USING THE INTERNAL LANGUAGE OF TOPOSES IN ALGEBRAIC GEOMETRY

INGO BLECHSCHMIDT

ABSTRACT. There are several important topoi associated to a scheme, for instance the petit and gros Zariski topoi. These support an internal mathematical language which closely resembles the usual formal language of mathematics, but is "local on the base scheme":

For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a certain condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

These expository notes give an introduction to this topic and show how the internal point of view can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry. No prior knowledge about topos theory and formal logic is assumed.

CONTENTS

1. Introduction

A *topos* is a category which shares certain categorical properties with the category of sets; the archetypical example is the the category of sets, and the most important example for the purposes of these notes is the category of set-valued sheaves on a topological space.

Any topos $\mathcal{E}$ supports an *internal language*. This is a device which allows one to *pretend* that the objects of $\mathcal{E}$ are plain sets and that the morphisms are plain maps between sets, even if in fact they are not. For instance, consider a morphism $\alpha : X \to Y$ in $\mathcal{E}$. From the *internal point of view*, this looks like a map between sets, and we can formulate the condition that this map is surjective; we write this as

$$\mathcal{E} \models \forall y : Y.\ \exists x : X.\ \alpha(x) = y.$$

The appearance of the colons instead of the usual element signs reminds us that this expression is not to be taken literally – X and Y are objects of $\mathcal{E}$ and thus not necessarily sets. The definition of the internal language is made in such a way so that the meaning of this internal statement is that α is an epimorphism. Similarly, the translation of the internal statement that α is injective is that α is a monomorphism.

Furthermore, we can *reason* with the internal language. There is a metatheorem to the effect that if some statement φ holds from the internal point of view of a topos $\mathcal{E}$ and if φ logically implies some further statement ψ, then ψ holds in $\mathcal{E}$ as well. As a simple example, consider the elementary fact that the composition of surjective maps is surjective. Interpreting this statement in the internal language of $\mathcal{E}$, we obtain the more abstract result that the composition of epimorphisms in $\mathcal{E}$ is epic.

There is, however, a slight caveat to this metatheorem. Namely, the internal language of a topos is in general only *intuitionistic*, not *classical*. This means that internally, one can not use the law of excluded middle ($\varphi \vee \neg\varphi$), nor the law of double negation elimination ($\neg\neg\varphi \Rightarrow \varphi$), nor the axiom of choice. For instance, one rendition of the axiom of choice is that any surjection splits. But it need not be the case that an epimorphism in a topos splits.

We apply this internal language to algebraic geometry as follows. If X is a scheme, the structure sheaf $\mathcal{O}_X$ is a sheaf of rings, i. e. the sets of local sections carry ring structures and these ring structures are compatible with restriction. From the internal point of view of the topos of set-valued sheaves on X, $\mathcal{O}_X$ looks much simpler: It looks just like a plain ring (and not a sheaf of rings). Similarly, a sheaf of $\mathcal{O}_X$-modules looks just like a plain module over that ring.

This allows to import notions and facts from basic linear and commutative algebra into the sheaf setting. For instance, it turns out that a sheaf of $\mathcal{O}_X$-modules is of finite type if and only if, from the internal perspective, it is finitely generated as an $\mathcal{O}_X$-module. Now consider the following fact of linear algebra: If in a short exact sequence of modules the two outer ones are finitely generated, then the middle one is too. The usual proof of this fact is intuitionistically acceptable and can thus be interpreted in the internal language. It then *automatically* yields the following more advanced proposition: If in a short exact sequence of sheaves of $\mathcal{O}_X$-modules the two outer ones are of finite type, then the middle one is too.

The internal language machinery thus allows us to understand the basic notions and statements of scheme theory as notions and statements of linear and commutative algebra, interpreted in a suitable sheaf topos. This brings conceptual clarity and reduces technical overhead.

In these notes, we explain how the internal language works and then develop a *dictionary* between common notions of scheme theory and corresponding notions of algebra. Once built, this dictionary can be used arbitrarily often.

Two highlights of our approach are the following. Let X be a reduced scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules of finite type. Then it is well-known that $\mathcal{F}$ is locally free on some dense open subset of X; for instance, this is stated in Vakil's lecture notes as an "important hard exercise" [12, exercise 13.7.K]. In fact, this fact is just the interpretation of the following statement of intuitionistic linear algebra in the sheaf topos: Any finitely generated vector space is *not not* free. The proof of this statement is entirely straightforward.[1]

The second highlight is that we can shed light on the phenomenon that sometimes, truth of a property at a point x spreads to some open neighbourhood of x; and in particular that sometimes, truth of a property at the generic point spreads to some dense open subset. For instance, if the stalk of a sheaf of finite type is zero at some point, the sheaf is even zero on some open neighbourhood; but this spreading does not occur for general sheaves which may fail to be of finite type.

We formalize this by introducing a *modal operator* $\Box$ into the internal language, such that the internal statement $\Box\varphi$ means that φ holds on some open neighbourhood of x. Furthermore, we introduce a simple operation on formulas, the $\Box$-*translation* $\varphi \mapsto \varphi^\Box$, such that $\varphi^\Box$ means that φ holds at the point x. The question whether truth at x spreads to truth on a neighbourhood can thus be formulated in the following way: Does $\varphi^\Box$ intuitionistically imply $\Box\varphi$?

This allows to deal with the question in a simpler, more logical way, with the technicalities of sheaves blinded out. We can also give a metatheorem which covers a wide range of cases. Namely, spreading occurs for all those properties which can be formulated in the internal language without using "$\Rightarrow$", "$\forall$", and "$\neg$".

To illustrate the example above, consider that the property of a module $\mathcal{F}$ being the zero module is formulated as $(\forall x : \mathcal{F}.\ x = 0)$ in the internal language. Because of the appearance of "$\forall$", the metatheorem is not applicable to this statement. But if $\mathcal{F}$

[1]Intuitionistically, the statement that any finitely generated vector space is free is stronger than the doubly negated version and can not be shown. It would imply that any sheaf of finite type is not only locally free on some dense open subset, but locally free on the whole space.

is of finite type, there are generators $x_1, \ldots, x_n : \mathcal{F}$ from the internal point of view, and the condition can be reformulated as $x_1 = 0 \wedge \cdots \wedge x_n = 0$; the metatheorem is applicable to this statement.

- dictionary; intuitionistic logic; microscope/telescope into another universe; types instead of sets; (dependent types to encompass almost all mathematics)
- explain that with the internal language business, it becomes more transparent where scheme condition enters
- discuss limitations (here or somewhere else)
- note that in-depth knowledge of formal logic or topos theory is not necessary for applications
- give pointers to introductory literature

2. The internal language of a sheaf topos

2.1. **Internal statements.** Let X be a topological space. Later, X will be the underlying space of a scheme. The meaning of internal statements is given by a set of rules, the *Kripke–Joyal semantics* of the topos of sheaves on X.

Definition 2.1. The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ over U is given by the rules listed in table 1, recursively in the structure of φ. In a *formula over U* there may appear sheaves defined on U as domains of quantifications, U-sections of sheaves as terms and morphisms of sheaves on U as function symbols. The symbols “$\top$” and “$\bot$” denote truth and falsehold, respectively. The universal and existential quantifiers come in two flavors: for bounded and unbounded quantification. The translation of $U \models \neg\varphi$ does not have to be defined, since negation can be expressed using other symbols: $\neg\varphi :\equiv (\varphi \Rightarrow \bot)$. If we want to emphasize the particular topos, we write

$$\mathrm{Sh}(X) \models \varphi \quad :\Longleftrightarrow \quad X \models \varphi.$$

Remark 2.2. The last two rules in table 1, concerning *unbounded quantification*, and are not part of the classical Kripke–Joyal semantics, but instead of Mike Shulman's stack semantics [9], a slight extension. They are needed so that we can formulate universal properties in the internal language.

Example 2.3. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is a monomorphism of sheaves if and only if, from the internal perspective, α is simply an injective map:

$$\begin{aligned}
& X \models \ulcorner \alpha \text{ is injective} \urcorner \\
\Longleftrightarrow\; & X \models \forall s : \mathcal{F}.\ \forall t : \mathcal{F}.\ \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\; & \text{for all open } U \subseteq X, \text{ sections } s \in \Gamma(U, \mathcal{F}): \\
& \text{for all open } V \subseteq U, \text{ sections } t \in \Gamma(V, \mathcal{F}): \\
& \qquad V \models \alpha(s) = \alpha(t) \Rightarrow s = t \\
\Longleftrightarrow\; & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F}): \\
& \qquad U \models \alpha(s) = \alpha(t) \Rightarrow s = t
\end{aligned}$$

$U \models s = t : \mathcal{F}$	$:\Longleftrightarrow$	$s\vert_U = t\vert_U \in \Gamma(U, \mathcal{F})$
$U \models s \in \mathcal{G}$	$:\Longleftrightarrow$	$s\vert_U \in \Gamma(U, \mathcal{G})$ ($\mathcal{G}$ a subsheaf of $\mathcal{F}$, s a section of $\mathcal{F}$)
$U \models \top$	$:\Longleftrightarrow$	$U = U$ (always fulfilled)
$U \models \bot$	$:\Longleftrightarrow$	$U = \emptyset$
$U \models \varphi \wedge \psi$	$:\Longleftrightarrow$	$U \models \varphi$ and $U \models \psi$
$U \models \bigwedge_{j \in J} \varphi_j$	$:\Longleftrightarrow$	for all $j \in J$: $U \models \varphi_j$ (J an index set)
$U \models \varphi \vee \psi$	$:\Longleftrightarrow$	~~$U \models \varphi$ or $U \models \psi$~~
		there exists a covering $U = \bigcup_i U_i$ such that for all i:
		$U_i \models \varphi$ or $U_i \models \psi$
$U \models \bigvee_{j \in J} \varphi_j$	$:\Longleftrightarrow$	~~$U \models \varphi_j$ for some $j \in J$~~ (J an index set)
		there exists a covering $U = \bigcup_i U_i$ such that for all i:
		$U_i \models \varphi_j$ for some $j \in J$
$U \models \varphi \Rightarrow \psi$	$:\Longleftrightarrow$	for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$
$U \models \forall s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	for all sections $s \in \Gamma(V, \mathcal{F})$, open $V \subseteq U$: $V \models \varphi(s)$
$U \models \exists s : \mathcal{F}.\ \varphi(s)$	$:\Longleftrightarrow$	~~there exists a section $s \in \Gamma(U, \mathcal{F})$ such that $U \models \varphi(s)$~~
		there exists an open covering $U = \bigcup_i U_i$ such that for all i:
		there exists $s_i \in \Gamma(U_i, \mathcal{F})$ such that $U_i \models \varphi(s_i)$
$U \models \forall \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	for all sheaves $\mathcal{F}$ on V, open $V \subseteq U$: $V \models \varphi(\mathcal{F})$
$U \models \exists \mathcal{F}.\ \varphi(\mathcal{F})$	$:\Longleftrightarrow$	there exists an open covering $U = \bigcup_i U_i$ such that for all i:
		there exists a sheaf $\mathcal{F}_i$ on U_i such that $U_i \models \varphi(\mathcal{F}_i)$

TABLE 1. The Kripke–Joyal semantics of a sheaf topos.

$$
\begin{aligned}
&\Longleftrightarrow \text{ for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F}): \\
&\qquad\qquad \text{for all open } W \subseteq U: \\
&\qquad\qquad\qquad\qquad \alpha_W(s|_W) = \alpha_W(t|_W) \text{ implies } s|_W = t|_W \\
&\Longleftrightarrow \text{ for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F}): \\
&\qquad\qquad \alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U \\
&\Longleftrightarrow \alpha \text{ is a monomorphism of sheaves}
\end{aligned}
$$

The corner quotes "$\ulcorner \ldots \urcorner$" indicate that translation into formal language is left to the reader. Similarly, α is an epimorphism of sheaves if and only if, from the internal perspective, α is a surjective map. Notice that injectivity and surjectivity are notions of a simple element-based language, and the Kripke–Joyal semantics takes care to properly handle *all* sections, not only global ones.

The rules are not all arbitrary. They are finely concerted to make the following propositions true, which are crucial for a proper appreciation of the internal language.

Proposition 2.4 (Locality of the internal language)**.** *Let $U = \bigcup_i U_i$ be covered by open subsets. Let φ be a formula over U. Then*

$$U \models \varphi \qquad \text{iff} \qquad U_i \models \varphi \text{ for each } i.$$

Proof. Induction on the structure of φ. Note that the canceled rules would make this proposition false. □

As a corollary, one may restrict the open coverings and universal quantifications in the the definition of the Kripke–Joyal semantics (table 1) to open subsets of some basis of the topology. For instance, if X is a scheme, one may restrict to affine open subsets.

