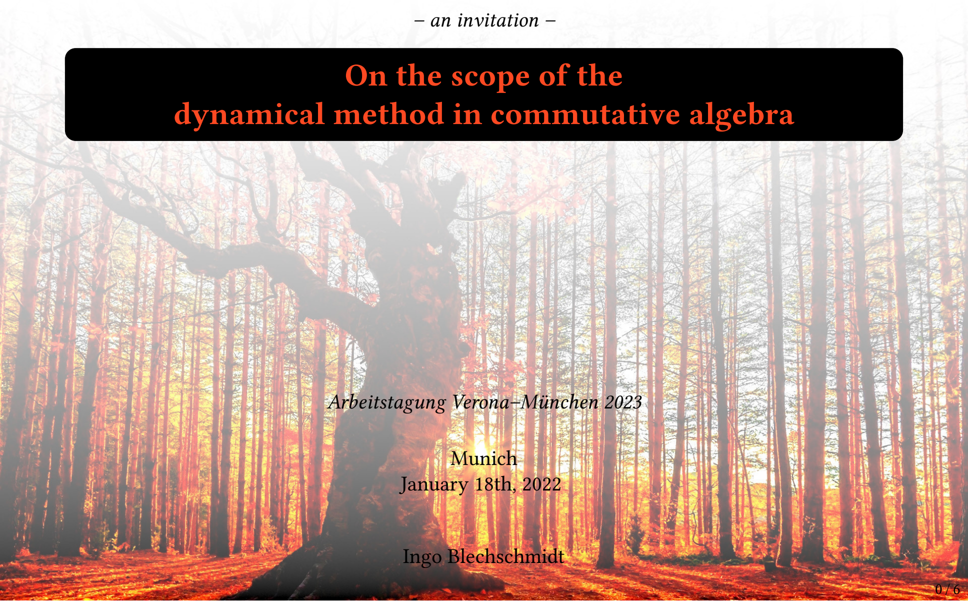

– an invitation –

On the scope of the dynamical method in commutative algebra

Arbeitstagung Verona–München 2023

Munich
January 18th, 2022

Ingo Blechschmidt

A primer to the dynamical method

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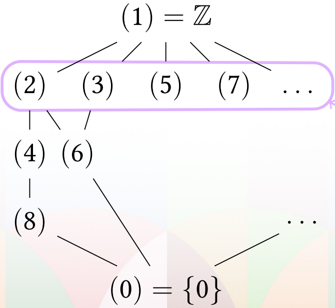

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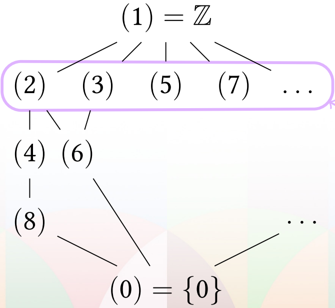

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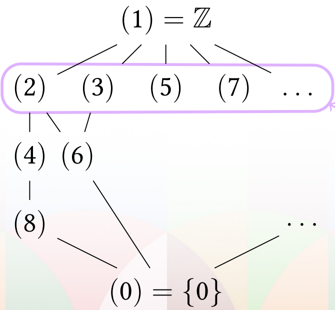

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1. *Is there also a constructive proof?*

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   Yes, by completeness for coherent logic.
2. *Do we know a constructive proof?*
   Write $M = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. By surjectivity, have u, v with $u \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $v \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Hence $1 = (vy)(ux) = (uy)(vx) = 0$.

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3. *Can the constructive proof be* ***extracted*** *from the classical one?*
   Yes, by the dynamical method (and others).

