REFLECTIONS ON REFLECTION FOR INTUITIONISTIC SET THEORIES

Abstract. We study the well-known reflection principle of Zermelo–Fraenkel set theory in the context of intuitionistic Zermelo–Fraenkel set theory IZF. We show that the reflection principle is equivalent to RRS_2, a strengthened version of Aczel's relation reflection scheme. As applications, we give a new proof that relativized dependent choice is equivalent to the conjunction of the relation reflection scheme and dependent choice and we present an intuitionistic version of Feferman's ZFC/S, a conservative extension of ZFC which is useful as a foundation for category theory.

1. Introduction

The basic form of the reflection principle for Zermelo–Fraenkel set theory ZF is the following.

Theorem 1.1. *Let $\varphi(x_1, \ldots, x_n)$ be a formula in the language of set theory with some of its free variables as indicated (and further free variables allowed). Then* ZF *proves*

$$\forall M.\ \exists S \supseteq M.\ \forall x_1, \ldots, x_n \in S.\ \varphi(x_1, \ldots, x_n) \Leftrightarrow \varphi^S(x_1, \ldots, x_n),$$

where φ^S is the S-relativization of φ, obtained by substituting any occurrence of "$\forall x$" and "$\exists x$" by "$\forall x \in S$" and "$\exists x \in S$". Furthermore, the resulting set S may be supposed to be transitive, to be closed under subsets or even to be a stage V_α of the cumulative hierarchy; and given not a single formula φ but a finite list $\varphi_1, \ldots, \varphi_s$ of formulas, we may suppose that S reflects all of them.

The reflection principle expresses that truth of any formula can already be checked in an initial segment of the universe. This observation is important both for philosophical and for practical reasons: Philosophically, it tells us that the set-theoretic universe cannot be distinguished from its initial segments by any set-theoretical property. Practically, it allows to transfer results obtained for a restricted class of objects to all such objects. For instance, if we manage to verify (the S-relativization of) some group-theoretic statement for all groups contained in an arbitrary set X, then we may deduce by the reflection principle that the statement holds for all groups in the universe. Examples from sheaf cohomology and more generally category theory abound; an example from set theory is presented in Proposition 3.6.

The reflection principle has been used by Feferman to construct ZFC/S ("ZFC with smallness"), a conservative extension of ZFC which provides a useful foundation of category theory [8]. This system extends ZFC by a new constant symbol $\mathbb{S}$ together with axioms stating that $\mathbb{S}$ is transitive, closed under subsets and reflective with respect to every formula $\varphi(x_1, \ldots, x_n)$ of the original language:

$$\forall x_1, \ldots, x_n \in \mathbb{S}.\ \varphi(x_1, \ldots, x_n) \Leftrightarrow \varphi^{\mathbb{S}}(x_1, \ldots, x_n). \qquad (\star)$$

The system ZFC/S is conservative over ZFC because any given proof in ZFC/S uses only a finite number of instances of the axioms ($\star$), whereby the reflection principle can be used to yield an honest set S which validates just the same equivalences and can hence be used in place of $\mathbb{S}$.

Elements of $\mathbb{S}$ are deemed "small", so that $\mathbb{S}$ is the set of all small sets. The system ZFC/S is useful as a foundation for category theory because it supports, without requiring new set-theoretical commitments such as the existence of Grothendieck universes, a native treatment of large structures. For instance, the category of all small sets can be formed entirely within ZFC/S, without resorting to classes. We invite the reader wanting to learn more about the merits of ZFC/S to study the survey by Shulman [12, Section 11].

To our naive eyes, the passage from ZFC to ZFC/S looked sufficiently innocent so that we set out to develop an intuitionistic version of ZFC/S: To our mind, size issues were entirely different concerns than issues of constructivity, and we hence opined that they should be dealt with separately. Such a separation would not only lead to improved mental hygiene and better understanding, but would also allow us to use the benefits of ZFC/S in situations where the law of excluded middle and the axiom of choice are not available, such as in realizability semantics, sheaf semantics or quite generally topos semantics.

