A GENERAL NULLSTELLENSATZ FOR GENERALIZED SPACES

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Abstract. We give a general Nullstellensatz for the generic model of a geometric theory, useful as a source of nongeometric sequents validated by the generic model, and characterize the first-order and higher-order formulas validated by the genetric model.

1. Introduction

Generic models. Let $\mathbb{T}$ be a geometric theory, such as the theory of rings, of local rings or of intervals. We follow Caramello's terminology [4] to mean by *geometric theory* a system given by a set of sorts, a set of finitary function symbols, a set of finitary relation symbols and a set of axioms, consisting of geometric sequents (sequents of the form $\varphi \vdash_{\vec{x}} \psi$ where φ and ψ are geometric formulas, that is formulas built from equality and the relation symbols by the logical connectives $\top \perp \wedge \vee \exists$ and by arbitrary set-indexed disjunctions $\bigvee$). By (infinitary) *first-order formula* we will mean a formula which may contain, additionally to the connectives allowed for geometric formulas, the connectives $\Rightarrow$ and $\forall$.

A fundamental result is that there is a *generic model* $U_{\mathbb{T}}$ of $\mathbb{T}$, a model such that for any geometric sequent σ, the following notions coincide:

(1) The sequent σ is provable modulo $\mathbb{T}$.
(2) The sequent σ holds for any $\mathbb{T}$-model in any Grothendieck topos.
(3) The sequent σ holds for $U_{\mathbb{T}}$.

One could argue that it is this model which we implicitly refer to when we utter the phrase "Let M be a $\mathbb{T}$-model.". It can typically not be realized as a set-theoretic model, consisting of a set for each sort, a function for each function symbol and so on; instead it is a model in a custom-tailored syntactically constructed Grothendieck topos, the *classifying topos* $\mathrm{Set}[\mathbb{T}]$ of $\mathbb{T}$, hence consists of an object of $\mathrm{Set}[\mathbb{T}]$ for each sort, a morphism for each function symbol and so on.

To state what it means for a $\mathbb{T}$-structure in a topos $\mathcal{E}$ to verify the axioms of $\mathbb{T}$, rendering it a model, the *internal language* of $\mathcal{E}$ is used, roughly reviewed in Section 2.1 below. We write "$\mathcal{E} \models \alpha$" to mean that a formula α holds from the internal point of view of $\mathcal{E}$. Since this language is a form of a higher-order intuitionistic extensional dependent type theory, the classifying topos $\mathrm{Set}[\mathbb{T}]$ can be regarded as a higher-order completion of the geometric theory $\mathbb{T}$. The generic model enjoys the universal property that any $\mathbb{T}$-model in any (Grothendieck) topos $\mathcal{E}$ is the pullback of $U_{\mathbb{T}}$ along an essentially unique geometric morphism $\mathcal{E} \to \mathrm{Set}[\mathbb{T}]$.

Nongeometric sequents. Crucially, the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) relating provability and truth in $\mathrm{Set}[\mathbb{T}]$ only pertains to geometric sequents. The generic model may validate additional nongeometric sequents which are not provable from the axioms of $\mathbb{T}$ in first-order or even higher-order logic, and these nongeometric sequents may be quite surprising and have useful consequences.

One of the most celebrated such sequents arises in the case that $\mathbb{T}$ is the theory of local rings. In this case, the classifying topos $\mathrm{Set}[\mathbb{T}]$ is also known as the *big Zariski topos* of $\mathrm{Spec}(\mathbb{Z})$ from algebraic geometry, the topos of sheaves over the site of schemes locally of finite presentation, and the generic model is the functor $\underline{\mathbb{A}}^1$ of points of the affine line, the functor which maps any (l.o.f.p.) scheme X to $\mathrm{Hom}(X, \mathbb{A}^1) = \mathcal{O}_X(X)$.

From the point of view of the topos, the ring object $\underline{\mathbb{A}}^1$ is not only a local ring, but even a field in the sense that any nonzero element is invertible. As this condition is of nongeometric form, it is not inherited by arbitrary local rings, which are indeed typically not fields. However, any intuitionistic consequence of this condition which is of geometric form is inherited by any local ring in any topos. Hence we may, when verifying a general fact about local rings which is expressible as a geometric sequent, suppose without loss of generality that the given ring is a field. This observation is due to Kock [6], who exploited it to develop projective geometry over local rings, and was further used by Reyes to prove a Jacobian criterion for étale morphisms [7].

A related nongeometric sequent is valid in the little Zariski topos of the spectrum of a ring A, the classifying topos of local localizations of A. If A is reduced, the generic model validates the dual condition that any noninvertible element is zero. This property has been used to give a short and even constructive proof of Grothendieck's generic freeness lemma, substantially improving on previously published proofs [2].

In time, further nongeometric sequents holding in the big Zariski topos of an arbitrary base scheme have been found [3, Section 18.4]. These include:

- $\underline{\mathbb{A}}^1$ is *anonymously algebraically closed* in the sense that any monic polynomial $p : \underline{\mathbb{A}}^1[T]$ of degree at least one does *not not* have a zero.
- The Nullstellensatz holds: Let $f_1, \ldots, f_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ be polynomials without a common zero in $(\underline{\mathbb{A}}^1)^n$. Then there are polynomials $g_1, \ldots, g_m \in \underline{\mathbb{A}}^1[X_1, \ldots, X_n]$ such that $\sum_i g_i f_i = 1$.
- Any function $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is given by a unique polynomial.
- $\underline{\mathbb{A}}^1$ is microaffine: Let $\Delta = \{\varepsilon : \underline{\mathbb{A}}^1 \mid \varepsilon^2 = 0\}$. Let $f : \Delta \to \underline{\mathbb{A}}^1$ be an arbitrary function. Then there are unique elements $a, b : \underline{\mathbb{A}}^1$ such that $f(\varepsilon) = a + b\varepsilon$ for all $\varepsilon : \Delta$.
- $\underline{\mathbb{A}}^1$ is *synthetically quasicoherent*: For any finitely presentable $\underline{\mathbb{A}}^1$-algebra A, the canonical homomorphism $A \to (\underline{\mathbb{A}}^1)^{\mathrm{Spec}(A)}$, where $\mathrm{Spec}(A)$ is defined as the set of $\underline{\mathbb{A}}^1$-algebra homomorphisms $A \to \underline{\mathbb{A}}^1$, is bijective.

All of these nongeometric sequents are useful for the purposes of synthetic algebraic geometry, the desire to carry out algebraic geometry in a language close to the simple language on the 19th and the beginning of the 20th century while still being fully rigorous and fully general, working over arbitrary base schemes instead of restricting to the field of complex numbers.

Characterizing nongeometric sequents. Referring to one of the previous examples, Tierney remarked around the time that those sequents were first studied that "[it] is surely important, though its precise significance is still somewhat obscure—as is the case with many such nongeometric formulas" [8, p. 209]. In view of their importance, is there a way to discover nongeometric sequents in a systematic fashion?

