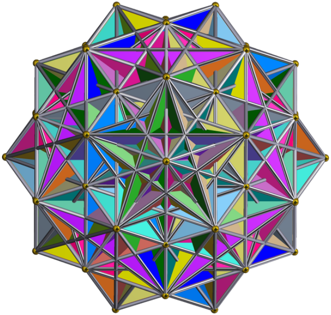
$$V_n = \int_0^1 \int_0^{2\pi} V_{n-2}(\sqrt{1-r^2})^{n-2}\, r\, d\theta\, dr$$

The curious world of four-dimensional geometry

Ingo Blechschmidt and Matthias Hutzler
with thanks to Sven Prüfer
Universität Augsburg

December 29th, 2016



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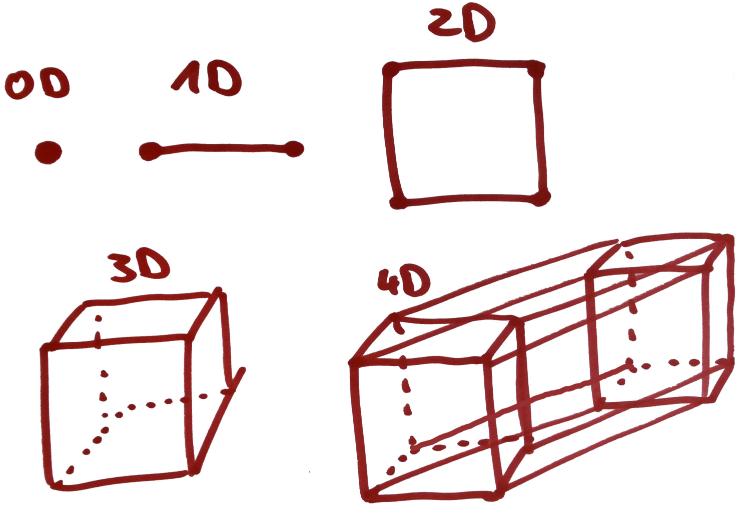
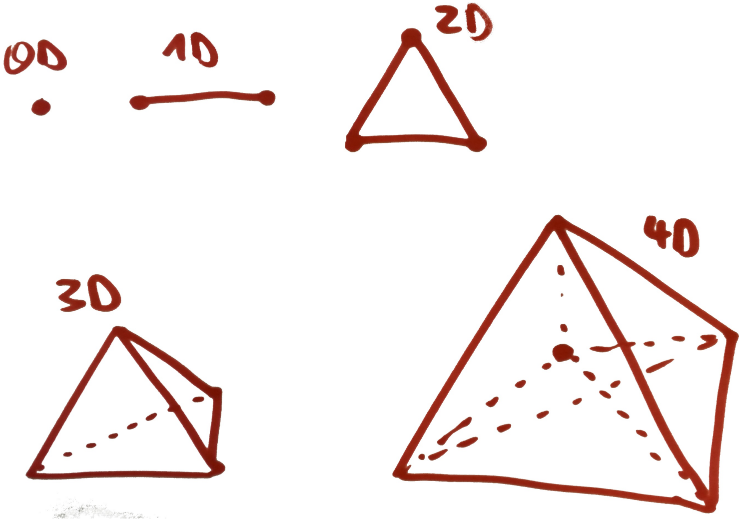
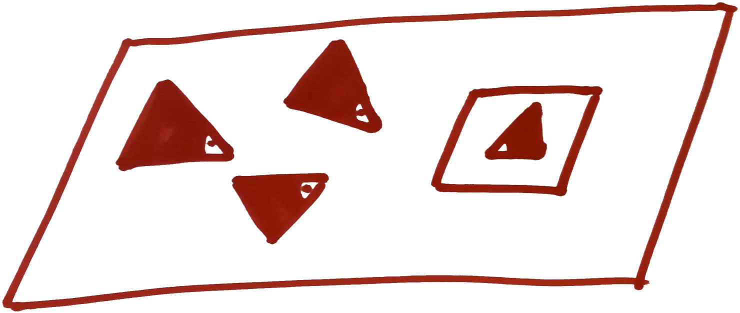
Four dimensions?



Four dimensions?

