USING THE INTERNAL LANGUAGE OF TOPOSES IN ALGEBRAIC GEOMETRY

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ABSTRACT. There are several important topoi associated to a scheme, for instance the petit and gros Zariski topoi. These support an internal mathematical language which closely resembles the usual formal language of mathematics, but is "local on the base scheme":

For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a certain condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

These expository notes give an introduction to this topic and show how the internal point of view can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry. No prior knowledge about topos theory and formal logic is assumed.

CONTENTS

1. Introduction	1
2. The internal language of a sheaf topos	4
3. Sheaves of rings	10
4. Sheaves of modules	16
5. Upper semicontinuous functions	24
6. Modalities	28
7. Rational functions and Cartier divisors	44
8. Compactness principles	51
9. Quasicoherent sheaves of modules	54
10. Subschemes	58
11. Relative spectrum	62
12. Unsorted	65
Appendix A. Noetherianity	66
Appendix B. Dictionary between internal and external notions	67
References	68

1. INTRODUCTION

Internal language of toposes. A *topos* is a category which shares certain categorical properties with the category of sets; the archetypical example is the category of sets, and the most important example for the purposes of these notes is the category of set-valued sheaves on a topological space.

Any topos $\mathcal{E}$ supports an *internal language*. This is a device which allows one to *pretend* that the objects of $\mathcal{E}$ are plain sets and that the morphisms are plain maps between sets, even if in fact they are not. For instance, consider a morphism $\alpha : X \to Y$ in $\mathcal{E}$. From the *internal point of view*, this looks like a map between sets, and we can formulate the condition that this map is surjective; we write this as

$$\mathcal{E} \models \forall y\!:\!Y.\ \exists x\!:\!X.\ \alpha(x) = y.$$

The appearance of the colons instead of the usual element signs reminds us that this expression is not to be taken literally – X and Y are objects of $\mathcal{E}$ and thus not necessarily sets. The definition of the internal language is made in such a way so that the meaning of this internal statement is that α is an epimorphism. Similarly, the translation of the internal statement that α is injective is that α is a monomorphism.

Furthermore, we can *reason* with the internal language. There is a metatheorem to the effect that if some statement φ holds from the internal point of view of a topos $\mathcal{E}$ and if φ logically implies some further statement ψ, then ψ holds in $\mathcal{E}$ as well. As a simple example, consider the elementary fact that the composition of surjective maps is surjective. Interpreting this statement in the internal language of $\mathcal{E}$, we obtain the more abstract result that the composition of epimorphisms in $\mathcal{E}$ is epic.

There is, however, a slight caveat to this metatheorem. Namely, the internal language of a topos is in general only *intuitionistic*, not *classical*. This means that internally, one can not use the law of excluded middle ($\varphi \vee \neg\varphi$), nor the law of double negation elimination ($\neg\neg\varphi \Rightarrow \varphi$), nor the axiom of choice. For instance, one rendition of the axiom of choice is that any surjection splits. But it need not be the case that an epimorphism in a topos splits. **XXX: the translation of splitting would be that locally (!), there is a right inverse.**

Algebraic geometry. We apply this internal language to algebraic geometry as follows. If X is a scheme, the structure sheaf $\mathcal{O}_X$ is a sheaf of rings, i. e. the sets of local sections carry ring structures and these ring structures are compatible with restriction. From the internal point of view of the topos of set-valued sheaves on X, denoted "$\mathrm{Sh}(X)$" in the following, the structure sheaf $\mathcal{O}_X$ looks much simpler: It looks just like a plain ring (and not a sheaf of rings). Similarly, a sheaf of $\mathcal{O}_X$-modules looks just like a plain module over that ring.

This allows to import notions and facts from basic linear and commutative algebra into the sheaf setting. For instance, it turns out that a sheaf of $\mathcal{O}_X$-modules is of finite type if and only if, from the internal perspective, it is finitely generated as an $\mathcal{O}_X$-module. Now consider the following fact of linear algebra: If in a short exact sequence of modules the two outer ones are finitely generated, then the middle one is too. The usual proof of this fact is intuitionistically acceptable and can thus be interpreted in the internal language. It then *automatically* yields the following more advanced proposition: If in a short exact sequence of sheaves of $\mathcal{O}_X$-modules the two outer ones are of finite type, then the middle one is too.

The internal language machinery thus allows us to understand the basic notions and statements of scheme theory as notions and statements of linear and commutative algebra, interpreted in a suitable sheaf topos. This brings conceptual clarity and reduces technical overhead.

In these notes, we explain how the internal language works and then develop a *dictionary* between common notions of scheme theory and corresponding notions of algebra. Once built, this dictionary can be used arbitrarily often.

Two highlights of our approach are the following. Let X be a reduced scheme and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules of finite type. Then it is well-known that $\mathcal{F}$ is locally free on some dense open subset of X; for instance, this is stated in Ravi Vakil's lecture notes as an "important hard exercise" [14, exercise 13.7.K]. In fact, this fact is just the interpretation of the following statement of intuitionistic linear algebra in the sheaf topos: Any finitely generated vector space is *not not* free. The proof of this statement is entirely straightforward.[1]

The second highlight is that we can shed light on the phenomenon that sometimes, truth of a property at a point x spreads to some open neighbourhood of x; and in particular that sometimes, truth of a property at the generic point spreads to some dense open subset. For instance, if the stalk of a sheaf of finite type is zero at some point, the sheaf is even zero on some open neighbourhood; but this spreading does not occur for general sheaves which may fail to be of finite type.

We formalize this by introducing a *modal operator* $\Box$ into the internal language, such that the internal statement $\Box\varphi$ means that φ holds on some open neighbourhood of x. Furthermore, we introduce a simple operation on formulas, the $\Box$-*translation* $\varphi \mapsto \varphi^\Box$, such that $\varphi^\Box$ means that φ holds at the point x. The question whether truth at x spreads to truth on a neighbourhood can thus be formulated in the following way: Does $\varphi^\Box$ intuitionistically imply $\Box\varphi$?

This allows to deal with the question in a simpler, more logical way, with the technicalities of sheaves blinded out. We also give a metatheorem which covers a wide range of cases. Namely, spreading occurs for all those properties which can be formulated in the internal language without using "$\Rightarrow$", "$\forall$", and "$\neg$".

To illustrate the example above, consider the property of a module $\mathcal{F}$ being the zero module. In the internal language, it can be formulated as $(\forall x\,{:}\,\mathcal{F}.\ x = 0)$. Because of the appearance of "$\forall$", the metatheorem is not applicable to this statement. But if $\mathcal{F}$ is of finite type, there are generators $x_1, \ldots, x_n : \mathcal{F}$ from the internal point of view, and the condition can be reformulated as $x_1 = 0 \wedge \cdots \wedge x_n = 0$; the metatheorem is applicable to this statement.

Limitations. The internal language is *local*, in the sense that if $X = \bigcup_i U_i$ is an open covering and an internal statement holds in the sheaf toposes $\mathrm{Sh}(U_i)$, it holds in $\mathrm{Sh}(X)$ as well. On the one hand, this property is very useful. But on the other hand, it gives an inherent limitation of the internal language: Global properties of sheaves of modules like "being generated by global sections" or "being ample" and global properties of schemes like "being quasicompact" or "having vanishing sheaf cohomology" can *not* be expressed in the internal language.

Thus for global considerations, the internal language of $\mathrm{Sh}(X)$ is only useful in that local subparts can be simplified. Also, some global features reflect themselves in certain meta properties of the internal language (for instance, a scheme is quasicompact if and only if the internal language fulfills a weak version of the so-called disjunction property of mathematical logic).

[1] Intuitionistically, the statement that any finitely generated vector space is *free* is stronger than the doubly negated version and can not be shown. It would imply that any sheaf of finite type is not only locally free on some dense open subset, but locally free on the whole space.

Introductory literature and related work. These notes are intended to be self-contained, requiring only basic knowledge of scheme theory. In particular, we assume no prior familiarity with topos theory or formal logic. But if the interested reader is so inclined, she will find a gentle introduction to topos theory in an article by Tom Leinster [7]. Standard references for the internal language of a topos include the book of Saunders Mac Lane and Ieke Moerdijk [8, chapter VI] and part D of Peter Johnstone's Elephant [5]. In the 1970s, there was a flurry of activity on applications of the internal language. An article by Christopher Mulvey [10] of this time gives a very accessible introduction to the topic, culminating in an internal proof of the Serre–Swan theorem (with just one external ingredient needed).

The internal language of toposes was applied to algebraic geometry before. For instance, Gavin Wraith used it to construct (and verify the universal property of) the big étale topos of a scheme by internally developing the theory of strict henselization [18]. However, to the best of my knowledge, a systematic creation of a dictionary between external and **XXX: word "creation"** internal notions has not been attempted before, and the use of modal operators to study the spreading of properties from points to neighbourhoods seems to be new as well.

In other branches of mathematics, the internal language is used as well. For instance, there is an ongoing effort in mathematical physics to understand quantum mechanical systems from an internal point of view: To any quantum mechanical system, one can associate a so-called Bohr topos containing an internal mirror image of the system. This mirror image looks like a system of classical mechanics from the internal perspective, and therefore tools like Gelfand duality can be used to construct an internal phase space for the system [3, 4].

- dictionary; intuitionistic logic; microscope/telescope into another universe; types instead of sets; (dependent types to encompass almost all mathematics)
- explain that with the internal language business, it becomes more transparent where scheme condition enters
- note that in-depth knowledge of formal logic or topos theory is not necessary for applications
- give pointers to introductory literature
- here or somewhere else: Note that the internal language is classical if and only if the space is discrete (assuming T1)
- note that parts of constructive mathematics get new fields of application

2. The internal language of a sheaf topos

2.1. **Internal statements.** Let X be a topological space. Later, X will be the underlying space of a scheme. The meaning of internal statements is given by a set of rules, the *Kripke–Joyal semantics* of the topos of sheaves on X.

Definition 2.1. The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ over U is given by the rules listed in table 1, recursively in the structure of φ. In a *formula over U* there may appear sheaves defined on U as domains of quantifications, U-sections of sheaves as terms and morphisms of sheaves on U as function symbols. The symbols "$\top$" and "$\bot$" denote truth and falsehold, respectively. The universal and existential quantifiers come in two flavors: for bounded and unbounded quantification. The translation of $U \models \neg\varphi$

$U \models s = t : \mathcal{F}$	$:\Longleftrightarrow$	$s\|_U = t\|_U \in \Gamma(U, \mathcal{F})$
$U \models s \in \mathcal{G}$	$:\Longleftrightarrow$	$s\|_U \in \Gamma(U, \mathcal{G})$ ($\mathcal{G}$ a subsheaf of $\mathcal{F}$, s a section of $\mathcal{F}$)
$U \models \top$	$:\Longleftrightarrow$	$U = U$ (always fulfilled)
$U \models \bot$	$:\Longleftrightarrow$	$U = \emptyset$
$U \models \varphi \wedge \psi$	$:\Longleftrightarrow$	$U \models \varphi$ and $U \models \psi$
$U \models \bigwedge_{j \in J} \varphi_j$	$:\Longleftrightarrow$	for all $j \in J$: $U \models \varphi_j$ (J an index set)
$U \models \varphi \vee \psi$	$:\Longleftrightarrow$	~~$U \models \varphi$ or $U \models \psi$~~
		there exists a covering $U = \bigcup_i U_i$ such that for all i:
		$U_i \models \varphi$ or $U_i \models \psi$
$U \models \bigvee_{j \in J} \varphi_j$	$:\Longleftrightarrow$	~~$U \models \varphi_j$ for some $j \in J$~~ (J an index set)
		there exists a covering $U = \bigcup_i U_i$ such that for all i:
		$U_i \models \varphi_j$ for some $j \in J$
$U \models \varphi \Rightarrow \psi$	$:\Longleftrightarrow$	for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$
$U \models \forall s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	for all sections $s \in \Gamma(V, \mathcal{F})$, open $V \subseteq U$: $V \models \varphi(s)$
$U \models \exists s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	~~there exists a section $s \in \Gamma(U, \mathcal{F})$ such that $U \models \varphi(s)$~~
		there exists an open covering $U = \bigcup_i U_i$ such that for all i:
		there exists $s_i \in \Gamma(U_i, \mathcal{F})$ such that $U_i \models \varphi(s_i)$
$U \models \forall \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	for all sheaves $\mathcal{F}$ on V, open $V \subseteq U$: $V \models \varphi(\mathcal{F})$
$U \models \exists \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	there exists an open covering $U = \bigcup_i U_i$ such that for all i:
		there exists a sheaf $\mathcal{F}_i$ on U_i such that $U_i \models \varphi(\mathcal{F}_i)$

TABLE 1. The Kripke–Joyal semantics of a sheaf topos.

does not have to be separately defined, since negation can be expressed using other symbols: $\neg\varphi :\equiv (\varphi \Rightarrow \bot)$. If we want to emphasize the particular topos, we write

$$\mathrm{Sh}(X) \models \varphi \quad :\Longleftrightarrow \quad X \models \varphi.$$

Remark 2.2. The last two rules in table 1, concerning *unbounded quantification*, and are not part of the classical Kripke–Joyal semantics. They are part of Mike Shulman's stack semantics [11], a slight extension. They are needed so that we can formulate universal properties in the internal language.

Example 2.3. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is a monomorphism of sheaves if and only if, from the internal perspective, α is simply an injective map:

$$\begin{aligned}
& X \models \ulcorner \alpha \text{ is injective} \urcorner \\
\Longleftrightarrow\ & X \models \forall s : \mathcal{F}.\ \forall t : \mathcal{F}.\ \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\ & \text{for all open } U \subseteq X, \text{ sections } s \in \Gamma(U, \mathcal{F}): \\
& \text{for all open } V \subseteq U, \text{ sections } t \in \Gamma(V, \mathcal{F}): \\
& \qquad V \models \alpha(s) = \alpha(t) \Rightarrow s = t
\end{aligned}$$

$\iff$ for all open $U \subseteq X$, sections $s, t \in \Gamma(U, \mathcal{F})$:

$$U \models \alpha(s) = \alpha(t) \Rightarrow s = t$$

$\iff$ for all open $U \subseteq X$, sections $s, t \in \Gamma(U, \mathcal{F})$:

for all open $W \subseteq U$:

$$\alpha_W(s|_W) = \alpha_W(t|_W) \text{ implies } s|_W = t|_W$$

$\iff$ for all open $U \subseteq X$, sections $s, t \in \Gamma(U, \mathcal{F})$:

$$\alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U$$

$\iff$ α is a monomorphism of sheaves

The corner quotes "$\ulcorner \ldots \urcorner$" indicate that translation into formal language is left to the reader. Similarly, α is an epimorphism of sheaves if and only if, from the internal perspective, α is a surjective map. Notice that injectivity and surjectivity are notions of a simple element-based language, and the Kripke–Joyal semantics takes care to properly handle *all* sections, not only global ones.

The rules are not all arbitrary. They are finely concerted to make the following propositions true, which are crucial for a proper appreciation of the internal language.

Proposition 2.4 (Locality of the internal language)**.** *Let $U = \bigcup_i U_i$ be covered by open subsets. Let φ be a formula over U. Then*

$$U \models \varphi \qquad \text{iff} \qquad U_i \models \varphi \text{ for each } i.$$

Proof. Induction on the structure of φ. Note that the canceled rules would make this proposition false. □

As a corollary, one may restrict the open coverings and universal quantifications in the the definition of the Kripke–Joyal semantics (table 1) to open subsets of some basis of the topology. For instance, if X is a scheme, one may restrict to affine open subsets.

Furthermore, the proposition shows that the internal language is monotone in the following sense: If $U \models \varphi$, and V is an open subset of U, then $V \models \varphi$. (This follows by applying the proposition to the trivial covering $U = V \cup U$.)

Proposition 2.5 (Soundness of the internal language)**.** *If a formula φ implies a further formula ψ in intuitionistic logic, then $U \models \varphi$ implies $U \models \psi$.*

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that on each U_i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in [5, p. D1.3.1]. □

XXX: Put rules into an appendix and give some explanation regarding contexts etc. Don't forget the rules for $\in$, $\bigwedge$, $\bigvee$.

XXX: XXX: Is it clear that any constructively valid statement yields a valid sheaf statement?

Because of the multitude of quantifiers, literal translations of internal statements can sometimes get slightly unwieldy. There are simplification rules for certain often-occuring special cases:

Proposition 2.6.

$$\begin{aligned}
U \models \forall s\,{:}\,\mathcal{F}.\ \forall t\,{:}\,\mathcal{G}.\ \varphi(s,t) \quad &\Longleftrightarrow \quad \textit{for all open } V \subseteq U, \\
&\qquad \textit{sections } s \in \Gamma(V,\mathcal{F}),\ t \in \Gamma(V,\mathcal{G})\textit{: } V \models \varphi(s,t) \\
U \models \forall s\,{:}\,\mathcal{F}.\ \varphi(s) \Rightarrow \psi(s) \quad &\Longleftrightarrow \quad \textit{for all open } V \subseteq U,\ \textit{sections } s \in \Gamma(V,\mathcal{F})\textit{:} \\
&\qquad\qquad V \models \varphi(s) \textit{ implies } V \models \psi(s) \\
U \models \exists! s\,{:}\,\mathcal{F}.\ \varphi(s) \quad &\Longleftrightarrow \quad \textit{for all open } V \subseteq U, \\
&\qquad \textit{there is exactly one section } s \in \Gamma(V,\mathcal{F}) \textit{ with:} \\
&\qquad\qquad V \models \varphi(s)
\end{aligned}$$

Proof. Straightforward. By way of example, we prove the existence claim in the "only if" direction of the last rule. (Note that this rule formalizes the saying "unique existence implies global existence".) By definition of $\exists!$, it holds that

$$U \models \exists s\,{:}\,\mathcal{F}.\ \varphi(s)$$

and

$$U \models \forall s,t\,{:}\,\mathcal{F}.\ \varphi(s) \wedge \varphi(t) \Rightarrow s = t.$$

Let $V \subseteq U$ be an arbitrary open subset. Then there exist local section $s_i \in \Gamma(V_i,\mathcal{F})$ such that $V_i \models \varphi(s_i)$, where $V = \bigcup_i V_i$ is an open covering. By the locality of the internal language, on intersections it holds that $V_i \cap V_j \models \varphi(s_i)$, so by the uniqueness assumption, it follows that the local sections agree on intersections. They therefore glue to a section $s \in \Gamma(V,\mathcal{F})$. Since $V_i \models \varphi(s)$ for all i, the locality of the internal language allows us to conclude that $V \models \varphi(s)$. □

Remark 2.7. Note that $\mathrm{Sh}(X) \models \neg\varphi$ is in general a much stronger statement that merely supposing that $\mathrm{Sh}(X) \models \varphi$ does not hold: The former always implies the latter (unless $X = \emptyset$, in which case *any* internal statement is true), but the converse does not hold: The former statement means that $U = \emptyset$ is the *only* open subset on which φ holds.

XXX: Find appropriate place for this remark.

2.2. **Internal constructions.** The Kripke–Joyal semantics defines the interpretation of internal *statements*. The interpretation of internal *constructions* is given by the following definition.

Definition 2.8. The interpretation of an internal construction T is denoted by $[\![T]\!] \in \mathrm{Sh}(X)$ and given by the following rules.

- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \times \mathcal{G}]\!]$ is the categorical product of $\mathcal{F}$ and $\mathcal{G}$ (i. e. their product as presheaves).
- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \amalg \mathcal{G}]\!]$ is the categorical coproduct of $\mathcal{F}$ and $\mathcal{G}$, i. e. the sheafification of the presheaf $U \mapsto \Gamma(U,\mathcal{F}) \amalg \Gamma(U,\mathcal{G})$.

- If $\mathcal{F}$ is a sheaf, the interpretation $[\![\mathcal{P}(\mathcal{F})]\!]$ of the power set construction is the sheaf given by
$$U \subseteq X \text{ open} \longmapsto \{\mathcal{G} \hookrightarrow \mathcal{F}|_U\},$$
i. e. sections on an open set U are subsheaves of $\mathcal{F}|_U$ (either literally or isomorphism classes of general monomorphisms into $\mathcal{F}|_U$).
- If $\mathcal{F}$ is a sheaf and $\varphi(s)$ is a formula containing a free variable $s : \mathcal{F}$, the interpretation $[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]$ is given by the subpresheaf of $\mathcal{F}$ defined by
$$U \subseteq X \text{ open} \longmapsto \{s \in \Gamma(U, \mathcal{F}) \mid U \models \varphi(s)\}.$$
Note that by the locality of the internal language, this presheaf is in fact a sheaf.

The definition is made in such a way that, from the internal perspective, the constructions enjoy their expected properties. For instance, it holds that
$$\mathrm{Sh}(X) \models \big[\forall x : [\![\{s : \mathcal{F} \mid \varphi(s)\}]\!].\ \psi(x)\big] \iff \big[\forall x : \mathcal{F}.\ \varphi(x) \Rightarrow \psi(x)\big].$$
We gloss over several details here. See [5, chapter D4.1] for a proper treatment.

To be able to fully express all constructions of "usual mathematics" in the internal language (i. e. not those specifically designed to test the limitations of the ambient logical framework), **XXX: weaken this phrase** we need *dependent types*. **XXX: short explanation, for instance by an example "A common example are coproducts indexed by elements of some sheaf"** In these notes, we do not describe how to deal with those. However, everything carries over to the more general setting, and we refer to an article by Awodey and Bauer [1] for a review of dependent types and their categorical semantics.

XXX: Construction of morphisms

2.3. **Geometric formulas and constructions.** In categorical logic, so-called geometric formulas play a special role, because their meaning is preserved under pullback with geometric morphisms. **XXX: gibberish!**

Definition 2.9. A formula is *geometric* if and only if it consists only of
$$= \quad \in \quad \top \quad \bot \quad \wedge \quad \vee \quad \bigvee \quad \exists,$$
but not "$\bigwedge$" nor "$\Rightarrow$" nor "$\forall$" (and thus not "$\neg$" either, since this is defined using "$\Rightarrow$"). A *geometric implication* is a formula of the form
$$\forall \cdots \forall.\ (\cdots) \Rightarrow (\cdots)$$
with the bracketed subformulas being geometric.

The *parameters* of a formula φ are the sheaves being quantified over, sections of sheaves appearing as terms, and morphisms of sheaves appearing as function symbols in φ. We say that a formula φ holds *at a point* $x \in X$ if and only if the formula obtained by substituting all parameters in φ with their stalks at x holds in the usual mathematical sense.

Lemma 2.10. *Let $x \in X$ be a point. Let φ be a geometric formula (over some open neighbourhood of x). Then φ holds at x if and only if there exists an open neighbourhood $U \subseteq X$ of x such that φ holds on U.*

Proof. This is a very general instance of the phenomenom that sometimes, truth at a point spreads to truth on a neighbourhood. It can be proven by induction on the structure of φ, but we will give a more conceptual proof later (corollary 6.25). $\square$

This lemma is in fact a very useful metatheorem. We will properly discuss its significance in section 6.7. For now, we just use it to prove a simple criterion for the internal truth of a geometric implication; we will apply this criterion many times.

Corollary 2.11. *A geometric implication holds on X if and only if it holds at every point of X.*

Proof. For notational simplicity, we consider a geometric implication of the form

$$\forall s : \mathcal{F}.\ \varphi(s) \Rightarrow \psi(s).$$

For the "only if" direction, assume that this formula holds on X and let $x \in X$ be an arbitrary point. Let $s_x \in \mathcal{F}_x$ be the germ of an arbitrary local section s of $\mathcal{F}$ and assume that $\varphi(s)$ holds at x. Then by the lemma, it follows that $\varphi(s)$ holds on some open neighbourhood of x. By assumption, $\psi(s)$ holds on this neighbourhood as well. Again by the lemma, $\psi(s)$ holds at x.

For the "if" direction, assume that the geometric implication holds at every point. Let $U \subseteq X$ be an arbitrary open subset and let $s \in \Gamma(U, \mathcal{F})$ be a local section such that $\varphi(s)$ holds on U. By the lemma and the locality of the internal language, to show that $\psi(s)$ holds on U, it suffices to show that $\psi(s)$ holds at every point of U. This is clear, since again by the lemma, $\varphi(s)$ holds at every point of U. $\square$

Example 2.12. Injectivity and surjectivity are geometric implications (surjectivity can be spelled $\forall y : \mathcal{G}.\ (\top \Rightarrow \exists x : \mathcal{F}.\ \alpha(x) = y)$). Thus the corollary gives a deeper reason for the well-known fact that a morphism of sheaves is a monomorphism resp. an epimorphism if and only if it is stalkwise injective resp. surjective.

A construction is *geometric* if and only if it commutes with pullback under arbitrary geometric morphisms. We do not want to discuss the notion of geometric morphisms here; suffice it to say that calculating the stalk at a point $x \in X$ is an instance of such a pullback. Among others, the following constructions are geometric:

- finite product: $(\mathcal{F} \times \mathcal{G})_x \cong \mathcal{F}_x \times \mathcal{G}_x$
- finite coproduct: $(\mathcal{F} \amalg \mathcal{G})_x \cong \mathcal{F}_x \amalg \mathcal{G}_x$
- arbitrary coproduct: $(\coprod_i \mathcal{F}_i)_x \cong \coprod_i (\mathcal{F}_i)_x$
- set comprehension with respect to a *geometric* formula φ:
$$[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]_x \cong \{[s] \in \mathcal{F}_x \mid \varphi(s) \text{ holds at } x\}$$
- free module: $(\mathcal{R}\langle\mathcal{F}\rangle)_x \cong \mathcal{R}_x\langle\mathcal{F}_x\rangle$ ($\mathcal{R}$ a sheaf of rings, $\mathcal{F}$ a sheaf of sets)
- localization of a module: $\mathcal{F}[\mathcal{S}^{-1}]_x \cong \mathcal{F}_x[\mathcal{S}_x^{-1}]$

XXX: compatibility with stalks is not sufficient for geometricity. This list makes one think it is.

The following constructions are not in general geometric:

- arbitrary product
- set comprehension with respect to a non-geometric formula
- powerset
- internal Hom: $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})_x \ncong \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$
- crash course on intuitionistic logic

3. Sheaves of rings

Recall that a *sheaf of rings* can be categorically described as a sheaf of sets $\mathcal{R}$ together with maps of sheaves $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and global elements $0, 1$ such that certain axioms hold. For instance, the axiom on the commutativity of addition is rendered in diagrammatic form as follows:

$$\begin{tikzcd} \mathcal{R} \times \mathcal{R} \arrow[rr, "\mathrm{swap}"] \arrow[dr, "+"'] & & \mathcal{R} \times \mathcal{R} \arrow[dl, "+"] \\ & \mathcal{R} & \end{tikzcd}$$

From the internal perspective, a sheaf of rings looks just like a plain ring. This is the content of the following proposition:

Proposition 3.1. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of sets on X. Let $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be maps of sheaves and let $0, 1$ be global elements of $\mathcal{R}$. Then these data define a sheaf of rings if and only if, from the internal perspective, these data fulfill the usual equational ring axioms.*

Proof. We only discuss the commutativity axiom. The internal statement

$$\mathrm{Sh}(X) \models \forall x, y : \mathcal{R}.\ x + y = y + x$$

means that for any open subset $U \subseteq X$ and any local sections $x, y \in \Gamma(U, \mathcal{R})$, it holds that $x + y = y + x \in \Gamma(U, \mathcal{R})$. This is precisely the external commutativity condition. $\square$

Lemma 3.2. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of rings on X. Let f be a global section of $\mathcal{R}$. Then the following statements are equivalent:*

(1) *f is invertible from the internal point of view, i. e.* $\mathrm{Sh}(X) \models \exists g : \mathcal{R}.\ fg = 1$.
(2) *f is invertible in all stalks $\mathcal{R}_x$.*
(3) *f is invertible in $\Gamma(X, \mathcal{R})$.*

Proof. Since invertibility is a geometric implication, the equivalence of the first two statements is clear. Also, it is obvious that the third statement implies the other two. For the remaining direction, note that the uniqueness of inverses in rings can be proven intuitionistically. Therefore, if f is invertible from the internal point of view, it actually holds that

$$\mathrm{Sh}(X) \models \exists! g : \mathcal{R}.\ fg = 1.$$

Since unique internal existence implies global existence (proposition 2.6), this shows that the first statement implies the third. $\square$

3.1. Reducedness.

Recall that a scheme X is *reduced* if and only if all stalks $\mathcal{O}_{X,x}$ are reduced rings. Since the condition on a ring R to be reduced is a geometric implication,

$$\forall s : R.\ \Big(\bigvee_{n \geq 0} s^n = 0 \Big) \Longrightarrow s = 0,$$

we immediately obtain the following characterization of reducedness in the internal language:

Proposition 3.3. *A scheme X is reduced iff, from the internal point of view, the ring $\mathcal{O}_X$ is reduced.*

3.2. **Locality.** Recall the usual definition of a local ring: a ring possessing exactly one maximal ideal. This is a so-called *higher-order condition* this it involves quantification over subsets. It is also not of a geometric form. Therefore, for our purposes, it is better to adopt the following elementary definition of a local ring.

