Fibered categories

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Fibered and indexed categories formalize objects and morphisms which can be pulled back along morphisms of a base category. One reason they are important is that stacks are fibered categories with certain extra properties.

These are informal notes prepared for the October 2014 meeting of the *Kleine Bayerische AG* at the University of Regensburg. The notes summarize parts of chapter 3 of Angelo Vistoli's *Notes on Grothendieck topologies, fibered categories and descent theory*.

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1 Indexed categories

Definition 1.1. Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-indexed category* is a pseudofunctor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$.

The notion of a *pseudofunctor* is a slight weakening of the notion of an ordinary functor. It is best explained by looking at a example. By Cat we mean the (2-)category of categories, functors, and natural transformations. We ignore all set-theoretical issues.

Example 1.2. Let $\mathcal{C}$ be a category of spaces (topological spaces, manifolds, schemes, schemes over a fixed scheme, ...). Then there is a $\mathcal{C}$-indexed category of vector bundles:

$$\begin{array}{rcl} X & \longmapsto & \mathrm{VB}(X) = \text{category of vector bundles on } X \\ (f : X \to Y) & \longmapsto & (f^* : \mathrm{VB}(Y) \to \mathrm{VB}(X)) = \text{pullback along } f \end{array}$$

Pullback of vector bundles is not associative on the nose; for maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ it is not the case that $(g \circ f)^* = f^* \circ g^*$ as functors $\mathrm{VB}(Z) \to \mathrm{VB}(X)$. In fact, recall from general category theory that it does not make sense to compare functors on equality, since this involves comparing objects on equality. A pseudofunctor does not need to satisfy the functor axioms on the nose, it suffices that they are satisfied up to coherent natural isomorphisms. We give the precise definition below.

Example 1.3. Let $\mathcal{C}$ be a category of schemes. Then there is a $\mathcal{C}$-indexed category of quasicoherent sheaves of modules, defined by $X \mapsto \mathrm{QCoh}(X)$.

Example 1.4. There is a Ring-indexed category of modules, given by $R \mapsto \mathrm{Mod}(R)$ (the category of R-modules) and the restriction of scalars functors.

Example 1.5. Let $\mathcal{C}$ be a category with pullbacks. There is a canonical $\mathcal{C}$-indexed category $\mathbb{C}$, the *self-indexing of* $\mathcal{C}$:

$$\begin{array}{rcl} X & \longmapsto & \mathcal{C}/X \\ (f : X \to Y) & \longmapsto & (f^* : \mathcal{C}/Y \to \mathcal{C}/X) \end{array}$$

As usual, $\mathcal{C}/X$ is the slice category of morphisms to X; its objects are morphisms $T \to X$, where $T \in \mathcal{C}$ is arbitrary, and its morphisms are commutative triangles. The functor f^* is defined by sending an object $(T \to Y)$ to some chosen pullback $(T \times_Y X \to X)$.

Remark 1.6. A morphism $T \xrightarrow{\pi} X$ in the category of sets can be thought of as an X-indexed family of sets, namely the fibers $\pi^{-1}[\{x\}]$, where x ranges over X. By analogy, we visualize a morphism $T \to X$ in an arbitrary category $\mathcal{C}$ as an X-indexed family of objects in $\mathcal{C}$. Thus $\mathcal{C}/X$ can be thought of as the category of X-indexed families of objects in $\mathcal{C}$, and the functors f^* *reindex families.*

Example 1.7. Any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ (that is, an ordinary functor) gives rise to a $\mathcal{C}$-indexed category by postcomposing with the embedding $\mathrm{Set} \to \mathrm{Cat}$, which associates to any set its induced discrete category. In particular, any object $A \in \mathcal{C}$ gives rise to a *representable* $\mathcal{C}$-indexed category $\underline{A}$ given by $T \mapsto \mathrm{Hom}_{\mathcal{C}}(T, A)$.

Stacks are special kinds of indexed categories, so indexed categories should be geometric in some sense. Where is the geometry? Where are the topological spaces, the points?

