The secret of the number 5

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32th Chaos Communication Congress

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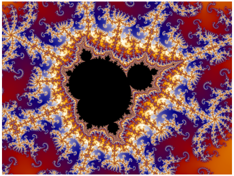
Dedicated to Prof. Dr. Jost-Hinrich Eschenburg.



Outline

A design pattern in nature

A curious fraction

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = \,?$$

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Multiplying by the denominator, we obtain $1 = 2x + x^2$,

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Multiplying by the denominator, we obtain $1 = 2x + x^2$, so we only have to solve the quadratic equation $0 = x^2 + 2x - 1$, thus

$$x = \frac{-2 + \sqrt{8}}{2} = -1 + \sqrt{2} \quad \text{or} \quad x = \frac{-2 - \sqrt{8}}{2} = -1 - \sqrt{2}.$$

It's the positive possibility.

More examples

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = \sqrt{2}$$

$$2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ddots}}} = \sqrt{5}$$

$$3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \ddots}}} = \sqrt{10}$$

More examples

$$[1; 2, 2, 2, \ldots] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} = \sqrt{2}$$

$$[2; 4, 4, 4, \ldots] = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \ddots}}} = \sqrt{5}$$

$$[3; 6, 6, 6, \ldots] = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \ddots}}} = \sqrt{10}$$

More examples

1. $\sqrt{2} = [1; 2, 2, 2, 2, 2, 2, \ldots]$
2. $\sqrt{5} = [2; 4, 4, 4, 4, 4, 4, \ldots]$
3. $\sqrt{10} = [3; 6, 6, 6, 6, 6, 6, \ldots]$
4. $\sqrt{6} = [2; 2, 4, 2, 4, 2, 4, \ldots]$
5. $\sqrt{14} = [3; 1, 2, 1, 6, 1, 2, \ldots]$

The Euclidean algorithm

Recall $\sqrt{2} = [1; 2, 2, 2, \ldots] = 1.41421356\ldots$

$$\begin{aligned}
1.41421356\ldots &= 1 \cdot 1.00000000\ldots + .41421356\ldots \\
1.00000000\ldots &= 2 \cdot 0.41421356\ldots + 0.17157287\ldots \\
0.41421356\ldots &= 2 \cdot 0.17157287\ldots + 0.07106781\ldots \\
0.17157287\ldots &= 2 \cdot 0.07106781\ldots + 0.02943725\ldots \\
0.07106781\ldots &= 2 \cdot 0.02943725\ldots + 0.01219330\ldots \\
0.02943725\ldots &= 2 \cdot 0.01219330\ldots + 0.00505063\ldots \\
&\vdots
\end{aligned}$$

Why does the Euclidean algorithm give the continued fraction coefficients? Let's write

$$
\begin{aligned}
x &= a_0 \cdot 1 \ + r_0 \\
1 &= a_1 \cdot r_0 + r_1 \\
r_0 &= a_2 \cdot r_1 + r_2 \\
r_1 &= a_3 \cdot r_2 + r_3
\end{aligned}
$$

and so on, where the numbers a_n are natural numbers and the residues r_n are smaller than the second factor of the respective adjacent product. Then:

$$
\begin{aligned}
x &= a_0 + r_0 = a_0 + 1/(1/r_0) \\
&= a_0 + 1/(a_1 + r_1/r_0) = a_0 + 1/(a_1 + 1/(r_0/r_1)) \\
&= a_0 + 1/(a_1 + 1/(a_2 + r_2/r_1)) = \cdots
\end{aligned}
$$

In the beautiful language Haskell, the code for lazily calculating the infinite continued fraction expansion is only one line long (the type declaration is optional).

```
cf ::  Double -> [Integer]
cf x = a :  cf (1 / (x - fromIntegral a)) where a = floor x
```

So the continued fraction expansion of a number x begins with a, the integral part of x, and continues with the continued fraction expansion of $1/(x-a)$.

Note that because of floating-point inaccuracies, only the first few terms of the expansion are reliable. For instance, `cf (sqrt 6)` could yield `[2,2,4,2,4,2,4,2,4,2,4,2,4,2,4,2,2,1,48,2,4,6,1,...]`.

Best approximations using continued fractions

Theorem

Cutting off the infinite fraction expansion of a number x yields a fraction a/b which is closest to x under all fractions with denominator $\leq b$.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}} \rightsquigarrow 1 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}} = \frac{17}{12} \approx 1.42$$

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Bonus. The bigger the coefficient after the cut-off is, the better is the approximation a/b.