Definition 3.4. A *local ring* is a ring R such that $1 \neq 0$ in R and for all $x, y \in R$

$$x + y \text{ invertible} \quad \Longrightarrow \quad x \text{ invertible} \ \vee \ y \text{ invertible}.$$

In classical logic, it is an easy exercise to show that this definition is equivalent to the usual one. In intuitionistic logic, we would need to be more precise in order to even state the question of equivalence, since intuitionistically, the notion of a maximal ideal bifurcates into several non-equivalent notions.[2]

Proposition 3.5. *In the internal language of a scheme X (or a locally ringed space), the ring $\mathcal{O}_X$ is a local ring.*

Proof. The stated locality condition is a conjunction of two geometric implications (the first one being $1 = 0 \Rightarrow \bot$, the second being the displayed one) and holds on each stalk. $\square$

3.3. **Field properties.** From the internal point of view, the structure sheaf $\mathcal{O}_X$ of a scheme X is *almost* a field, in the sense that any element which is not invertible is nilpotent. This is a genuine property of schemes, not shared with general locally ringed spaces. It is also a specific feature of the internal universe: Neither the local rings $\mathcal{O}_{X,x}$ nor the rings of local sections $\Gamma(U, \mathcal{O}_X)$ have this property in general.

Proposition 3.6. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall s \colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Rightarrow \ulcorner s \text{ nilpotent} \urcorner.$$

Proof. By the locality of the internal language and since X can be covered by open affine subsets, it is enough to show that for any affine scheme $X = \operatorname{Spec} A$ and global function $s \in \Gamma(X, \mathcal{O}_X) = A$ it holds that

$$X \models \neg(\ulcorner s \text{ invertible} \urcorner) \quad \text{implies} \quad X \models \ulcorner s \text{ nilpotent} \urcorner.$$

The meaning of the antecedent is that any open subset on which s is invertible is empty. This implies in particular that the standard open subset $D(s)$ is empty. This means that s is an element of any prime ideal of A, thus nilpotent, and therefore implies the a priori weaker statement $X \models \ulcorner s \text{ nilpotent} \urcorner$ (which would allow s to have different indices of nilpotency on an open covering). $\square$

Remark 3.7. In classical logic, the statement "not invertible implies nilpotent" is equivalent to "any element is invertible or nilpotent". However, in intuitionistic logic, the latter is strictly stronger than the former: We will see in the next section (corollary 3.13) that the structure sheaf of a scheme fulfills the latter condition if and only if the scheme is zero-dimensional.

Corollary 3.8. *Let X be a scheme. If X is reduced, the ring $\mathcal{O}_X$ is a field from the internal point of view, in the sense that*

$$\mathrm{Sh}(X) \models \forall s \colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Rightarrow s = 0.$$

The converse holds as well.

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[2] For instance, should a maximal ideal $\mathfrak{m}$ be such that if $\mathfrak{n}$ is any ideal with $\mathfrak{m} \subseteq \mathfrak{n} \subsetneq (1)$, then $\mathfrak{m} = \mathfrak{n}$? Or should the condition be that if $\mathfrak{n}$ is any ideal with $\mathfrak{m} \subseteq \mathfrak{n}$, then $\mathfrak{m} = \mathfrak{n}$ or $\mathfrak{n} = (1)$? Intuitionistically, the latter condition is stronger than the former.

Proof. We can prove this purely in the internal language: It suffices to give an intuitionistic proof of the fact that a local ring which satisfies the condition of the previous proposition fulfills the stated field condition if and only if it is reduced. This is straightforward. □

This field property is very useful. We will put it to good use when giving a simple proof of the fact that $\mathcal{O}_X$-modules of finite type on a reduced scheme are locally free on a dense open subset (lemma 5.10). Note that the field property only holds precisely as stated in the corollary; the classically equivalent condition that any element is invertible or zero is intuitionistically stronger. This is a common phenomenom in intuitionistic mathematics: Classically equivalent notions may bifurcate into related but inequivalent notions intuitionistically.

The following proposition says that we can deduce a certain unconditional statement from the premise that an element $s : \mathcal{O}_X$ is zero under the assumption that some further element $f : \mathcal{O}_X$ is invertible. This is interesting on its own, but will be of particular importance in understanding quasicoherence from the internal point of view (section 9).

Proposition 3.9. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \forall s : \mathcal{O}_X.\ (\ulcorner f \text{ inv.} \urcorner \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0.$$

Proof. It is enough to show that for any affine scheme $X = \operatorname{Spec} A$ and global functions $f, s \in A$ such that

$$X \models (\ulcorner f \text{ inv.} \urcorner \Rightarrow s = 0),$$

it holds that $X \models \bigvee_{n \geq 0} f^n s = 0$. This indeed follows, since by assumption such a function s is zero on $D(f)$, i. e. s is zero as an element of $A[f^{-1}]$. □

3.4. **Krull dimension.** Recall that the *Krull dimension* of a ring is usually defined as the supremum of the lengths of strictly ascending chains of prime ideals. Like with the classical definition of a local ring, this definition does not lead to a well-behaved notion in an intuitionistic context. Furthermore, it is a higher-order condition, so interpreting it with the Kripke–Joyal semantics is a bit unwieldy.

Luckily, there is an elementary definition of the Krull dimension which works intuitionistically and which is classically equivalent to the usual notion. It was found by **XXX: authors** and simplified by Thierry Coquand **XXX: reference**.

Definition 3.10. Let R be a ring. A *complementary sequence* for a sequence $(a_0, \ldots, a_n)$ of elements of R is a sequence $(b_0, \ldots, b_n)$ such that the following inclusions of radical ideals hold:

- $\sqrt{(1)} \subseteq \sqrt{(a_0, b_0)}$
- $\sqrt{(a_0 b_0)} \subseteq \sqrt{(a_1, b_1)}$
- $\sqrt{(a_1 b_1)} \subseteq \sqrt{(a_2, b_2)}$
- $\vdots$
- $\sqrt{(a_{n-1} b_{n-1})} \subseteq \sqrt{(a_n, b_n)}$
- $\sqrt{(a_n b_n)} = \sqrt{(0)}$

The ring R is *constructively of Krull dimension* $\leq n$ if and only if for any sequence $(a_0, \ldots, a_n)$ there exists a complementary sequence. (The ring R is trivial if and only if it is constructively of Krull dimension $\leq (-1)$.)

Note that unlike the usual definition, this definition posits only a condition on elements and not on ideals. It is thus of a simpler logical form. (The radical ideals appear only for convenience. We will dispose of them in the proof of proposition 3.12.)

For the following, no intuition about this definition is needed; however, we feel that some motivation might be of use. Recall that we can picture inclusions of radical ideals geometrically by considering standard open subsets $D(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}$: The inclusion $\sqrt{(f)} \subseteq \sqrt{(g,h)}$ holds if and only if $D(f) \subseteq D(g) \cup D(h)$, and intersections are calculated by products, i. e. $D(f) \cap D(g) = D(fg)$.

The condition that $(b_0, \ldots, b_n)$ is complementary to $(a_0, \ldots, a_n)$ thus means that $D(a_0)$ and $D(b_0)$ cover the whole of $\operatorname{Spec} R$; that their intersection is covered by $D(a_1)$ and $D(b_1)$; that in turn their intersection is covered by $D(a_2)$ and $D(b_2)$; $\ldots$; and that finally, the intersection of $D(a_n)$ and $D(b_n)$ is empty.

For the special case $n = 0$, the condition that R is constructively of Krull dimension ≤ 0 means that for any element a_0 there exists an element b_0 such that $D(a_0)$ and $D(b_0)$ cover $\operatorname{Spec} R$ and are disjoint. This implies that $\operatorname{Spec} R$ is zero-dimensional in an intuitive sense.

Theorem 3.11. *Let R be a ring.*

(1) *In classical logic, the ring R is constructively of Krull dimension $\leq n$ if and only if the usually defined Krull dimension of R is less than or equal to n.*
(2) *If the ring R is constructively of Krull dimension $\leq n$, the radical of any finitely generated ideal is equal to the radical of some ideal which can be generated by $n+1$ elements. This holds intuitionistically, and there is an explicit algorithm for computing the reduced set of generators from the given ones.*

Proof. **XXX: Give reference.** □

We can apply the constructive theory of Krull dimension to the structure sheaf $\mathcal{O}_X$ of a scheme X as follows. Note that the condition that a scheme X has dimension exactly n is not local – the dimension may vary on an open cover; therefore it is not possible to characterize this condition in the internal language. However, the condition that the dimension of X is less than or equal to n *is* local, thus there is hope that it can be internalized. And indeed, this is the case.

Proposition 3.12. *Let X be a scheme. Then:*

$$\dim X \leq n \quad \Longleftrightarrow \quad \mathrm{Sh}(X) \models \ulcorner \mathcal{O}_X \text{ is constructively of Krull dimension } \leq n \urcorner$$

Proof. A condition of the form "$\sqrt{(f)} \subseteq \sqrt{(g,h)}$" like in the constructive definition of the Krull dimension is not a geometric formula when taken on face value. However, it is equivalent to a geometric condition, namely to

$$\exists a, b \colon \mathcal{O}_X.\ \bigvee_{m \geq 0} f^m = ag + bh \quad \wedge \qquad \exists u \colon \mathcal{O}_X.\ \bigvee_{p \geq 0} g^p = uf \quad \wedge \quad \exists v \colon \mathcal{O}_X.\ \bigvee_{q \geq 0} h^q = vf.$$

Therefore the condition $\ulcorner \mathcal{O}_X$ is constructively of Krull dimension $\leq n \urcorner$ is (equivalent to) a geometric implication and thus holds internally if and only if it holds at every point $x \in X$. This in turn means that the Krull dimension of any stalk $\mathcal{O}_{X,x}$

is less than or equal to n. This is equivalent to the (Krull) dimension of X being less than or equal to n. □

For the following corollary, note that if a scheme X is not empty, $\dim X \leq 0$ is equivalent to $\dim X = 0$.

Corollary 3.13. *Let X be a scheme. Then:*

$$\dim X \leq 0 \quad \Longleftrightarrow \quad \mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \ulcorner s \text{ inv.}\urcorner \vee \ulcorner s \text{ nilpotent}\urcorner.$$

If furthermore X is reduced, this is further equivalent to $\mathcal{O}_X$ being a field in the (stronger) sense that any element of $\mathcal{O}_X$ is invertible or zero. ***XXX: short explanation***

Proof. By the proposition and the fact that $\mathcal{O}_X$ is a local ring from the internal perspective, this is an immediate consequence of interpreting the following standard fact of ring theory in the internal language of $\mathrm{Sh}(X)$: A local ring R is constructively of Krull dimension ≤ 0 if and only if any element of R is invertible or nilpotent.

It is well-known that this holds classically; to make sure that it holds intuitionistically as well (so that it can be used in the internal universe), we give a proof of the "only if" direction. Let $a : R$ be arbitrary. By assumption on the Krull dimension, there exists an element $b : R$ such that $\sqrt{(1)} \subseteq \sqrt{(a, b)}$ and $\sqrt{(ab)} = \sqrt{(0)}$. The latter means that ab is nilpotent. Since R is local, the former implies that a is invertible or that b is invertible. In the first case, we are done. In the second case, it follows that a is nilpotent, so we are done as well. □

3.5. **Integrality.** In intuitionistic logic, the notion of an integral domain bifurcates into several inequivalent notions. The following two are important for our purposes:

Definition 3.14. A ring R is an *integral domain in the weak sense* if and only if $1 \neq 0$ in R and

$$\forall x, y : R.\ xy = 0 \Longrightarrow x = 0 \vee y = 0.$$

A ring R is an *integral domain in the strong sense* if and only if $1 \neq 0$ in R and

$$\forall x : R.\ x = 0 \vee \ulcorner x \text{ is regular}\urcorner,$$

where $\ulcorner x \text{ is regular}\urcorner$ means that $xy = 0$ implies $y = 0$ for any $y : R$.

For the following result, recall that a scheme X (or ringed space) is *integral at a point $x \in X$* if and only if $\mathcal{O}_{X,x}$ is an integral domain (in either sense, since we have adopted a classical metatheory).

Proposition 3.15. *Let X be a ringed space. Then:*

1. *X is integral at all points if and only if, internally, $\mathcal{O}_X$ is an integral domain in the weak sense.*
2. *If X is even a locally Noetherian scheme, then $\mathcal{O}_X$ is an integral domain in the weak sense iff it is an integral domain in the strong sense from the internal point of view.*

Proof. The condition on a ring to be an integral domain in the weak sense is a conjunction of two geometric implications, "$1 = 0 \Rightarrow \bot$" and the implication displayed in the definition. Therefore the first statement is obvious.

For the second statement, note that the condition on a function $f \in \Gamma(U, \mathcal{O}_X)$ to be regular from the internal perspective is open: It holds at a point $x \in U$ if and only if it holds on some open neighbourhood of x. We will give a proof of this specific feature of locally Noetherian schemes later on, when we have developed appropriate machinery to do so easily (proposition 7.4). In any case, this openness property was the essential ingredient for the equivalence between "holding internally" and "holding at every point" (corollary 2.11). Therefore $\mathcal{O}_X$ is an integral domain in the strong sense from the internal point of view if and only if all local rings $\mathcal{O}_{X,x}$ are integral domains. By the first statement, this is equivalent to $\mathcal{O}_X$ being an integral domain in the weak sense from the internal point of view. □

Lemma 3.16. *Let $X = \operatorname{Spec} A$ be an affine scheme. Let $f \in A$. Then f is a regular element of A if and only if f is a regular element of $\mathcal{O}_X$ from the internal perspective.*

Proof. The Kripke–Joyal translation of internal regularity is:

> For any open subset $U \subseteq X$ and any function $g \in \Gamma(U, \mathcal{O}_X)$, $fg = 0$ in $\Gamma(U, \mathcal{O}_X)$ implies $g = 0$ in $\Gamma(U, \mathcal{O}_X)$.

So the "only if" direction is clear (use $U := X$). For the "if" direction, note that $\Gamma(U, \mathcal{O}_X)$ is a localization of A and that regular elements remain regular in localizations. □

3.6. **Bézout property.** Recall that a *Bézout ring* is a ring in which any finitely generated ideal is a principal ideal. In intuitionistic mathematics, this is a better notion than that of a principal ideal ring: The requirement that *any* ideal is a principal ideal is far too strong. Intuitively, this is because without any given generators to begin with, one cannot hope to explicitly pinpoint a principal generator. One can (provably) not even verify this property for the ring $\mathbb{Z}$.[3]

Proposition 3.17. *Let X be a scheme (or ringed space).*

1. *$\mathcal{O}_X$ is a Bézout ring from the internal perspective if and only if all rings $\mathcal{O}_{X,x}$ are Bézout rings.*
2. *$\mathcal{O}_X$ is such that, from the internal perspective, of any two elements, one divides the other if and only if all rings $\mathcal{O}_{X,x}$ are such.*

Proof. Both properties can be formulated as geometric implications:

$$(1) \quad \forall f, g : \mathcal{O}_X.\ \top \Rightarrow \exists d : \mathcal{O}_X.\ (\exists a, b : \mathcal{O}_X.\ d = af + bg) \wedge (\exists u : \mathcal{O}_X.\ f = ud) \wedge (\exists v : \mathcal{O}_X.\ g = vd)$$

$$(2) \quad \forall f, g : \mathcal{O}_X.\ \top \Rightarrow (\exists u : \mathcal{O}_X.\ f = ug) \vee (\exists u : \mathcal{O}_X.\ g = uf)$$

□

[3]Assume that any ideal of $\mathbb{Z}$ is finitely generated. Let φ be an arbitrary proposition; we want to intuitionistically deduce $\varphi \vee \neg\varphi$. Consider the ideal $\mathfrak{a} := \{x \in \mathbb{Z} \mid (x = 0) \vee \varphi\} \subseteq \mathbb{Z}$. The definition is such that φ holds if and only if $\mathfrak{a}$ contains an element other than zero; and that $\neg\varphi$ holds if and only if zero is the only element of $\mathfrak{a}$. By assumption, $\mathfrak{a}$ is finitely generated. Since $\mathbb{Z}$ is a Bézout ring, it is therefore even principal: $\mathfrak{a} = (x_0)$ for some $x_0 \in \mathbb{Z}$. Even intuitionistically we have $(x_0 = 0) \vee (x_0 \neq 0)$ (for the natural numbers, this can be proven by induction). In the first case, it follows that $\mathfrak{a}$ contains only zero; in the second case, it follows that $\mathfrak{a}$ contains an element other than zero. Thus $\neg\varphi \vee \varphi$. This mode of reasoning is called *exhibiting a Brouwerian counterexample*. The definition of $\mathfrak{a}$ may look slightly dubious, considering that φ does not depend on x; but we will see that such definitions actually have a clear geometric meaning – they can be used to define extensions of sheaves by zero in the internal language (lemma 10.2).

Corollary 3.18. *Let X be a Dedekind scheme, i. e. a locally Noetherian normal scheme of dimension ≤ 1. Then, from the internal perspective, any matrix over $\mathcal{O}_X$ can be put into Smith canonical form, i. e. is equivalent to a (rectangular) diagonal matrix with diagonal entries $a_1|a_2|\cdots|a_n$ successively dividing each other.*

Proof. It is well-known that such a scheme has principal ideal domains as local rings $\mathcal{O}_{X,x}$. For local domains, the Bézout condition is equivalent to the property that of any two elements, one divides the other. Therefore all local rings have this property, and by the previous proposition, the internal ring $\mathcal{O}_X$ has it as well. The statement thus follows from interpreting the following fact of linear algebra in the internal universe: Let R be a ring such that of any two elements, one divides the other. Then any matrix over R can be put into Smith canonical form.

The usual proof of this fact is indeed intuitionistically acceptable: Let a matrix over R be given. By induction, one can show that for any finite family of ring elements, one divides all the others. So some matrix entry is a factor of all the others. We can put this entry to the upper left by row and column transformations and then kill the other entries of the first row and the first column. After these operations, it is still the case that the entry in the first row and column is a factor of all other entries. Continuing in this fashion, we arrive at a diagonal matrix. Its diagonal entries already fulfill the divisibility condition. □

Note that phrases such as "if by chance the entry in the upper left divides all the others, we can directly proceed with the next step; otherwise, some other entry must be a factor of all entries, so ..." may not be included in a proof which is intended to be intuitionistically acceptable: Those phrases assume that one may do**XXX: word** the case distinction that for any two ring elements x, y, either x divides y or not. Fortunately, those case distinctions are in fact superfluous.

A consequence of the corollary is that internal to the sheaf topos of a Dedekind scheme, the usual structure theorem on finitely presented $\mathcal{O}_X$-modules is available. We will exploit this in **XXX: ref**.

3.7. **Normality.** We will discuss the property of a ring to be *normal*, i. e. to be integrally closed in its total field of fractions, in section 7.3, after giving an internal characterization of the sheaf of rational functions.

4. Sheaves of modules

From the internal perspective, a sheaf of $\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of rings, looks just like a plain module over the plain ring $\mathcal{R}$. This is proven just as the correspondence between sheaf of rings and internal rings (proposition 3.1).

4.1. **Local finite freeness.** Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally finitely free* if and only if there exists a covering of X by open subsets U such that on each such U, the restricted module $\mathcal{F}|_U$ is isomorphic as an $\mathcal{O}_X|_U$-module to $(\mathcal{O}_X|_U)^n$ for some natural number n (which may depend on U).

Proposition 4.1. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is locally finitely free if and only if, from the internal perspective, $\mathcal{F}$ is a finitely free module, i. e.*

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \ulcorner \mathcal{F} \cong (\mathcal{O}_X)^n \urcorner,$$

or more elementarily

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_i a_i x_i.$$

Proof. By the expression "$(\mathcal{O}_X)^n$" in the internal language we mean the internally constructed object $\mathcal{O}_X \times \cdots \times \mathcal{O}_X$ with its componentwise $\mathcal{O}_X$-module structure. This coincides with the sheaf $(\mathcal{O}_X)^n$ as usually understood.

It is clear that the two stated internal conditions are equivalent, since the corresponding proof in linear algebra is intuitionistically acceptable. The equivalence with the external notion of locally finite freeness follows because the interpretation of the first condition with the Kripke–Joyal semantics is the following: There exists a covering of X by open subsets U such that for each such U, there exists a natural number n and a morphism of sheaves $\varphi : \mathcal{F}|_U \to (\mathcal{O}_X|_U)^n$ such that

$$U \models \ulcorner \varphi \text{ is } \mathcal{O}_X\text{-linear} \urcorner \quad \text{and} \quad U \models \ulcorner \varphi \text{ is bijective} \urcorner.$$

The first subcondition means that φ is a morphism of sheaves of $\mathcal{O}_X|_U$-modules and the second one means that φ is an isomorphism of sheaves. □

4.2. **Finite type, finite presentation, coherence.** Recall the conditions of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or ringed space) to be of finite type, of finite presentation and to be coherent:

- $\mathcal{F}$ is *of finite type* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

  of $\mathcal{O}_X|_U$-modules.
- $\mathcal{F}$ is *of finite presentation* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^m \longrightarrow (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

- $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type and the kernel of any $\mathcal{O}_X|_U$-linear morphism $(\mathcal{O}_X|_U)^n \to \mathcal{F}|_U$, $U \subseteq X$ any open subset, is of finite type.

The following proposition gives translations of these definitions into the internal language.

Proposition 4.2. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then:*

- *$\mathcal{F}$ is of finite type if and only if $\mathcal{F}$, considered as an ordinary module from the internal perspective, is finitely generated, i. e. if*

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{F}.\ x = \sum_i a_i x_i.$$

- *$\mathcal{F}$ is of finite presentation if and only if $\mathcal{F}$ is a finitely presented module from the internal perspective, i. e. if*

$$\mathrm{Sh}(X) \models \bigvee_{n,m \geq 0} \ulcorner \text{there is a short exact sequence } \mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0 \urcorner.$$

- *$\mathcal{F}$ is coherent if and only if $\mathcal{F}$ is a coherent module from the internal perspective, i. e. if*

$$\begin{aligned}\mathrm{Sh}(X) \models\ & \ulcorner \mathcal{F} \text{ is finitely generated} \urcorner \wedge \\ & \bigwedge_{n \geq 0} \forall \varphi : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \ulcorner \ker \varphi \text{ is finitely generated} \urcorner.\end{aligned}$$

Proof. Straightforward: The translations of the internal statements using the Kripke–Joyal semantics are precisely the corresponding external statements. □

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *generated by global sections* if and only if there exist global sections $s_i \in \Gamma(X, \mathcal{F})$ such that for any $x \in X$, the stalk $\mathcal{F}_x$ is generated by the germs of the s_i. This condition is of course not local on the base space. Therefore there cannot exist a formula φ such that for any space X and any $\mathcal{O}_X$-module $\mathcal{F}$ it holds that $\mathcal{F}$ is generated by global sections if and only if $\mathrm{Sh}(X) \models \varphi(\mathcal{F})$. But still, global generation can be characterized by a mixed internal/external statement:

Proposition 4.3. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is generated by global sections if and only if there exist global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that*

$$\mathrm{Sh}(X) \models \forall x : \mathcal{F}.\ \bigvee\nolimits_{J = \{i_1, \ldots, i_n\} \subseteq I \text{ finite}} \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_j a_j x_{i_j}.$$

Furthermore, $\mathcal{F}$ is generated by finitely many global sections if and only if there exist global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{F})$ such that

$$\mathrm{Sh}(X) \models \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_j a_j x_j.$$

Proof. The given internal statements are geometric implications, their validity can thus be checked stalkwise. □

4.3. **Tensor product and flatness.** Recall that the tensor product of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X (or ringed space) is usually constructed as the sheafification of the presheaf

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$

From the internal point of view, $\mathcal{F}$ and $\mathcal{G}$ look like ordinary modules, so that we can consider their tensor product as usually constructed in commutative algebra, as a certain quotient of the free module on the elements of $\mathcal{F} \times \mathcal{G}$:

$$\mathcal{O}_X\langle x \otimes y \,|\, x : \mathcal{F}, y : \mathcal{G} \rangle / R,$$

where R is the submodule generated by

$$\begin{gathered}(x + x') \otimes y - x \otimes y - x' \otimes y, \\ x \otimes (y + y') - x \otimes y - x \otimes y', \\ (sx) \otimes y - s(x \otimes y), \\ x \otimes (sy) - s(x \otimes y)\end{gathered}$$

with $x, x' : \mathcal{F}$, $y, y' : \mathcal{G}$, $s : \mathcal{O}_X$. This internal construction will give rise to the same sheaf of modules as the externally defined tensor product:

Proposition 4.4. *Let X be scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the internally constructed tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ coincides with the external one.*

Proof. Since the proof of the corresponding fact of commutative algebra is intuitionistically acceptable, the internally defined tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ has the following universal property: For any $\mathcal{O}_X$-module H, any $\mathcal{O}_X$-bilinear map $\mathcal{F} \times \mathcal{G} \to H$ uniquely factors over the canonical map $\mathcal{F} \times \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Interpreting this property with the Kripke–Joyal semantics, we see that the internally constructed tensor product has the following external property: For any open subset $U \subseteq X$ and any $\mathcal{O}_X|_U$-module $\mathcal{H}$ on U, any $\mathcal{O}_X|_U$-bilinear morphism $\mathcal{F}|_U \times \mathcal{G}|_U \to \mathcal{H}$ uniquely factors over the canonical morphism $\mathcal{F} \times \mathcal{G} \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U$.

In particular, for $U = X$, this property is well-known to be the universal property of the externally constructed tensor product. Therefore the claim follows. □

By the internal construction, a description of the stalks of the tensor product follows purely by considering the logical form of the construction:

Corollary 4.5. *Let X be scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the stalks of the tensor product coincide with the tensor products of the stalks: $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.*

Proof. We constructed the tensor product using the following operations: product of two sets, free module on a set, quotient module with respect to a submodule; submodule generated by a set of elements given by a geometric formula. All of these operations are geometric, so the tensor product construction is geometric as well. Hence taking stalks commutes with performing the construction. □

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if and only if all stalks $\mathcal{F}_x$ are flat $\mathcal{O}_{X,x}$-modules. We can characterize flatness in the internal language.

Proposition 4.6. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat if and only if, from the internal perspective, $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Recall that flatness of an A-module M can be characterized without reference to tensor products by the following condition (using suggestive vector notation): For any natural number p, any p-tuple $m : M^p$ of elements of M and any p-tuple $a : A^p$ of elements of A, it should hold that

$$a^T m = 0 \implies \bigvee_{q \geq 0} \exists n : M^q, B : A^{p \times q}.\ Bn = m \wedge a^T B = 0.$$

The equivalence of this condition with tensoring being exact holds intuitionistically as well [9, theorem III.5.3]. This formulation of flatness has the advantage that it is the conjunction of geometric implications (one for each $p \geq 0$); therefore it holds internally if and only if it holds at any stalk. □

4.4. **Support.** Recall that the *support* of an $\mathcal{O}_X$-module $\mathcal{F}$ is the subset $\operatorname{supp} \mathcal{F} := \{x \in X \mid \mathcal{F}_x \neq 0\} \subseteq X$. If $\mathcal{F}$ is of finite type, this set is closed, since its complement is then open by a standard lemma. (We will give an internal proof of this fact in lemma 6.30.)

Proposition 4.7. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the interior of the complement of the support of $\mathcal{F}$ can be characterized as the largest open subset of X on which the internal statement $\mathcal{F} = 0$ holds.*

Proof. For any open subset $U \subseteq X$, it holds that:

$$\begin{aligned} & U \subseteq \operatorname{int}(X \setminus \operatorname{supp} \mathcal{F}) \\ \Longleftrightarrow\ & U \subseteq X \setminus \operatorname{supp} \mathcal{F} \\ \Longleftrightarrow\ & U \subseteq \{x \in X \mid \forall s \in \mathcal{F}_x .\ s = 0\} \\ \Longleftrightarrow\ & U \models \forall s : \mathcal{F}.\ s = 0 \\ \Longleftrightarrow\ & U \models \ulcorner \mathcal{F} = 0 \urcorner \end{aligned}$$

The second to last equivalence is because "$\forall s : \mathcal{F}.\ s = 0$" is a geometric implication and can thus be checked stalkwise. □

Remark 4.8. The support of a sheaf of *sets* $\mathcal{F}$ is defined as the subset $\{x \in X \mid \mathcal{F}_x \text{ is not a singleton}\}$. A similar proof shows that the interior of its complement can be characterized as the largest open subset of X where the internal statement $\ulcorner \mathcal{F}$ is a singleton$\urcorner$ holds.

4.5. **Torsion.** Let R be a ring. Recall that the *torsion submodule* M_{tors} of an R-module M is defined as

$$M_{\text{tors}} := \{x : M \mid \exists a : R.\ \ulcorner a \text{ regular} \urcorner \wedge ax = 0\} \subseteq M.$$

This definition is meaningful even if R is not an integral domain (in the weak sense). An R-module M is *torsion-free* if and only if M_{tors} is the zero submodule; an R-module M is a *torsion module* if and only if $M_{\text{tors}} = M$.