Let $\mathcal{C}$ be a category of spaces and F a $\mathcal{C}$-indexed category. We can infuse geometric content by imagining, for any space $A \in \mathcal{C}$, the category $F(A)$ to be *the category of maps* $A \to F$. So even though F does not define topological structure directly, we can *probe* it by test spaces. This is the general idea of Grothendieck's *functor* *of points* philosophy (see `http://ncatlab.org/nlab/show/motivation+for+sheaves,+cohomology+and+higher+stacks` for a beautiful exposition).

How good is this idea? We expect to be able to precompose morphisms $f : A \to F$ by morphisms $g : B \to A$ in $\mathcal{C}$ to obtain morphisms $f \circ g : B \to F$. This is indeed possible: $f \circ g$ can be realized as $g^*(f)$.

But the resulting notion fails in general to be local: If $A = \bigcup_i U_i$ is covered by open subsets, we expect to be able to define maps $A \to F$ by glueing compatible maps $U_i \to F$. This is not possible with an arbitrary indexed category – the notion of a topology does not enter the definition in any way. A *stack* is an indexed category where this kind of pathology does not arise.

Example 1.8. The *points* of the indexed category VB of vector bundles (Example 1.2) are by definition (and by analogy with the classical situation) the maps pt $\to$ VB. So for any natural number n, there is a point of VB (corresponding to a vector space of dimension n). These points have lots of automorphisms, namely the invertible $(n \times n)$-matrices.

Remark 1.9. In speaking of morphisms from a test space A to an indexed category F we commit a type error. This can be fixed by replacing A with its induced indexed category $\underline{A}$. A 2-categorical Yoneda lemma then gives a canonical equivalence $\mathrm{Hom}(\underline{A}, F) \simeq F(A)$.

We are now ready to appreciate a precise definition of a pseudofunctor.

Definition 1.10. Let $\mathcal{C}$ be a category. A *pseudofunctor* $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ consists of

- a category $F(X)$ for each object X of $\mathcal{C}$,
- a functor $F(f) =: f^* : F(Y) \to F(X)$ for each morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$,
- a natural isomorphism $\eta_X : (\mathrm{id}_X)^* \Rightarrow \mathrm{Id}_{F(X)}$ for each object X in $\mathcal{C}$, and
- a natural isomorphism $\alpha_{f,g} : f^* \circ g^* \Rightarrow (g \circ f)^*$ for each diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$

such that the following *coherence conditions* hold:

- For any $X \xrightarrow{f} Y$ in $\mathcal{C}$, $\alpha_{\mathrm{id}_X,f} = \eta_X f^* : \mathrm{id}_X^* \circ f^* \Rightarrow f^*$.
- For any $X \xrightarrow{f} Y$ in $\mathcal{C}$, $\alpha_{f,\mathrm{id}_Y} = f^* \eta_Y : f^* \circ \mathrm{id}_Y^* \Rightarrow f^*$.
- For any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{C}$, the diagram commutes.

XXX diagram

The coherence conditions are needed for the following reason: When dealing with vector bundles or similar objects which can be pulled back, we are used to identify iterated pullbacks with pullbacks along compositions ("$f^* \circ g^* = (g \circ f)^*$"). This is justified only because between any two pullback expressions, e. g.

$$(f^* \circ (h \circ g)^* \circ p^* \circ q^*)(E) \quad \text{and} \quad ((g \circ f)^* \circ (q \circ p \circ h)^*)(E),$$

we have *exactly one* agreed-upon canonical isomorphism (of course, there are lots of other non-canonical isomorphisms as well). In a general indexed category, we can construct these canonical isomorphisms using the given η_X and $\alpha_{f,g}$; the coherence conditions ensure that – no matter in which way we do this – we always obtain the same canonical isomorphism.

Remark 1.11. Indexed categories are also important in categorical logic and topos theory. For instance, any geometric morphism $\mathcal{F} \xrightarrow{f} \mathcal{E}$ of toposes induces an $\mathcal{E}$-indexed category $\mathbb{F}$, given by $X \mapsto \mathcal{F}/f^*X$. From the point of view of the internal language of $\mathcal{E}$, an $\mathcal{E}$-indexed category looks like an ordinary (large, but locally small) category. This is useful for doing *relative category theory.*

2 Fibered categories

Recall that an I-indexed family $(X_i)_{i\in I}$ of sets can equivalently be given by

- a map $I \to \operatorname{Ob}\mathrm{Set}$, $i \mapsto X_i$, or
- a map $\coprod_{i\in I} X_i \xrightarrow{\pi} I$. (Recover X_i as the fiber of π over i.)

