EXPLORING MATHEMATICAL OBJECTS FROM CUSTOM-TAILORED MATHEMATICAL UNIVERSES

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ABSTRACT. XXX

CONTENTS

Toposes can be pictured as mathematical universes in which we can do mathematics. Most mathematicians spend all their professional life in just a single topos, the so-called *standard topos*. However, besides the standard topos, there is a colorful host of alternate toposes which are just as worthy of mathematical study and in which mathematics plays out slightly differently (Figure 1).

For instance, there are toposes in which the axiom of choice and the intermediate value theorem from undergraduate calculus fail, toposes in which any function $\mathbb{R} \to \mathbb{R}$ is continuous and toposes in which infinitesimal numbers exist.

The purpose of this contribution is twofold.

(1) We give a glimpse of the toposophic landscape, presenting several specific toposes and exploring their peculiar properties.
(2) We explicate how toposes provides distinct lenses through which the usual mathematical objects of the standard topos can be viewed.

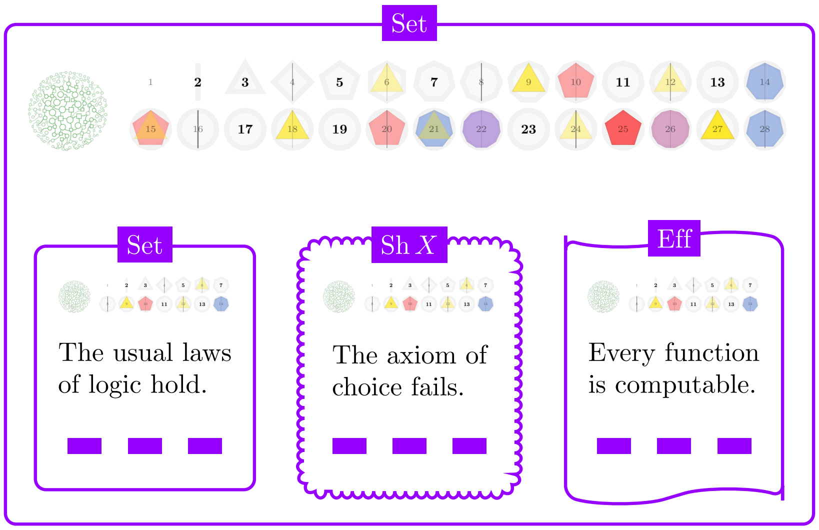

FIGURE 1. A glimpse of the toposophic landscape, displaying alongside the standard topos Set two further toposes.

Viewed through such a lens, a given mathematical object can have different properties than when considered normally. In particular, it can have better properties for the purposes of specific applications, especially if the topos is custom-tailored to the object in question. This change of perspective has been used in mathematical practice and demonstrates that toposes go much beyond being logicians' testbeds. To give just a taste of what is possible, through the lens provided by an appropriate topos, any given ring can look like a field and hence mathematical techniques for fields also apply, through the lens, to rings.

We argue that toposes and specifically the change in perspective provided by toposes are ripe for philosophical analysis. In particular, there are the following connections with topics in the philosophy of mathematics:

(1) Toposes enrich the realism/anti-realism debate in that they paint the larger picture that the platonic heaven of mathematical objects is not unique: besides the standard heaven of the standard topos, we can fathom the alternate heavens of all other toposes, all embedded in a second-order heaven.
(2) Mathematics is not only about studying mathematical objects, but also about studying the relations between mathematical objects. The distinct view on mathematical objects provided by any topos uncovers relations which otherwise remain hidden.
(3) In some cases, a mathematical relation can be expressed quite succinctly using the language of a specific topos and not so succinctly using the language of the standard topos. This phenomenon showcases the importance of *appropriate language*.
(4) Toposes provide new impetus to study constructive mathematics and intuitionistic logic, in particular also to restrict to intuitionistic logic on the meta level and to consider the idea that the platonic heaven might be governed by intuitionistic logic.

This note touches on these topics and invites further research.

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1. Toposes as alternate mathematical universes

Formally, a topos is a certain kind of *category*, containing objects and morphisms between those objects. The formal definition, recorded here only for reference, requires some amount of category theory, but, as will be outlined in the following sections, exploring the mathematical universe of a given topos does not.

Definition 1.1. A *topos* (more precisely *elementary topos* or *logos* with a natural numbers object) is a category which has all finite limits, is cartesian closed, has a subobject classifier and contains a natural numbers object.

Put briefly, these axioms state that a topos should share several categorical properties with the category of sets; they ensure that each topos contains its own versions of familiar mathematical objects such as natural numbers, real numbers, groups and manifolds, and is closed under the usual constructions such as cartesian products or quotients. The prototypical topos is the standard topos:

Definition 1.2. The *standard topos* Set is the category which contains all sets as its objects and all maps between sets as morphisms.

Given a topos $\mathcal{E}$, we write "$\mathcal{E} \models \varphi$" to denote that a mathematical statement φ *holds in* $\mathcal{E}$. The meaning of "$\mathcal{E} \models \varphi$" is defined by recursion on the structure of φ following the so-called *Kripke–Joyal translation rules*. For instance, the rule for translating conjunction reads

$$\mathcal{E} \models (\alpha \wedge \beta) \qquad \text{iff} \qquad \mathcal{E} \models \alpha \quad \text{and} \quad \mathcal{E} \models \beta.$$

The remaining translation rules are more involved; we do not list them here for the case of a general topos $\mathcal{E}$, but we will state them in the next sections for several specific toposes.

In the definition of $\mathcal{E} \models \varphi$, the statement φ can be any statement in the language of a general version of higher-order predicate calculus with dependent types. In practice almost any mathematical statement can be interpreted in a given topos.[1]

It is by the Kripke–Joyal translation rules that we can access the alternate universe of a topos. In the special case of the standard topos Set, the definition of "Set $\models \varphi$" unfolds to φ for any statement φ. Hence a statement holds in the standard topos if and only if it holds in the usual mathematical sense.

1.1. The logic of toposes.

By their definition as special kinds of categories, toposes are merely algebraic structures not unlike groups or vector spaces. Hence we need to argue why we picture toposes as mathematical universes while we do not elevate other kinds of algebraic structures in the same way. For us, this usage is justified by the following metatheorem:

[1] The main exceptions are statements from set theory, which typically make substantial use of a global membership predicate "$\in$". Toposes only support a typed *local* membership predicate, where we may write "$x \in A$" only in the context of some fixed type M such that x is of type M and A is of type $P(M)$, the power type of M. We refer to [**XXX**] for ways around this restriction.

Theorem 1.3. *Let $\mathcal{E}$ be a topos and let φ be a statement such that $\mathcal{E} \models \varphi$. If φ intuitionistically entails a further statement ψ (that is, if it is provable in intuitionistic logic that φ entails ψ), then $\mathcal{E} \models \psi$.*

This metatheorem allows us to *reason* in toposes. When first exploring a new topos $\mathcal{E}$, we need to employ the Kripke–Joyal translation rules each time we want to check whether a statement holds in $\mathcal{E}$. But as soon as we have amassed a stock of statements known to be true in $\mathcal{E}$, we can find more by deducing their logical consequences.