Recall also that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an integral scheme X, there is a unique subsheaf $\mathcal{F}_{\text{tors}} \subseteq \mathcal{F}$ with the property that $\Gamma(U, \mathcal{F}_{\text{tors}}) = \Gamma(U, \mathcal{F})_{\text{tors}}$ for all affine open subsets $U \subseteq X$. The content of the following proposition is that internally constructing the torsion submodule of $\mathcal{F}$, regarded as a plain module from the internal perspective, gives exactly that subsheaf. There is therefore no harm in using the same notation "$\mathcal{F}_{\text{tors}}$" for the result of the internal construction.

Proposition 4.9. *Let X be an integral scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $U = \operatorname{Spec} A \subseteq X$ be an affine open subset. Let $s \in \Gamma(U, \mathcal{F})$ be a local section. Then*

$$s \in \Gamma(U, \mathcal{F})_{\text{tors}} \quad \textit{if and only if} \quad U \models s \in \mathcal{F}_{\text{tors}}.$$

Proof. The "only if" direction is trivial in light of lemma 3.16: If s is a torsion element of $\Gamma(U, \mathcal{F})$, there exists a regular element $a \in \Gamma(U, \mathcal{O}_X)$ such that $as = 0$. By the lemma, this element is regular from the internal perspective as well, so $U \models \ulcorner a \text{ regular} \urcorner \wedge as = 0$.

For the "if" direction, we may assume that there exists an open covering $X = \bigcup_i U_i$ be standard open subsets $U_i = D(f_i)$ such that there are sections $a_i \in \Gamma(U_i, \mathcal{O}_X) = A[f_i^{-1}]$ with $U_i \models \ulcorner a_i \text{ regular} \urcorner \wedge a_i s = 0$. Without loss of generality, we may assume that the denominators of the a_i's are ones, that the f_i are finite in number, and that the f_i are regular (i. e. nonzero, since A is an integral domain). By lemma 3.16, the a_i are regular in $A[f_i^{-1}]$ and by regularity of the f_i also regular in A. Therefore their product $\prod_i a_i \in A$ is regular in A as well and annihilates s. □

Proposition 4.10. *Let X be a locally Noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $x \in X$ be a point. Then $(\mathcal{F}_{\text{tors}})_x = (\mathcal{F}_x)_{\text{tors}}$.*

Proof. This would be obvious if the condition on an element $s : \mathcal{F}$ to belong to $\mathcal{F}_{\mathrm{tors}}$ were a geometric formula. Because of the universal quantifier, it is not:

$$\exists a : \mathcal{O}_X.\ (\forall b : \mathcal{O}_X.\ ab = 0 \Rightarrow b = 0) \wedge as = 0.$$

But since X is assumed to be locally Noetherian, regularity is an open property nonetheless (see proposition 7.4 for an internal proof of this fact). Thus the claim still follows, just like in the proof of proposition 3.15. ☐

4.6. **Internal proofs of common lemmas.**

Lemma 4.11. *Let X be a scheme (or ringed space). Let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{H}$ are of finite type, so is $\mathcal{G}$; similarly, if $\mathcal{F}$ and $\mathcal{H}$ are locally finitely free, so is $\mathcal{G}$.

Proof. From the internal perspective, we are given a short exact sequence of modules with the outer ones being finitely generated (resp. finitely free) and we have to show that the middle one is finitely generated (resp. finitely free) as well. It is well-known that this follows; and since the usual proof of this fact is intuitionistically acceptable, we are done. ☐

XXX: discuss alternative proofs

Lemma 4.12. *Let X be a scheme (or ringed space).*

- *Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If two of the three modules are coherent, so is the third.*
- *Let $\mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules such that $\mathcal{F}$ is of finite type and $\mathcal{G}$ is coherent. Then its kernel is of finite type as well.*
- *If $\mathcal{F}$ is a finitely presented $\mathcal{O}_X$-module and $\mathcal{G}$ is a coherent $\mathcal{O}_X$-module, the $\mathcal{O}_X$-modules $\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})$ and $\mathcal{F} \otimes \mathcal{G}$ are coherent as well.*

Proof. These statements follow directly from interpreting the corresponding standard proofs of commutative algebra in the internal language. For those standard proofs, see for instance the lecture notes of Ravi Vakil [14, section 13.8], where they are given as a series of exercises. ☐

Lemma 4.13. *Let X be a scheme (or locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of locally finitely free $\mathcal{O}_X$-modules. Then the kernel of α is locally finitely free as well.*

Proof. It suffices to give an intuitionistic proof of the following statement: The kernel of a matrix over a local ring, which as a linear map is surjective, is finitely free.

Let $M : R^{n \times m}$ be such a matrix. Since by the surjectivity assumption some linear combination of the columns is e_1 (the first canonical basis vector), some linear combination of the entries of the first row of M is 1. By locality of R, at least one entry of the first row is invertible. By applying appropriate column and row transformations, we may assume that M is of the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \widetilde{M} & \\ 0 & & & \end{array}\right)$$

with the submatrix $\widetilde{M}$ fulfilling the same condition as M. Continuing in this way, it follows that $m \geq n$ and that we may assume that M is of the form

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right).$$

The kernel of such a matrix is obviously freely generated by the canonical basis vectors corresponding to the zero columns. In particular, the rank of the kernel is $m - n$. ☐

Remark 4.14. The internal language machinery gives no reason to believe that the dual statement is true, i. e. that the cokernel of a monomorphism of locally finitely free $\mathcal{O}_X$-modules is locally finitely free: This would follow from an intuitionistic proof of the statement that the cokernel of an injective map between finitely free modules over a local ring is finitely free. But this statement is false, as the following example shows.

$$0 \longrightarrow \mathbb{Z}_{(2)} \xrightarrow{\cdot 2} \mathbb{Z}_{(2)} \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

Lemma 4.15. *Let X be a scheme (or locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of locally finitely free $\mathcal{O}_X$-modules of the same rank. Then α is an isomorphism.*

Proof. It suffices to give an intuitionistic proof of the following statement: A square matrix over a local ring, which as a linear map is surjective, is invertible.

This follows from the proof of the previous lemma, since it shows that the kernel of such a matrix is finitely free of rank zero. ☐

Lemma 4.16. *Let X be a scheme (or ringed space). Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Then* $\operatorname{cl}\operatorname{supp}\mathcal{G} = \operatorname{cl}\operatorname{supp}\mathcal{F} \cup \operatorname{cl}\operatorname{supp}\mathcal{H}$.

Proof. Switching to complements, we have to prove that

$$\operatorname{int}(X \setminus \operatorname{supp}\mathcal{G}) = \operatorname{int}(X \setminus \operatorname{supp}\mathcal{F}) \cap \operatorname{int}(X \setminus \operatorname{supp}\mathcal{H}).$$

By proposition 4.7, it suffices to prove

$$\operatorname{Sh}(X) \models (\mathcal{G} = 0 \Longleftrightarrow \mathcal{F} = 0 \wedge \mathcal{H} = 0);$$

this is a basic observation in linear algebra, valid intuitionistically. ☐

XXX: This is kind of a lame example.

Lemma 4.17. *Let X be a scheme (or locally ringed space). Let $\mathcal{L}$ be a line bundle on X, i. e. an $\mathcal{O}_X$-module locally free of rank 1. Let $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L})$ be global sections. Then these sections globally generate $\mathcal{F}$ if and only if*

$$\operatorname{Sh}(X) \models \bigvee_i \ulcorner \alpha(s_i) \text{ is invertible for some isomorphism } \alpha : \mathcal{L} \to \mathcal{O}_X \urcorner.$$

Proof. It suffices to give an intuitionistic proof of the following fact: Let R be a local ring. Let L be a free R-module of rank 1. Let $s_1, \ldots, s_n : L$ be given elements. Then L is generated as an R-module by these elements if and only if for some i, the image of s_i under some isomorphism $L \to R$ is invertible.

Note that the choice of such an isomorphism does not matter, since any two such isomorphisms $\alpha, \beta : L \to R$ differ by a unit of R: $\alpha(x) = \alpha(\beta^{-1}(1)) \cdot \beta(x)$ for any $x : L$, and $\alpha(\beta^{-1}(1)) \cdot \beta(\alpha^{-1}(1)) = 1$ in R.

For the "if" direction, we have that some $\alpha(s_i)$ is a generator of R. Since α is an isomorphism, it follows that s_i generates L, and thus in particular, the family $s_1, \ldots, s_n$ generates L.

For the "only if" direction, we have that the unit of R can be expressed as a linear combination of the $\alpha(s_i)$, where $\alpha : L \to R$ is some isomorphism (whose existence is assured by the assumption on the rank of L). Since R is a local ring, it follows that one of the summands and thus one of the $\alpha(s_i)$ is invertible. $\square$

Remark 4.18. Note that the canonical ring homomorphism $\mathcal{O}_{X,x} \twoheadrightarrow k(x)$ is local. Therefore a germ in $\mathcal{O}_{X,x}$ is invertible if and only if its image in $k(x)$ is not zero. From this one can follow that global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{F})$ generate $\mathcal{F}$ if and only if, for any point $x \in X$, the images $s_i \in \mathcal{F}|_x$ in the fibers do not vanish simultaneously.

Lemma 4.19. *Let X be a scheme (or ringed space). Let $\mathcal{L}$ be a locally finitely free $\mathcal{O}_X$-module. Then $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.*

Proof. Recall that the dual is defined by $\mathcal{L}^\vee := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. Since "$\mathcal{H}$om" looks like "Hom" from the internal point of view, the dual sheaf $\mathcal{L}^\vee$ looks just like the ordinary dual module. However, to prove the claim, it does *not* suffice to give an intuitionistic proof of the following fact of linear algebra: Let L be a free R-module of rank 1. Then there exists an isomorphism $L^\vee \otimes_R L \to R$. Since the interpretation of "$\exists$" using the Kripke–Joyal semantics is local existence, this would only show that $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L}$ is *locally* isomorphic to $\mathcal{O}_X$.

Instead, we have to actually *write down* (i. e. explicitly give) an isomorphism in the internal language – not using the assumption that L is free of rank 1, as this would introduce an existential quantifier again (see **XXX: ref**). So we have to prove the following fact: Let L be an R-module. Then there explicitly exists a linear map $L^\vee \otimes_R L \to R$ such that this map is an isomorphism if L is free of rank 1.

This is done as usual: Define $\alpha : L^\vee \otimes_R L \to R$ by $\lambda \otimes x \mapsto \lambda(x)$. Since L is free of rank 1, there is an isomorphism $L \cong R$. Precomposing α with the induced isomorphism $R^\vee \otimes_R R \to L^\vee \otimes_R L$, we obtain the linear map $R^\vee \otimes_R R \to R$ given by the same term: $\lambda \otimes x \mapsto \lambda(x)$. One can check that an inverse is given by $x \mapsto \mathrm{id}_R \otimes x$. $\square$

Lemma 4.20. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then:*

1. *Assume X to be a locally Noetherian scheme. Then $\mathcal{F}$ is torsion-free (meaning $\mathcal{F}_{\mathrm{tors}} = 0$) if and only if all stalks $\mathcal{F}_x$ are torsion-free.*
2. *The quotient sheaf $\mathcal{F}/\mathcal{F}_{\mathrm{tors}}$ is torsion-free and the torsion submodule $\mathcal{F}_{\mathrm{tors}}$ is a torsion module.*
3. *The dual sheaf $\mathcal{F}^\vee$ is torsion-free.*
4. *If $\mathcal{F}$ is reflexive (meaning that the canonical morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism), it is torsion-free.*
5. *If $\mathcal{F}$ is locally finitely free, it is reflexive.*
6. *Assume X to be a Dedekind scheme and $\mathcal{F}$ to be of finite presentation. If $\mathcal{F}$ is torsion-free, then it is locally finitely free.*

Proof. The first statement follows from the observation that $(\mathcal{F}_{\mathrm{tors}})_x = (\mathcal{F}_x)_{\mathrm{tors}}$ (proposition 4.10). All the others follow simply by interpreting the corresponding facts of linear algebra in the internal universe. For concreteness, we give intuitionistic proofs of the last three statements.

So let M be an reflexive R-module. We have to show that M is torsion-free. To this end, let an element $x : M$ and a regular element $a : R$ such that $ax = 0$ be given. For any $\vartheta : M^\vee$, it follows that $\vartheta(x) = 0$, since $a\vartheta(x) = \vartheta(ax) = \vartheta(0) = 0$ and a is regular. Thus the image of x under the canonical map $M \to M^{\vee\vee}$ is zero. By reflexivity, this implies that x is zero.

For statement (4), we have to prove that R-modules of the form R^n are reflexive. This is obvious, the required inverse map is $(R^n)^{\vee\vee} \to R^n,\ \lambda \mapsto \sum_i \lambda(\vartheta_i)$ where $\vartheta_i : R^n \to R,\ (x_j)_j \mapsto x_i$.

In light of corollary 3.18 we can put matrices over $\mathcal{O}_X$ into Smith canonical form, if X is a Dedekind scheme. Therefore it suffices to give an intuitionistic proof of the following fact: Let R be an integral domain in the strong sense such that matrices over R can be put into Smith canonical form. Let M be a finitely presented torsion-free R-module. Then M is finitely free.

This goes as follows: Since M is finitely presented, it is the cokernel of some matrix. Without loss of generality, we may assume that it is a diagonal matrix, so M is isomorphic to some (finite) direct sum $\bigoplus_i R/(a_i)$. Since M is torsion-free, all the summands $R/(a_i)$ are torsion-free as well. Since R is an integral domain in the strong sense, this holds if and only if the a_i are either zero or invertible. Thus $R/(a_i)$ is isomorphic to R or to the zero module. In any case, $R/(a_i)$ is finitely free and therefore M is finitely free as well. □

- basic lemmas: filtered colimits, flatness, ...

5. Upper semicontinuous functions

5.1. **Interlude on natural numbers.** In classical logic, the natural numbers are complete in the sense that any inhabited set of natural numbers possesses a minimal element. This statement can not be proven intuitionistically – intuitively, this is because one cannot explicitly pinpoint the (classically existing) minimal element of an arbitrary inhabited set. In intuitionistic logic, this principle can be salvaged in two essentially different ways: either be strengthening the premise, or by weakening the conclusion. **XXX: Give sheaf-theoretic interpretation of the failure.**

Lemma 5.1. *Let $U \subseteq \mathbb{N}$ be an inhabited subset of the natural numbers.*

1. *Assume U to be* detachable*, i. e. assume that for any natural number n, either $n \in U$ or $n \notin U$. Then U possesses a minimal element.* ***XXX: Give sheaf-theoretic interpretation of detachability. With this interpretation, it should be totally clear that U possesses a minimal element.***
2. *In any case, U does* not not *possess a minimal element.*

Proof. The first statement can be proven by induction on the witness of inhabitation, i. e. the given number n such that $n \in U$. Details omitted, since we will not need this statement.

For the second statement, we give a careful proof since logical subtleties matter. To simplify the exposition, we assume that U is upward-closed, i. e. that any number larger than some element of U lies in U as well. Any subset can be closed in this way (by considering $\{n \in \mathbb{N} \mid \exists m \in U.\ n \geq m\}$) and a minimal element of the closure will be a minimal element for U as well.

We induct on the number $n \in U$ given by the assumption that U is inhabited. In the case $n = 0$ we are done since 0 is a minimal element of U. For the induction step $n \to n + 1$, the weak law of excluded middle gives

$$\neg\neg(n \in U \vee n \notin U).$$

If we can show that $n \in U \vee n \notin U$ implies the conclusion, we're done by **XXX: ref**. So assume $n \in U \vee n \notin U$. If $n \in U$, then U does not not possess a minimal element by the induction hypothesis. If $n \notin U$, then $n + 1$ is a minimal element (and so, in particular, U does *not not* possess a minimal element): For if m is any element of U, we have $m \geq n + 1$ or $m \leq n$. In the first case, we're done. In the second case, it follows that $n \in U$ because U is upward-closed and so we obtain a contradiction. From this contradiction we trivially can deduce $m \geq n + 1$ as well. □

If we want to work with a complete set of natural numbers in intuitionistic logic, we have to construction a suitable completion.

Definition 5.2. The partially ordered set of *completed natural numbers* is the set $\widehat{\mathbb{N}}$ of all inhabited upward-closed subsets of $\mathbb{N}$, ordered by reverse inclusion.

Lemma 5.3. *The poset of completed natural numbers is the least partially ordered set containing $\mathbb{N}$ and possessing minima of arbitrary inhabited subsets.*

Proof. The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \longmapsto \uparrow(n) := \{m \in \mathbb{N} \,|\, m \geq n\}.$$

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

The proof of the universal property is left to the reader. □

Remark 5.4. In classical logic, the map $\widehat{\mathbb{N}} \to \mathbb{N}$, $U \mapsto \min U$ is a well-defined isomorphism of partially ordered sets. In fact, it is the inverse of the canonical embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$. In intuitionistic logic, this embedding is still injective, but it can not be shown to be surjective: It is only the case that any element of $\widehat{\mathbb{N}}$ does *not not* possess a preimage (by lemma 5.1).

5.2. **A geometric interpretation.** We are interested in the completed natural numbers for the following reason: A completed natural number in the topos of sheaves on a topological space X is the same as an upper semicontinuous function $X \to \mathbb{N}$.

Lemma 5.5. *Let X be a topological space. The sheaf $\widehat{\mathbb{N}}$ of completed natural numbers on X is canonically isomorphic to the sheaf of upper semicontinuous $\mathbb{N}$-valued functions on X.*

Proof. When referring to the natural numbers in the internal language, we actually refer to the constant sheaf $\underline{\mathbb{N}}$ on X. (This is because the sheaf $\underline{\mathbb{N}}$ fulfills the axioms of a natural numbers object, cf. [8, section VI.1].) Recall that its sections on an open subset $U \subseteq X$ are continuous functions $U \to \mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology.

Therefore, a section of $\widehat{\mathbb{N}}$ on an open subset $U \subseteq X$ is given by a subsheaf $\mathcal{A} \hookrightarrow \underline{\mathbb{N}}|_U$ such that

$$U \models \exists n : \underline{\mathbb{N}}.\ n \in \mathcal{A} \quad \text{and} \quad U \models \forall n, m : \underline{\mathbb{N}}.\ n \geq m \wedge n \in \mathcal{A} \Rightarrow m \in \mathcal{A}.$$

Since these conditions are geometric implications, they are satisfied if and only if any stalk $\mathcal{A}_x$ is an inhabited upward-closed subset of $\underline{\mathbb{N}}_x \cong \mathbb{N}$. The association

$$x \in X \longmapsto \min\{n \in \mathbb{N} \mid n \in \mathcal{A}_x\}$$

thus defines a map $X \to \mathbb{N}$. This map is indeed upper semicontinuous, since if $n \in \mathcal{A}_x$, there exists an open neighbourhood V of x such that the constant function with value n is an element of $\Gamma(V, \mathcal{A})$ and therefore $n \in \mathcal{A}_y$ for all $y \in V$.

Conversely, let $\alpha : U \to \mathbb{N}$ be a upper semi-continous function. Then

$$V \subseteq U \text{ open} \longmapsto \{f : V \to \mathbb{N} \mid f \text{ continuous, } f \geq \alpha \text{ on } V\}$$

is a subobject of $\underline{\mathbb{N}}|_U$ which internally is inhabited and upward-closed. Further details are left to the reader. $\square$

Under the correspondence given by the lemma, locally *constant* functions map exactly to the (image of the) *ordinary* internal natural numbers (in the completed natural numbers).

Remark 5.6. In a similar vein, the sheaf given by the internal construction of the set of *all* upward-closed subsets of the natural numbers (not only the inhabited ones) is canonically isomorphic to the sheaf of upper semicontinuous functions with values in $\mathbb{N} \cup \{+\infty\}$.

5.3. **The upper semicontinuous rank function.** Recall that the rank of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or locally ringed space) at a point $x \in X$ is defined as the $k(x)$-dimension of the vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. If we assume that $\mathcal{F}$ is of finite type around x, this dimension is finite and equals the minimal number of elements needed to generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module (by Nakayama's lemma).

In the internal language, we can define an element of $\widehat{\mathbb{N}}$ by

$$\operatorname{rank} \mathcal{F} := \min\{n \in \mathbb{N} \mid \ulcorner\text{there is a gen. family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner\} \in \widehat{\mathbb{N}}.$$

If $\mathcal{F}$ is locally finitely free, it will be a finitely free module from the internal point of view and the rank defined in this way will be an actual natural number (see below); but in general, the rank is really an element of the completion.

Proposition 5.7. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type on a scheme X (or locally ringed space). Under the correspondence given by the previous lemma, the internally defined rank maps to the rank function of $\mathcal{F}$.*

Proof. We have to show that for any point $x \in X$ and natural number n, there exists a generating family for $\mathcal{F}_x$ consisting of n elements if and only if there exists an open neighbourhood U of x such that

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

The "if" direction is obvious. For the "only if" direction, consider (liftings to local sections of a) generating family $s_1, \ldots, s_n$ of $\mathcal{F}_x$. Since $\mathcal{F}$ is of finite type, there also exist sections $t_1, \ldots, t_m$ on some neighbourhood V of x which generate any stalk $\mathcal{F}_y$, $y \in V$. Since the t_i can be expressed as a linear combination of the s_j in $\mathcal{F}_x$, the same is true on some open neighbourhood $U \subseteq V$ of x. On this neighbourhood, the s_j generate any stalk $\mathcal{F}_y$, $y \in U$, so by geometricity we have

$$U \models \ulcorner s_1, \ldots, s_n \text{ generate } \mathcal{F}\urcorner. \qquad \square$$

Remark 5.8. Once we understand when properties holding at a stalk spread to a neighbourhood, we will be able to give a simpler proof of the proposition (see lemma 6.32).

Lemma 5.9. *Let X be a scheme (or a locally ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. If $\mathcal{F}$ is locally finitely free, its rank function is locally constant. The converse holds if X is a reduced scheme.*

Proof. The rank function is locally constant if and only if internally, the rank of $\mathcal{F}$ is an actual natural number. Also recall that the structure sheaf fulfills a certain field condition if X is a reduced scheme (corollary 3.8). Therefore it suffices to give a proof of the following fact of intuitionistic linear algebra: Let R be a local ring. Let M be a finitely generated R-module. If M is finitely free, its rank is an actual natural number. The converse holds if R fulfills the field condition that any element which is not invertible is zero.

So assume that such a module M is finitely free. Then it is isomorphic to R^n for some actual natural number n; by the internal proof in lemma 4.13, the rank of M is therefore this number n (for any surjection $R^m \twoheadrightarrow R^n$ it holds that $m \geq n$).

Conversely, assume that the rank of M is an actual natural number. Then there exists a minimal generating family $x_1, \ldots, x_n : M$. This family is linearly independent (and thus a basis, demonstrating that M is finitely free): Let $\sum_i a_i x_i = 0$ with $a_i : R$. If any a_i were invertible, the family $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ would too generate M, contradicting the minimality. So each a_i is not invertible. Since R fulfills the appropriate field condition, each a_i is zero. □

Lemma 5.10. *Let X be a reduced scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then $\mathcal{F}$ is locally free on a dense open subset.*

Proof. Since "dense open" translates to "not not" in the internal language (proposition 6.4), it suffices to give an intuitionistic proof of the following fact: Let R be a local ring which fulfills an appropriate field condition. Let M be a finitely generated R-module. Then R is *not not* finitely free.

By remark 5.4, the rank of such a module M is *not not* an actual natural number. By the last part of the previous proof, it thus follows that M is *not not* finitely free. □

Remark 5.11. Note that besides basics on natural numbers in an intuitionistic setting and some dictionary terms ("reduced", "locally finitely free", "finite type", "dense open"), this proof does not depend on any further tools. In particular, Nakayama's lemma and facts about semicontinuous functions do not enter. For the (more complex) standard proof of this fact, see for instance [14], where the claim is dubbed an "important hard exercise" (exercise 13.7.K).

5.4. **The upper semicontinuous dimension function.** Recall that the dimension of a topological space X at a point $x \in X$ is defined as the infimum

$$\dim_x X := \inf\{\dim U \mid U \text{ open neighbourhood of } x\}.$$

One may restrict to open *connected* neighbourhoods of X, since any arbitrary open neighbourhood of x contains such a one and the dimension decreases (weakly) on subsets.

The map $X \to \mathbb{N} \cup \{+\infty\}, x \mapsto \dim_x X$ is upper semicontinuous and thus corresponds to an internal completed (possibly unbounded) natural number. The following proposition shows that this number has an explicit description.

Proposition 5.12. *Let X be a scheme. Then the upper semicontinuous function associated to the internal number "Krull dimension of $\mathcal{O}_X$" is the dimension function $x \mapsto \dim_x X$.*

Proof. Internally, we define the Krull dimension of $\mathcal{O}_X$ as the infimum over all natural numbers n such that $\mathcal{O}_X$ is constructively of Krull dimension $\leq n$. This infimum need not exist in the natural numbers, of course; so we really mean the upward-closed set $\mathcal{A}$ of all those numbers. (It is inhabited if and only if, from the external perspective, the dimension of X is locally finite. In this case, it defines a completed natural number.)

We thus have to show for any point $x \in X$:

$$\inf\{n \in \mathbb{N} \cup \{+\infty\} \,|\, n \in \mathcal{A}_x\} = \dim_x X.$$

The condition on n can be expressed as follows, where we write "$\underline{n}$" to denote the constant function with value n:

$$\begin{aligned}
& n \in \mathcal{A}_x \\
\Longleftrightarrow\ & \text{for some open neighbourhood } U \text{ of } x,\ \underline{n} \in \Gamma(U, \mathcal{A}) \\
\Longleftrightarrow\ & \text{for some open connected neighbourhood } U \text{ of } x,\ \underline{n} \in \Gamma(U, \mathcal{A}) \\
\Longleftrightarrow\ & \text{for some open connected neighbourhood } U \text{ of } x, \\
& \qquad U \models \ulcorner \mathcal{O}_X \text{ is constructively of Krull dimension} \leq n \urcorner \\
\Longleftrightarrow\ & \text{for some open connected neighbourhood } U \text{ of } x, \\
& \qquad \dim U \leq n
\end{aligned}$$

We thus have:

$$\begin{aligned}
\inf\{n \,|\, n \in \mathcal{A}_x\} &= \inf\{\inf\{n \,|\, \dim U \leq n\} \,|\, U \text{ open connected neighbourhood of } x\} \\
&= \inf\{\dim U \,|\, U \text{ open connected neighbourhood of } x\} \\
&= \dim_X X.
\end{aligned}$$

□

6. Modalities

6.1. **Basics on truth values and modal operators.**

Definition 6.1. The *set of truth values* Ω is the powerset of the singleton set $1 := \{\star\}$, where $\star$ is a formal symbol.

XXX: customary in topos theory, do not confuse with Kähler stuff

In classical logic, any subset of $\{\star\}$ is either empty or inhabited, so that Ω contains exactly two elements, the empty set ("false") and $\{\star\}$ ("true"). But in intuitionistic logic, this can not be shown; indeed, if we interpret the definition in the topos of sheaves on a space X, we obtain a (large) sheaf Ω with

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \Omega) = \{V \subseteq U \,|\, V \text{ open}\}.$$

(This is because by definition of Ω as the power object of the terminal sheaf 1, sections of Ω on an open subset U correspond to subsheaves $\mathcal{F} \hookrightarrow 1|_U$, and those are given by the greatest open subset $V \subseteq U$ such that $\Gamma(V, \mathcal{F})$ is inhabited.) **XXX: remark on the nontriviality/bigness of this sheaf**

The *truth value* of a formula φ is by definition the subset $\{x \in 1 \,|\, \varphi\} \in \Omega$, where "$x$" is a fresh variable not appearing in φ. This subset is inhabited if and only if φ holds and is empty if and only if $\neg\varphi$ holds. Conversely, we can associate to a subset $F \subseteq 1$ the proposition $\ulcorner F$ is inhabited$\urcorner$. **XXX: give sheaf-theoretic interpretation as the largest open subset...**

Under this correspondence of formulas with truth values, logical operations like $\wedge$ and $\vee$ map to set-theoretic operations like $\cap$ and $\cup$ – for instance, we have

$$\{x \in 1 \,|\, \varphi\} \cap \{x \in 1 \,|\, \psi\} = \{x \in 1 \,|\, \varphi \wedge \psi\}.$$

This justifies a certain abuse of notation: We will sometimes treat elements of Ω as propositions and use logical instead of set-theoretic connectives. In particular, if φ and ψ are elements of Ω, we will write "$\varphi \Rightarrow \psi$" to mean $\varphi \subseteq \psi$; "$\bot$" to mean $\emptyset$; and "$\top$" to mean 1.

Definition 6.2. A *modal operator* is a map $\Box : \Omega \to \Omega$ such that for all $\varphi, \psi \in \Omega$,

(1) $\varphi \Longrightarrow \Box\varphi$,
(2) $\Box\Box\varphi \Longrightarrow \Box\varphi$,
(3) $\Box(\varphi \wedge \psi) \Longleftrightarrow \Box\varphi \wedge \Box\psi$.

The intuition is that $\Box\varphi$ is a certain weakening of φ, where the precise meaning of "weaker" depends on the modal operator. By the second axiom, weakening twice is the same as weakening once.

