underline{\mathbb{A}}^1_S[X,Y] \to [P, \underline{\mathbb{A}}^1_S]$ *is bijective.*

Proof. The synthetic punctured plane can be expressed as the pushout

$$P \cong D(X) \amalg_{D(X)\cap D(Y)} D(Y).$$

Therefore we have the chain of isomorphisms

$$\begin{aligned} [P, \underline{\mathbb{A}}^1_S] &\cong [D(X) \amalg_{D(X)\cap D(Y)} D(Y), \underline{\mathbb{A}}^1_S] \\ &\cong [D(X), \underline{\mathbb{A}}^1_S] \times_{[D(X)\cap D(Y), \underline{\mathbb{A}}^1_S]} [D(Y), \underline{\mathbb{A}}^1_S] \\ &\cong \underline{\mathbb{A}}^1_S[X, X^{-1}] \times_{\underline{\mathbb{A}}^1_S[XY,(XY)^{-1}]} \underline{\mathbb{A}}^1_S[Y, Y^{-1}] \\ &\cong \underline{\mathbb{A}}^1_S[X,Y]. \end{aligned}$$

The penultimate isomorphism exploits the synthetic quasicoherence of $\underline{\mathbb{A}}^1_S$, which ensures that the canonical map

$$\underline{\mathbb{A}}^1_S[X, X^{-1}] \longrightarrow [\mathrm{Spec}(\underline{\mathbb{A}}^1_S[X, X^{-1}]), \underline{\mathbb{A}}^1_S] \cong [D(X), \underline{\mathbb{A}}^1_S]$$

is bijective. The ultimate isomorphism rests on the purely algebraic argument that elements of $\underline{\mathbb{A}}^1_S[X, X^{-1}]$ and $\underline{\mathbb{A}}^1_S[Y, Y^{-1}]$ which agree as elements of $\underline{\mathbb{A}}^1_S[(XY), (XY)^{-1}]$ are both given by an element of $\underline{\mathbb{A}}^1_S[X,Y]$ and in fact by the same element. □

Corollary 20.4. *The punctured plane is not affine.*

Proof. The canonical map $P \to \mathrm{Spec}([P, \underline{\mathbb{A}}^1_S])$ is isomorphic to the strict inclusion $P \hookrightarrow (\underline{\mathbb{A}}^1_S)^2$ and therefore not bijective. □

20.2. Cohomology of projective space. Let $\mathcal{V}$ be a finite locally free $\mathcal{O}_S$-module. In this section, we give internal descriptions of the projectivization of $\mathcal{V}$ and of Serre's twisting bundles on $\mathbb{P}(V)$, and show how their cohomology groups can be calculated internally. Let $V := \mathcal{V}^{\mathrm{Zar}}$.

Definition 20.5. The *synthetic projective space* is the set

$$\mathbb{P}(V) := \{\ell \subseteq V \mid \ulcorner \ell \text{ is a one-dimensional subspace} \urcorner\}.$$

The twisting bundles are modules over the affine line of $\mathbb{P}(V)$. As such they are $\mathbb{P}(V)$-indexed families of $\underline{\mathbb{A}}^1_S$-modules, as in Section 19.4. We set $X := \mathbb{P}(V)$.

Definition 20.6. For $m \in \mathbb{Z}$, the *twisting bundle* $\mathcal{O}(m)$ is the $\underline{\mathbb{A}}^1_X$-module $(\ell^{\otimes(-m)})_{\ell:X}$. In particular, the *tautological bundle* $\mathcal{O}(-1)$ is $(\ell)_{\ell:X}$ and its dual is $\mathcal{O}(1) = (\ell^\vee)_{\ell:X}$.

We stress that Definition 20.6 formalizes the common geometric intuition about the twisting bundles. In algebraic geometry, one often pictures a bundle as the collection of its fibers, well knowing that this picture is formally incomplete and that a bundle is not really determined by the collection of its fibers. In our account of synthetic algebraic geometry, the statement that a bundle is given by its collection of fibers is *just true*, and in fact is so by definition.

Proposition 20.7.
(1) *The canonical map* $\underline{\mathbb{A}}^1_S \to \Gamma(X, \mathcal{O}(0)) = [X, \underline{\mathbb{A}}^1_S]$ *is an isomorphism.*
(2) *The canonical map* $V^\vee \to \Gamma(X, \mathcal{O}(1))$, $\vartheta \mapsto (\vartheta|_\ell)_{\ell:X}$ *is an isomorphism.*
(3) *The module* $\Gamma(X, \mathcal{O}(-1))$ *is zero.*

Proof. For notational convenience, we assume that V is isomorphic to $(\underline{\mathbb{A}}^1_S)^2$. In this case $X = \{[x:y] \mid x \neq 0 \vee y \neq 0\}$. Let $X = U_0 \cup U_1$ be the standard open covering with

$$U_0 = \{[x:y] \mid x \neq 0\} = \{[1:y] \mid y : \underline{\mathbb{A}}^1_S\} \cong \underline{\mathbb{A}}^1_S$$

and similarly with U_1.

For the first claim, let $f : X \to \underline{\mathbb{A}}^1_S$ be an arbitrary function. Since $\underline{\mathbb{A}}^1_S$ is synthetically quasicoherent, the canonical morphism $\underline{\mathbb{A}}^1_S[T] \longrightarrow [U_0, \underline{\mathbb{A}}^1_S]$ is bijective. Hence there is a unique polynomial $p_0 : \underline{\mathbb{A}}^1_S[T]$ such that $f([x:y]) = p_0(y/x)$ for all $[x:y] \in U_0$. Analogously, there is a unique polynomial $p_1 : \underline{\mathbb{A}}^1_S[T]$ such that $f([x:y]) = p_1(x/y)$ for all $[x:y] \in U_1$.

Since $\underline{\mathbb{A}}^1_S$ is synthetically quasicoherent, the canonical morphism

$$\underline{\mathbb{A}}^1_S[T, T^{-1}] \longrightarrow [U_0 \cap U_1, \underline{\mathbb{A}}^1_S]$$

is bijective. Because $p_0(y/x) = p_1(x/y)$ as functions on $U_0 \cap U_1$, the polynomials $p_0(T)$ and $p_1(1/T)$ agree as elements of $\underline{\mathbb{A}}^1_S[T, T^{-1}]$. Hence p_0 and p_1, and thus f, are constant.

This shows that the canonical morphism $\underline{\mathbb{A}}^1_S \to [X, \underline{\mathbb{A}}^1_S]$ is surjective. This morphism is trivially injective since X is inhabited.

The other claims are verified similarly. □

In the same spirit as in the proof of Proposition 20.7, we can calculate the cohomology of the twisting bundles, if we define it as the cohomology of the Čech complex associated to the standard open covering. We want to indicate how the general idea of the usual proof can be carried out in the synthetic account.

For notational convenience, we assume that V is isomorphic to $(\underline{\mathbb{A}}^1_S)^2$. The Čech complex for calculating $H^\bullet(X, \mathcal{O}(-2))$ is

$$0 \longrightarrow \Gamma(U_0, \mathcal{O}(-2)) \times \Gamma(U_1, \mathcal{O}(-2)) \longrightarrow \Gamma(U_0 \cap U_1, \mathcal{O}(-2)) \longrightarrow 0.$$

The differential maps (f, g) to $g|_{U_0 \cap U_1} - f|_{U_0 \cap U_1}$.

The verification that $H^0(X, \mathcal{O}(-2))$ is zero proceeds similarly to the proof of Proposition 20.7. To verify that $H^1(X, \mathcal{O}(-2)) \cong \underline{\mathbb{A}}^1_S$, we explicitly describe the isomorphism. Given a function $h : \Gamma(U_0 \cap U_1, \mathcal{O}(-2))$, there is for any number $y : (\underline{\mathbb{A}}^1_S)^\times$ a unique value $h_0(y)$ such that

$$h([1:y]) = h_0(y) \cdot \left(\begin{pmatrix} 1 \\ y \end{pmatrix} \otimes \begin{pmatrix} 1 \\ y \end{pmatrix} \right) : \operatorname{span}\left(\begin{pmatrix} 1 \\ y \end{pmatrix} \right)^{\otimes 2}.$$

By the principle of unique choice, the mapping $y \mapsto h_0(y)$ defines a well-defined function on $(\underline{\mathbb{A}}^1_S)^\times$ as the notation suggests. Because $\underline{\mathbb{A}}^1_S$ is synthetically quasicoherent, the canonical map

$$\underline{\mathbb{A}}^1_S[T, T^{-1}] \longrightarrow [U_0 \cap U_1, \underline{\mathbb{A}}^1_S]$$

is bijective. Hence there is a unique Laurent polynomial $p_0 \in \underline{\mathbb{A}}^1_S[T, T^{-1}]$ such that $h_0(y) = p_0(y)$ for all $y \in (\underline{\mathbb{A}}^1_S)^\times$. The sought isomorphism maps h to the coefficient of T^{-1} in p_0.

20.3. Grassmannian. Let $\mathcal{V}$ be a finite locally free $\mathcal{O}_S$-module. We want to illustrate the synthetic approach by verifying the basic fact that the Grassmannian $\operatorname{Gr}(\mathcal{V}, r)$ of rank-r locally free quotients of $\mathcal{V}$, defined as a certain functor of points, is representable by a locally finitely presented S-scheme using the internal language of $\operatorname{Zar}(S)$.

Definition 20.8. The *Grassmannian* $\operatorname{Gr}(\mathcal{V}, r)$ is the functor which associates to an S-scheme $f : T \to S$ the set

$$\operatorname{Gr}(\mathcal{V}, r)(T) := \{U \subseteq f^*\mathcal{V} \text{ sub-}\mathcal{O}_T\text{-module} \mid (f^*\mathcal{V})/U \text{ is locally free of rank } r\}.$$

Definition 20.9. The *synthetic Grassmannian* of rank-r quotients of a module V is the set

$$\mathrm{Gr}(V,r) := \{U \subseteq V \text{ submodule} \mid V/U \text{ is free of rank } r\}.$$

We could just as well define the synthetic Grassmannian somewhat more directly as the set of free rank r-quotients (up to isomorphism). This set is canonically isomorphic to the Grassmannian as we chose to define it, by mapping a quotient $\pi : V \twoheadrightarrow Q$ to the kernel of π.

Proposition 20.10. *The synthetic Grassmannian of $\mathcal{V}$, as constructed by the internal language of* $\mathrm{Zar}(S)$ *where $\mathcal{V}$ looks like an ordinary free module, coincides with the functorially defined Grassmannian.*

Proof. Immediate from Definition 2.8 and Proposition 18.1. □

Having established that the internally constructed synthetic Grassmannian actually describes the external Grassmannian which we're interested in, we can switch to a fully internal perspective. We'll reflect this switch notationally by referring to the $\underline{\mathbb{A}}^1_S$-module $V := \mathcal{V}^{\mathrm{Zar}}$ instead of $\mathcal{V}$.