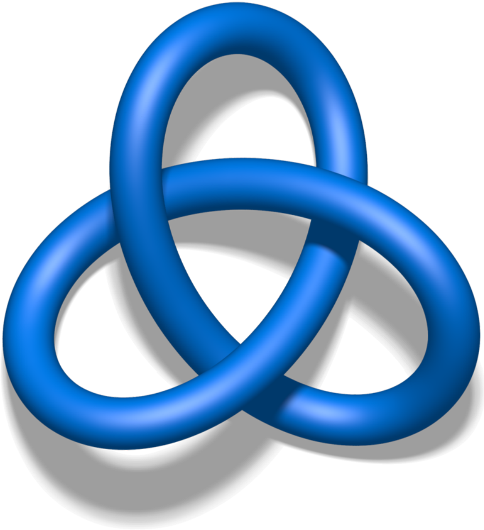


Four dimensions?



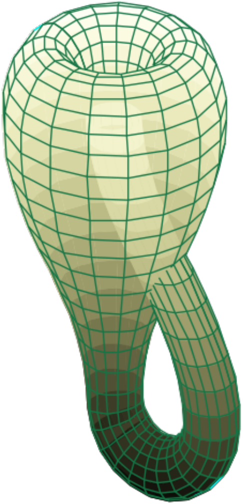
- On the previous slide you see two-dimensional projections of the three-dimensional cube and the four-dimensional hypercube (tesseract).
- We're talking about four spatial dimensions. This is not related to four-dimensional spacetime or eleven-dimensional string theory.
- A flatlander can be imprisoned by enclosing them with a square. But we, as three-dimensional beings, can free them by grabbing them, lifting them up in the third dimension, moving them a little to the side, and putting them back into flatland.
- Similarly, a four-dimensional being could free us if we were imprisoned in a three-dimensional cube.

Tying your shoelaces

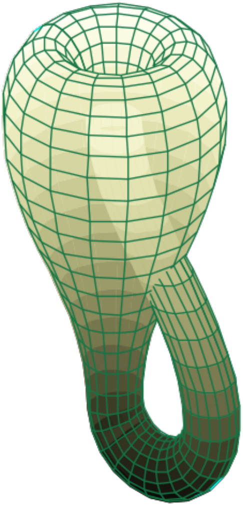


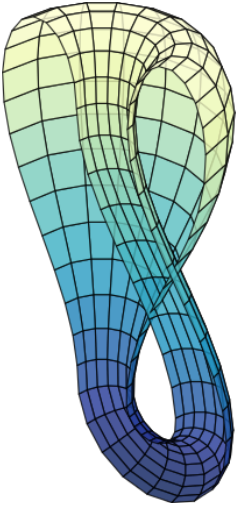
- You can untie any knot in four dimensions. Two linked one-dimensional strings can always be separated in four dimensions.
- But it's possible to tangle an one-dimensional string with the two-dimensional surface of a sphere in four dimensions.
- More generally, in n dimensions, one can tangle a-dimensional objects with b-dimensional objects provided that $a + b \geq n - 1$.

The Klein bottle



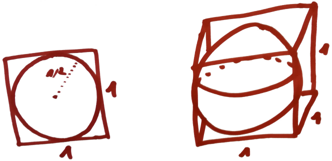
The Klein bottle





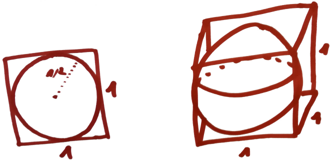
- The familiar torus (donut) can be obtained from a cylinder by glueing the two bounding circles together.
- The Klein bottle can be obtained in the same way, but with flipping one of the bounding circles first.
- In three dimensions, the Klein bottle can only be realized with a self-intersection. Only in four dimensions it's possible to exhibit the true Klein bottle.
- Like the Möbius strip, the Klein bottle is not orientable: It has only one side. Unlike the Möbius strip, it doesn't have a boundary.
- A mathematician named Klein
  Thought the Möbius band was divine.
  Said he: "If you glue
  The edges of two,
  You'll get a weird bottle like mine."
  – Leo Moser

Hypervolume of hyperballs



dimension	hypervolume	
$n = 2$	$\pi/4$	≈ 0.785
$n = 3$	$\pi/6$	≈ 0.524
$n = 4$	$\pi^2/32$	≈ 0.308
$n = 5$	$\pi^2/60$	≈ 0.164
$n = 6$	$\pi^3/384$	≈ 0.081
$n = 7$	$\pi^3/840$	≈ 0.037
$n \to \infty$	$\to 0$	

Hypervolume of hyperballs



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$n = 7$	$\pi^3/840$	≈ 0.037
$n \to \infty$	$\to 0$	
$n = 0$	1	≈ 1.000
$n = 1$	1	≈ 1.000

- The portion of the n-dimensional unit hypercube which is occupied by the inscribed n-dimensional hyperball gets arbitrary small in sufficiently high dimensions.