In classical logic, where $\Omega = \{\bot, \top\}$, there are only two modal operators: the identity map and the constant map with value $\top$. Both of these are not very interesting: The identity operator does not weaken propositions at all, while the constant operator weakens every proposition to the trivial statement $\top$.

In intuitionistic logic, there can potentially exist further modal operators. For applications to algebraic geometry, the following four operators will have a clear geometric meaning and be of particular importance:

(1) $\Box\varphi :\equiv (\alpha \Rightarrow \varphi)$, where α is a fixed proposition.
(2) $\Box\varphi :\equiv (\varphi \vee \alpha)$, where α is a fixed proposition.
(3) $\Box\varphi :\equiv \neg\neg\varphi$ (the *double negation modality*).
(4) $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition.

Lemma 6.3. *Any modal operator $\Box$ is monotonic, i. e. if $\varphi \Rightarrow \psi$, then $\Box\varphi \Rightarrow \Box\psi$. Furthermore, there holds a modus ponens rule: If $\Box\varphi$ holds, and φ implies $\Box\psi$, then $\Box\psi$ holds as well.*

Proof. Assume $\varphi \Rightarrow \psi$. This is equivalent to supposing $\varphi \wedge \psi \Leftrightarrow \varphi$. We are to show that $\Box\varphi \Rightarrow \Box\psi$, i. e. that $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box\varphi$. The statement follows since by the third axiom on a modal operator, we have $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box(\varphi \wedge \psi)$.

For the second statement, consider that if $\varphi \Rightarrow \Box\psi$, by monotonicity and the second axiom on a modal operator it follows that $\Box\varphi \Rightarrow \Box\Box\psi \Rightarrow \Box\psi$. $\Box$

The modus ponens rule justifies the following proof scheme: When showing that a boxed statement $\Box\psi$ holds given that a further boxed statement $\Box\varphi$ holds, we may assume that indeed φ holds. We can also write this principle in the following way:

$$(\Box\varphi \Rightarrow \Box\psi) \Longleftrightarrow (\varphi \Rightarrow \Box\psi).$$

6.2. **Geometric meaning.** Let X be a topological space. As discussed above, an open subset $U \subseteq X$ defines an internal truth value (a global section of the sheaf Ω). We also denote it by "U", such that

$$V \models U \quad \Longleftrightarrow \quad V \subseteq U$$

for any open subset $V \subseteq X$. (Shortcutting the various intermediate steps, this can also be taken as a definition of "$V \models U$".) If $A \subseteq X$ is a closed subset, there is thus an internal truth value A^c corresponding to the open subset $A^c = X \setminus A$. If $x \in X$ is a point, we define "$!x$" to denote the truth value corresponding to $\mathrm{int}(X \setminus \{x\})$, such that

$$V \models\, !x \quad \Longleftrightarrow \quad V \subseteq \mathrm{int}(X \setminus \{x\}) \quad \Longleftrightarrow \quad x \notin V.$$

Proposition 6.4. *Let $U \subseteq X$ be a fixed open and $A \subseteq X$ be a fixed closed subset. Let $x \in X$. Then, for any open subset $V \subseteq X$, it holds that:*

$$\begin{aligned}
&V \models (U \Rightarrow \varphi) && \Longleftrightarrow && V \cap U \models \varphi. \\
&V \models (\varphi \vee A^c) && \Longleftrightarrow && \text{there is an open subset } W \subseteq V \\
&&&&& \qquad \text{containing } A \cap V \text{ such that } W \models \varphi. \\
&V \models \neg\neg\varphi && \Longleftrightarrow && \text{there is a dense open subset } W \subseteq V \text{ s. th. } W \models \varphi. \\
&V \models ((\varphi \Rightarrow\, !x) \Rightarrow\, !x) && \Longleftrightarrow && x \notin V \text{ or there is an open neighbourhood } W \subseteq V \\
&&&&& \qquad \text{of } x \text{ such that } W \models \varphi.
\end{aligned}$$

Proof. (1) Omitted.

(2) Let $V \models \varphi \vee A^c$. Then there exists an open covering $V = \bigcup_i V_i$ such that for each i, $V_i \models \varphi$ or $V_i \subseteq A^c$. Let $W \subseteq V$ be the union of those V_i such that $V_i \models \varphi$. Then $W \models \varphi$ by the locality of the internal language and $A \cap V \subseteq W$.

Conversely, let $W \subseteq V$ be an open subset containing $A \cap V$ such that $W \models \varphi$. Then $V = W \cup (V \cap A^c)$ is an open covering attesting $V \models \varphi \vee A^c$.

(3) For the "only if" direction, let $W \subseteq V$ be the largest open subset on which φ holds, i. e. the union of all open subsets of V on which φ holds. For the "if" direction, we may assume that the given W is also the largest open subset on which φ holds (by enlarging W if necessary). The claim then follows by the following chain of equivalences:

$$\begin{aligned}
&V \models \neg\neg\varphi \\
\Longleftrightarrow\ & \forall Y \subseteq V \text{ open.} \left(\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow Z = \emptyset\right) \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ & \forall Y \subseteq V \text{ open.} \left(\forall Z \subseteq Y \text{ open. } Z \subseteq W \Rightarrow Z = \emptyset\right) \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ & \forall Y \subseteq V \text{ open. } Y \cap W = \emptyset \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ & W \text{ is dense in } V.
\end{aligned}$$

(4) Straightforward, since the interpretation of the internal statement with the Kripke–Joyal semantics is

$$\forall Y \subseteq V \text{ open.} \left(\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow x \notin Z\right) \Longrightarrow x \notin Y.$$

□

Modal operator	associated nucleus	$j(V) = X$ iff ...	subspace
$\Box\varphi :\equiv (U \Rightarrow \varphi)$	$j(V) = \mathrm{int}(U^c \cup V)$	$U \subseteq V$	U
$\Box\varphi :\equiv (\varphi \vee A^c)$	$j(V) = V \cup A^c$	$A \subseteq V$	A
$\Box\varphi :\equiv \neg\neg\varphi$	$j(V) = \mathrm{int}(\mathrm{cl}(V))$	V is dense in X	smallest dense sublocale of X
$\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$	$j(V) = \begin{cases} X \setminus \mathrm{cl}\{x\}, & \text{if } x \notin V \\ X, & \text{if } x \in V \end{cases}$	$x \in V$	$\{x\}$

Table 2. List of important modal operators and their associated nuclei (notation as in proposition 6.4).

6.3. **The subspace associated to a modal operator.** Any modal operator $\Box : \Omega \to \Omega$ in the sheaf topos of X induces on global sections a map

$$j : \mathrm{Op}(X) \to \mathrm{Op}(X),$$

where $\mathrm{Op}(X) = \Gamma(X, \Omega)$ is the set of open subsets of X. By the axioms on a modal operator, the map j fulfills similar axioms: For any open subsets $U, V \subseteq X$,

(1) $U \subseteq j(U)$,
(2) $j(j(U)) \subseteq j(U)$,
(3) $j(U \cap V) = j(U) \cap j(V)$.

Such a map is called a *nucleus* on $\mathrm{Op}(X)$. Table 2 lists the nuclei associated to the four modal operators of proposition 6.4.

Any nucleus j defines a subspace X_j of X, with a small caveat: In general, the subspace X_j can not be realized as a topological subspace, but only as a so-called *sublocale*; the notion of a locale is a slight generalization of the notion of a topological space, in which an underlying set of points is not part of the definition. Instead, a locale is simply given by a lattice of general *opens* – these may, but do not necessarily have to, be sets of points. Sheaf theory carries over to locales essentially unchanged, since the notions of presheaves and sheaves only need opens and coverings. **XXX: give introductory reference**

Definition 6.5. Let j be a nucleus on $\mathrm{Op}(X)$. Then the sublocale X_j of X is given by the lattice of opens $\mathrm{Op}(X_j) := \{U \in \mathrm{Op}(X) \,|\, j(U) = U\}$.

If j is induced by a modal operator $\Box$, we also write "$X_\Box$" for X_j. In three of the four cases listed in table 2, the sublocale $X_\Box$ can indeed be realized as a topological subspace. The only exception is the sublocale $X_{\neg\neg}$ associated to the double negation modality. It can be also be described as the *smallest dense sublocale* of X; this is obviously a true locale-theoretic notion, since a topological space does not have (in general) a smallest dense topological subspace (consider $\mathbb{R}$ and its dense subsets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$). **XXX: word "true" XXX: give introductory references, i.e. Johnstone**

The inclusion $i : X_j \hookrightarrow X$ can not in general be described on the level of points, since X_j might not be realizable as a topological subspace. But for sheaf-theoretic purposes, it suffices to describe i on the level of opens. This is done as follows:

$$i^{-1} : \mathrm{Op}(X) \longrightarrow \mathrm{Op}(X_j), \quad U \longmapsto j(U).$$

Thus we can relate the toposes of sheaves on X_j and X by the usual pullback and pushforward functors.

$$\begin{aligned} i^{-1}\mathcal{F} &= \text{sheafification of } (U \mapsto \operatorname{colim}_{U \preceq i^{-1}V} \Gamma(V, \mathcal{F})) \\ i_*\mathcal{G} &= (U \mapsto \Gamma(i^{-1}U, \mathcal{G})) = (U \mapsto \Gamma(j(U), \mathcal{G})) \end{aligned}$$

As familiar from honest topological subspace inclusions, the pushforward functor $i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X)$ is fully faithful and the composition $i^{-1} \circ i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X_j)$ is (canonically isomorphic to) the identity.

6.4. **Internal sheaves and sheafification.** It turns out that the image of the pushforward functor $i_* : \mathrm{Sh}(X_\Box) \to \mathrm{Sh}(X)$, where $\Box$ is a modal operator in $\mathrm{Sh}(X)$, can be explicitly described: Namely, it consists exactly of those sheaves which from the internal point of view are so-called $\Box$-*sheaves*, a notion explained below. **XXX: state this as a proposition later as well?**

Furthermore, if we identify $\mathrm{Sh}(X_\Box)$ with its image in $\mathrm{Sh}(X)$, the pullback functor is given by an internal sheafification process with respect to the modality $\Box$. Thus the external situation of pushforward/pullback translates to forget/sheafify. This broadens the scope of the internal language: It can not only be used to talk about sheaves on X in a simple, element-based language, but also to talk about sheaves on arbitrary subspaces of X.

To describe the notion of $\Box$-sheaves and related ones, we switch to the internal perspective and thus forget X; we are simply given a model operator $\Box : \Omega \to \Omega$ and have to take care that our proofs are intuitionistically acceptable. A reference for the material in this subsection is a preprint by Fer-Jan de Vries [17][4].

Definition 6.6. A set F is $\Box$-*separated* if and only if

$$\forall x, y : F.\ \Box(x = y) \Longrightarrow x = y.$$

A set F is a $\Box$-*sheaf* if and only if it is $\Box$-separated and

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton} \urcorner) \Longrightarrow \exists x : F.\ \Box(x \in S).$$

The two conditions can be combined: A set F is a $\Box$-sheaf if and only if

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton} \urcorner) \Longrightarrow \exists! x : F.\ \Box(x \in S).$$

XXX: explain how to read these definitions

Definition 6.7. The *plus construction* of a set F with respect to $\Box$ is the set

$$F^+ := \{S \subseteq F \mid \Box(\ulcorner S \text{ is a singleton} \urcorner)\}/\sim,$$

where the equivalence relation is defined by $S \sim T :\Leftrightarrow \Box(S = T)$. There is a canonical map $F \to F^+$ given by $x \mapsto [\{x\}]$. The $\Box$-*sheafification* of a set F is the set F^{++}.

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[4]Note that on page 5 of that preprint, there is a slight typing error: Fact 2.1(i) gives the characterization of j-closedness, not j-denseness. The correct characterization of j-denseness in that context is $\forall b \in B.\ j(b \in A)$.

If F is $\Box$-separated, then for any subset $S \subseteq F$ it holds that

$$\Box(\ulcorner S \text{ is a singleton}\urcorner) \quad \Longleftrightarrow \quad \ulcorner S \text{ is a subsingleton}\urcorner \wedge \Box(\ulcorner S \text{ is inhabited}\urcorner).$$

XXX: introduce "singleton", "subsingleton"

Remark 6.8. The topos of *pre*sheaves on a topological space X admits an internal language as well [8, section VI.7, discussion after theorem 1]. In it, there exists a modal operator $\Box$ reflecting the topology of X. A presheaf on X is separated in the usual sense if, from the internal perspective of $\mathrm{PSh}(X)$, it is $\Box$-separated; and it is a sheaf if, from the internal perspective, it is a $\Box$-sheaf. Furthermore, the $\Box$-sheafification of a presheaf (considered as a set from the internal perspective) coincides with the usual sheafification.

Example 6.9. Any singleton set is a $\Box$-sheaf. The empty set is always $\Box$-separated (trivially) and is a $\Box$-sheaf if and only if $\Box\bot \Rightarrow \bot$.

Lemma 6.10. *For any set F, it holds that:*

1. *F^+ is $\Box$-separated.*
2. *The canonical map $F \to F^+$ is injective if and only if F is $\Box$-separated.*
3. *If F is $\Box$-separated, F^+ is a $\Box$-sheaf.*
4. *If F is a $\Box$-sheaf, the canonical map $F \to F^+$ is bijective.*

Let "$\mathrm{Sh}_\Box(\mathrm{Set})$" denote the full subcategory of Set *consisting of the $\Box$-sheaves. Then it holds that:*

5. *The functor $(\_\_)^+ : \mathrm{Set} \to \mathrm{Set}$ is left exact.*
6. *The functor $(\_\_)^{++} : \mathrm{Set} \to \mathrm{Sh}_\Box(\mathrm{Set})$ is left exact and left adjoint to the forgetful functor $\mathrm{Sh}_\Box(\mathrm{Set}) \to \mathrm{Set},\ F \mapsto F$.*

Proof. These are all straightforward, and it fact simpler than their classical counterparts, since there are no colimit constructions which would have to be dealt with. $\Box$

Remark 6.11. As is to be expected from the familiar inclusion of sheaves in presheaves on topological spaces, the forgetful functor $\mathrm{Sh}_\Box(\mathrm{Set}) \to \mathrm{Set}$ does not in general preserve colimits. It is instructive to see why epimorphisms in $\mathrm{Sh}_\Box(\mathrm{Set})$ need not be epimorphisms in Set: A map $f : A \to B$ between $\Box$-sheaves is an epimorphism in $\mathrm{Sh}_\Box(\mathrm{Set})$ if and only if

$$\forall y : B.\ \Box(\exists x : X.\ f(x) = y),$$

i. e. preimages do not need to exist, but merely need to "$\Box$-exist". (Using results about the $\Box$-translation, to be introduced below, this characterization will be obvious.) This condition is intuitionistically weaker than the condition that f is an epimorphism in Set, i. e. that f is surjective. Compare this to the failure of the forgetful functor $\mathrm{Sh}(X) \to \mathrm{PSh}(X)$ to preserve epimorphisms: A morphism of sheaves does not need to have preimages for any local section in order to be an epimorphism. Instead, it suffices for any local section to *locally* have preimages.

6.5. Sheaves for the double negation modality. XXX: introductory words

Proposition 6.12. *Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X. Then:*

1. *$\mathcal{F}$ is $\neg\neg$-separated if and only if it is sufficient for local sections to be equal to agree on a dense open subset of their common domain.*

(2) *$\mathcal{F}$ is a $\neg\neg$-sheaf if and only if it is $\neg\neg$-separated and for any open subset $U \subseteq X$ and any open subset $V \subseteq U$ dense in U, any V-section of $\mathcal{F}$ extends to a U-section of $\mathcal{F}$.*
(3) *If $\mathcal{F}$ is $\neg\neg$-separated, the sections of $\mathcal{F}^+$ on an open subset $U \subseteq X$ can be described by pairs (V, s), where V is a dense open subset of U and s is a section of $\mathcal{F}$ on V. Two such pairs $(V, s), (V', s')$ give the same element in $\Gamma(U, \mathcal{F}^+)$ if and only if s and s' agree on $V \cap V'$.*

Proof. The first statement is obvious from the definition of $\neg\neg$-separatedness (definition 6.6 for $\Box = \neg\neg$) and the geometric interpretation of double negation (proposition 6.4).

For the second statement, it suffices to show that if $\mathcal{F}$ is $\neg\neg$-separated, $\mathcal{F}$ has the extension property if and only if

$$\begin{aligned} \mathrm{Sh}(X) \models \forall \mathcal{S} : \mathcal{P}(\mathcal{F}).\ \ulcorner\mathcal{S} \text{ is a subsingleton}\urcorner \wedge \neg\neg(\ulcorner\mathcal{S} \text{ is inhabited}\urcorner) \Longrightarrow \\ \exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S}). \end{aligned}$$

Note that a section $\mathcal{S} \in \Gamma(U, \mathcal{P}(\mathcal{F}))$ which internally is a subsingleton and *not not* inhabited is precisely a subsheaf $\mathcal{S} \hookrightarrow \mathcal{F}$ such that all stalks $\mathcal{S}_x$, $x \in U$ are subsingletons and such that for some dense open subset $V \subseteq U$, the stalks $\mathcal{S}_x$, $x \in V$ are inhabited. This is precisely the datum of a section of $\mathcal{F}$ defined on some dense open subset of U: Consider the gluing of the unique germs in $\mathcal{S}_x$ for those points x such that $\mathcal{S}_x$ is inhabited. (Conversely, a section $s \in \Gamma(V, \mathcal{F})$ defines a subsheaf $\mathcal{S}$ by setting $\Gamma(W, \mathcal{S}) := \{s|_W \mid W \subseteq V\}$.)

In view of this explicit description and the observation that the existence in question ("$\exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S})$") is actually a question of unique existence, the second statement follows. **XXX: this is hard to understand**

For the third statement, one can check that the presheaf on X defined by

$$U \subseteq X \text{ open} \longmapsto \{(V, s) \mid V \subseteq U \text{ dense open},\ s \in \Gamma(V, \mathcal{F})\}/\sim$$

is in fact a sheaf (with respect to the topology of X), internally a $\neg\neg$-sheaf, and that it has the universal property of the $\neg\neg$-sheafification of $\mathcal{F}$. ☐

XXX: introduce notion of Box-stability

6.6. **The $\Box$-translation.** There is certain well-known transformation $\varphi \mapsto \varphi^{\neg\neg}$ on formulas, the *double negation translation*, with the following curious property: A formula φ is derivable in classical logic if and only if its translation $\varphi^{\neg\neg}$ is derivable in intuitionistic logic. The translation $\varphi^{\neg\neg}$ is obtained from φ by putting "$\neg\neg$" before any subformula, i. e. before any "$\exists$" and "$\forall$", around any logical connective and around any atomic statement ("$x = y$", "$x \in A$").

We will describe a slight generalization of the double negation translation, the $\Box$-translation for any modal operator $\Box$. It will be central to understand truth of formulas on subspaces from the internal language of the ambient space. **XXX: sentence is rubbish XXX: check http://arxiv.org/pdf/1101.5442.pdf for references to cite**

Definition 6.13. The $\Box$-*translation* is recursively defined as follows.

$$\begin{aligned}
(f = g)^\Box &:\equiv \Box(f = g) \\
(x \in A)^\Box &:\equiv \Box(x \in A) \\
\top^\Box &:\equiv \Box\top \quad (\Leftrightarrow \top) \\
\bot^\Box &:\equiv \Box\bot \\
(\varphi \wedge \psi)^\Box &:\equiv {\color{gray}\Box}(\varphi^\Box \wedge \psi^\Box) & (\textstyle\bigwedge_i \varphi_i)^\Box &:\equiv {\color{gray}\Box}(\textstyle\bigwedge_i \varphi_i^\Box) \\
(\varphi \vee \psi)^\Box &:\equiv \Box(\varphi^\Box \vee \psi^\Box) & (\textstyle\bigvee_i \varphi_i)^\Box &:\equiv \Box(\textstyle\bigvee_i \varphi_i^\Box) \\
(\varphi \Rightarrow \psi)^\Box &:\equiv {\color{gray}\Box}(\varphi^\Box \Rightarrow \psi^\Box) \\
(\forall x : X.\ \varphi)^\Box &:\equiv {\color{gray}\Box}(\forall x : X.\ \varphi^\Box) & (\forall X.\ \varphi)^\Box &:\equiv {\color{gray}\Box}(\forall X.\ \varphi^\Box) \\
(\exists x : X.\ \varphi)^\Box &:\equiv \Box(\exists x : X.\ \varphi^\Box) & (\exists X.\ \varphi)^\Box &:\equiv \Box(\exists X.\ \varphi^\Box)
\end{aligned}$$

Lemma 6.14.
(1) *Formulas in the image of the $\Box$-translation are $\Box$-stable, i. e. for any formula φ it holds that $\Box(\varphi^\Box) \Longrightarrow \varphi^\Box$.*
(2) *In the definition of the $\Box$-translation, one may omit the boxes printed in gray.*

Proof. The first statement is obvious, since one of the axioms on a modal operator demands that $\Box\Box\varphi \Rightarrow \Box\varphi$ for any formula φ. The second statement follows by an induction on the formula structure. By way of example, we prove the case for "$\Rightarrow$":

$$\begin{aligned}
&(\varphi \Rightarrow \psi)^{\Box \text{ with the gray parts}} \\
\Longleftrightarrow\ &\Box(\varphi^{\Box \text{ with the gray parts}} \Rightarrow \psi^{\Box \text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box \text{ with the gray parts}} \Rightarrow \psi^{\Box \text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box \text{ without the gray parts}} \Rightarrow \psi^{\Box \text{ without the gray parts}}) \\
\Longleftrightarrow\ &(\varphi \Rightarrow \psi)^{\Box \text{ without the gray parts}}
\end{aligned}$$

The first step is by definition; the second by $\Box$-stability of $\psi^{\Box \text{ with the gray parts}}$; the third by the induction hypothesis; the fourth by definition. $\Box$

Lemma 6.15. *The $\Box$-translation is sound with respect to intuitionistic logic: Assume that there is an intuitionistic proof of an implication $\varphi \Rightarrow \psi$. Then there is also an intuitionistic proof of the translated implication $\varphi^\Box \Rightarrow \psi^\Box$.*

Proof. This follows by an induction on the structure of intuitionistic proofs. We have to verify that we can mirror any inference rule of intuitionistic logic in the translation. For instance, one of the disjunction rules justifies the following proof scheme: In order to prove $\varphi \vee \psi \Rightarrow \chi$, it suffices to give proofs of $\varphi \Rightarrow \chi$ and $\psi \Rightarrow \chi$. We have to justify the translated proof scheme: In order to prove $(\varphi \vee \psi)^\Box \Rightarrow \chi^\Box$, it suffices to give proofs of $\varphi^\Box \Rightarrow \chi^\Box$ and $\psi^\Box \Rightarrow \chi^\Box$.

So assume that proofs of the two implications are given. Further assume $(\varphi \vee \psi)^\Box$, i. e. $\Box(\varphi^\Box \vee \psi^\Box)$. We want to show $\chi^\Box$. Since this is a $\Box$-stable statement, we may assume that in fact $\varphi^\Box \vee \psi^\Box$ holds. Then the claim is obvious by the two given proofs.

The cases for the other rules (see appendix **??** for a list) are similar and left to the reader. $\Box$

Remark 6.16. The reader well-versed in formal logic will have noticed that we are mixing syntax and semantics here. The proper way to state the lemma would be to formally adjoin a box operator to the language of intuitionistic logic, governed by three inference rules which are modeled on the three axioms on a modal operator. This formal box operator could then be instantiated by any concrete modal operator $\Box : \Omega \to \Omega$.

Soundness of the $\Box$-translation is important for the following reason (among others). If φ and φ' are equivalent formulas, we are accustomed to be able to freely substitute φ by φ' anywhere we want. Since a modal operator $\Box$ is semantically defined as a map $\Omega \to \Omega$, it is trivially justified that $\Box\varphi$ and $\Box\varphi'$ are equivalent: The formulas φ and φ' give rise to the *same* element $\{x \in 1 \,|\, \varphi\} = \{x \in 1 \,|\, \varphi'\}$ of Ω, and therefore their images under $\Box$ are equal as well.

However, it is *not* clear that the translated formulas $\varphi^\Box$ and $(\varphi')^\Box$ are equivalent. This follows only by applying the soundness lemma (two times, once for each direction). We should stress that to apply this lemma, it is not enough to merely *know* that φ and φ' are equivalent; instead, there has to be an intuitionistic proof of this equivalence. The difference might not be apparent at the top level (how should we know that the equivalence holds if not by a proof?[5]), but it becomes evident when considering hypothetical situations: To prove an implication $\psi \Rightarrow \chi$, we may give a proof of χ under the additional assumption that ψ holds; however, we may not suppose that there is a proof of ψ – indeed, ψ might later turn out to be false. **XXX: is this a good (and correct!) explanation?**

Lemma 6.17. *Let φ be a formula such that for any subformulas ψ appearing as antecedents of implications, it holds that $\psi^\Box \Rightarrow \Box\psi$. (In particular, this condition is satisfied if there are no "$\Rightarrow$" signs in φ.) Then $\Box\varphi \Rightarrow \varphi^\Box$.*

Proof. We prove this by an induction on the formula structure. All cases except for "$\Rightarrow$" are obvious. For this case, assume $\Box(\psi \Rightarrow \chi)$; we are to show that $(\psi^\Box \Rightarrow \chi^\Box)$. Since this is a $\Box$-stable statement, we can in fact assume that $(\psi \Rightarrow \chi)$. We then have

$$\psi^\Box \Longrightarrow \Box\psi \Longrightarrow \Box\chi \Longrightarrow \chi^\Box,$$

with the first step being by the requirement on antecedents, the second by the monotonicity of $\Box$, and the third by the induction hypothesis. $\Box$

Lemma 6.18. *Let φ be a geometric formula. Then $\varphi^\Box \Rightarrow \Box\varphi$.*

Proof. By induction on the formula structure. By way of example, we prove the case for "$\bigvee$". So assume $\Box(\bigvee_i \varphi_i^\Box)$; we are to show that $\Box(\bigvee_i \varphi_i)$. Since this is a boxed statement, we may in fact assume $\bigvee_i \varphi_i^\Box$, so for some index j, it holds that $\varphi_j^\Box$. By the induction hypothesis, it follows that $\Box\varphi_j$. By $\varphi_j \Rightarrow \bigvee_i \varphi_i$ and the monotonicity of $\Box$, it follows that that $\Box(\bigvee_i \varphi_i)$. $\Box$

Remark 6.19. In the special case that $\Box$ is the double negation modality, the lemma holds with slightly weaker hypotheses: Namely, implications may occur in φ, provided that for their antecedents ψ it holds that $\psi \Rightarrow \psi^\Box$. This is because for the double negation modality, the formula $\Box(\psi \Rightarrow \chi)$ is equivalent to $\psi \Rightarrow \Box\chi$. (In general, only the former implies the latter.) The case for "$\Rightarrow$" in the inductive proof then goes as follows: Assume $(\psi \Rightarrow \chi)^\Box$. Then $\psi \Rightarrow \psi^\Box \Rightarrow \chi^\Box \Rightarrow \Box\chi$, so $\Box(\psi \Rightarrow \chi)$.

[5]Of course, considering logical incompleteness theorems, this questions is slightly naive.

Lemma 6.20. *Let φ, φ', ψ be formulas. Assume that:*

- *The formula φ' is geometric. (More generally, it suffices for $(\varphi')^\Box$ to imply $\Box\varphi'$.)*
- *There is an intuitionistic proof that φ and φ' are equivalent under the (only) hypothesis ψ.*
- *Both $\Box\psi$ and $\psi^\Box$ hold.*

Then $\varphi^\Box \Rightarrow \Box\varphi$.

Proof. Assume $\varphi^\Box$. Since $\psi^\Box$, $(\varphi \wedge \psi)^\Box$. Because the $\Box$-translation is sound with respect to intuitionistic logic (lemma 6.15) it follows that $(\varphi')^\Box$. As φ' is geometric, it follows that $\Box\varphi'$. Since $\Box\psi$ holds, it follows that $\Box\varphi$. □

Example 6.21. Let M be an R-module. Then the statement that M is zero is not geometric: $\varphi :\equiv (\forall x : M.\ x = 0)$. But if M is generated by some finite family $x_1, \ldots, x_n : M$, then φ is equivalent to the statement $\varphi' :\equiv (x_1 = 0 \wedge \cdots \wedge x_n = 0)$ which is geometric; and there is an intuitionistic proof of this equivalence. Since no implication signs occur in $\psi :\equiv \ulcorner M$ is generated by $x_1, \ldots, x_n \urcorner$, the lemma is applicable and shows that $\varphi^\Box$ implies $\Box\varphi$. This example will gain geometric meaning in lemma 6.30.

Lemma 6.22. *For the modality $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition, the $\Box$-translation of the law of excluded middle holds. In particular, this applies to the double negation modality $\Box = \neg\neg$, where $\alpha = \bot$.*

Proof. We are to show that $(\varphi \vee \neg\varphi)^\Box$, i. e. that

$$(((\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)) \Longrightarrow \alpha) \Longrightarrow \alpha.$$

So assume that the antecedent holds. If $\varphi^\Box$ would hold, then in particular $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α would hold. Therefore it follows that $(\varphi^\Box \Rightarrow \alpha)$. This implies $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α. □

6.7. **Truth at stalks vs. truth on neighbourhoods.** We now state the crucial property of the $\Box$-translation. Recall that "$X_\Box$" denotes the sublocale of X induced by $\Box$ (definition 6.5).