For instance, in any topos where the statement "any map $\mathbb{R} \to \mathbb{R}$ is continuous" is true, also the statement "any map $\mathbb{R} \to \mathbb{R}^2$ is continuous" is, since there is an intuitionistic proof that a map into a higher-dimensional Euclidean space is continuous if its individual components are.

The only caveat of Theorem 1.3 is that toposes generally only support intuitionistic reasoning, not the full power of the ordinary *classical reasoning.* That is, within most toposes, the law of excluded middle ($\varphi \vee \neg\varphi$) and the law of double negation elimination ($\neg\neg\varphi \Rightarrow \varphi$) are not available.

While it may appear that these two laws pervade any mathematical theory, in fact a substantial amount of mathematics can be developed intuitionistically (see for instance [**mines-richman-ruitenburg:constructive-algebra**, **lombardi-quitte:constructive-algebra**] for constructive algebra, [**bishop-bridges:constructive-analysis**] for constructive analysis and [**bauer:int-mathematics**, **bauer:video**, **melikhov:intuitionistic-logic**] for further references) and hence the alternate universes provided by toposes cannot be too strange: In any topos, there are infinitely many prime numbers, the square root of two is not rational and the powerset of the naturals is uncountable.

That said, intuitionistic logic still allows for a considerable amount of freedom, and in many toposes statements are true which are baffling if one has only received training in mathematics based on classical logic. For instance, on first sight it looks like the signum function

$$\mathrm{sgn} : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0, \end{cases}$$

is an obvious counterexample to the statement "any map $\mathbb{R} \to \mathbb{R}$ is continuous". However, closer inspection reveals that the signum function cannot be proven to be a total function $\mathbb{R} \to \mathbb{R}$ if only intuitionistic logic is available. The domain of the signum function is the subset $\{x \in \mathbb{R} \,|\, x < 0 \vee x = 0 \vee x > 0\} \subseteq \mathbb{R}$, and in intuitionistic logic this subset cannot be shown to coincide with $\mathbb{R}$.

Sections 2 to 4 present several examples for such anti-classical statements and explain how to make sense of them. Some toposes are closer to the standard topos and do not validate such anti-classical statements:

Definition 1.4. A topos $\mathcal{E}$ is *boolean* if and only if the laws of classical logic are true in $\mathcal{E}$.

Since exactly those statements hold in the standard topos which hold on the meta level, the standard topos is boolean if and only if, as is commonly supposed, the laws of classical logic hold on the meta level. Most toposes of interest are not boolean, irrespective of one's philosophical commitments about the meta level, and conversely some toposes are boolean even if classical logic is not available on the meta level.

Remark 1.5. The axiom of choice (which is strictly speaking not part of classical logic, but of classical set theory) is also not available in most toposes. By *Diaconescu's theorem*, the axiom of choice implies the law of excluded middle in presence of other axioms which are available in any topos.

1.2. **Relation to models of set theory.** In set theory, philosophy and logic, models of set theories are studied. These are structures $(M, \in)$ validating the axioms of some set theory such as Zermelo–Fraenkel set theory with choice ZFC, and they can be pictured as "universes in which we can do mathematics" in much the same way as toposes.

In fact, to any model $(M, \in)$ of a set theory such as ZF or ZFC, there is a topos Set_M such that a statement holds in Set_M if and only if it holds in M.[2]

Example 1.6. The topos Set_V associated to the universe V of all sets (if this structure is available in one's chosen ontology) coincides with the standard topos Set.

In set theory, we use forcing and other techniques to construct new models of set theory from given ones, thereby exploring the set-theoretic multiverse. There are similar techniques available for constructing new toposes from given ones, and some of these correspond to the techniques from set theory.

However, there are also important differences between the notion of mathematical universes as provided by toposes and as provided by models of set theory, both regarding the subject matter and the reasons for why we are interested in them.

Firstly, toposes are more general than models of set theory. By definition, a model of ZFC will always satisfy the axioms of ZFC; in contrast, most toposes do not even validate the law of excluded middle, much less so the axiom of choice.

Secondly, there is a shift in emphasis. An important philosophical objective for studying models of set theory is to explore which notions of sets are coherent: Does the cardinality of the reals need to be the cardinal directly succeeding $\aleph_0$, the cardinality of the naturals? No, there are models of set theory in which the continuum hypothesis fails. Do non-measurable sets of reals need to exist? No, in models of ZF + AD, Zermelo–Fraenkel set theory plus the axiom of determinacy, it is a theorem that every subset of $\mathbb{R}^n$ is Lebesgue-measurable. Can the axiom of choice be added to the axioms of ZF without causing inconsistency? Yes, if M is a model of ZF then L^M, the structure of the constructible sets of M, forms a model of ZFC.

Toposes can be used for similar such purposes, and indeed have been, especially to explore the various intuitionistic notions of sets. However, an important aspect of topos theory is that toposes are used to explore the *standard* mathematical universe: truth in the effective topos tells us what is computable; truth in sheaf toposes tells us what is true locally; toposes adapted to synthetic differential geometry can be used to rigorously work with infinitesimals. All of these examples will be presented in more detail in the next sections.

In a sense which can be made precise, toposes allow us to study the usual objects of mathematics from a different point of view – one such view for every topos –

[2]The topos Set_M can be described as follows: Its objects are the elements of M, that is the entities which M believes to be sets, and its morphisms are those entities which M believes to be maps. The topos Set_M validates the axioms of the structural set theory ETCS [**XXX**], and models are isomorphic if and only if their associated toposes are equivalent as categories.

and it is a beautiful and intriguing fact that with the sole exception of the law of excluded middle, the laws of logic apply to mathematical objects also when viewed through the lens of a specific topos.

1.3. **A glimpse of the toposophic landscape.** There is a proper class of toposes. Figure 1 depicts three toposes side by side: the standard topos, a sheaf topos and the effective topos. Each of these toposes tells a different story of mathematics, and any topos which is not the standard topos invites us to ponder alternative ways how mathematics could unfold.

Some of the most prominent toposes are the following.

(1) The *trivial topos*. In the trivial topos, any statement whatsoever is true. The trivial topos is not interesting on its own, but its existence streamlines the theory and it can be an interesting question whether a given topos coincides with the trivial topos.
(2) Set, the *standard topos*. A statement is true in Set iff it is true in the ordinary mathematical sense.
(3) Set_M, the topos associated to any model $(M, \in)$ of ZF.
(4) Eff, the *effective topos*. A statement is true in Eff iff if it has a *computable witness* as detailed in Section 2. In Eff, any function $\mathbb{N} \to \mathbb{N}$ is computable, any function $\mathbb{R} \to \mathbb{R}$ is continuous and the countable axiom of choice holds (even if it does not on the meta level).
(5) $\mathrm{Sh}(X)$, the *topos of sheaves* over any space X. A statement is true in $\mathrm{Sh}(X)$ iff it holds *locally on* X, as detailed in Section 3. For most choices of X, the axiom of choice and the intermediate value theorem fail in $\mathrm{Sh}(X)$, and this failure is for geometric reasons.
(6) $\mathrm{Zar}(A)$, the *Zariski topos* of a ring A presented in Section 4. This topos contains a mirror image of A which is a field, even if A is not.
(7) $\mathrm{Bohr}(A)$, the *Bohr topos* associated to a noncommutative C^*-algebra A. This topos contains a mirror image of A which is commutative. In this sense, quantum mechanical systems (which are described by noncommutative C^*-algebras) can be regarded as classical mechanical systems (which are described by commutative algebras) [**butterfield-hamilton-isham:bohr**, **heunen-landsman-spitters:aqt**].
(8) $\mathrm{Set}[\mathbb{T}]$, the *classifying topos* of a geometric theory $\mathbb{T}$.[3] This topos contains the *generic* $\mathbb{T}$*-model*. For instance, the classifying topos of the theory of groups contains the *generic group*. Arguably it is this group which we implicitly refer to when we utter the phrase “Let G be a group.”. The generic group has has exactly those properties which are shared by any group whatsoever.[4]
(9) $T(\mathcal{L}_0)$, the *free topos*. A statement is true in the free topos iff it is intuitionistically provable. Lambek and Scott proposed that the free topos can