Proposition 2.5 (Soundness of the internal language)**.** *If a formula φ implies a further formula ψ in intuitionistic logic, then $U \models \varphi$ implies $U \models \psi$.*

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that on each U_i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in [3, p. D1.3.1]. □

Because of the multitude of quantifiers, literal translations of internal statements can sometimes get slightly unwieldy. There are simplification rules for certain often-occuring special cases:

Proposition 2.6.

$$U \models \forall s\,{:}\,\mathcal{F}.\ \forall t\,{:}\,\mathcal{G}.\ \varphi(s,t) \iff \textit{for all open } V \subseteq U, \textit{ sections } s \in \Gamma(V,\mathcal{F}),\ t \in \Gamma(V,\mathcal{G}){:}\ V \models \varphi(s,t)$$

$$U \models \forall s\,{:}\,\mathcal{F}.\ \varphi(s) \Rightarrow \psi(s) \iff \textit{for all open } V \subseteq U, \textit{ sections } s \in \Gamma(V,\mathcal{F}){:}\ V \models \varphi(s) \textit{ implies } V \models \psi(s)$$

$$U \models \exists! s\,{:}\,\mathcal{F}.\ \varphi(s) \iff \textit{for all open } V \subseteq U, \textit{ there is exactly one section } s \in \Gamma(V,\mathcal{F}) \textit{ with: } V \models \varphi(s)$$

Proof. Straightforward. By way of example, we prove the existence claim in the "only if" direction of the last rule. (Note that this rule formalizes the saying "unique existence is global existence".) By definition of $\exists!$, it holds that

$$U \models \exists s\,{:}\,\mathcal{F}.\ \varphi(s)$$

and

$$U \models \forall s,t\,{:}\,\mathcal{F}.\ \varphi(s) \wedge \varphi(t) \Rightarrow s = t.$$

Let $V \subseteq U$ be an arbitrary open subset. Then there exist local section $s_i \in \Gamma(V_i,\mathcal{F})$ such that $V_i \models \varphi(s_i)$, where $V = \bigcup_i V_i$ is an open covering. By the locality of the internal language, on intersections it holds that $V_i \cap V_j \models \varphi(s_i)$, so by the uniqueness assumption, it follows that the local sections agree on intersections. They therefore glue to a section $s \in \Gamma(V, \mathcal{F})$. Since $V_i \models \varphi(s)$ for any i, the locality of the internal language allows us to conclude that $V \models \varphi(s)$. $\square$

Remark 2.7. Note that $\mathrm{Sh}(X) \models \neg\varphi$ is in general a much stronger statement that merely supposing that $\mathrm{Sh}(X) \models \varphi$ does not hold: The former always implies the latter (unless $X = \emptyset$, in which case *any* internal statement is true), but the converse does not hold: The former statement means that $U = \emptyset$ is the *only* open subset on which φ holds.

2.2. **Internal constructions.** The Kripke–Joyal semantics defines the interpretation of internal statements. The interpretation of internal constructions is given by the following definition.

Definition 2.8. The interpretation of an internal construction T is denoted by $[\![T]\!] \in \mathrm{Sh}(X)$ and given by the following rules.

- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \times \mathcal{G}]\!]$ is the categorical product of $\mathcal{F}$ and $\mathcal{G}$ (i. e. their product as presheaves).
- If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $[\![\mathcal{F} \amalg \mathcal{G}]\!]$ is the categorical coproduct of $\mathcal{F}$ and $\mathcal{G}$, i. e. the sheafification of the presheaf $U \mapsto \Gamma(U, \mathcal{F}) \amalg \Gamma(U, \mathcal{G})$.
- If $\mathcal{F}$ is a sheaf, the interpretation $[\![\mathcal{P}(\mathcal{F})]\!]$ of the power set construction is the sheaf given by
$$U \subseteq X \text{ open} \longmapsto \{\mathcal{G} \hookrightarrow \mathcal{F}|_U\},$$
i. e. sections on an open set U are subsheaves of $\mathcal{F}|_U$ (either literally or isomorphism classes of general monomorphisms into $\mathcal{F}|_U$).
- If $\mathcal{F}$ is a sheaf and $\varphi(s)$ is a formula containing a free variable $s : \mathcal{F}$, the interpretation $[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]$ is given by the subsheaf of $\mathcal{F}$ defined by
$$U \subseteq X \text{ open} \longmapsto \{s \in \Gamma(U, \mathcal{F}) \mid U \models \varphi(s)\}.$$

The definition is made in such a way that, from the internal perspective, the constructions enjoy their expected properties. For instance, it holds that
$$\mathrm{Sh}(X) \models \big[\forall x : [\![\{s : \mathcal{F} \mid \varphi(s)\}]\!].\ \psi(x)\big] \iff \big[\forall x : \mathcal{F}.\ \varphi(x) \Rightarrow \psi(x)\big].$$
We gloss over several details here. See [3, ???] for a proper treatment.

2.3. **Geometric formulas and constructions.** In categorical logic, so-called geometric formulas play a special role, because their meaning is preserved under pullback with geometric morphisms.

Definition 2.9. A formula is *geometric* if and only if it consists only of
$$= \quad \in \quad \top \quad \bot \quad \wedge \quad \vee \quad \bigvee \quad \exists,$$
but not "$\bigwedge$" nor "$\Rightarrow$" nor "$\forall$" (and thus not "$\neg$" either, since this is defined using "$\Rightarrow$"). A *geometric implication* is a formula of the form
$$\forall \cdots \forall.\ (\cdots) \Rightarrow (\cdots)$$
with the bracketed subformulas being geometric.

We say that a formula φ holds *at a point* $x \in X$ if and only if the formula obtained by substituting all parameters in φ (sheaves being quantified over, sections of sheaves appearing as terms and morphisms of sheaves appearing as function symbols) with their stalks at x holds in the usual mathematical sense.

Lemma 2.10. *Let $x \in X$ be a point. Let φ be a geometric formula (over some open neighbourhood of x). Then φ holds at x if and only if there exists an open neighbourhood $U \subseteq X$ of x such that φ holds on U.*

Proof. This is a very general instance of the phenomenom that sometimes, truth at a point spreads to truth on a neighbourhood. It can be proven by induction on the structure of φ, but we will give a more conceptual proof later (corollary 8.22). □

This lemma is in fact a very useful metatheorem. We will properly discuss its significance in section 8.7. For now, we just use it to prove a simple criterion for the internal truth of a geometric implication; we will apply this criterion many times.

Corollary 2.11. *A geometric implication holds on X if and only if it holds at every point of X.*

Proof. For notational simplicity, we consider a geometric implication of the form

$$\forall s : \mathcal{F}.\ \varphi(s) \Rightarrow \psi(s).$$

For the “only if” direction, assume that this formula holds on X and let $x \in X$ be an arbitrary point. Let $s_x \in \mathcal{F}_x$ be the germ of an arbitrary local section s of $\mathcal{F}$ and assume that $\varphi(s)$ holds at x. Then by the lemma, it follows that $\varphi(s)$ holds on some open neighbourhood of x. By assumption, $\psi(s)$ holds on this neighbourhood as well. Again by the lemma, $\psi(s)$ holds at x.

For the “if” direction, assume that the geometric implication holds at every point. Let $U \subseteq X$ be an arbitrary open subset and let $s \in \Gamma(U, \mathcal{F})$ be a local section such that $\varphi(s)$ holds on U. By the lemma and the locality of the internal language, to show that $\psi(s)$ holds on U, it suffices to show that $\psi(s)$ holds at every point of U. This is clear, since again by the lemma, $\varphi(s)$ holds at every point of U. □

Example 2.12. Injectivity and surjectivity are geometric implications (surjectivity can be spelled $\forall y : \mathcal{G}.\ \top \Rightarrow \exists x : \mathcal{F}.\ \alpha(x) = y$). Thus the corollary gives a deeper reason for the well-known fact that a morphism of sheaves is a monomorphism resp. an epimorphism if and only if it is stalkwise injective resp. surjective.

A construction is *geometric* if and only if it commutes with pullback under arbitrary geometric morphisms. We do not want to discuss the notion of geometric morphisms here; suffice it to say that calculating the stalk at a point $x \in X$ is an instance of such a pullback. Among others, the following constructions are geometric:

- finite product: $(\mathcal{F} \times \mathcal{G})_x \cong \mathcal{F}_x \times \mathcal{G}_x$
- finite coproduct: $(\mathcal{F} \amalg \mathcal{G})_x \cong \mathcal{F}_x \amalg \mathcal{G}_x$
- arbitrary coproduct: $(\coprod_i \mathcal{F}_i)_x \cong \coprod_i (\mathcal{F}_i)_x$
- set comprehension with respect to a *geometric* formula φ:
$$[\![\{s : \mathcal{F} \mid \varphi(s)\}]\!]_x \cong \{[s] \in \mathcal{F}_x \mid \varphi(s) \text{ holds at } x\}$$
- free module: $(\mathcal{R}\langle\mathcal{F}\rangle)_x \cong \mathcal{R}_x\langle\mathcal{F}_x\rangle$ ($\mathcal{R}$ a sheaf of rings, $\mathcal{F}$ a sheaf of sets)
- localization of a module: $\mathcal{F}[\mathcal{S}^{-1}]_x \cong \mathcal{F}_x[\mathcal{S}_x^{-1}]$

The following constructions are not in general geometric:

- arbitrary product
- set comprehension with respect to a non-geometric formula
- powerset
- internal Hom: $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \ncong \mathrm{Hom}(\mathcal{F}_x, \mathcal{G}_x)$

- crash course on intuitionistic logic

3. Sheaves of rings

Recall that a *sheaf of rings* can be categorically described as a sheaf of sets $\mathcal{R}$ together with maps of sheaves $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and global elements $0, 1$ such that certain axioms hold. For instance, the axiom on the commutativity of addition is rendered in diagrammatic form as follows:

$$\begin{tikzcd} \mathcal{R} \times \mathcal{R} \arrow[rr, "\mathrm{swap}"] \arrow[dr, "+"'] & & \mathcal{R} \times \mathcal{R} \arrow[dl, "+"] \\ & \mathcal{R} & \end{tikzcd}$$

From the internal perspective, a sheaf of rings looks just like a plain ring. This is the content of the following proposition:

Proposition 3.1. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of sets on X. Let $+, \cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be maps of sheaves and let $0, 1$ be global elements of $\mathcal{R}$. Then these data define a sheaf of rings if and only if, from the internal perspective, these data fulfill the usual equational ring axioms.*

Proof. We only discuss the commutativity axiom. The internal statement

$$\mathrm{Sh}(X) \models \forall x, y : \mathcal{R}.\ x + y = y + x$$

means that for any open subset $U \subseteq X$ and any local sections $x, y \in \Gamma(U, \mathcal{R})$, it holds that $x + y = y + x \in \Gamma(U, \mathcal{R})$. This is precisely the external commutativity condition. □

Lemma 3.2. *Let X be a topological space. Let $\mathcal{R}$ be a sheaf of rings on X. Let f be a global section of $\mathcal{R}$. Then the following statements are equivalent:*

1. *f is invertible from the internal point of view, i. e.* $\mathrm{Sh}(X) \models \exists g : \mathcal{R}.\ fg = 1$.
2. *f is is invertible in all stalks $\mathcal{R}_x$.*
3. *f is in invertible in $\Gamma(X, \mathcal{R})$.*

Proof. Since invertibility is a geometric implication, the equivalence of the first two statements is clear. Also, it's obvious that the third statement implies the other two. For the remaining direction, note that the uniqueness of inverses in rings can be proven intuitionistically. Therefore, if f is invertible from the internal point of view, it actually holds that

$$\mathrm{Sh}(X) \models \exists! g : \mathcal{R}.\ fg = 1.$$

Since unique internal existence implies global existence (proposition 2.6), this shows that the first statement implies the third. □

3.1. **Reducedness.** Recall that a scheme X is *reduced* if and only if all stalks $\mathcal{O}_{X,x}$ are reduced rings. Since the condition on a ring R to be reduced is a geometric implication,

$$\forall s : R.\ s^2 = 0 \Longrightarrow s = 0,$$

we immediately obtain the following characterization of reducedness in the internal language:

Proposition 3.3. *A scheme X is reduced iff, from the internal point of view, the ring $\mathcal{O}_X$ is reduced.*

3.2. **Locality.** Recall the usual definition of a local ring: a ring possessing exactly one maximal ideal. This is a higher-order condition and in particular not of a geometric form. Therefore, for our purposes, it's better to adopt the following elementary definition of a local ring.

Definition 3.4. A *local ring* is a ring R such that $1 \neq 0$ in R and for all $x, y \in R$

$$x + y \text{ invertible} \quad \Longrightarrow \quad x \text{ invertible} \ \vee \ y \text{ invertible}.$$

In classical logic, it's an easy exercise to show that this definition is equivalent to the usual one. In intuitionistic logic, we would need to be more precise in order to even state the question of equivalence, since intuitionistically, the notion of a maximal ideal bifurcates into several non-equivalent notions.

Proposition 3.5. *In the internal language of a scheme X (or a locally ringed space), the ring $\mathcal{O}_X$ is a local ring.*

Proof. The stated locality condition is a conjunction of two geometric implications (the first one being $1 = 0 \Rightarrow \bot$, the second being the displayed one) and holds on each stalk. □

3.3. **Field properties.** From the internal point of view, the structure sheaf $\mathcal{O}_X$ of a scheme X is *almost* a field, in the sense that any element which is not invertible is nilpotent. This is a genuine property of schemes, not shared with general locally ringed spaces.

Proposition 3.6. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall s \colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Rightarrow \ulcorner s \text{ nilpotent} \urcorner.$$

Proof. By the locality of the internal language and since X can be covered by open affine subsets, it's enough to show that for any affine scheme $X = \operatorname{Spec} A$ and global function $s \in \Gamma(X, \mathcal{O}_X) = A$ it holds that

$$X \models \neg(\ulcorner s \text{ invertible} \urcorner) \quad \text{implies} \quad X \models \ulcorner s \text{ nilpotent} \urcorner.$$

The meaning of the antecedent is that any open subset on which s is invertible is empty. So in particular, the standard open subset $D(s)$ is empty. Therefore s is an element of any prime ideal of A and thus nilpotent. This implies the a priori weaker statement $X \models \ulcorner s \text{ nilpotent} \urcorner$ (which would allow s to have different indices of nilpotency on an open covering). □

Corollary 3.7. *Let X be a scheme. If X is reduced, the ring $\mathcal{O}_X$ is a field from the internal point of view, in the sense that*

$$\mathrm{Sh}(X) \models \forall s \colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Rightarrow s = 0.$$

The converse holds as well.

Proof. We can prove this purely in the internal language: It suffices to give an intuitionistic proof of the fact that a local ring which satisfies the condition of the previous proposition fulfills the stated field condition if and only if it is reduced. This is straightforward. □

This field property is very useful. We will put it to good use when giving a simple proof of the fact that $\mathcal{O}_X$-modules of finite type on a reduced scheme are locally free on a dense open subset (proposition **??**).