Theorem 6.23. *Let X be a topological space. Let $\Box$ be a modal operator in $\mathrm{Sh}(X)$. Let φ be a formula over X. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(X_\Box) \models \varphi,$$

where on the right hand side, all parameters occuring in φ were pulled back to $X_\Box$ along the inclusion $X_\Box \hookrightarrow X$.

XXX: think about powersets appearing as domains of quantification

We have not yet explicitly stated the Kripke–Joyal semantics for a sheaf topos over a locale, which $X_\Box$ is in general. The definition is exactly the same as in the case for sheaf toposes over a topological space, only that any mention of "open sets" has to be substituted by the more general "opens" and any mention of the union operator "$\bigcup$" has to be interpreted by the supremum operator in the lattice of opens of the locale. For $X_\Box$, this is $\sup U_i = j(\bigcup_i U_i)$. Before giving a proof of the theorem, we want to discuss some of its consequences.

Corollary 6.24. *Let X be a topological space.*

(1) *Let $U \subseteq X$ be an open subset and let $\Box\varphi :\equiv (U \Rightarrow \varphi)$. Then*
$$\mathrm{Sh}(X) \models \varphi^{\Box} \quad \textit{iff} \quad \mathrm{Sh}(U) \models \varphi.$$
(2) *Let $A \subseteq X$ be a closed subset and let $\Box\varphi :\equiv (\varphi \vee A^c)$. Then*
$$\mathrm{Sh}(X) \models \varphi^{\Box} \quad \textit{iff} \quad \mathrm{Sh}(A) \models \varphi.$$
(3) *Let $\Box\varphi :\equiv \neg\neg\varphi$. Then*
$$\mathrm{Sh}(X) \models \varphi^{\Box} \quad \textit{iff} \quad \mathrm{Sh}(X_{\neg\neg}) \models \varphi.$$
(4) *Let $x \in X$ be a point and let $\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$. Then*
$$\mathrm{Sh}(X) \models \varphi^{\Box} \quad \textit{iff} \quad \varphi \textit{ holds at } x.$$

Proof. Combine theorem 6.23 and table 2. □

We want to discuss the last case of the corollary in more detail. Let x be a point of a topological space X and let φ be a formula. Let $\Box$ be the modal operator given in the corollary. Then φ *holds at x* if and only if, from the internal perspective of $\mathrm{Sh}(X)$, the translated formula $\varphi^{\Box}$ holds; and φ *holds on some open neighbourhood of x* if and only if, from the internal perspective, the formula $\Box\varphi$ holds.

Thus the question whether the truth of φ at the point x spreads to some open neighbourhood can be formulated in the following way:

Does $\varphi^{\Box}$ imply $\Box\varphi$ in the internal language of $\mathrm{Sh}(X)$*?*

Phrased this way, technicalities like appropriately shrinking open neighbourhoods are blinded out. A purposefully trivial example to illustrate this is the following. Let X be a scheme (or ringed space). Let $f, g \in \Gamma(X, \mathcal{O}_X)$ be global functions. Suppose that the germs of f and g are zero in some stalk $\mathcal{O}_{X,x}$; we want to show that they are zero on a common open neighbourhood of x.

Usual proof. Since the germ of f vanishes in $\mathcal{O}_{X,x}$, there is an open neighbourhood U_1 of x such that $f|_{U_1} = 0$ in $\Gamma(U_1, \mathcal{O}_X)$. Since furthermore the germ of g vanishes in the same stalk, there exists an open neighbourhood U_2 of x such that $g|_{U_2} = 0$. The intersection of both neighbourhoods is still an open neighbourhood of x; on this it holds that f and g both vanish. □

Proof in the internal language. We may suppose that $(f = 0 \wedge g = 0)^{\Box}$, i. e. $\Box(f = 0) \wedge \Box(g = 0)$, and have to prove that $\Box(f = 0 \wedge g = 0)$. (To this end, we could simply invoke the third axiom on a modal operator, but we want to stay close to the given external proof.) So by assumption, both $\Box(f = 0)$ and $\Box(g = 0)$ hold. Since our goal is to prove a boxed statement, we may in fact assume that $f = 0$ and $g = 0$. Thus $f = 0 \wedge g = 0$. □

By using the internal language with its modal operators, we can thus reduce basic facts of scheme theory which deal with stalks and neighbourhoods to facts of algebra in a *modal intuitionistic context*. As with using the internal language in its basic form without modalities, this brings conceptual clarity and reduced technical overhead. There are however two more distinctive advantages. Firstly, many internal proofs do not require specific properties of the modal operator and thus work with any modal operator. By interpreting such a proof using different operators, one obtains a whole family of external statements without any additional work.

Secondly, the following corollary gives a general metatheorem which is applicable to a wide range of cases. It allows to decide whether spreading will occur (or is likely not to occur) simply by looking at the *logical form* of the statement is question.

Corollary 6.25. *Let X be a topological space. Let φ be a formula. If φ is geometric, truth of φ at a point $x \in X$ implies truth of φ on some open neighbourhood of x, and vice versa.*

Proof. By the purely logical lemmas of the previous section, it holds that $\varphi^\Box \Leftrightarrow \Box\varphi$. ☐

Corollary 6.26. *Let X be a topological space. Let φ be a formula. If φ is geometric, the property "φ holds at a point $x \in X$" is open.*

Proof. This is just a reformulation of the previous corollary: If φ holds at a point $x \in X$, it holds on some open neighbourhood U of x as well. Going back to stalks, it follows that φ holds at every point of U. ☐

Example 6.27. Let X be a scheme (or a ringed space). Since the condition for a function $f : \mathcal{O}_X$ to be nilpotent is geometric (it is $\bigvee_{n \geq 0} f^n = 0$), nilpotency of f at a point is equivalent to nilpotency on some open neighbourhood.

Combined with lemma 6.20, this metatheorem is quite useful. We will illustrate it with many examples in the next subsection.

An important special case of spreading from stalks to neighbourhoods is the case of spreading from the generic point to a dense open subset. Recall that the *generic point* $\xi \in X$ has the property that $\mathrm{cl}\{\xi\} = X$. Such a point exists and is unique if X is an irreducible scheme, and need not exist otherwise.

Lemma 6.28. *Let X be a topological space and $\xi \in X$ a point such that $\mathrm{cl}\{\xi\} = X$. Then the modal operator $\Box :\equiv ((\_\_ \Rightarrow {!\xi}) \Rightarrow {!\xi})$ coincides with the double negation modality.*

Proof. The semantics of the formula $!\xi$ was defined by the equivalence

$$U \models {!\xi} \quad \Longleftrightarrow \quad \xi \notin U.$$

By the assumption on ξ, this is equivalent to requiring $U = \emptyset$. Thus for any open subset U the formulas $!\xi$ and $\bot$ have the same meaning; they are therefore logically equivalent from the internal point of view. The given modal operator thus simplifies to

$$\Box\varphi \equiv ((\varphi \Rightarrow {!\xi}) \Rightarrow {!\xi}) \Leftrightarrow ((\varphi \Rightarrow \bot) \Rightarrow \bot) \Leftrightarrow \neg\neg\varphi.$$ ☐

In particular, if X contains a generic point ξ in the sense of the lemma, we can describe the sublocale $X_{\neg\neg}$ in very explicit terms: In this case, it coincides with the subspace $\{\xi\}$. **XXX: discuss that sheafifying is the same as calculating the generic stalk; and that pushing forward is the same as calculating the constant sheaf. maybe do this in the section discussing negneg-sheaves. Then continue with the story here...**

Proof of theorem 6.23. A fancy proof goes as follows. First, one shows intuitionistically that for a modal operator $\Box$ in Set, it holds that

$$\mathrm{Set} \models \varphi^\Box \quad \Longleftrightarrow \quad \mathrm{Sh}_\Box(\mathrm{Set}) \models \varphi.$$

This can be done by an easy and nontechnical induction on the structure of formulas φ. Then one interprets this result in the sheaf topos $\mathrm{Sh}(X)$:

$$\begin{aligned}
& \mathrm{Sh}(X) \models \varphi^{\Box} \\
\iff & \mathrm{Sh}(X) \models \ulcorner \mathrm{Set} \models \varphi^{\Box} \urcorner && \text{by idempotency} \\
\iff & \mathrm{Sh}(X) \models \ulcorner \mathrm{Sh}_{\Box}(\mathrm{Set}) \models \varphi \urcorner && \text{by the first step} \\
\iff & \mathrm{Sh}_{\Box}(\mathrm{Sh}(X)) \models \varphi && \text{by idempotency} \\
\iff & \mathrm{Sh}(X_{\Box}) \models \varphi && \text{since } \mathrm{Sh}_{\Box}(\mathrm{Sh}(X)) \simeq \mathrm{Sh}(X_{\Box})
\end{aligned}$$

By *idempotency*, we mean that internally employing the Kripke–Joyal semantics to interpret doubly-internal statements is the same as using the Kripke–Joyal semantics once. However, we do not want to discuss this here any further; some details can be found in the original article on the stack semantics [11, lemma 7.20], but the lemma given there is not general enough to justify the second use of idempotency above. For this, one would have to extend the stack semantics to support internal statements about locally internal categories like $\mathrm{Sh}(X_{\Box}) \hookrightarrow \mathrm{Sh}(X)$ (which then look like locally small categories from the internal point of view). This is worthwhile for other reasons too, but shall not be pursued in these notes.

Therefore, we give a more explicit proof. By induction, we are going to prove that for any open subset $U \subseteq X$ and any formula φ over U, it holds that

$$U \models_X \varphi^{\Box} \quad \iff \quad j(U) \models_{X_{\Box}} \varphi,$$

where the internal statements are to be interpreted by the Kripke–Joyal semantics of X and $X_{\Box}$ respectively and j is the nucleus associated to $\Box$. We may assume that any sheaves occuring in φ as domains of quantifications are in fact $\Box$-sheaves; we justify this with a separate lemma below.

The cases $\varphi \equiv \top$, $\varphi \equiv (\psi \wedge \chi)$, and $\varphi \equiv \bigwedge_i \psi_i$ are trivial. For $\varphi \equiv \bot$, the claim is that $U \models_X \Box\varphi$ if and only if $j(U) \models_{X_{\Box}} \bot$. The former means $U \subseteq j(\emptyset)$ and the latter means $j(U) = \sup \emptyset = j(\emptyset)$, so the claim follows from the first two axioms on a nucleus. $\Box$

Lemma 6.29. *Let $\Box$ be a modal operator. Let φ be a formula. Let $\psi :\equiv \varphi^{\Box}$ be the $\Box$-translation of φ. Let ψ' be the formula obtained from ψ by substituting any occuring domain of quantification by its $\Box$-sheafification. Then φ and φ' are equivalent.*

Proof. For any formula φ, we denote by "$\varphi^{\boxplus}$" the result of first applying the $\Box$-translation to φ and then substituting any set F occuring in φ as a domain of quantification by the plus construction F^+. Recall that for any such F there is a canonical map $F \to F^+, x \mapsto [\{x\}]$. We are going to show by induction that for any formula $\varphi(x_1, \ldots, x_n)$ in which elements $x_i : F_i$ may occur as terms, it holds that $\varphi^{\Box}(x_1, \ldots, x_n)$ is equivalent to $\varphi^{\boxplus}([\{x_1\}], \ldots, [\{x_n\}])$. This suffices to prove the lemma.

The cases for

$$\top \quad \bot \quad \wedge \quad \bigwedge \quad \vee \quad \bigvee \quad \Longrightarrow$$

are trivial. The cases for unbounded "$\forall$" and "$\exists$" are trivial as well. The case for "$=$" is slightly more interesting; let $\varphi(x, y) \equiv (x = y)$. Then we are to show that $\varphi^{\Box}(x, y) \equiv \Box(x = y)$ (equality in some set F) is equivalent to $\varphi^{\boxplus}([\{x\}], [\{y\}]) \equiv$

$\Box([\{x\}] = [\{y\}])$ (equality in F^+). This follows by the definition of the plus construction. The case for "$\in$" is similar.

Let $\varphi \equiv (\exists x : F.\ \psi(x))$, where we have dropped further variables occuring in ψ for simplicity. Then we are to show that $\varphi^\Box \equiv \Box(\exists x : F.\ \psi^\Box(x))$ is equivalent to $\varphi^\boxplus \equiv \Box(\exists \bar{x} : F^+.\ \psi^\boxplus(\bar{x}))$. The "only if" direction is trivial (set $\bar{x} := [\{x\}]$). For the "if" direction, we may assume that there exists $\bar{x} : F^+$ such that $\psi^\boxplus(\bar{x})$ since we want to prove a boxed statement. By definition of the plus construction, it holds that $\Box(\ulcorner\bar{x}$ is a singleton$\urcorner)$. So, again since we want to prove a boxed statement, we may assume that $\bar{x}$ is actually a singleton. Therefore there exists $x : F$ such that $\bar{x} = [\{x\}]$ and that $\psi^\boxplus([\{x\})$ holds. By the induction hypothesis, it follows that $\psi^\Box(x)$. From this the claim follows.

The case for "$\forall$" is similar. ∎

6.8. **Internal proofs of common lemmas.**

Lemma 6.30. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.*

- *Let $x \in X$ be a point. Then the stalk $\mathcal{F}_x$ is zero if and only if $\mathcal{F}$ is zero on some open neighbourhood of x.*
- *Let $A \subseteq X$ be a closed subset. Then the restriction $\mathcal{F}|_A$ (i. e. the pullback of $\mathcal{F}$ to A) is zero if and only if $\mathcal{F}$ is zero on some open subset of X containing A.*

Proof. Both statements are simply internalizations of example 6.21, using the modal operators $\Box = (\_\_ \vee A^c)$ and $\Box = ((\_\_ \Rightarrow !x) \Rightarrow !x)$. ∎

Remark 6.31. Note that the proposition fails if one drops the hypothesis that $\mathcal{F}$ is of finite type. Indeed, in this case one cannot reformulate the condition that $\mathcal{F}$ is zero in a geometric way.

Lemma 6.32. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point. Let n be a natural number. Then the following statements are equivalent:*

(1) *There exists a generating family for $\mathcal{F}_x$ consisting of n elements.*
(2) *There exists an open neighbourhood U of x such that*

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

Proof. Using the modal operator $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$, we have to show that the following statements in the internal language are equivalent:

(1) $\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner^\Box$.
(2) $\Box(\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner)$.

By lemma 6.17, the second statement implies the first – note that in a formal spelling of the statement in quotes,

$$(\star) \qquad \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i,$$

no implication signs occur. To show the converse direction, we may assume that there is a generating family $y_1, \ldots, y_m : \mathcal{F}$ for $\mathcal{F}$ (since $\mathcal{F}$ is, externally speaking, of finite type). Then the $\Box$-translation of the statement that the y_i generate $\mathcal{F}$ holds as well (again by lemma 6.17). Since there is an intuitionistic proof of

$$\ulcorner y_1, \ldots, y_m \text{ generate } \mathcal{F}\urcorner \Longrightarrow \Big(\ulcorner\text{there exist } x_1, \ldots, x_n : \mathcal{F} \text{ which generate } \mathcal{F}\urcorner \Longleftrightarrow \exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m\times n}.\ \ulcorner \vec{y} = A\vec{x}\urcorner\Big),$$

we can substitute the non-geometric formula ($\star$) by the geometric formula

$$\exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m\times n}.\ \ulcorner \vec{y} = A\vec{x}\urcorner$$

(lemma 6.20). Thus the claim follows. □

XXX: give backreference

Lemma 6.33. *Let X be a scheme (or ringed space). Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be of finite type and assume that $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for some point $x \in X$. Then α is an epimorphism on some open neighbourhood of x.*

Proof. In the presence of generators $y_1, \ldots, y_n : \mathcal{G}$, the non-geometric surjectivity condition $(\forall y : \mathcal{G}.\ \exists x : \mathcal{F}.\ \alpha(x) = y)$ can be reformulated in a geometric way: $\bigwedge_{i=1}^n \exists x : \mathcal{F}.\ \alpha(x) = y_i$. Thus the claim follows by lemma 6.20. □

Lemma 6.34. *Let X be a scheme (or ringed space). Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Let $\mathcal{F}$ be of finite type and $\mathcal{G}$ be coherent. Suppose that α_x is injective at some point $x \in X$. Then α is a monomorphism on some open neighbourhood of x.*

Proof. The kernel of α is of finite type (by lemma 4.12) and zero at x. By the previous lemma, it is therefore zero on some open neighbourhood of x. □

Lemma 6.35. *Let $i : A \hookrightarrow X$ be a closed immersion of schemes (or ringed spaces). Let $\mathcal{F}$ be an $\mathcal{O}_A$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. Let $\Box$ be the modal operator defined by $\Box\varphi :\equiv (\varphi \vee A^c)$. From the internal perspective, we have a surjective ring homomorphism $i^\sharp : \mathcal{O}_X \to \mathcal{O}_A$, where we omit the forgetful functor i_* from $\Box$-sheaves to arbitrary sets in the notation, and an $\mathcal{O}_A$-module $\mathcal{F}$. Furthermore, we may assume that $\mathcal{F}$ is a $\Box$-sheaf. We can regard $\mathcal{F}$ as an $\mathcal{O}_X$-module by $i^\sharp$.

Note that $A^c \Rightarrow (\mathcal{F} = 0)$, by $\Box$-separatedness of $\mathcal{F}$.

We are to show that $\mathcal{F}$ is a finitely generated $\mathcal{O}_X$-module if and only if the $\Box$-translation of "$\mathcal{F}$ is a finitely generated $\mathcal{O}_A$-module" holds. In explicit terms, we have to show the equivalence of the following statements:

(1) $\bigvee_{n\geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_i i^\sharp(a_i)x_i$.
(2) $\Box(\bigvee_{n\geq 0} \Box(\exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \Box(\exists b_1, \ldots, b_n : \mathcal{O}_A.\ \Box(x = \sum_i b_i x_i))))$.

It is clear that the first statement implies the second. For the converse direction, we just have to repeatedly use the observation that $\Box\varphi$ implies $\varphi \vee (\mathcal{F} = 0)$ (once for each occurence of $\Box$). So in each step, we either obtain the statement we want or may assume that $\mathcal{F}$ is the trivial module, in which case any subclaim trivially follows. By surjectivity of $i^\sharp$, we may write any $b : \mathcal{O}_A$ as $b = i^\sharp(a)$ for some $a : \mathcal{O}_X$. □

Lemma 6.36. *Let X be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. Then $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.*

Proof. It suffices to give an intuitionistic proof of the following fact: The construction $\mathrm{Hom}_R(M, \_\_)$ is geometric if M is a finitely presented R-module. So assume that M is the cokernel of a presentation matrix $(a_{ij}) \in R^{n \times m}$. Then we can calculate the Hom with any R-module N as

$$\mathrm{Hom}_R(M, N) \cong \Big\{ x : N^n \;\Big|\; \bigwedge_{j=1}^{m} \sum_{i=1}^{n} a_{ij} x_i = 0 : N \Big\},$$

and this construction is patently geometric (set comprehension with respect to a geometric formula). □

Lemma 6.37. *Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $x \in X$. Then the stalk $\mathcal{F}_x$ is a finitely free $\mathcal{O}_{X,x}$-module if and only if $\mathcal{F}$ is locally free on some open neighbourhood of x.*

Proof. The internal statement that $\mathcal{F}$ is a free module is not geometric:

$$\bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i.$$

But it can equivalently be reformulated as

$$\bigvee_{n \geq 0} \exists \alpha : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n).\ \exists \beta : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \alpha \circ \beta = \mathrm{id} \wedge \beta \circ \alpha = \mathrm{id}.$$

This reformulation is geometric, therefore it holds at x if and only if it holds on some open neighbourhood of x. The claim follows since, by the previous proposition, taking stalks commutes with calculating $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \_\_)$ resp. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \_\_)$; thus the pulled back formula indeed expresses that $\mathcal{O}_{X,x}$ is finitely free. □

XXX: Find proper place for the following lemma. XXX: Also, generalize to arbitrary schemes.

Lemma 6.38. *Let X be an integral scheme. Let φ be any formula over X. Then*

$$\mathrm{Sh}(X) \models \neg\neg\varphi \Longrightarrow \exists f : \mathcal{O}_X.\ \neg\neg(\ulcorner f \text{ inv.} \urcorner) \wedge (\ulcorner f \text{ inv.} \urcorner \Rightarrow \varphi).$$

Proof. We may assume that X is the spectrum of an integral domain A and that there is a dense open subset $U \subseteq X$ on which φ holds. The open set U may be covered by standard open subsets $D(f_i)$; by the integrality hypothesis, at least one of these is nonempty and thus itself dense. We may take this f_i as the required f. □

Lemma 6.39. *Let X be an integral scheme with generic point ξ. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is a torsion module if and only if its generic stalk $\mathcal{F}_\xi$ vanishes.*

Proof. The generic stalk vanishes if and only if the internal statement "$(\mathcal{F} = 0)^{\neg\neg}$" holds. Therefore it suffices to give an intuitionistic proof of the following internal statement: The module $\mathcal{F}$ is torsion if and only if any element of $\mathcal{F}$ is *not not* zero.

For the "only if" direction, let $x : \mathcal{F}$ be an arbitrary element. Since $\mathcal{F}$ is a torsion module, there exists a regular element $a : \mathcal{O}_X$ such that $ax = 0$. Since X is reduced, regularity is equivalent to not-not-invertibility. Since we want to verify the $\Box$-stable statement "$\neg\neg(x = 0)$", we may in fact assume that a is invertible. Then $x = 0$ obviously follows.

For the "if" direction, let $x : \mathcal{F}$ be an arbitrary element; we may assume that x is *not not* zero. Since X is integral, lemma 6.38 is applicable. Therefore there exists an element $a : \mathcal{O}_X$ such that a is *not not* invertible and such that invertibility of a implies $x = 0$. This $\mathcal{F}$ is quasicoherent, for some natural number n it holds that $a^n x = 0$. Since a is *not not* invertible, it is regular **XXX: give reference** and a^n are regular. So $x \in \mathcal{F}_{\text{tors}}$. $\square$

- general explanation of modalities (as for instance in philosophy)
- explain that for some modal operators, the $\Box$-translation of the law of excluded middle is valid; explain consequences
- spreading of properties from stalk to neighbourhood: give many examples
- give proof of the expressions for the nuclei listed in the table
- baby version of Barr's theorem

7. Rational functions and Cartier divisors

7.1. **The sheaf of rational functions.** Recall that the sheaf $\mathcal{K}_X$ of rational functions on a scheme X (or ringed space) can be defined as the sheaf associated to the presheaf

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S})^{-1}],$$

where $\Gamma(U, \mathcal{S})$ is the multiplicative set of those sections of $\mathcal{O}_X$ on U, which are regular in each stalk $\mathcal{O}_{X,x}$, $x \in U$. Recall also that there are some wrong definitions in the literature [6].

Using the internal language, we can give a simpler definition of $\mathcal{K}_X$. Recall that we can associate to any ring R its total quotient ring, i. e. its localization at the multiplicative subset of regular elements. Since from the internal perspective $\mathcal{O}_X$ is an ordinary ring, we can associate to it its total quotient ring $\mathcal{O}_X[\mathcal{S}^{-1}]$, where $\mathcal{S}$ is internally defined by the formula

$$\mathcal{S} := \{ s : \mathcal{O}_X \mid \ulcorner s \text{ is regular} \urcorner \} \subseteq \mathcal{O}_X.$$

Externally, this ring is the sheaf $\mathcal{K}_X$.

Proposition 7.1. *Let X be a scheme (or a ringed space). The sheaf of rings defined in the internal language by localizing $\mathcal{O}_X$ at its set of regular elements is (canonically isomorphic to) the sheaf $\mathcal{K}_X$ of rational functions.*

Proof. Internally, the ring $\mathcal{O}_X[\mathcal{S}^{-1}]$ has the following universal property: For any ring R and any homomorphism $\mathcal{O}_X \to R$ which maps the elements of $\mathcal{S}$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}] \to R$ which makes the evident diagram commute.

$$\begin{tikzcd}
\mathcal{O}_X \arrow[rr] \arrow[dr] & & R \\
& \mathcal{O}_X[\mathcal{S}^{-1}] \arrow[ur, dashed] &
\end{tikzcd}$$

The translation using the Kripke–Joyal semantics gives the following universal property: For any open subset $U \subseteq X$, any sheaf of rings $\mathcal{R}$ on U and any homomorphism $\mathcal{O}_X|_U \to \mathcal{R}$ which maps all elements of $\Gamma(V, \mathcal{S})$, $V \subseteq U$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}]|_U \to \mathcal{R}$ which makes the evident diagram commute. It is well-known that the sheaf $\mathcal{K}_X$ as usually defined has this universal property as well. $\square$

Proposition 7.2. *Let X be a scheme (or ringed space). Then the stalks of $\mathcal{K}_X$ are given by*

$$\mathcal{K}_{X,x} = \mathcal{O}_{X,x}[\mathcal{S}_x^{-1}].$$

The elements of $\mathcal{S}_x$ are exactly the germs of those local sections which are regular not only in $\mathcal{O}_{X,x}$, but in all rings $\mathcal{O}_{X,y}$ where y ranges over some neighbourhood of x (depending on the section).

Proof. Since localization is a geometric construction, the first statement is entirely trivial with our framework. The second statement follows since

$$\Gamma(U, \mathcal{S}) = \{s \in \Gamma(U, \mathcal{O}_X) \mid U \models \ulcorner s \text{ is regular}\urcorner\}$$

and regularity is a geometric implication, so that $U \models \ulcorner s \text{ is regular}\urcorner$ if and only if the germ s_y is regular in $\mathcal{O}_{X,y}$ for all $y \in U$. □

Remark 7.3. Speaking internally, the multiplicative set $\mathcal{S}$ is saturated. Therefore an element $s/t : \mathcal{K}_X$ is invertible in $\mathcal{K}_X$ if and only if the numerator s belongs to $\mathcal{S}$, i. e. is an regular element of $\mathcal{O}_X$.

7.2. **Regularity of local functions.** It is well known that on a locally Noetherian scheme, regularity spreads from stalks to neighbourhoods, i. e. a section of $\mathcal{O}_X$ is regular in $\mathcal{O}_{X,x}$ if and only if it is regular on some neighbourhood on x. This fact has a simple proof in the internal language.

Proposition 7.4. *Let X be a locally Noetherian scheme. Let $s \in \Gamma(U, \mathcal{O}_X)$ be a local function on X. Let $x \in U$. Then the following statements are equivalent:*

(1) *The section s is regular in $\mathcal{O}_{X,x}$.*
(2) *The section s is regular in all local rings $\mathcal{O}_{X,y}$ where y ranges over some neighbourhood of x.*

Proof. Let $\Box$ be the modal operator defined by $\Box\varphi :\equiv ((\varphi \Rightarrow\ !x) \Rightarrow\ !x)$. By corollary 6.24, we are to show that the following statements of the internal language are equivalent:

(1) $(\ulcorner s \text{ is regular}\urcorner)^{\Box}$, i. e. $\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow \Box(t = 0)$.
(2) $\Box(\ulcorner s \text{ is regular}\urcorner)$, i. e. $\Box(\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow t = 0)$.

It is clear that the second statement implies the first – in fact, this is true without any assumptions on X: Let $t : \mathcal{O}_X$ be such that $st = 0$. Since we want to prove the boxed statement $\Box(t = 0)$, we may assume that s is regular and prove $t = 0$. This is immediate. (This direction also follows simply by examining the logical form and applying lemma 6.17).

For the converse direction, consider the annihilator of s, i. e. the ideal

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X.$$

This ideal satisfies the quasicoherence condition (example 9.6), thus I is a quasicoherent submodule of a finitely generated module. Since X is locally Noetherian, it follows that I is finitely generated as well. By assumption, each generator $x_i : I$ fulfills $\Box(x_i = 0)$. Since we want to prove a boxed statement, we may in fact assume $x_i = 0$. Thus $I = (0)$ and the assertion that s is regular follows. □

Corollary 7.5. *Let X be a locally Noetherian scheme. Then the stalks $\mathcal{K}_{X,x}$ of the sheaf of rational functions are given by the total quotient rings of the local rings $\mathcal{O}_{X,x}$.*

Proof. Combine proposition 7.2 and proposition 7.4. □

7.3. **Normality.** Recall that a ring R is *normal* if and only if it is integrally closed in its total quotient ring. Recall also that a scheme X (or ringed space) is *normal* if and only if all rings $\mathcal{O}_{X,x}$ are normal.