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$$\mathfrak{m}_0 = \{0\}, \qquad \mathfrak{m}_{n+1} = \begin{cases} \mathfrak{m}_n + (x_n), & \text{if } 1 \not\in \mathfrak{m}_n + (x_n), \\ \mathfrak{m}_n, & \text{else.} \end{cases}$$

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"a bad joke"

"non-informative"

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The quotient $A/\mathfrak{m}$ is a residue field: noninvertible implies zero.
$\neg(1 \in \mathfrak{m})$
$\neg(1 \in \mathfrak{m} + (x)) \Longrightarrow x \in \mathfrak{m}$

Forcing

▶ Forcing in commutative algebra

$A \rightsquigarrow A[X]$ adjoining an indeterminate
$A \rightsquigarrow A[x^{-1}]$ forcing an element to become invertible
$A \rightsquigarrow A/(x)$ forcing an element to become zero

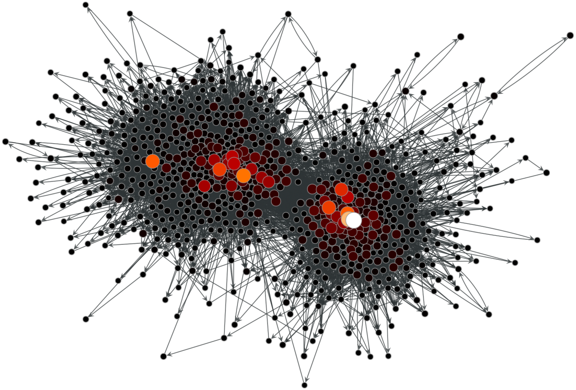
▶ Forcing in classical set theory

$V \rightsquigarrow V[G]$ adjoining a generic filter of a forcing poset $\mathbb{P}$
e.g. adding a cardinal between $\aleph_0$ and $\mathfrak{c}$,
adding a random real,
collapsing two cardinals, …

▶ Forcing in constructive mathematics

$V \rightsquigarrow V^{\neg\neg}$ forcing LEM
$V \rightsquigarrow \mathrm{Sh}(X)$ adjoining a generic point of X
$V \rightsquigarrow V[\mathbb{T}]$ adjoining a generic $\mathbb{T}$-model

Parametrized mathematics



local “Every real symmetric matrix does have an eigenvector.” ✓

global “For every continuous family of symmetric matrices, eigenvectors can locally be picked continuously.” ?

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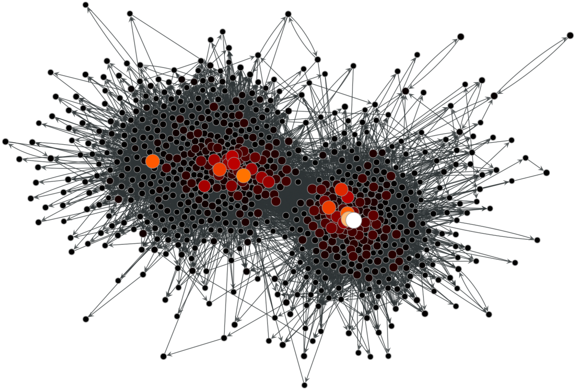

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“Let X be a topological space and let $A : X \to M_n^{\mathrm{sym}}(\mathbb{R})$ be a continuous map to the space of symmetric $(n \times n)$-matrices. Then there is an open covering $\bigcup_{i \in I} U_i$ of X such that or all indices $i \in I$, there is a continuous map $v : U_i \to \mathbb{R}^n$ such that for all $x \in U_i$, the vector $v(x)$ is an eigenvector of $A(x)$.”

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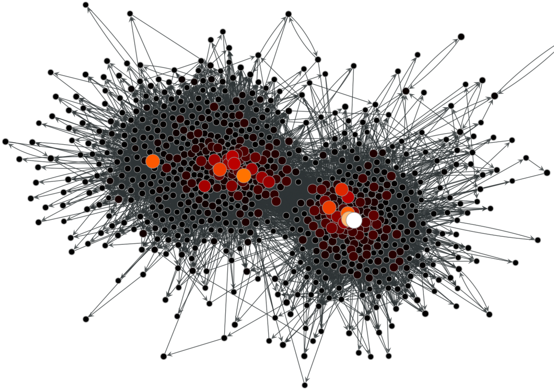