The following proposition says that one can deduce a certain unconditional statement from the premise that an element $s : \mathcal{O}_X$ is zero under the assumption that some further element $f : \mathcal{O}_X$ is invertible.

Proposition 3.8. *Let X be a scheme. Then*

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \forall s : \mathcal{O}_X.\ (\ulcorner f \text{ inv.} \urcorner \Rightarrow s = 0) \Longrightarrow \bigvee_{n \geq 0} f^n s = 0.$$

Proof. It's enough to show that for any affine scheme $X = \operatorname{Spec} A$ and global functions $f, s \in A$ such that

$$X \models (\ulcorner f \text{ inv.} \urcorner \Rightarrow s = 0),$$

it holds that $X \models \bigvee_{n \geq 0} f^n s = 0$. This is obvious, since by assumption such a function s is zero on $D(f)$, i. e. s is zero as an element of $A[f^{-1}]$. □

- Remark that intuitionistically, the notion of a field bifurcates into several inequivalent notions
- normal rings, principal ideal domains, ...
- discreteness

4. Sheaves of modules

From the internal perspective, a sheaf of $\mathcal{R}$-modules, where $\mathcal{R}$ is a sheaf of rings, looks just like a plain module over the plain ring $\mathcal{R}$. This is proven just as the correspondence between sheaf of rings and internal rings (proposition 3.1).

4.1. **Local finite freeness.** Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally finitely free* if and only if there exists a covering of X by open subsets U such that on each such U, the restricted module $\mathcal{F}|_U$ is isomorphic as an $\mathcal{O}_X|_U$-module to $(\mathcal{O}_X|_U)^n$ for some natural number n (which may depend on U).

Proposition 4.1. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is locally finitely free if and only if, from the internal perspective, $\mathcal{F}$ is a finitely free module, i. e.*

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \ulcorner \mathcal{F} \cong (\mathcal{O}_X)^n \urcorner$$

or more elementary

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_i a_i x_i.$$

Proof. By the expression "$(\mathcal{O}_X)^n$" in the internal language we mean the internally constructed object $\mathcal{O}_X \times \cdots \times \mathcal{O}_X$ with its pointwise $\mathcal{O}_X$-module structure. This coincides with the sheaf $(\mathcal{O}_X)^n$ as usually understood.

It is clear that the two stated internal conditions are equivalent, since the corresponding proof in linear algebra is intuitionistic. The equivalence with the external notion of locally finite freeness is obvious, since the interpretation of the first condition with the Kripke–Joyal semantics is the following: There exists a covering of X by open subsets U such that for each such U, there exists a natural number n and a morphism of sheaves $\varphi : \mathcal{F}|_U \to (\mathcal{O}_X|_U)^n$ such that

$$U \models \ulcorner \varphi \text{ is } \mathcal{O}_X\text{-linear} \urcorner \quad \text{and} \quad U \models \ulcorner \varphi \text{ is bijective} \urcorner.$$

The first subcondition means that φ is a morphism of sheaves of $\mathcal{O}_X$-modules and the second one means that φ is an isomorphism of sheaves. □

4.2. **Finite type, finite presentation, coherence.** Recall the conditions of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or ringed space) to be of finite type, of finite presentation and to be coherent:

- $\mathcal{F}$ is *of finite type* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence
$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$
of $\mathcal{O}_X|_U$-modules.
- $\mathcal{F}$ is *of finite presentation* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence
$$(\mathcal{O}_X|_U)^m \longrightarrow (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$
- $\mathcal{F}$ is *coherent* if and only if $\mathcal{F}$ is of finite type and the kernel of any $\mathcal{O}_X|_U$-linear morphism $(\mathcal{O}_X|_U)^n \to \mathcal{F}|_U$, $U \subseteq X$ any open subset, is of finite type.

The following proposition gives translations of these definitions into the internal language.

Proposition 4.2. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then:*

- *$\mathcal{F}$ is of finite type if and only if $\mathcal{F}$, considered as an ordinary module from the internal perspective, is finitely generated, i. e. if*
$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{F}.\ x = \sum_i a_i x_i.$$
- *$\mathcal{F}$ is of finite presentation if and only if $\mathcal{F}$ is a finitely presented module from the internal perspective, i. e. if*
$$\mathrm{Sh}(X) \models \bigvee_{n,m \geq 0} \ulcorner\text{there is a short exact sequence } \mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathcal{F} \to 0\urcorner.$$
- *$\mathcal{F}$ is coherent if and only if*
$$\mathrm{Sh}(X) \models \ulcorner\mathcal{F} \text{ is finitely generated}\urcorner \wedge \bigwedge_{n \geq 0} \forall \varphi : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \ulcorner\ker\varphi \text{ is finitely generated}\urcorner.$$

Proof. Straightforward: The translations of the internal statements using the Kripke–Joyal semantics are precisely the corresponding external statements. □

4.3. **Tensor product.** Recall that the tensor product of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X (or ringed space) is usually constructed as the sheafification of the presheaf
$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$
From the internal point of view, $\mathcal{F}$ and $\mathcal{G}$ look like ordinary modules, so that we can consider their tensor product as usually constructed in commutative algebra, as a certain quotient of a free module on the elements of $\mathcal{F} \times \mathcal{G}$:
$$\mathcal{O}_X\langle x \otimes y \,|\, x : \mathcal{F}, y : \mathcal{G}\rangle / R,$$

where R is the submodule generated by

$$\begin{gathered}(x+x')\otimes y - x\otimes y - x'\otimes y,\\ x\otimes(y+y') - x\otimes y - x\otimes y',\\ (sx)\otimes y - s(x\otimes y),\\ x\otimes(sy) - s(x\otimes y)\end{gathered}$$

with $x, x' : \mathcal{F}$, $y, y' : \mathcal{G}$, $s : \mathcal{O}_X$. This internal construction will give rise to the same sheaf of modules as the externally defined tensor product:

Proposition 4.3. *Let X be scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the internally constructed tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ coincides with the external one.*

Proof. Since the proof of the corresponding fact of commutative algebra is intuitionistic, the internally defined tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ fulfills the following universal property: For any $\mathcal{O}_X$-module $\mathcal{H}$, any $\mathcal{O}_X$-bilinear map $\mathcal{F} \times \mathcal{G} \to \mathcal{H}$ uniquely factors over the canonical morphism $\mathcal{F} \times \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

Interpreting this property with the Kripke–Joyal semantics, we see that the internally constructed tensor product fulfills the following external property: For any open subset $U \subseteq X$ and any $\mathcal{O}_X|_U$-module $\mathcal{H}$ on U, any $\mathcal{O}_X|_U$-bilinear map $\mathcal{F}|_U \times \mathcal{G}|_U \to \mathcal{H}$ uniquely factors over the canonical morphism $\mathcal{F} \times \mathcal{G} \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U$.

In particular, for $U = X$, this property is well-known to be the universal property satisfied by the externally constructed tensor product. Therefore the claim follows. ☐

By the internal construction, a description of the stalks of the tensor product follows purely by considering the logical form of the construction:

Corollary 4.4. *Let X be scheme (or a ringed space). Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then the stalks of the tensor product coincide with the tensor products of the stalks: $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x \cong \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$.*

Proof. We constructed the tensor product using the following operations: product of two sets, free module on a set, quotient module with respect to a submodule; submodule generated by a set of elements given by a geometric formula. All of these operations are geometric, so the tensor product construction is geometric as well. Hence taking stalks commutes with performing the construction. ☐

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *flat* if and only if all stalks $\mathcal{F}_x$ are flat $\mathcal{O}_{X,x}$-modules. We can characterize flatness in the internal language.

Proposition 4.5. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is flat if and only if, from the internal perspective, $\mathcal{F}$ is a flat $\mathcal{O}_X$-module.*

Proof. Recall that flatness of an A-module M can be characterized without reference to tensor products by the following condition (using suggestive vector notation): For any natural number p, any p-tuple $m : M^p$ of elements of M and any p-tuple $a : A^p$ of elements of A,

$$a^T m = 0 \implies \bigvee_{q \geq 0} \exists n : M^q, B : A^{p \times q}.\ Bn = m \wedge a^T B = 0.$$

This formulation of flatness has the advantage that it is the conjunction of geometric implications; therefore it holds internally if and only if it holds at any stalk. ☐

4.4. **Support.** Recall that the *support* of an $\mathcal{O}_X$-module $\mathcal{F}$ is the subset $\operatorname{supp} \mathcal{F} := \{x \in X \mid \mathcal{F}_x \neq 0\} \subseteq X$. If $\mathcal{F}$ is of finite type, this set is closed, since its complement is then open by XXX.

Proposition 4.6. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the interior of the complement of the support of $\mathcal{F}$ can be characterized as the largest open subset of X on which the internal statement $\mathcal{F} = 0$ holds.*

Proof. For any open subset $U \subseteq X$, it holds that:

$$\begin{aligned} & U \subseteq \operatorname{int}(X \setminus \operatorname{supp} \mathcal{F}) \\ \Longleftrightarrow\ & U \subseteq X \setminus \operatorname{supp} \mathcal{F} \\ \Longleftrightarrow\ & U \subseteq \{x \in X \mid \forall s \in \mathcal{F}_x.\ s = 0\} \\ \Longleftrightarrow\ & U \models \forall s \colon \mathcal{F}.\ s = 0 \\ \Longleftrightarrow\ & U \models \ulcorner \mathcal{F} = 0 \urcorner \end{aligned}$$

The second to last equivalence is because "$\forall s \colon \mathcal{F}.\ s = 0$" is a geometric implication and can thus be checked stalkwise. □

4.5. **Internal proofs of common lemmas.**

Lemma 4.7. *Let X be a scheme (or ringed space). Let*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}$ and $\mathcal{H}$ are of finite type, so is $\mathcal{G}$; similarly, if $\mathcal{F}$ and $\mathcal{H}$ are locally finitely free, so ist $\mathcal{G}$.

Proof. From the internal perspective, we are given a short exact sequence of modules with the outer ones being finitely generated (resp. finitely free) and we have to show that the middle one is finitely generated (resp. finitely free) as well. It is well-known that this follows; and since the usual proof of this fact is intuitionistic, we are done. □

Lemma 4.8. *Let X be a scheme (or locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of locally finitely free $\mathcal{O}_X$-modules. Then the kernel of α is locally finitely free as well.*

Proof. It suffices to give an intuitionistic proof of the following statement: The kernel of a matrix over a local ring, which as a linear map is surjective, is finitely free.

Let $M \in R^{n \times m}$ be such a matrix. Since by the surjectivity assumption some linear combination of the columns is e_1 (the first canonical basis vector), some linear combination of the entries of the first row of M is 1. By locality of R, at least one entry of the first row is invertible. By applying appropriate column and row transformations, we may assume that M is of the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \widetilde{M} & \\ 0 & & & \end{array}\right)$$

with the submatrix $\widetilde{M}$ fulfilling the same condition as M. Continuing in this way, it follows that $m \geq n$ and that we may assume that M is of the form

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \end{array}\right).$$

The kernel of such a matrix is obviously freely generated by the canonical basis vectors corresponding to the zero columns. In particular, the rank of the kernel is $m - n$. ☐

Remark 4.9. The internal language machinery gives no reason to believe that the dual statement is true, i. e. that the cokernel of a monomorphism of locally finitely free $\mathcal{O}_X$-modules is locally finitely free: This would follow from an intuitionistic proof of the statement that the cokernel of an injective map between finitely free modules over a local ring is finitely free. But this statement is false, as the following example shows.

$$0 \longrightarrow \mathbb{Z}_{(2)} \xrightarrow{\cdot 2} \mathbb{Z}_{(2)} \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

Lemma 4.10. *Let X be a scheme (or locally ringed space). Let $\alpha : \mathcal{G} \to \mathcal{H}$ be an epimorphism of locally finitely free $\mathcal{O}_X$-modules of the same rank. Then α is an isomorphism.*

Proof. It suffices to give an intuitionistic proof of the following statement: A square matrix over a local ring, which as a linear map is surjective, is invertible.

This follows from the proof of the previous lemma, since it shows that the kernel of such a matrix is finitely free of rank zero. ☐

Lemma 4.11. *Let X be a scheme (or ringed space). Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. Then $\operatorname{cl}\operatorname{supp}\mathcal{G} = \operatorname{cl}\operatorname{supp}\mathcal{F} \cup \operatorname{cl}\operatorname{supp}\mathcal{H}$.*

Proof. Switching to complements, we have to prove that

$$\operatorname{int}(X \setminus \operatorname{supp}\mathcal{G}) = \operatorname{int}(X \setminus \operatorname{supp}\mathcal{F}) \cap \operatorname{int}(X \setminus \operatorname{supp}\mathcal{H}).$$

By proposition 4.6, it suffices to prove

$$\operatorname{Sh}(X) \models (\mathcal{G} = 0 \Longleftrightarrow \mathcal{F} = 0 \wedge \mathcal{H} = 0);$$

this is a basic observation in linear algebra, valid intuitionistically. ☐

- basic lemmas: finite type in exact sequences, filtered colimits, flatness, surjective implies iso, ...
- important hard exercise
- torsion (check Liu p. 174)
- reorganize? place lemmas directly in their corresponding sections?

5. Upper semicontinuous functions

5.1. **Interlude on natural numbers.** In classical logic, the natural numbers are complete in the sense that any inhabited set of natural numbers possesses a minimal element. This statement can not be proven intuitionistically – intuitively, this is because one cannot explicitly pinpoint the (classically existing) minimal element of an arbitrary inhabited set. In intuitionistic logic, this principle can be salvaged in two essentially different ways: either be strengthening the premise, or by weakening the conclusion.