Proposition 7.6. *A locally Noetherian scheme is normal if and only if the ring $\mathcal{O}_X$ is normal from the internal perspective.*

Proof. The condition of normality can be put into a form which is almost a geometric implication:

$$\forall s,t : \mathcal{O}_X.\ \ulcorner t \text{ regular} \urcorner \wedge (\exists a_0,\ldots,a_{n-1} : \mathcal{O}_X.\ s^n + a_{n-1}ts^{n-1} + \cdots + a_1 t^{n-1}s + a_0 t^n = 0) \Longrightarrow \\ \exists u : \mathcal{O}_X.\ s = ut.$$

The only non-geometric subpart is the condition on t to be regular. However, by proposition 7.4, for the purposes of comparing its truth at points vs. on neighbourhoods, it behaves just like a geometric formula. Therefore the claim follows. ☐

7.4. **Geometric interpretation of rational functions.** Recall that on integral schemes, rational functions (i. e. sections of $\mathcal{K}_X$) are the same thing as regular functions defined on dense open subsets. This amounts to saying that $\mathcal{K}_X$ *is the* $\neg\neg$-*sheafification of* $\mathcal{O}_X$ (see proposition 6.12). We want to rederive this result, as far as possible in the internal language, and generalize it to arbitrary (not necessarily locally Noetherian) schemes.

Lemma 7.7. *Let X be a reduced scheme. Then:*

(1) *$\mathcal{O}_X$ is $\neg\neg$-separated.*
(2) *Internally, an element $s : \mathcal{O}_X$ is regular if and only if it is* not not *invertible.*

Proof. Recall from corollary 3.8 that

(1) $$\mathrm{Sh}(X) \models \forall s : \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Leftrightarrow s = 0.$$

From this we can deduce that $\mathcal{O}_X$ is $\neg\neg$-separated: Assume $\neg\neg(s = 0)$ for $s : \mathcal{O}_X$. If s were invertible, we would have $\neg\neg(1 = 0)$ and thus $\bot$. Therefore s is not invertible and thus zero.

For the "only if" direction of the second statement, note that a regular element is not zero (if it were, then the true statement $0 \cdot 0 = 0 \cdot 1$ would imply the false statement $0 = 1$) and thus *not not* invertible. For the "if" direction, let $st = 0$ in $\mathcal{O}_X$. Since s is *not not* invertible, it follows that t is *not not* zero. Since $\mathcal{O}_X$ is $\neg\neg$-separated, this implies that t really is zero. ☐

Proposition 7.8. *Let X be a reduced scheme. Then $\mathcal{K}_X$ is the $\neg\neg$-sheafification of $\mathcal{O}_X$.*

Proof. We first show that $\mathcal{K}_X$ is $\neg\neg$-separated, so assume $\neg\neg(a/s = 0)$ for $a/s : \mathcal{K}_X$. Since $\mathcal{K}_X$ is obtained from $\mathcal{O}_X$ by localizing at regular elements, it holds that $a/s = 0$ in $\mathcal{K}_X$ if and only if $a = 0$ in $\mathcal{O}_X$. Thus it follows that $\neg\neg(a = 0)$ in $\mathcal{O}_X$ and thus $a = 0$ in $\mathcal{O}_X$; in particular, $a/s = 0$ in $\mathcal{K}_X$.

We defer the proof that $\mathcal{K}_X$ is a $\neg\neg$-sheaf to the end and first verify the universal property of $\neg\neg$-sheafification. So let G be a $\neg\neg$-sheaf and let $\alpha : \mathcal{O}_X \to G$ be a map. We can define an extension $\bar{\alpha} : \mathcal{K}_X \to G$ in the following way: Let $f : \mathcal{K}_X$. Define the subsingleton $S := \{x : G \mid \exists b : \mathcal{O}_X.\ f = b/1 \wedge x = \alpha(b)\} \subseteq G$. Since f can be written in the form a/s with s *not not* invertible, it follows that S is *not not* inhabited. Since G

is a $\neg\neg$-sheaf, there exists a unique $x : G$ such that $\neg\neg(x \in S)$. We declare $\bar{\alpha}(f)$ to be this x. It is straightforward to check that the composition $\mathcal{O}_X \to \mathcal{K}_X \to G$ equals α and that $\bar{\alpha}$ is unique with this property.

Up to this point, the proof did not need that X is a scheme – it was enough for X to be a ringed space such that equivalence (1) holds and such that $\neg(0 = 1)$ in $\mathcal{O}_X$. Only now, in showing that $\mathcal{K}_X$ is a $\neg\neg$-sheaf, the scheme condition enters. To this end, we first reformulate the sheaf condition in a way such that it only refers to $\mathcal{O}_X$, not $\mathcal{K}_X$: The quotient ring $\mathcal{K}_X$ is a $\neg\neg$-sheaf if and only if

$$\mathrm{Sh}(X) \models \forall T \subseteq \mathcal{O}_X.\ \ulcorner T \text{ subsingleton}\urcorner \wedge \neg\neg(\ulcorner T \text{ inhabited}\urcorner) \Longrightarrow \\ \exists a, b : \mathcal{O}_X.\ \ulcorner b \text{ regular}\urcorner \wedge \neg\neg(b^{-1}a \in T).$$

This is done just as in the proof of theorem 9.3. Note that "b^{-1}" refers to the inverse of b which indeed exists in a doubly negated context, since b is assumed regular. More explicitly, we should write

$$\neg\neg(\exists c : \mathcal{O}_X.\ bc = 1 \wedge ca \in T) \quad \text{instead of} \quad \neg\neg(b^{-1}a \in T).$$

To prove the Kripke–Joyal interpretation of the rewritten sheaf condition, let an affine open subset $U = \operatorname{Spec} A \subseteq X$ and a subsheaf $T \hookrightarrow \mathcal{O}_X|_U$ be given, such that T is internally a subsingleton and *not not* inhabited. We may thus glue the unique germs in the inhabited stalks of T to obtain a section $s \in \Gamma(V, \mathcal{O}_X)$ where $V \subseteq U$ is a dense open subset. We may write $V = \bigcup_i D(f_i)$ for some (possibly infinitely many) functions $f_i \in A$. Since V is dense in U, these have the property that

$$\forall g \in A.\ (\forall i.\ f_i g \text{ nilpotent in } A) \Longrightarrow g \text{ nilpotent in } A.$$

Since $\Gamma(V, \mathcal{O}_X) = A[S_V^{-1}]$ with $S_V = \{x \in A \mid f_i \in \sqrt{(x)} \text{ for all } i\}$, we can write $s = a/b$ with $a, b \in A$ and $f_i^{n_i} = c_i b$ for some $c_i \in A$, $n_i \geq 0$. **XXX: $\Gamma(V, \mathcal{O}_X) = A[S_V^{-1}]$ is wrong.**

The function b is a regular element of A: Let $bg = 0$ for $g \in A$. Then $c_i bg = f_i^{n_i} g = 0 \in A$. Thus $f_i g$ is nilpotent in A. Since this holds for any i, g is nilpotent in A. Since A is reduced, g is in fact zero.

By lemma 3.16, the function b is also regular as an element of $\mathcal{O}_X$ from the internal point of view. Note that b is invertible on V, since $V \subseteq D(b)$. It follows that on the dense open subset $V \subseteq U$, the sections s and $b^{-1}a$ agree. This observation concludes the proof. □

Corollary 7.9. *Let X be a reduced scheme. Then $\mathcal{K}_X$ is the result of pulling back $\mathcal{O}_X$ to the sublocale $X_{\neg\neg}$ and then pushing forward again. If X is irreducible with generic point ξ, then $\mathcal{K}_X$ is the constant sheaf associated to the set $\mathcal{O}_{X,\xi}$.*

Proof. Recall from section 6.4 that pulling back to $X_{\neg\neg}$ is equivalent to sheafifying with respect to the double negation modality; and that pushing forward is equivalent to forgetting the sheaf property. Therefore the first statement holds.

For the second statement, recall from lemma 6.28 that the sublocale $X_{\neg\neg}$ is given by the subspace $\{\xi\}$; that the sheafification functor $\mathrm{Sh}(X) \to \mathrm{Sh}(\{\xi\}) \simeq \mathrm{Set}$ is given by calculating the stalk at ξ; and that the inclusion functor $\mathrm{Set} \simeq \mathrm{Sh}(\{\xi\}) \hookrightarrow \mathrm{Sh}(X)$ is given by the constant sheaf construction. □

XXX: maybe reorder qcoh before this

If X is a general scheme (not necessarily integral, reduced, or locally Noetherian), we can describe $\mathcal{K}_X$ in a similar way as the sheafification of $\mathcal{O}_X$; specifically, it is the sheafification with respect to the modal operator defined by

$$\Box\varphi :\equiv \ulcorner \mathcal{O}_X \text{ is } (\varphi \Rightarrow \_\_)\text{-separated}\urcorner$$

in the internal language of $\mathrm{Sh}(X)$.

XXX: is $\Box$ the largest operator such that $\mathcal{O}_X$ is $\Box$-separated?

Proposition 7.10. *Let X be a ringed space. Then:*

(1) *The operator $\Box$ fulfills the axioms on a modal operator.*
(2) *$\mathcal{O}_X$ is $\Box$-separated.*
(3) *$\mathcal{K}_X$ is $\Box$-separated.*
(4) *Internally, it holds that $\Box(\ulcorner f \text{ inv.}\urcorner)$ implies that f is regular for any $f : \mathcal{O}_X$.*

Suppose furthermore that X is a scheme. Then:

(5) *The converse in (4) holds.*
(6) *$\mathcal{K}_X$ is the $\Box$-sheafification of $\mathcal{O}_X$.*

Proof. The first four properties are entirely formal; we thus skip over some details. For the first property, we verify the second axiom on a modal operator. So we assume $\Box\Box\varphi$ and have to show $\Box\varphi$. To this end, let $s : \mathcal{O}_X$ be arbitrary such that $\varphi \Rightarrow (s = 0)$; we have to prove that $s = 0$. If $\mathcal{O}_X$ were separated with respect to the modal operator $(\varphi \Rightarrow \_\_)$, it would follow that $s = 0$. So unconditionally it holds that $\Box\varphi \Rightarrow (s = 0)$. Since by assumption $\mathcal{O}_X$ is $\Box\varphi$-separated, the claim follows.

For the second property, let $s : \mathcal{O}_X$ be arbitrary such that $\Box(s = 0)$. Obviously it holds that $(s = 0) \Rightarrow (s = 0)$. Thus, since $\mathcal{O}_X$ is separated with respect to $((s = 0) \Rightarrow \_\_)$, it follows that $s = 0$. The proof of the third property is similar.

For the fourth property, assume $\Box(\ulcorner f \text{ inv.}\urcorner)$ and let $h : \mathcal{O}_X$ be arbitrary such that $fh = 0$. Then, trivially, it holds that $\ulcorner f \text{ inv.}\urcorner \Rightarrow h = 0$. Since $\mathcal{O}_X$ is separated with respect to $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$, it follows that $h = 0$.

We may now suppose that X is a scheme. To verify the fifth property, let a regular function $f : \mathcal{O}_X$ be given. We have to show that $\mathcal{O}_X$ is separated with respect to the modality $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$. So assume that $\ulcorner f \text{ inv.}\urcorner \Rightarrow (s = 0)$ for some $s : \mathcal{O}_X$. By proposition 3.9, it follows that $f^n s = 0$ for some natural number n. Since f is regular, it follows that $s = 0$.

The verification of the universal property of $\mathcal{K}_X$ is done analogously as in the case that X is reduced: For the proof of proposition 7.8, it was critical that regular elements of $\mathcal{O}_X$ are *not not* invertible. We now need (and have) that regular elements of $\mathcal{O}_X$ are $\Box(\ulcorner\text{invertible}\urcorner)$.

Thus it only remains to verify that $\mathcal{K}_X$ is a $\Box$-sheaf. We may again imitate the proof of proposition 7.8; using the same notation, we may now suppose that V is an open subset such that $U \models \Box V$ (previously, we supposed that $U \models \neg\neg V$). The proof that the denominator b is regular (as seen from the internal perspective, as an element of $\mathcal{O}_X$) now goes as follows: We have $V \subseteq D(b)$. Thus $U \models \Box V$ implies $U \models \Box(\ulcorner f \text{ inv.}\urcorner)$. By the fourth property, it follows that $U \models \ulcorner f \text{ is regular}\urcorner$. $\Box$

In the special case that X is a reduced scheme, proposition 7.10 recovers the result of proposition 7.8.

Proposition 7.11. *Let X be a reduced scheme. Then the modal operator $\Box$ coincides with the double negation modality.*

Proof. We show that from the internal point of view, $\Box\varphi$ is equivalent to $\neg\neg\varphi$ for any formula φ. For the "only if" direction, note that $\neg\varphi$ is equivalent to $\varphi \Rightarrow (1 = 0)$. Since by assumption $\mathcal{O}_X$ is separated with respect to the $(\varphi \Rightarrow \_\_)$-modality, this in turn is equivalent to $1 = 0 : \mathcal{O}_X$, i. e. to $\bot$. Thus $\neg\neg\varphi$.

For the "if" direction, let $\varphi \Rightarrow (s = 0)$ for some $s : \mathcal{O}_X$; we have to show that in fact $s = 0$. Since by assumption $\neg\neg\varphi$, it follows that s is *not not* zero. Since X is reduced, $\mathcal{O}_X$ is $\neg\neg$-separated, so this implies that s is really zero. $\square$

Corollary 7.12. *Let X be a scheme. Then $\mathcal{K}_X$ is the result of pulling back $\mathcal{O}_X$ to the sublocale $X_\Box$ associated to the modal operator $\Box$ and then pushing forward again. If X is locally Noetherian, this sublocale is the subspace of associated points in X.*

In formulas, the corollary says that the canonical map

$$\mathcal{K}_X \longrightarrow i_* i^{-1} \mathcal{O}_X$$

is an isomorphism, where $i : X_\Box \hookrightarrow X$ is the inclusion of the sublocale $X_\Box$. This result holds without any Noetherian hypotheses. These are only needed if we want to describe $X_\Box$ as a certain explicitly given subspace.

Proof. The first statement follows trivially by the results of section 6.4 and the fact that $\mathcal{K}_X$ is the $\Box$-sheafification of $\mathcal{O}_X$.

For the second statement, we need to verify that the nucleus $j_{\mathrm{Ass}(\mathcal{O}_X)}$ associated to the subspace of associated points coincides with the nucleus $j_\Box$ associated to the modal operator $\Box$. Recall from remark **XXX: ref** that the former is defined by

$$j_{\mathrm{Ass}(\mathcal{O}_X)}(U) = \bigcup \{ V \subseteq X \mid V \text{ open},\ V \cap \mathrm{Ass}(\mathcal{O}_X) \subseteq U \}$$

and recall from **XXX: ref** that the latter is given by

$$\begin{aligned} j_\Box(U) &= \text{largest open subset of } X \text{ on which } \Box U \text{ holds} \\ &= \bigcup \{ V \subseteq X \mid V \text{ open},\ V \models \Box U \}. \end{aligned}$$

The equivalence thus follows from a standard result on the set of associated points on locally Noetherian schemes:

$$\begin{aligned} & V \cap \mathrm{Ass}(\mathcal{O}_X) \subseteq U \\ \Longleftrightarrow\ & \mathrm{Ass}(\mathcal{O}_V) \subseteq U \\ \Longleftrightarrow\ & \text{the canonical morphism } \mathcal{O}_V \to i_* \mathcal{O}_{U \cap V} \text{ (with } i : U \cap V \hookrightarrow V\text{) is injective} \\ & \qquad \text{(this steps requires the Noetherian assumption)} \\ \Longleftrightarrow\ & V \models \ulcorner \mathcal{O}_X \to \mathcal{O}_X^{++} \text{ is injective} \urcorner \\ & \qquad \text{(where the plus construction is with respect to} \\ & \qquad \text{the modality } (U \Rightarrow \_\_)) \\ \Longleftrightarrow\ & V \models \ulcorner \mathcal{O}_X \to \mathcal{O}_X^{+} \text{ is injective} \urcorner \\ & \qquad \text{(by the factorization } \mathcal{O}_X \to \mathcal{O}_X^+ \to \mathcal{O}_X^{++}) \\ \Longleftrightarrow\ & V \models \ulcorner \mathcal{O}_X \text{ is } (U \Rightarrow \_\_)\text{-separated} \urcorner \\ \Longleftrightarrow\ & V \models \Box U. \end{aligned}$$

$\square$

XXX: explain that $X \models \Box U$ means that U is scheme-theoretically dense in X. Give internal proof that scheme-theoretical denseness implies topological denseness, and sometimes vice versa.

7.5. **Cartier divisors.** Let X be a scheme (or ringed space). Recall that a *Cartier divisor* on x is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$. This sheaf can be constructed internally, with the same notation: It is the quotient of the group of invertible elements of the ring $\mathcal{K}_X$ by the subgroup of invertible elements of the ring $\mathcal{O}_X$. So an arbitrary section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ is internally of the form $[s/t]$ with $s, t : \mathcal{O}_X$ being regular elements; this is a simpler description than the usual external one (as a family $(f_i)_i$ of functions $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ such that $f_i^{-1}|_{U_i \cap U_j} \cdot f_j|_{U_i \cap U_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ for all i, j).

We can sketch the basic theory of Cartier divisors completely from the internal perspective. In accordance with common practice, we will write the group operation of $\mathcal{K}_X^*/\mathcal{O}_X^*$ (which is induced by multiplication of elements in $\mathcal{K}_X^*$) additively.

Definition 7.13. A Cartier divisor is *effective* if and only if, from the internal perspective, it can be written in the form $[s/1]$ with $s : \mathcal{O}_X$ being a regular element.

Thus a Cartier divisor $[s/t]$ is effective if and only if s is an $\mathcal{O}_X$-multiple of t.

Definition 7.14. A Cartier divisor D is *principal* if and only if there exists a global section $f \in \Gamma(X, \mathcal{K}_X^*)$ such that internally, $D = [f]$. Two Cartier divisors are *linearly equivalent* if and only if their difference is a principal divisor.

Note that decidedly, principality is a global notion: For *any* divisor D it is true that locally there exists sections f of $\mathcal{K}_X^*$ such that $D = [f]$.

Definition 7.15. The *line bundle associated to a Cartier divisor* D is the $\mathcal{O}_X$-submodule

$$\mathcal{O}_X(D) := \{g \in \mathcal{K}_X \mid gD \in \mathcal{O}_X\} = D^{-1}\mathcal{O}_X \subseteq \mathcal{K}_X$$

of $\mathcal{K}_X$. Here we are abusing language for "$gD \in \mathcal{O}_X$" to mean that $gf \in \mathcal{O}_X$ if $D = [f]$ with $f : \mathcal{K}_X$; and for "$D^{-1}\mathcal{O}_X$" to mean $f^{-1}\mathcal{O}_X$. This condition resp. submodule does not depend on the representative f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^*$.

The submodule $\mathcal{O}_X(D)$ is indeed locally free of rank 1, since internally f^{-1} gives an one-element basis. Note that D is effective if and only if $\mathcal{O}_X(-D)$ is a subset of $\mathcal{O}_X$ from the internal perspective. In this case, we can define the *support* of D to be the closed subscheme of X associated to the sheaf of ideals $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$.

Definition 7.16. The *Cartier divisor associated to a free submodule $\mathcal{L} \subseteq \mathcal{K}_X$ of rank 1* is $D := [f^{-1}]$, where $f : \mathcal{K}_X$ is the unique element of some one-element-basis of $\mathcal{L}$.

The basis element $f : \mathcal{K}_X$ does indeed lie in $\mathcal{K}_X^*$: Write $f = s/t$ with $s, t : \mathcal{O}_X$. It suffices to show that s is a regular element of $\mathcal{O}_X$. So let $h : \mathcal{O}_X$ such that $sh = 0$ in $\mathcal{O}_X$. Then in particular $hf = 0$ in $\mathcal{K}_X$. By linear independence, it follows that $h = 0$ in $\mathcal{K}_X$ and thus $h = 0$ in $\mathcal{O}_X$.

Furthermore, the associated divisor does not depend on the choice of f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^*$: If $f\mathcal{O}_X = g\mathcal{O}_X \subseteq \mathcal{K}_X$, then there exist $u, v : \mathcal{O}_X$ such that $fu = g$ and $gv = f$ in $\mathcal{K}_X$. It follows that $uv = fuvf^{-1} = gvf^{-1} = ff^{-1} = 1$ in $\mathcal{K}_X$ and thus in $\mathcal{O}_X$, by injectivity of the canonical map $\mathcal{O}_X \to \mathcal{K}_X$. Therefore u and v are elements of $\mathcal{O}_X^*$.

Lemma 7.17. *Let D and D' be divisors on X. Then $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \cong \mathcal{O}_X(D + D')$.*

Proof. The wanted morphism of sheaves $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D') \to \mathcal{O}_X(D + D')$ is given by multiplication. That this is well-defined and an isomorphism can be checked from the internal point of view, where the claims are obvious. $\square$

Proposition 7.18. *The association $D \mapsto \mathcal{O}_X(D)$ defines an one-to-one correspondence between Cartier divisors on X and rank-one submodules of $\mathcal{K}_X$. This correpondence descends to an one-to-one correspondence between Cartier divisiors up to linear equivalance and rank-one submodules of $\mathcal{K}_X$ up to isomorphism.*

Proof. The first statement is obvious from the definitions. For the second statement, it suffices to show that $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$ if and only if D is principal. A given isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ gives a global section $f \in \mathcal{K}_X^*$ (by considering the image of the unit element) such that internally, $D = [f^{-1}]$; this shows that D is principal. The converse is similar. $\square$

Remark 7.19. Locally principal subschemes (closed subschemes which are locally the vanishing subscheme of a regular section of $\mathcal{O}_X$) up to isomorphisms of subschemes are in one-to-one correspondence with rank-1 submodules of $\mathcal{O}_X$ (see **XXX: ref**). Thus locally principal subschemes (up to isomorphisms of abstract schemes) are in one-to-one correspondence with effective Cartier divisors (up to linear equivalence).

XXX: check this.

Proposition 7.20. *Assume that X is an integral scheme. Then any line bundle on X is (uncanonically) a submodule of $\mathcal{K}_X$.*

XXX: weaker hypothesis possible?

Proof. Let ξ be the generic point of X and let $\Box := \neg\neg$ denote the modal operator such that internal sheafification with respect to $\Box$ is the same as pulling back to $\{\xi\}$ and then pushing forward to X again (see **XXX: ref**). Let $\mathcal{L}$ be a line bundle on X. Since $\mathcal{L}_\xi \cong \mathcal{O}_{X,\xi}$ (uncanonically), there is some injection $\mathcal{L}_\xi \to \mathcal{K}_{X,\xi}$; this corresponds internally to an injection $\mathcal{L}^{++} \to \mathcal{K}_X^{++}$. Since $\mathcal{K}_X$ is already a $\Box$-sheaf (see proposition 7.8) and $\mathcal{L}$ is $\Box$-separated (being isomorphic to $\mathcal{O}_X$), we have the global injection

$$\mathcal{L} \hookrightarrow \mathcal{L}^{++} \hookrightarrow \mathcal{K}_X^{++} \xrightarrow{(\cong)^{-1}} \mathcal{K}_X. \qquad \square$$

- on reduced schemes, $\mathcal{K}_X$ is the sheaf of meromorphic functions
- show $\mathcal{K}_X = j_*(\mathcal{O}_X)$?
- divisor associated to rational sections

8. Compactness principles

As stated in the introduction, quasicompactness of a space can not be detected by the internal language: There cannot exist a formula φ such that a topological space is quasicompact if and only if $\mathrm{Sh}(X) \models \varphi$, since the latter is always a local property on X while quasicompactness is not. However, quasicompactness can be characterized by a *metaproperty* of the internal language.

This result is best stated in a way which does not explicitly refer to a notion of finiteness. So recall that quasicompactness of a topological space X can be phrased in the following way: For any directed set I and any monotone family $(U_i)_{i\in I}$ of open subsets, if $X = \bigcup_i U_i$ then $X = U_i$ for some $i \in I$. As usual, a *directed set* is an inhabited partially ordered set such that for any two elements, there exists a common upper bound. A family $(U_i)_{i\in I}$ is *monotone* if and only if $i \preceq j$ implies $U_i \subseteq U_j$.

Proposition 8.1. *Let X be a topological space. Then X is quasicompact if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any directed set I and any monotone family $(\varphi_i)_{i\in I}$ of formulas over X,*

$$\mathrm{Sh}(X) \models \bigvee_{i\in I} \varphi_i \quad \textit{implies} \quad \textit{for some } i \in I,\ \mathrm{Sh}(X) \models \varphi_i.$$

The monotonicity condition means that $\mathrm{Sh}(X) \models (\varphi_i \Rightarrow \varphi_j)$ for any $i \preceq j$ in I.

Stated more succintly, a topological space X is quasicompact if and only if "$\mathrm{Sh}(X) \models$" commutes with directed "$\bigvee_{i\in I}$"'s.

Proof. For the "only if" direction, let such a family of formulas be given. Declare U_i to be the largest open subset of X where φ_i holds. Then by assumption, the U_i form a monotone family and cover X. By quasicompactness of X, some single U_i covers X as well, such that the corresponding formula φ_i holds on X.

For the "if" direction, note that a monotone family (U_i) of open subsets induces a monotone family of formulas by defining $\varphi_i :\equiv U_i$. This correspondence is such that $\mathrm{Sh}(X) \models \bigvee_i \varphi_i$ holds if and only if $X = \bigcup_i U_i$ and such that $\mathrm{Sh}(X) \models \varphi_i$ if and only if $X = U_i$. With these observations the claim is obvious. □

XXX: formulate for locally constant index sheaves I as well.

Example 8.2. Let X be a quasicompact scheme (or quasicompact ringed space). Let $f \in \Gamma(X, \mathcal{O}_X)$ be a global function. Endow the set of natural numbers with the usual ordering. Then the family of formulas given by $(f^n = 0)_{n\in\mathbb{N}}$ is monotone. Thus, if it internally holds that f is nilpotent, then f is nilpotent as an element of $\Gamma(X, \mathcal{O}_X)$ as well.

Proposition 8.3. *Let X be a topological space. Let $K \subseteq X$ be an open subset which is* locally quasicompact *in the sense that there exists an open covering $X = \bigcup_j U_j$ such that each $K \cap U_j$ is quasicompact. Then the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any directed set I and monotone family $(\varphi_i)_{i\in I}$ of formulas over X it holds that*

$$\mathrm{Sh}(X) \models \big(K \Rightarrow \bigvee_i \varphi_i\big) \quad \textit{implies} \quad \mathrm{Sh}(X) \models \bigvee_i (K \Rightarrow \varphi_i).$$

If additionally for any open subset $V \subseteq X$ the set $K \cap V$ is locally quasicompact in V, the following stronger and purely internal statement holds:

$$\mathrm{Sh}(X) \models \big(K \Rightarrow \bigvee_i \varphi_i\big) \Longrightarrow \bigvee_i (K \Rightarrow \varphi_i).$$

Proof. Assume that $\mathrm{Sh}(X) \models (K \Rightarrow \bigvee_i \varphi_i)$. This is equivalent to $K \models \bigvee_i \varphi_i$. By the locality of the internal language, it follows that $K \cap U_j \models \bigvee_i \varphi_i$ for each j. Since $K \cap U_j$ is quasicompact, it follows by the previous proposition that there exists an index $i_j \in I$ such that $K \cap U_j \models \varphi_{i_j}$. This is equivalent to $U_j \models (K \Rightarrow \varphi_{i_j})$. In particular, it holds that $U_j \models \bigvee_i (K \Rightarrow \varphi_i)$. Since this is true for any j, it follows that $X \models \bigvee_i (K \Rightarrow \varphi_i)$, again by the locality of the internal language.

The second statement is a corollary of the first one. □

Example 8.4. Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ be a global function. Then the open set $D(f) = \{x \in X \mid f_x \text{ is invertible in } \mathcal{O}_{X,x}\}$ is locally quasicompact in the sense of the proposition, even in the stronger sense: Let $V \subseteq X$ be any open set and consider a covering $V = \bigcup_i U_i$ by open affine subsets $U_i = \operatorname{Spec} A_i$. Then $D(f) \cap U_i \cong \operatorname{Spec} A_i[f^{-1}]$ is quasicompact.

From this example, it will trivially follow that the nilradical $\sqrt{(0)} \subseteq \mathcal{O}_X$ of a scheme and indeed the radical of any quasicoherent ideal sheaf is quasicoherent (lemma 9.7).

Remark 8.5. In applications, the open set K of the proposition is often given as the largest open subset on which some formula ψ holds. Then the conclusion of the proposition is that *assuming that ψ holds commutes with directed disjunctions.*

A stronger condition on a topological space X than quasicompactness is locality: A topological space is *local* if and only if for any open covering $X = \bigcup_i U_i$ (not necessarily directed) a certain single U_i covers X as well. Locality has a similar characterization as a metaproperty of $\mathrm{Sh}(X)$:

Proposition 8.6. *Let X be a topological space. Then X is local if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any set I and any family $(\varphi_i)_{i\in I}$ of formulas over X, it holds that*

$$\mathrm{Sh}(X) \models \bigvee_{i\in I} \varphi_i \quad \textit{implies} \quad \textit{for some } i \in I,\ \mathrm{Sh}(X) \models \varphi_i.$$

In this case, the internal language has additionally the following (weaker) metaproperty: For any sheaf $\mathcal{F}$ on X and any formula $\varphi(s)$ containing a variable $s : \mathcal{F}$, it holds that

$$\mathrm{Sh}(X) \models \exists s : \mathcal{F}.\ \varphi(s) \quad \textit{implies} \quad \textit{for some } s \in \Gamma(X, \mathcal{F}),\ \mathrm{Sh}(X) \models \varphi(s).$$

Proof. The proof of the first part is very similar to the proof of the previous proposition. For the "only if" direction of the second part, note that the antecedent implies that there exist local section $s_i \in \Gamma(U_i, \mathcal{F})$ such that $U_i \models \varphi(s_i)$ for some open covering $X = \bigcup_i U_i$. By locality of X, one such U_i suffices to cover X; so the corresponding section s_i is actually a global section and verifies $X \models \varphi(s_i)$.