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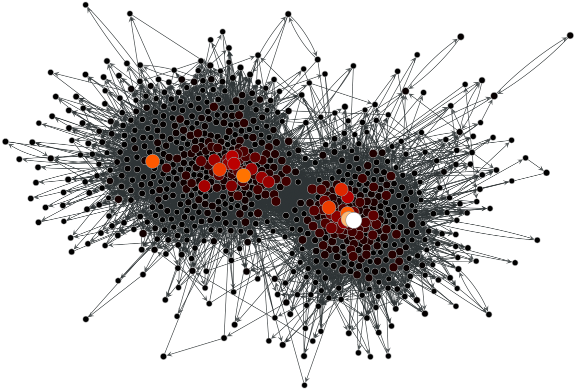

local “Every real symmetric matrix does **not not** have an eigenvector.” ✓

global “For every continuous family of symmetric matrices, **on a dense open** eigenvectors can locally be picked continuously.” ✓

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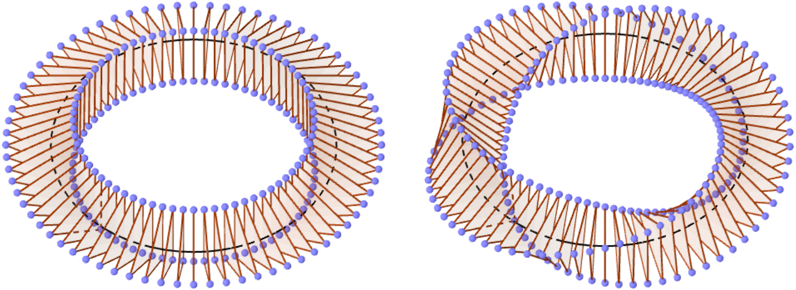

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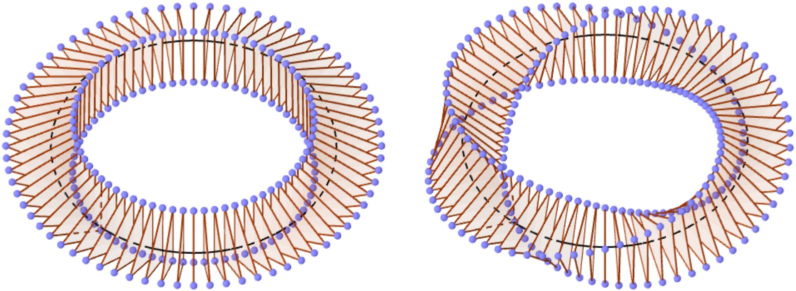

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“Let M be a finitely generated module over an arbitrary commutative ring A. Then there is a partition $1 = f_1 + \cdots + f_n \in A$ of unity such that, for each index i, the localized module $M[f_i^{-1}]$ is finite free over $A[f_i^{-1}]$.”

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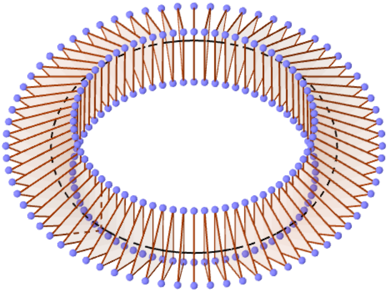

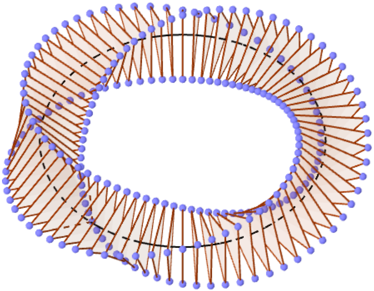

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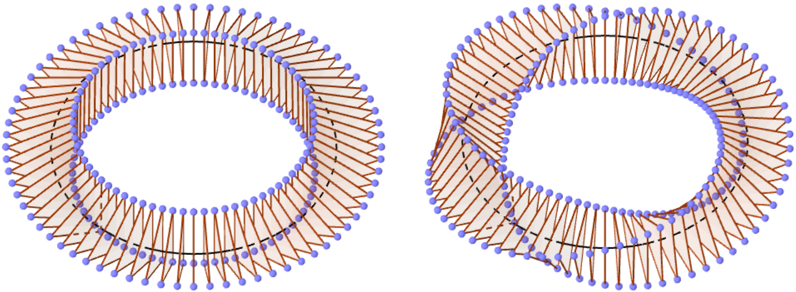

local “Let M be a finitely generated module over a field k.
Then M is **not not** finite free.” ✓

global “Let M be a finitely generated module over a ring A.
Then $M^\sim$ is finite locally free **on a dense open**.” ✓

“Let M be a finitely generated module over an arbitrary commutative ring A. **If $f = 0$ is the only element of A** such that $M[f^{-1}]$ is finite free over $A[f^{-1}]$, **then $1 = 0$ in A**.”