Lemma 5.1. *Let $U \subseteq \mathbb{N}$ be an inhabited subset of the natural numbers.*

(1) *Assume U to be* detachable*, i. e. assume that for any natural number n, either $n \in U$ or $n \notin U$. Then U possesses a minimal element.*
(2) *In any case, U does* not not *possess a minimal element.*

Proof. (1) By induction on the witness of inhabitation, i. e. the given number n such that $n \in U$. Details omitted, since we will not need this statement.

(2) We give a careful proof since logical subtleties matter. To simplify the exposition, we assume that U is upward-closed, i. e. that any number larger than some element of U lies in U as well. Any subset can be closed in this way (by considering $\{n \in \mathbb{N} \mid \exists m \in U.\, n \geq m\}$) and a minimal element of the closure will be a minimal element for U as well.

We induct on the number $n \in U$ given by the assumption that U is inhabited. In the case $n = 0$ we are done since 0 is a minimal element of U. For the induction step $n \to n + 1$, the weak law of excluded middle gives

$$\neg\neg(n \in U \vee n \notin U).$$

If we can show that $n \in U \vee n \notin U$ implies the conclusion, we're done by XXX. So assume $n \in U \vee n \notin U$. If $n \in U$, then U does not not possess a minimal element by the induction hypothesis. If $n \notin U$, then $n + 1$ is a minimal element (and so, in particular, U does not not possess a minimal element): For if m is any element of U, we have $m \geq n+1$ or $m \leq n$. In the first case, we're done. In the second case, it follows that $n \in U$ because U is upward-closed and so we obtain a contradiction. From this contradiction we can deduce $m \geq n + 1$. □

If we want to work with a complete set of natural numbers in intuitionistic logic, we have to construction a completion.

Definition 5.2. The partially ordered set of *completed natural numbers* is the set $\widehat{\mathbb{N}}$ of all inhabited upward-closed subsets of $\mathbb{N}$, ordered by reverse inclusion.

Lemma 5.3. *The poset of completed natural numbers is the least partially ordered set containing $\mathbb{N}$ and possessing minima of arbitrary inhabited subsets.*

Proof. The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \longmapsto \uparrow(n) := \{m \in \mathbb{N} \mid m \geq n\}.$$

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

The proof of the universal property is left to the reader. □

Remark 5.4. In classical logic, the map $\widehat{\mathbb{N}} \to \mathbb{N}$, $U \mapsto \min U$ is a well-defined isomorphism of partially ordered sets.

5.2. **A geometric interpretation.** We are interested in the completed natural numbers for the following reason: A completed natural number of the topos of sheaves on a topological space X is the same as an upper semicontinuous function $X \to \mathbb{N}$.

Lemma 5.5. *Let X be a topological space. The sheaf $\widehat{\mathbb{N}}$ of completed natural numbers on X is canonically isomorphic to the sheaf of upper semicontinuous $\mathbb{N}$-valued functions on X.*

Proof. When referring to the natural numbers in the internal language, we actually refer to the constant sheaf $\underline{\mathbb{N}}$ on X. (This is because the sheaf $\underline{\mathbb{N}}$ fulfills the axioms of a natural numbers object, cf. [**johnstone:elephant/moerdijk**].) Recall that its sections on an open subset $U \subseteq X$ are continuous functions $U \to \mathbb{N}$, where $\mathbb{N}$ is equipped with the discrete topology.

Therefore, a section of $\widehat{\mathbb{N}}$ on an open subset $U \subseteq X$ is given by a subsheaf $\mathcal{A} \hookrightarrow \underline{\mathbb{N}}|_U$ such that

$$U \models \exists n\,{:}\,\underline{\mathbb{N}}.\ n \in \mathcal{A} \quad \text{and} \quad U \models \forall n, m\,{:}\,\underline{\mathbb{N}}.\ n \geq m \wedge n \in \mathcal{A} \Rightarrow m \in \mathcal{A}.$$

Since these conditions are geometric, they are satisfied if and only if any stalk $\mathcal{A}_x$ is an inhabited upward-closed subset of $\underline{\mathbb{N}}_x \cong \mathbb{N}$. The association

$$x \in X \longmapsto \min\{n \in \mathbb{N} \mid n \in \mathcal{A}_x\}$$

thus defines a map $X \to \mathbb{N}$. This map is indeed upper semicontinuous, since if $n \in \mathcal{A}_x$, there exists an open neighbourhood V of x such that the constant function with value n is an element of $\Gamma(V, \mathcal{A})$ and therefore $n \in \mathcal{A}_y$ for all $y \in V$.

Conversely, let $\alpha : U \to \mathbb{N}$ be a upper semi-continous function. Then

$$V \subseteq X \text{ open} \longmapsto \{f : V \to \mathbb{N} \mid f \text{ continuous, } f \geq \alpha \text{ on } V\}$$

is a subobject of $\underline{\mathbb{N}}|_U$ which internally is inhabited and upward-closed. Further details are left to the reader. ☐

Under the correspondence given by the lemma, locally *constant* functions map exactly to the (image of the) *ordinary* internal natural numbers (in the completed natural numbers).

Remark 5.6. In a similar vein, the sheaf given by the internal construction of the set of *all* upward-closed subsets of the natural numbers (not only the inhabited ones) is canonically isomorphic to the sheaf of upper semicontinuous functions with values in $\mathbb{N} \cup \{+\infty\}$.

5.3. **The upper semicontinuous rank function.** Recall that the rank of an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X (or locally ringed space) at a point $x \in X$ is defined as the $k(x)$-dimension of the vector space $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$. If we assume that $\mathcal{F}$ is of finite type around x, this dimension is finite and equals the minimal number of elements needed to generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module (by Nakayama's lemma).

In the internal language, we can define an element of $\widehat{\mathbb{N}}$ by

$$\operatorname{rank} \mathcal{F} := \min\{n \in \mathbb{N} \mid \ulcorner\text{there is a gen. family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner\} \in \widehat{\mathbb{N}}.$$

If $\mathcal{F}$ is locally finitely free, it will be a finitely free module from the internal point of view and the rank defined in this way will be an actual natural number; but in general, the rank is really an element of the completion.

Proposition 5.7. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type on a scheme X (or locally ringed space). Under the correspondence given by the previous lemma, the internally defined rank maps to the rank function of $\mathcal{F}$.*

Proof. We have to show that for any point $x \in X$ and natural number n, there exists a generating family for $\mathcal{F}_x$ consisting of n elements if and only if there exists an open neighbourhood U of x such that

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

The "if" direction is obvious. For the "only if" direction, consider (liftings to local sections of a) generating family $s_1, \ldots, s_n$ of $\mathcal{F}_x$. Since $\mathcal{F}$ is of finite type, there also exist sections $t_1, \ldots, t_m$ on some neighbourhood V of x which generate any stalk $\mathcal{F}_y$, $y \in V$. Since the t_i can be expressed as a linear combination of the s_j in $\mathcal{F}_x$, the same is true on some open neighbourhood $U \subseteq V$ of x. On this neighbourhood, the s_j generate any stalk $\mathcal{F}_y$, $y \in U$, so we have

$$U \models \ulcorner s_1, \ldots, s_n \text{ generate } \mathcal{F} \urcorner. \qquad \square$$

Remark 5.8. Once we understand when properties holding at a stalk spread to a neighbourhood, we will be able to give a simpler proof of the proposition (see proposition 8.24).

6. Rational functions and Cartier divisors

6.1. **The sheaf of rational functions.** Recall that the sheaf $\mathcal{K}_X$ of rational functions on a scheme X (or ringed space) can be defined as the sheaf associated to the presheaf

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \mathcal{O}_X)[\Gamma(U, \mathcal{S})^{-1}],$$

where $\Gamma(U, \mathcal{S})$ is the multiplicative set of those sections of $\mathcal{O}_X$ on U, which are regular in each stalk $\mathcal{O}_{X,x}$, $x \in U$. Recall also there are some wrong definitions in the literature [4].

Using the internal language, we can give a simpler definition of $\mathcal{K}_X$. Recall that we can associate to any ring R its total quotient ring, i. e. its localization at the multiplicative subset of regular elements. Since from the internal perspective $\mathcal{O}_X$ is an ordinary ring, we can associate to it its total quotient ring $\mathcal{O}_X[\mathcal{S}^{-1}]$, where $\mathcal{S}$ is internally defined by the formula

$$\mathcal{S} := \{ s : \mathcal{O}_X \mid \ulcorner s \text{ is regular} \urcorner \} \subseteq \mathcal{O}_X.$$

Externally, this ring is the sheaf $\mathcal{K}_X$.

Proposition 6.1. *Let X be a scheme (or a ringed space). The sheaf of rings defined in the internal language by localizing $\mathcal{O}_X$ at its set of regular elements is (canonically isomorphic to) the sheaf $\mathcal{K}_X$ of rational functions.*

Proof. Internally, the ring $\mathcal{O}_X[\mathcal{S}^{-1}]$ fulfills the following universal property: For any ring R and any homomorphism $\mathcal{O}_X \to R$ which maps the elements of $\mathcal{S}$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}] \to R$ which makes the evident diagram commute.

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr] \arrow[dr] & & R \\ & \mathcal{O}_X[\mathcal{S}^{-1}] \arrow[ur, dashed] & \end{tikzcd}$$

The translation using the Kripke–Joyal semantics gives the following universal property: For any open subset $U \subseteq X$, any sheaf of rings $\mathcal{R}$ on U and any homomorphism $\mathcal{O}_X|_U \to \mathcal{R}$ which maps all elements of $\Gamma(V, \mathcal{S})$, $V \subseteq U$ to units, there exists exactly one homomorphism $\mathcal{O}_X[\mathcal{S}^{-1}]|_U \to \mathcal{R}$ which makes the evident diagram commute. It is well-known [**???**] that the sheaf $\mathcal{K}_X$ as usually defined satisfies this universal property as well. $\square$

Proposition 6.2. *Let X be a scheme (or ringed space). Then the stalks of $\mathcal{K}_X$ are given by*

$$\mathcal{K}_{X,x} = \mathcal{O}_{X,x}[\mathcal{S}_x^{-1}].$$

The elements of $\mathcal{S}_x$ are exactly the germs of those local sections which are regular not only in $\mathcal{O}_{X,x}$, but in all rings $\mathcal{O}_{X,y}$ where y ranges over some neighbourhood of x (depending on the section).

Proof. Since localization is a geometric construction, the first statement is entirely trivial. The second statement follows since

$$\Gamma(U, \mathcal{S}) = \{s \in \Gamma(U, \mathcal{O}_X) \,|\, U \models \ulcorner s \text{ is regular} \urcorner\}$$

and regularity is a geometric implication, so that $U \models \ulcorner s \text{ is regular} \urcorner$ if and only if the germ s_y is regular in $\mathcal{O}_{X,y}$ for all $y \in U$. □

Remark 6.3. Speaking internally, the multiplicative set $\mathcal{S}$ is saturated. Therefore an element $s/t : \mathcal{K}_X$ is invertible in $\mathcal{K}_X$ if and only if the numerator s belongs to $\mathcal{S}$, i. e. is an regular element of $\mathcal{O}_X$.

6.2. **Regularity of local functions.** It is well known that on a locally Noetherian scheme, regularity spreads from stalks to neighbourhoods, i. e. a section of $\mathcal{O}_X$ is regular in $\mathcal{O}_{X,x}$ if and only if it is regular on some neighbourhood on x. This fact has a simple proof in the internal language:

Proposition 6.4. *Let X be a locally Noetherian scheme. Let $s \in \Gamma(U, \mathcal{O}_X)$ be a local function on X. Let $x \in U$. Then the following statements are equivalent:*

(1) *The section s is regular in $\mathcal{O}_{X,x}$.*
(2) *The section s is regular in all local rings $\mathcal{O}_{X,y}$ where y ranges over some neighbourhood of x.*

Proof. Let $\Box$ be the modal operator defined by $\Box(\varphi) :\equiv ((\varphi \Rightarrow\, !x) \Rightarrow\, !x)$. By XXX, we are to show that the following statements of the internal language are equivalent:

(1) $(\ulcorner s \text{ is regular} \urcorner)^{\Box}$, i. e. $\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow \Box(t = 0)$.
(2) $\Box(\ulcorner s \text{ is regular} \urcorner)$, i. e. $\Box(\forall t : \mathcal{O}_X.\ st = 0 \Rightarrow t = 0)$.

It is clear that the second statement implies the first – in fact, this is true without any assumptions on X: Let $t : \mathcal{O}_X$ be such that $st = 0$. Since we want to prove the boxed statement $\Box(t = 0)$, we may assume that s is regular and prove $t = 0$. This follows by definition.

For the converse direction, consider the annihilator of s, i. e. the ideal

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \,|\, st = 0\} \subseteq \mathcal{O}_X.$$

This ideal satisfies the quasicoherence condition (example 9.4), thus I is a quasicoherent submodule of a finitely generated module. Since X is locally Noetherian, it follows that I is finitely generated as well. By assumption, each generator $x_i : I$ fulfills $\Box(x_i = 0)$. Since we want to prove a boxed statement, we may in fact assume $x_i = 0$. Thus $I = (0)$ and the assertion, that s is regular, follows. □

Corollary 6.5. *Let X be a locally Noetherian scheme. Then the stalks $\mathcal{K}_{X,x}$ of the sheaf of rational functions are given by the total quotient rings of the local rings $\mathcal{O}_{X,x}$.*

Proof. Combine proposition 6.2 and proposition 6.4. □

6.3. **Geometric interpretation of rational functions.**

Proposition 6.6. *Let X be a reduced scheme. Then $\mathcal{K}_X$ is the $\neg\neg$-sheafification of $\mathcal{O}_X$.*

Proof. Recall from corollary 3.7 that

$$\mathrm{Sh}(X) \models \forall s \colon \mathcal{O}_X.\ \neg(\ulcorner s \text{ invertible} \urcorner) \Leftrightarrow s = 0.$$

From this we can deduce that $\mathcal{O}_X$ is $\neg\neg$-separated: Assume $\neg\neg(s = 0)$ for $s \colon \mathcal{O}_X$. If s were invertible, we would have $\neg\neg(1 = 0)$ and thus $\perp$. Therefore s is not invertible and thus zero.

