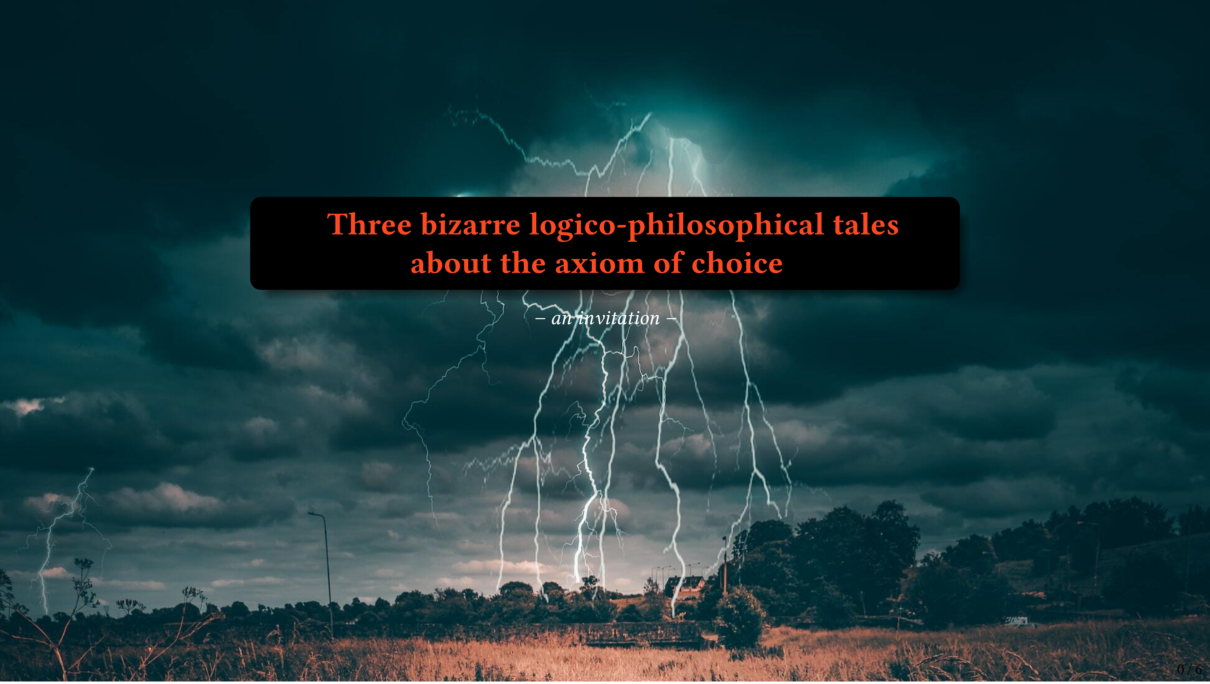
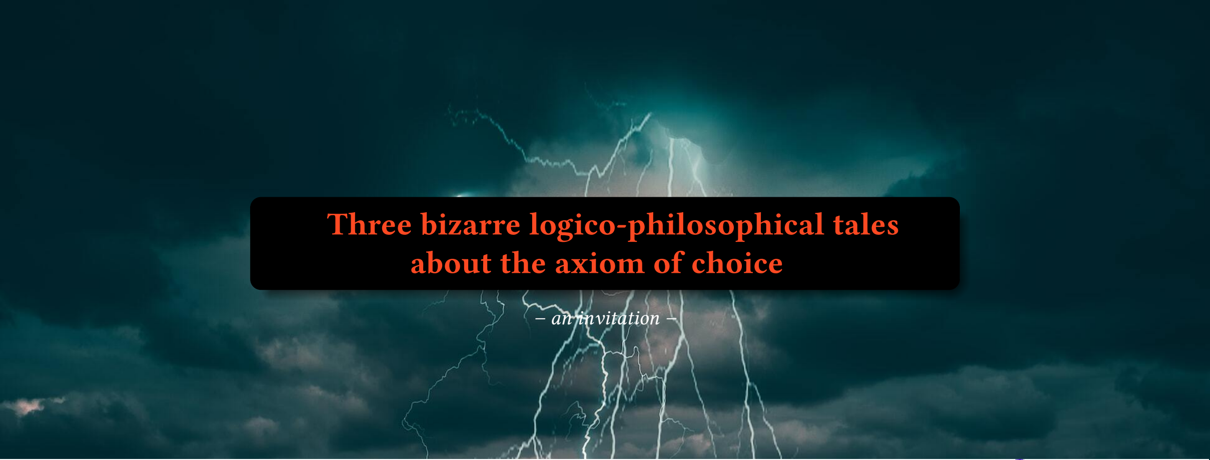


