– *an invitation* –

# Exploring hypercomputation with the effective topos


IMJ-PRG, Paris
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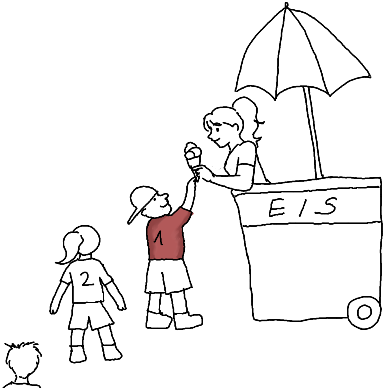
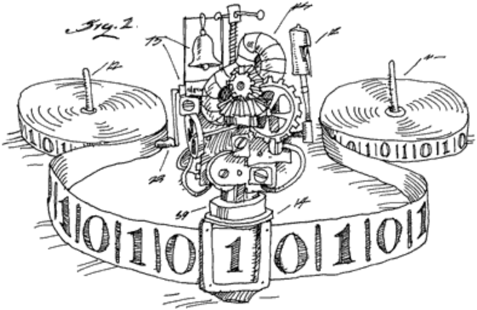
# Part I

## A crash course on ordinal numbers



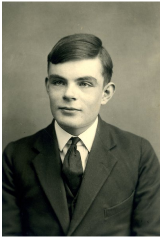
# Part II

## (Super) Turing machines

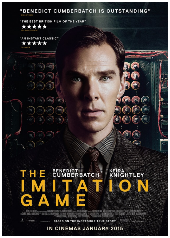


# Basics on Turing machines

- Turing machines are idealised computers operating on an **infinite tape** according to a **finite list** of rules.
- The concept is astoundingly robust.
- A subset of $\mathbb{N}$ is **enumerable by a Turing machine** if and only if it's a $\Sigma_1$-set.



Alan Turing
(* 1912, † 1954)




worth watching

Alison Bechdel
(* 1960)

# Super Turing machines

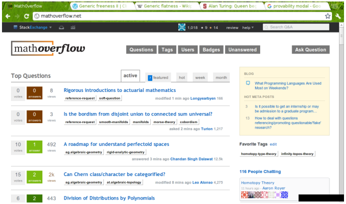
With super Turing machines, the time axis is more interesting:

- normal: $0,\ 1,\ 2,\ \ldots$
- super: $0,\ 1,\ 2,\ \ldots,\ \omega,\ \omega+1,\ \ldots,\ \omega\cdot 2,\ \omega\cdot 2+1,\ \ldots\ldots\ldots$

On reaching a limit ordinal time step like $\omega$ or $\omega\cdot 2$,

- the machine is put into a designated state,
- the read/write head is moved to the start of the tape, and
- the tape is set to the “lim sup” of all its previous contents.

Joel David Hamkins




MathOverflow

Andy Lewis

# A question for you

What's the behaviour of this super Turing machine?

In the start state and the limit state, check whether the current cell contains a "1".

- If yes, then stop.
- If not, then flash that cell: set it to "1", then reset it to "0". Then unremittingly move the head rightwards.

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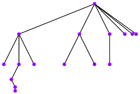
- If yes, then stop.
- If not, then flash that cell: set it to "1", then reset it to "0". Then unremittingly move the head rightwards.

This machine halts after time step $\omega^2$.

**Super Turing machines can break out of (some kinds of) infinite loops.**

# What can super Turing machines do?

- Everything ordinary Turing machines can do.
- Verify number-theoretic statements.
- Decide whether a given ordinary Turing machine halts.
- Simulate super Turing machines.
- Decide $\Pi^1_1$- and $\Sigma^1_1$-statements:
  - "For every function $\mathbb{N} \to \mathbb{N}$ it holds that …"
  - "There is a function $\mathbb{N} \to \mathbb{N}$ such that …"



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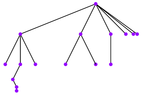
**But:** Super Turing machines can't compute all functions and can't write every 0/1-sequence to the tape.



# Fun facts

- Every super Turing machine either halts or gets caught in an unbreakable infinite loop after **countably many steps**.
- An ordinal number $\alpha$ is **clockable** iff there is a super Turing machine which halts precisely after time step $\alpha$.
  - Speed-up Lemma: If $\alpha + n$ is clockable, then so is $\alpha$.
  - Big Gaps Theorem
  - Many Gaps Theorem
  - Gapless Blocks Theorem
- **Lost Melody Theorem:** There are 0/1-sequences which a super Turing machine can recognise, but not write to the tape.

# Part III

## The effective topos

# The effective topos

| | statement | in Set | in Eff(TM) | in Eff(STM) |
|---|---|---|---|---|
| 1 | Every number is prime or not prime. | ✓ (trivially) | ✓ | ✓ |
| 2 | Beyond every number there is a prime. | ✓ | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ? | ? |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ? | ? |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

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| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ? | ? |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

"Eff(TM) $\models$ 1" amounts to: There is a Turing machine which determines of any given number whether it is prime or not.