Furthermore, we can can deduce that internally, an element $s \colon \mathcal{O}_X$ is regular if and only if it is *not not* invertible: For the "only if" direction, note that a regular element is not zero (if it were, then the true statement $0 \cdot 0 = 0 \cdot 1$ would imply the false statement $0 = 1$). For the "if" direction, let $st = 0$ in $\mathcal{O}_X$. Since s is *not not* invertible, it follows that t is *not not* zero. Since $\mathcal{O}_X$ is $\neg\neg$-separated, this implies that t really is zero.

With these observations, we can proceed to that $\mathcal{K}_X$ is $\neg\neg$-separated. So assume $\neg\neg(a/s = 0)$ for $a/s \colon \mathcal{K}_X$. Since $\mathcal{K}_X$ is obtained from $\mathcal{O}_X$ by localizing at regular elements, it holds that $a/s = 0$ in $\mathcal{K}_X$ if and only if $a = 0$ in $\mathcal{O}_X$. Thus it follows that $\neg\neg(a = 0)$ in $\mathcal{O}_X$ and thus $a = 0$ in $\mathcal{O}_X$; in particular, $a/s = 0$ in $\mathcal{K}_X$.

We defer the proof that $\mathcal{K}_X$ is a $\neg\neg$-sheaf to the end and first verify the universal property of $\neg\neg$-sheafification. So let G be a $\neg\neg$-sheaf and let $\alpha : \mathcal{O}_X \to G$ be a map. We can define an extension $\bar{\alpha} : \mathcal{K}_X \to G$ in the following way: Let $f \colon \mathcal{K}_X$. Define the subsingleton $S := \{x \colon G \mid \exists b \colon \mathcal{O}_X.\ f = b/1 \wedge x = \alpha(b)\} \subseteq G$. Since f can be written in the form a/s with s *not not* invertible, it follows that S is *not not* inhabited. Since G is a $\neg\neg$-sheaf, there exists a unique $x \colon G$ such that $\neg\neg(x \in S)$. We declare $\bar{\alpha}(f)$ to be this x. It is straightforward to check that the composition $\mathcal{O}_X \to \mathcal{K}_X \to G$ equals α and that $\bar{\alpha}$ is unique with this property.

Up to this point, the proof did not need that X is a scheme – it was enough for X to be a ringed space such that the display equivalence above holds and such that $\neg(0 = 1)$ in $\mathcal{O}_X$. Only now, to show that $\mathcal{K}_X$ is a $\neg\neg$-sheaf, the scheme condition enters. □

6.4. **Cartier divisors.**

Let X be a scheme (or ringed space). Recall that a *Cartier divisor* on x is a global section of the sheaf of groups $\mathcal{K}_X^*/\mathcal{O}_X^*$. This sheaf can be constructed internally, with the same notation: It is the quotient of the group of invertible elements of the ring $\mathcal{K}_X$ by the subgroup of invertible elements of the ring $\mathcal{O}_X$. So an arbitrary section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ is internally of the form $[s/t]$ with $s, t \colon \mathcal{O}_X$ being regular elements; this is a simpler description than the usual external one (as a family $(f_i)_i$ of functions $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ such that $f_i^{-1}|_{U_i \cap U_j} \cdot f_j|_{U_i \cap U_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ for all i, j).

We can sketch the basic theory of Cartier divisors completely from the internal perspective. In accordance with common practice, we will write the group operation of $\mathcal{K}_X^*/\mathcal{O}_X^*$ (which is induced by multiplication of elements in $\mathcal{K}_X^*$) additively.

Definition 6.7. A Cartier divisor is *effective* if and only if, from the internal perspective, it can be written in the form $[s/1]$ with $s \colon \mathcal{O}_X$ being a regular element.

Thus a Cartier divisor $[s/t]$ is effective if and only if s is an $\mathcal{O}_X$-multiple of t.

Definition 6.8. A Cartier divisor D is *principal* if and only if there exists a global section $f \in \Gamma(X, \mathcal{K}_X^*)$ such that internally, $D = [f]$. Two Cartier divisors are *linearly equivalent* if and only if their difference is a principal divisor.

Note that decidedly, principality is a global notion: For any divisor D it is true that locally there exists sections f of $\mathcal{K}_X^*$ such that $D = [f]$.

Definition 6.9. The *line bundle associated to a Cartier divisor* D is the $\mathcal{O}_X$-submodule

$$\mathcal{O}_X(D) := \{g \in \mathcal{K}_X \mid gD \in \mathcal{O}_X\} = D^{-1}\mathcal{O}_X \subseteq \mathcal{K}_X$$

of $\mathcal{K}_X$. Here we are abusing language for "$gD \in \mathcal{O}_X$" to mean that $gf \in \mathcal{O}_X$ if $D = [f]$ with $f : \mathcal{K}_X$; and for "$D^{-1}\mathcal{O}_X$" to mean $f^{-1}\mathcal{O}_X$. This condition resp. submodule does not depend on the representative f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^*$.

The submodule $\mathcal{O}_X(D)$ is indeed locally free of rank 1, since internally f^{-1} gives an one-element basis. Note that D is effective if and only if $\mathcal{O}_X(-D)$ is a subset of $\mathcal{O}_X$ from the internal perspective. In this case, we can define the *support* of D to be the closed subscheme of X associated to the sheaf of ideals $\mathcal{O}_X(-D) \subseteq \mathcal{O}_X$.

Definition 6.10. The *Cartier divisor associated to a free submodule* $\mathcal{L} \subseteq \mathcal{K}_X$ *of rank 1* is $D := [f^{-1}]$, where $f : \mathcal{K}_X$ is the unique element of some one-element-basis of $\mathcal{L}$.

The basis element $f : \mathcal{K}_X$ does indeed lie in $\mathcal{K}_X^*$: Write $f = s/t$ with $s, t : \mathcal{O}_X$. It suffices to show that s is a regular element of $\mathcal{O}_X$. So let $h : \mathcal{O}_X$ such that $sh = 0$ in $\mathcal{O}_X$. Then in particular $hf = 0$ in $\mathcal{K}_X$. By linear independence, it follows that $h = 0$ in $\mathcal{K}_X$ and thus $h = 0$ in $\mathcal{O}_X$.

Furthermore, the associated divisor does not depend on the choice of f, since f is well-defined up to multiplication by an element of $\mathcal{O}_X^*$: If $f\mathcal{O}_X = g\mathcal{O}_X \subseteq \mathcal{K}_X$, then there exist $u, v : \mathcal{O}_X$ such that $fu = g$ and $gv = f$ in $\mathcal{K}_X$. It follows that $uv = fuvf^{-1} = gvf^{-1} = ff^{-1} = 1$ in $\mathcal{K}_X$ and thus in $\mathcal{O}_X$, by injectivity of the canonical map $\mathcal{O}_X \to \mathcal{K}_X$. Therefore u and v are elements of $\mathcal{O}_X^*$.

Lemma 6.11. *Let D and D' be divisors on X. Then $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D') \cong \mathcal{O}_X(D + D')$.*

Proof. The wanted morphism of sheaves $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D') \to \mathcal{O}_X(D + D')$ is given by multiplication. That this is well-defined and an isomorphism can be checked from the internal point of view, where the claims are obvious. □

Proposition 6.12. *The association $D \mapsto \mathcal{O}_X(D)$ defines an one-to-one correspondence between Cartier divisors on X and rank-one submodules of $\mathcal{K}_X$. This correpondence descends to an one-to-one correspondence between Cartier divisiors up to linear equivalance and rank-one submodules of $\mathcal{K}_X$ up to isomorphism.*

Proof. The first statement is obvious from the definitions. For the second statement, it suffices to show that $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$ if and only if D is principal. A given isomorphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ gives a global section $f \in \mathcal{K}_X^*$ (by considering the image of the unit element) such that internally, $D = [f^{-1}]$; this shows that D is principal. The converse is similar. □

Remark 6.13. Locally principal subschemes (closed subschemes which are locally the vanishing subscheme of a regular section of $\mathcal{O}_X$) up to isomorphisms of subschemes are in one-to-one correspondence with rank-1 submodules of $\mathcal{O}_X$ (see XXX). Thus locally principal subschemes (up to isomorphisms of abstract schemes) are in one-to-one correspondence with effective Cartier divisors (up to linear equivalence).

Proposition 6.14. *Assume that X is an integral scheme. Then any line bundle on X is (uncanonically) a submodule of $\mathcal{K}_X$.*

Proof. Let ξ be the generic point of X and let $\Box := \neg\neg$ denote the modal operator such that internal sheafification with respect to $\Box$ is the same as pulling back to $\{\xi\}$ and then pushing forward to X again (see XXX). Let $\mathcal{L}$ be a line bundle on X. Since $\mathcal{L}_\xi \cong \mathcal{O}_{X,\xi}$ (uncanonically), there is some injection $\mathcal{L}_\xi \to \mathcal{K}_{X,\xi}$; this corresponds internally to an injection $\mathcal{L}^{++} \to \mathcal{K}_X^{++}$. Since $\mathcal{K}_X$ is already a $\Box$-sheaf (see XXX) and $\mathcal{L}$ is $\Box$-separated (being isomorphic to $\mathcal{O}_X$), we have the global injection

$$\mathcal{L} \hookrightarrow \mathcal{L}^{++} \hookrightarrow \mathcal{K}_X^{++} \xrightarrow{(\cong)^{-1}} \mathcal{K}_X. \qquad \Box$$

- on reduced schemes, $\mathcal{K}_X$ is the sheaf of meromorphic functions
- show $\mathcal{K}_X = j_*(\mathcal{O}_X)$?
- divisor associated to rational sections

7. Relative spectrum

- ...

8. Modalities

8.1. Basics on truth values and modal operators.

Definition 8.1. The *set of truth values* Ω is the powerset of the singleton set $1 := \{\star\}$, where $\star$ is a formal symbol.

In classical logic, any subset of $\{\star\}$ is either empty or inhabited, so that Ω contains exactly two elements, the empty set ("false") and $\{\star\}$ ("true"). But in intuitionistic logic, this can not be shown; indeed, if we interpret the definition in the topos of sheaves on a space X, we obtain a sheaf Ω with

$$U \subseteq X \text{ open} \longmapsto \Gamma(U, \Omega) = \{V \subseteq U \,|\, V \text{ open}\}.$$

(This is because by definition of Ω as the power object of the terminal sheaf 1, sections of Ω on an open subset U correspond to subsheaves $\mathcal{F} \hookrightarrow 1|_U$, and those are given by the greatest open subset $V \subseteq U$ such that $\Gamma(V, \mathcal{F})$ is inhabited.)

The *truth value* of a formula φ is by definition the subset $\{x \in 1 \,|\, \varphi\} \in \Omega$, where "$x$" is a fresh variable not appearing in φ. This subset is inhabited if and only if φ holds and is empty if and only if $\neg\varphi$ holds. Conversely, we can associate to a subset $F \subseteq 1$ the formula $\ulcorner F$ is inhabited$\urcorner$.

Under this correspondence of formulas with truth values, logical operations like $\wedge$ and $\vee$ map to set-theoretic operations like $\cap$ and $\cup$ – for instance, we have

$$\{x \in 1 \,|\, \varphi\} \cap \{x \in 1 \,|\, \psi\} = \{x \in 1 \,|\, \varphi \wedge \psi\}.$$

This justifies a certain abuse of notation: We will sometimes treat elements of Ω as propositions and use logical instead of set-theoretic connectives. In particular, if φ

and ψ are elements of Ω, we will write "$\varphi \Rightarrow \psi$" to mean $\varphi \subseteq \psi$; "$\perp$" to mean $\emptyset$; and "$\top$" to mean 1.

Definition 8.2. A *modal operator* is a map $\Box : \Omega \to \Omega$ such that for all $\varphi, \psi \in \Omega$,

(1) $\varphi \Longrightarrow \Box\varphi$,
(2) $\Box\Box\varphi \Longrightarrow \Box\varphi$,
(3) $\Box(\varphi \wedge \psi) \Longleftrightarrow \Box\varphi \wedge \Box\psi$.

The intuition is that $\Box\varphi$ is a certain weakening of φ, where the precise meaning of "weaker" depends on the modal operator. By the second axiom, weakening twice is the same as weakening once.

In classical logic, where $\Omega = \{\perp, \top\}$, there are only two modal operators: the identity function and the constant function with value $\top$. Both of these are not very interesting: The identity operator does not weaken propositions at all, while the constant operator weakens every proposition to the trivial statement $\top$.

In intuitionistic logic, there can potentially exist further modal operators. For applications to algebraic geometry, the following four operators will have a clear geometric meaning and be of particular importance:

(1) $\Box\varphi :\equiv (\alpha \Rightarrow \varphi)$, where α is a fixed proposition.
(2) $\Box\varphi :\equiv (\varphi \vee \alpha)$, where α is a fixed proposition.
(3) $\Box\varphi :\equiv \neg\neg\varphi$ (the *double negation modality*).
(4) $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition.

Lemma 8.3. *Any modal operator $\Box$ is monotonic, i. e. if $\varphi \Rightarrow \psi$, then $\Box\varphi \Rightarrow \Box\psi$. Furthermore, there holds a modus ponens rule: If $\Box\varphi$ holds, and φ implies $\Box\psi$, then $\Box\psi$ holds as well.*

Proof. Assume $\varphi \Rightarrow \psi$. This is equivalent to supposing $\varphi \wedge \psi \Leftrightarrow \varphi$. We are to show that $\Box\varphi \Rightarrow \Box\psi$, i. e. that $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box\varphi$. The statement follows since by the third axiom on a modal operator, we have $\Box\varphi \wedge \Box\psi \Leftrightarrow \Box(\varphi \wedge \psi)$.