We expected this modification to be entirely straightforward. However, the situation turned out to be more subtle and we failed in our original goal of verifying the reflection principle in intuitionistic Zermelo–Fraenkel set theory IZF. The situation for CZF, the predicative subsystem of IZF commonly heralded as the largest common denominator of all flavors of constructive set theory, remains even more elusive.

However, we succeeded in verifying the reflection principle in only a slight extension of IZF:

Theorem 1.2. *The reflection principle is equivalent, over* IZF*, to the strong relation reflection scheme* RRS_2*.*

We also give a weaker version of the reflection principle which is equivalent to Aczel's original reflection scheme RRS. Both RRS and RRS_2 will be reviewed below. They are validated not only by ZFC, but also by ZF, and furthermore by all known models of IZF, hence might be regarded as not entirely unconstructive, even though they are conjectured to be independent of IZF. As a result, the question whether the reflection principle holds for IZF remains open (though conjectured to be false), and for the stronger system $\mathrm{IZF} + \mathrm{RRS}_2$ we can give a variant "with smallness" which can serve as a set-theoretic foundation for category theory.

This note is organized as follows. Section 2 reviews the classical proof of the reflection principle in the context of ZF set theory. Section 3 reviews Aczel's relation reflection scheme and its variants. Our main result is presented in Section 4. We conclude in Section 5 and Section 6 with two short applications.

Acknowledgments. XXX

2. Review of the classical reflection principle

A basic proof of the reflection principle in ZF runs as follows. Our proof of the reflection principle in $\mathrm{IZF} + \mathrm{RRS}_2$ will follow the same outline.

Lemma 2.1. *Let $\varphi(u_1, \ldots, u_n, x)$ be a formula in the language of set theory. Then* ZF *proves*

$$\begin{aligned} &\forall M.\ \exists S \supseteq M.\ \forall u_1, \ldots, u_n \in S. \\ &\qquad\qquad (\exists x.\ \varphi(u_1, \ldots, u_n, x)) \Longrightarrow (\exists x \in S.\ \varphi(u_1, \ldots, u_n, x)). \end{aligned}$$

Furthermore: (1) We may suppose that S is transitive. (2) We may suppose that S is closed under subsets. (3) Given a finite list $\varphi_1, \ldots, \varphi_s$ of formulas instead of the single formula φ, we may suppose that S bounds all of the formulas φ_i.

Proof. Given a class X (as commonly understood as the comprehension of a formula), we denote by $X^\sim$ its subclass $\{x \in X \mid \forall y \in X.\ \mathrm{rank}(x) \leq \mathrm{rank}(y)\}$, where the rank function refers to the stage in the cumulative hierarchy. The two fundamental properties of this construction are: This subclass is equal to a set,[1] and it is inhabited if and only if X is.

Starting with $S_0 := M$, we construct S_{k+1} from S_k as the union

$$S_{k+1} := S_k \cup \bigcup_{u_1, \ldots, u_n \in S_k} \{x \mid \varphi(u_1, \ldots, u_n, x)\}^\sim.$$

It is then easy to check that $S := \bigcup_{k \in \mathbb{N}} S_k$ is a set with the required property.

For addendum (1), we change the definition of S_{k+1} to be the transitive closure of what is was before. To further accommodate addendum (2), we change this definition again, to be P^ω of what is was before, where $P^\omega(X) := \bigcup_{\ell \in \mathbb{N}} P^{(\ell)}(X)$ is the union of iterated powersets. For addendum (3), we change the definition of S_{k+1} to include one summand for each formula φ_i. □

The proof of Lemma 2.1 is mostly constructive; however, there is one issue with nontrivial ramifications: While IZF does show that the subclass $X^\sim$ of a given class X is a set, it does not verify that $X^\sim$ is inhabited if X is. This would amount to the constructive taboo that any inhabited set contains a rank-minimal element; and in fact, by a result of Friedman and Scedrov [**XXX**], no definable substitute for $X^\sim$ exists. The remedy presented in Section 4 will construct $X^\sim$ in a non-unique fashion and then deal with the resulting fallout that taking the union requires additional care.