[3]A geometric theory is a theory in many-sorted first-order logic whose axioms can be put as *geometric sequents*, sequents of the form $\varphi \vdash_{\vec{x}} \psi$ where φ and ψ are geometric formulas (formulas built from equality and specified relation symbols by the logical connectives $\top \perp \wedge \vee \exists$ and by arbitrary set-indexed disjunctions $\bigvee$).

[4]More precisely, this is only true for those properties which can be formulated as geometric sequents. For arbitrary properties φ, the statements “the generic group has property φ” and “all groups have property φ” need not be equivalent. This situation is explored in [**blechschmidt:nullstellensatz**].

reconcile moderate platonism (because this topos has a certain universal property which can be used to single it out among the plenitude of toposes), moderate formalism (because it is constructed in a purely syntactic way) and moderate logicism (because, as a topos, it supports an intuitionistic type theory) [**lambek:incompatible**, **couture-lambek:reflections**].

There are several constructions which produce a new toposes from a given topos $\mathcal{E}$. A non-exhaustive list is the following.

(1) Given an object X of $\mathcal{E}$, the *slice topos* $\mathcal{E}/X$ contains a *generic element* x_0 of X. This generic element can be pictured as the element we implicitly refer to when we utter the phrase "Let x be an element of X.". A statement $\varphi(x_0)$ about x_0 is true in $\mathcal{E}/X$ if and only if in $\mathcal{E}$ the statement $\forall x : X.\ \varphi(x)$ is true.

For instance, the topos $\mathrm{Set}/\mathbb{Q}$ contains the generic rational number x_0. Neither the statement "x_0 is zero" nor the statement "x_0 is not zero" hold in $\mathrm{Set}/\mathbb{Q}$, as it is neither the case that any rational number in Set is zero nor that any rational number in Set is not zero. Like any rational number, the number x_0 can be written as a fraction $\frac{a}{b}$. Just as x_0 itself, the numbers a and b are quite indetermined.

(2) Given a statement φ (which may contain objects of $\mathcal{E}$ as parameters but which must be formalizable as a geometric sequent), there is a largest subtopos of $\mathcal{E}$ in which φ holds. This construction is useful if neither φ nor $\neg\varphi$ hold in $\mathcal{E}$ and we want to force φ to be true. If $\mathcal{E} \models \neg\varphi$, then the resulting topos is the trivial topos.

(3) There is a "smallest dense" subtopos $\mathrm{Sh}_{\neg\neg}(\mathcal{E})$. This topos is always boolean, even if $\mathcal{E}$ and the meta level are not. For a mathematician who employs intuitionistic logic on their meta level, the nonconstructive results of their classical colleagues do not appear to make sense in Set, but they hold in $\mathrm{Sh}_{\neg\neg}(\mathrm{Set})$. If classical logic holds on the meta level, then Set and $\mathrm{Sh}_{\neg\neg}(\mathrm{Set})$ coincide.

The topos $\mathrm{Sh}_{\neg\neg}(\mathcal{E})$ is related to the *double negation translation* $\varphi \mapsto \varphi^{\neg\neg}$ from classical logic into intuitionistic logic: A statement φ holds in $\mathrm{Sh}_{\neg\neg}(\mathcal{E})$ if and only if $\varphi^{\neg\neg}$ holds in $\mathcal{E}$. XXX reference

Toposes are still mathematical structures, and as long as we study toposes within the usual setup of mathematics, our toposes are all part of the standard topos. This is why Figure 1 pictures the standard topos twice. Hence the toposes which we can study in mathematics do not tell us all possible stories how mathematics could unfold, only those which appear coherent from the point of view of the standard topos, and the topos-theoretic multiverse which we have access to is just a small part of an even larger landscape.[5]

To obtain just a hint of how the true landscape looks like, we can study topos theory from the inside of toposes; the resulting picture can look quite different than the picture which emerges from within the standard topos.

For instance, from within the standard topos, we can write down the construction which yields the standard topos and the construction which yields the effective

[5]This paragraph employs an overly narrow conception of "mathematics", for instance excluding any predicative flavors of mathematics. Toposes are impredicative in the sense that any object of a topos is required to have a powerobject. A predicative cousin of toposes are the *arithmetic universes* introduced by Joyal which have recently been an important object of consideration by Maietti and Vickers [**maietti:au**, **maietti-vickers:induction**, **vickers:sketches**].

topos Eff and observe that the resulting toposes are not at all equivalent: In Eff, any function $\mathbb{R} \to \mathbb{R}$ is continuous while Set abounds with discontinuous functions (at least if we assume a classical meta level). In contrast, if we carry out these two constructions from within the effective topos, we obtain toposes which are elementarily equivalent. More precisely, for any statement φ of higher order arithmetic,

$$\mathrm{Eff} \models (\mathrm{Set} \models \varphi) \qquad \text{iff} \qquad \mathrm{Eff} \models (\mathrm{Eff} \models \varphi).$$

In this sense the construction which yields the effective topos is *idempotent* [**oosten:realizability**].

2. The effective topos, a convenient home for computability theory

A basic question in computability theory is: Which computational tasks are solvable in principle by computer programs? For instance, there is an algorithm for computing the greatest common divisor of any pair of natural numbers, and hence we say "any pair of natural numbers *effectively* has a greatest common divisor" or "the function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(n, m) \mapsto \gcd(n, m)$ is *computable*".

In such questions of computability, practical issues such as resource constraints and hardware malfunctions are ignored; we work with the theoretical notion of *Turing machines*, a mathematical abstraction of the computers of the real world.

A basic observation in computability theory is that there are computational tasks which are not solvable even for these idealized Turing machines. The paramount example is the *halting problem*: Given a turing machine M, determine whether M terminates (comes to a stop after having carried out a finite number of computational steps) or not.

A Turing machine H which would solve this problem, that is read the description of a Turing machine M as input and output one or zero depending on whether M terminates or not, would be called a *halting oracle*, and a basic result is that there are no halting oracles. If we fix some effective enumeration of all Turing machines, then we can express the undecidability of the halting problem also by saying that the *halting function*

$$h : \mathbb{N} \longrightarrow \mathbb{N}, \; n \longmapsto \begin{cases} 1, & \text{if the } n\text{-th Turing machine terminates,} \\ 0, & \text{otherwise,} \end{cases}$$

is not computable.

The *effective topos* Eff is a convenient home for computability theory. A statement is true in Eff if and only if it has a *computable witness*. For instance, a computable witness of a statement of the form "$\forall x.\ \exists y.\ \varphi(x, y)$" is a Turing machine which, when given an input x, computes an output y together with a computable witness for $\varphi(x, y)$.