To characterize the nongeometric sequents holding in classifying toposes? To this end, Wraith put forward a specific conjecture [9, p. 336]:

> *The problem of characterising all the non-geometric properties of a generic model appears to be difficult. If the generic model of a geometric theory $\mathbb{T}$ satisfies a sentence α then any geometric consequence of $\mathbb{T} + \alpha$ has to be a consequence of $\mathbb{T}$. We might call α $\mathbb{T}$-redundant. Does the generic $\mathbb{T}$-model satisfy all $\mathbb{T}$-redundant sentences?*

This question was recently answered in the negative by Bezem, Buchholtz and Coquand [1]; hence the characterization we propose is necessarily more nuanced.

Our starting point was the empirical observation [3, p. 164] that in the case of the big Zariski topos, every true known nongeometric sequent followed from just a single such, namely the synthetic quasicoherence of the generic model, and in earlier work we surmised that one could formulate an appropriate metatheorem explaining this observation and generalizing it to arbitrary classifying toposes [3, Speculation 22.1]. This hope turned out to be true, in the sense we will now indicate.

A general Nullstellensatz. To explain the relevant background, the somewhat vague question "to which extent does the classifying topos $\mathrm{Set}[\mathbb{T}]$ realize that it is the classifying topos for $\mathbb{T}$?" is a useful guiding principle. This is easiest to visualize with a concrete example for $\mathbb{T}$, such as the theory of rings.

Let A be a ring. A simple version of the classical Nullstellensatz states: *For any polynomials f and g over A, if any zero of f is also a zero of g, then there is a polynomial h such that $g = hf$.* The polynomial h can be regarded as an "algebraic certificate" of the hypothesis. This principle holds for instance in the case that A is an algebraically closed field and g is the unit polynomial. We will see below that it is also true, without any restriction on g, for the generic ring.

We could try to generalize the Nullstellensatz to arbitrary geometric theories $\mathbb{T}$ as follows: *For any geometric sequent σ, if σ holds for a given $\mathbb{T}$-model M then σ is provable modulo $\mathbb{T}$.* In place of the algebraic certificate we now have a logical certificate, a *proof* of σ. However, this generalized statement is typically false, even for the generic model $U_{\mathbb{T}}$: The statement

$$\begin{aligned} \mathrm{Set}[\mathbb{T}] \models \ulcorner &\text{for any geometric sequent } \sigma, \\ &\qquad \text{if } \sigma \text{ holds for } U_{\mathbb{T}} \text{ then } \underline{\mathbb{T}} \text{ proves } \sigma \urcorner \end{aligned}$$

does not hold.[1] In this sense $\mathrm{Set}[\mathbb{T}]$ does not believe that $U_{\mathbb{T}}$ is the generic $\underline{\mathbb{T}}$-model.

A concrete counterexample is as follows. Let $\mathbb{T}$ be the theory of rings and let σ be the sequent $(\top \vdash 1 + 1 = 0)$. Since there is an intuitionistic proof that $\mathbb{T}$ does not prove σ and toposes are sound with respect to intuitionistic logic, the statement $\ulcorner \underline{\mathbb{T}}$ proves $\sigma \urcorner$ is false from the internal point of view of $\mathrm{Set}[\mathbb{T}]$. However, it is not the case that the statement $\ulcorner 1 + 1 = 0$ in $U_{\mathbb{T}} \urcorner$ is false from the internal

[1] Here $\underline{\mathbb{T}}$ is the internal geometric theory induced by $\mathbb{T}$, obtained by pulling back the set of sorts, the set of function symbols and so on along the geometric morphism $\mathrm{Set}[\mathbb{T}] \to \mathrm{Set}$. For instance, if $\mathbb{T}$ is the theory of rings, then from the internal point of view of $\mathrm{Set}[\mathbb{T}]$ the theory $\underline{\mathbb{T}}$ will again be the theory of rings. More details will be given in Section 2.2. The corner quotes indicate that for sake of readability, the translation into formal language is to be carried out by the reader.

The displayed statement is much stronger than the statement that for any geometric sequent σ, if $\mathrm{Set}[\mathbb{T}] \models \ulcorner \sigma$ holds for $U_{\mathbb{T}} \urcorner$ then $\mathbb{T}$ proves σ. This latter statement, where the universal quantifier and the "if ... then" have been pulled out, is true.

point of view. In fact, this statement holds in a nontrivial slice of $\mathrm{Set}[\mathbb{T}]$, the open subtopos coinciding with the classifying topos of the theory of rings of characteristic two.

Intuitively, the problem is that while the meaning of $\ulcorner \underline{\mathbb{T}}$ proves $\sigma \urcorner$ is fixed, the meaning of $\ulcorner \sigma$ holds for $U_{\mathbb{T}} \urcorner$ can vary with the slice. This problem can be solved by passing from $\underline{\mathbb{T}}$ to a varying theory, the internal theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ defined in Section 3. If $\mathbb{T}$ is the theory of rings, then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is the $\mathrm{Set}[\mathbb{T}]$-theory of $U_{\mathbb{T}}$-algebras. Unlike $\underline{\mathbb{T}}$, this theory is not the pullback of an external geometric theory. We then have, subject to some qualifications made precise in Section 3, the following general Nullstellensatz:

Theorem 1.1. *Let $\mathbb{T}$ be a geometric theory. Then, internally to* $\mathrm{Set}[\mathbb{T}]$*:*

$$\textit{A geometric}^{*} \textit{ sequent } \sigma \textit{ holds for } U_{\mathbb{T}} \textit{ if and only if } \underline{\mathbb{T}}/U_{\mathbb{T}} \textit{ proves}^{*}\ \sigma. \qquad (\ddagger)$$

To illustrate Theorem 1.1, let $\mathbb{T}$ be the theory of rings and let σ be the sequent $(f(x) = 0 \vdash_x g(x) = 0)$ for some polynomials f and g. To say that σ holds for $U_{\mathbb{T}}$ amounts to saying that any zero $x : U_{\mathbb{T}}$ of f is also a zero of g, and to say that $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves σ amounts to saying that in $U_{\mathbb{T}}[X]/(f(X))$, the free $U_{\mathbb{T}}$-algebra on one generator X subject to the relation $f(X) = 0$, the relation $g([X]) = 0$ holds. Hence we obtain

$$\mathrm{Set}[\mathbb{T}] \models \forall f, g : U_{\mathbb{T}}[X].\ \big((\forall x : U_{\mathbb{T}}.\ f(x) = 0 \Rightarrow g(x) = 0) \Longleftrightarrow \exists h : U_{\mathbb{T}}[X].\ g = hf\big).$$

The statement $(\ddagger)$ is not a geometric sequent. Therefore it is not to be expected that it passes from $\mathrm{Set}[\mathbb{T}]$ to a subtopos $\mathrm{Set}[\mathbb{T}']$ corresponding to a quotient theory $\mathbb{T}'$ of $\mathbb{T}$, and indeed in general it does not. However, there is still a useful substitute, which we formulate as Theorem 3.9. This substitute substantially broadens the scope of the Nullstellensatz.

Summarizing, the situation is as follows.

- The generic model $U_{\mathbb{T}}$ is a conservative $\mathbb{T}$-model.
- The topos $\mathrm{Set}[\mathbb{T}]$ does not believe that $U_{\mathbb{T}}$ is a conservative $\underline{\mathbb{T}}$-model.
- The topos $\mathrm{Set}[\mathbb{T}]$ does believe that $U_{\mathbb{T}}$ is a conservative* $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model.