- The volume of such a hyperball is the answer to the following question: What is the probability that we managed to hit the hyperball with an dart, provided that we managed to hit the enclosing hyperball?

- Wikipedia gives derivations for these formulas.

- You can use the *power of negative thinking* to motivate that the formula for the n-dimensional volume of the n-dimensional hyperball does *not* contain π^n (but rather $\pi^{\lfloor n/2 \rfloor}$): Think about the zero- and one-dimensional case.

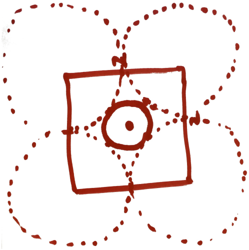
  A zero-dimensional ball is just a point. Its zero-dimensional volume is 1.

  An one-dimensional ball is just a line segment. Its one-dimensional volume is its length.

Love is important.

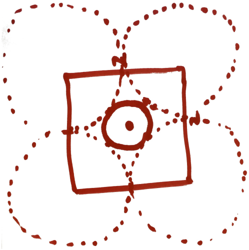
♡

Kissing hyperspheres



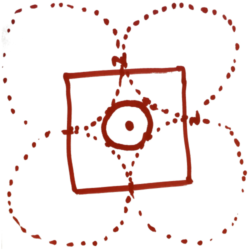
dimension	radius of the inner hypersphere
$n = 2$	

Kissing hyperspheres



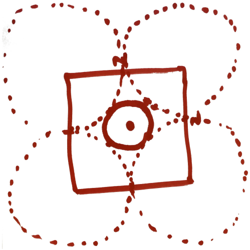
dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$

Kissing hyperspheres



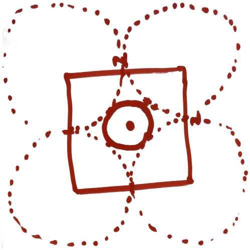
dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	

Kissing hyperspheres



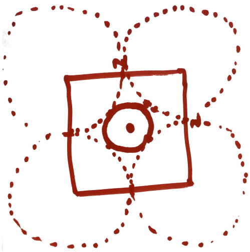
dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	$\sqrt{3} - 1$

Kissing hyperspheres



dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	$\sqrt{3} - 1$
$n = 4$	$\sqrt{4} - 1$

Kissing hyperspheres



dimension	radius of the inner hypersphere
$n = 2$	$\sqrt{2} - 1$
$n = 3$	$\sqrt{3} - 1$
$n = 4$	$\sqrt{4} - 1$
n	$\sqrt{n} - 1$

The distance to the corners gets bigger and bigger.

- In two dimensions, the distance of a point (x, y) to the origin is $\sqrt{x^2 + y^2}$ (by the Pythagorean theorem).

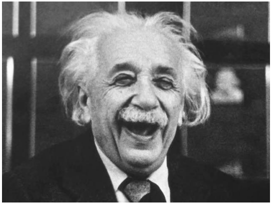
- In three dimensions, the distance of a point (x, y, z) to the origin is $\sqrt{x^2 + y^2 + z^2}$.

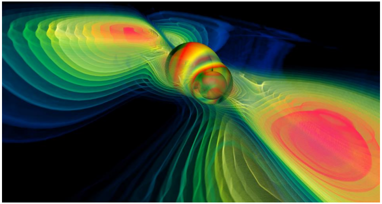
- The pattern continues to arbitrary dimensions.

- In four dimensions, the "small hypersphere in the middle" has exactly the same size as the hyperspheres at the 16 vertices of the hypercube.

- In even greater dimensions, the hyperspheres at the vertices are so small that the "small hypersphere in the middle" is bigger than them and in fact bigger than the hypercube!