For the converse direction, note that given a family $(U_i)_{i\in I}$ of open subsets, we can define a sheaf on X by the internal definition

$$\mathcal{F} := \{M \in \Omega \mid \bigvee_{i\in I} (M = U_i)\}.$$

Recall that Ω is the object of truth values, internally defined as the power set of $1 := \{\star\}$ and that "$M = U_i$" is abuse of notation and means $M = \{x \in 1 \mid U_i\}$. **XXX: the global sections of $\mathcal{F}$ are not those which I expected! Is the claim true?** ☐

XXX: lead-in

Proposition 8.7. *A topological space X is irreducible if and only if the internal language of $\mathrm{Sh}(X)$ has the following metaproperty: For any formulas φ and ψ*

$$\mathrm{Sh}(X) \models \neg(\varphi \wedge \psi) \quad \textit{implies} \quad \mathrm{Sh}(X) \models \neg\varphi \textit{ or } \mathrm{Sh}(X) \models \neg\psi.$$

Furthermore, in this case the following internal logical principle holds:

$$\mathrm{Sh}(X) \models \forall \alpha, \beta \in \Omega.\ \neg(\alpha \wedge \beta) \Rightarrow (\neg\alpha \vee \neg\beta).$$

Proof. The statement "$\mathrm{Sh}(X) \models \neg(\varphi \wedge \psi)$" means that $U \cap V =$, where U and V are the largest open subsets on which φ respectively ψ hold. The disjunction "$\mathrm{Sh}(X) \models \neg\varphi$ or $\mathrm{Sh}(X) \models \neg\psi$ means that $U =$ or $V =$.

Therefore, if X is irreducible, then the internal language has the claimed metaproperty. The converse can be seen by instantiating φ and ψ with the formulas associated to given open subsets having empty intersection. It then follows that one of these formulas is false in the internal language; thus the associated subset is empty.

The stated logical principle holds since open subsets of irreducible spaces are irreducible. □

8.1. **Internal proofs of common lemmas.**

Lemma 8.8. *Let X be an irreducible reduced scheme. Then all local rings $\mathcal{O}_{X,x}$ are integral domains.*

Proof. It suffices to give a proof of the following statement: Let R be a local ring such that elements which are not invertible are nilpotent. Further assume that R is reduced. Then R is an integral domain in the weak sense.

This proof may, additionally to the rules of intuitionistic logic, use the classical axiom given by proposition 8.7.

So let arbitrary elements $x, y : R$ with $xy = 0$ be given. Then it is not the case that x and y are both invertible: If they were, their product xy would be invertible as well, contradicting $1 \neq 0$. By the classicality principle, it follows that x is not invertible or that y is not invertible. Thus x or y is nilpotent and therefore zero. □

9. Quasicoherent sheaves of modules

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space X is *quasicoherent* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^J \longrightarrow (\mathcal{O}_X|_U)^I \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules, where I and J are arbitrary sets (which may depend on U).

If X is indeed a scheme, quasicoherence can also be characterized in terms of inclusions of distinguished open subsets of affines: An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any open affine subscheme $U = \operatorname{Spec} A$ of X and any function $f \in A$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}),\ \tfrac{s}{f^n} \longmapsto f^{-n} s|_{D(f)}$$

is an isomorphism of $A[f^{-1}]$-modules. Here $D(f) \subseteq U$ denotes the standard open subset $\{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$. Both conditions can be internalized.

Proposition 9.1. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \exists I, J \text{ lc. } \ulcorner\text{there exists an exact sequence } \mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner.$$

*The "*lc*" indicates that when interpreting this internal statement with the Kripke–Joyal semantics, I and J should only be instantiated with* locally constant *sheaves.*

Proof. We only sketch the proof. The translation of the internal statement is that there exists a covering of X by open subsets U such that for each such U, there exist sets I, J and an exact sequence

$$(\mathcal{O}_X|_U)^{\underline{J}} \longrightarrow (\mathcal{O}_X|_U)^{\underline{I}} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

where $\underline{I}$ and $\underline{J}$ are the constant sheaves associated to I respectively J. The term "$(\mathcal{O}_X|_U)^{\underline{I}}$" refers to the internally defined free $\mathcal{O}_X$-module with basis the elements of $\underline{I}$. By exploiting that $\underline{I}$ is a discrete set from the internal point of view (i. e. any two elements are either equal or not), one can show that this is the same as $(\mathcal{O}_X|_U)^I$; similarly for J. With this observation, the statement follows. $\square$

Remark 9.2. The restriction to locally constant sheaves is really necessary: The internal statement $\mathrm{Sh}(X) \models \exists I, J.\ \ulcorner$there exists an exact sequence $\mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner$ is true for *any* $\mathcal{O}_X$-module $\mathcal{F}$. This is because the usual proof of the fact that any module admits a resolution by (not necessarily finitely) free modules is intuitionistically acceptable and thus also valid in the internal universe.

In practice, the internal condition given by the proposition is not very useful, since at the moment, we do not know of any internal characterization of locally constant sheaves. The internal condition given by the following proposition does not have this defect.

Theorem 9.3. *Let X be scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if, from the internal perspective, the localized module $\mathcal{F}[f^{-1}]$ is a sheaf for the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$ for any $f : \mathcal{O}_X$.*

In detail, the internal condition is that for any $f : \mathcal{O}_X$, it holds that

$$\forall s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow s = 0$$

and for any subsingleton $\mathcal{S} \subseteq \mathcal{F}[f^{-1}]$ it holds that

$$(\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner \mathcal{S} \text{ inhabited}\urcorner) \Longrightarrow \exists s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{S}).$$

Unlike with the internalizations of finite type, finite presentation and coherence, this condition is *not* a standard condition of commutative algebra. In fact, in classical logic, this condition is always satisfied – for trivial logical reasons if f is invertible and because $\mathcal{F}[f^{-1}]$ is the zero module if f is not invertible (since then, it is nilpotent). This is to be expected: *Any* module M in commutative algebra is quasicoherent in the sense that the associated sheaf of modules $M^\sim$ is quasicoherent. **XXX: More to the point, commutative algebra does not deal with quasicoherence, since quasicoherence is an interesting condition only on on arbitrary schemes, not on affine schemes.**

We give the proof below, after first discussing some examples and consequences. The proof will explain the origin of this condition.

Example 9.4. The zero $\mathcal{O}_X$-module is quasicoherent, since (it and) all localizations of it are singleton sets from the internal perspective and thus $\square$-sheaves for any modal operator $\square$ (example 6.9).

Corollary 9.5. *Let X be a scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subseteq \mathcal{F}$ be a submodule. Then $\mathcal{G}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \forall s : \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{G}) \Longrightarrow \bigvee_{n \geq 0} f^n s \in \mathcal{G}.$$

Proof. We can give a purely internal proof. Let $f : \mathcal{O}_X$. Since subpresheaves of separated sheaves are separated, the module $\mathcal{G}[f^{-1}]$ is in any case separated with respect to the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$.

Now suppose that $\mathcal{G}$ is quasicoherent. Let $f : \mathcal{O}_X$. Let $s : \mathcal{F}$ and assume that if f were invertible, s would be an element of $\mathcal{G}$. Define the subsingleton $S :=$

$\{t : \mathcal{G}[f^{-1}] \mid \ulcorner f \text{ inv.}\urcorner \wedge t = s/1\}$. Then S would be inhabited by $s/1$ if f were invertible. Since $\mathcal{G}[f^{-1}]$ is a sheaf, it follows that there exists an element u/f^n of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be the case that $u/f^n = s/1 \in \mathcal{G}[f^{-1}] \subseteq \mathcal{F}[f^{-1}]$. Since $\mathcal{F}[f^{-1}]$ is separated, it follows that it actually holds that $u/f^n = s/1 \in \mathcal{F}[f^{-1}]$. Therefore there exists $m : \mathbb{N}$ such that $f^m f^n s = f^m u \in \mathcal{F}$. Thus $f^{m+n} s$ is an element of $\mathcal{G}$.

For the converse direction, assume that $\mathcal{G}$ fulfills the stated condition. Let$f : \mathcal{O}_X$. Let $S \subseteq \mathcal{G}[f^{-1}]$ be a subsingleton which would be inhabited if f were invertible. By regarding S as a subset of $\mathcal{F}[f^{-1}]$, it follows that there exists an element $u/f^n \in \mathcal{F}[f^{-1}]$ such that, if f were invertible, u/f^n would be an element of S. In particular, u would be an element of $\mathcal{G}$. By assumption it follows that there exists $m : \mathbb{N}$ such that $f^m u \in G$. Thus $(f^m u)/(f^m f^n)$ is an element of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be an element of S. □

Example 9.6. Let X be a scheme and s be a global section of $\mathcal{O}_X$. Then the annihilator of s, i. e. the sheaf of ideals internally defined by the formula

$$I := \operatorname{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X$$

is quasicoherent. To prove this in the internal language, it suffices to verify the condition of the proposition. So let $f : \mathcal{O}_X$ and $t : \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.}\urcorner \Rightarrow t \in I$, i. e. assume that if f were invertible, st would be zero. By proposition 3.9 it follows that $f^n st = 0$ for some $n : \mathbb{N}$, i. e. that $f^n t \in I$.

Example 9.7. Let X be a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasicoherent ideal sheaf. Then the radical of $\mathcal{I}$, internally definable as

$$\sqrt{\mathcal{I}} := \Big\{ s : \mathcal{O}_X \;\Big|\; \bigvee_{n \geq 0} s^n \in \mathcal{I} \Big\},$$

is quasicoherent as well: Let $f : \mathcal{O}_X$ and $s : \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \sqrt{\mathcal{I}}$, i. e. assume that if f were invertible, some power s^n would be an element of $\mathcal{I}$. Since *assuming that f is invertible commutes with directed disjunctions* (example 8.4), it follows that for some natural number n, it holds that $\ulcorner f \text{ inv.}\urcorner \Rightarrow s^n \in \mathcal{I}$. By quasicoherence of $\mathcal{I}$, we may deduce that for some natural number m, it holds that $f^m s^n \in \mathcal{I}$. Thus $fs \in \sqrt{\mathcal{I}}$.

Proposition 9.8. *Let X be a scheme of dimension ≤ 0. Then any $\mathcal{O}_X$-module is quasicoherent.*

Proof. By corollary 3.13, any element $f : \mathcal{O}_X$ is invertible or nilpotent. Therefore the quasicoherence condition of theorem 9.3 is for any $\mathcal{O}_X$-module trivially satisfied. □

Remark 9.9. The notion of quasicoherence as given by the internal condition of theorem 9.3 is not usually studied in constructive mathematics. Indeed, outside of the internal universe of a scheme, the notion does not seem to be very interesting: For many important rings, there a few quasicoherent modules in this sense. For instance, let M be a module over a ring R such that every element of R is invertible or not invertible. (The ring $\mathbb{Z}$ is such a ring.) Then M is quasicoherent if and only if for any $f : R$ which is not invertible, the localized module $M[f^{-1}]$ is the zero module, i. e. any element of M is annihilated by some power f^n. As a concrete example, any $\mathbb{Z}$-submodule of $\mathbb{Z}$ which contains a nonzero element fails to be quasicoherent.

Proof of theorem 9.3. By the well-known characterization of quasicoherence in terms of inclusions of distinguished open subsets, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any affine open subset $U \subseteq X$ and function $f \in \Gamma(U, \mathcal{O}_U)$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}),\ \ s/f^n \longmapsto f^{-n} s|_{D(f)} \tag{2}$$

is bijective. We will see that this map is injective for all such U and f if and only if from the internal perspective, the set $\mathcal{F}[f^{-1}]$ is a separated presheaf with respect to the modal operator $(\ulcorner f \text{ inv.}\urcorner \Rightarrow \_\_)$ for any $f : \mathcal{O}_X$; and we will see that in this case, the map is additionally surjective for all such U and f if the full sheaf condition is fulfilled.

Since the sheaf $\mathcal{F}[f^{-1}]$ does not appear in the stated characterization, we will first reformulate the separatedness and the sheaf condition in terms of $\mathcal{F}$ instead of $\mathcal{F}[f^{-1}]$. To this end, note that the separatedness condition is equivalent to

$$\forall f : \mathcal{O}_X.\ \forall s : \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0 : \mathcal{F}) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0 : \mathcal{F}. \tag{3}$$

The equivalence can easily be proven in the internal language. The sheaf condition is equivalent to (the conjunction of the separatedness condition and)

$$\forall f : \mathcal{O}_X.\ \forall \mathcal{K} \subseteq \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner K \text{ is a singleton}\urcorner) \Longrightarrow \bigvee_{n \geq 0} \exists s : \mathcal{F}.\ \ulcorner f \text{ inv.}\urcorner \Rightarrow f^{-n} s \in \mathcal{K}. \tag{4}$$

In one direction, a set $\mathcal{S} \subseteq \mathcal{F}[f^{-1}]$ is given; construct $K := \{s : \mathcal{F} \mid s/1 \in \mathcal{S}\} \subseteq \mathcal{F}$. In the other direction, a set $\mathcal{K} \subseteq \mathcal{F}$ is given; construct $S := \{s : \mathcal{F}[f^{-1}] \mid \exists s' : \mathcal{F}.\ s' \in \mathcal{K} \wedge s = s'/1\} \subseteq \mathcal{F}[f^{-1}]$. The remaining details can easily be filled out.

We now interpret the internal statement (3) using the Kripke–Joyal semantics: Using the simplification rules, the external meaning is that for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$ the following condition is satisfied: For any section $s \in \Gamma(U, \mathcal{F})$ it should hold that

$$U \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \quad \text{implies} \quad U \models \bigvee_{n \geq 0} f^n s = 0.$$

The antecedent is equivalent to saying that s is zero in $\Gamma(D(f), \mathcal{F})$. The consequent is (by quasicompactness of U, see example 8.2) equivalent to saying that for some $n \geq 0$, the section $f^n s$ is zero in $\Gamma(U, \mathcal{F})$, i. e. that s is zero in $\Gamma(U, \mathcal{F})[f^{-1}]$. So this condition is precisely the injectivity of the canonical map (2).

The external meaning of statement (4) is that for any affine open subset $U \subseteq X$ and any function $f \in \Gamma(U, \mathcal{O}_U)$ the following condition is satisfied: For any subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ it should hold that

$$U \models (\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner \mathcal{K} \text{ is a singleton}\urcorner) \quad \text{implies} \quad U \models \bigvee_{n \geq 0} \exists s : \mathcal{F}.\ \ulcorner f \text{ inv.}\urcorner \Rightarrow f^{-n} s \in \mathcal{K}.$$

Given the injectivity of the canonical map (2) (for any affine open subset, not only U), this condition is equivalent to its surjectivity: To see that surjectivity is sufficient, let a subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ verifying the antecedent be given. Since $\mathcal{K}|_{D(f)}$ is a singleton sheaf, we can consider its unique section $u \in \Gamma(D(f), \mathcal{K}) \subseteq \Gamma(D(f), \mathcal{F})$.

By surjectivity, there exists a preimage, i. e. a fraction $s/f^n \in \Gamma(U, \mathcal{F})[f^{-1}]$ such that $u = f^{-n}s|_{D(f)}$ in $\Gamma(D(f), \mathcal{F})$. Thus $U \models f^{-n}s \in \mathcal{K}$ holds and the consequent is verified.

To see that surjectivity is necessary, let a section $u \in \Gamma(D(f), \mathcal{F})$ be given. Define a subsheaf $\mathcal{K} \subseteq \mathcal{F}|_U$ by setting $\Gamma(V, \mathcal{K}) := \{u|_V \mid V \subseteq D(f)\}$. Then $\mathcal{K}$ verifies the antecedent. Thus the consequent holds: There exists an open covering $U = \bigcup_i U_i$ such that for each i, there exists a natural number n_i and a section $s_i \in \Gamma(U_i, \mathcal{F})$ such that $f^{-n_i}s_i = u$ on $U_i \cap D(f)$. Without loss of generality, we may assume that the U_i are distinguished open subsets $D(g_i) \subseteq U$; that they are finite in number; and that the natural numbers n_i agree with each other and thus equal some number n. Since $s_i = s_j$ in $\Gamma(U_i \cap U_j \cap D(f), \mathcal{F})$, injectivity of the canonical map (2) (on the affine set $U_i \cap U_j = D(g_i g_j)$) implies that $s_i = s_j$ in $\Gamma(U_i \cap U_j, \mathcal{F})[f^{-1}]$. Thus for any i, j there exists a natural number m_{ij} such that $f^{m_{ij}}s_i = f^{m_{ij}}s_j$ in $\Gamma(U_i \cap U_j, \mathcal{F})$. We may assume that the numbers m_{ij} equal some common number m; thus the local sections $f^m s_i$ glue to a section $s \in \Gamma(U, \mathcal{F})$. The sought preimage of u is the fraction s/f^{n+m}, since $f^{-(n+m)}s|_{D(f)}$ equals u in $\Gamma(D(f), \mathcal{F})$ (as this is true on the covering $D(f) = \bigcup_i (D(f) \cap U_i)$). □

- is the condition good enough to show that modules of finite type are quasicoherent? To show that cokernels are quasicoherent?
- discussion meaning of the sheaf condition in external language
- give more examples: (h), ...
- Noetherian hypotheses: for example, that any quasicoherent submodule of a module of finite type is of finite type as well

10. Subschemes

10.1. **Sheaves on open and closed subspaces. XXX: Remind the reader of abuse of notation like "U" for formulas vs. open sets.**

Lemma 10.1. *Let X be a topological space. Let $j : U \hookrightarrow X$ be the inclusion of an open subspace. Then there is a canonical functor $j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ called* extension by the empty set *with the following properties:*

1. *The functor $j_!$ is left adjoint to the restriction functor $j^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$.*
2. *The composition $j^{-1} \circ j_! : \mathrm{Sh}(U) \to \mathrm{Sh}(U)$ is (canonically isomorphic to) the identity.*
3. *The essential image of $j_!$ consists of exactly those sheaves $\mathcal{F}$ on X whose stalks are empty at all points of U^c. In this case, it holds that $j_! j^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Internally, for a set $\mathcal{F}$, we can define $j_!(\mathcal{F})$ simply be the set comprehension

$$j_!(\mathcal{F}) := \{x : \mathcal{F} \mid U\}.$$

Externally, the sections of the thusly defined sheaf on an open subset $V \subseteq X$ are given by $\{x \in \Gamma(V, \mathcal{F}) \mid V \subseteq U\}$, i. e. the whole of $\Gamma(V, \mathcal{F})$ if $V \subseteq U$ and the empty set otherwise. With this short internal description, all of the stated properties can be easily verified in the internal language.

For instance, recall that internally the functor j^{-1} is given by sheafifying with respect to the modal operator $\Box :\equiv (U \Rightarrow \_\_)$. Thus, to show the second statement, we have to give a bijection $(j_!(\mathcal{F}))^{++} \to \mathcal{F}$ for any $\Box$-sheaf $\mathcal{F}$. (This map has to be given explicitly, to not only show a weaker statement about a local isomorphism, see discussion at **XXX: ref**). To this end, we can use the composition

$$(j_!(\mathcal{F}))^{++} \hookrightarrow \mathcal{F}^{++} \xrightarrow{(\cong)^{-1}} \mathcal{F},$$

where the first map is injective since sheafifying is exact. It is also surjective, since the $\Box$-translation of the statement $\ulcorner j_!(F) \to \mathcal{F}$ is surjective$\urcorner$ holds: For any element $x : \mathcal{F}$, it holds that $\Box(\ulcorner x$ possesses a preimage$\urcorner)$.

For the third property, note that a sheaf $\mathcal{F}$ on X fulfills the stated condition on stalks if and only if, from the internal perspective, it holds that $U \Rightarrow \ulcorner \mathcal{F}$ is inhabited$\urcorner$. We omit further details. $\square$

Lemma 10.2. *Let X be a ringed space. Let $j : U \hookrightarrow X$ be the inclusion of an open subspace. Then there is a canonical functor $j_! : \mathrm{Mod}_U(\mathcal{O}_U) \to \mathrm{Mod}_X(\mathcal{O}_X)$ called* extension by zero *with the following properties:*

(1) *The functor $j_!$ is left adjoint to the restriction functor $j^{-1} : \mathrm{Mod}_X(\mathcal{O}_X) \to \mathrm{Mod}_U(\mathcal{O}_U)$.*
(2) *The composition $j^{-1} \circ j_! : \mathrm{Mod}_U(\mathcal{O}_U) \to \mathrm{Mod}_U(\mathcal{O}_U)$ is (canonically isomorphic to) the identity.*
(3) *The essential image of $j_!$ consists of exactly those $\mathcal{O}_X$-modules whose stalks are zero at all points of U^c. In this case, it holds that $j_! j^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Internally, a sheaf of modules on $\mathcal{O}_U$ is simply a module on $\mathcal{O}_X^{++}$ which is a $\Box$-sheaf, where $\Box :\equiv (U \Rightarrow \_\_)$. The suitable internal definition for the extension by zero of such a module $\mathcal{F}$ is

$$j_!(\mathcal{F}) := \{x : \mathcal{F} \mid (x = 0) \vee U\}.$$

With this description, all necessary verifications are easy. Note that an $\mathcal{O}_X$-module $\mathcal{F}$ fulfills the stated condition on stalks if and only if internally, it holds that $\forall x : \mathcal{F}.\ (x = 0) \vee U$. **XXX: add parenthesis** $\square$

Lemma 10.3. *Let X be a topological space. Let $i : A \hookrightarrow X$ be the inclusion of a closed subspace. The essential image of the inclusion $i_* : \mathrm{Sh}(A) \to \mathrm{Sh}(X)$ consists of exactly those sheaves $\mathcal{F}$ whose support is a subset of A. In this case, it holds that $i_* i^{-1}\mathcal{F} \cong \mathcal{F}$ (canonically).*

Proof. Recall that the modal operator associated to A is $\Box\varphi :\equiv (\varphi \vee A^c)$, and that by section 6.4 the essential image of i_* consists of exactly those sheaves which are $\Box$-sheaves from the internal perspective. Let $\mathcal{F}$ be a sheaf on X. Then it holds that

$$\operatorname{supp}\mathcal{F} \subseteq A \quad\Longleftrightarrow\quad A^c \subseteq X \setminus \operatorname{supp}\mathcal{F} \quad\Longleftrightarrow\quad A^c \subseteq \operatorname{int}(X \setminus \operatorname{supp}\mathcal{F}).$$

Since the interior of the complement of $\operatorname{supp}\mathcal{F}$ can be characterized as the largest open subset of X on which the internal statement "$\mathcal{F}$ is a singleton" holds (remark 4.8), the condition on the support is fulfilled if and only if

$$Sh(X) \models (A^c \Rightarrow \ulcorner \mathcal{F} \text{ is a singleton}\urcorner).$$

We thus have to show that this internal condition is equivalent to $\mathcal{F}$ being a $\Box$-sheaf. For the "if" direction, assume A^c. Then the empty subset $S \subseteq \mathcal{F}$ trivially verifies the condition that $\Box(\ulcorner S$ is a singleton$\urcorner)$. There thus exists an element $x : \mathcal{F}$ (such that $\Box(x \in S)$). If we're given a further element $y : \mathcal{F}$, it trivially holds that $\Box(x = y)$.

By $\Box$-separatedness, it thus follows that $x = y$. Thus $\mathcal{F}$ is the singleton $\{x\}$. The proof of the "only if" direction is similar.

The second statement says that internally, sheafifying a $\Box$-sheaf with respect to the modal operator $\Box$ and then forgetting that the result is a $\Box$-sheaf amounts to doing nothing. This is obvious. $\Box$

10.2. **Closed subschemes.** Let X be a ringed space. Recall that an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$ defines a closed subset $V(\mathcal{I}) = \{x \in X \mid \mathcal{I}_x \neq (1) \subseteq \mathcal{O}_{X,x}\}$, a sheaf of rings $\mathcal{O}_X/\mathcal{I}$, and a ringed space $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ where $\mathcal{O}_{V(\mathcal{I})}$ is the pullback of $\mathcal{O}_X/\mathcal{I}$ to $V(\mathcal{I})$. In the internal universe, we can reify $V(\mathcal{I})$ by giving a modal operator $\Box$ such that externally, the subspace $X_\Box$ coincides with $V(\mathcal{I})$.

Proposition 10.4. *Let X be a ringed space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. Then:*

(1) *The subspace of X associated to the modal operator $\Box$ defined by $\Box\varphi :\equiv (\varphi \vee (1 \in \mathcal{I}))$ is $V(\mathcal{I})$.*
(2) *The support of $\mathcal{O}_X/\mathcal{I}$ is exactly $V(\mathcal{I})$.*
(3) *The canonical morphism $i : V(\mathcal{I}) \to X$ is a closed immersion of ringed spaces.*

Proof. For any open subset $U \subseteq X$, it holds that $U \models 1 \in \mathcal{I}$ if and only if $U \subseteq D(\mathcal{I}) = X \setminus V(\mathcal{I})$. Thus $D(\mathcal{I})$ can be characterized as the largest open subset on which "$1 \in \mathcal{I}$" holds. According to table 2 on page 31, the stated modal operator thus defines the subspace $D(\mathcal{I})^c$, i. e. $V(\mathcal{I})$.

For the second statement, note that since $\mathcal{O}_X/\mathcal{I}$ is a sheaf of rings, its support is closed. Therefore the largest open subset of X where the internal statement "$\mathcal{O}_X/\mathcal{I} = 0$" holds is the complement of the support (proposition 4.7). Since $D(\mathcal{I})$ is the largest open subset where the internal statement "$\mathcal{I} = (1)$" holds, it suffices to show that internally, $\mathcal{O}_X/\mathcal{I} = 0$ if and only if $\mathcal{I} = (1)$. This is obvious.

The topological part of the third statement is clear. For the ring-theoretic part, we have to show that the canonical ring homomorphism $\mathcal{O}_X \to i_*\mathcal{O}_{V(\mathcal{I})}$, that is the canonical projection $\mathcal{O}_X \to \mathcal{O}_X/(\mathcal{I})$, is an epimorphism of sheaves. This is obvious. $\Box$

By lemma 10.3, the sheaf $\mathcal{O}_X/\mathcal{I}$ is thus a $\Box$-sheaf from the internal perspective.

Proposition 10.5. *Let X be a locally ringed space. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. Then the ringed space $(V(\mathcal{I}), \mathcal{O}_{V(\mathcal{I})})$ is too locally ringed.*

Proof. We have to show that

$$\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is a local ring} \urcorner.$$

By theorem 6.23, this is equivalent to

$$\mathrm{Sh}(X) \models (\ulcorner \mathcal{O}_X/\mathcal{I} \text{ is a local ring} \urcorner)^{\Box},$$

where $\Box$ is the modal operator given by $\Box\varphi :\equiv (\varphi \vee (1 \in \mathcal{I}))$. We therefore have to give an intuitionistic proof of the fact

$$\forall x, y : \mathcal{O}_X/\mathcal{I}.\ \ulcorner x + y \text{ inv.} \urcorner \Longrightarrow \Box(\ulcorner x \text{ inv.} \urcorner \vee \ulcorner y \text{ inv.} \urcorner).$$

So let $x = [s], y = [t] : \mathcal{O}_X/\mathcal{I}$ such that $x + y$ is invertible in $\mathcal{O}_X/\mathcal{I}$. This means that there exists $u : \mathcal{O}_X$ and $v : \mathcal{I}$ such that $us + ut + v = 1$ in $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a local ring, it follows that us, ut, or v is invertible. In the first two cases, it follows that x respectively y are invertible in $\mathcal{O}_X/\mathcal{I}$. In the third case, it follows that $1 \in \mathcal{I}$ and thus any boxed statement is trivially true. $\Box$

If X is a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ is an ideal sheaf, it is well-known that the locally ringed space $V(\mathcal{I})$ is a scheme if and only if $\mathcal{I}$ is quasicoherent. We cannot give an internal proof of this fact since we lack an internal characterization of being a scheme.