Theorem 1.1 is a source of nongeometric sequents. Indeed, it is the universal such source in the sense that any first-order formula which holds for $U_{\mathbb{T}}$ can be deduced from $(\ddagger)$:

Theorem 1.2. *Let $\mathbb{T}$ be a geometric theory. Let α be a first-order formula over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) *The formula α holds for $U_{\mathbb{T}}$.*
(2) *The formula α is provable in first-order intuitionistic logic modulo the axioms of $\mathbb{T}$ and the additional axiom $(\ddagger)$.*

Theorem 1.2 characterizes the first-order formulas which hold for the generic model. We could of course wish for a more explicit characterization; but since even the characterization of geometric sequents holding for the generic model (they are precisely those which are provable in geometric logic modulo $\mathbb{T}$) is of a rather implicit nature, this wish appears unfounded.

We stress that our characterization is more explicit than the tautologous characterization ("a first-order formula holds for $U_{\mathbb{T}}$ iff it is provable modulo $\mathbb{T}'$, where $\mathbb{T}'$ is the first-order theory whose set of axioms is the set of first-order formulas satisfied by $U_{\mathbb{T}}$") and the (incorrect) characterization "a first-order formula holds for $U_{\mathbb{T}}$ iff it is $\mathbb{T}$-redundant". Indeed, if $\mathbb{T}$ happens to be coherent and recursively axiomatizable, then in stating Theorem 1.2 we may restrict to coherent existential fixed-point logic, and the resulting theory will again be recursively axiomatizable.

Acknowledgments. XXX

2. Background

2.1. Background on the internal language of Grothendieck toposes.

2.2. Background on internal geometric theories.

Given an internal signature Σ internal to a Grothendieck topos $\mathcal{E}$ (or elementary topos with a natural numbers object), we can successively build the object of contexts (the object of lists of sorts); the object of terms (equipped with a morphism to the object of contexts); the object of atomic propositions (again equipped with such a morphism); the object of geometric formulas (again so); the object of geometric sequents (again so); and, given an internal geometric theory $\mathbb{T}$ over Σ, the object of proof trees of $\mathbb{T}$.

From the internal point of view of $\mathcal{E}$, these objects can be obtained by simply carrying out the usual constructions of the set of contexts, the set of terms and so on. Having the object of proof trees at hand, we can define the notion of provability:

Definition 2.1. Let $\mathbb{T}$ be a geometric theory internal to a Grothendieck topos $\mathcal{E}$. Internally to $\mathcal{E}$, an element σ of the object of geometric sequents is *provable modulo* $\mathbb{T}$ if and only if there is an element of the object of proof trees of $\mathbb{T}$ which has σ as its conclusion.

Example 2.2. An ordinary geometric theory is the same as a geometric theory internal to the topos Set. A geometric sequent is provable in the ordinary sense if and only if, from the internal point of view of Set, it is provable in the sense of Definition 2.1.

Disjunctions appearing in internal geometric formulas may be indexed by arbitrary objects of the topos, just like disjunctions appearing in ordinary external geometric formulas over an ordinary signature may be indexed by arbitrary sets. If $\mathcal{E}$ is a Grothendieck topos, the object of geometric formulas over an internal signature Σ in $\mathcal{E}$ has an important subobject, the subobject of those formulas such that locally, any appearing disjunction is indexed by a constant sheaf. Such internal geometric formulas will be called *geometric** *formulas*.

Correspondingly, there is a subobject of the object of proof trees, the object of *proof** *trees* where any occurring geometric formula is a geometric* formula. The shape of such a proof* tree is locally given by the shape of an ordinary external proof tree.

Definition 2.3. Let $\mathbb{T}$ be a geometric theory internal to a Grothendieck topos $\mathcal{E}$. Internally to $\mathcal{E}$, an element σ of the object of geometric* sequents is *provable** *modulo* $\mathbb{T}$ if and only if there is an element of the object of proof* trees of $\mathbb{T}$ which has σ as its conclusion.

The following lemma shows that for coherent theories, there is no difference between provability and provability*.

Lemma 2.4. *Let $\mathcal{E}$ be a Grothendieck topos. Let $\mathbb{T}$ be a coherent theory internal to $\mathcal{E}$. Let σ be a coherent sequent over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of externally finite shape$\urcorner$.
(2) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of arbitrary external shape$\urcorner$, *that is* $\mathcal{E} \models \ulcorner\sigma$ is provable* modulo $\mathbb{T}\urcorner$.
(3) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of arbitrary internal shape$\urcorner$, *that is* $\mathcal{E} \models \ulcorner\sigma$ is provable modulo $\mathbb{T}\urcorner$.
(4) $\mathcal{E} \models \ulcorner$There is a $\mathbb{T}$-derivation of σ of internally finite shape$\urcorner$.

Proof. It is trivial that (1) implies (2) implies (3).

To verify that (3) implies (4), we can mimic the usual proof of this fact in the internal language of $\mathcal{E}$: There is a variant of the syntactic site of $\mathbb{T}$ which is built using only coherent sequents and finitary derivability [4, Section 1.4]. The sheaf topos over this site is another model for the classifying topos of $\mathbb{T}$, and still validates, like any Grothendieck topos, full infinitary logic. Hence, if σ is $\mathbb{T}$-derivable by a proof tree of arbitrary shape, then σ holds in this model of the classifying topos. By the analogue of Proposition 2.8 for this model, we obtain that σ is $\mathbb{T}$-derivable by a proof tree of finite shape.

That (4) implies (1) is a routine exercise exploiting that

$$\mathcal{E} \models \forall X.\ \ulcorner X \text{ is Kuratowski-finite}\urcorner \Leftrightarrow \bigvee_{n\geq 0} \exists x_1,\ldots,x_n : X.\ \forall x : X.\ \bigvee_{i=1}^{n} x = x_i. \quad \square$$

2.3. Background on classifying toposes.

Definition 2.5. The *syntactic site* $\mathcal{C}_\mathbb{T}$ of a geometric theory $\mathbb{T}$ has:

(1) as objects "geometric formulas in contexts" $\{x_1 : X_1, \ldots, x_n : X_n.\ \varphi\}$ where φ is a geometric formula over the signature of $\mathbb{T}$ in the displayed context;
(2) as set of morphisms $\mathrm{Hom}_{\mathcal{C}_\mathbb{T}}(\{\vec{x}.\ \varphi\}, \{\vec{y}.\ \psi\})$ the set of formulas θ in the context $\vec{x}, \vec{y}$ which are $\mathbb{T}$-provably functional, modulo $\mathbb{T}$-provable equivalence of such formulas;
(3) as covering families those families $(\{\vec{x}_i.\ \varphi_i\} \xrightarrow{\theta_i} \{\vec{y}.\ \psi\})_i$ for which $\mathbb{T}$ proves $(\psi \vdash_{\vec{y}} \bigvee_i \exists \vec{x}_i.\ \theta_i)$.

Definition 2.6. The *classifying topos* $\mathrm{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$ is the topos of set-valued sheaves on $\mathcal{C}_\mathbb{T}$.