We define for any free submodule $W \subseteq V$ of rank r the subset

$$G_W := \{U \in \mathrm{Gr}(V,r) \mid W \to V \to V/U \text{ is bijective}\}.$$

This sets admits a more concrete description, since it is in canonical bijection to the set

$$G'_W := \{\pi : V \to W \mid \pi \circ \iota = \mathrm{id}\}$$

of all splittings of the inclusion $\iota : W \hookrightarrow V$: An element $U \in G_W$ corresponds to the splitting $V \twoheadrightarrow V/U \xrightarrow{(\cong)^{-1}} W$. Conversely, a splitting π corresponds to $U := \ker(\pi) \in G_W$.

Proposition 20.11. *The union of the subsets G_W is* $\mathrm{Gr}(V,r)$.

Proof. Let $U \in \mathrm{Gr}(V,r)$. Then there exists a basis $([v_1],\ldots,[v_r])$ of V/U. The family $(v_1,\ldots,v_r)$ is linearly independent in V, therefore the submodule $W := \mathrm{span}(v_1,\ldots,v_r) \subseteq V$ is free of rank r. The canonical linear map $W \hookrightarrow V \twoheadrightarrow V/U$ maps the basis $(v_i)_i$ to the basis $([v_i])_i$ and is therefore bijective. Thus $U \in G_W$. □

Proposition 20.12. *The sets G_W are (quasicompact-)open subsets of* $\mathrm{Gr}(V,r)$.

Proof. Let $U \in \mathrm{Gr}(V,r)$. Then $U \in G_W$ if and only if the canonical linear map $W \hookrightarrow V \twoheadrightarrow V/U$ is bijective. Since W and V/U are both free modules of rank r, this map is given by an $(r \times r)$-matrix M over $\underline{\mathbb{A}}^1_S$; therefore it's bijective if and only if the determinant of M is invertible.

Thus we've found a number which is invertible if and only if $U \in G_W$. By Corollary 19.14, the truth value of "$U \in G_W$" is open. □

Proposition 20.13. *The sets G_W are synthetic affine schemes. Moreover, the algebras which the G_W are spectra of are finitely presented.*

Proof. The set of all linear maps $V \to W$ is the spectrum of the $\underline{\mathbb{A}}^1_S$-algebra $A := \mathrm{Sym}(\mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(V,W)^\vee)$, since

$$\begin{aligned}
\mathrm{Spec}(A) &= \mathrm{Hom}_{\mathrm{Alg}(\underline{\mathbb{A}}^1_S)}(\mathrm{Sym}(\mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V,W)^\vee), \underline{\mathbb{A}}^1_S) \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(\mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V,W)^\vee, \underline{\mathbb{A}}^1_S) \\
&= \mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V,W)^{\vee\vee} \\
&\cong \mathrm{Hom}_{\mathrm{Mod}(\underline{\mathbb{A}}^1_S)}(V,W).
\end{aligned}$$

In the last step the assumption that not only W, but also V is a free module of finite rank enters. (This is the first time in this development that we need this assumption.)

The set G'_W is a closed subset of this spectrum, namely the locus where the generic linear map $V \to W$ is a splitting of the inclusion $\iota : W \hookrightarrow V$. If we choose bases of V and W, whereby $\mathrm{Sym}(\mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(V, W)^\vee)$ is isomorphic to $\underline{\mathbb{A}}^1_S[M_{11}, \ldots, M_{rn}]$, we can be more explicit: The set G'_W is isomorphic to

$$\mathrm{Spec}(k[M_{11}, \ldots, M_{rn}]/(MN - I)),$$

where I is the $(r \times r)$ identity matrix, M is the generic matrix $M = (M_{ij})_{ij}$, and N is the matrix of ι with respect to the chosen bases. The notation "$(MN - I)$" denotes the ideal generated by the entries of $MN - I$. □

Corollary 20.14. *The Grassmannian* $\mathrm{Gr}(V, r)$ *is a locally finitely presented quasi-compact synthetic scheme.*

Proof. We need to verify that $\mathrm{Gr}(V, r)$ admits a finite covering by spectra of finitely presented $\underline{\mathbb{A}}^1_S$-algebras. We already know that $\mathrm{Gr}(V, r)$ can be covered by the open subsets G_W and that these sets are spectra of finitely presented algebras. Therefore it remains to prove that finitely many of these subsets suffice to cover $\mathrm{Gr}(V, r)$.

In fact, if we choose an isomorphism $V \cong (\underline{\mathbb{A}}^1_S)^n$, we see that $\binom{n}{r}$ of these subsets suffice: namely those where W is one of the standard submodules of $(\underline{\mathbb{A}}^1_S)^n$ (generated by the standard basis vectors). For if $U \in \mathrm{Gr}((\underline{\mathbb{A}}^1_S)^n, r)$, the surjection $V \to V/U$ maps the basis of at least one of these standard submodules to a basis and is therefore bijective. This is because from any surjective $(r \times n)$-matrix over a local ring one can select r columns which form an linearly independent family. □

Proposition 20.15. *Let* $U \in \mathrm{Gr}(V, r)$*. Then the tangent space at* U *is given by* $T_U \mathrm{Gr}(V, r) \cong \mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(U, V/U)$*.*

Proof. For notational simplicity, we verify the claim in the case $r = 1$, in which case $\mathrm{Gr}(V, r)$ is the projectivization of V. Let $\gamma : \Delta \to \mathbb{P}(V)$ be a tangent vector with base point $\ell := \gamma(0) : \mathbb{P}(V)$. By Corollary 19.21 and Corollary 20.14, there's a lift $\overline{\gamma} : \Delta \to V$ such that $\gamma(\varepsilon) = \mathrm{span}(\overline{\gamma}(\varepsilon))$ for all $\varepsilon : \Delta$. We define a linear map $\alpha : \ell \to V/\ell$ by setting

$$x \longmapsto \alpha(x) = [x/\overline{\gamma}(0) \cdot \overline{\gamma}'(0)].$$

The expression "$x/\overline{\gamma}(0)$" should be read as follows: The vector x, being an element of ℓ, is some multiple λ of $\overline{\gamma}(0)$. The expression "$x/\overline{\gamma}(0)$" denotes this unique number λ. It can be checked that the vector $\alpha(x) : V/\ell$ does not depend on the choice of the lifting $\overline{\gamma}$. The element α is therefore a well-defined element of $\mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(\ell, V/\ell)$.

Conversely, let an element $\alpha : \mathrm{Hom}_{\underline{\mathbb{A}}^1_S}(\ell, V/\ell)$ be given. We choose vectors $x_0 : V$ and $z : V$ such that $\ell = \mathrm{span}(x_0)$ and $\alpha(x_0) = [z]$ and define $\gamma : \Delta \to \mathbb{P}(V)$ by setting

$$\varepsilon \longmapsto \gamma(\varepsilon) = \mathrm{span}(x_0 + \varepsilon z).$$

The definition of $\gamma(\varepsilon)$ is invariant under scaling of x_0 and also under changing z to some other vector $z + \lambda x_0$ in its equivalence class:

$$\begin{aligned} \mathrm{span}(x_0 + \varepsilon(z + \lambda x_0)) &= \mathrm{span}((1 + \varepsilon\lambda)x_0 + \varepsilon z) \\ &= \mathrm{span}(x_0 + \varepsilon/(1 + \varepsilon\lambda)z) \\ &= \mathrm{span}(x_0 + \varepsilon z), \end{aligned}$$

since $\varepsilon/(1 + \varepsilon\lambda) = \varepsilon(1 - \varepsilon\lambda) = \varepsilon$. Therefore γ is a well-defined element of $T_\ell \mathbb{P}(V)$ which only depends on α and not on the arbitrary choices of x_0 and z.

One can check that the two described constructions are mutually inverse. □

Remark 20.16. The arguments given in this section are intended to be applied internally to the big Zariski topos of a base scheme. However, one can also apply them internally to well-adapted models for synthetic differential geometry. In this way, one almost obtains that the differential-geometric Grassmannian can be represented as a manifold; only a verification of smoothness is missing.

21. Beyond the Zariski topology

The Zariski topology is of course not the only interesting topology on Sch/S. For any finer topology τ, such as the Nisnevich, étale, or fppf topology (a valuable hyperlinked chart of the various topologies is located at [20]), the big τ-topos of S, that is the topos of sheaves on Sch/S with respect to τ, is a subtopos of the big Zariski topos. Therefore there is a modal operator $\Box_\tau$ in $\mathrm{Zar}(S)$ reflecting the topology τ. Explicitly, for an S-scheme T and a formula φ over T, the meaning of

$$T \models \Box_\tau \varphi$$

is that there exists a τ-covering $(T_i \to T)_i$ of T such that $T_i \models \varphi$ for all i (where parameters appearing in φ have to be pulled back along $T_i \to T$). Succinctly, the formula "$\Box_\tau \varphi$" means that φ holds τ-locally. Generalizing Theorem 6.31 from sheaves on locales to sheaves on arbitrary Grothendieck sites we also have

$$\mathrm{Zar}(S) \models \varphi^{\Box_\tau} \qquad \text{iff} \qquad \mathrm{Sh}((\mathrm{Sch}/S)_\tau) \models \varphi.$$

21.1. The étale topology. A basic illustration of these modal operators is provided by the Kummer sequence, that is the short sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_\mathrm{m} \xrightarrow{(\_)^n} \mathbb{G}_\mathrm{m} \longrightarrow 1$$

of multiplicatively-written commutative group objects in $\mathrm{Zar}(S)$. With the internal description of μ_n and $\mathbb{G}_\mathrm{m}$, there is a purely internal and straightforward proof that this sequence is exact at the first two terms. But except for trivial cases, the n-th power map $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ will fail to be an epimorphism; internally speaking, the statement

$$\forall f : (\underline{\mathbb{A}}^1_S)^\times. \qquad \exists g : (\underline{\mathbb{A}}^1_S)^\times.\ f = g^n$$

is false in general. However, if n is invertible in $\Gamma(S, \mathcal{O}_S)$, the internal statement

$$\forall f : (\underline{\mathbb{A}}^1_S)^\times.\ \Box_\text{ét}(\exists g : (\underline{\mathbb{A}}^1_S)^\times.\ f = g^n)$$

is true. In fact, the more general statement

$$\forall p : \underline{\mathbb{A}}^1_S[X].\ \ulcorner p \text{ is monic, of positive degree, and separable} \urcorner \Longrightarrow$$
$$\Box_\text{ét}(\exists x : \underline{\mathbb{A}}^1_S.\ p(x) = 0 \wedge \ulcorner p'(x) \text{ inv.} \urcorner)$$

is true from the internal point of view, where a polynomial p is called *separable* if and only if there exists a Bézout representation $ap + bp' = 1$. After simplifying, the interpretation of that statement with the Kripke–Joyal semantics is that for any S-scheme T and any monic separable polynomial $p \in \Gamma(T, \mathcal{O}_T)[X]$ of positive degree there exists an étale covering $(T_i \to T)_i$ of T such that the pullbacks of p to each of the T_i possess a simple zero. The required covering is given by the canonical surjective étale map $\underline{\mathrm{Spec}}_T\, \mathcal{O}_T[X]/(p) \to T$.