Indexed categories are like the first approach, while fibered categories are like the second approach.

Definition 2.1. Let $\mathcal{C}$ be a category. A *category fibered over* $\mathcal{C}$ is a category $\mathcal{F}$ together with a functor $\mathcal{F} \xrightarrow{p} \mathcal{C}$ such that "pullbacks exist": For any morphism $Y \xrightarrow{f} X$ in $\mathcal{C}$ and any object E in the fiber category $\mathcal{F}(X)$ (the subcategory of $\mathcal{F}$ consisting of those objects which map to X and those morphisms which map to id_X under p), there should exist an object E' of $\mathcal{F}(X)$ and a *cartesian morphism* $E' \xrightarrow{\phi} E$ with $p(\phi) = f$.

$$
\begin{tikzcd}
E' \arrow[r, "\phi"] \arrow[d, no head] & E \arrow[d, no head] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}
$$

The notion of a cartesian morphism is modeled on the universal property of usual pullbacks: A morphism $\phi : E' \to E$ with $p(\phi) = f$ as in the definition is cartesian if and only if, for any morphism $g : Z \to Y$, for any object E'' in $\mathcal{F}(Z)$, and any morphism $\psi : E'' \to E$ such that $p(\psi) = f\circ g$, there exists a unique morphism $\alpha : E'' \to E'$ such that $p(\alpha) = g$ and $\phi \circ \alpha = \psi$.

XXX diagram

Remark 2.2. The definition of a fibered category as stated is *evil* in the technical sense that it requires certain equalities between objects to hold (namely, $p(E') = Y$). Since important examples of categories which ought to be fibered categories actually fulfill these evil equalities, this defect is tolerable in practice. But for some purposes, a non-evil variant is necessary; see for instance `http://ncatlab.org/nlab/show/Grothendieck+fibration#StreetFibration`.

Example 2.3. The incarnation of Example 1.2 as a fibered category is the category VB of vector bundles on arbitrary base spaces. The projection functor $\mathrm{VB} \to \mathcal{C}$ associates to any vector bundle its base space.

Definition 2.4. An *indexed functor* $\mathcal{F} \to \mathcal{G}$ between fibered categories over $\mathcal{C}$ is an ordinary functor $\mathcal{F} \to \mathcal{G}$ which commutes with the projection functors to $\mathcal{C}$ and which carries cartesian morphisms to cartesian morphisms.

3 Equivalence between indexed and fibered categories

Proposition 3.1. *There is a pseudoequivalence between the 2-category of indexed categories, indexed functors, and indexed natural transformations; and the 2-category of fibered categories, fibered functors, and fibered natural transformations.*

From fibered categories to indexed categories. If suitable choice principles are available, any fibered category $\mathcal{F} \to \mathcal{C}$ defines a $\mathcal{C}$-indexed category $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ by setting $F(X) := \mathcal{F}(X)$. For morphisms $f : Y \to X$ in $\mathcal{C}$, the pullback functors $f^* : F(X) \to F(Y)$ are defined by associating to any object E a chosen pullback E' as in the definition of a fibered category. The action of f^* on morphisms is given by the universal property of cartesian morphisms.

One can check that this construction indeed gives rise to a pseudofunctor. In fact, the proof is very similar to the proof that the self-indexing $X \mapsto \mathcal{C}/X$ is a pseudofunctor.

From indexed categories to fibered categories. In the opposite direction, any $\mathcal{C}$-indexed category $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ induces a fibered category $\int F \to \mathcal{C}$ via the *Grothendieck construction*. Mimicking the case of families of sets, the objects of $\int F$ are pairs $(X \in \mathcal{C}, E \in F(X))$. A morphism $(X, E) \to (Y, F)$ is a pair $(f : X \to Y,\ \alpha : E \to f^*(F))$. The projection functor $\int F \to \mathcal{C}$ sends an object (X, E) to X and a morphism (f, α) to f.