Theorem 2.2. *Let $\varphi(x_1, \ldots, x_n)$ be a formula in the language of set theory. Then* ZF *proves*

$$\forall M.\ \exists S \supseteq M.\ \forall x_1, \ldots, x_n \in S.\ \varphi(x_1, \ldots, x_n) \Leftrightarrow \varphi^S(x_1, \ldots, x_n).$$

Furthermore, the resulting set S may be supposed to be transitive and to be closed under subsets; and given not a single formula φ but a finite list $\varphi_1, \ldots, \varphi_s$ of formulas, we may suppose that S reflects all of them.

Proof. Let a set M be given. We obtain S by applying Lemma 2.1 to the list of all subformulas of φ which start with an existential quantifier. That the resulting set S has the required property can then be checked by an induction on the structure of φ. The cases "$=$", "$\in$", "$\top$", "$\bot$", "$\wedge$", "$\vee$", "$\Rightarrow$" follow trivially from the induction hypothesis. The case "$\forall$" does not need to be treated since we may assume without loss of generality that all universal quantifiers in φ have been rewritten as "$\neg\exists\neg$".

[1] If X is empty, then this claim is trivial; if X is inhabited by some element x_0, then $X^\sim$ can be obtained using separation from $V_{\mathrm{rank}(x_0)+1}$; and in fact, by an argument using the set of truth values and unbounded separation, the claim can also be proven in IZF.

The remaining case is where φ is of the form $\varphi \equiv (\exists x.\ \psi(u_1, \ldots, u_m, x))$. In this case, the claim is that

$$\forall u_1, \ldots, u_m \in S.\ (\exists x.\ \psi(u_1, \ldots, u_m, x)) \Longleftrightarrow (\exists x \in S.\ \psi^S(u_1, \ldots, u_m, x)).$$

This claim follows by the property of S guaranteed by Lemma 2.1 and by the induction hypothesis concerning the subformula $\psi^S(u_1, \ldots, u_m, x)$. $\square$

The proof of Theorem 2.2 is constructive with the sole exceptions of treating the case of universal quantifiers by appealing to the law of excluded middle and referring to the unconstructive Lemma 2.1. For the constructive proof in Section 4, we will solve both issues by instead appealing to a strengthened version of Lemma 2.1.

3. Aczel's relation reflection axiom scheme and its variants

We follow the usual convention that IZF is the set theory with the following axioms: extensionality, pair, union, empty set, infinity, unbounded separation, collection, powerset and $\in$-induction. We direct the reader wishing for a survey of IZF and other nonclassical set theories to Crosilla's entry in the SEP [7].

We write "$R : X \rightrightarrows Y$" to mean "$\forall x \in X.\ \exists y \in Y.\ \langle x, y\rangle \in R$".

Definition 3.1. Let X and $R \subseteq X \times X$ be classes. Let $x_0 \in X$.

- DC *dependent choice*
  If X and R are sets and if $R : X \rightrightarrows X$, then there is a function $f : \mathbb{N} \to X$ such that $f(0) = x_0$ and $\langle f(k), f(k+1)\rangle \in R$ for all numbers k.
- RDC *relativized dependent choice*
  If $R : X \rightrightarrows X$, then there is a function $f : \mathbb{N} \to X$ such that $f(0) = x_0$ and $\langle f(k), f(k+1)\rangle \in R$ for all numbers k.
- RRS *Aczel's relation reflection scheme*
  If $R : X \rightrightarrows X$, then there is a set B such that $x_0 \in B \subseteq X$ and $R : B \rightrightarrows B$.
- MDC *Palmgren's multivalued dependent choice*
  If $R : X \rightrightarrows X$, then there is a function $f : \mathbb{N} \to P(X)$ (the class of all subsets of X) such that $x_0 \in f(0)$ and such that $\forall x \in f(k).\ \exists y \in f(k+1).\ \langle x, y\rangle \in R$ for every number k.