The following theorem shows that the modal operator $\Box_\text{ét}$ corresponding to the étale topology admits a purely internal characterization in $\mathrm{Zar}(S)$, which furthermore resonates well with the intuition about the étale topology.

Theorem 21.1. *Let S be a scheme. Employ one of the parsimonious sites to define $\mathrm{Zar}(S)$. The modal operator $\Box_{\acute{e}t}$ in $\mathrm{Zar}(S)$ corresponding to the étale topology is the smallest operator $\Box$ such that the $\Box$-translation of the statement "$\underline{\mathbb{A}}^1_S$ is separably closed" holds.*

Here, a ring A is *separably closed* if and only if

$$\forall p : A[X].\ \ulcorner p \text{ is monic, of positive degree, and unramifiable}\urcorner \Longrightarrow \\ \exists x : A.\ p(x) = 0 \wedge \ulcorner p'(x) \text{ inv.}\urcorner.$$

We call a polynomial p over a ring A *unramifiable* if and only if it admits at least one simple root in every algebraically closed field over A. Since quantifying over algebraically closed fields raises red flags from an intuitionistic point of view, just as quantifying over maximal ideals does, this condition has to be formulated in a sensible way. One possibility is to use the *hyperdiscriminants* of p, i. e. the elementary symmetric polynomials in the values of p' at the roots of p, resulting in a simple existential statement involving only the coefficients of p; in particular, the condition for a polynomial to be unramifiable is a geometric formula. See [136, p. 751] for the precise formulation.

In more detail, the claim is that firstly $\Box_{\text{ét}}$ is a modal operator such that the displayed formula holds and that secondly, if $\Box$ is any modal operator verifying the formula, internally it holds that $\Box_{\text{ét}}\varphi \Rightarrow \Box\varphi$ for any truth value $\varphi : \Omega$.

Proof. For the proof we require some familiarity with the concept of classifying toposes. We are grateful to Felix Geißler for contributing a key step of the argument.

To verify the first statement, we observe that the displayed formula is a geometric implication and that the big étale topos $\text{Ét}(S)$ has *enough points*. Therefore it suffices to show that for any S-scheme T and any geometric point $\bar{t}$ of T, the stalk $\mathcal{O}_{T,\bar{t}}$ is separably closed. It is well-known that this is true.

For the second statement we may assume without loss of generality that $S = \operatorname{Spec} A$ is affine. It is well-known that, for any cocomplete topos $\mathcal{E}$, geometric morphisms $\mathcal{E} \to \operatorname{Zar}(\operatorname{Spec} A)$ are in canonical one-to-one correspondence with local algebras over $\underline{A}$ in $\mathcal{E}$ (where $\underline{A}$ denotes the pullback of A along the unique geometric morphism $\mathcal{E} \to \mathrm{Set}$) and that geometric morphisms $\mathcal{E} \to \text{Ét}(\operatorname{Spec} A)$ are in canonical one-to-one correspondence with algebras over $\underline{A}$ which are local and separably closed from the internal point of view of $\mathcal{E}$; see [86, Section VIII.6] and [5].

Therefore a geometric morphism $\mathcal{E} \to \operatorname{Zar}(\operatorname{Spec} A)$ factors over the geometric embedding $\text{Ét}(\operatorname{Spec} A) \hookrightarrow \operatorname{Zar}(\operatorname{Spec} A)$ if and only if the pullback of $\underline{\mathbb{A}}^1_{\operatorname{Spec} A}$ along $\mathcal{E} \to \operatorname{Zar}(\operatorname{Spec} A)$ is separably closed.

Let $\Box$ be a modal operator in $\operatorname{Zar}(\operatorname{Spec} A)$ such that the $\Box$-translation of "$\underline{\mathbb{A}}^1_{\operatorname{Spec} A}$ is separably closed" holds. Then the pullback of $\underline{\mathbb{A}}^1_{\operatorname{Spec} A}$ along $\operatorname{Zar}(\operatorname{Spec} A)_\Box \hookrightarrow \operatorname{Zar}(\operatorname{Spec} A)$ is separably closed and therefore this geometric embedding factors over $\text{Ét}(\operatorname{Spec} A) \hookrightarrow \operatorname{Zar}(\operatorname{Spec} A)$. This shows that any $\Box$-sheaf is also a $\Box_{\text{ét}}$-sheaf.

The claim that $\Box_{\text{ét}}\varphi \Rightarrow \Box\varphi$ for any truth value $\varphi : \Omega$ then follows by combining the following two basic observations of the theory of modal operators, valid for any modal operator $\Box$:

(1) $\Box\varphi \Longleftrightarrow \forall \psi : \Omega.\ ((\Box\psi \Rightarrow \psi) \wedge (\varphi \Rightarrow \psi)) \Rightarrow \psi$.
(2) $(\Box\psi \Rightarrow \psi) \Longleftrightarrow \ulcorner \{x \in 1 \mid \psi\} \text{ is a } \Box\text{-sheaf}\urcorner$. $\Box$

21.2. The fppf topology. The big Zariski topos of a scheme S is the classifying topos of local rings over S, when employing one of the parsimonious sites. The big fppf topos of S is a particular subtopos of the big Zariski topos; it therefore classifies a particular quotient theory of the theory of local rings of S, obtained by adding certain further axioms [30]. What are these axioms?

The analogous question for the big étale topos has been answered by Wraith, building upon work by Hakim [58]: The big étale topos classifies separably closed local rings [136]. Since the big fppf topos is a subtopos of the big étale topos (any étale covering being in particular an fppf covering), the sought axioms need to at least imply the axioms for separably closed rings.

Wraith conjectured that the big fppf topos classifies algebraically closed local rings (local rings for which any monic polynomial of positive degree has a zero, also called absolutely integrally closed local rings). We neither confirm nor refute his conjecture, but we are able to give an alternative explicit description: The big fppf topos of a scheme classifies fppf-local rings over S.

These kinds of rings where studied by Schröer and independently by Gabber and Kelly [110, 51]; we'll review the notion below. Every fppf-local ring is algebraically closed, therefore the theory of fppf-local rings encompasses the theory of algebraically closed local rings. It is an open question whether these two theories coincide.

As discussed in Section 15, one can only expect the big fppf topos to classify a simple theory if one employs parsimonious sites.

Definition 21.2. The *big fppf topos* of a scheme S is the topos of sheaves over the category $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ of locally finitely presented S-schemes, where a family $(f_i : X_i \to X)_i$ of morphisms is deemed a covering if and only if the morphisms f_i are flat, locally of finite presentation, and jointly surjective.

The condition that the morphisms f_i are locally of finite presentation is automatically satisfied in our setup, since we require that source and target are locally of finite presentation [118, Tag 02FV]. One can equivalently define the big fppf topos as the topos of sheaves over the category $(\mathrm{Aff}/S)_{\mathrm{lfp}}$ of locally finitely presented S-schemes which are affine as absolute schemes.

Definition 21.3. Let A be a ring.

(1) An *fppf-algebra over A* is an A-algebra B such that the structure morphism $A \to B$ is faithfully flat and of finite presentation.
(2) A *basic fppf-algebra over A* is an A-algebra which is finite free of positive rank as an A-module.

Since algebras which are free as modules are also finitely presented as algebras, a basic fppf-algebra is also an fppf-algebra and in fact an integral fppf-algebra. Conversely, an integral fppf-algebra over a local ring is a basic fppf-algebra, since integral algebras which are finitely presented as algebras are also finitely presented as modules [118, Tag 0564], finitely presented flat modules are projective [118, Tag 058R], and finitely generated projective modules over local rings are finite free [118, Tag 00NX]. This equivalence even holds intuitionistically.

Definition 21.4. An *fppf-local ring* is a local ring A such that any finite system of polynomial equations over A which has a solution in some basic fppf-algebra over A admits a solution in A.

An fppf-local ring is algebraically closed, since the A-algebra $A[X]/(f)$ is a basic fppf-algebra whenever f is a monic polynomial of positive degree.

We refer to basic fppf-algebras instead of arbitrary integral fppf-algebras in Definition 21.4 in order to ensure that the condition for a ring to be fppf-local is a geometric implication; we'll expand on this below. The standard definition of fppf-locality refers to arbitrary (not necessarily integral) fppf-algebras [110, Definition 4.1]. The following proposition establishes the equivalence of our definition with the standard one.

Proposition 21.5. *Let A be a local ring. The following statements are equivalent.*

(1) *The ring A is fppf-local.*
(1') *Any finite system of polynomial equations over A which has a solution in some fppf-algebra over A admits a solution in A.*

(2) *The structure morphism $A \to B$ of any basic fppf-algebra has a retraction.*
(2') *The structure morphism $A \to B$ of any fppf-algebra has a retraction.*
(3) *The functor*

$$\mathrm{Sch} \longrightarrow \mathrm{Set},\ X \longmapsto \mathrm{Hom}(\mathrm{Spec}(A), X)$$

maps fppf coverings to jointly surjective families. That is, the canonical map $\coprod_i \mathrm{Hom}(\mathrm{Spec}(A), X_i) \to \mathrm{Hom}(\mathrm{Spec}(A), X)$ is surjective for any fppf covering $(X_i \to X)_i$.

Furthermore, for any scheme S and any morphism $\mathrm{Spec}(A) \to S$, the following statement is equivalent to the others:

(4) *The functor $\mathrm{Sch}/S \to \mathrm{Set}, X \mapsto \mathrm{Hom}_S(\mathrm{Spec}(A), X)$ maps fppf coverings to jointly surjective families.*

Proof. The directions (1') $\Rightarrow$ (1), (2') $\Rightarrow$ (2), and (3) $\Leftrightarrow$ (4) are trivial.

For verifying (1) $\Rightarrow$ (2), let a basic fppf-algebra B over A be given. Writing $B \cong A[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$, we that the polynomial system "$f_1 = 0, \ldots, f_m = 0$" has the tautologous solution $([X_1], \ldots, [X_n])$ in B. Since A is fppf-local, it therefore has a solution in A. Such a solution gives rise to an A-algebra homomorphism $A[X_1, \ldots, X_n]/(f_1, \ldots, f_m) \to A$, so to an retraction of the structure morphism $A \to B$.

