♥ P vs. NP ♥

the biggest open question in computer science

– an invitation –

36th Chaos Communication Congress

Questions are very much welcome! Please interrupt me mid-sentence.

Ingo Blechschmidt
University of Augsburg

The landscape of complexity classes

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Prop. Every P-problem is also in NP: P ⊆ NP.

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P ≠ EXP, hence P ≠ NP or NP ≠ PSPACE or PSPACE ≠ EXP.

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Thm. For some B, $\mathrm{P}^B = \mathrm{NP}^B$; and for some B, $\mathrm{P}^B \neq \mathrm{NP}^B$.
Proof, first part. Pick for B some problem in PSPACE-C. Then $\mathrm{PSPACE} \subseteq P^B \subseteq \mathrm{NP}^B \subseteq \mathrm{PSPACE}^B \subseteq \mathrm{PSPACE}$.
Proof, second part. Pick for B a zero/one **random oracle**. Then the problem "do n consecutive ones occur in the first 2^n drawings of B?" is in NP^B but not in P^B.