Section 2.1 presents several examples to convey an intuitive understanding of truth in the effective topos; the precise translation rules are displayed in Table 1. A precise definition of the effective topos requires notions of category theory which we do not want to suppose here; it is included only for reference.

Definition 2.1. (1) An *assembly* is a set X together with a relation $(\Vdash_X) \subseteq \mathbb{N} \times X$ such that for every element $x \in X$, there is a number n such that $n \Vdash_X x$.

(2) A *morphism of assemblies* $(X, \Vdash_X) \to (Y, \Vdash_Y)$ is a map $f : X \to Y$ which is *tracked* by a Turing machine, that is for which there exists a Turing

machine M such that for any element $x \in X$ and any number n such that $n \Vdash x$, the computation $M(n)$ terminates and $M(n) \Vdash f(x)$.

A number n such that $n \Vdash_X x$ is called a *realizer* for x and can be pictured as a concrete representation of the abstract element x. The *assembly of natural numbers* is the assembly $(\mathbb{N}, =_\mathbb{N})$ and the *assembly of functions* $\mathbb{N} \to \mathbb{N}$ is the assembly $(X, \Vdash)$ where X is the set of computable functions $\mathbb{N} \to \mathbb{N}$ and $n \Vdash f$ if and only if the n-th Turing machine computes f.

Definition 2.2. The *effective topos* Eff is the ex/reg completion of the category of assemblies.

2.1. **Exploring the effective topos.** Due to its computational nature, truth in the effective topos is quite different from truth in the standard topos. This section explores the following examples:

Statement	in Set	in Eff
Any natural number is prime or not prime.	✓ (trivially)	✓
There are infinitely many primes.	✓	✓
Any function $\mathbb{N} \to \mathbb{N}$ is constantly zero or not.	✓ (trivially)	✗
Any function $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)
Any function $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓
Markov's principle holds.	✓ (trivially)	✓
Heyting arithmetic is categorical.	✗	✓

"Any natural number is prime or not." Even without knowing what a prime number is, one can safely judge this statement to be true in the standard topos, since it is just an instance of the law of excluded middle.

By the Kripke–Joyal semantics, stating that this statement is true in the effective topos amounts to stating that there is a Turing machine which, when given a natural number n as input, terminates with a correct judgment whether n is prime or not. Such a Turing machine indeed exists – writing such a program is often a first exercise in programming courses. Hence the statement is also true in the effective topos, but for the nontrivial reason that such a machine exists.

"There are infinitely many primes." A first-order formalization of this statement is "for any natural number n, there is a prime number p which is greater than n", and is known to be true in the standard topos by any of the many proofs of this fact.

Its external meaning when interpreted in the effective topos is that there exists a Turing machine M which, when given a natural number n as input, terminates with a prime number $p > n$ as output. Such a Turing machine exists, hence the statement is true in the effective topos.[6]

"Any function $\mathbb{N} \to \mathbb{N}$ is constantly zero or not." More formally, the statement is

$$\forall f : \mathbb{N}^\mathbb{N}.\ \big((\forall n : \mathbb{N}.\ f(n) = 0) \vee \neg(\forall n : \mathbb{N}.\ f(n) = 0)\big).$$

[6]More precisely, the machine M should also output the description of a Turing machine which witnesses that p is prime. However, the statement "p is prime" is $\neg\neg$-stable (even decidable), and for those statements witnesses are redundant.

By the law of excluded middle, this statement is trivially true in the standard topos.

Its meaning when interpreted in the effective topos is that there exists a Turing machine M which, when given the description of a Turing machine F which computes a function $f : \mathbb{N} \to \mathbb{N}$ as input, terminates with a correct judgment of whether f is the zero function or not. Such a machine M does not exist, hence the statement is false in the effective topos.

Intuitively, the issue is the following. Turing machines are able to simulate other Turing machines, hence M could simulate F on various inputs to search the list of function values $f(0), f(1), \ldots$ for a nonzero number. In case that after a certain number of steps a nonzero function value is found, the machine M can correctly output the judgment that f is not the zero function. But if the search only turned up zero values, it cannot come to any verdict – it cannot rule out that a nonzero function value will show up in the as yet unexplored part of the function.

A rigorous proof that such a machine M does not exist reduces its assumed existence to the undecidability of the halting problem.

"Any function $\mathbb{N} \to \mathbb{N}$ is computable." The preceding examples give the impression that what is true in the effective topos is simply a subset of what is true in the standard topos. The example of this subsection shows that the relation between the two toposes is more nuanced.

As recalled above, in the standard topos there are functions $\mathbb{N} \to \mathbb{N}$ which are not computable by a Turing machine. Cardinality arguments even show that most functions $\mathbb{N} \to \mathbb{N}$ are not computable: There are $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ functions $\mathbb{N} \to \mathbb{N}$, but only $\aleph_0$ Turing machines and hence only $\aleph_0$ functions which are computable by a Turing machine.

In contrast, in the effective topos, any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine. The external meaning of this internal statement is that there exists a Turing machine M which, when given a description of a Turing machine F computing a function $f : \mathbb{N} \to \mathbb{N}$, outputs a description of a Turing machine computing f. It is trivial to program such a machine M: the machine M simply has to echo its input back to the caller.

To avert a paradox, we should point out where the usual proof of the existence of noncomputable functions theory employs nonconstructive reasoning, for if the proof would not go beyond intuitionistic reasoning, it would also hold internally to the effective topos, in contradiction to the fact that in the effective topos all functions $\mathbb{N} \to \mathbb{N}$ are computable.

The usual proof sets up the halting function $h : \mathbb{N} \to \mathbb{N}$, defined using the case distinction

$$h : n \mapsto \begin{cases} 1, & \text{if the } n\text{-th Turing machine terminates,} \\ 0, & \text{if the } n\text{-th Turing machine does not terminate,} \end{cases}$$

and proceeds to show that h is not computable. However, in the effective topos, this definition does not give rise to a total function from $\mathbb{N}$ to $\mathbb{N}$. The actual domain is the subset of those natural numbers n for which the n-th Turing machine terminates or does not terminate. Assuming the law of excluded middle, this is a trivial condition; but intuitionistically, it is not.

"Any function $\mathbb{R} \to \mathbb{R}$ is continuous." In the standard topos, this statement is plainly false, with the signum and Heaviside functions being prominent counterexamples. In the effective topos, this statement is true and independently due to Kreisel–Lacombe–Shoenfield [**kreisel-lacombe-shoenfield:cont**] and Ceĭtin [**ceitin:cont**]. A rigorous proof is not entirely straightforward (a textbook reference is [**longley-normann:higher-order-computability**]), but an intuitive explanation is as follows.

What the effective topos believes to be a real number is, from the external point of view, a Turing machine X which outputs, when called with a natural number n as input, a rational approximation $X(n)$. These approximations are required to be *consistent* in the sense that $|X(n) - X(m)| \leq 2^{-n} + 2^{-m}$. Intuitively, such a machine X denotes the real number $\lim_{n\to\infty} X(n)$, and the approximations $X(n)$ must be within 2^{-n} of the limit.