The proof of the converse direction is similar: A solution of a polynomial system of equations in a basic fppf-algebra B can be transported along a retraction to yield a solution in A.

The directions (1') $\Rightarrow$ (2') and (2') $\Rightarrow$ (1') can be verified in exactly the same way.

We now verify (2) $\Rightarrow$ (3). Let $(f_i : X_i \to X)_i$ be an fppf covering and let a morphism $g : \mathrm{Spec}(A) \to X$ be given. We want to show that g factors over one of the morphisms f_i. Since the fppf topology is generated by Zariski coverings and singleton coverings $(T \to W)$ where $T \to W$ is a surjective finite locally free morphism [118, Tag 05WN], we may assume that the given covering $(X_i \to X)_i$ consists entirely of open immersions or is such a singleton covering.

In the first case, the morphism g lifts to one of the open subschemes X_i since the preimages $g^{-1}X_i$ cover $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A)$ is a local topological space.

In the second case, we may assume that W is affine and that f is not only finite locally free, but finite free. The left morphism in the pullback diagram

$$\begin{tikzcd}
\mathrm{Spec}(A) \times_W T \arrow[r] \arrow[d] & T \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & W
\end{tikzcd}$$

admits a section since it is the induced morphism on spectra of a basic fppf-algebra. The sought lift is then the composite $\mathrm{Spec}(A) \to \mathrm{Spec}(A) \times_W T \to T$.

Finally, we verify (3) $\Rightarrow$ (2'). Let an fppf-algebra B over A be given. The singleton family $(\mathrm{Spec}(B) \to \mathrm{Spec}(A))$ is an fppf covering. Therefore the identity morphism on $\mathrm{Spec}(A)$ lifts to $\mathrm{Spec}(B)$. This lift yields the desired retraction. $\square$

Remark 21.6. A scheme T such that the functor $\mathrm{Hom}(T, \_\_) : \mathrm{Sch} \to \mathrm{Set}$ maps Zariski coverings to jointly surjective families is already the spectrum of a local ring. Thus Proposition 21.5 implies that a scheme T such that $\mathrm{Hom}(T, \_\_)$ maps fppf coverings to jointly surjective families is the spectrum of an fppf-local ring.

Remark 21.7. The proof of Proposition 21.5 is intuitionistically valid, with the possible exception of the part (2) $\Rightarrow$ (3). This part of the proof crucially rests upon the description of the fppf topology given in the Stacks Project [118, Tag 05WN], which is proved in the usual language involving prime ideals and is therefore not obviously intuitionistically valid.

Lemma 21.8. (1) *The condition for a finitely presented algebra to be a basic fppf-algebra can be expressed as a geometric formula.*
(2) *The condition for a ring to be fppf-local can be expressed as a countable conjunction of geometric implications.*

Proof. A finitely presented A-algebra $B := A[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ is a basic fppf-algebra if and only if there exists a number $r \geq 1$, polynomials $g_1, \ldots, g_r : A[X_1, \ldots, X_n]$, vectors $v_1, \ldots, v_n : A^r$, a vector $u : A^r$, and vectors $w_{ij} : A^r$ for $i, j = 1, \ldots, r$ such that

- the multiplication defined on A^r by $(e_i, e_j) \mapsto w_{ij}$ is associative, commutative, and has u as neutral element,
- the map $B \to A^r$ given by $[X_k] \mapsto v_k$ is well-defined, that is $f_l(v_1, \ldots, v_n) = 0$ for $l = 1, \ldots, m$, and
- the map $B \to A^r$ and the map $A^r \to B$ sending e_i to $[g_i]$ are inverse to each other.

Each of these conditions can be expressed by geometric formulas involving the components of the data g_i, v_k, u, w_{ij}.

A ring A is fppf-local if and only if it is local, and for any numbers $n \geq 0, m \geq 0$ and polynomials $f_1, \ldots, f_m \in A[X_1, \ldots, X_n]$ the implication

$$\ulcorner A[X_1, \ldots, X_n]/(f_1, \ldots, f_m) \text{ is a basic fppf-algebra}\urcorner \Longrightarrow \exists x_1, \ldots, x_n : A.\ \bigwedge_{l=1}^{m} f_l(v_1, \ldots, v_n) = 0$$

holds. Since the antecedent can be expressed as a geometric formula, this formula is a geometric implication. $\square$

Theorem 21.9. *The big fppf topos of a scheme S is the largest subtopos of* $\mathrm{Zar}(S)$ *where $\underline{\mathbb{A}}^1_S$ is fppf-local.*

Proof. Let $\Box_{\mathrm{fppf}}$ be the modal operator associated to the fppf topology; the big fppf topos of S is the subtopos $\mathrm{Zar}(S)_{\Box_{\mathrm{fppf}}} \hookrightarrow \mathrm{Zar}(S)$. We verify that

$$\begin{aligned}\mathrm{Zar}(S) \models \bigwedge_{n\geq 0} \bigwedge_{m \geq 0} \forall f_1, \ldots, f_m : \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n].\ & \\ \ulcorner \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]/(f_1, \ldots, f_m) \text{ is a basic fppf-algebra}\urcorner \Longrightarrow & \\ \Box_{\mathrm{fppf}}(\exists x_1, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \bigwedge_{j=1}^{m} f_j(x_1, \ldots, x_n) = 0) &\end{aligned}$$

and we show that if $\Box$ is any modal operator with this property, then

$$\mathrm{Zar}(S) \models \forall \varphi : \Omega.\ \Box_{\mathrm{fppf}} \varphi \Rightarrow \Box \varphi.$$

For the first part we may assume, by Lemma 21.8, that $S = \mathrm{Spec}(A)$ is affine and that we're given polynomials $f_1, \ldots, f_m \in A[X_1, \ldots, X_n]$ such that $B := A[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ is a basic fppf-algebra. Then, trivially,

$$\mathrm{Spec}(B) \models \exists x_1, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \bigwedge_{j=1}^{m} f_j(x_1, \ldots, x_n) = 0.$$

Since $(\mathrm{Spec}(B) \to \mathrm{Spec}(A))$ is an fppf covering, we have

$$\mathrm{Zar}(S) \models \Box_{\mathrm{fppf}}(\exists x_1, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \bigwedge_{j=1}^{m} f_j(x_1, \ldots, x_n) = 0)$$

as claimed.

For the second part, let an fppf covering $(X_i \to X)_i$ be given such that $X_i \models \varphi$ for all i. We want to show that $X \models \Box\varphi$. Since the fppf topology is generated by Zariski coverings and singleton coverings $(T \to W)$ where $T \to W$ is a surjective finite locally free morphism [118, Tag 05WN] and the internal language of $\mathrm{Zar}(S)$ is (Zariski-)local, we may assume that the given covering is such a singleton covering. Moreover, we may assume that W is affine and that T is of the form $\Gamma(W, \mathcal{O}_W)[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$. Then

$$W \models \ulcorner \underline{\mathbb{A}}^1_S[X_1, \ldots, X_n]/(f_1, \ldots, f_m) \text{ is a basic fppf-algebra} \urcorner \text{ and}$$
$$W \models \forall x_1, \ldots, x_n \colon \underline{\mathbb{A}}^1_S.\ \Big(\bigwedge_{j=1}^{m} f_j(x_1, \ldots, x_n) = 0 \Big) \Rightarrow \varphi.$$

The latter is because for any $\Gamma(W, \mathcal{O}_W)$-algebra R such that there are elements $x_1, \ldots, x_n \in R$ with $f_j(x_1, \ldots, x_n) = 0$ for $j = 1, \ldots, m$, the structure morphism $\mathrm{Spec}(R) \to W$ factors over $T \to W$. The assumption on the modal operator $\Box$ implies $W \models \Box\varphi$. ☐

Corollary 21.10. *The big fppf topos of a scheme S is the classifying topos of the theory of fppf-local rings over S.*

Proof. The ring object $\underline{\mathbb{A}}^1_S$ of the big fppf topos of S is an fppf-local ring by Theorem 21.9. Equipped with the morphism $\flat\underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S$, it is thus an fppf-local ring over S.

Let $\mathcal{E}$ be an arbitrary cocomplete topos containing an fppf-local ring A over S. As detailed in Remark 16.4, this comprises an fppf-local ring A, a model of the theory which $\mathrm{Sh}(S)$ classifies (yielding a geometric morphism $f : \mathcal{E} \to \mathrm{Sh}(S)$), and a local homomorphism $\alpha : f^{-1}\mathcal{O}_S \to A$. By Theorem 16.1, this homomorphism gives rise to a unique geometric morphism $g : \mathcal{E} \to \mathrm{Zar}(S)$ over $\mathrm{Sh}(S)$ such that $g^{-1}\underline{\mathbb{A}}^1_S \cong A$ and such that the induced morphism $g^{-1}(\flat\underline{\mathbb{A}}^1_S) \to g^{-1}\underline{\mathbb{A}}^1_S$ coincides with α.

The geometric morphism g factors over the inclusion of the big fppf topos by Theorem 21.9, yielding a geometric morphism from $\mathcal{E}$ to the big fppf topos such that the pullback of $\underline{\mathbb{A}}^1_S$ coincides with A as rings over S.

Uniqueness of the geometric morphism constructed in this way follows already from the universal property of the big Zariski topos. ☐

Corollary 21.11. *The points of the big fppf topos of a scheme S are in canonical one-to-one correspondence with the fppf-local rings over S, that is fppf-local rings A equipped with a morphism* $\mathrm{Spec}(A) \to S$.

Proof. By Proposition 16.3, the points of $\mathrm{Zar}(S)$ are in canonical one-to-one correspondence with the local rings over S. Such a point is contained in the big fppf topos (that is, the associated geometric morphism $\mathrm{Set} \to \mathrm{Zar}(S)$ factors over the inclusion of the big fppf topos) if and only if the underlying local ring is fppf-local. ☐

We wish to record some algebraic and logical facts about fppf-local rings, hoping that they entice the reader to tackle the question whether any algebraically closed ring is already fppf-local.

- For any prime ideal $\mathfrak{p}$ of an algebraically closed ring, it holds that $\mathfrak{p}^2 = \mathfrak{p}$, since any element of $\mathfrak{p}$ possesses a square root.
- If an algebraically closed ring A is Noetherian, it is already a field, since by Krull's intersection theorem $(0) = \bigcap_{n \geq 0} \mathfrak{m}_A^n = \mathfrak{m}_A$.
- Algebraically closed fields K are fppf-local: Let $K \to B$ be a basic fppf-algebra. Since $B \neq 0$, there exists a maximal ideal $\mathfrak{n} \subseteq B$. The quotient ring $B/\mathfrak{n}$ is an algebraic extension of K. Since K is algebraically closed,

the identity morphism $K \to K$ can be extended to a morphism $B/\mathfrak{n} \to K$ of K-algebras, yielding the desired retraction. This statement can also be verified intuitionistically, see Proposition 21.21 below.