Definition 3.2. Let X and $R \subseteq X \times X \times X$ be classes. Let $x_0 \in X$.

- RRS$_2$ If $R : X \times X \rightrightarrows X$, then there is a set B such that $x_0 \in B \subseteq X$ and $R : B \times B \rightrightarrows B$.
- MDC$_2$ If $R : X \times X \rightrightarrows X$, then there is a function $f : \mathbb{N} \to P(X)$ such that $x_0 \in f(0)$ and such that $\forall x \in f(k).\ \forall x' \in f(k').\ \exists y \in f(\max\{k, k'\}+1).\ \langle x, x', y\rangle \in R$ for any numbers k, k'.

Aczel's relation reflection scheme RRS first surfaced in the theory of coinductive definitions of classes and enjoys substantial stability properties, as it passes from the meta theory to XXX[all kinds of] models. Background on RRS can be found in Aczel's original article introducing it [1], and more information on choice axioms in general is contained in the book draft by Aczel and Rathjen [3, Section 10]. RRS is equivalent, over CZF and a fortiori over IZF, to Palmgren's multivalued dependent choice [11]; this observation makes the relationship to dependent choice more visible.

Remark 3.3. Most published renderings of RRS and MDC do not refer to an initial element $x_0 \in X$, but to a set $A \subseteq X$ and then require instead of that $x_0 \in B$, that $A \subseteq B$. This difference is immaterial, thanks in one direction to the existence of singleton sets and in the other to strong collection. The same is true for RRS_2 and MDC_2, though the proof is not as easy.

Aczel's RRS should not in itself be regarded as a choice principle; however in its presence, ordinary DC entails RDC. This result is due to Aczel [1, Theorem 2.4], who proved the equivalence $\mathrm{RDC} = \mathrm{RRS} + \mathrm{DC}$ over CZF^-, and we give a new proof of this equivalence over the stronger theory IZF, in Section 5.

The axiom scheme RRS_2 does not appear to have been studied much. Apart from the PhD thesis by Ziegler [13], where it is called the *strong relation reflection principle*, we have not been able to track down further mentions of it in the literature; hence it seems prudent to verify some of its basic properties here.

In the presence of DC, RRS is equivalent to RRS_2:

Proposition 3.4. *Over* CZF^-, RDC *is equivalent to* $\mathrm{RRS}_2 + \mathrm{DC}$.

Proof. Trivially, RDC entails DC, and $\mathrm{RRS}_2 + \mathrm{DC}$ entails RDC by Aczel's result since RRS_2 entails RRS.

To verify that RDC entails RRS_2, let classes X and $R \subseteq X \times X \times X$ be given, let $x_0 \in X$ be an element and assume $R : X \times X \rightrightarrows X$. Let $\mathrm{List}(X)$ be the class of finite lists with entries in X. We declare a class $\widehat{X}$ by

$$\widehat{X} := \{\langle i, j, v\rangle \mid i, j \in \mathbb{N}, v \in \mathrm{List}(X), i, j < \mathrm{length}(v)\}$$

and a relation $\widehat{R} \subseteq \widehat{X} \times \widehat{X}$ by defining $\langle\langle i, j, v\rangle, \langle i', j', v'\rangle\rangle \in \widehat{R}$ to be equivalent to

there exists an element $y \in X$ such that
(1) $\langle v!i, v!j, y\rangle \in R$ (where $v!k$ is the element of the list v at position k),
(2) v' is obtained from v by adding the single element y at the end and
(3) $\langle i', j'\rangle$ is the next point after $\langle i, j\rangle$ in some fixed enumeration of $\mathbb{N}^2$ which, for each number n, first visits all points $\langle k, l\rangle$ with $k, l < n$ before it visits any of the other points.