A function $f : \mathbb{R} \to \mathbb{R}$ in the effective topos is therefore given by a Turing machine M which, when given the description of such a Turing machine X as input, outputs the description of a similar such Turing machine Y. To compute a rational approximation $Y(n)$, the machine Y may simulate X and can therefore determine arbitrarily many rational approximations $X(m)$. However, within in a finite amount of time, the machine Y can only learn finitely many such approximations. Hence a function such as the signum function, for which even rough rational approximations of $\operatorname{sgn}(x)$ require infinite precision in the input x, do not exist in the effective topos.

"Markov's principle holds." Markov's principle is the following statement:

$$\forall f : \mathbb{N}^{\mathbb{N}}.\ \big((\neg\neg\exists n : \mathbb{N}.\ f(n) = 0) \Longrightarrow \exists n : \mathbb{N}.\ f(n) = 0\big). \tag{MP}$$

It is an instance of the law of double negation elimination and hence trivially true in the standard topos, at least if we subscribe to classical logic on the meta level.

The effective topos inherits Markov's principle from the meta level. The statement "Eff $\models$ (MP)" means that there is a Turing machine M which, when given the description of a Turing machine F computing a function $f : \mathbb{N} \to \mathbb{N}$, outputs the description of a Turing machine S_F which, when given a witness of "$\neg\neg\exists n : \mathbb{N}.\ f(n) = 0$", outputs a witness of "$\exists n : \mathbb{N}.\ f(n) = 0$" (up to trivial conversions, a number n such that $f(n) = 0$).

By the translation rules listed in Table 1, a number e realizes "$\neg\neg\exists n : \mathbb{N}.\ f(n) = 0$" if and only if it is *not not* the case that there is some number e' such that e' realizes "$\exists n : \mathbb{N}.\ f(n) = 0$". Hence, if "$\exists n : \mathbb{N}.\ f(n) = 0$" is realized at all, then any number is a witness of "$\neg\neg\exists n : \mathbb{N}.\ f(n) = 0$".

As a consequence, the input given to machine S_F is entirely uninformative and S_F cannot make direct computational use of it, but it ensures that an *unbounded search* will not fail: The machine S_F can simulate F in order to compute, in turn, the values $f(0), f(1), f(2), \ldots$, and stop with output n as soon as it determines that some function value $f(n)$ is zero.

"Heyting arithmetic is categorical." In addition to the standard model $\mathbb{N}$, the standard topos contains uncountably many nonstandard models of Peano arithmetic (at least if we assume a classical meta level). By a theorem of Benno van den Berg and Jaap van Oosten, the situation is quite different in the effective topos [**berg-oosten:arithmetic**]:

(1) Heyting arithmetic, the intuitionistic cousin of Peano arithmetic, has exactly one model up to isomorphism, namely $\mathbb{N}$.

$\mathrm{Eff} \models \varphi$	iff there is a natural number e such that $e \Vdash \varphi$.

In the following, we write "$e \cdot n \downarrow$" to mean that running the e-th Turing machine on input n terminates, and in this case denote the result by "$e \cdot n$".

$e \Vdash s = t$	iff $s = t$.
$e \Vdash \top$	iff $1 = 1$.
$e \Vdash \bot$	iff $1 = 0$.
$e \Vdash (\varphi \wedge \psi)$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ and $e \cdot 0 \Vdash \varphi$ and $e \cdot 1 \Vdash \psi$.
$e \Vdash (\varphi \vee \psi)$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ and if $e \cdot 0 = 0$ then $e \cdot 1 \Vdash \varphi$, and if $e \cdot 0 \neq 0$ then $e \cdot 1 \Vdash \psi$.
$e \Vdash (\varphi \Rightarrow \psi)$	iff for any number $r \in \mathbb{N}$ such that $r \Vdash \varphi$, $e \cdot r \downarrow$ and $e \cdot r \Vdash \psi$.
$e \Vdash (\forall n\,{:}\,\mathbb{N}.\ \varphi(n))$	iff for any number $n_0 \in \mathbb{N}$, $e \cdot n_0 \downarrow$ and $e \cdot n_0 \Vdash \varphi(n_0)$.
$e \Vdash (\exists n\,{:}\,\mathbb{N}.\ \varphi(n))$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ and $e \cdot 1 \Vdash \varphi(e \cdot 0)$.
$e \Vdash (\forall f\,{:}\,\mathbb{N}^{\mathbb{N}}.\ \varphi(f))$	iff for any function $f_0 : \mathbb{N} \to \mathbb{N}$ and any number r_0 such that f_0 is computed by the r_0-th Turing machine, $e \cdot r_0 \downarrow$ and $e \cdot r_0 \Vdash \varphi(f_0)$.
$e \Vdash (\exists f\,{:}\,\mathbb{N}^{\mathbb{N}}.\ \varphi(f))$	iff $e \cdot 0 \downarrow$ and $e \cdot 1 \downarrow$ and the $(e \cdot 0)$-th Turing machine computes a function $f_0 : \mathbb{N} \to \mathbb{N}$ and $e \cdot 1 \Vdash \varphi(f_0)$.

Table 1. A (fragment of) the translation rules defining the meaning of statements internal to the effective topos.

(2) In fact, even the finitely axiomatizable subsystem of Heyting arithmetic where the induction scheme is restricted to Σ_1-formulas has exactly one model up to isomorphism, again $\mathbb{N}$. As a consequence, Heyting arithmetic is finitely axiomatizable.
(3) Peano arithmetic is "quasi-inconsistent" in that it does not have any models, for any model of Peano arithmetic would also be a model of Heyting arithmetic, but the only model of Heyting arithmetic is $\mathbb{N}$ and $\mathbb{N}$ does not validate the theorem "any Turing machine terminates or does not terminate" of Peano arithmetic.

As a consequence, Gödel's completeness theorem fails in the effective topos: In the effective topos, Peano arithmetic is consistent (because it is equiconsistent to Heyting arithmetic, which has a model) but does not have a model.

The reason for (1) is related to the fact that no nonstandard model of Peano arithmetic in the standard topos is computable [**tennenbaum:models**].

2.2. **Variants of the effective topos.** The effective topos belongs to a wider class of *realizability toposes.* These can be obtained by repeating the construction of the effective topos with any other reasonable model of computation in place of Turing machines. The resulting toposes will in general not be equivalent and reflect higher-order properties of the employed models. Two of these further toposes are of special philosophical interest.

Hypercomputation. Firstly, in place of ordinary Turing machines, one can employ the *infinite-time Turing machines* pioneered by Hamkins and Lewis [**hamkins-lewis:ittm**]. These machines model *hypercomputation* in that they can run for "longer than infinity"; more precisely, the computational steps are indexed by the ordinal numbers instead of the natural numbers. For instance, an infinite-time Turing machine can trivially decide the twin prime conjecture, by simply walking along the natural number line and recording any twin primes it finds. Then, on day ω, it can observe whether it has found infinitely many twins or not.

In the realizability topos constructed using infinite-time Turing machines, the full law of excluded middle still fails, but some instances which are wrong in the effective topos do hold in this topos. For instance, the instance "any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not" does: Its external meaning is that there is an infinite-time Turing machine M which, when given the description of an infinite-time Turing machine F computing a function $f : \mathbb{N} \to \mathbb{N}$ as input, terminates (at some ordinal time step) with a correct judgment of whether f is the zero function or not. Such a machine M indeed exists: It simply has to simulate F on all inputs $0, 1, \ldots$ in order and check whether one of the resulting function values is not zero. This will require a transfinite amount of time (not least because simulating F on just one input might require a transfinite amount of time), but as an infinite-time Turing machine, M is capable of carrying out this procedure.