- Since the condition that a ring is fppf-local is a conjunction of geometric implications, a proof that any algebraically closed ring is fppf-local using classical logic and the axiom of choice implies that (nonconstructively) there also exists an intuitionistic proof of this statement.
- The proof of Lemma 21.8 yields a way to phrase fppf-locality of a local ring in the language of linear algebra: A local ring A is fppf-local if and only if for any number $n \geq 1$ and any pairwise commuting $(n \times n)$-matrices $W_1, \ldots, W_n$ over A such that the first matrix is the identity matrix and such that the i-th row of W_j coincides with the j-th row of W_i, there is a common eigenvector of the matrices $W_1, \ldots, W_n$ whose first component is the unit of A.

  An equivalent condition is the following: A local ring A is fppf-local if and only if for any matrices $W_1, \ldots, W_n$ as above there are elements $\lambda_1, \ldots, \lambda_n \in A$ such that $\lambda_1 = 1$ and such that $W_i \cdot (\lambda_1, \ldots, \lambda_n)^T = \lambda_i \cdot (\lambda_1, \ldots, \lambda_n)^T$.

  The condition is nontrivial only for $n \geq 3$.
- In order to show that any algebraically closed local ring is fppf-local, it would suffice to show that commuting matrices over algebraically closed local rings admit a common nontrivial eigenvector. However, we suspect that this stronger statement is false, since it would imply that any matrix over an algebraically closed local ring admits a nontrivial eigenvector.

  If there is a classical proof of this statement, then there is also a constructive proof; however, there is probably no such constructive proof, since it would imply that, internally to any topos, matrices over algebraically closed local rings admit a nontrivial eigenvector. Since the complex numbers (constructed using Cauchy reals) form an algebraically closed local ring [107, Theorem 3.13], this would imply that complex matrices admit nontrivial eigenvectors. Since eigenvectors are in general uncomputable [137, Proposition 12], there is probably a suitable realizability topos in which this statement fails.

21.3. The fpqc topology.

Definition 21.12. The *big fpqc topos* of a scheme S is the topos of sheaves over the category $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ of locally finitely presented S-schemes, where a family $(f_i : X_i \to X)_i$ of morphisms is deemed a covering if and only if the morphisms f_i are flat, jointly surjective, and each affine open subset of X is the union of the images of finitely many affine open subsets of some of the X_i.

Every fppf covering is an fpqc covering. Conversely, since we employ the parsimonious sites, every fpqc covering is an fppf covering. Therefore in our setup there is no difference between the big fppf topos and the big fpqc topos.

21.4. The ph topology. The goal of this section is to give an explicit description of the theory which the big ph topos of a scheme classifies, conditional on a conjecture we didn't prove.

Definition 21.13.
1. A *standard ph covering* of an S-scheme X is a family of the form $(U_i \hookrightarrow T \to X)_i$, where $T \to X$ is a proper surjective morphism and $T = \bigcup_i U_i$ is an open affine covering.
2. A family $(f_i : X_i \to X)_i$ of morphisms between locally finitely presented S-schemes if a *covering for the ph topology* if and only if for any affine open subset $U \subseteq X$ the family $(X_i \times_X U \to U)_i$ can be refined by a standard ph covering.

(3) The *big ph topos* $\mathrm{Zar}(S)_{\mathrm{ph}}$ of a scheme S is the topos of sheaves over the category $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ of locally finitely presented S-schemes equipped with the ph coverings.

The ph topology is not subcanonical, therefore we mean by "the affine line in the big ph topos" the sheafification of $\underline{\mathbb{A}}^1_S$. We write "$\Box_{\mathrm{ph}}$" for the modal operator of $\mathrm{Zar}(S)$ corresponding to the ph topology.

Definition 21.14. A *valuation ring* is an integral domain R in the weak sense (see Definition 3.16) such that, for any elements $a, b : R$, $a \mid b$ or $b \mid a$.

With this definition, valuation rings are local rings.

Proposition 21.15. *From the point of view of* $\mathrm{Zar}(S)_{\mathrm{ph}}$*, the affine line is an algebraically closed valuation ring and a field in the sense that nonzero elements are invertible.*

Proof. Since $\mathrm{Zar}(S)_{\mathrm{ph}}$ is a subtopos of $\mathrm{Zar}(S)_{\mathrm{fppf}}$, the affine line is algebraically closed (and even fppf-local) from the internal point of view of $\mathrm{Zar}(S)$. The proof that nonzero elements are invertible proceeds just as in Proposition 14.3, exploiting that an S-scheme for which the empty family is a covering is empty.

To show that $\underline{\mathbb{A}}^1_S$ is an integral domain in the weak sense, we employ the ph covering $(V(a) \to X, V(b) \to X)$ for S-schemes X and functions $a, b \in \Gamma(X, \mathcal{O}_X)$ such that $ab = 0$.

To show that for any given functions $a, b \in \Gamma(X, \mathcal{O}_X)$ one divides the other, we employ the ph covering $(\underline{\mathrm{Proj}}_X\, \mathcal{O}_X[U, V]/(bU - aV) \to X)$. This covering ensures that, from the internal point of view, the set $\{[u : v] \mid bu - av = 0\}$ is inhabited. If u is invertible, then $a \mid b$; if v is invertible, then $b \mid a$. $\Box$

Proposition 21.16. *The affine line of* $\mathrm{Zar}(S)$ *has the following closure property: Any finite system of homogeneous polynomial equations which the projective Nullstellensatz predicts to have a nontrivial solution* $\Box_{\mathrm{ph}}$*-has a solution. Formally,*

$$\begin{aligned} \mathrm{Zar}(S) \models \bigwedge_{n,m\geq 0} & \forall f_1, \ldots, f_m : \underline{\mathbb{A}}^1_S[X_0, \ldots, X_n]. \\ & \Big(\ulcorner\text{the } f_j \text{ are homogeneous}\urcorner \wedge \neg\big((X_0, \ldots, X_n) \subseteq \sqrt{(f_1, \ldots, f_m)}\big)\Big) \Longrightarrow \\ & \qquad \Box_{\mathrm{ph}}\Big(\exists x_0, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \bigvee_{i=0}^{n} \ulcorner x_i \text{ inv.}\urcorner \wedge \bigwedge_{j=1}^{m} f_j(x_0, \ldots, x_n) = 0\Big). \end{aligned}$$

Proof. Let homogeneous polynomials $f_1, \ldots, f_m \in \Gamma(T, \mathcal{O}_T)[X_0, \ldots, X_n]$ be given such that

$$T \models \neg\big((X_0, \ldots, X_n) \subseteq \sqrt{(f_1, \ldots, f_m)}\big).$$

Then the projection morphism $T' := \underline{\mathrm{Proj}}_T\, \mathcal{O}_T[X_0, \ldots, X_n]/(f_1, \ldots, f_m) \to T$ is surjective. Since

$$T' \models \exists x_0, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \bigvee_{i=0}^{n} \ulcorner x_i \text{ inv.}\urcorner \wedge \bigwedge_{j=1}^{m} f_j(x_0, \ldots, x_n) = 0$$

and since $(T' \to T)$ is a ph covering, it follows that

$$T \models \Box_{\mathrm{ph}}\Big(\exists x_0, \ldots, x_n : \underline{\mathbb{A}}^1_S.\ \bigvee_{i=0}^{n} \ulcorner x_i \text{ inv.}\urcorner \wedge \bigwedge_{j=1}^{m} f_j(x_0, \ldots, x_n) = 0\Big). \qquad \Box$$

To proceed further, we need to assume the following statement.

Conjecture 21.17. For any natural numbers n and m and any finite set of variables organized to yield the coefficients of m homogeneous polynomials $f_1, \ldots, f_m$ in $n+1$ variables, there is a geometric formula $\alpha(f_1, \ldots, f_m)$ which is equivalent to the formula

$$\neg\Big(\bigwedge_{i=0}^{n} X_i \in \sqrt{(f_1, \ldots, f_m)}\Big)$$

in the intuitionistic first-order theory of algebraically closed valuation rings which are fields in the sense that an element is invertible if and only if its nonzero.

We believe that this conjecture can be verified by using classical resultant theory. For instance, the conjecture holds in the easy case of two linear homogeneous polynomials in two variables, where $\alpha(f,g)$ can be chosen as $\mathrm{Res}(f,g) = 0$.

Proposition 21.18. *Let S be a locally Noetherian scheme. Assuming the existence of geometric formulas $\alpha(f,g)$ as in Conjecture 21.17, the big ph topos of S classifies algebraically closed valuation rings with the extra property that*

$$\alpha(f_1, \ldots, f_m) \Longrightarrow \exists x_0, \ldots, x_n.\ \bigvee_{i=0}^{n} \ulcorner x_i \text{ inv.}\urcorner \wedge \bigwedge_{j=1}^{m} f_j(x_0, \ldots, x_n) = 0.$$

Proof. We show that $\mathrm{Zar}(S)_{\mathrm{ph}}$ is the largest subtopos of $\mathrm{Zar}(S)$ where $\underline{\mathbb{A}}^1_S$ has the properties mentioned in the statement of the proposition. The claim then follows in the same way as Corollary 21.10 follows from Theorem 21.9.

Applying Proposition 21.16, one can see that the affine line has these properties internally in $\mathrm{Zar}(S)_{\mathrm{ph}}$.

Conversely, let a modal operator $\Box$ be given such that $\underline{\mathbb{A}}^1_S$ has these properties in $\mathrm{Zar}(S)_\Box$. We want to verify that $\mathrm{Zar}(S) \models \forall \varphi : \Omega.\ \Box_{\mathrm{ph}} \varphi \Rightarrow \Box\varphi$. Let a ph covering $(X_i \to X)_i$ of an S-scheme X be given such that $X_i \models \varphi$ for all i. We are to show that $X \models \Box\varphi$.

