

Using the internal language of toposes in algebraic geometry

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Summary

With the internal language of toposes, we can

- express sheaf-theoretic concepts in a simple, element-based language and thus understand them in a more conceptual way,
- mechanically obtain corresponding sheaf-theoretic theorems for any (intuitionistic) theorem of linear and commutative algebra, and
- understand which properties spread from points to neighbourhoods.

What is a topos?

A *topos* is a category which has finite limits, is cartesian closed and has a subobject classifier. Intuitively, a topos is a category which has similar properties to the category of sets.

Important examples of toposes are the category of sets and the category of sheaves on a topological space.

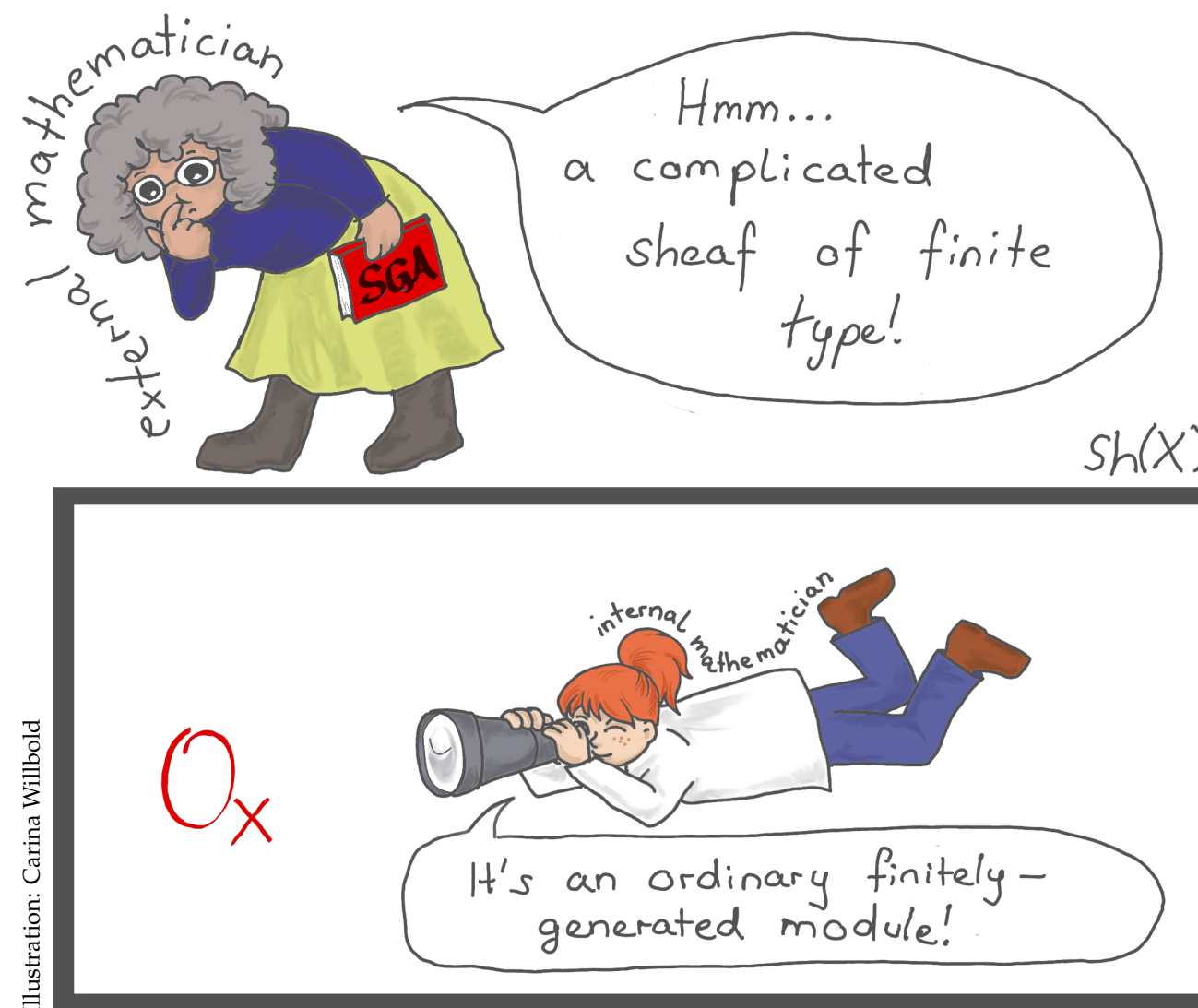
What is the internal language?