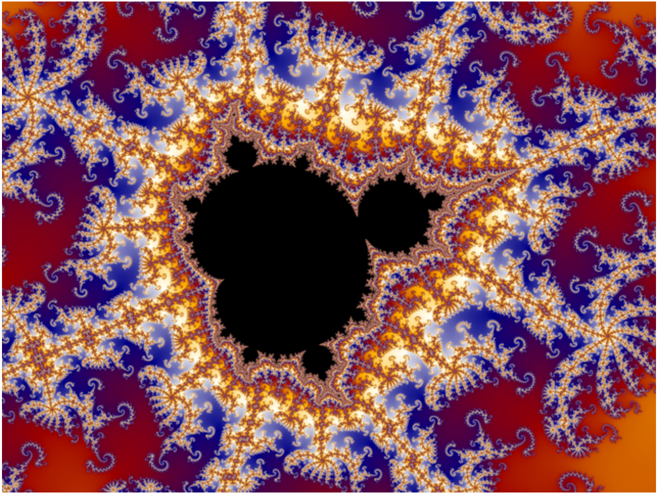
More precisely, the bonus statement is that the distance from x to a/b is less than $1/(a_n a_{n+1})$, where a_n is the last coefficient to be included in the cut-off and a_{n+1} is the first coefficient after the cut-off.

Approximations of π

$$\pi = 3.1415926535\ldots = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ddots}}}}$$

1. 3
2. $[3; 7] = 22/7 = \underline{3.14}28571428\ldots$
3. $[3; 7, 15] = 333/106 = \underline{3.1415}094339\ldots$
4. $[3; 7, 15, 1] = 355/113 = \underline{3.141592}9203\ldots$

The Mandelbrot fractal



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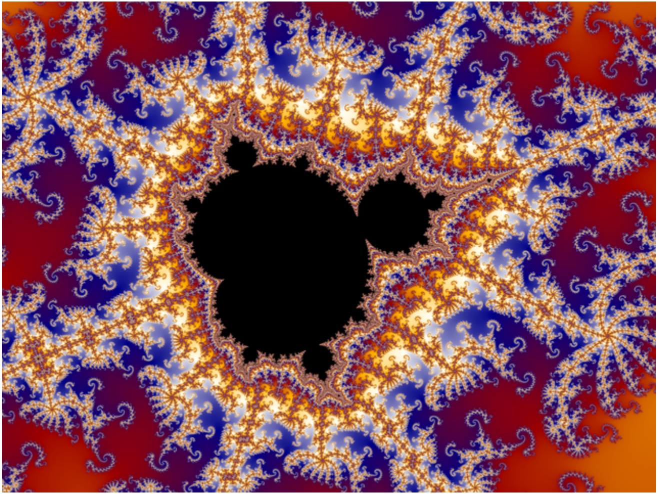
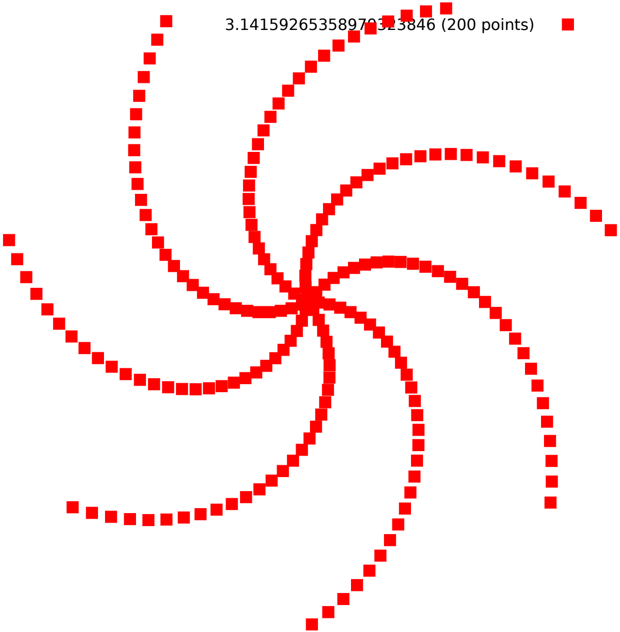
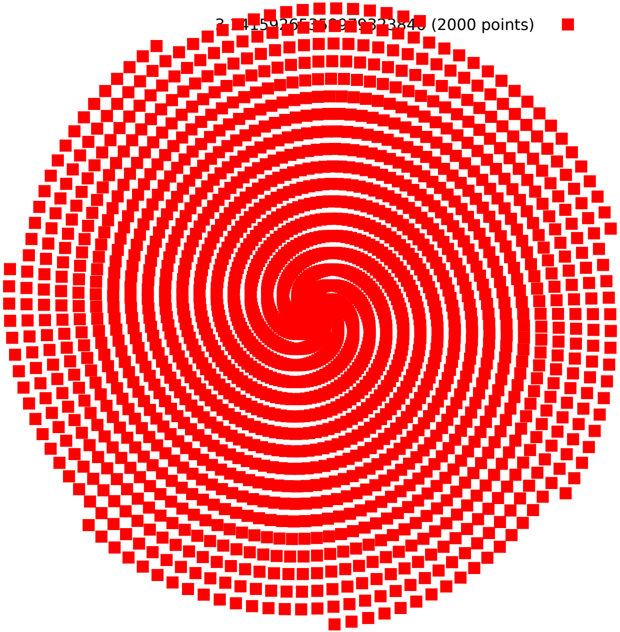