Without loss of generality, we may assume that X is affine and that the covering is a standard ph covering. Furthermore, we may assume that the covering is a singleton covering $(T \to X)$ where $T \to X$ is a proper surjective morphism. By Chow's lemma, we can further assume that $T \to X$ is the canonical projection of a closed subscheme of some projective space $\mathbb{P}^n_X$ to X. Since X is locally Noetherian, the defining sheaf of ideals is of finite type. Thus we may assume $T = \underline{\mathrm{Proj}}_X\, \mathcal{O}_X[U_0, \ldots, U_n]/(f_1, \ldots, f_m)$. Since $T \models \varphi$, we have

$$X \models \ulcorner\text{the system “}f_1 = \cdots = f_m = 0\text{” has a nontrivial solution}\urcorner \Rightarrow \varphi.$$

Since $T \to X$ is surjective, we have

$$X \models \neg\Big(\bigwedge_{i=0}^{n} U_i \in \sqrt{(f_1, \ldots, f_m)}\Big)$$

and therefore $X \models \Box(\alpha(f_1, \ldots, f_m))$ (we might not have $X \models \alpha(f_1, \ldots, f_m)$, since the proof of equivalence may assume that $\underline{\mathbb{A}}^1_S$ is an algebraically closed valuation ring, which $\underline{\mathbb{A}}^1_S \in \mathrm{Zar}(S)$ is only in the pathological case $S = \emptyset$). Combining these, we see that $X \models \Box\varphi$. $\Box$

A corollary of Proposition 21.18 (assuming Conjecture 21.17) is that the points of the big ph topos are those algebraically closed valuation rings over S which satisfy the condition on solvability of systems of homogeneous polynomial equations. Goodwillie and Lichtenbaum have already determined the points of the big ph topos to be the valuation rings with algebraically closed field of fractions over S [56, Proposition 2.2], without any extra condition on solvability of systems of equations.

This mismatch can be explained as follows. In classical logic one can show, using the valuative criterion for properness, that the rings studied by Goodwillie and Lichtenbaum automatically satisfy the condition on solvability of systems of equations. However, we don't believe that the proof can be made intuitionistic. Assuming this, it's no surprise that the theory which the big ph topos classifies contains further axioms.

21.5. The surjective topology.

Definition 21.19. A family $(f_i : X_i \to X)_i$ of morphisms between locally finitely presented S-schemes is a *covering for the surjective topology* if and only if the morphisms f_i are jointly surjective and each affine open subset of X is the union of the images of finitely many affine open subsets of some of the X_i.

Equivalently, a family $(X_i \to X)_i$ is a covering for the surjective topology if and only if any affine open subset of X is the image of a quasicompact open subset under the induced morphism $\coprod_i X_i \to X$.

Definition 21.20. An *algebraically closed geometric field* is a ring such that $1 \neq 0$, any element is zero or invertible, and that any monic polynomial of positive degree has a zero.

In contrast with other field conditions in intuitionistic mathematics, the condition for a ring to be an algebraically closed geometric field is the (countable conjunction of) geometric implications.

Proposition 21.21. *Intuitionistically, an algebraically closed geometric field is fppf-local.*

Proof. Over geometric fields, the kernel of any matrix admits a finite basis. Moreover, the kernel of any matrix of determinant zero contains a vector which has at least one invertible component. Therefore the usual proof that commuting matrices admit a common eigenvector (with at least one invertible component) applies. This fact can be used to show that the linear algebra problem stated on page 192 which characterizes fppf-locality is solvable. $\square$

Theorem 21.22. *The topos of sheaves over* $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ *for the surjective topology is the largest subtopos of* $\mathrm{Zar}(S)$ *where* $\underline{\mathbb{A}}^1_S$ *is an algebraically closed geometric field.*

Proof. Let $\Box_{\mathrm{surj}}$ be the modal operator associated to the surjective topology. We verify that

$$\mathrm{Zar}(S) \models \big(\forall s : \underline{\mathbb{A}}^1_S.\ \Box_{\mathrm{surj}}(s = 0 \vee \ulcorner s \text{ inv.}\urcorner)\big) \wedge \bigwedge_{n \geq 0} \forall a_0, \ldots, a_{n-1} : \underline{\mathbb{A}}^1_S.\ \Box_{\mathrm{surj}}(\exists x : \underline{\mathbb{A}}^1_S.\ x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0)$$

and we show that if $\Box$ is any modal operator with this property, then

$$\mathrm{Zar}(S) \models \forall \varphi : \Omega.\ \Box_{\mathrm{surj}}\varphi \Rightarrow \Box\varphi.$$

For the first part it suffices to prove the following two statements: If $s \in A$ is an element of a ring, then there is a covering $(X_i \to \mathrm{Spec}(A))_i$ for the surjective topology such that, for each i, $X_i \models s = 0$ or $X_i \models \ulcorner s \text{ inv.}\urcorner$. If $p \in A[X]$ is a monic polynomial of positive degree over a ring, then there is a covering $(X_i \to \mathrm{Spec}(A))_i$ for the surjective topology such that, for each i, $X_i \models \exists x : \underline{\mathbb{A}}^1_S.\ p(x) = 0$.

For the first claim, we may use the covering $(D(s) \to \mathrm{Spec}(A), V(s) \to \mathrm{Spec}(A))$. For the second claim, we may use the singleton covering $(\mathrm{Spec}(A[X]/(p)) \to \mathrm{Spec}(A))$.

For the second part, let a covering $(X_i \to X)_i$ of the surjective topology be given such that $X_i \models \varphi$ for each i. We want to show that $X \models \Box\varphi$. Without loss of generality, we may assume that X is affine and that the given covering is a singleton covering $(Y \to X)$ where Y is affine and therefore $Y \to X$ is of finite presentation.

By the lemma on the existence of a flattening stratification [118, Tag 0ASY], there exist finitely many locally closed subschemes $E_j = D(f_j) \cap V(g_{j1}, \ldots, g_{j,m_j}) \subseteq X$ such that the pullback $Y_j := Y \times_X E_j \to E_j$ is flat (and, being surjective, therefore faithfully flat). Since $Y_j \models \varphi$ and since $(Y_j \to E_j)$ is an fppf covering, we have $E_j \models \Box_{\mathrm{fppf}}\varphi$. By Proposition 21.21 and Theorem 21.9, we also have $E_j \models \Box\varphi$.

It's easily checked that $E \models (\ulcorner f_j \text{ inv.}\urcorner \wedge g_{j1} = 0 \wedge \cdots \wedge g_{j,m_j} = 0) \Rightarrow \varphi$, for each j. To conclude that $X \models \Box\varphi$, it therefore suffices to verify that

$$X \models \Box\Big(\bigvee_{j=1}^{n} (\ulcorner f_j \text{ inv.}\urcorner \wedge g_{j1} = 0 \wedge \cdots \wedge g_{j,m_j} = 0)\Big).$$

This claim follows from a basic combinatorial lemma[41] and the elementary reformulation of the statement that $X = \bigcup_{j=1}^{n} E_j$: For any finite subset $J \subseteq \{1, \ldots, n\}$ and any indices $k_j \in \{1, \ldots, m_j\}$ for $j \in J$,

$$\prod_{j \in J} g_{j,k_j} \in \sqrt{(f_j)_{j \in \{1,\ldots,n\} \setminus J}}. \qquad \Box$$

Corollary 21.23. *The topos of sheaves over* $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ *for the surjective topology is the classifying topos of algebraically closed geometric fields over S. The points of that topos are the algebraically closed geometric fields over S.*

Proof. Follows from Theorem 21.22 in the same way as Corollary 21.10 and Corollary 21.11 follow from Theorem 21.9. $\Box$

21.6. The double negation topology. As in Section 6.3, let $\mathrm{Zar}(S)_{\neg\neg}$ be the smallest dense subtopos of $\mathrm{Zar}(S)$ (defined using the parsimonious sites). It is the topos of sheaves over $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ for the double negation topology and Boolean. The following proposition describes the double negation topology in explicit terms.

Proposition 21.24. *The subtopos* $\mathrm{Zar}(S)_{\neg\neg}$ *can be presented as the topos of sheaves over the site* $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ *whose covering families* $(X_i \to X)_i$ *are precisely the families such that*

$$T \times_X X_i = \emptyset \text{ for all } i \quad \textit{implies} \quad T = \emptyset \qquad (\star)$$

for all locally finitely presented X-schemes T.

In case that S is locally of finite type over a field, condition $(\star)$ is satisfied if and only if for every closed point $x \in X$ there is a finite field extension $K \mid k(x)$ such that x has a K-valued preimage in one of the X_i.

Proof. For the first claim, we note that by a general fact $\mathrm{Zar}(S)_{\neg\neg}$ can be presented as the topos of sheaves over $(\mathrm{Sch}/S)_{\mathrm{lfp}}$ whose covering families $(f_i : X_i \to X)_i$ are precisely the families such that $(\underline{X_i}^{++} \to \underline{X}^{++})_i$ is a jointly epimorphic family in $\mathrm{Zar}(S)_{\neg\neg}$, where $(\_\_)^{++} : \mathrm{Zar}(S) \to \mathrm{Zar}(S)_{\neg\neg}$ is the sheafification functor.

[41] Let statements ψ_{jk} where $j = 1, \ldots, n$, $k = 1, \ldots, r_j$ be given. We picture this situation as a ragged matrix of statements with ψ_{jk} located at column k of row j. Assume that, for any selection of one statement from each row, at least one of the selected statements holds. Then there is a row all of whose statements hold.
The usual proof of this lemma proceeds by contradiction: If no row contains only true statements, then on each row there is some statement which is false. Since we want to use this lemma in the topos $\mathrm{Zar}(S)_{\Box}$, it's important that there is also an intuitionistic proof. Such a proof can, for instance, proceed by induction. The base case $n = 1$ is trivial. In the induction step $n \to n + 1$, we apply the induction hypothesis to the statements $\psi'_{jk} := (\psi_{jk} \vee \bigwedge_{l=1}^{r_{n+1}} \psi_{n+1,l})$ where $j = 1 \ldots, n$, $k = 1, \ldots, r_j$.

This is the case if and only if the morphisms $\underline{X_i}^{++} \to \underline{X}^{++}$ are jointly surjective from the internal point of view of $\mathrm{Zar}(S)_{\neg\neg}$. By the generalization of Theorem 6.31 from locales to toposes, this is the case if and only if

$$\mathrm{Zar}(S) \models \forall x : \underline{X}.\ \neg\neg\big(\bigvee_i \exists u : \underline{X_i}.\ \underline{f_i}(u) = x\big).$$

As detailed in Section 16.3, this is the case if and only if

$$\mathrm{Zar}(X) \models \neg\neg\big(\bigvee_i \ulcorner \underline{X_i} \text{ is inhabited} \urcorner\big).$$

Similarly to the (easy part of) Proposition 19.37, this in turn is equivalent to condition $(\star)$.