Then $\widehat{R} : \widehat{X} \rightrightarrows \widehat{X}$. Applying RDC with initial value $\langle 0, 0, [x_0]\rangle \in \widehat{X}$ yields a function $f : \mathbb{N} \to \widehat{X}$. Let B be the set of all entries of the lists contained in the tuples $f(k)$. Then $x_0 \in B \subseteq X$ and $R : B \times B \rightrightarrows B$. □

A consequence of Proposition 3.4 is that RRS_2 holds in Aczel's type-theoretic "sets as trees" model of CZF [2], since that model validates RDC.

Proposition 3.5. *Over* CZF^-, RRS_2 *is equivalent to* MDC_2.

Proof. The proof in [11] carries over. □

Proposition 3.6. ZF *proves* RRS_2.

Proof. Let X and $R \subseteq X \times X$ be classes. Let $x_0 \in X$ and assume $R : X \rightrightarrows X$. By the reflection principle, there is a set $S \ni x_0$ such that $R : X \cap S \rightrightarrows X \cap S$. This concludes the proof as the class $B := X \cap S$ is a set by separation. □

4. Constructivizing the reflection principle

Our constructive rendition of the reflection principle will require the axiom scheme RRS_2 displayed in Definition 3.2. This result is the best possible, as we verify in Theorem 4.6 that conversely the reflection principle entails RRS_2.

Even though superficially similar, the following lemma is *not* yet a constructivization of Lemma 2.1; these two lemmas differ in the set from which $u_1, \ldots, u_n$ are drawn.

Lemma 4.1. *Let $\varphi(u_1, \ldots, u_n, x)$ be a formula in the language of set theory. Then* IZF *proves*

$$\begin{aligned}
&\forall H.\ \exists H' \supseteq H.\ \forall u_1, \ldots, u_n \in H. \\
&\qquad (\exists x.\ \varphi(u_1, \ldots, u_n, x)) \implies (\exists x \in H'.\ \varphi(u_1, \ldots, u_n, x)) \quad \wedge \\
&\qquad (\forall x.\ \varphi(u_1, \ldots, u_n, x)) \impliedby (\forall x \in H'.\ \varphi(u_1, \ldots, u_n, x)).
\end{aligned}$$

Furthermore: (1) We may suppose that H' is transitive. (2) We may suppose that H' is closed under subsets. (3) Given a finite list $\varphi_1, \ldots, \varphi_s$ of formulas instead of the single formula φ, we may suppose that H' has the displayed property for each of the formulas φ_i.

Proof. Let $\Omega := P(\{0\})$ be the set of truth values. For given elements $u_1, \ldots, u_n \in H$, we have

$$\begin{aligned}
&\forall a \in \{a \in \{0\} \mid \exists x.\ \varphi(u_1, \ldots, u_n, x)\}. \qquad \exists x.\ \varphi(u_1, \ldots, u_n, x) \quad \text{and} \\
&\forall p \in \{p \in \Omega \mid \exists x.\ (0 \in p \Leftrightarrow \varphi(u_1, \ldots, u_n, x))\}.\ \exists x.\ (0 \in p \Leftrightarrow \varphi(u_1, \ldots, u_n, x)).
\end{aligned}$$

Hence, by collection, there are sets C and D such that

$$\begin{aligned}
(\exists x.\ \varphi(u_1, \ldots, u_n, x)) &\implies (\exists x \in C.\ \varphi(u_1, \ldots, u_n, x)) \quad \text{and} \\
(\forall x.\ \varphi(u_1, \ldots, u_n, x)) &\impliedby (\forall x \in D.\ \varphi(u_1, \ldots, u_n, x)).
\end{aligned}$$

The union $C \cup D$ satisfies both of these conditions at once.