General relativity





Einstein's celebrated **field equation** states that

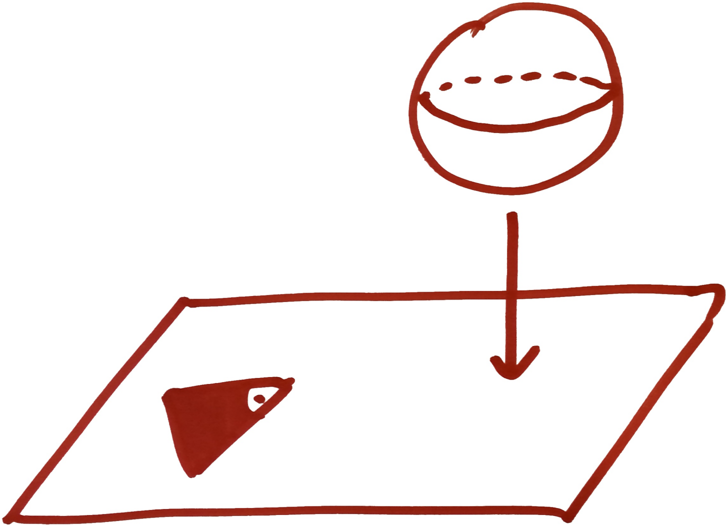
$$G = \kappa \cdot T,$$

where

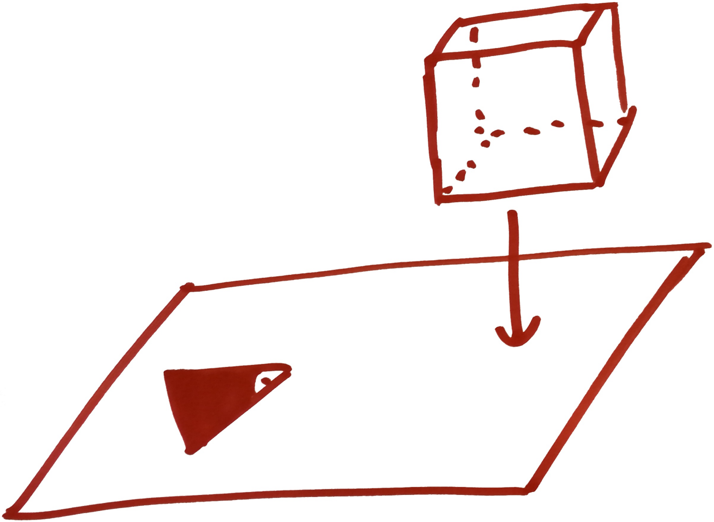
- G relates to the **curvature** of space,
- T measures the **mass distribution**, and
- κ is a constant.

In $2+1$ dimensions, the equation implies $T = 0$. The theory is nontrivial only in four or more dimensions.

A hyperball arrives



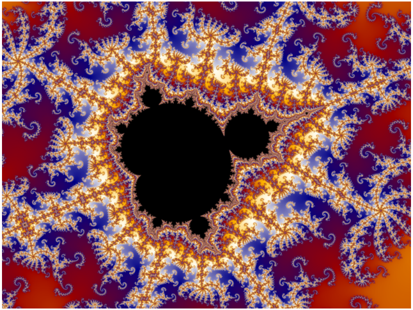
A tesseract arrives



A four-dimensional fractal

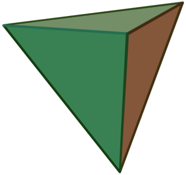
You know the Mandelbrot set. Maybe you also know the Julia sets associated to any point of the plane.

But did you know that these infinitely many fractals are just two-dimensional cuts of an unifying four-dimensional fractal? We invite you to play with it.