Three bizarre logico-philosophical tales about the axiom of choice

– an invitation –



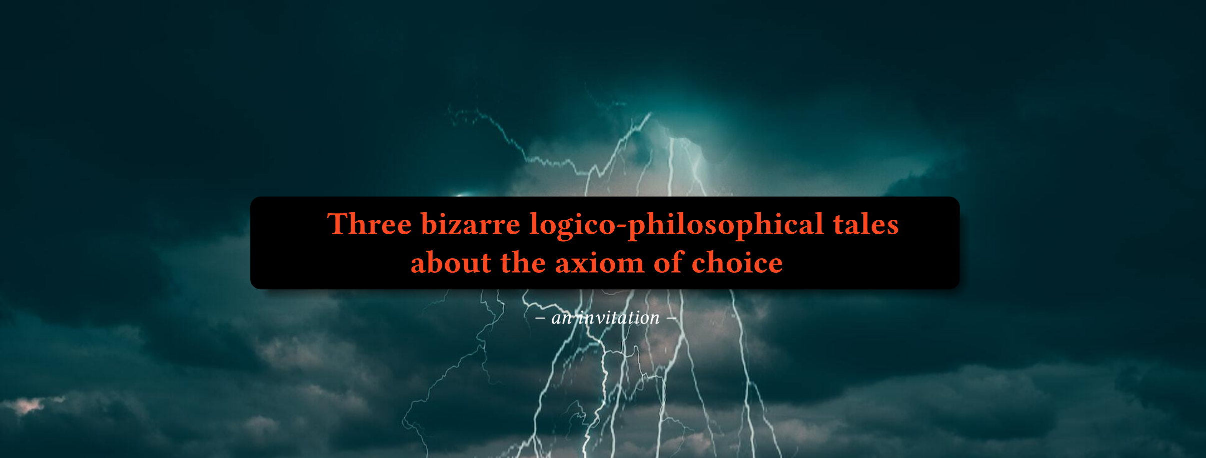
Three bizarre logico-philosophical tales about the axiom of choice

– an invitation –

An evil monster conjures numbers atop all members of an infinite team of mathematicians. They can see and cognitively process the numbers placed on everyone else, but are strictly forbidden to peek at their own number. Instead, the monster will ask every one of them to privately venture a guess regarding the value of their number.

Is there a strategy which, if followed by the mathematicians, ensures that *only finitely many* of them guess their number incorrectly? The strategy must be universal, applicable regardless of the specific numbers assigned by the monster.

Communication among the team is allowed only beforehand. Nothing is known about the distribution of numbers chosen by the monster. Also note that infinitely many mathematicians being right does not yet mean that only finitely many guess incorrectly. For instance, if every second guess is right, then every other second guess is incorrect.



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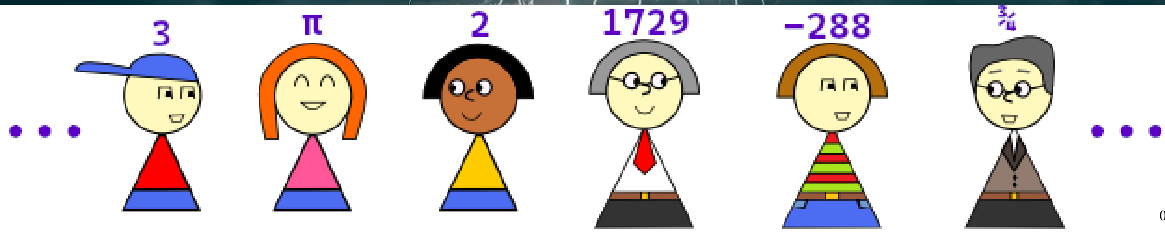

An evil monster conjures numbers atop all members of an infinite team of mathematicians. They can see and cognitively process the numbers placed on everyone else, but are strictly forbidden to peek at their own number. Instead, the monster will ask every one of them to privately venture a guess regarding the value of their number.