Writing $よ : \mathcal{C}_\mathbb{T} \to \mathrm{Set}[\mathbb{T}]$ for the Yoneda embedding, the *generic model* $U_\mathbb{T}$ of $\mathbb{T}$ interprets a sort X of $\mathbb{T}$ as the sheaf $よ\{x : X.\ \top\}$, a function symbol $f : X_1 \cdots X_n \to Y$ as the morphism given by the $\mathbb{T}$-provably functional formula $f(x_1, \ldots, x_n) = y$ and a relation symbol $R \rightarrowtail X_1 \cdots X_n$ by the subobject $よ\{\vec{x}.\ R(\vec{x})\} \rightarrowtail よ\{\vec{x}.\ \top\}$.

Theorem 2.7. *The generic model is universal in the sense that for any Grothendieck topos $\mathcal{E}$, the functor*

$$\textit{(category of geometric morphisms } \mathcal{E} \to \mathrm{Set}[\mathbb{T}]\textit{)} \longrightarrow \textit{(category of } \mathbb{T}\textit{-models in } \mathcal{E}\textit{)}$$

given by $f \mapsto f^* U_\mathbb{T}$ is an equivalence of categories.

Proposition 2.8. *Let α and φ be geometric formulas in a context $\vec{x}$ over the signature of a geometric theory $\mathbb{T}$. Then the following statements are equivalent:*

(1) $\mathrm{Set}[\mathbb{T}] \models \forall \vec{x}.\ (\alpha \Rightarrow \varphi)$.
(2) $\{\vec{x}.\ \alpha\} \models \varphi$, *where the free variables in φ are interpreted as their generic values over $\{\vec{x}.\ \alpha\}$, that is the projection maps $\{\vec{x}.\ \alpha\} \to \{x_i : X_i.\ \top\}$).*
(3) $\mathbb{T}$ *proves* $(\alpha \vdash_{\vec{x}} \varphi)$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows immediately by unrolling the Kripke–Joyal semantics. The equivalence (2) $\Leftrightarrow$ (3) is by induction on the structure of φ. $\square$

3. Proofs of the main theorems

Given a geometric theory $\mathbb{T}$, the main theorems reference a certain internal theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ internal to $\mathrm{Set}[\mathbb{T}]$. This theory is defined as follows.

Definition 3.1. Let $\mathbb{T}$ be a geometric theory. The theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is the geometric theory internal to $\mathrm{Set}[\mathbb{T}]$ which arises from the pulled-back theory $\underline{\mathbb{T}}$ by adding additional constant symbols e_x of the appropriate sorts, one for each element $x : U_{\mathbb{T}}$, axioms $(\top \vdash f(e_{x_1}, \ldots, e_{x_n}) = e_{f(x_1,\ldots,x_n)})$ for each function symbol f and n-tuple of elements of $U_{\mathbb{T}}$ (of the appropriate sorts), and axioms $(\top \vdash R(e_{x_1}, \ldots, e_{x_n}))$ for each relation symbol R and n-tuple $(x_1, \ldots, x_n)$ (of the appropriate sorts) such that $R(x_1, \ldots, x_n)$ holds for $U_{\mathbb{T}}$.

From the point of view of $\mathrm{Set}[\mathbb{T}]$, a model of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is a model of $\underline{\mathbb{T}}$ equipped with a $\underline{\mathbb{T}}$-homomorphism from $U_{\mathbb{T}}$. In particular, the identity $(U_{\mathbb{T}} \to U_{\mathbb{T}})$ is a model of $\underline{\mathbb{T}}/U_{\mathbb{T}}$. This is what we mean when we say that $U_{\mathbb{T}}$ is in a canonical way a $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model.

Example 3.2. Let $\mathbb{T}$ be the theory of rings. Then $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is, from the internal point of view of $\mathrm{Set}[\mathbb{T}]$, the theory of $U_{\mathbb{T}}$-algebras.

Example 3.3. Let $\mathbb{T}$ be a geometric theory. Let M be a model of $\mathbb{T}$ in the category of sets. Let $f : \mathrm{Set} \to \mathrm{Set}[\mathbb{T}]$ be the corresponding geometric morphism. Then $f^*(\underline{\mathbb{T}}/U_{\mathbb{T}})$ is the theory of M-algebras ($\mathbb{T}$-models equipped with a $\mathbb{T}$-homomorphism from M). This is because $f^*\underline{\mathbb{T}} = \mathbb{T}$, $f^*U_{\mathbb{T}} = M$ and because the construction of the theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ is geometric.

Remark 3.4. From the internal point of view of $\mathrm{Set}[\mathbb{T}]$, we can construct the classifying topos of $\underline{\mathbb{T}}/U_{\mathbb{T}}$. Externally, this construction gives rise to a bounded topos over $\mathrm{Set}[\mathbb{T}]$, hence to a Grothendieck topos. Using for instance the technique described in [**blechschmidt-hutzler-oldenziel:composition**], one can show that this topos classifies the theory of homomorphisms between $\mathbb{T}$-models. It can also be obtained as the lax pullback $(\mathrm{Set}[\mathbb{T}] \Rightarrow_{\mathrm{Set}[\mathbb{T}]} \mathrm{Set}[\mathbb{T}])$.

Lemma 3.5. *Let $\mathbb{T}$ be a geometric theory. Let α be a geometric formula over the signature of $\mathbb{T}$ in a context $x_1 : X_1, \ldots, x_n : X_n$. Then*

$$\{\vec{x}.\ \alpha\} \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves } (\top \vdash_{[]} \alpha)\urcorner,$$

where the free variables $\vec{x}$ occurring in α are interpreted as in Proposition 2.8.

Proof. By induction on the structure of α. The cases of "$\top$" and "$\wedge$" are trivial; the cases of "$\bigvee$" and "$\exists$" follow from passing to suitable coverings; and the case of atomic propositions is by definition of $\underline{\mathbb{T}}/U_{\mathbb{T}}$. $\square$

Lemma 3.6. *Let $\mathbb{T}$ be a geometric theory. Let φ be a section of the sheaf of geometric* formulas over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ over a stage $A \in \mathcal{C}_{\mathbb{T}}$. Then there is a covering $(A_i \to A)_i$ of A such that for each index i, there is a formula φ_i over the signature of $\mathbb{T}/U_{\mathbb{T}}(A_i)$ such that $A_i \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* $(\varphi \dashv\vdash \varphi_i)\urcorner$.*

Proof. By passing to a covering, we may suppose that φ is given by an (external) geometric formula over the signature of $\underline{\mathbb{T}}(A)/U_{\mathbb{T}}(A)$.

Any function symbol and relation symbol of $\underline{\mathbb{T}}(A)$ occurring in φ is locally given by a symbol of $\mathbb{T}$. Hence the claim would be trivial if φ were a coherent formula, for in this case we would just have to pass to further coverings, one for each occurring symbol, a finite number of times in total.

However, in general, we cannot conclude as easily. Write $A = \{\vec{x}.\ \alpha\}$. Let R be a relation symbol of $\underline{\mathbb{T}}(A)$ occurring in φ. By the explicit description of constant sheaves as sheaves of locally constant maps, there is a covering $(\{\vec{y}_j.\ \alpha_j\} \xrightarrow{[\theta_j]} \{\vec{x}.\ \alpha\})_j$ such that, restricted to $\{\vec{y}_j.\ \alpha_j\}$, R is given by a relation symbol R_j of $\mathbb{T}$. To construct the desired formula φ', we replace any such occurrence $R(\ldots)$ in φ by

$$\bigvee_j \big((\exists \vec{y}_j.\ \theta_j) \wedge R_j(\ldots)\big).$$

In a similar vein we treat any occurrence of function symbols.