The internal language of a topos \mathcal{E} allows us to construct objects and morphisms of the topos, formulate statements about them, and prove such statements in a *naive element-based* language. The translation of internal statements and proofs into external ones is facilitated by an easy mechanical procedure, the *Kripke–Joyal semantics*. *Special case:* The language of the topos of sets is the usual formal mathematical language.

external point of view	internal point of view
objects of \mathcal{E}	sets
morphisms of \mathcal{E}	maps of sets
monomorphisms in \mathcal{E}	surjective maps
epimorphisms in \mathcal{E}	injective maps

The small Zariski topos

Let X be a scheme. Let $\mathrm{Sh}(X)$ be the small Zariski topos, i. e. the topos of set-valued sheaves on X . From the point of view of $\mathrm{Sh}(X)$, the structure sheaf \mathcal{O}_X looks like an *ordinary ring* (instead of a sheaf of rings), and sheaves of \mathcal{O}_X -modules look like *ordinary modules* on that ring.



Basic example

Let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be a short exact sequence of sheaves of \mathcal{O}_X -modules. It is well-known that if \mathcal{F}' and \mathcal{F}'' are of finite type, then \mathcal{F} is as well.

A sheaf is of finite type if and only if, internally, it is a finitely generated module. Therefore the proposition follows *immediately* by interpreting the analogous statement of linear algebra in the little Zariski topos: Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M . We can thus recognize notions and statements of scheme theory as notions and statements of non-sheafy linear algebra. *Caveat:* Non-intuitionistic proofs by contradiction can not be interpreted with the internal language.

Locally free sheaves

Let X be a reduced scheme. The structure sheaf \mathcal{O}_X looks like a *field* from the internal point of view. Recall that neither the rings of local sections nor the stalks are fields. Let \mathcal{F} be a finite type sheaf of \mathcal{O}_X -modules. Then it is well-known that \mathcal{F} is locally free on a dense open subset of X . (Important hard exercise in Ravi Vakil's notes.)

This is an *immediate* application of the following easy lemma of intuitionistic linear algebra: Let M be a finitely generated vector space. Then M is *not not* finite free.

Rational functions

The sheaf \mathcal{K}_X of rational functions can internally simply be defined as the total quotient ring of \mathcal{O}_X .

Spreading of properties

The following metatheorem covers a wide range of cases: Let φ be a property which can be formulated without using $\Rightarrow, \neg, \forall$. Then φ holds at a point if and only if it holds on some open neighbourhood of the point.

For instance, a sheaf of modules \mathcal{F} is zero if and only if, from the internal perspective, " $\forall x \in \mathcal{F}: x = 0$ ". Because of the " \forall ", a stalk may be zero without the sheaf being zero on a neighbourhood.

But if \mathcal{F} is of finite type, the condition can be reformulated using generators as " $x_1 = 0 \wedge \dots \wedge x_n = 0$ ". The metatheorem is applicable to this statement and thus a stalk is zero if and only if \mathcal{F} is zero on a neighbourhood.

Dictionary of external vs. internal notions

Expository notes are available at <http://tiny.cc/topos> (work in progress).

Contents: Tensor product of sheaves = internal ordinary tensor product, quasicoherent sheaves = internal ordinary modules satisfying an interesting condition, internal Cartier divisors, more metatheorems about spreading of properties, pullback along immersions = internal sheafification, relative spectrum = internal spectrum, scheme dimension = internal Krull dimension of \mathcal{O}_X , dense = not not, modal operators, other toposes, group schemes = groups, ...