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The challenge posed by the monster seems impossible to satisfy: The monster is free in its distribution of numbers; observing the numbers hovering on all the other mathematicians does not restrict the amount of possibilities for your own number in any way.

Surprisingly, despite appearances, there *is* a suitable strategy—if and only if (a certain instance of) the axiom of choice holds.

Choice functions

The axiom of choice (AC) asserts:

"For **every** collection of inhabited sets, there is a **choice function** picking **representatives** from each set."

The slides uses the terms "collection" and "set" interchangeably. A set is called "inhabited" if and only if it contains at least one element.

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Examples for functions:

1. sine function: $x \mapsto \sin(x)$
2. squaring function: $x \mapsto x^2$, so $1 \mapsto 1,\ 2 \mapsto 4,\ 3 \mapsto 9,\ \ldots$
3. computeAreaOfCircle: $r \mapsto \pi r^2$, so $1 \mapsto \pi,\ 2 \mapsto 4\pi,\ \ldots$
4. document.getElementById

The slides uses the terms "collection" and "set" interchangeably. A set is called "inhabited" if and only if it contains at least one element.

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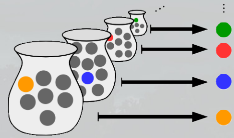

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The two examples for choice functions provided on the slide don't compute arbitrary representatives, but representatives which are singled out by special properties—being the mayor or being the youngest member. However, choice functions are also allowed to return representatives with no discernible special attributes.

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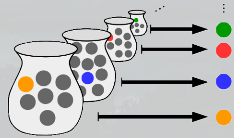

Note. The axiom of choice is **superfluous** for …

A. **finite** collections
B. collections of inhabited decidable sets of natural numbers

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The two examples for choice functions provided on the slide don't compute arbitrary representatives, but representatives which are singled out by special properties—being the mayor or being the youngest member. However, choice functions are also allowed to return representatives with no discernible special attributes.

In case A, we could just write down a full specification of a choice function by randomly drawing representatives. In order for the resulting choice function to be deterministic, as required for functions, the random sampling needs to be done once, beforehand, not anew for each call.

In case B, we can write down a choice function by picking the smallest possible representatives.

What a choice function can do for us in the riddle

For the collection of **sets of almost-identical scenarios**, a choice function could look like this:

$$\left\{\begin{array}{c}\cdots\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\cdots\\ \cdots\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{1}\,\boxed{3}\,\boxed{3}\,\boxed{3}\cdots\\ \cdots\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{1}\,\boxed{1}\,\boxed{3}\,\boxed{3}\cdots\\ \cdots\boxed{3}\,\boxed{3}\,\boxed{9}\,\boxed{9}\,\boxed{9}\,\boxed{3}\,\boxed{3}\cdots\\ \text{(and many more scenarios)}\end{array}\right\} \longmapsto \cdots\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\,\boxed{3}\cdots$$

$$\left\{\begin{array}{c}\cdots\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1}\cdots\\ \cdots\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{4}\,\boxed{1}\,\boxed{2}\,\boxed{1}\cdots\\ \cdots\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{5}\,\boxed{1}\,\boxed{2}\,\boxed{1}\cdots\\ \cdots\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{6}\,\boxed{1}\,\boxed{2}\,\boxed{1}\cdots\\ \text{(and many more scenarios)}\end{array}\right\} \longmapsto \cdots\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1}\,\boxed{2}\,\boxed{1}\cdots$$

(and many more sets) (and many more representatives)

If the players use a **common choice function** to make their guesses,
only finitely many will be incorrect.

For the purposes of the slide, two scenarios being "almost-identical" means that they differ at most at finitely many positions.

As every team member knows the numbers of all the others, every team member can identify with certainty the correct set of almost-identical scenarios. If they would now each pick an inhabitant of that set independently of each other, nothing would be gained. But if they all know a common choice function, they can use it to coordinate their guesses without violating the rule of no-communication.