For the second statement, consider that if $\varphi \Rightarrow \Box\psi$, by monotonicity and the second axiom on a modal operator it follows that $\Box\varphi \Rightarrow \Box\Box\psi \Rightarrow \Box\psi$. $\Box$

The modus ponens rule justifies the following proof scheme: When showing that a boxed statement $\Box\psi$ holds given that a further boxed statement $\Box\varphi$ holds, we may assume that indeed φ holds.

8.2. **Geometric meaning.** Let X be a topological space. As discussed above, an open subset $U \subseteq X$ defines an internal truth value (a global section of the sheaf Ω) also denoted by "U" such that

$$V \models U \quad \Longleftrightarrow \quad V \subseteq U$$

for any open subset $V \subseteq X$. (Shortcutting the various intermediate steps, this can also be taken as a definition of "$V \models U$".) If $A \subseteq X$ is a closed subset, there is thus an internal truth value A^c corresponding to the open subset $A^c = X \setminus A$. If $x \in X$ is a point, we define "$!x$" to denote the truth value corresponding to $\operatorname{int}(X \setminus \{x\})$, such that

$$V \models\ !x \quad \Longleftrightarrow \quad V \subseteq \operatorname{int}(X \setminus \{x\}) \quad \Longleftrightarrow \quad x \notin V.$$

Proposition 8.4. *Let $U \subseteq X$ be a fixed open and $A \subseteq X$ be a fixed closed subset. Let $x \in X$. Then, for any open subset $V \subseteq X$, it holds that:*

$$\begin{aligned}
&V \models (U \Rightarrow \varphi) &&\Longleftrightarrow\quad V \cap U \models \varphi. \\
&V \models (\varphi \vee A^c) &&\Longleftrightarrow\quad \text{there is an open subset } W \subseteq V \\
& && \qquad\qquad \text{containing } A \cap V \text{ such that } W \models \varphi. \\
&V \models \neg\neg\varphi &&\Longleftrightarrow\quad \text{there is a dense open subset } W \subseteq V \text{ s.th. } W \models \varphi. \\
&V \models ((\varphi \Rightarrow !x) \Rightarrow !x) &&\Longleftrightarrow\quad x \notin V \text{ or there is an open neighbourhood } W \subseteq V \\
& && \qquad\qquad \text{of } x \text{ such that } W \models \varphi.
\end{aligned}$$

Proof.
(1) Omitted.
(2) Let $V \models \varphi \vee A^c$. Then there exists an open covering $V = \bigcup_i V_i$ such that for each i, $V_i \models \varphi$ or $V_i \subseteq A^c$. Let $W \subseteq V$ be the union of those V_i such that $V_i \models \varphi$. Then $W \models \varphi$ by the locality of the internal language and $A \cap V \subseteq W$.

Conversely, let $W \subseteq V$ be an open subset containing $A \cap V$ such that $W \models \varphi$. Then $V = W \cup (V \cap A^c)$ is an open covering attesting $V \models \varphi \vee A^c$.
(3) For the "only if" direction, let $W \subseteq V$ be the largest open subset on which φ holds, i. e. the union of all open subsets of V on which φ holds. For the "if" direction, we may assume that the given W is also the largest open subset on which φ holds (by enlarging W if necessary). The claim then follows by the following chain of equivalences:

$$\begin{aligned}
&V \models \neg\neg\varphi \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open. } [\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow Z = \emptyset] \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open. } [\forall Z \subseteq Y \text{ open. } Z \subseteq W \Rightarrow Z = \emptyset] \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &\forall Y \subseteq V \text{ open. } Y \cap W = \emptyset \Longrightarrow Y = \emptyset \\
\Longleftrightarrow\ &W \text{ is dense in } V.
\end{aligned}$$

(4) Straightforward, since the interpretation of the internal statement with the Kripke–Joyal semantics is

$$\forall Y \subseteq V \text{ open. } [\forall Z \subseteq Y \text{ open. } Z \models \varphi \Rightarrow x \notin Z] \Longrightarrow x \notin Y. \qquad \square$$

8.3. **The subspace associated to a modal operator.** Any modal operator $\Box : \Omega \to \Omega$ in the sheaf topos of X induces on global sections a map

$$j : \mathrm{Op}(X) \to \mathrm{Op}(X),$$

where $\mathrm{Op}(X) = \Gamma(X, \Omega)$ is the set of open subsets of X. By the axioms on a modal operator, the map j fulfills similar axioms: For any open subsets $U, V \subseteq X$,

(1) $U \subseteq j(U)$,
(2) $j(j(U)) \subseteq j(U)$,
(3) $j(U \cap V) = j(U) \cap j(V)$.

Such a map is called a *nucleus* on $\mathrm{Op}(X)$. Table 2 lists the nuclei associated to the four modal operators of proposition 8.4.

Any nucleus j defines a subspace X_j of X, with a small caveat: In general, the subspace X_j can not be realized as a topological subspace, but only as a so-called *sublocale*; the notion of a locale is a slight generalization of the notion of a topological

Modal operator	associated nucleus	$j(V) = X$ iff ...	subspace
$\Box\varphi :\equiv (U \Rightarrow \varphi)$	$j(V) = \mathrm{int}(U^c \cup V)$	$U \subseteq V$	U
$\Box\varphi :\equiv (\varphi \vee A^c)$	$j(V) = V \cup A^c$	$A \subseteq V$	A
$\Box\varphi :\equiv \neg\neg\varphi$	$j(V) = \mathrm{int}(\mathrm{cl}(V))$	V is dense in X	(see text)
$\Box\varphi :\equiv ((\varphi \Rightarrow !x) \Rightarrow !x)$	$j(V) = \begin{cases} X \setminus \mathrm{cl}\{x\}, & \text{if } x \notin V \\ X, & \text{if } x \in V \end{cases}$	$x \in V$	$\{x\}$

TABLE 2. List of important modal operators and their associated nuclei (notation as in proposition 8.4).

space, in which an underlying set of points is not part of the definition. Instead, a locale is simply given by a lattice of general *opens* – these may, but do not necessarily have to, be sets of points. Sheaf theory carries over to locales essentially unchanged, since the notions of presheaves and sheaves only need opens and coverings.

Definition 8.5. Let j be a nucleus on $\mathrm{Op}(X)$. Then a sublocale X_j of X is given by the lattice of opens $\mathrm{Op}(X_j) := \{U \in \mathrm{Op}(X) \mid j(U) = U\}$.

If j is induced by a modal operator $\Box$, we also write "$X_\Box$" for X_j. In three of the four cases listed in table 2, the sublocale $X_\Box$ can indeed be realized as a topological subspace. The only exception is the sublocale $X_{\neg\neg}$ associated to the double negation modality. It can be also be described as the *smallest dense sublocale* of X; this is obviously a true locale-theoretic notion, since a topological space does not have (in general) a smallest dense topological subspace (consider $\mathbb{R}$ and its dense subsets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$).

The inclusion $i : X_j \hookrightarrow X$ can not in general be described on the level of points, since X_j might not be realizable as a topological subspace. But for sheaf-theoretic purposes, it suffices to describe i on the level of opens. This is done as follows:

$$i^{-1} : \mathrm{Op}(X) \longrightarrow \mathrm{Op}(X_j), \quad U \longmapsto j(U).$$

Thus we can relate the toposes of sheaves on X_j and X by the usual pullback and pushforward functors.

$$\begin{aligned} i^{-1}\mathcal{F} &= \text{sheafification of } (U \mapsto \mathrm{colim}_{U \preceq i^{-1}V}\, \Gamma(V, \mathcal{F})) \\ i_*\mathcal{G} &= (U \mapsto \Gamma(i^{-1}U, \mathcal{G}) = \Gamma(j(U), \mathcal{G})) \end{aligned}$$

As familiar from honest topological subspace inclusions, the pushforward functor $i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X)$ is fully faithful and the composition $i^{-1} \circ i_* : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X_j)$ is (canonically isomorphic to) the identity.

8.4. **Internal sheaves and sheafification.** It turns out that the image of the pushforward functor $i_* : \mathrm{Sh}(X_\Box) \to \mathrm{Sh}(X)$, where $\Box$ is a modal operator in $\mathrm{Sh}(X)$, can be explicitly described: Namely, it consists exactly of those sheaves which from the internal point of view are so-called $\Box$-*sheaves*, a notion explained below.

Furthermore, if we identify $\mathrm{Sh}(X_\Box)$ with its image in $\mathrm{Sh}(X)$, the pullback functor is given by an internal sheafification process with respect to the modality $\Box$. Thus the external situation of pushforward/pullback translates to forget/sheafify. This broadens the scope of the internal language: It can not only be used to talk about sheaves on X in a simple, element-based language, but also to talk about sheaves on arbitrary subspaces of X.

To describe the notion of $\Box$-sheaves and related ones, we switch to the internal perspective and thus forget X; we are simply given a model operator $\Box : \Omega \to \Omega$ and have to take care that our proofs are intuitionistic.

Definition 8.6. A set F is $\Box$-*separated* if and only if

$$\forall x, y : F.\ \Box(x = y) \Longrightarrow x = y.$$

A set F is a $\Box$-*sheaf* if and only if it is $\Box$-separated and

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton}\urcorner) \Longrightarrow \exists x : F.\ \Box(x \in S).$$

The two conditions can be combined: A set F is a $\Box$-sheaf if and only if

$$\forall S \subseteq F.\ \Box(\ulcorner S \text{ is a singleton}\urcorner) \Longrightarrow \exists! x : F.\ \Box(x \in S).$$

Definition 8.7. The *plus construction* of a set F with respect to $\Box$ is the set

$$F^+ := \{S \subseteq F \mid \Box(\ulcorner S \text{ is a singleton}\urcorner)\}/\sim,$$

where the equivalence relation is defined by $S \sim T :\Leftrightarrow \Box(S = T)$. There is a canonical map $F \to F^+$ given by $x \mapsto [\{x\}]$. The $\Box$-*sheafification* of a set F is the set F^{++}.

If F is $\Box$-separated, then for any subset $S \subseteq F$ it holds that

$$\Box(\ulcorner S \text{ is a singleton}\urcorner) \quad \Longleftrightarrow \quad \ulcorner S \text{ is a subsingleton}\urcorner \wedge \Box(\ulcorner S \text{ is inhabited}\urcorner).$$

Remark 8.8. The topos of *pre*sheaves on a topological space X admits an internal language as well [6, ???]. In it, there exists a modal operator $\Box$ reflecting the topology of X. A presheaf is separated in the usual sense if, from the internal perspective of $\mathrm{PSh}(X)$, it is $\Box$-separated; and it is a sheaf if, from the internal perspective, it is a $\Box$-sheaf. Furthermore, the $\Box$-sheafification of a presheaf (considered as a set from the internal perspective) coincides with the usual sheafification.

Lemma 8.9. *For any set F, it holds that:*

(1) *F^+ is $\Box$-separated.*
(2) *The canonical map $F \to F^+$ is injective if and only if F is $\Box$-separated.*
(3) *If F is $\Box$-separated, F^+ is a $\Box$-sheaf.*
(4) *If F is a $\Box$-sheaf, the canonical map $F \to F^+$ is bijective.*

Let "$\mathrm{Sh}_\Box(\mathrm{Set})$" denote the full subcategory of Set *consisting of the $\Box$-sheaves. Then it holds that:*

(5) *The functor $(\_\_)^+ : \mathrm{Set} \to \mathrm{Set}$ is left exact.*
(6) *The functor $(\_\_)^{++} : \mathrm{Set} \to \mathrm{Sh}_\Box(\mathrm{Set})$ is left exact and left adjoint to the forgetful functor $\mathrm{Sh}_\Box(\mathrm{Set}) \to \mathrm{Set},\ F \mapsto F$.*

Proof. These are all straightforward, and it fact simpler than their classical counterparts, since there are no colimit constructions which would have to be dealt with. $\Box$

8.5. **Sheaves for the double negation modality.**

Proposition 8.10. *Let X be a topological space. Let $\mathcal{F}$ be a sheaf on X. Then:*

(1) *$\mathcal{F}$ is $\neg\neg$-separated if and only if it is sufficient for local sections to be equal to agree on a dense open subset of their common domain.*
(2) *$\mathcal{F}$ is a $\neg\neg$-sheaf if and only if it is $\neg\neg$-separated and for any open subset $U \subseteq X$ and any open subset $V \subseteq U$ dense in U, any V-section of $\mathcal{F}$ extends to a U-section of $\mathcal{F}$.*
(3) *If $\mathcal{F}$ is $\neg\neg$-separated, the sections of $\mathcal{F}^+$ on an open subset $U \subseteq X$ can be described as pairs (V, s), where V is a dense open subset of U and s is a section of $\mathcal{F}$ on V. Two such pairs $(V, s), (V', s')$ give the same element in $\Gamma(U, \mathcal{F}^+)$ if and only if s and s' agree on $V \cap V'$.*

Proof. The first statement is obvious from the definition of $\neg\neg$-separatedness (definition 8.6 for $\Box = \neg\neg$) and the geometric interpretation of double negation (proposition 8.4).