The resulting formula φ' is a geometric formula over the signature of $\mathbb{T}/U_{\mathbb{T}}(A)$. The verification of $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* $(\varphi \dashv\vdash \varphi')\urcorner$ rests on the observation

$$A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}} \text{ proves}^\star \ \big((\exists \vec{y}_k.\ \theta_k) \vdash_{[]} \bigvee_\ell \{\top \mid (\exists \vec{y}_\ell.\ \theta_\ell) \text{ holds for } U_{\mathbb{T}}\}\big)\urcorner$$

which in turn can be checked on the covering $(\{\vec{y}_j.\ \alpha_j\} \xrightarrow{[\theta_j]} \{\vec{x}.\ \alpha\})_j$, applying Lemma 3.5 and using that $\mathbb{T}$ (and hence $\underline{\mathbb{T}}$) proves $(\exists \vec{y}_j.\ \theta_j) \wedge (\exists \vec{y}_k.\ \theta_k) \vdash_{\vec{x}} \bigvee \{\top \mid j = k\}$. □

Theorem 3.7. *Let $\mathbb{T}$ be a geometric theory. Then, internally to* Set[$\mathbb{T}$]*, for any geometric* sequent σ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, the following statements are equivalent:*

1. *The sequent σ holds for $U_{\mathbb{T}}$.*
2. *The sequent σ is provable* modulo $\underline{\mathbb{T}}/U_{\mathbb{T}}$.*

Proof. The direction (2) ⇒ (1) is immediate because $U_{\mathbb{T}}$ is, from the internal point of view of Set[$\mathbb{T}$], a $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model. Hence even the following stronger statement holds internally: *For any geometric sequent σ, if $\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves σ, then σ holds for $U_{\mathbb{T}}$.*

For the direction (1) ⇒ (2) we have to verify that, given any stage $A \in \mathcal{C}_{\mathbb{T}}$ and any section σ of the sheaf of geometric* sequents over A, if $A \models \ulcorner\sigma$ holds for $U_{\mathbb{T}}\urcorner$ then $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* $\sigma\urcorner$. By Lemma 3.6 we may suppose that σ is an (external) geometric sequent over the signature of $\mathbb{T}/U_{\mathbb{T}}(A)$.

Writing $A = \{\vec{x}.\ \alpha\}$ and $\sigma = (\varphi \vdash_{\vec{y}} \psi)$, we have $\{\vec{x}.\ \alpha\} \models \forall \vec{y}.\ (\varphi \Rightarrow \psi)$, hence $\{\vec{x}, \vec{y}.\ \alpha \wedge \varphi\} \models \psi$. Thus $\mathbb{T}$ proves $(\alpha \wedge \varphi \vdash_{\vec{x},\vec{y}} \psi)$. This proof can be pulled back from Set to Set[$\mathbb{T}$]$/ \text{よ}A$ to show $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* $(\alpha \wedge \varphi \vdash_{\vec{x},\vec{y}} \psi)\urcorner$. By Lemma 3.5, we also have $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* $(\top \vdash_{[]} \alpha)\urcorner$ (where the free variables occurring in α are interpreted as the generic values available over A), hence $A \models \ulcorner \underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* $(\varphi \vdash_{\vec{y}} \psi)\urcorner$. □

Remark 3.8. Theorem 3.7 cannot be strengthened to arbitrary geometric sequents. For instance, in the case that $\mathbb{T}$ is the theory of rings, the internal geometric sequent $(\top \vdash_{x : U_{\mathbb{T}}} \bigvee_{a : U_{\mathbb{T}}} (x = e_a))$ trivially holds for $U_{\mathbb{T}}$. However, it is not provable modulo $\mathbb{T}/U_{\mathbb{T}}$, as for instance the polynomial algebra $U_{\mathbb{T}}[X]$ does not validate it.

Theorem 3.9. *Let $\mathbb{T}$ be a geometric theory. Let $\mathbb{T}'$ be a quotient theory of $\mathbb{T}$. Assume that the generic model $U_{\mathbb{T}}$ is a sheaf for the topology on* Set[$\mathbb{T}$] *cutting out* *the subtopos* $\mathrm{Set}[\mathbb{T}']$. *Then the following statement holds internally to* $\mathrm{Set}[\mathbb{T}']$*:*

A geometric* *sequent* σ *with Horn consequent holds for* $U_{\mathbb{T}'}$ *iff* $\underline{\mathbb{T}}/U_{\mathbb{T}}$ *proves** σ.

Proof. In general, the generic model of $\mathbb{T}'$ is the pullback of the generic model of $\mathbb{T}$ to the subtopos $\mathrm{Set}[\mathbb{T}']$ [5, Lemma 2.3]. By the sheaf assumption, the objects $U_{\mathbb{T}'}$ and $U_{\mathbb{T}}$ actually agree, that is $U_{\mathbb{T}}$ is contained in the subtopos and has the universal property of $U_{\mathbb{T}'}$.

The "if" direction is trivial, as $U_{\mathbb{T}'}$ is a $\underline{\mathbb{T}}/U_{\mathbb{T}}$-model.

For the "only if" direction, we use that a statement holds in $\mathrm{Set}[\mathbb{T}']$ if and only if its ∇-translation holds in $\mathrm{Set}[\mathbb{T}]$, where ∇ is the modal operator associated to the topology cutting out $\mathrm{Set}[\mathbb{T}']$ [3, Theorem 6.31]. Exploiting some of the simplification rules of the ∇-translation [3, Section 6.6], it hence suffices to verify, internally to $\mathrm{Set}[\mathbb{T}]$, that:

For any geometric* *sequent* $\sigma = (\varphi \vdash_{\vec{x}} \psi)$ *where* ψ *is a Horn formula,*
if $\forall x_1, \ldots, x_n : U_{\mathbb{T}}.\ (\varphi \Rightarrow \nabla\psi)$, *then* $\underline{\mathbb{T}}/U_{\mathbb{T}}$ *proves** σ.

Since ∇ commutes with finite conjunctions and since the sheaf assumption implies that $\nabla(s = t)$ is equivalent to $s = t$ and that, for relation symbols R, the statement $\nabla(R(s_1, \ldots, s_m))$ is equivalent to $R(s_1, \ldots, s_m)$, the statement $\nabla\psi$ is equivalent to ψ. Hence the claim follows from Theorem 3.7. □

A situation in which the sheaf assumption of Theorem 3.9 is satisfied is when $\mathbb{T}$ is a Horn theory and the topology cutting out $\mathrm{Set}[\mathbb{T}']$ is subcanonical. For instance, this is the case if $\mathbb{T}$ is the theory of rings and $\mathrm{Set}[\mathbb{T}']$ is one of several well-known toposes in algebraic geometry such as the big Zariski topos, the big étale topos or the big fppf topos.

