# Three bizarre logico-philosophical tales about the axiom of choice

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# Choice functions

*The axiom of choice asserts:*

"For **every** collection of inhabited sets, there is a **choice function** picking **representatives** from each set."

Examples for functions:

1. sine function: $x \mapsto \sin(x)$
2. squaring function: $x \mapsto x^2$, so $1 \mapsto 1,\ 2 \mapsto 4,\ 3 \mapsto 9,\ \ldots$
3. computeAreaOfCircle: $r \mapsto \pi r^2$, so $1 \mapsto \pi,\ 2 \mapsto 4\pi,\ \ldots$
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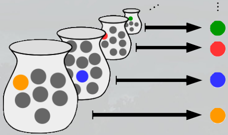
4. document.getElementById
5. lookupMayorOfCity
6. getYoungestStudentOfClass

"choice functions"



**Note.** The axiom of choice is **superfluous** for …

A. **finite** collections

B. collections of inhabited decidable sets of natural numbers

# What a choice function can do for us in the riddle

For the collection of **sets of almost-identical scenarios**, a choice function could look like this:

$$\left\{\begin{array}{c}
\cdots \boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3} \cdots \\
\cdots \boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{1}\,\boxed{3}\,\boxed{3}\,\boxed{3} \cdots \\
\cdots \boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{1}\,\boxed{1}\,\boxed{3}\,\boxed{3} \cdots \\
\cdots \boxed{3}\,\boxed{3}\,\boxed{9}\,\boxed{9}\,\boxed{9}\,\boxed{3}\,\boxed{3} \cdots \\
\vdots
\end{array}\right\} \longmapsto \cdots \boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3} \cdots$$

$$\left\{\begin{array}{c}
\cdots \boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1} \cdots \\
\cdots \boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{4}\,\boxed{1}\,\boxed{2}\,\boxed{1} \cdots \\
\cdots \boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{5}\,\boxed{1}\,\boxed{2}\,\boxed{1} \cdots \\
\cdots \boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{6}\,\boxed{1}\,\boxed{2}\,\boxed{1} \cdots \\
\vdots
\end{array}\right\} \longmapsto \cdots \boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1} \cdots$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

If the players use a **common choice function** to make their guesses, **only finitely many** will be incorrect.

# Consequences of the axiom of choice

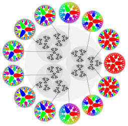
## “Weird”:

Vitali fractal

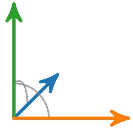
Banach–Tarski paradox

Prophecy

## “Good/procrastinatory”:

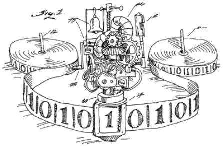


Every field has an algebraic closure.



Every vector space has a basis.

# An alternative universe: the effective topos



| | statement | in Std | in Eff |
|---|---|---|---|
| 1 | Every number is prime or not prime. | ✓ (trivially) | ✓ |
| 2 | After every number there is a prime. | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ? |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ? |
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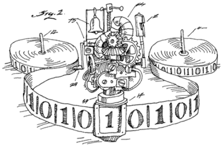

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"1" in the effective topos amounts to: There is a machine which determines of any given number whether it is prime or not.

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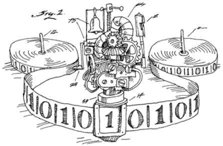


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"2" in the effective topos amounts to: There is a machine which, given a number $n$, computes a prime larger than $n$.

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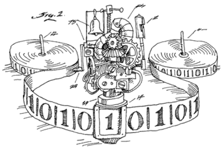

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"3" in the effective topos amounts to: There is a machine which, given a machine computing a map $f : \mathbb{N} \to \mathbb{N}$, determines whether $f$ has a zero or not.

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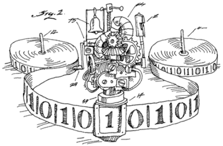

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"4" in the effective topos amounts to: There is a machine which, given a machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs a machine computing $f$.

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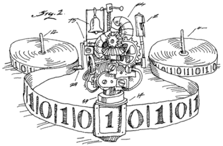

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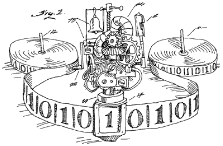

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In Eff, there is **no choice function** for the collection of **sets of behaviourally identical programs**.

# A counterexample to the axiom of choice

A **choice function** for the collection of **sets of behaviourally identical programs** would look like this:

$$\left\{\begin{array}{l}\texttt{while True: pass}\\ \texttt{while 2 == 1 + 1: pass}\\ \texttt{s = "a"; while len(s) > 0: s = s + "a"}\\ \vdots\end{array}\right\} \longmapsto \texttt{while True: pass}$$

$$\left\{\begin{array}{l}\texttt{print(2+2)}\\ \texttt{print(4)}\\ \texttt{print(len("37c3"))}\\ \vdots\end{array}\right\} \longmapsto \texttt{print(4)}$$

$$\vdots \qquad\qquad \vdots$$

With such a choice function c, a **halting oracle** can be built:
A program $p$ loops if and only if $\mathrm{c}(p) = \mathrm{c}(\text{"while True: pass"})$.

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4. Much more severe than the axiom of choice is the **powerset axiom**. 😱