Applying collection again, there is a set X such that for any $u_1, \ldots, u_n \in H$ there exists a set $E \in X$ such that

$$\begin{aligned}
(\exists x.\ \varphi(u_1, \ldots, u_n, x)) &\implies (\exists x \in E.\ \varphi(u_1, \ldots, u_n, x)) \quad \text{and} \\
(\forall x.\ \varphi(u_1, \ldots, u_n, x)) &\impliedby (\forall x \in E.\ \varphi(u_1, \ldots, u_n, x)).
\end{aligned}$$

Hence the set $H' := M \cup \bigcup X$ has the required property.

To ensure that H' is transitive and closed under subsets, we pass first to its transitive closure and then compute the union of all its finitely-iterated powersets.

In order to accommodate more than a single formula φ, we add one summand in the definition of H' for each formula φ_i. ☐

The proof of Lemma 4.1 makes crucial use of unbounded separation and the powerset axiom. Hence we do not believe that it can be improved to work over CZF. Since we do not have any uniqueness guarantee on the "$\exists x$" quantifiers in the proof, it also does not work over IZF_{Rep}, the variant of IZF with replacement instead of collection.

Lemma 4.2. *Let $\varphi(u_1, \ldots, u_n, x)$ be a formula in the language of set theory. Then* $\text{IZF} + \text{RRS}_2$ *proves*

$$\begin{aligned}
&\forall M.\ \exists S \supseteq M.\ \forall u_1, \ldots, u_n \in S. \\
&\qquad (\exists x.\ \varphi(u_1, \ldots, u_n, x)) \implies (\exists x \in S.\ \varphi(u_1, \ldots, u_n, x)) \quad \wedge \\
&\qquad (\forall x.\ \varphi(u_1, \ldots, u_n, x)) \impliedby (\forall x \in S.\ \varphi(u_1, \ldots, u_n, x)).
\end{aligned}$$

Furthermore: (1) We may suppose that S is transitive. (2) We may suppose that S is closed under subsets. (3) Given a finite list $\varphi_1, \ldots, \varphi_s$ of formulas instead of the single formula φ, we may suppose that S bounds all of the formulas φ_i.

Proof. By RRS_2, there is a set B such that $M \in B$ and such that for any $H_1, H_2 \in B$, there is a set $H' \in B$ such that $H := H_1 \cup H_2$ and H' are related as in the conclusion of Lemma 4.1.

We set $S := \bigcup B$. This set has the required property. To verify this, it is useful to observe that given $u_1, \ldots, u_n \in S$, there is a common set $H \in B$ such that $u_1, \ldots, u_n \in H$.

In order to ensure addendum (1) and (2), we apply RRS_2 in a slightly different way to guarantee that for any sets $H_1, H_2 \in B$, there is a set $H' \in B$ such that $H := H_1 \cup H_2$ and H' are related as in the conclusion of Lemma 4.1 and such that furthermore H' is transitive and closed under subsets. Even though it cannot be expected that any particular set $H \in B$ will be transitive and closed under subsets, the union S will. Addendum (3) can be ensured because of addendum (3) of Lemma 4.1. $\square$

Remark 4.3. In the special case $n = 1$, the proof of Lemma 4.2 can be simplified to only use RRS instead of RRS_2, because in this case it suffices for the set B to be such that for any $H \in B$, there is a set $H' \in B$ such that H and H' are related as in the conclusion of Lemma 4.1.

Theorem 4.4. *Let $\varphi(x_1, \ldots, x_n)$ be a formula in the language of set theory. Then* $\mathrm{IZF} + \mathrm{RRS}_2$ *proves*

$$\forall M.\ \exists S \supseteq M.\ \forall x_1, \ldots, x_n \in S.\ \varphi(x_1, \ldots, x_n) \Leftrightarrow \varphi^S(x_1, \ldots, x_n),$$

where φ^S is the S-relativization of φ. Furthermore, the resulting set S may be supposed to be transitive and to be closed under subsets; and given not a single formula φ but a finite list $\varphi_1, \ldots, \varphi_s$ of formulas, we may suppose that S reflects all of them.