Theorem 3.7 cannot be strengthened to arbitrary first-order (or first-order* or even just coherent) formulas in place of geometric* sequents. For instance, in the case that $\mathbb{T}$ is the theory of local rings, the generic model $U_{\mathbb{T}}$ validates the coherent formula ⌜any element which is not zero is invertible⌝, but $\mathbb{T}/U_{\mathbb{T}}$ does not prove this fact, as it is for instance not validated by the polynomial algebra $U_{\mathbb{T}}[X]$. However, Theorem 3.7 still plays an important role in understanding first-order formulas:

Theorem 3.10. *Let* $\mathbb{T}$ *be a geometric theory. Let* α *be a first-order formula over the signature of* $\mathbb{T}$. *Then the following statements are equivalent.*

(1) *The formula* α *holds for* $U_{\mathbb{T}}$.
(2) *The formula* α *is provable in first-order intuitionistic logic modulo the axioms of* $\mathbb{T}$ *and the additional axiom scheme* (‡).

Proof. The direction (2) ⇒ (1) is immediate, as the internal language of $\mathrm{Set}[\mathbb{T}]$ validates first-order intuitionistic logic and as it validates the axiom scheme (‡) by Theorem 3.7.

For the converse direction, we note that, given a geometric sequent σ over the signature of $\mathbb{T}$, the statement ⌜$\underline{\mathbb{T}}/U_{\mathbb{T}}$ proves* σ⌝ can be expressed as a geometric formula over the signature of $\mathbb{T}$. Applying this observation successively to subformulas of the given formula α, Theorem 3.7 on the semantic side and the axiom scheme (‡) on the syntactic side imply that we may assume that α is in fact a geometric formula. Hence we are reduced to the basic fact that, for geometric formulas φ, $\mathrm{Set}[\mathbb{T}] \models \varphi$ implies that $\mathbb{T}$ proves φ. □

4. The special case of Horn theories

Throughout this section, let $\mathbb{T}$ be a Horn theory.

Lemma 4.1. *Let X be a set equipped with a morphism $X \to S$ to the set of sorts of the signature Σ of $\mathbb{T}$. Let R be a set of atomic propositions in which the elements of X may appear as new constants of the respective sorts. Then there is $\mathbb{T}\langle X|R\rangle$, the free $\mathbb{T}$-model on the generators X modulo the relations R.*

Proof. The desired model can be constructed as a term algebra. As a set, it consists of the terms (in the empty context) of the signature $\Sigma + X$ modulo the equivalence relation identifying two terms if and only if $\mathbb{T} + R$ proves them to be equal. The function symbols f of Σ are interpreted by declaring $[\![f]\!]([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ and the relation symbols S are interpreted by declaring $([t_1], \ldots, [t_n]) \in [\![S]\!] \Leftrightarrow (\mathbb{T} + R \vdash S(t_1, \ldots, t_n))$.

We omit the required verifications and only remark that while the same construction could be carried out if $\mathbb{T}$ was a general geometric theory, the resulting object would in general not be a model of $\mathbb{T}$. ☐

Lemma 4.2. *The category of $\mathbb{T}$-models is complete and cocomplete.*

Proof. Limits are computed as the limits of the underlying sets, colimits are computed by using the construction of Lemma 4.1. For instance, the coproduct of $\mathbb{T}\langle X|R\rangle$ and $\mathbb{T}\langle X'|R'\rangle$ is $\mathbb{T}\langle X \amalg X' \,|\, R, R'\rangle$. ☐

Having the special case of the theory of rings in mind, we write the coproduct in the category of $\mathbb{T}$-models as "$\otimes$".

Lemma 4.3. *Let $\sigma = (\varphi_1 \wedge \cdots \wedge \varphi_n \vdash_{x_1,\ldots,x_k} \psi_1 \wedge \cdots \wedge \psi_m)$ be a Horn sequent over the signature of $\mathbb{T}$. Then the following statements are equivalent.*

(1) *The theory $\mathbb{T}$ proves σ.*
(2) *In $\mathbb{T}\langle x_1, \ldots, x_k \,|\, \varphi_1, \ldots, \varphi_n\rangle$, the propositions $\psi_1, \ldots, \psi_m$ hold for the k-tuple $([x_1], \ldots, [x_k])$.*

Proof. By construction of the term algebra. ☐

Lemma 4.4. *A $\mathbb{T}$-model is finitely presentable as an object of the category of $\mathbb{T}$-models if and only if it is isomorphic to a model of the form $\mathbb{T}\langle X|R\rangle$ where X is Bishop-finite and R is Kuratowski-finite.*

Proof. It is an instructive exercise to verify that models of the stated form are compact. Conversely, let a $\mathbb{T}$-model M be given. Then $\mathbb{T}$ is the filtered colimit of all models over M which are of the stated form. If M is compact, the identity on M factors over such a model. Hence M is a retract of such a model and hence itself isomorphic to a model of this form. ☐

Any $\mathbb{T}$-model A has a mirror image in the topos $\mathrm{Set}[\mathbb{T}]$, namely the functor $A^\sim : \mathbb{T}\text{-mod}_{\mathrm{fp}} \to \mathrm{Set}$ given by $T \mapsto A \otimes T$. This object is in a canonical way a $\mathbb{T}$-model over $U_\mathbb{T}$, hence from the point of view of $\mathrm{Set}[\mathbb{T}]$ a $\underline{\mathbb{T}}/U_\mathbb{T}$-model.

Lemma 4.5. *The functor $(\cdot)^\sim$ from $\mathbb{T}$-models to $\underline{\mathbb{T}}/U_\mathbb{T}$-models in $\mathrm{Set}[\mathbb{T}]$ is left adjoint to the functor $\Gamma = \mathrm{Hom}(1, \cdot)$ computing global elements.*

Proof. A $U_{\mathbb{T}}$-algebra homomorphism $\alpha : A^{\sim} \to M$ yields the $\mathbb{T}$-model homomorphism $\alpha_0 : A \to M(0) = \Gamma(M)$, where 0 is the initial $\mathbb{T}$-model. Conversely, a $\mathbb{T}$-model homomorphism $\beta : A \to \Gamma(M)$ yields a $U_{\mathbb{T}}$-algebra homomorphism by summing $A \to M(0) \to M(T)$ with the structure morphism $T = U_{\mathbb{T}}(T) \to M(T)$. □

Definition 4.6. The *spectrum* $\mathrm{Spec}(M)$ of a $U_{\mathbb{T}}$-algebra M in $\mathrm{Set}[\mathbb{T}]$ is the result of constructing, internally to $\mathrm{Set}[\mathbb{T}]$, the set of $U_{\mathbb{T}}$-algebra homomorphisms $M \to U_{\mathbb{T}}$.