The existence of such a choice function is guaranteed by the axiom of choice. Hence, assuming the axiom of choice, there at least *exists* a winning strategy for the mathematicians. (Whether they have access to this strategy is a different question.)

Consequences of the axiom of choice

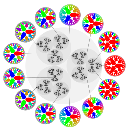
“Weird”:

Vitali fractal

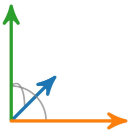
Banach–Tarski paradox

Prophecy

“Good/procrastinatory”:



Every field has an algebraic closure.



Every vector space has a basis.

To properly assess a proposed axiom for mathematics, we should not only philosophically connect with its statement, but also explore its consequences. One reason that the axiom of choice is somehow contested is that it entails both consequences which are commonly considered “bad” and consequences which are considered “good” (but which I personally prefare to reframe as “procrastinatory”).

Among the “bad” consequences are the following counterintuitive results:

1. There is a winning strategy for the mathematicians challenged by the evil monster.
2. There are shapes in the usual three-dimensional space of such weird form that, provably so, there is no reasonable way of assigning them a volume (not even the extremal values 0 or ∞ cubic units).
3. A solid three-dimensional ball can be disassembled into five number of pieces in such a way that these pieces can be reassembled (after rotation and translation) to form two disjoint copies of the original ball, each of the same size as the original ball.
4. A certain form of prophecy is possible (see link).

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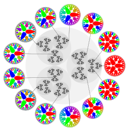

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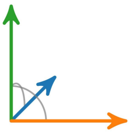


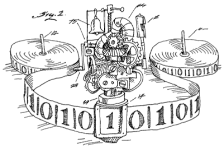
Every vector space has a basis.

In many cases, a more detailed analysis enables us to cope with losing the consequences usually deemed "good". For instance:

1. While AC is required to ensure that every field has an algebraic closure, the following fact can be established without it: Every field has an algebraic closure *in a certain extension of the mathematical universe*. This "sheaf-theoretic substitute" can serve similar purposes as a true algebraic closure existing in the same universe. *Some results otherwise obtained using the axiom of choice can be recovered if we are prepared to travel the toposophic multiverse and pass to extensions of the universe.*

2. While AC is required for several infrastructural tools, concrete results obtained with these tools can often be recovered without appealing to AC. This is ascertained by certain meta theorems concerning Gödel's sandbox (introduced below).

An alternative universe: the effective topos



	statement	in Std	in Eff
1	Every number is prime or not prime.	✓ (trivially)	✓
2	After every number there is a prime.	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?
6	Every map $\mathbb{N} \to \mathbb{N}$ which does *not not* have a zero has a zero.	✓ (trivially)	?

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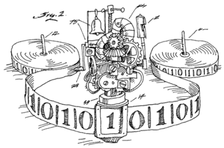

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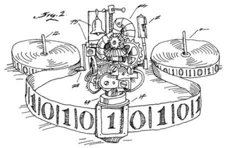
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?
6	Every map $\mathbb{N} \to \mathbb{N}$ which does *not not* have a zero has a zero.	✓ (trivially)	?

"1" in the effective topos amounts to: There is a machine which determines of any given number whether it is prime or not.

An alternative universe: the effective topos



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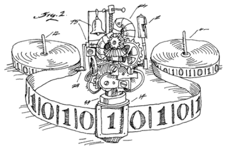
"2" in the effective topos amounts to: There is a machine which, given a number n, computes a prime larger than n.

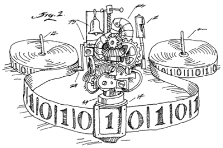
An alternative universe: the effective topos



	statement	in Std	in Eff
1	Every number is prime or not prime.	✓ (trivially)	✓
2	After every number there is a prime.	✓	✓
3	Every map $\mathbb{N} \to \mathbb{N}$ has a zero or not.	✓ (trivially)	✗
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	?
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?
6	Every map $\mathbb{N} \to \mathbb{N}$ which does *not not* have a zero has a zero.	✓ (trivially)	?