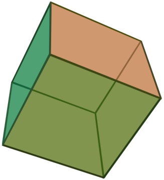


Platonic solids in 3D

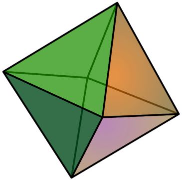
Tetrahedron
4 faces, 4 vertices



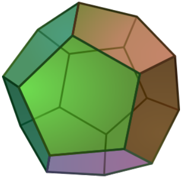
Hexahedron
6 faces, 8 vertices



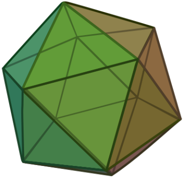
Octahedron
8 faces, 6 vertices



Dodecahedron
12 faces, 20 vertices

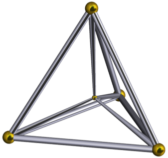


Icosahedron
20 faces, 12 vertices

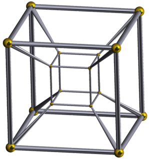


Platonic solids in 4D

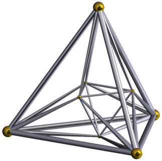
Pentachoron
5v, 10e, 10f, 5c



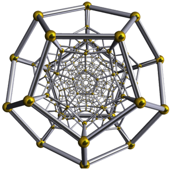
Octachoron
16v, 32e, 24f, 8c



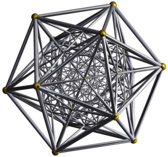
Hexadecachoron
8v, 24e, 32f, 16c



Hecatonicosachoron
600v, 1200e, 720f, 120c

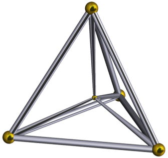


Hexacosichoron
120v, 720e, 1200f, 600c



Platonic solids in 4D

Pentachoron
5v, 10e, 10f, 5c



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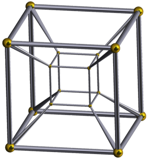
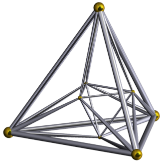
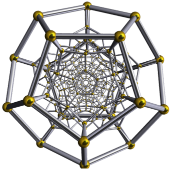


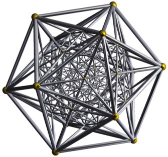
Hexadecachoron
8v, 24e, 32f, 16c



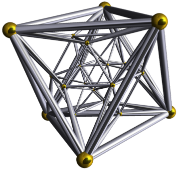
Hecatonicosachoron
600v, 1200e, 720f, 120c



Hexacosichoron
120v, 720e, 1200f, 600c



Icositetrachoron
24v, 96e, 96f, 24c



Platonic solids in arbitrary dimensions

dimension	number of Platonic solids
$n = 1$	1 (just the line segment)
$n = 2$	∞ (triangle, square, $\ldots$; any regular polygon)
$n = 3$	5
$n = 4$	6
$n > 4$	3 (just the simplex, the hypercube and its dual)

Glueing four-dimensional shapes

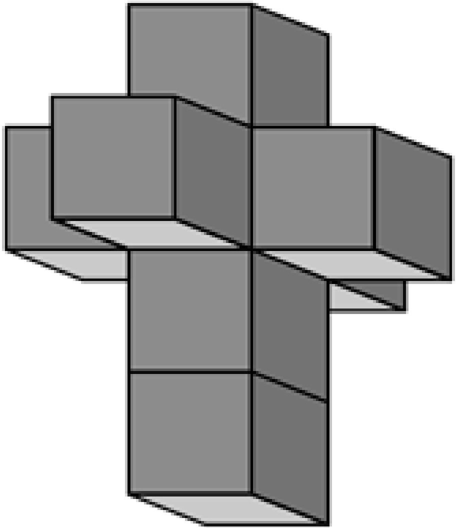


Image sources

Miscellaneous pictures:
https://commons.wikimedia.org/wiki/File:Blue_Trefoil_Knot.png
http://www.gnuplotting.org/figs/klein_bottle.png
http://4.bp.blogspot.com/_TbkIC-eqFNM/S-dK9dd1cUI/AAAAAAAAFjA/d8qdTHhKy1U/s320/tesseract+unfolded.png
https://en.wikipedia.org/wiki/File:Tetrahedron.svg
https://en.wikipedia.org/wiki/File:Hexahedron.svg
https://en.wikipedia.org/wiki/File:Octahedron.svg
https://en.wikipedia.org/wiki/File:Dodecahedron.svg
https://en.wikipedia.org/wiki/File:Icosahedron.svg

Rendered images of four-dimensional bodies created by Robert Webb with his Stella software:
https://en.wikipedia.org/wiki/File:Ortho_solid_011-uniform_polychoron_53p-t0.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_5-cell.png
https://en.wikipedia.org/wiki/File:Schlegel_wireframe_8-cell.png
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