# The effective topos

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| 2 | Beyond every number there is a prime. | ✓ | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ? | ? |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ? | ? |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

"Eff(TM) $\models$ 2" amounts to: There is a Turing machine which, given a number $n$, computes a prime larger than $n$.

# The effective topos

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| 1 | Every number is prime or not prime. | ✓ (trivially) | ✓ | ✓ |
| 2 | Beyond every number there is a prime. | ✓ | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ? | ? |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ? | ? |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

"$\mathrm{Eff}(\mathrm{TM}) \models$ 3" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, determines whether $f$ has a zero or not.

# The effective topos

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|---|---|---|---|---|
| 1 | Every number is prime or not prime. | ✓ (trivially) | ✓ | ✓ |
| 2 | Beyond every number there is a prime. | ✓ | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ? |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ? | ? |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

"$\mathrm{Eff}(\mathrm{TM}) \models$ 4" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs a Turing machine computing $f$.

# The effective topos

| | statement | in Set | in Eff(TM) | in Eff(STM) |
|---|---|---|---|---|
| 1 | Every number is prime or not prime. | ✓ (trivially) | ✓ | ✓ |
| 2 | Beyond every number there is a prime. | ✓ | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ? | ? |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

"$\mathrm{Eff}(\mathrm{STM}) \models$ 4" amounts to: There is a super Turing machine which, given a *super* Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs an (*ordinary*) Turing machine computing $f$.

# The effective topos

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| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ? |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

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| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ✗ |
| 6 | Markov's principle holds. | ✓ (trivially) | ? | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
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| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ✗ |
| 6 | Markov's principle holds. | ✓ (trivially) | ✓ (if MP) | ? |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

"$\mathrm{Eff}(\mathrm{TM}) \models$ 6" amounts to: There is a Turing machine which, given a Turing machine computing a map $f : \mathbb{N} \to \mathbb{N}$ and given the promise that it is *not not* the case that $f$ has a zero, determines a zero of $f$.

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| 2 | Beyond every number there is a prime. | ✓ | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ✗ |
| 6 | Markov's principle holds. | ✓ (trivially) | ✓ (if MP) | ✓ (if MP) |
| 7 | Countable choice holds. | ✓ | ? | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

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| 2 | Beyond every number there is a prime. | ✓ | ✓ | ✓ |
| 3 | Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not. | ✓ (trivially) | ✗ | ✓ |
| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ✗ |
| 6 | Markov's principle holds. | ✓ (trivially) | ✓ (if MP) | ✓ (if MP) |
| 7 | Countable choice holds. | ✓ | ✓ (always!) | ? |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ? | ? |

"Eff(TM) $\models$ 7" amounts to: There is a Turing machine which, given a Turing machine computing for every $x \in \mathbb{N}$ some $y \in A$ together with a witness of $\varphi(x, y)$, outputs a Turing machine computing a suitable choice function $\mathbb{N} \to A$.

# The effective topos

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| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ✗ |
| 6 | Markov's principle holds. | ✓ (trivially) | ✓ (if MP) | ✓ (if MP) |
| 7 | Countable choice holds. | ✓ | ✓ (always!) | ✓ (always!) |
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| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ✗ |
| 6 | Markov's principle holds. | ✓ (trivially) | ✓ (if MP) | ✓ (if MP) |
| 7 | Countable choice holds. | ✓ | ✓ (always!) | ✓ (always!) |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ✗ | ? |

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| 4 | Every map $\mathbb{N} \to \mathbb{N}$ is computable. | ✗ | ✓ (trivially) | ✗ |
| 5 | Every map $\mathbb{R} \to \mathbb{R}$ is continuous. | ✗ | ✓ (if MP) | ✗ |
| 6 | Markov's principle holds. | ✓ (trivially) | ✓ (if MP) | ✓ (if MP) |
| 7 | Countable choice holds. | ✓ | ✓ (always!) | ✓ (always!) |
| 8 | There is an injection $\mathbb{R} \hookrightarrow \mathbb{N}$. | ✗ | ✗ | ✓ |

# Curious size phenomena

$\mathrm{Eff}(\mathrm{STM}) \models$ "There exists an injection $\mathbb{N}^{\mathbb{N}} \hookrightarrow \mathbb{N}$."

means:

> There is a super Turing machine which inputs the source of a super Turing machine $A$ computing a function $\mathbb{N} \to \mathbb{N}$ and outputs a number $n(A)$ such that $n(A) = n(B)$ if and only if $A$ and $B$ compute the same function.

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This statement is witnessed by following super Turing machine:

| |
|---|
| Read the source of a super Turing machine $A$ from the tape. Simulate all super Turing machines in a dovetailing fashion. As soon a machine is found which has the same input/output behaviour as $A$, output the number of this machine and halt. |

# Wrapping up

- Effective toposes are a good vehicle for studying the nature of computation.
- Effective toposes build links between constructive mathematics and programming.
- Toposes allow for curious dream axioms.
- Toposes also have a geometric flavour: points, subtoposes, continuous maps between toposes.

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**There is more to mathematics than the standard topos.**