The realizability topos given by infinite-time Turing machines provides an intriguing environment challenging many mathematical intuitions shaped by classical logic. For instance, while from the point of view of this topos the reals are still uncountable in the sense that there is no surjection $\mathbb{N} \to \mathbb{R}$, there is an injection $\mathbb{R} \to \mathbb{N}$ [**bauer:injection**].

Machines of the physical world. A second variant of the effective topos is obtained by using machines of the physical world instead of abstract Turing machines. In doing so, we of course leave the realm of mathematics, as real-world machines are not objects of mathematical study, but still it is interesting to see which commitments about the nature of the physical world imply which internal statements of the resulting topos.

For instance, Andrej Bauer showed that inside this topos any function $\mathbb{R} \to \mathbb{R}$ is continuous if, in the physical world, only finitely many computational steps can be carried out in finite time and if it is possible to form tamper-free private communication channels [**bauer:int-mathematics**].

3. Toposes of sheaves, a convenient home for local truth

Associated to any topological space X (such as Euclidean space), there is the *topos of sheaves over* X, $\mathrm{Sh}(X)$. To a first approximation, a statement is true in $\mathrm{Sh}(X)$ if and only if it "holds locally on X"; what $\mathrm{Sh}(X)$ believes to be a set is a "continuous family of sets, one set for each point of X". The precise rules of the Kripke–Joyal semantics of $\mathrm{Sh}(X)$ are listed in Table 2.

3.1. **A geometric interpretation of double negation.**

In intuitionistic logic, the double negation $\neg\neg\varphi$ of a statement φ is a slight weakening of φ; while $(\varphi \Rightarrow \neg\neg\varphi)$ is an intuitionistic tautology, the converse can only be shown for some specific statements. The internal language of $\mathrm{Sh}(X)$ gives geometric meaning to this logical peculiarity.

Namely, it is an instructive exercise that $\mathrm{Sh}(X) \models \neg\neg\varphi$ is equivalent to the existence of a *dense open* U of X such that $U \models \varphi$. If $\mathrm{Sh}(X) \models \varphi$, that is if $X \models \varphi$, then there obviously exists such a dense open, namely X itself; however the converse usually fails.

The only case that the law of excluded middle does hold internally to $\mathrm{Sh}(X)$ is when the only dense open of X is X itself; assuming classical logic in the metatheory, this holds if and only if every open is also closed. This is essentially only satisfied if X is discrete.

An important special case is when X is the one-point space. In this case $\mathrm{Sh}(X)$ is equivalent (as categories and hence toposes) to the standard topos. To the extent that mathematics within $\mathrm{Sh}(X)$ can be described as "mathematics over X", this observation justifies the slogan that "ordinary mathematics is mathematics over the point".

3.2. **Real numbers.** As detailed in Section 2.1, what the effective topos believes to be a real number is actually a Turing machine computing arbitrarily-good consistent rational approximations. A similarly drastic shift in meaning, though in an orthogonal direction, occurs with $\mathrm{Sh}(X)$. What $\mathrm{Sh}(X)$ believes to be a (Dedekind) real number a is actually a continuous family of real numbers on X, that is, a continuous function $a : X \to \mathbb{R}$.

Such a function is everywhere positive on X if and only if, from the internal point of view $\mathrm{Sh}(X)$, the number a is positive; it is everywhere zero if and only if, internally, the number a is zero; and it is everywhere negative if and only if, internally, the number a is negative.

The law of trichotomy, stating that any real number is either negative, zero or positive, generally fails in $\mathrm{Sh}(X)$. By the Kripke–Joyal semantics, the external meaning of the internal statement "$\forall a : \mathbb{R}.\ a < 0 \vee a = 0 \vee a > 0$" is that for any continuous function $a : U \to \mathbb{R}$ defined on any open U of X, there is an open covering $U = \bigcup_i U_i$ such that on each member U_i of this covering, the function a is either everywhere negative on U_i, everywhere zero on U_i or everywhere positive on U_i. But this statement is, for most base spaces X, false. Figure 2(c) shows a counterexample.

The weaker statement that for any real number a it is *not not* the case that $a < 0$ or $a = 0$ or $a > 0$ does hold in $\mathrm{Sh}(X)$, for this statement is an intuitionistic tautology. Its meaning is that there exists a dense open U such that U can be covered by opens on which a is either everywhere negative, everywhere zero or everywhere positive. In the example given in Figure 2(c), this open U could be taken as X with the unique zero of a removed.

3.3. **Real functions.** Let $(f_x)_{x \in X}$ be a continuous family of continuous real-valued functions; that is, each of the individual functions $f_x : \mathbb{R} \to \mathbb{R}$ should be continuous and moreover the map $\mathbb{R} \times X \to \mathbb{R}, (a, x) \mapsto f_x(a)$ should be continuous. From the point of view of $\mathrm{Sh}(X)$, this family looks like a single continuous function $f : \mathbb{R} \to \mathbb{R}$.

The internal statement "$f(-1) < 0$" means that $f_x(-1) < 0$ for all $x \in X$, and similarly so for being positive. More generally, if a and b are continuous functions $X \to \mathbb{R}$ (hence real numbers from the internal point of view), the internal statement "$f(a) < b$" means that $f_x(a(x)) < b(x)$ for all $x \in X$.

The internal statement "f possesses a zero", that is "there exists a number a such that $f(a) = 0$", means that all the functions f_x each possess a zero and that moreover, these zeros can locally be picked in a continuous fashion. More precisely, this statement means that there is an open covering $X = \bigcup_i U_i$ such that, for

$\mathrm{Sh}(X) \models \varphi$	iff $X \models \varphi$.
$U \models a = b$	iff $a = b$ on U.
$U \models \top$	is true for any open U.
$U \models \bot$	iff U is the empty open.
$U \models (\varphi \wedge \psi)$	iff $U \models \varphi$ and $U \models \psi$.
$U \models (\varphi \vee \psi)$	iff there is an open covering $U = \bigcup_i U_i$ such that, for each index i, $U_i \models \varphi$ or $U_i \models \psi$.
$U \models (\varphi \Rightarrow \psi)$	iff for every open $V \subseteq U$, $V \models \varphi$ implies $V \models \psi$.
$U \models (\forall a : \mathbb{R}.\ \varphi(a))$	iff for every open $V \subseteq U$ and any continuous function $a_0 : V \to \mathbb{R}$, $V \models \varphi(a_0)$.
$U \models (\exists a : \mathbb{R}.\ \varphi(a))$	iff there is an open covering $U = \bigcup_i U_i$ such that, for each index i, there exists a continuous function $a_0 : U_i \to \mathbb{R}$ with $U_i \models \varphi(a_0)$.

TABLE 2. A (fragment of) the translation rules defining the meaning of statements internal to $\mathrm{Sh}(X)$, the topos of sheaves over a topological space X.