The second claim follows from Proposition 17.1. □

Proposition 21.25. *Let p be a point of* $\mathrm{Zar}(S)$. *By Proposition 16.3, the point p corresponds to a local ring A over S. If p is even a point of* $\mathrm{Zar}(S)_{\neg\neg}$, *then A is an algebraically closed geometric field.*

Proof. By Proposition 18.27 and Proposition 18.32, the $\neg\neg$-translation of "$\underline{\mathbb{A}}^1_S$ is an algebraically closed geometric field" holds in $\mathrm{Zar}(S)$. Therefore $\underline{\mathbb{A}}^1_S$ is an algebraically closed geometric field from the internal point of view of $\mathrm{Zar}(S)_{\neg\neg}$. Since being an algebraically closed geometric field is a (conjunction of) geometric implications, this property is preserved under pullback along geometric morphisms. This suffices to establish the claim. □

Since $\underline{\mathbb{A}}^1_S$ is an algebraically closed geometric field from the internal point of view of $\mathrm{Zar}(S)_{\neg\neg}$, Theorem 21.22 implies that $\mathrm{Zar}(S)_{\neg\neg}$ is a subtopos of the subtopos $\mathrm{Zar}(S)_{\mathrm{surj}}$.[42]

These toposes don't coincide, however, for $\mathrm{Zar}(S)_{\neg\neg}$ is Boolean while $\mathrm{Zar}(S)_{\mathrm{surj}}$ is not. One way to see this is to appeal to the syntactic characterization of when the classifying topos of a coherent theory is Boolean [67, Theorem D3.4.6]. In the coherent theory of an algebraically closed geometric field (which is the theory which $\mathrm{Zar}(S)_{\mathrm{surj}}$ classifies in the special case $S = \mathrm{Spec}(\mathbb{Z})$), the formulas "$p \cdot 1 = 0$" for prime numbers p are pairwise non-equivalent. By the cited characterization, the existence of an infinite family of pairwise non-equivalent formulas is sufficient to ensure that the classifying topos is not Boolean.

This observation settles a question by Madore [87, entry 2002-03-16:036].

We don't know much more about $\mathrm{Zar}(S)_{\neg\neg}$. Far from knowing which theory $\mathrm{Zar}(S)_{\neg\neg}$ classifies, we don't even have a full description of its points. In the case that S is $\mathrm{Spec}(\mathbb{Z})$, determining this theory would solve the purely algebro-logical problem of describing the Booleanization of the theory of a local ring.

22. Outlook

We believe that the approach of using internal language in algebraic geometry is already in its current form as developed in Part II and Part III useful to working algebraic geometers and interesting for topos theorists and logicians.

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[42] This can also be seen more directly by observing that $\mathrm{Zar}(S)_{\mathrm{surj}}$ is a dense subtopos of $\mathrm{Zar}(S)$. Because $\mathrm{Zar}(S)_{\neg\neg}$ is the smallest dense subtopos of $\mathrm{Zar}(S)$, this observation implies that $\mathrm{Zar}(S)_{\neg\neg}$ is a subtopos of $\mathrm{Zar}(S)_{\mathrm{surj}}$. Denseness of $\mathrm{Zar}(S)_{\mathrm{surj}}$ can be checked by verifying that the internal statement $\mathrm{Zar}(S) \models \neg\Box_{\mathrm{surj}}\bot$ holds; this amounts to verifying that a scheme which admits a cover in the surjective topology by empty schemes, is empty.

Further axioms for synthetic algebraic geometry. We showed in Section 18.4 and in Section 19 that from the single axiom

"The ring $\underline{\mathbb{A}}^1_S$ is local and synthetically quasicoherent."

a surprising number of properties can be deduced and that the core of a synthetic account for algebraic geometry, comprising notions such as affine schemes, open and closed immersions, surjective, universally closed and proper morphisms, and quasicompact synthetic schemes, can be built around it.

However, we still don't feel that we have a full understanding of the notion of quasicoherence.

Firstly, there are several properties of synthetically quasicoherent modules which we couldn't account for internally. For instance, the tensor product of synthetically quasicoherent modules is synthetically quasicoherent, but our only proof of this fact is external (Lemma 18.25). It doesn't seem prudent to just add statements like these as axioms on a case-by-case basis. We rather need to find further suitable general axioms from which these statements can be deduced.

Secondly, we know of no properties of $\underline{\mathbb{A}}^1_S$, formulated purely within the first-order language of rings, which wouldn't follow from the axiom that $\underline{\mathbb{A}}^1_S$ is local and synthetically quasicoherent. Have we just not looked hard enough, or does this observation have a deeper reason? Determining the first-order properties of the universal model of a geometric theory is a well-known problem; maybe having a look at this special case can shed light on this problem.

While speculating, we can just as well remark that the question can be generalized considerably:

Speculation 22.1. Let $\mathbb{T}$ be an equational theory. Let $\mathbb{T}'$ be a geometric theory obtained from $\mathbb{T}$ by adding further axioms. Then every first-order sequent which is true for the universal model U of $\mathbb{T}'$ can be deduced from the following quasicoherence condition: "For any finitely presented $\mathbb{T}$-model A, the canonical map

$$A \longrightarrow [[A, U]_{\mathrm{Mod}(\mathbb{T})}, U]$$

is bijective."

To make this speculation into a rigorous conjecture, we would have to specify what kind of deductions from the quasicoherence condition (which is a higher-order statement) are allowed.

Characterizing Zariski toposes among arbitrary toposes. When is an arbitrary topos the big Zariski topos of a scheme? How can this property be detected in the internal language, or in otherwise intrinsic terms? An answer to this question would also tell us which axioms should be stipulated for synthetic algebraic geometry.

Conversely, we proposed on page 152 a definition of what could be called the big Zariski topos of an arbitrary locally ringed locale S (or even a locally ringed topos): the classifying $\mathrm{Sh}(S)$-topos of the $\mathrm{Sh}(S)$-theory of local $\mathcal{O}_S$-algebras which are local over $\mathcal{O}_S$. Internal to any such topos, one can try to conduct synthetic algebraic geometry. Which properties of S, shared by all schemes, are needed for which results in synthetic algebraic geometry?

Proper morphisms. In Section 19.11, we characterized proper morphisms in the internal language of the big Zariski topos by mimicking the classical definition as a separated universally closed morphism (fulfilling a finiteness condition). Although we didn't tell so in Part II, a characterization along similar lines is possible in the internal language of the little Zariski topos.

However, the synthetic approach should facilitate a characterization which is closer to the intuitive way of thinking about proper morphisms: that any one-parameter family has a unique limit. This intuition can be formally expressed by the valuative criterion for properness, which states that a morphism $X \to S$ of schemes is proper if and only if it is of finite type and if for every valuation ring A with field of fractions K, any solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

can be filled by a unique dashed morphism. It is evident from this formulation of the valuative criterion that it depends only on the functor of points of X; therefore it should be perfectly suited to internalization in the big Zariski topos of S.

However, as of yet, we failed to do so. One difficulty is that we don't know of a description of the sheaf of rational functions in the internal language of the big Zariski topos. We showed in Section 9 that $\mathcal{K}_S$ is, from the point of view of the little Zariski topos, just the localization of $\mathcal{O}_S$ at the filter of regular elements. This description can't be carried over to the big Zariski topos: Since regular elements of $\underline{\mathbb{A}}^1_S$ are already invertible by Proposition 18.27, the localization of $\underline{\mathbb{A}}^1_S$ at the filter of regular elements is just $\underline{\mathbb{A}}^1_S$.

Moreover, depending on the site used, the synthetic spectrum of $\mathcal{K}_S^{\mathrm{Zar}}$ can be empty from the internal point of view. A specific example is given by $S = \mathrm{Spec}(\mathbb{Z})$ and the parsimonious sites. In this case, the synthetic spectrum of $\mathcal{K}_S^{\mathrm{Zar}}$ coincides with the functor of points of the $\mathbb{Z}$-scheme $\mathrm{Spec}(\mathbb{Q})$, which is empty from the internal point of view by Proposition 11.19.

A related problem is that we don't know how to properly describe punctured germs of curves. For instance, a candidate for what could be considered a germ of a curve is the synthetic spectrum of the formal power series ring $\underline{\mathbb{A}}^1_S[\![T]\!]$. This spectrum is canonically isomorphic to the subset of $\underline{\mathbb{A}}^1_S$ consisting of those elements which are *not not* zero.[43] Removing the origin from this subset yields, however, the empty set, and not some nontrivial punctured germ.

We strongly believe that these hurdles can be overcome and that there exists an characterization of proper maps in the big Zariski topos which is close to geometric intuition.

Cohomology, intersection theory, derived categories. Real algebraic geometry starts where the basics of scheme theory end. Therefore there should be an internal treatment of advanced tools like cohomology, intersection theory, and bounded derived categories of coherent sheaves. Such an account would be particularly interesting in the big Zariski topos, where the largest simplications can be expected.

Stacks and other kinds of generalized schemes. These notes only deal with classical 1-categorical schemes. An internal treatment of higher stacks and derived schemes is also desirable; it would probably rest upon a version of homotopy type theory as the internal language of ∞-toposes.

If one wants to stay in the 1-categorical setting, then one could extend the internal language to the various accounts of scheme theory over the field with one element [96].

[43]Given an element $a : \underline{\mathbb{A}}^1_S$ which is *not not* zero, the evaluation map $\underline{\mathbb{A}}^1_S[\![T]\!] \to \underline{\mathbb{A}}^1_S,\ f \mapsto f(a)$ is a well-defined element of $\mathrm{Spec}(\underline{\mathbb{A}}^1_S[\![T]\!])$ since a is nilpotent by Proposition 18.29. Conversely, Conversely, let $\varphi : \underline{\mathbb{A}}^1_S[\![T]\!] \to \underline{\mathbb{A}}^1_S$ be a homomorphism of $\underline{\mathbb{A}}^1_S$-algebras. If $\varphi(T)$ is invertible, then so is $\varphi(\varphi(T) - T) = \varphi(T) - \varphi(T) = 0$, since $\varphi(T) - T$ is invertible in $\underline{\mathbb{A}}^1_S[\![T]\!]$. Hence $\varphi(T)$ is not invertible and hence *not not* zero.

Applications to constructive algebra. We demonstrated in Section 11.5 using the example of Grothendieck's generic freeness lemma how the internal language of the little Zariski topos can be used to derive results in constructive algebra. Particularly useful is the property

$$\mathrm{Sh}(S) \models \forall f : \mathcal{O}_S.\ \neg\neg(f = 0) \Longrightarrow f = 0,$$

valid for reduced schemes, which allows us to use classical logic to some extent and still obtain intuitionistically valid results. The internal language is therefore a a tool for constructively pretending that a reduced ring is a local ring with $\neg\neg$-stable equality and the field property $\neg(\ulcorner f \text{ inv.}\urcorner) \Rightarrow f = 0$.

Any of the related toposes, such as the toposes of sheaves for related topologies such as the constructible or the flat topology, or the various subtoposes of the big Zariski topos, can be used for similar purposes. Which statements of constructive algebra profit from the internal language of theses toposes? Are there further useful toposes?

Constructive algebraic geometry. The book on homotopy type theory states [128, Section 3.4]:

> Thus, contrary to how it may appear on the surface, doing mathematics "constructively" does not usually involve giving up important theorems, but rather finding the best way to state the definitions so as to make the important theorems constructively provable. That is, we may freely use the [law of excluded middle] when first investigating a subject, but once that subject is better understood, we can hope to refine its definitions and proofs so as to avoid that axiom.