Proof. The proof of Theorem 2.2 carries over. The only difference is that instead of Lemma 2.1, Lemma 4.2 has to be used, and that the case for the universal quantifier has to be treated just as the case for the existential quantifier has to. $\square$

Scholium 4.5. *Let $\varphi(x_1, \ldots, x_n)$ be a formula in the language of set theory. Assume that the surrounding scope of any unbounded quantifier in φ contains at most one free variable. Then* IZF + RRS *proves the same conclusion as stated in Theorem 4.4.*

Proof. The condition on the number of free variables allows the proof of Theorem 4.4 to be adapted to employ the version of Lemma 4.2 outlined in Remark 4.3, which requires only IZF + RRS instead of $\mathrm{IZF} + \mathrm{RRS}_2$. $\square$

Theorem 4.6. *Over* IZF*, each instance of Aczel's relation reflection scheme* RRS_2 *can be deduced from suitable instances of the assumption that, given a finite list of formulas, for every set M there is a set $S \supseteq M$ reflecting the given formulas.*

Proof. Let X and $R \subseteq X \times X \times X$ be classes. Let $x_0 \in X$ be an element and suppose $R : X \times X \rightrightarrows X$.

By assumption, there is a set $S \ni x_0$ which reflects the three formulas "$x \in X$", "$\langle x, x', y\rangle \in R$" and "$\exists y \in X.\ \langle x, x', y\rangle \in R$".

The class $B := X \cap S$ is a set by separation and contains x_0. Given $x, x' \in B$, there is a set y such that $y \in X$ and $\langle x, y, y'\rangle \in R$. By the reflecting property of S, we can assume that such an element y exists in S.

Hence $R : B \times B \rightrightarrows B$. $\square$

5. A NEW PROOF OF RDC = RRS + DC

When he introduced RRS, Aczel proved that over CZF without subset collection, relative dependent choice RDC is equivalent to the conjunction of RRS and dependent choice DC [1, Theorem 2.4]. Using reflection, we can provide a new proof of this fact, although over the much stronger base theory IZF instead of CZF$^-$. The idea is that reflection allows to reduce RDC to DC.

Proposition 5.1. *Over* IZF, RDC *is equivalent to* RRS + DC.

Proof. Trivially, RDC implies DC, and RDC implies RRS by a similar, though much simpler, argument as in the proof of Proposition 3.4.

Conversely, assume RRS and DC. In order to verify RDC, let classes X and $R \subseteq X \times X$ be given. Let $x_0 \in X$ and assume $\forall x \in X.\ \exists y \in X.\ \langle x, y\rangle \in R$.

We cannot apply Scholium 4.5 to the formula "$\forall x \in X.\ \exists y \in X.\ \langle x, y\rangle \in R$" since unbounded quantifiers implicitly appearing in the formulas "$x \in X$" and "$\langle x, y\rangle \in R$" (recalling that X and R are classes) may violate the condition on the number of free variables. However, we can opt to leave these subformulas to be untranslated when carrying out the S-relativization; with this understanding, Scholium 4.5 can be applied to yield a set $S \ni x_0$ such that

$$\forall x \in S.\ (x \in X \Rightarrow \exists y \in S.\ (y \in X \wedge \langle x, y\rangle \in R)).$$

Hence $\forall x \in X \cap S.\ \exists y \in X \cap S.\ \langle x, y\rangle \in R$. By DC, there is a choice function $f : \mathbb{N} \to X \cap S$ such that $f(0) = x_0$ and $\langle f(k), f(k+1)\rangle \in R$ for all numbers k. This is a function of the kind required by RDC. ☐

6. AN INTUITIONISTIC VERSION OF FEFERMAN'S ZFC/S

Definition 6.1. The system $(\text{IZF} + \text{RRS}_2)/\text{S}$ is obtained from $\text{IZF} + \text{RRS}_2$ by adding a constant symbol $\mathbb{S}$ together with axioms stating that $\mathbb{S}$ is transitive, closed under subsets and reflective for all formulas of the original language.