"3" in the effective topos amounts to: There is a machine which, given a machine computing a map $f : \mathbb{N} \to \mathbb{N}$, determines whether f has a zero or not.

An alternative universe: the effective topos



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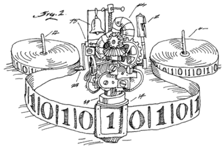
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	?
6	Every map $\mathbb{N} \to \mathbb{N}$ which does *not not* have a zero has a zero.	✓ (trivially)	?

"4" in the effective topos amounts to: There is a machine which, given a machine computing a map $f : \mathbb{N} \to \mathbb{N}$, outputs a machine computing f. *cat!*

An alternative universe: the effective topos



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5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)
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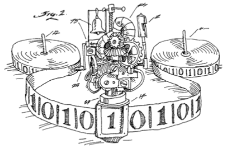
An alternative universe: the effective topos



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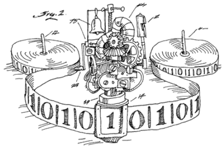
4	Every map $\mathbb{N} \to \mathbb{N}$ is computable.	✗	✓ (trivially)
5	Every map $\mathbb{R} \to \mathbb{R}$ is continuous.	✗	✓ (if MP)
6	Every map $\mathbb{N} \to \mathbb{N}$ which does *not not* have a zero has a zero.	✓ (trivially)	✓ (if MP)

"6" in the effective topos amounts to: There is a machine which, given a machine computing a map $f : \mathbb{N} \to \mathbb{N}$ and given the promise that it is *not not* the case that f has a zero, determines a zero of f.

unbounded search!

An alternative universe: the effective topos



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In Eff, there is **no choice function** for the collection of **sets of behaviourally identical programs**.

A counterexample to the axiom of choice

A **choice function** for the collection of **sets of behaviourally identical programs** would look like this:

$$\left\{\begin{array}{l}\text{while True: pass}\\ \text{while 2 == 1 + 1: pass}\\ \text{s = "a"; while len(s) > 0: s = s + "a"}\\ \vdots\end{array}\right\} \longmapsto \text{while True: pass}$$

$$\left\{\begin{array}{l}\text{print(2+2)}\\ \text{print(4)}\\ \text{print(len("37c3"))}\\ \vdots\end{array}\right\} \longmapsto \text{print(4)}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

With such a choice function c, a **halting oracle** can be built:
A program p loops if and only if $\mathrm{c}(p) = \mathrm{c}(\text{"while True: pass"})$.

A varied perspective on the axiom of choice

1. There is an alternate opposing axiom, the **axiom of determinacy** (AD):
"Every instance of the **infinite sequence game** is **determined**."
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2. Introducing the axiom of choice does **not** yield **new** inconsistencies (if ZFC is inconsistent, then ZF is as well—provably so in weak metatheories such as PRA).
   Hence worries about inconsistency arising from the axiom of choice are unfounded.

The standard foundational system for mathematics commonly accepted by logicians is ZFC, Zermelo–Fraenkel set theory with the axiom of choice. ZF is the variant without AC. PRA is a certain base theory so weak that it is contested just by ultrafinitists and hence often used for metamathematical pursuits.

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3. Even if AC fails, it always holds in L, **Gödel's sandbox**. Amazingly, Std's and L's $\mathbb{N}$ coincide, hence Std and L share the same **arithmetic truths** and hence from every proof of such a truth, any appeals to AC can be **mechanically eliminated**.
   *Thus AC can be regarded as **convenient fiction**, similar to how negative numbers are useful but we could always make do with tracking assets and debts separately. AC is required for certain general infrastructural tools, but superfluous for arithmetic consequences of such tools.*

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Much more severe than the axiom of choice is the **powerset axiom**, commonly adopted but enabling **impredicative reasoning**. 😱

Provably so, a sandbox for emulating the powerset axiom in case it is not assumed on the meta level is **impossible**: Unlike AC (or the also-debated law of excluded middle, which incidentally is implied by AC), the powerset axiom vastly increases proof-theoretic strength.

The strength of predicative set theories can be looked up on Wikipedia, whereas precisely calibrating the strength of impredicative formal systems such as ZF is currently well out of reach.