We believe that algebraic geometry has definitely reached the necessary maturity alluded to in this quote and that it will be a rich and interesting endeavor to try to give a completely constructive account of algebraic geometry, including nontrivial results of current interest. We sketched why one might be interested in such an account in Section 12.9.

The Stacks Project [118] already takes care to formulate statements in proper generality, not needlessly requiring Noetherian hypotheses or demanding that fields are algebraically closed. Having such a careful treatment is a very valuable first steps towards a fully constructive development. Additionally, a constructive account of commutative algebra is readily available [92, 83]; a constructive account of algebraic geometry is therefore entirely within reach.

Appendix

23. Dictionary relating external notions and notions internal to the little Zariski topos

External	Internal	Reference
Sheaves of sets		
sheaf of sets	set	
$\alpha : \mathcal{F} \to \mathcal{G}$ monomorphism	α injective	Ex. 2.3
$\alpha : \mathcal{F} \to \mathcal{G}$ epimorphism	α surjective	Ex. 2.3
$\operatorname{int}(X \setminus \operatorname{supp} \mathcal{F})$	truth value of "$\mathcal{F}$ is a singleton"	Rem. 4.11
$f : X \to \mathbb{N}$ upper semicont.	element of $\widehat{\mathbb{N}}$	Lemma 5.5
$f : X \to \mathbb{N}$ locally constant	element of $\mathbb{N}$	Lemma 5.5
Sheaves of rings		
sheaf of rings	ring	Prop. 3.1
local sheaf of rings	local ring	Prop. 3.5
X is reduced	$\mathcal{O}_X$ is reduced (and $\neg$invertible $\Rightarrow$ zero)	Prop. 3.3
$\dim X \leq n$	Krull dimension of $\mathcal{O}_X$ is $\leq n$	Prop. 3.13
X is integral at all points	$\mathcal{O}_X$ is a integral domain	Prop. 3.17
X is locally Noetherian	$\mathcal{O}_X$ is processly Noetherian	Prop. 3.24 (only "$\Rightarrow$" holds)
X is normal	$\mathcal{O}_X$ is normal (assuming that X is locally Noetherian)	Prop. 9.6
Sheaves of modules		
sheaf of modules	module	
$\mathcal{F}$ is finite locally free	$\mathcal{F}$ is finite free	Prop. 4.1
$\mathcal{F}$ is of finite type	$\mathcal{F}$ is finitely generated	Prop. 4.3
$\mathcal{F}$ is of finite presentation	$\mathcal{F}$ is finitely presented	Prop. 4.3
$\mathcal{F}$ is coherent	$\mathcal{F}$ is coherent	Prop. 4.3
$\mathcal{F}$ is quasicoherent	$\mathcal{F}[f^{-1}]$ is a sheaf wrt. ($\ulcorner f$ inv.$\urcorner \Rightarrow$ __) for $f : \mathcal{O}_X$	Thm. 8.3
$\mathcal{F}$ is flat	$\mathcal{F}$ is flat	Prop. 4.9
$\mathcal{F}$ is torsion	$\mathcal{F}$ is torsion	Prop. 4.12
$M^{\sim}$	$\underline{M}[\mathcal{F}^{-1}]$ (localization at generic filter)	Prop. 11.6
tensor product $\mathcal{F} \otimes \mathcal{G}$	tensor product $\mathcal{F} \otimes \mathcal{G}$	Prop. 4.7
dual $\mathcal{F}^{\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$	dual $\mathcal{F}^{\vee} = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$	

$\operatorname{int}(X \setminus \operatorname{supp}\mathcal{F})$	truth value of "$\mathcal{F} = 0$"	Prop. 4.10
quasicoherator of $\mathcal{I}$	$\{s : \mathcal{O}_X \mid \ulcorner s \text{ inv.}\urcorner \Rightarrow s \in \mathcal{I}\}$ ($\mathcal{I}$ a radical ideal)	Prop. 8.11
rank function of $\mathcal{F}$	minimal number of generators for $\mathcal{F}$	Prop. 5.6
Subspaces ($i : A \hookrightarrow X$ closed immersion, $j : U \hookrightarrow X$ open immersion)		
sheaf supported on A	$\Box$-sheaf, where $\Box = (\_\_ \vee A^c)$	Lemma 10.3
sheaf of the form $j_*(\mathcal{F})$	$\Box$-sheaf, where $\Box = (U \Rightarrow \_\_)$	
extension of $\mathcal{F}$ by the empty set	$j_!(\mathcal{F}) = \{x : \mathcal{F} \mid U\}$	Lemma 10.1
extension of $\mathcal{F}$ by zero	$j_!(\mathcal{F}) = \{x : \mathcal{F} \mid (x = 0) \vee U\}$	Lemma 10.2
sheaf with empty/zero stalks on U^c	sheaf of the form $j_!(\mathcal{F})$	
sections of $\mathcal{F}$ are equal if they agree on dense open	$\mathcal{F}$ is $\neg\neg$-separated	Prop. 6.15
sheaf of sections of $\mathcal{F}$ defined on dense open subsets	$\mathcal{F}^{++}$ with respect to $\Box = \neg\neg$ (assuming that $\mathcal{F}$ is $\neg\neg$-separated)	Prop. 6.15
U is dense	$\neg\neg U$	Prop. 6.5
U is scheme-theoretically dense	$\widehat{\Box}U$, i.e. $\mathcal{O}_X$ is separated wrt. $(U \Rightarrow \_\_)$	Lemma 9.11
$V(\mathcal{I})$ is reduced	$\mathcal{I}$ is a radical ideal	Lemma 10.6
$\mathcal{O}_{X_{\mathrm{red}}}$	$\mathcal{O}_X / \sqrt{(0)}$	Lemma 10.7
Rational functions and Cartier divisors		
$\mathcal{K}_X$	total quotient ring of $\mathcal{O}_X$	Prop. 9.1
Cartier divisor	element of $\mathcal{K}_X^\times / \mathcal{O}_X^\times$	
effective Cartier divisor	$[s/1]$ with $s : \mathcal{O}_X$ regular	Def. 9.21
line bundle $\mathcal{O}_X(D)$	$D^{-1}\mathcal{O}_X \subseteq \mathcal{K}_X$	Def. 9.23
Topological properties		
X is quasicompact	"$\mathrm{Sh}(X) \models$" commutes with directed disjunctions	Prop. 7.1
X is local	"$\mathrm{Sh}(X) \models$" commutes with arbitrary disjunctions	Prop. 7.7
X is irreducible	if $\neg(\varphi \wedge \psi)$, then $\neg\varphi$ or $\neg\psi$	Prop. 7.9

24. The inference rules of intuitionistic logic

Structural rules

$$\frac{}{\varphi \vdash_{\vec{x}} \varphi} \qquad \frac{\varphi \vdash_{\vec{x}} \psi}{\varphi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}]} \qquad \frac{\varphi \vdash_{\vec{x}} \psi \qquad \psi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \chi}$$

Rules for nullary and binary conjunction

$$\frac{}{\varphi \vdash_{\vec{x}} \top} \qquad \frac{}{\varphi \wedge \psi \vdash_{\vec{x}} \varphi} \qquad \frac{}{\varphi \wedge \psi \vdash_{\vec{x}} \psi} \qquad \frac{\varphi \vdash_{\vec{x}} \psi \qquad \varphi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \psi \wedge \chi}$$

Rules for nullary and binary disjunction

$$\frac{}{\bot \vdash_{\vec{x}} \varphi} \qquad \frac{}{\varphi \vdash_{\vec{x}} \varphi \vee \psi} \qquad \frac{}{\psi \vdash_{\vec{x}} \varphi \vee \psi} \qquad \frac{\varphi \vdash_{\vec{x}} \chi \qquad \psi \vdash_{\vec{x}} \chi}{\varphi \vee \psi \vdash_{\vec{x}} \chi}$$

Rules for arbitrary set-indexed conjunction and disjunction

$$\frac{}{\bigwedge_{i\in I}\varphi_i \vdash_{\vec{x}} \varphi_j \text{ for all } j\in I} \qquad \frac{\varphi \vdash_{\vec{x}} \psi_j \text{ for all } j\in I}{\varphi \vdash_{\vec{x}} \bigwedge_{i\in I}\psi_i}$$

$$\frac{}{\varphi_j \vdash_{\vec{x}} \bigvee_{i\in I}\varphi_i \text{ for all } j\in I} \qquad \frac{\varphi_j \vdash_{\vec{x}} \psi \text{ for all } j\in I}{\bigvee_{i\in I}\varphi_i \vdash_{\vec{x}} \psi}$$

Double rule for implication

$$\frac{\varphi\wedge\psi \vdash_{\vec{x}} \chi}{\varphi \vdash_{\vec{x}} \psi\Rightarrow\chi}$$

Double rules for bounded and unbounded quantification

$$\frac{\varphi \vdash_{\vec{x},y} \psi}{\exists y:Y.\,\varphi \vdash_{\vec{x}} \psi}\ (y \text{ not occurring in } \psi) \qquad \frac{\varphi \vdash_{\vec{x},y} \psi}{\varphi \vdash_{\vec{x}} \forall y:Y.\,\psi}\ (y \text{ not occurring in } \varphi)$$

$$\frac{\varphi \vdash_{\vec{x},Y} \psi}{\exists Y.\,\varphi \vdash_{\vec{x}} \psi}\ (Y \text{ not occurring in } \psi) \qquad \frac{\varphi \vdash_{\vec{x},Y} \psi}{\varphi \vdash_{\vec{x}} \forall Y.\,\psi}\ (Y \text{ not occurring in } \varphi)$$

Rules for equality

$$\frac{}{\top \vdash_x x=x} \qquad \frac{}{(\vec{x}=\vec{y})\wedge\varphi \vdash_{\vec{z}} \varphi[\vec{y}/\vec{x}]}$$

(“$\vec{x}=\vec{y}$” is short for “$x_1=y_1\wedge\cdots\wedge x_n=y_n$”.)

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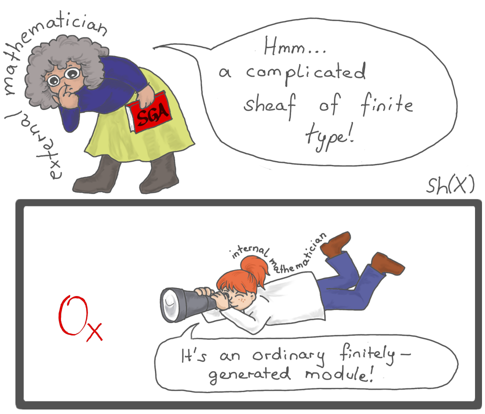

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