Lemma 4.7. *Let A be a $\mathbb{T}$-model. Then $\mathrm{Spec}(A^{\sim})$ coincides with $\text{よ}A$, the functor $\mathrm{Hom}_{\mathbb{T}\text{-mod}}(A, \cdot)$.*

Proof. By the Yoneda lemma, the sections of the sheaf $\mathrm{Spec}(A^{\sim}) : \mathbb{T}\text{-mod}_{\mathrm{fp}} \to \mathrm{Set}$ on an object T are given by the set

$$\begin{aligned}\mathrm{Spec}(A^{\sim})(T) &\cong \mathrm{Hom}(\text{よ}T, \mathrm{Spec}(A^{\sim})) = \mathrm{Hom}(\text{よ}T, [A^{\sim}, U_{\mathbb{T}}]_{U_{\mathbb{T}}}) \\ &\cong \mathrm{Hom}(\text{よ}T \times A^{\sim}, U_{\mathbb{T}})_{U_{\mathbb{T}}\text{-algebra homomorphism in second argument}} \\ &\cong \mathrm{Hom}_{U_{\mathbb{T}}}(A^{\sim}, (U_{\mathbb{T}})^{\text{よ}T}) \cong \mathrm{Hom}_{U_{\mathbb{T}}}(A^{\sim}, U_{\mathbb{T}}|T),\end{aligned}$$

where $[A^{\sim}, U_{\mathbb{T}}]_{U_{\mathbb{T}}}$ is the object of $U_{\mathbb{T}}$-algebra homomorphisms from $A^{\sim}$ to $U_{\mathbb{T}}$ (a subobject of the internal Hom $U_{\mathbb{T}}^{A^{\sim}}$); $\mathrm{Hom}_{U_{\mathbb{T}}}$ denotes the set of $U_{\mathbb{T}}$-algebra homomorphisms; $(U_{\mathbb{T}})^{\text{よ}T}$ is the object of morphisms from $\text{よ}T$ to $U_{\mathbb{T}}$; and $U_{\mathbb{T}}|T$ is the functor $U_{\mathbb{T}}(T \times \cdot)$, that is $S \mapsto T \otimes S$.

An arbitrary element $f \in (\text{よ}A)(T)$, that is an arbitrary $\mathbb{T}$-model homomorphism $f : A \to T$, induces a $U_{\mathbb{T}}$-algebra homomorphism $g : A^{\sim} \to U_{\mathbb{T}}|T$ by setting $g_S := f \otimes \mathrm{id}_S : A \otimes S \to T \otimes S$. The given homomorphism f can be recovered by $f = g_0$, the component of g at the initial model.

Conversely, a $U_{\mathbb{T}}$-algebra homomorphism $g : A^{\sim} \to U_{\mathbb{T}}|T$ induces a $\mathbb{T}$-model homomorphism $f : A \to T$ by setting $f := g_0$. Because g is a natural transformation and because g is compatible with the structure morphisms $U_{\mathbb{T}} \to A^{\sim}$ and $U_{\mathbb{T}} \to U_{\mathbb{T}}|T$, the morphism g is determined by f. □

Lemma 4.8. *Let A be a finitely presentable $\mathbb{T}$-model. Then the canonical morphism*

$$A^{\sim} \longrightarrow (U_{\mathbb{T}})^{\mathrm{Spec}(A^{\sim})}$$

is an isomorphism of $U_{\mathbb{T}}$-algebras.

Proof. By Lemma 4.7, the functor $\mathrm{Spec}(A^{\sim})$ coincides with $\text{よ}A$. Since by assumption A is contained in the site defining $\mathrm{Set}[\mathbb{T}]$, the exponential $(U_{\mathbb{T}})^{\text{よ}A}$ coincides with $U_{\mathbb{T}}|A$ (notation as in the proof in Lemma 4.7), that is, the $U_{\mathbb{T}}$-algebra $A^{\sim}$. □

Corollary 4.9. *Let A and B be $\mathbb{T}$-models. Assume that B is finitely presentable. Then the canonical morphism*

$$\mathrm{Hom}_{U_{\mathbb{T}}}(A^{\sim}, B^{\sim}) \longrightarrow \mathrm{Spec}(A^{\sim})^{\mathrm{Spec}(B^{\sim})}$$

is an isomorphism.

Proof. We have the chain of isomorphisms

$$\begin{aligned}\mathrm{Spec}(A^{\sim})^{\mathrm{Spec}(B^{\sim})} &= ([A^{\sim}, U_{\mathbb{T}}]_{U_{\mathbb{T}}})^{\mathrm{Spec}(B^{\sim})} \cong [\mathrm{Spec}(B^{\sim}) \times A^{\sim}, U_{\mathbb{T}}]_{U_{\mathbb{T}}} \\ &\cong [A^{\sim}, U_{\mathbb{T}}^{\mathrm{Spec}(B^{\sim})}]_{U_{\mathbb{T}}} \cong [A^{\sim}, B^{\sim}],\end{aligned}$$

where the final isomorphism is by Lemma 4.8. □

Theorem 4.10. *The generic $\mathbb{T}$-model is* quasicoherent *in the following sense: From the point of view of* $\mathrm{Set}[\mathbb{T}]$*, for any finitely presentable $U_{\mathbb{T}}$-algebra A (finitely presented object in the category of $U_{\mathbb{T}}$-algebras), the canonical $U_{\mathbb{T}}$-algebra homomorphism*

$$A \longrightarrow (U_{\mathbb{T}})^{\mathrm{Spec}(A)}$$

is an isomorphism.

Proof. The proof of Lemma 4.4 is constructive and thus valid in the internal language of $\mathrm{Set}[\mathbb{T}]$. Hence we can apply it, internally, to the theory $\underline{\mathbb{T}}/U_{\mathbb{T}}$ to deduce that a $U_{\mathbb{T}}$-algebra A is finitely presentable if and only if it is isomorphic to a $U_{\mathbb{T}}$-algebra of the form $(\underline{\mathbb{T}}/U_{\mathbb{T}})\langle X|R\rangle$ with X Bishop-finite and R Kuratowski-finite.

We therefore have to verify the following internal statement: *For any number n, for any sorts $X_1, \ldots, X_n$ of $\underline{\mathbb{T}}/U_{\mathbb{T}}$, for any number m, for any atomic propositions $R_1, \ldots, R_m$ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ extended by constants $e_1 : X_1, \ldots, e_n : X_n$, the canonical map $A \to (U_{\mathbb{T}})^{\mathrm{Spec}(A)}$ where $A := (\underline{\mathbb{T}}/U_{\mathbb{T}})\langle e_1 : X_1, \ldots, e_n : X_n \mid R_1, \ldots, R_m\rangle$ is an isomorphism.*

Following the Kripke–Joyal translation of this statement, let a stage $T \in \mathbb{T}\text{-mod}_{\mathrm{fp}}$, T-elements $X_1, \ldots, X_n$ of the object of sorts of the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ (that is the constant sheaf on the set of sorts of $\mathbb{T}$), and T-elements $R_1, \ldots, R_m$ of the object of atomic propositions over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ be given. By passing to a covering of T, we may assume that the X_i are given by sorts of $\mathbb{T}$ and that the R_j are given by atomic propositions over the signature of $\mathbb{T}/U_{\mathbb{T}}(T)$.

Since the slice $\mathrm{Set}[\mathbb{T}]/\text{よ}T$ is equivalent to $\mathrm{Set}[\mathbb{T}/T]$, hence again the classifying topos of a Horn theory, we may without loss of generality assume that T is the initial $\mathbb{T}$-model.

In this case the claim follows from Lemma 4.8, since the result of constructing, internally to $\mathrm{Set}[\mathbb{T}]$, the model $(\underline{\mathbb{T}}/U_{\mathbb{T}})\langle e_1 : X_1, \ldots, e_n : X_n \mid R_1, \ldots, R_m\rangle$ coincides with the $U_{\mathbb{T}}$-algebra $(\mathbb{T}\langle e_1 : X_1, \ldots, e_n : X_n \mid R_1, \ldots, R_m\rangle)^{\sim}$. $\square$

5. The generalization to the higher-order case

By *extended geometric logic* we mean the extension of geometric logic where we are allowed to form, in addition to the basic sorts supplied by a given signature, finite limits of sorts and set-indexed colimits of sorts. By (intuitionistic) *higher-order logic*, we mean the further extension where we may also form powersorts. These derived sorts come with respective term constructors (tuple formers, coprojections, set comprehension) and the usual rules governing these constructors.