Finite approximations to ideal objects

- Approximate **maps** $\mathbb{N} \to X$ by their finite prefixes.
  Given a finite list σ, be prepared to …
  1. make it more defined: $\{\sigma ::^r x \mid x \in X\}$

- Approximate **enumerations** $\mathbb{N} \twoheadrightarrow X$ by their finite prefixes.
  Given a finite list σ, be prepared to …
  1. make it more defined: $\{\sigma ::^r x \mid x \in X\}$
  2. ensure that a value x occurs: $\{\sigma +\!\!+ \tau \mid \tau \in X^*, x \in \sigma +\!\!+ \tau\}$

- Approximate **prime ideals** by finitely generated ideals.
  Given a f.g. ideal $\mathfrak{a}$, be prepared to …
  1. add the individual factors in case $xy \in \mathfrak{a}$: $\{\mathfrak{a} + (x), \mathfrak{a} + (y)\}$
  2. collapse in case $1 \in \mathfrak{a}$: $\emptyset$

- Approximate **local algebras** by finitely presented rings.
  Given a f.p. ring A, be prepared to …
  1. invert the individual summands in case $x + y$ is invertible in A: $\{A[x^{-1}], A[y^{-1}]\}$
  2. collapse in case $1 = 0$ in A: $\emptyset$

The generic enumeration

For any monotone predicate P on finite lists, we inductively define what it means that

> *no matter how a given list σ evolves to a better approximation σ', eventually $P(\sigma')$ will hold*

by the following clauses.

- **0** If $P(\sigma)$, then $P \mid \sigma$.
- **1** If $P \mid \sigma ::^r x$ for all $x \in X$, then $P \mid \sigma$.
- **2** If $P \mid \sigma +\!\!\!+ \tau$ for all $\tau \in X^*$ such that $x \in \sigma +\!\!\!+ \tau$, then $P \mid \sigma$.

Notation. Write "$\nabla\sigma.\, P(\sigma)$" for $P \mid \sigma$.

Examples. (in case $X = \mathbb{R}$)

- ✓ $\nabla\sigma.\, \mathrm{length}(\sigma) \geq 5$
- ✗ $\nabla\sigma.\, \mathrm{length}(\sigma) \geq 2 \wedge \sigma[0] = \sigma[1]$
- ✓ $\forall x \in \mathbb{R}.\, \nabla\sigma.\, \exists n \in \mathbb{N}.\, \sigma[n] = x \wedge (\nabla\sigma.\, \exists m \in \mathbb{N}.\, \sigma[m] = \sin(n))$

Soundness of the ∇-translation. If $\Gamma \vdash \varphi$, then $\Gamma^\nabla \vdash \varphi^\nabla$.

```
module _ (A : Set) where

open import Data.List
open import Data.List.Membership.Propositional
open import Data.Product

data Eventually (P : List A → Set) : List A → Set where
  now
    : {σ : List A}
    → P σ
    → Eventually P σ
  later
    : {σ : List A} {a : A}
    → ((τ : List A) → a ∈ (σ ++ τ) → Eventually P (σ ++ τ))
    → Eventually P σ

State : (List A → Set) → (List A → Set)
State P σ = ((τ : List A) → Σ[ υ ∈ List A ] P (σ ++ τ ++ υ))
U:**-  Countable.agda   All L1    <N>    (Agda:Checked +5 Undo-Tree)

U:%*-  *All Done*   All L1    <M>    (AgdaInfo Undo-Tree)
```

Agda formalization available.