Proposition 6.2. *The system* $(\text{IZF} + \text{RRS}_2)/\text{S}$ *is conservative over* $\text{IZF} + \text{RRS}_2$.

Proof. Because the reflection principle is available in $\text{IZF} + \text{RRS}_2$, the same argument as for ZFC/S applies. ☐

Just as ZFC/S can serve as a set-theoretic foundation for category theory in a classical context, we argue that $(\text{IZF} + \text{RRS}_2)/\text{S}$ can serve as such a foundation in an intuitionistic context (provided, of course, one is willing to accept RRS_2).

The system $(\text{IZF} + \text{RRS}_2)/\text{S}$ is also interesting from the point of view of topos theory. We recall that any topos supports an internal language which can be used to reason about the objects and morphisms of the topos in a naive element-based language, allowing us to pretend that the objects are plain sets (or types) and that the morphisms are plain maps between those sets ([5, Chapter 6], [6, Section 1.3], [9, Chapter 14], [10, Chapter VI]). We refer to [4, Sections 1 and 2] for a short introduction and a review of some of the applications of the internal language.

Given a topos $\mathcal{E}$ and a formula φ in its internal language, we write "$\mathcal{E} \models \varphi$" to mean that φ holds in $\mathcal{E}$. As a special case, truth in the topos Set, the category of all sets, coincides with truth in the background theory; symbolically: $\text{Set} \models \varphi$ iff φ.

However, in the context of (IZF or) ZFC, it is difficult to make this claim precise. Because in ZFC the category Set of all sets can not be coded as a set, ZFC cannot define truth in Set. We must therefore resort to a meta theory in order to express this claim, for instance by stating that primitive recursive arithmetic PRA proves that for any formula φ of ZFC, ZFC proves "($\mathrm{Set} \models \varphi) \Leftrightarrow \varphi$", where "$\mathrm{Set} \models \varphi$" is to be unrolled by PRA.

An alternative is offered by ZFC+I, ZFC plus the existence of a strongly inaccessible cardinal. In this system, there is a Grothendieck universe U; we can form the category Set_U of all sets of U as an honest set; can define truth in Set_U; and prove, within the system, that for any formula φ, $(\mathrm{Set}_U \models \varphi) \Leftrightarrow (U \models \varphi)$. This even holds for formulas of the full infinitary language of toposes, which allows infinite disjunctions and infinite conjunctions; this extended language could not be treated by resorting to PRA as indicated above.

However, truth in a Grothendieck universe U need not be related to actual truth. A solution to this problem is provided by ZFC/S and by $(\mathrm{IZF} + \mathrm{RRS}_2)/\mathrm{S}$. In these systems, we can form the topos $\mathrm{Set}_{\mathbb{S}}$, define truth in it, prove for all formulas φ that $(\mathrm{Set}_{\mathbb{S}} \models \varphi) \Leftrightarrow (\mathbb{S} \models \varphi)$; and reflection for $\mathbb{S}$ ensures that for each (external, standard) formula φ, the system proves "$(\mathrm{Set}_{\mathbb{S}} \models \varphi) \Leftrightarrow \varphi$".

7. Outlook

We proved that, over IZF, the reflection principle is equivalent to RRS_2. This gives credence to the claim that the reflection principle is not provable over IZF alone. However, the following question remains open:

Question 7.1. Does IZF prove RRS_2?

If the answer is in the negative, as is most likely, then no conservative extension could include a constant symbol $\mathbb{S}$ should that $\mathbb{S}$-relativized truth is the same as absolute truth.

However, truth in toposes is a more flexible notion than S-relativized truth for any set S. Hence one might hope that even in this case, the following question does have a positive answer:

Question 7.2. Is there a conservative extension IZF' of IZF, containing a constant symbol $\mathbb{E}$, such that IZF' proves that $\mathbb{E}$ is an elementary topos and such that IZF' proves "$(\mathbb{E} \models \varphi) \Leftrightarrow \varphi$" for any formula φ?

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