An *extended geometric formula* is a formula of extended geometric logic built from equality and relation symbols by the logical connectives $\top \perp \wedge \vee \exists$ and by arbitrary set-indexed disjunctions $\bigvee$. The existential quantification can be over any sorts of extended geometric logic, including the derived sorts. An *extended geometric sequent* is a sequent of the form $(\varphi \vdash_{\vec{x}} \psi)$ where φ and ψ are extended geometric formulas and the sorts of the variables $\vec{x}$ may be derived sorts.

It is possible to extend the Kripke–Joyal semantics so that higher-order logic can be interpreted in any Grothendieck topos. The truth of a higher-order sequent $(\varphi \vdash_{\vec{x}} \psi)$ is in general not preserved under pullback along geometric morphisms, even if φ and ψ do not contain $\forall$ and $\Rightarrow$, since powerobjects are in general not preserved under pullback. However, as can be deduced from the following lemma, the truth of extended geometric sequents is preserved; as is folklore, extended geometric logic is just a thin layer over ordinary geometric logic.

Lemma 5.1. *Let σ be an extended geometric sequent over the signature of a geometric theory $\mathbb{T}$. Then there is a set-indexed family $(\sigma_i)_i$ of ordinary geometric sequents over the same signature, so that σ is provable in extended geometric logic if and only if all the sequents σ_i are provable in ordinary geometric logic.*

Proof. Any existential quantification of the form "$\exists p : X \times Y$" can be replaced by the string "$\exists x : X.\ \exists y : Y$", and similarly for free variables of product sorts appearing in the context of σ. In a similar vein more general finite limits are treated.

An existential quantification of the form "$\exists x : \coprod_i X_i$" can be replaced by the string "$\bigvee_i \exists x : X_i$".

Finally, for any occurrence of a free variable $x : \coprod_i X_i$ in the context of σ, we can replace σ by the family of sequents $(\sigma_i)_i$, where the sequent σ_i is the same as σ only that the free variable x is changed to be of sort X_i (and the corresponding change in the consequent and the antecedent is applied, applying the appropriate coprojection).

After carrying out these steps, the free variables are only of the basic sorts supplied by the signature of $\mathbb{T}$ and existential quantifications only range over the basic sorts. However, in the consequents and antecedents, still tuple formers and coprojections may appear. These can be replaced as suggested by the rules governing these. For instance, an equation "$\langle x, y\rangle = \langle x', y'\rangle$" can be replaced by the conjunction "$x = x' \wedge y = y'$", and an equation "$\iota_i(x) = \iota_j(y)$" (where ι_i and ι_j are coprojections associated with coproduct sorts) can be replaced by the subsingleton-indexed disjunction "$\bigvee\{x = y \mid i = j\}$". $\square$

Theorem 5.2. *Let $\mathbb{T}$ be a geometric theory. Let $x_1 : X_1, \ldots, x_n : X_n$ be a context over the signature of $\mathbb{T}$. Then the canonical morphism*

$$\mathrm{Form}^{\star}_{\vec{x}}(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash_{\vec{x}}) \longrightarrow P(X_1 \times \cdots \times X_n)$$

sending, internally speaking, the equivalence class of a geometric* formula φ over the signature of $\underline{\mathbb{T}}/U_{\mathbb{T}}$ in the context $\vec{x}$ to the subset $\{(x_1, \ldots, x_n) \mid \varphi\}$ is an isomorphism.

Proof. Injectivity is by the Nullstellensatz of Theorem 3.7. Surjectivity is by the definability result [5, Theorem 2.2], exploiting that the internal statement localizes well by Lemma 3.5. $\square$

Corollary 5.3. *Let $\mathbb{T}$ be a geometric theory. Let $\{\vec{x}.\ \varphi\}$ and $\{\vec{y}.\ \psi\}$ be geometric formulas in given contexts. Then, internally to* $\mathrm{Set}[\mathbb{T}]$*, the canonical map from the set of equivalence classes of $\underline{\mathbb{T}}/U$-provably* functional geometric* formulas from $\{\vec{x}.\ \varphi\}$ to $\{\vec{y}.\ \psi\}$ to the set of maps $\{(\vec{y}) \mid \psi\}^{\{(\vec{x}) \mid \varphi\}}$ is a bijection.*

Proof. We argue internally to $\mathrm{Set}[\mathbb{T}]$. The canonical map sends an equivalence class $[\theta]$ to the unique map $f : \{(\vec{x}) \mid \varphi\} \to \{(\vec{y}) \mid \psi\}$ whose graph is given by the set $\{(\vec{x}, \vec{y}) \mid \theta\}$.

For verifying surjectivity, let a map $f : \{(\vec{x}) \mid \varphi\} \to \{(\vec{y}) \mid \psi\}$ be given. Then its graph is a subset of $\vec{X} \times \vec{Y}$, hence by Theorem 5.2 given by a geometric* formula θ. Because f is a map, this formula is functional; and by the Nullstellensatz, it is $\underline{\mathbb{T}}/U_{\mathbb{T}}$-provably* so.

For verifying injectivity, let θ and θ' be $\underline{\mathbb{T}}/U_{\mathbb{T}}$-provably* functional formulas which give rise to identical maps. Then they also give rise to identical graphs, hence are $\underline{\mathbb{T}}/U_{\mathbb{T}}$-provably* equivalent by Theorem 5.2. $\square$

Theorem 5.4. *Let $\mathbb{T}$ be a geometric theory. Then, internally to* $\mathrm{Set}[\mathbb{T}]$*, for any extended geometric** *sequent σ over the signature of $\underline{\mathbb{T}}/U$, the following statements are equivalent:*

(1) *The sequent σ holds for $U_{\mathbb{T}}$.*
(2) *The sequent σ is provable** *modulo $\mathbb{T}/U_{\mathbb{T}}$ in extended geometric logic.*

Proof. The implication (2) $\Rightarrow$ (1) is immediate since $U_{\mathbb{T}}$ is a model of $\mathbb{T}/U_{\mathbb{T}}$. The converse direction is by Lemma 5.1, which holds internally in $\mathrm{Set}[\mathbb{T}]$ as the proof we supplied is constructive, and the Nullstellensatz for ordinary geometric logic of Theorem 3.7. □

Theorem 5.5. *Let $\mathbb{T}$ be a geometric theory. Let α be a higher-order formula over the signature of $\mathbb{T}$. Then the following statements are equivalent:*

(1) *The formula α holds for $U_{\mathbb{T}}$.*
(2) *The formula α is provable in higher-order intuitionistic logic modulo the axioms of $\mathbb{T}$ and the additional axiom scheme XXX.*

Proof. Theorem 5.2 on the semantic side and the axiom scheme on the syntactic side allow us to replace any mention of a powersort $P(X)$ in α by $\mathrm{Form}^\star_{x:X}(\underline{\mathbb{T}}/U_{\mathbb{T}})/(\dashv\vdash_x)$. Then we can argue as in the proof of Theorem 3.10, noting that the axiom scheme indeed entails the axiom scheme posited by Theorem 3.10. □

6. Applications

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