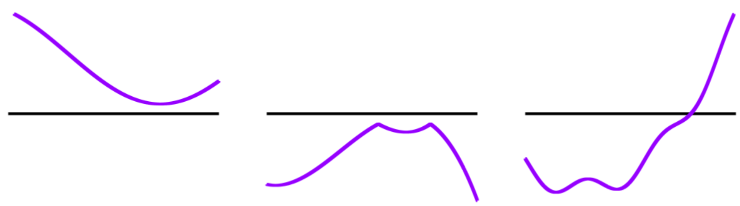

FIGURE 2. Three examples for what the topos $\mathrm{Sh}(X)$ each believes to be a single real number, where the base space X is the unit interval. (a) A positive real number. (b) A negative real number. (c) A number which is neither negative nor zero nor positive. Externally speaking, there is no covering of the unit interval by open subsets on which the depicted function a is either everywhere negative, everywhere zero or everywhere positive.

each index i, there is a continuous function $a : U_i \to \mathbb{R}$ such that $f_x(a(x)) = 0$ for all $x \in U_i$. (On overlaps $U_i \cap U_j$, the zero-picking functions a need not agree.)

From these observations we can deduce that the intermediate value theorem of undergraduate calculus does in general not hold in $\mathrm{Sh}(X)$ and hence does not allow for an intuitionistic proof. The intermediate value theorem states: "If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(-1) < 0$ and $f(1) > 0$, there exists a number a such that $f(a) = 0$." The external meaning of this statement is that in any continuous family $(f_x)_x$ of continuous functions with $f_x(-1) < 0$ and $f_x(1) > 0$ for all $x \in X$, it is locally possible to pick zeros of the family in a continuous fashion. Figure **??** shows a counterexample to this claim.

4. Topos adapted to synthetic differential geometry

The idea of *infinitesimal numbers* – numbers which lie between $-\frac{1}{n}$ and $\frac{1}{n}$ for any natural number n – has a long and rich history. They are not part of today's standard setup of the reals, but they are still intriguing as calculational tools and as a device to bring mathematical intuition and mathematical formalism closer together.

For instance, employing numbers ε such that $\varepsilon^2 = 0$, we can compute derivatives blithely as follows, without requiring the notion of limits:

$$\begin{aligned}(x+\varepsilon)^2 - x^2 &= x^2 + 2x\varepsilon + \varepsilon^2 - x^2 = 2x\varepsilon \\ (x+\varepsilon)^3 - x^3 &= x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 - x^3 = 3x^2\varepsilon\end{aligned} \tag{$\star$}$$

In each case, the derivative is visible as the coefficient of ε in the result. A further example is from geometry: Having a nontrivial set Δ of infinitesimal numbers available allows us to define a *tangent vector* to a manifold M to be a map $\gamma : \Delta \to M$. This definition precisely captures the intuition that a tangent vector is an infinitesimal curve.

4.1. **Hyperreal numbers.** There are several ways for introducing infinitesimals into rigorous mathematics. One is Robinson's *nonstandard analysis*, where we enlarge the field $\mathbb{R}$ of real numbers to a field ${}^\star\mathbb{R}$ of *hyperreal numbers* by means of a non-principal ultrafilter.

The hyperreals contain an isomorphic copy of the ordinary reals as the so-called *standard elements*, and they also contain infinitesimal numbers and their inverses, transfinite numbers. Additionally, they support a powerful *transfer principle*: Any statement which does not refer to standardness is true for the hyperreals if and only if it is true for the ordinary reals.

In the "if" direction, the transfer principle is useful for importing knowledge about the ordinary reals in the hyperreal realm. For instance, addition of hyperreals is commutative because addition of reals is. By the "only if" direction, a theorem established for the hyperreals also holds for the ordinary reals. In this way, the infinitesimal numbers of nonstandard analysis can be viewed as a convenient fiction, generating a conservative extension of the usual setup of mathematics.

However, the realization of this fiction crucically rests on a non-principal ultrafilter, whose existence requires principles which go beyond the means of Zermelo–Fraenkel set theory ZF.[7] Non-principal ultrafilters cannot be described in explicit terms, and they are also not at all canonical structures: ZFC proves that there are $2^{2^{\aleph_0}}$ many [**pospisil:ultrafilters**].

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[7]A hyperreal number is represented by an infinite sequence $(x_0, x_1, x_2, \ldots)$ of ordinary real numbers. For instance, the sequence $(1, 1, 1, \ldots)$ represents the hyperreal version of the number 1, the sequence $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ represents an infinitesimal number and its inverse $(1, 2, 3, \ldots)$ represents a transfinite number. The sequence $(1, 1, 1, \ldots)$ is deemed positive, and so is $(-1, 1, 1, 1, \ldots)$, which differs from the former only in finitely many places. But should $(1, -1, 1, -1, \ldots)$ be deemed positive or negative? Whatever the answer, our decision has consequences for other sequences. For instance $(-1, 1, -1, 1, \ldots)$ should be assigned the opposite sign and $(\tan(1), \tan(-1), \tan(1), \tan(-1), \ldots)$ the same. A non-principal ultrafilter is a set-theoretic gadget which fixes all such decisions once and for all in a coherent manner. Having such an ultrafilter available, a sequence $(x_0, x_1, x_2, \ldots)$ is deemed positive if and only if the set $\{i \in \mathbb{N} \mid x_i > 0\}$ is part of the ultrafilter.

A practical consequence of this nonconstructivity is that it can be hard to unwind proofs which employ hyperreal numbers to direct proofs, and even where possible there is no general procedure for doing so.

4.2. **Topos-theoretic alternatives to the hyperreal numbers.** Topos theory provides several constructive alternatives for realizing infinitesimals. One such is "cheap nonstandard analysis" by Terence Tao [**tao:cheap-nsa**]. It is to Robinson's nonstandard analysis what potential infinity is to actual infinity: Instead of appealing to the axiom of choice to obtain a completed ultrafilter, cheap nonstandard analysis constructs larger and larger approximations to an ideal ultrafilter on the go.

The following section presents a (variant of a) topos used in *synthetic differential geometry* [**kock:sdg**, **kock:new-methods**]. This subject is a further topos-theoretic approach to infinitesimals which is suited to illustrate the philosophy of toposes as lenses. A major motivation for the development of synthetic differential geometry was to devise a rigorous context in which the writings of Sophus Lie, who freely employed infinitesimals in his seminal works, can be effortlessly interpreted, staying close to the original and requiring no coding.

4.3. **The Zariski topos.** The starting point is the observation that while the field $\mathbb{R}$ of ordinary real numbers does not contain infinitesimals (except for zero), the ring $\mathbb{R}[\varepsilon]/(\varepsilon^2)$ of *dual numbers* does. This ring has the cartesian product $\mathbb{R}\times\mathbb{R}$ as its underlying set and the ring operations are defined such that $\varepsilon^2 = 0$, where $\varepsilon := \langle 0, 1\rangle$:

$$\langle a, b\rangle + \langle a', b'\rangle := \langle a + a', b + b'\rangle \qquad \langle a, b\rangle \cdot \langle a', b'\rangle := \langle aa', ab' + a'b\rangle$$

We write $\langle a, b\rangle$ more clearly as $a + b\varepsilon$.

The dual numbers contain infinitesimal numbers, more precisely *nilsquare numbers*, namely all numbers of the form $b\varepsilon$ for $b \in \mathbb{R}$, and they are sufficient to rigorously reproduce derivative computations of polynomials such as $(\star)$.