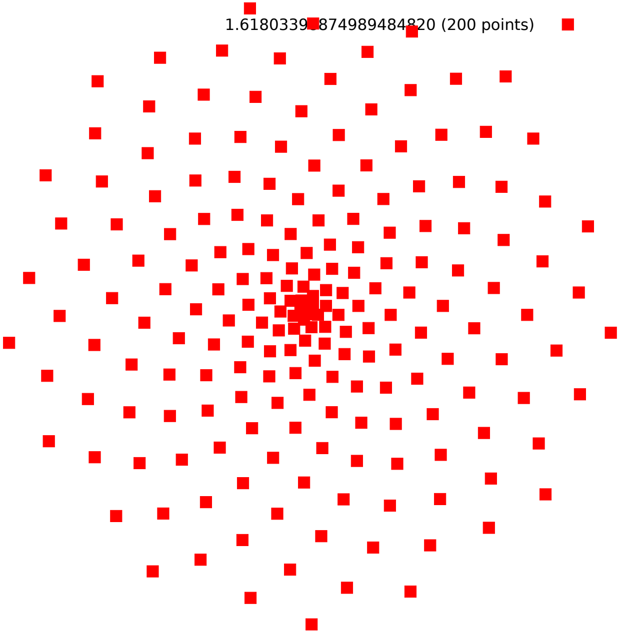
The Fibonacci numbers show up in the Mandelbrot fractal.

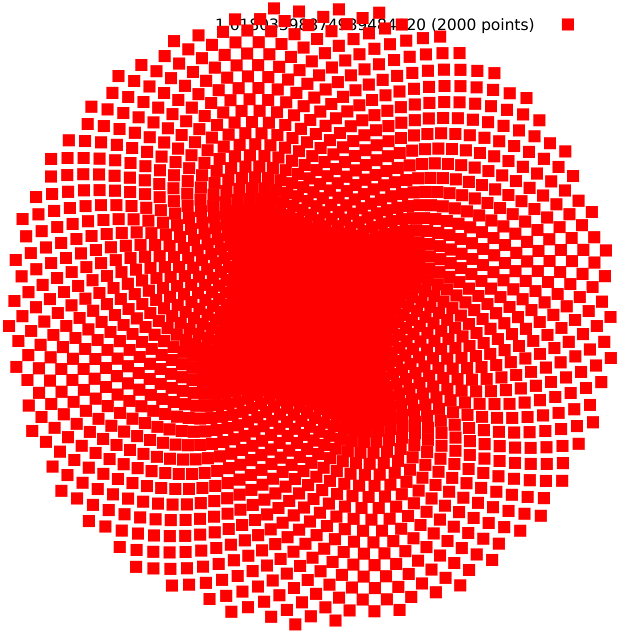
See `http://math.bu.edu/DYSYS/FRACGEOM2/node7.html` for an explanation of why the Fibonacci numbers show up in the Mandelbrot fractal.

Spirals in nature


3.14159265358979323846 (200 points)


(2000 points)


1.61803398874989484820 (200 points)


20 (2000 points)

The ananas from SpongeBob SquarePants

By Vi Hart, recreational mathemusician.