Lemma 10.6. *Let X be a scheme (or ringed space). Let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. The ringed space $V(\mathcal{I})$ is reduced if and only if, from the internal perspective of $\mathrm{Sh}(X)$, the ideal $\mathcal{I}$ is a radical ideal.*

Proof. The following chain of equivalences holds:

$$\begin{aligned}
&\mathrm{Sh}(V(\mathcal{I})) \models \ulcorner \mathcal{O}_{V(\mathcal{I})} \text{ is a reduced ring} \urcorner \\
\Longleftrightarrow\ &\mathrm{Sh}(V(\mathcal{I})) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_{V(\mathcal{I})}.\ s^n = 0 \Longrightarrow s = 0 \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \Big(\bigwedge_{n \geq 0} \forall s : \mathcal{O}_X/\mathcal{I}.\ s^n = 0 \Rightarrow s = 0 \Big)^{\Box} \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X/\mathcal{I}.\ s^n = 0 \Rightarrow \Box(s = 0) \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X.\ s^n \in \mathcal{I} \Rightarrow \Box(s \in \mathcal{I}) \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall s : \mathcal{O}_X.\ s^n \in \mathcal{I} \Rightarrow s \in \mathcal{I} \\
\Longleftrightarrow\ &\mathrm{Sh}(X) \models \ulcorner \mathcal{I} \text{ is a radical ideal} \urcorner
\end{aligned}$$

In the second-to-last step, we used that $\Box(s \in \mathcal{I}) \equiv ((s \in \mathcal{I}) \vee (1 \in \mathcal{I}))$ implies $s \in \mathcal{I}$. This is trivial in both cases of the disjunction. $\Box$

Lemma 10.7. *Let X be a scheme (or ringed space).*

(1) *There exists a reduced closed sub-ringed space $X_{\mathrm{red}} \hookrightarrow X$ having the same underlying topological space as X with the following universal property: Any morphism $Y \to X$ of (ringed or locally ringed) spaces such that Y is reduced factors uniquely over the closed immersion $X_{\mathrm{red}} \hookrightarrow X$.*
(2) *Let $A \subseteq X$ be a closed subset. Then there exists a structure of a reduced closed ringed subspace on A with a similar universal property.*

Proof. For the first statement, let $\mathcal{N} \subseteq \mathcal{O}_X$ be the nilradical of $\mathcal{O}_X$. This can internally be simply defined by $\mathcal{N} := \sqrt{(0)} = \{s : \mathcal{O}_X \mid \bigvee_{n \geq 0} s^n = 0\}$. Define X_{red} as the closed subspace associated to this ideal sheaf. This ringed space is reduced by the previous lemma. The proof of the universal property can also be done in the internal language, by using that the well-known fact of locale theory that the category of locales over X is equivalent to internal locales in $\mathrm{Sh}(X)$; but we do not want to discuss this further. **XXX: mention result on N(quasicompact)**

For the second statement, internally define the ideal $\mathcal{I} := \sqrt{\{s : \mathcal{O}_X \mid s = 0 \vee A^c\}} \subseteq \mathcal{O}_X$. Then $1 \in \mathcal{I}$ if and only if A^c, thus by proposition 10.4 the closed ringed subspace defined by $\mathcal{I}$ has A as underlying topological space. It is reduced since $\mathcal{I}$ is a radical ideal. **XXX: is $\mathcal{I}$ quasicoherent, if X is a scheme?** $\Box$

- open subschemes
- Koszul resolution

11. Relative spectrum

Recall that if $\mathcal{A}$ is a quasicoherent $\mathcal{O}_X$-algebra on a scheme X, one can construct the *relative spectrum* $\mathrm{RelSpec}_X \mathcal{A}$ by appropriately gluing the spectra $\operatorname{Spec} \Gamma(U, \mathcal{A})$ where U ranges over the affine opens of X. This relative spectrum comes equipped with a canonical morphism $\mathrm{RelSpec}_X \mathcal{A} \to X$. We can give a simple construction of the relative spectrum in the internal language of $\mathrm{Sh}(X)$, imitating the usual construction of the spectrum of a ring (as opposed to a sheaf of rings).

11.1. Internal locales.

Let X be a topological space (or a locale). A fundamental fact in the theory of locales is that there is a canonical equivalence between the category of *locales over* X – that is locales Y equipped with a morphism $Y \to X$ – and *internal locales in* $\mathrm{Sh}(X)$. An internal locale in a topos $\mathcal{E}$ is given by an object L of $\mathcal{E}$ (the internal lattice of opens of the locale) together with a binary relation $(\preceq) \hookrightarrow L \times L$ such that the axioms on a locale hold from the internal point of view. (In these notes, we do not need a precise wording of these axioms.)

The equivalence is described as follows: A locale $f : Y \to X$ over X induces an internal locale $I(Y)$ with object of opens given by $\mathrm{Op}(I(Y)) := f_* \Omega_{\mathrm{Sh}(Y)} \in \mathrm{Sh}(X)$, where f_* is the pushforward functor and $\Omega_{\mathrm{Sh}(Y)}$ is the object of truth values in the topos of sheaves on Y. Conversely, an internal locale $\mathcal{L} \in \mathrm{Sh}(X)$ induces an (external) locale $E(\mathcal{L})$ with lattice of opens given by $\mathrm{Op}(E(\mathcal{L})) := \Gamma(X, \mathcal{L})$. This comes equipped with a canonical morphism $Y \to X$ of locales which we do not need to describe explicitly. **XXX: give reference**

As a special case, the internalization of the trivial locale $\mathrm{id} : X \to X$ over X has as lattice of opens the object $\mathrm{id}_* \Omega_{\mathrm{Sh}(X)} = \Omega_{\mathrm{Sh}(X)} = \mathcal{P}(1)$. This is precisely the lattice of opens of the one-point space. Thus $I(X) \cong \mathrm{pt}$. This illustrates the intuition behind working internally in $\mathrm{Sh}(X)$ very well: From the perspective of $\mathrm{Sh}(X)$, the space X looks like the one-point space (even if in fact it is not).

One can associate to an internal locale L in a topos $\mathcal{E}$ a topos of internal sheaves on it: $\mathrm{Sh}_{\mathcal{E}}(L)$. The correspondence is made in such a way that the topos of sheaves on a locale Y over X is equivalent to the topos of sheaves on the internal locale $I(Y)$: $\mathrm{Sh}(Y) \simeq \mathrm{Sh}_{\mathrm{Sh}(X)}(I(Y))$.

There is no similarly nice correspondence between topological spaces over X and internal topological spaces in $\mathrm{Sh}(X)$. This is one of the reasons why locales are better suited for working internally and switching between internal and external perspectives.

XXX: give references

11.2. The spectrum of a ring as a locale.

Recall that the spectrum of a ring A is usually constructed as the set

$$\operatorname{Spec} A := \{\mathfrak{p} \subseteq A \mid \mathfrak{p} \text{ is a prime ideal}\}$$

endowed with a certain topology and a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$. From an intuitionistic (and thus internal) point of view, this construction does not work well: Prime ideals are intuitionistically much more elusive than classically, where one can appeal to Zorn's lemma to obtain maximal (and thus prime) ideals. More to the point, one can not show that this construction of the spectrum as a topological space verifies the expected universal property.

On the other hand, the lattice of opens of $\operatorname{Spec} A$ admits a simple description not needing the notion of prime ideals:

$$\operatorname{Op}(\operatorname{Spec} A) \cong \{\mathfrak{a} \subseteq A \,|\, \mathfrak{a} \text{ is a radical ideal}\}.$$

An open subset $U \subseteq \operatorname{Spec} A$ corresponds to the radical ideal $\{h \in A \,|\, D(h) \subseteq U\}$ (so in particular, the open subset $D(f)$ corresponds to the radical ideal $\sqrt{(f)}$); conversely, a radical ideal $\mathfrak{a}$ corresponds to the open subset $\bigcup_{h\in\mathfrak{a}} D(h)$.

Thus, in an intuitionistic context, we will construct the spectrum of a ring A only as a locale, not as a topological space; we will use the lattice of radical ideals as the lattice opens. This construction has the expected universal property, namely that it is adjoint to the global sections functor:

$$\operatorname{Hom}_{\mathrm{LRL}}(X, \operatorname{Spec} A) \cong \operatorname{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X)).$$

Here, "LRL" refers to the category of *locally ringed locales*, i. e. locales X equipped with a sheaf of rings $\mathcal{O}_X$ such that from the internal point of view of $\operatorname{Sh}(X)$, the ring $\mathcal{O}_X$ is a local ring. A morphism $Y \to X$ of locally ringed locales consists of a locale morphism $f : Y \to X$ and a morphism $f^\sharp : f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ of sheaves of rings on Y such that, from the internal point of view of $\operatorname{Sh}(Y)$, the ring homomorphism $f^\sharp$ is a local homomorphism. The notion of a locally ringed locale is thus a straightforward generalization of that of a locally ringed space.

For later use, we study the question when the spectrum is the one-point space. The answer is well-known classically, but since we want to use this result in an internal context, we have to give an intuitionistic proof.

Lemma 11.1. *Let A be a ring with $0 \neq 1$. Its spectrum is a one-point space (as a locale) if and only if any element of A is nilpotent or invertible.*

Proof. The locale $\operatorname{Spec} A$ is a one-point space if and only if the canonical map

$$\begin{array}{rcl} \Omega = \mathcal{P}(1) & \longrightarrow & \operatorname{Op}(\operatorname{Spec} A) \\ \varphi & \longmapsto & \mathfrak{a}_\varphi := \sup\{\sqrt{(1)} \,|\, \varphi\} = \{x : A \,|\, \ulcorner x \text{ nilpotent}\urcorner \vee \varphi\} \end{array}$$

is bijective. It is always injective: If $\mathfrak{a}_\varphi = \mathfrak{a}_\psi$, then $(1 \in \mathfrak{a}_\varphi) \Leftrightarrow (1 \in \mathfrak{a}_\psi)$. Since the unit of A is not nilpotent, this amounts to $\varphi \Leftrightarrow \psi$.

The map is surjective if and only if for any radical ideal $\mathfrak{a} \subseteq A$, it holds that $\mathfrak{a} = \{x : A \,|\, \ulcorner x \text{ nilpotent}\urcorner \vee \varphi\}$ for some proposition φ. By considering the condition "$1 \in \mathfrak{a}$", it follows that this proposition φ must be equivalent to the proposition "$1 \in \mathfrak{a}$" (if it is at all possible to write $\mathfrak{a}$ in such a way).

So the map is surjective if and only if for any radical ideal $\mathfrak{a} \subseteq A$ and any element x of A it holds that

$$x \in \mathfrak{a} \quad \Longleftrightarrow \quad \ulcorner x \text{ nilpotent}\urcorner \vee (1 \in \mathfrak{a}).$$

The "if" direction always holds. If any element of A is nilpotent or invertible, the "only if" direction holds as well (for any $\mathfrak{a}$ and any x). Considering the radical ideal $\sqrt{(f)}$ for an element $f : A$, one verifies that the converse holds as well. ☐

11.3. **The relative spectrum as an ordinary spectrum from the internal point of view.** Let X be a scheme and $\mathcal{A}$ be a quasicoherent $\mathcal{O}_X$-algebra. Since $\mathcal{A}$ looks like a plain algebra from the internal perspective of $\operatorname{Sh}(X)$, we can consider its internally defined spectrum. This is a locale internal to $\mathrm{Sh}(X)$; we might hope that its externalization is precisely the relative spectrum of $\mathcal{A}$ (considered as a locale):

$$E(\operatorname{Spec}\mathcal{A}) \overset{?}{\cong} \operatorname{RelSpec}_X \mathcal{A}.$$

However, this turns out to be too naive. The externalization of the internally defined spectrum has the universal property

$$\begin{aligned}\operatorname{Hom}_{\mathrm{LRL}/E(\operatorname{Spec}\mathcal{O}_X)}(Y, E(\operatorname{Spec}\mathcal{A})) &\cong \operatorname{Hom}_{\mathrm{LRL}_{\mathrm{Sh}(X)}/\operatorname{Spec}\mathcal{O}_X}(I(Y), \operatorname{Spec}\mathcal{A}) \\ &\cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mu_*\mathcal{O}_Y)\end{aligned}$$

for all locally ringed locales Y over $E(\operatorname{Spec}\mathcal{O}_X)$. Here, μ is the structure morphism $Y \to \operatorname{Spec}\mathcal{O}_X$; $E(\operatorname{Spec}\mathcal{O}_X)$ is the locally ringed locale associated to the internally defined spectrum of $\mathcal{O}_X$; and $\mathrm{LRL}_{\mathrm{Sh}(X)}$ is the category of locally ringed locales internal to $\mathrm{Sh}(X)$. However, the relative spectrum has the different universal property

$$\begin{aligned}\operatorname{Hom}_{\mathrm{LRL}/X}(Y, \operatorname{RelSpec}_X\mathcal{A}) &\cong \operatorname{Hom}_{\mathrm{LRL}_{\mathrm{Sh}(X)}/(\mathrm{pt},\mathcal{O}_X)}(I(Y), I(\operatorname{RelSpec}_X\mathcal{A})) \\ &\cong \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mu_*\mathcal{O}_Y)\end{aligned}$$

for all locally ringed locales Y over X. The crucial difference is that in general, the internally defined locally ringed locale $\operatorname{Spec}\mathcal{O}_X$ does *not* coincide with the internal locally ringed locale $(\mathrm{pt}, \mathcal{O}_X)$ (which is simply $(X, \mathcal{O}_X)$ from the external point of view). More succinctly, the functor $E \circ \operatorname{Spec}$ is an adjoint to the global sections functor $\mathrm{LRL}/E(\operatorname{Spec}\mathcal{O}_X) \to \mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}}$, while the relative spectrum functor is an adjoint to the global sections functor $\mathrm{LRL}/X \to \mathrm{Alg}(\mathcal{O}_X)^{\mathrm{op}}$. **XXX: check details?**

The internal construction which correctly reflects the relative spectrum is the following.

Definition 11.2. Let A be an R-algebra which is quasicoherent in the sense that the condition given in theorem 9.3 is fulfilled. The lattice of opens of its *quasicoherent spectrum* is given by

$$\operatorname{Op}(\operatorname{Spec}_R^{\mathrm{qcoh}} A) := \{\mathfrak{a} \subseteq A \mid \mathfrak{a} \text{ is a quasicoherent radical ideal}\},$$

where again “quasicoherent” is meant as in theorem 9.3. If this lattice fulfills the axioms on a locale, we further define a structure sheaf on a base by $\Gamma(\sqrt{(f)}, \mathcal{O}_{\operatorname{Spec}_R^{\mathrm{qcoh}} A}) := A[f^{-1}]$.

Proposition 11.3. *Let X be a scheme and $\mathcal{A}$ a quasicoherent $\mathcal{O}_X$-algebra. Then:*

1. *The internal lattice $\operatorname{Op}(\operatorname{Spec}_{\mathcal{O}_X}^{\mathrm{qcoh}}\mathcal{A})$ fulfills the axioms on an internal locale.*
2. *The externalization $E(\operatorname{Spec}_{\mathcal{O}_X}^{\mathrm{qcoh}}\mathcal{A})$ coincides with the relative spectrum $\operatorname{RelSpec}_{\mathcal{O}_X}\mathcal{A}$.*

Proof. We first show that the internalization $I(\operatorname{RelSpec}_{\mathcal{O}_X}\mathcal{A})$ coincides with the internal lattice $\operatorname{Op}(\operatorname{Spec}_{\mathcal{O}_X}^{\mathrm{qcoh}}\mathcal{A})$. Both of these objects are sheaves on X; so to show that they are canonically isomorphic, it suffices to give compatible canonical isomorphisms on the base of X consisting of all affine opens. On such an open $U = \operatorname{Spec} R$, the quasicoherent algebra $\mathcal{A}$ is given by a tilde construction: $\mathcal{A}|_U = S^\sim$ for some R-algebra S. Thus it holds that

$$\begin{aligned}\Gamma(U, I(\operatorname{RelSpec}_{\mathcal{O}_X} \mathcal{A})) &\cong \operatorname{Op}(\operatorname{RelSpec}_{\mathcal{O}_X}(\mathcal{A}) \times_X U) \\ &\cong \operatorname{Op}(\operatorname{RelSpec}_{\mathcal{O}_X|_U} \mathcal{A}|_U) \\ &\cong \operatorname{Op}(\operatorname{Spec} S) \\ &\cong \{\mathfrak{a} \subseteq S \mid \mathfrak{a} \text{ is a radical ideal}\}, \\ \Gamma(U, \operatorname{Op}(\operatorname{Spec}^{\mathrm{qcoh}}_{\mathcal{O}_X} \mathcal{A})) &\cong \{\mathcal{I} \hookrightarrow \mathcal{A}|_U \mid \mathcal{I} \text{ is a quasicoherent radical ideal sheaf}\}.\end{aligned}$$

Clearly, these lattices are isomorphic (by $\mathfrak{a} \mapsto \mathfrak{a}^\sim$ and $\mathcal{I} \mapsto \Gamma(U, \mathcal{I})$).

This is already enough to prove the claims: By a general lemma of locale theory, **XXX: give reference** the internal lattice $\operatorname{Op}(\operatorname{Spec}^{\mathrm{qcoh}}_{\mathcal{O}_X} \mathcal{A})$ defines an internal locale (as the pushforward under $\operatorname{RelSpec}_{\mathcal{O}_X} \mathcal{A} \to X$ of a locale); and its externalization coincides with $\operatorname{RelSpec}_{\mathcal{O}_X} \mathcal{A}$. □

XXX: cite Hakim and Gillam ("Localization of ringed spaces")

Proposition 11.4. *Let X be a scheme. Then $E(\operatorname{Spec} \mathcal{O}_X) \cong X$ as locales over X if and only if X is the empty scheme or X has dimension zero.*

Proof. The externalization of $\operatorname{Spec} \mathcal{O}_X$ coincides with X if and only if from the internal point of view, the locale $\operatorname{Spec} \mathcal{O}_X$ coincides with the one-point locale. By interpreting lemma 11.1 in the internal language of $\mathrm{Sh}(X)$, it follows that this is the case if and only if

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\, \ulcorner f \text{ nilpotent} \urcorner \vee \ulcorner f \text{ invertible} \urcorner.$$

(Internally, it always holds that $\neg(1 = 0)$ in $\mathcal{O}_X$, even if X happens to be the empty scheme. Therefore the lemma is indeed applicable.) By corollary 3.13, this condition is equivalent to the dimension of X being less than or equal to zero, i. e. to X being empty or having dimension exactly zero. □

Corollary 11.5. *Let X be a scheme. Then the relative spectrum of quasicoherent $\mathcal{O}_X$-algebras can be calculated by the internal spectrum (instead of the internal quasicoherent spectrum) if and only if X is empty or zero-dimensional.*

Proof. The externalization of the internal spectrum of arbitrary quasicoherent $\mathcal{O}_X$-algebras $\mathcal{A}$ coincides with the relative spectrum if and only if it coincides in the special case $\mathcal{A} = \mathcal{O}_X$. This is apparent by the universal properties of both constructions. Thus the claim follows by the previous proposition. □

12. Unsorted

- "functoriality"
- Kähler differentials
- closed and open subschemes
- $j_! \mathcal{O}_U$ flat over $\mathcal{O}_X$, ...
- Koszul resolution; Beilinson resolution?
- meta properties: some lemmas about limits of modules
- locally small categories
- big Zariski topos
- open/closed immersions
- morphisms of schemes...
- proper maps...

- limits and colimits...

Appendix A. Noetherianity

Recall the usual notion of a Noetherian ring: Any sequence $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots$ of ideals should stabilize, i. e. there should exist a natural number n such that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$.

Intuitionistically, this definition has two problems. Firstly, without the axiom of dependent choice, it is often not possible to construct a *sequence* of ideals: Often, it is only possible to show that there *exists* a suitable ideal $\mathfrak{a}_{n+1}$ depending on $\mathfrak{a}_n$. But since in general, this successor ideal is not unique, the axiom of dependent choice would be required to collect those into a sequence, i. e. a function from $\mathbb{N}$ to the set of ideals.

Secondly, the conclusion that the sequence stabilizes is too strong: Intuitionistically, one cannot even show that a weakly descending sequence of natural numbers stabilizes in this sense; the statement that one could is equivalent to the *limited principle of omniscience for* $\mathbb{N}$. Intuitionistically, it is only true that a weakly descending sequence $a_0 \geq a_1 \geq \cdots$ of natural numbers eventually *halts* in the sense that there exists an index n such that $a_n = a_{n+1}$ (but $a_{n+1} > a_{n+2}$ is allowed).

We therefore adopt the following definitions.

Definition A.1. Let M be a partially ordered set. An *ascending process* with values in M is a function $f : \mathbb{N} \to \mathcal{P}(M)$ such that the subset $f(0)$ is inhabited and such that for any $x \in f(n)$, $n \in \mathbb{N}$, there exists an element $y \in f(n+1)$ with $x \preceq y$. (In particular, all subsets $f(n)$ are inhabited.) Such a process *halts* if and only if there exists a step n such that there is an element $x \in f(n) \cap f(n+1)$. The set M satisfies the *ascending process condition* if and only if any ascending process with values in M halts.

Definition A.2. A ring R is *processly Noetherian* if and only if the set of finitely generated ideals in R satisfies the ascending process condition.

Any ascending chain of elements $a_0 \preceq a_1 \preceq \cdots$ in a partially ordered set gives rise to an ascending process by setting $f(n) := \{a_n\}$. Conversely, the axiom of dependent choice would allow to construct an ascending chain out of an ascending process. In classical logic, a ring is processly Noetherian if and only if it is Noetherian in the usual sense.

The notion of a processly Noetherian ring works well in an intuitionistic context: Important rings such as $\mathbb{Z}$ and more generally $\mathcal{O}_K$ for any algebraic number field K are processly Noetherian, and matrices over Bézout rings which are integral domains in the weak sense and processly Noetherian can be put into Smith canonical form.

Proposition A.3. *A scheme X is locally Noetherian if and only if the ring $\mathcal{O}_X$ is processly Noetherian from the internal point of view.*

Proof. We only prove the "only if" direction. We may assume that $X = \operatorname{Spec} A$ is affine with A a Noetherian ring and that internally, we are given an ascending process on the set of finitely generated ideals of $\mathcal{O}_X$. Externally, this is a morphism $\underline{\mathbb{N}} \to \mathcal{P}(\mathcal{M})$ where $\mathbb{N}$ is the constant sheaf with value $\mathbb{N}$ and U-sections of $\mathcal{M}$ are finite type ideal sheaves of $\mathcal{O}_X|_U$.

Since $X \models \ulcorner f(0)$ is inhabited$\urcorner$, there exists an open covering $X = \bigcup_i U_i$ and finite type ideal sheaves $\mathcal{I}_i \hookrightarrow \mathcal{O}_X|_{U_i}$ such that $U_i \models \mathcal{I}_i \in f(0)$. Without loss of generality, we may assume that the open sets U_i are standard open sets $D(f_i)$ and that the covering is finite. Since the sheaves $\mathcal{I}_i$ are of finite type, they correspond to ideals $J_i \subseteq A[f_i^{-1}]$. (Note for future reference that $\mathcal{I}_i|_{D(g)}$ corresponds to the extension of J_i in the further localized ring $A[g^{-1}]$, if $D(g) \subseteq D(f_i)$.)

For each i, $D(f_i) \models \exists \mathfrak{a} \in f(1).\ \mathcal{I}_i \subseteq \mathfrak{a}$. Thus there exists an open covering $D(f_i) = \bigcup_j D(g_{ij})$ and finite type ideal sheaves $\mathcal{I}_{ij} \hookrightarrow \mathcal{O}_X|_{D(g_{ij})}$; these correspond to ideals $J_{ij} \subseteq A[g_{ij}^{-1}]$ such that $J_i \subseteq J_{ij}$ (where we have suppressed the localization morphism $A[f_i^{-1}] \to A[g_{ij}^{-1}]$ in the notation). Equivalently, writing $J_i' := A \cap J_i$ and $J_{ij}' := A \cap J_{ij}$ for the contractions, we have the inclusions $J_i' \subseteq J_{ij}'$ of ideals of A.

Continuing in this fashion, we obtain a tree of ideals $J'_{i_1 \cdots i_n}$. Each path in this tree is a chain of ascending ideals and thus stabilizes since A is Noetherian. Since only finitely many subtrees branch off at each node, there appear only finitely many distinct ideals in this tree (this is an application of the graph-theoretical *König's lemma*).

There thus exists a natural number n such that $J'_{i_1 \cdots i_n} = J'_{i_1 \cdots i_n i_{n+1}}$ for all appropriate indices $i_1, \ldots, i_n, i_{n+1}$. For this number n, the internal statement $X \models \exists \mathfrak{a} \in f(n) \cap f(n+1)$ holds; we leave further details to the reader. $\square$

Remark A.4. Let X be a locally Noetherian scheme. The proof shows that even the set of all quasicoherent ideals (instead of merely the finitely generated ones) fulfills the ascending process condition. We have not taken this property as the definition of a processly Noetherian ring, since it is a notion not usually studied in constructive mathematics (compare remark

Remark A.5. rem:qcoh-in-constructive-mathematics).

XXX: Find appropriate place for the material in this appendix
XXX: Give examples made possible by the internal Noetherianity

Appendix B. Dictionary between internal and external notions

external	internal	
sheaves of sets		
sheaf of sets	set	
$\alpha : \mathcal{F} \to \mathcal{G}$ epimorphism	α surjective	example 2.3
$\alpha : \mathcal{F} \to \mathcal{G}$ monomorphism	α injective	example 2.3
$\operatorname{int}(X \setminus \operatorname{supp} \mathcal{F})$	largest open subset on which $\mathcal{F}$ is a singleton	remark 4.8
$f : X \to \mathbb{N}$ upper semicontinuous	element of $\widehat{\mathbb{N}}$	lemma 5.5
$f : X \to \mathbb{N}$ locally constant	element of $\mathbb{N}$	discussion after lemma 5.5
sheaves of rings		
sheaf of rings	ring	proposition 3.1
local sheaf of rings	local ring	proposition 3.5

$\mathcal{O}_X$ is reduced	$\mathcal{O}_X$ is reduced	proposition 3.3; a field property holds: $\neg$invertible $\Rightarrow$ zero
sheaves of modules		
sheaf of modules	module	
$\mathcal{F}$ is locally finitely free	$\mathcal{F}$ is finitely free	proposition 4.1
$\mathcal{F}$ is of finite type	$\mathcal{F}$ is finitely generated	proposition 4.2
$\mathcal{F}$ is of finite presentation	$\mathcal{F}$ is finitely presented	proposition 4.2
$\mathcal{F}$ is coherent	$\mathcal{F}$ is coherent	proposition 4.2
$\mathcal{F}$ is quasicoherent	$\mathcal{F}[f^{-1}]$ is a sheaf wrt. ($\ulcorner f$ inv.$\urcorner \Rightarrow$ __) for any $f : \mathcal{O}_X$	theorem 9.3
$\mathcal{F}$ is flat	$\mathcal{F}$ is flat	proposition 4.6
tensor product $\mathcal{F} \otimes \mathcal{G}$	tensor product $\mathcal{F} \otimes \mathcal{G}$	proposition 4.4
dual $\mathcal{F}^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$	dual $\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$	
$\mathrm{int}(X \setminus \mathrm{supp}\,\mathcal{F})$	largest open subset on which $\mathcal{F} = 0$	proposition 4.7
rank function of $\mathcal{F}$	minimal number of generators for $\mathcal{F}$	proposition 5.7
subspaces		
sheaf supported on closed subset A	$\Box$-sheaf, where $\Box = (\_\_ \vee A^c)$	lemma 10.3
sheaf of the form $j_*(\mathcal{F})$	$\Box$-sheaf, where $\Box = (U \Rightarrow \_\_)$	$j : U \hookrightarrow X$ open inclusion
extension of $\mathcal{F}$ by the empty set	$j_!(\mathcal{F}) = \{x : \mathcal{F} \mid U\}$	lemma 10.1
extension of $\mathcal{F}$ by zero	$j_!(\mathcal{F}) = \{x : \mathcal{F} \mid (x = 0) \vee U\}$	lemma 10.2
sheaf with empty/zero stalks on U^c	sheaf of the form $j_!(\mathcal{F})$	$j : U \hookrightarrow X$
sections of $\mathcal{F}$ are equal if they agree on dense open	$\mathcal{F}$ is $\neg\neg$-separated	proposition 6.12
sheaf of sections of $\mathcal{F}$ defined on dense open subsets	$\mathcal{F}^{++}$ with respect to $\Box = \neg\neg$	proposition 6.12, assuming $\mathcal{F}$ is $\neg\neg$-separated
$V(\mathcal{I})$ is a reduced scheme	$\mathcal{I}$ is a radical ideal	lemma 10.6
$\mathcal{O}_{X_{\mathrm{red}}}$	$\mathcal{O}_X/\sqrt{(0)}$	lemma 10.7
rational functions and Cartier divisors		
$\mathcal{K}_X$	total quotient ring of $\mathcal{O}_X$	proposition 7.1
Cartier divisor	element of $\mathcal{K}_X^*/\mathcal{O}_X^*$	
effective Cartier divisor	$[s/1]$ with $s : \mathcal{O}_X$ regular	definition 7.13
line bundle $\mathcal{O}_X(D)$	$D^{-1}\mathcal{O}_X \subseteq \mathcal{K}_X$	definition 7.15

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XXX: "completed natural number" is a misnomer.

XXX: remark that for simplicity, we work in a classical metatheory

XXX: check whether the negneg translation "finitely generated ==¿ free" implies the hard but important exercise

XXX: simplification rule "box(phi) =¿ box(psi) iff phi =¿ box(psi)".

XXX: consistently use "i" for closed embeddings and "j" for open embeddings

XXX: explain use of ":" even outside the internal language

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