However, the dual numbers are severely lacking in other aspects. Firstly, they do not contain any nilcubic numbers which are not already nilsquare. These are required in order to extend calculations like $(\star)$ to second derivatives, as in

$$(x + \varepsilon)^3 - x^3 = 3x^2\varepsilon + \frac{1}{2!}6x\varepsilon^2.$$

Secondly, the dual numbers contain, up to scaling, only a single infinitesimal number. Further independent infinitesimals are required in order to deal with functions of several variables, as in

$$f(x + \varepsilon, y + \varepsilon') - f(x, y) = D_x f(x, y)\varepsilon + D_y f(x, y)\varepsilon'.$$

Thirdly, and perhaps most importantly, the ring of dual numbers fails to be a field. The only invertible dual numbers are the numbers of the form $a+b\varepsilon$ with a invertible in the reals; it is not true that any nonzero dual number is invertible.

The first deficiency could be fixed by passing from $\mathbb{R}[\varepsilon]/(\varepsilon^2)$ to $\mathbb{R}[\varepsilon]/(\varepsilon^3)$ (a ring whose elements are triples and whose ring operations are defined such that $\langle 0, 1, 0\rangle^3 = 0$) and the second by passing from $\mathbb{R}[\varepsilon]/(\varepsilon^2)$ to $\mathbb{R}[\varepsilon, \varepsilon']/(\varepsilon^2, \varepsilon'^2, \varepsilon\varepsilon')$. In a sense, both of these proposed replacements are better *stages* than the basic ring $\mathbb{R}[\varepsilon]/(\varepsilon)^2$ or even $\mathbb{R}$ itself. However, similar criticisms can be mounted against any of these better stages, and the problem that all these substitutes are not fields persists.

Introducing the topos. The *Zariski topos of* $\mathbb{R}$, $\mathrm{Zar}(\mathbb{R})$, meets all of these challenges. It contains a ring R, the so-called *ring of smooth numbers*, which reifies the real numbers, the dual numbers, the two proposed better stages and indeed any finitely presented $\mathbb{R}$-algebra into a single coherent entity. The Kripke–Joyal translations rules of $\mathrm{Zar}(\mathbb{R})$ are listed in Table **??**. Any evaluation of an internal statement starts out with the most basic stage of all, the ordinary reals $\mathbb{R}$; then, during the course of evaluation, the current stage is successively refined to better stages (further finitely presented $\mathbb{R}$-algebras).

For instance, universal quantification "$\forall x : R$" not only refers to all elements of the current stage, but also to any elements of any refinement of the current stage. Similarly, negation "$\neg\varphi$" does not only mean that φ would imply $1 = 0$ in the current stage, but also that it does so at any later stage.

For reference purposes only, we include the precise definition of the Zariski topos.

Definition 4.1. The *Zariski topos of* $\mathbb{R}$, $\mathrm{Zar}(\mathbb{R})$, is the category of functors from finitely presented $\mathbb{R}$-algebras to sets. The object R of $\mathrm{Zar}(\mathbb{R})$ is the tautologous functor $A \mapsto A$.

Properties of the smooth numbers. As a concrete example, the Kripke–Joyal translation of the statement that R is a field,

$$\mathrm{Zar}(\mathbb{R}) \models \forall x : R.\ (\neg(x = 0) \Rightarrow \exists y : R.\ xy = 1),$$

is this:

For any stage A and any element $x \in A$,
  if for any later stage B of A
    in which $x = 0$ holds
    also $1 = 0$ holds,
  then A can be covered by later stages B_i such that, for each index i,
    there is an element $y \in B_i$ such that $xy = 1$ in B_i.

And indeed, this statement is true. Let a stage A (a finitely presented $\mathbb{R}$-algebra) and an element $x \in A$ which fulfills the stated condition be given. Then $x = 0$ holds in the refinement $B := A/(x)$. Hence $1 = 0$ holds in B. By elementary algebra, this means that x is invertible in A. Hence the conclusion holds for the singleton covering of A by $B_1 := A$.

Within $\mathrm{Zar}(\mathbb{R})$, we set $\Delta := \{\varepsilon : R \mid \varepsilon^2 = 0\}$. Then R has the following properties:

(1) Law of cancellation: $\forall x : R.\ \forall y : R.\ \big((\forall \varepsilon : \Delta.\ x\varepsilon = y\varepsilon) \Rightarrow x = y\big)$
(2) Axiom of micro-affinity: $\forall f : R^\Delta.\ \exists! a : R.\ \forall \varepsilon : \Delta.\ f(\varepsilon) = f(0) + a\varepsilon$

The unique number a in the axiom of micro-affinity deserves to be called "$f'(0)$"; this is how we synthetically define the derivative in synthetic differential geometry.

Having motivated the Zariski topos by the desire to devise a universe with infinitesimals, the actual ontological status of the infinitesimal numbers in the Zariski topos is more nuanced. The law of cancellation implies that, within $\mathrm{Zar}(\mathbb{R})$, it is not the case that zero is the only nilsquare number. However, this does not mean that there actually *is* a nilsquare number in R. In fact, any nilsquare number cannot be nonzero, as nonzero numbers are invertible while nilsquare numbers are not. Hence any nilsquare number in R is *not not* zero. This state of affairs is only possible in an intuitionistic context.

Remark 4.2. The ring R of smooth numbers does not coincide with the Cauchy reals, the Dedekind reals or indeed any well-known construction of the reals within $\mathrm{Zar}(\mathbb{R})$. This observation explains why R can satisfy the law of cancellation even though it is an intuitionistic theorem that the only nilquadratic number in any flavor of the reals is zero.

4.4. **Well-adapted models.** The Zariski topos of $\mathbb{R}$ allows to compute with infinitesimals in a satisfying manner. However it is not suited as a home for synthetic differential geometry, a first indication being that in $\mathrm{Zar}(\mathbb{R})$, any function $R \to R$ is a polynomial function. Hence important functions such as the exponential function do not exist in $\mathrm{Zar}(\mathbb{R})$. More comprehensively, the Zariski topos is not a well-adapted model in the sense of the following definition.

Definition 4.3. A *well-adapted model* of synthetic differential geometry is a topos $\mathcal{E}$ together with a ring R in $\mathcal{E}$ such that:

(1) The ring R is a field.
(2) The ring R validates the axiom of micro-affinity and several related axioms.
(3) There is a fully faithful functor $i : \mathrm{Mnf} \to \mathcal{E}$ embedding the category of smooth manifolds into $\mathcal{E}$.
(4) The ring R coincides with $i(\mathbb{R}^1)$, the image of the real line in $\mathcal{E}$.

It is the culmination of a long line of research by several authors that several well-adapted models of synthetic differential geometry exist. By the conditions imposed in Definition 4.3, for any such topos $\mathcal{E}$ the following transfer principle holds: If $f, g : \mathbb{R} \to \mathbb{R}$ are smooth functions, then $f' = g$ (in the ordinary sense of the derivative) if and only if $i(f)' = i(g)$ in $\mathcal{E}$ (in the synthetic sense of the derivative).

Hence the nilquadratic infinitesimal numbers of synthetic differential geometry may freely be employed as a convenient fiction when computing derivatives. Because the theorem on the existence of well-adapted models has a constructive proof, any proof making use of these infinitesimals may mechanically be unwound to a (longer and more complex) proof which only refers to the ordinary reals.

4.5. **On the importance of language.** XXX translation unwieldy, lens, view, illustrates custom-tailored, ...

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