For the second statement, it suffices to show that if $\mathcal{F}$ is $\neg\neg$-separated, $\mathcal{F}$ has the extension property if and only if

$$\mathrm{Sh}(X) \models \forall \mathcal{S} : \mathcal{P}(\mathcal{F}).\ \ulcorner\mathcal{S} \text{ is a subsingleton}\urcorner \wedge \neg\neg(\ulcorner\mathcal{S} \text{ is inhabited}\urcorner) \Longrightarrow \exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S}).$$

Note that a section $\mathcal{S} \in \Gamma(U, \mathcal{P}(\mathcal{F}))$ which internally is a subsingleton and not not inhabited is precisely a subsheaf $\mathcal{S} \hookrightarrow \mathcal{F}$ such that all stalks $\mathcal{S}_x$, $x \in U$ are subsingletons and such that for some dense open subset $V \subseteq U$, the stalks $\mathcal{S}_x$, $x \in V$ are inhabited. This is precisely the datum of a section of $\mathcal{F}$ defined on some dense open subset of U: Consider the gluing of the unique germs in $\mathcal{S}_x$ for those points x such that $\mathcal{S}_x$ is inhabited. (Conversely, a section $s \in \Gamma(V, \mathcal{F})$ defines a subsheaf $\mathcal{S}$ by setting $\Gamma(W, \mathcal{S}) := \{s|_W \mid W \subseteq V\}$.)

In view of this explicit description and the observation that the existence in question ("$\exists x : \mathcal{F}.\ \neg\neg(x \in \mathcal{S})$") is actually a question of unique existence, the second statement follows.

For the third statement, one can check that the presheaf on X defined by

$$U \subseteq X \text{ open} \longmapsto \{(V, s) \mid V \subseteq U \text{ dense open},\ s \in \Gamma(V, \mathcal{F})\}/\sim$$

is in fact a sheaf (with respect to the topology of X), a $\neg\neg$-sheaf and that it fulfills the universal property of the $\neg\neg$-sheafification of $\mathcal{F}$. □

8.6. **The $\Box$-translation.** There is certain well-known transformation $\varphi \mapsto \varphi^{\neg\neg}$ on formulas, the *double negation translation*, with the following curious property: A formula φ is derivable in classical logic if and only if its translation $\varphi^{\neg\neg}$ is derivable in intuitionistic logic. The translation $\varphi^{\neg\neg}$ is obtained from φ by putting "$\neg\neg$" before any subformula, i. e. before any "$\exists$" and "$\forall$", around any logical connective and around any atomic statement ("$x = y$", "$x \in A$").

We will describe a slight generalization of the double negation translation, the $\Box$-translation for any modal operator $\Box$.

Definition 8.11. The $\Box$-*translation* is recursively defined as follows.

$$
\begin{aligned}
(f = g)^\Box &:\equiv \Box(f = g) \\
(x \in A)^\Box &:\equiv \Box(x \in A) \\
\top^\Box &:\equiv \Box\top \quad (\Leftrightarrow \top) \\
\bot^\Box &:\equiv \Box\bot \\
(\varphi \wedge \psi)^\Box &:\equiv \Box(\varphi^\Box \wedge \psi^\Box) & (\textstyle\bigwedge_i \varphi_i)^\Box &:\equiv \Box(\textstyle\bigwedge_i \varphi_i^\Box) \\
(\varphi \vee \psi)^\Box &:\equiv \Box(\varphi^\Box \vee \psi^\Box) & (\textstyle\bigvee_i \varphi_i)^\Box &:\equiv \Box(\textstyle\bigvee_i \varphi_i^\Box) \\
(\varphi \Rightarrow \psi)^\Box &:\equiv \Box(\varphi^\Box \Rightarrow \psi^\Box) \\
(\forall x : X.\ \varphi)^\Box &:\equiv \Box(\forall x : X.\ \varphi^\Box) & (\forall X.\ \varphi)^\Box &:\equiv \Box(\forall X.\ \varphi^\Box) \\
(\exists x : X.\ \varphi)^\Box &:\equiv \Box(\exists x : X.\ \varphi^\Box) & (\exists X.\ \varphi)^\Box &:\equiv \Box(\exists X.\ \varphi^\Box)
\end{aligned}
$$

Lemma 8.12.
(1) *Formulas in the image of the $\Box$-translation are $\Box$-stable, i. e. for any formula φ it holds that $\Box(\varphi^\Box) \Longrightarrow \varphi^\Box$.*
(2) *In the definition of the $\Box$-translation, one may omit the boxes printed in gray.*

Proof. The first statement is obvious, since one of the axioms on a modal operator demands that $\Box\Box\varphi \Rightarrow \varphi$ for any formula φ. The second statement follows by an induction on the formula structure. By way of example, we prove the case for "$\Rightarrow$":

$$
\begin{aligned}
&(\varphi \Rightarrow \psi)^{\Box \text{ with the gray parts}} \\
\Longleftrightarrow\ &\Box(\varphi^{\Box \text{ with the gray parts}} \Rightarrow \psi^{\Box \text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box \text{ with the gray parts}} \Rightarrow \psi^{\Box \text{ with the gray parts}}) \\
\Longleftrightarrow\ &(\varphi^{\Box \text{ without the gray parts}} \Rightarrow \psi^{\Box \text{ without the gray parts}}) \\
\Longleftrightarrow\ &(\varphi \Rightarrow \psi)^{\Box \text{ without the gray parts}}
\end{aligned}
$$

The first step is by definition; the second by $\Box$-stability of $\psi^{\Box \text{ with the gray parts}}$; the third by the induction hypothesis; the fourth by definition. $\Box$

Lemma 8.13. *Let φ be a formula such that for any subformulas ψ appearing as antecedents of implications, it holds that $\psi^\Box \Rightarrow \Box\psi$. (In particular, this condition is satisfied if there is no "$\Rightarrow$" in φ.) Then $\Box\varphi \Rightarrow \varphi^\Box$.*

Proof. We prove this by an induction on the formula structure. All cases except for "$\Rightarrow$" are obvious. For this case, assume $\Box(\varphi \Rightarrow \psi)$; we are to show that $(\varphi^\Box \Rightarrow \psi^\Box)$. Since this is a $\Box$-stable statement, we can in fact assume that $(\varphi \Rightarrow \psi)$. We thus have

$$\varphi^\Box \Longrightarrow \Box\varphi \Longrightarrow \Box\psi \Longrightarrow \psi^\Box,$$

with the first step being by the requirement on antecedents, the second by the monotonicity of $\Box$, and the third by the induction hypothesis. $\Box$

Lemma 8.14. *Let φ be a geometric formula. Then $\varphi^\Box \Rightarrow \Box\varphi$.*

Proof. By induction on the formula structure. By way of example, we prove the case for "$\bigvee$". So assume $\Box(\bigvee_i \varphi_i^\Box)$; we are to show that $\Box(\bigvee_i \varphi_i)$. Since this is a boxed statement, we may in fact assume $\bigvee_i \varphi_i^\Box$, so for some index j, it holds that $\varphi_j^\Box$. By the induction hypothesis, it follows that $\Box\varphi_j$. By $\varphi_j \Rightarrow \bigvee_i \varphi_i$ and the monotonicity of $\Box$, it follows that that $\Box(\bigvee_i \varphi_i)$. ☐

Remark 8.15. In the special case that $\Box$ is the double negation modality, the lemma holds with slightly weaker hypotheses: Namely, implications may occur in φ, provided that for their antecedents ψ it holds that $\psi \Rightarrow \psi^\Box$.

Lemma 8.16. *Let φ, φ', ψ be formulas. Assume that:*

- *The formula φ' is geometric. [It suffices for $(\varphi')^\Box$ to imply $\Box\varphi'$.]*
- *There is an intuitionistic proof that φ and φ' are equivalent under the (only) hypothesis ψ.*
- *Both $\Box\psi$ and $\psi^\Box$ hold.*

Then $\varphi^\Box \Rightarrow \Box\varphi$.

Proof. Assume $\varphi^\Box$. Since $\psi^\Box$, $(\varphi \wedge \psi)^\Box$. Because the $\Box$-translation is sound with respect to intuitionistic logic (see XXX), it follows that $(\varphi')^\Box$. As φ' is geometric, it follows that $\Box\varphi'$. Since $\Box\psi$ holds, it follows that $\Box\varphi$. ☐

Remark 8.17. The requirement that there exists an intuitionistic proof is stronger than merely *knowing* that the equivalence holds.

Example 8.18. Let M be an R-module. Then the statement that M is zero is not geometric: $\varphi :\equiv (\forall x\colon M.\ x = 0)$. But if M is generated by some finite family $x_1, \ldots, x_n \colon M$, then φ is equivalent to the statement $\varphi' :\equiv (x_1 = 0 \wedge \cdots \wedge x_n = 0)$ which is geometric; and there is an intuitionistic proof of this equivalence. Since no implication signs occur in $\psi :\equiv \ulcorner M$ is generated by $x_1, \ldots, x_n\urcorner$, the lemma is applicable and shows that $\varphi^\Box$ implies $\Box\varphi$.

Lemma 8.19. *For the modality $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow \alpha) \Rightarrow \alpha)$, where α is a fixed proposition, the $\Box$-translation of the law of excluded middle holds. In particular, this is for the $\Box = \neg\neg$, where $\alpha = \bot$.*

Proof. We are to show that $(\varphi \vee \neg\varphi)^\Box$, i. e. that

$$(((\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)) \Longrightarrow \alpha) \Longrightarrow \alpha.$$

So assume that the antecedent holds. If $\varphi^\Box$ would hold, then in particular $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α would hold. Therefore it follows that $(\varphi^\Box \Rightarrow \alpha)$. This implies $\varphi^\Box \vee (\varphi^\Box \Rightarrow \alpha)$ and thus α. ☐

8.7. **Truth at stalks vs. truth on neighbourhoods.** We now state the crucial property of the $\Box$-translation. Recall that "$X_\Box$" denotes the sublocale of X induced by $\Box$ (definition 8.5).

Theorem 8.20. *Let X be a topological space. Let $\Box$ be a modal operator in $\mathrm{Sh}(X)$. Let φ be a formula. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(X_\Box) \models \varphi.$$

Corollary 8.21. *Let X be a topological space.*

1. *Let $U \subseteq X$ be an open subset and let $\Box\varphi :\equiv (U \Rightarrow \varphi)$. Then*
$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(U) \models \varphi.$$
2. *Let $A \subseteq X$ be a closed subset and let $\Box\varphi :\equiv (\varphi \vee A^c)$. Then*
$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \mathrm{Sh}(A) \models \varphi.$$

(3) *Let $x \in X$ be a point and let $\Box\varphi :\equiv ((\varphi \Rightarrow {!x}) \Rightarrow {!x})$. Then*

$$\mathrm{Sh}(X) \models \varphi^\Box \quad \textit{iff} \quad \varphi \textit{ holds at } x.$$

Proof. Combine theorem 8.20 and table 2. ☐

We want to discuss the third case of the corollary in more detail. Let x be a point of a topological space X and let φ be a formula. Let $\Box$ be the modal operator given in the corollary. Then φ *holds at* x if and only if, from the internal perspective of $\mathrm{Sh}(X)$, the translated formula $\varphi^\Box$ holds; and φ *holds on some open neighbourhood of* x if and only if, from the internal perspective, the formula $\Box\varphi$ holds.

Thus the question whether the truth of φ at the point x spreads to some open neighbourhood can be formulated in the following way: *Does $\varphi^\Box$ imply $\Box\varphi$ in the internal language of* $\mathrm{Sh}(X)$*?*

Corollary 8.22. *Let X be a topological space. Let φ be a formula. If φ is geometric, truth of φ at a point $x \in X$ implies truth of φ on some open neighbourhood of x, and vice versa.*

Proof. By the purely logical lemmas of the previous section, it holds that $\varphi^\Box \Leftrightarrow \Box\varphi$. ☐

Proof of theorem 8.20. ... ☐

Example 8.23. Let X be a scheme (or a ringed space). Since the condition for a function $f : \mathcal{O}_X$ to be nilpotent is geometric (it is $\bigvee_{n \geq 0} f^n = 0$), nilpotency of f at a point is equivalent to nilpotency on some open neighbourhood.

Proposition 8.24. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Let $x \in X$ be a point. Let n be a natural number. Then the following statements are equivalent:*

(1) *There exists a generating family for $\mathcal{F}_x$ consisting of n elements.*
(2) *There exists an open neighbourhood U of x such that*

$$U \models \ulcorner\text{there exists a generating family for } \mathcal{F} \text{ consisting of } n \text{ elements}\urcorner.$$

Proof. Using the modal operator $\Box$ defined by $\Box\varphi :\equiv ((\varphi \Rightarrow {!x}) \Rightarrow {!x})$, we have to show that the following statements in the internal language are equivalent:

(1) $\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner^\Box$.
(2) $\Box(\ulcorner$there exists a generating family for $\mathcal{F}$ consisting of n elements$\urcorner)$.

By lemma 8.13, the second statement implies the first – note that in a formal spelling of the statement in quotes,

$$(\star) \qquad \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i,$$

no implication signs occur. To show the converse direction, we may assume that there is a generating family $y_1, \ldots, y_m : \mathcal{F}$ for $\mathcal{F}$ (since $\mathcal{F}$ is, externally speaking, of finite type). Then it the $\Box$-translation of the statement that the y_i generate $\mathcal{F}$ holds as well (again by lemma 8.13). Since there is an intuitionistic proof of

$$\ulcorner y_1, \ldots, y_m \text{ generate } \mathcal{F}\urcorner \Longrightarrow \Big(\ulcorner\text{there exist } x_1, \ldots, x_n : \mathcal{F} \text{ which generate } \mathcal{F}\urcorner \Longleftrightarrow \exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m \times n}.\ \ulcorner\vec{y} = A\vec{x}\urcorner\Big),$$

we can substitute the non-geometric formula ($\star$) by the geometric formula

$$\exists x_1, \ldots, x_n : \mathcal{F}.\ \exists A : \mathcal{O}^{m \times n}.\ \ulcorner \vec{y} = A\vec{x} \urcorner$$

(lemma 8.16). Thus the claim follows. $\square$

Proposition 8.25. *Let X be a scheme (or ringed space). Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.*

- *Let $x \in X$ be a point. Then the stalk $\mathcal{F}_x$ is zero if and only if $\mathcal{F}$ is zero on some open neighbourhood of x.*
- *Let $A \subseteq X$ be a closed subset. Then the restriction $\mathcal{F}|_A$ (i. e. the pullback of $\mathcal{F}$ to A) is zero if and only if $\mathcal{F}$ is zero on some open subset of X containing A.*

Proof. Both statements are simply internalizations of example 8.18, using the modal operators $\Box = (\_\_ \vee A^c)$ respectively $\Box = ((\_\_ \Rightarrow {!x}) \Rightarrow {!x})$. $\square$

Remark 8.26. Note that the proposition fails if one drops the hypothesis that $\mathcal{F}$ is of finite type. Indeed, in this case one cannot reformulate the condition that $\mathcal{F}$ is zero in a geometric way.

Proposition 8.27. *Let X be a scheme (or ringed space). Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of $\mathcal{O}_X$-modules. Let $\mathcal{G}$ be of finite type and assume that $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for some point $x \in X$. Then α is an epimorphism on some open neighbourhood of x.*

Proof. In the presence of generators $y_1, \ldots, y_n : \mathcal{G}$, the non-geometric surjectivity condition ($\forall y : \mathcal{G}.\ \exists x : \mathcal{F}.\ \alpha(x) = y$) can be reformulated in a geometric way: $\bigwedge_{i=1}^{n} \exists x : \mathcal{F}.\ \alpha(x) = y_i$. Thus the claim follows by lemma 8.16. $\square$

Proposition 8.28. *Let $i : A \hookrightarrow X$ be a closed immersion of schemes (or ringed spaces). Let $\mathcal{F}$ be an $\mathcal{O}_A$-module. Then $i_*\mathcal{F}$ is of finite type if and only if $\mathcal{F}$ is of finite type.*

Proof. $\square$

Proposition 8.29. *Let X be a scheme. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Let $x \in X$. Then $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$ if $\mathcal{F}$ is of finite presentation around x.*

Proof. It suffices to give an intuitionistic proof of the following fact: The construction $\mathrm{Hom}_R(M, \_\_)$ is geometric, if M is a finitely presented R-module. So assume that M is the cokernel of a presentation matrix $(a_{ij}) \in \mathcal{O}_X^{n \times m}$. Then we can calculate the Hom with any R-module N as

$$\mathrm{Hom}_R(M, N) \cong \Big\{ x : N^n \;\Big|\; \bigwedge_{j=1}^{m} \sum_{i=1}^{n} a_{ij} x_i = 0 : N \Big\},$$

and this construction is patently geometric (set comprehension with respect to a geometric formula). $\square$

Proposition 8.30. *Let X be a scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite presentation. Let $x \in X$. Then the stalk $\mathcal{F}_x$ is a finitely free $\mathcal{O}_{X,x}$-module if and only if $\mathcal{F}$ is locally free on some open neighbourhood of x.*

Proof. The internal statement that $\mathcal{F}$ is a free module is not geometric:

$$\bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists! a_1, \ldots, a_n : \mathcal{O}_X.\ x = \textstyle\sum_i a_i x_i.$$

But it can equivalently be reformulated as

$$\bigvee_{n \geq 0} \exists \alpha : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X^n).\ \exists \beta : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \mathcal{F}).\ \alpha \circ \beta = \mathrm{id} \wedge \beta \circ \alpha = \mathrm{id}.$$

This reformulation is geometric, therefore it holds at x if and only if it holds on some open neighbourhood of x. The claim follows since, by the previous proposition, taking stalks commutes with calculating $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \underline{\ \ })$ resp. $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X^n, \underline{\ \ })$. $\square$

- general explanation of modalities (as for instance in philosophy)
- explain that for some modal operators, the $\square$-translation of the law of excluded middle is valid; explain consequences
- spreading of properties from stalk to neighbourhood: give many examples
- give proof of the expressions for the nuclei listed in the table
- baby version of Barr's theorem

9. Quasicoherent sheaves of modules

Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space X is *quasicoherent* if and only if there exists a covering of X by open subsets U such that on each such U, there exists an exact sequence

$$(\mathcal{O}_X|_U)^J \longrightarrow (\mathcal{O}_X|_U)^I \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

of $\mathcal{O}_X|_U$-modules, where I and J are arbitrary sets (which may depend on U).

If X is indeed a scheme, quasicoherence can also be characterized in terms of inclusions of distinguished open subsets of affines: An $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent if and only if for any open affine subscheme $U = \operatorname{Spec} A$ of X and any function $f \in A$, the canonical map

$$\Gamma(U, \mathcal{F})[f^{-1}] \longrightarrow \Gamma(D(f), \mathcal{F}),\ \tfrac{s}{f^n} \longmapsto f^{-n} s|_{D(f)}$$

is an isomorphism of $A[f^{-1}]$-modules. Here $D(f) \subseteq U$ denotes the standard open subset $\{\mathfrak{p} \in \operatorname{Spec} A \,|\, f \notin \mathfrak{p}\}$. Both conditions can be internalized.

Proposition 9.1. *Let X be a ringed space. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \exists I, J \text{ lc. } \ulcorner\text{there exists an exact sequence } \mathcal{O}_X^J \to \mathcal{O}_X^I \to \mathcal{F} \to 0\urcorner.$$

The "lc" indicates that when interpreting this internal statement with the Kripke–Joyal semantics, I and J should only be instantiated with locally constant *sheaves.*

Proof. We only sketch the proof. The translation of the internal statement is that there exists a covering of X by open subsets U such that for each such U, there exist sets I, J and an exact sequence

$$(\mathcal{O}_X|_U)^{\underline{J}} \longrightarrow (\mathcal{O}_X|_U)^{\underline{I}} \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

where $\underline{I}$ and $\underline{J}$ are the constant sheaves associated to I respectively J. The term "$(\mathcal{O}_X|_U)^{\underline{I}}$" refers to the internally defined free $\mathcal{O}_X$-module with basis the elements of $\underline{I}$. By exploiting that $\underline{I}$ is a discrete set from the internal point of view (i. e. any two elements are either equal or not), one can show that this is the same as $(\mathcal{O}_X|_U)^I$; similarly for J. With this observation, the statement follows. $\square$

In practice, the internal condition given by the proposition is not very useful, since at the moment, we do not know of any internal characterization of locally constant sheaves. The internal condition given by the following proposition does not have this defect.

Proposition 9.2. *Let X be scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if, from the internal perspective, the localized module $\mathcal{F}[f^{-1}]$ is a sheaf for the modal operator ($\ulcorner f \text{ inv.}\urcorner \Rightarrow$ __) for any $f : \mathcal{O}_X$.*

In detail, the internal condition is that for any $f : \mathcal{O}_X$, it holds that

$$\forall s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s = 0) \Longrightarrow s = 0$$

and for any subsingleton $S \subseteq \mathcal{F}[f^{-1}]$ it holds that

$$(\ulcorner f \text{ inv.}\urcorner \Rightarrow \ulcorner S \text{ inhabited}\urcorner) \Longrightarrow \exists s : \mathcal{F}[f^{-1}].\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in S).$$

Unlike with the internalizations of finite type, finite presentation and coherence, this condition is *not* a standard condition of commutative algebra. In fact, in classical logic, this condition is always satisfied – for trivial logical reasons if f is invertible and because $\mathcal{F}[f^{-1}]$ is the zero module if f is not invertible (since then, it's nilpotent). This is to be expected: *Any* module M in commutative algebra is quasicoherent in the sense that the associated sheaf of modules $M^\sim$ is quasicoherent.

The proof will explain the origin of this condition.

Proof of proposition 9.2. ... □

Corollary 9.3. *Let X be a scheme. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module. Let $\mathcal{G} \subseteq \mathcal{F}$ be a submodule. Then $\mathcal{G}$ is quasicoherent if and only if*

$$\mathrm{Sh}(X) \models \forall f : \mathcal{O}_X.\ \forall s : \mathcal{F}.\ (\ulcorner f \text{ inv.}\urcorner \Rightarrow s \in \mathcal{G}) \Longrightarrow \bigvee_{n \geq 0} f^n s \in \mathcal{G}.$$

Proof. We can give a purely internal proof. Let $f : \mathcal{O}_X$. Since subpresheaves of separated sheaves are separated, the module $\mathcal{G}[f^{-1}]$ is in any case separated with respect to the modal operator ($\ulcorner f \text{ inv.}\urcorner \Rightarrow$ __).

Now suppose that $\mathcal{G}$ is quasicoherent. Let $f : \mathcal{O}_X$. Let $s : \mathcal{F}$ and assume that if f were invertible, s would be an element of $\mathcal{G}$. Define the subsingleton $S := \{t : \mathcal{G}[f^{-1}] \mid \ulcorner f \text{ inv.}\urcorner \wedge t = s/1\}$. Then S would be inhabited by $s/1$ if f were invertible. Since $\mathcal{G}[f^{-1}]$ is a sheaf, it follows that there exists an element u/f^n of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be the case that $u/f^n = s/1 \in \mathcal{G}[f^{-1}] \subseteq \mathcal{F}[f^{-1}]$. Since $\mathcal{F}[f^{-1}]$ is separated, it follows that it actually holds that $u/f^n = s/1 \in \mathcal{F}[f^{-1}]$. Therefore there exists $m : \mathbb{N}$ such that $f^m f^n s = f^m u \in \mathcal{F}$. Thus $f^{m+n} s$ is an element of $\mathcal{G}$.

For the converse direction, assume that $\mathcal{G}$ fulfills the stated condition. Let $f : \mathcal{O}_X$. Let $S \subseteq \mathcal{G}[f^{-1}]$ be a subsingleton which would be inhabited if f were invertible. By regarding S as a subset of $\mathcal{F}[f^{-1}]$, it follows that there exists an element $u/f^n \in \mathcal{F}[f^{-1}]$ such that, if f were invertible, u/f^n would be an element of S. In particular, u would be an element of $\mathcal{G}$. By assumption it follows that there exists $m : \mathbb{N}$ such that $f^m u \in G$. Thus $(f^m u)/(f^m f^n)$ is an element of $\mathcal{G}[f^{-1}]$ such that, if f were invertible, it would be an element of S. □

Example 9.4. Let X be a scheme and s be a global section of $\mathcal{O}_X$. Then the annihilator of s, i. e. the sheaf of ideals internally defined by the formula

$$I := \mathrm{Ann}_{\mathcal{O}_X}(s) = \{t : \mathcal{O}_X \mid st = 0\} \subseteq \mathcal{O}_X$$

is quasicoherent. To prove this in the internal language, it suffices to verify the condition of the proposition. So let $f : \mathcal{O}_X$ be arbitrary and assume $\ulcorner f \text{ inv.}\urcorner \Rightarrow t \in I$, i. e. assume that if f were invertible, st would be zero. By proposition 3.8 it follows that $f^n st = 0$ for some $n : \mathbb{N}$, i. e. that $f^n t \in I$.

- is the condition good enough to show that modules of finite type are quasicoherent? To show that cokernels are quasicoherent?
- discussion meaning of the sheaf condition in external language
- give more examples: $\sqrt{(0)}$, (h), ...
- Noetherian hypotheses: for example, that any quasicoherent submodule of a module of finite type is of finite type as well

10. Unsorted

- "functoriality"
- Kähler differentials
- closed and open subschemes
- reduced closed subscheme
- Koszul resolution
- meta properties, uses (e.g. nilpotent on stalks iff globally nilpotent, some lemmas about limits of modules)
- compactness principle for "f inv."
- locally small categories
- big Zariski topos
- open/closed immersions
- morphisms of schemes...
- proper maps...
- limits and colimits...
- related work: Mulvey/Burden, Wraith, Vickers, the Bohr topos crew, Awodey, ...

References

[1] S. Awodey and A. Bauer. "Propositions as [Types]". In: *J. Logic Comput.* 14.4 (2004), pp. 447–471.
[2] J. Bell. "The development of categorical logic". In: *Handbook of Philosophical Logic*. Ed. by D. Gabbay and F. Guenthner. 2nd ed. Vol. 12. Springer, 2005, pp. 279–361.
[3] P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium.* Oxford University Press, 2002.
[4] S. Kleiman. "Misconceptions about K_X". In: *Enseign. Math.* 25 (1979), pp. 203–206.
[5] T. Leinster. "An informal introduction to topos theory". In: *Publications of the nLab* 1.1 (2011).
[6] S. Mac Lane and I. Moerdijk. *Sheaves in Geometry and Logic: a First Introduction to Topos Theory.* Universitext. Springer, 1992.
[7] R. Mines, F. Richman, and W. Ruitenburg. *A Course in Constructive Algebra.* Universitext. Springer, 1988.

[8] C. Mulvey. “Intuitionistic algebra and representations of rings”. In: *Recent Advances in the Representation Theory of Rings and C^*-algebras by Continuous Sections*. Ed. by K. H. Hofmann and J. R. Liukkonen. Vol. 148. Mem. Amer. Math. Soc. American Mathematical Society, 1974.

[9] M. Shulman. “Stack semantics and the comparison of material and structural set theories”. 2010. URL: `http://arxiv.org/abs/1004.3802`.

[10] The Stacks Project Authors. *Stacks Project*. URL: `http://stacks.math.columbia.edu/`.

[11] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction*. North-Holland Publishing, 1988.

[12] Ravi Vakil. *Foundations of Algebraic Geometry*. 2013. URL: `http://math.stanford.edu/~vakil/216blog/`.

[13] S. Vickers. “Locales and Toposes as Spaces”. In: *Handbook of Spatial Logics*. Ed. by M. Aiello, I. Pratt-Hartmann, and J. van Benthem. Springer, 2007, pp. 429–496.

E-mail address: `iblech@web.de`