One can check that this category over $\mathcal{C}$ is indeed a fibered category. The cartesian morphisms $\phi : (Y, E') \to (X, E)$ required by the definition are the morphisms $(f : Y \to X,\ \mathrm{id} : f^*(E) \to f^*(E))$.

Example 3.2. Let $\mathcal{C}$ be a category with pullbacks. Then $\int \mathbb{C}$ (see Example 1.5) is equivalent to the *category of morphisms in* $\mathcal{C}$. This category has arbitrary morphisms $Y \to X$ as objects and commutative squares as morphisms. The equivalence is given by sending an object $(X \in \mathcal{C}, (Y \xrightarrow{f} X))$ to f.

Example 3.3. Let $\mathcal{C}$ be an arbitrary category and A a fixed object of $\mathcal{C}$. Then $\int \underline{A}$ (see Example 1.7) is equivalent to the slice category $\mathcal{C}/A$, by sending an object $(X \in \mathcal{C}, X \xrightarrow{f} A)$ to f.

4 Special types of indexed and fibered categories

Definition 4.1. A *category fibered in groupoids (sets) [equivalence relations]* is a fibered category $\mathcal{F}$ such that the fiber categories $\mathcal{F}(X)$ are all groupoids (discrete categories) [categories equivalent to discrete categories].

Equivalently, the associated pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ factors over $\mathrm{Grpd} \hookrightarrow \mathrm{Cat}$ ($\mathrm{Set} \hookrightarrow \mathrm{Cat}$) [$\mathrm{EqRel} \hookrightarrow \mathrm{Cat}$].

Proposition 4.2. *A functor $\mathcal{F} \xrightarrow{p} \mathcal{C}$ is ...*

- *a category fibered in groupoids if and only if any morphism in $\mathcal{F}$ is cartesian and for any morphism $Y \xrightarrow{f} X$ in $\mathcal{C}$ and any object E in $\mathcal{F}(X)$, there exists a morphism $E' \xrightarrow{\phi} E$ in $\mathcal{F}$ such that $p(\phi) = f$.*
- *a category fibered in sets if and only if for any morphism $Y \xrightarrow{f} X$ in $\mathcal{C}$ and any object E in $\mathcal{F}(X)$, there exists a unique morphism $E' \xrightarrow{\phi} E$ in $\mathcal{F}$ such that $p(\phi) = f$.*

5 An important example: quotient stacks

For this example, we fix a category $\mathcal{C}$ of spaces. We will be so sketchy that it does not matter which category $\mathcal{C}$ actually is. The source for this section is the article *Algebraic stacks* of Tomás Gómez.

Let G be a group acting on a space X. Assume that the action is *free*, i. e. for any $g \in G$ and any $x \in X$, $g \cdot x = x$ implies $g = 1$. Then recall that there is a well-defined projection $X \to X/G$; that this map is a G-principal bundle; and that this map is the *universal* G-principal bundle equipped with a G-equivariant map to X: For any G-principal bundle $P \to Y$ equipped with a G-equivariant map $P \to X$, there is a unique map $Y \to X/G$ such that the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X/G \end{tikzcd}$$

is a pullback diagram. Thus the space X/G has the pleasant property that maps $Y \to X/G$ correspond to G-principal bundles $P \to Y$ equipped with G-equivariant maps $Y \to X$. (If such a bundle is given, construct $Y \to X/G$ by selecting, for any $y \in Y$, a preimage under $P \to Y$, and applying $P \to X \to X/G$ to this preimage. One can check that the resulting element in X/G is independent of the choice.)

Unfortunately, if the action of G on X is not free, X/G fails to exist as an object of $\mathcal{C}$ or fails to have this universal property. But there is always a quotient *stack* $[X/G]$ (at least an indexed category) with this property, tautologically defined by sending a test object Y of $\mathcal{C}$ to the category of G-principal bundles $P \to Y$ equipped with G-equivariant maps $Y \to X$.

Indeed, one can canonically define a map $\underline{X} \to [X/G]$ and check that it is the universal bundle.

Proposition 5.1. *The stack $[X/G]$ is representable, i. e. of the form $\underline{A}$ for some space A in $\mathcal{C}$, if and only if the action of G on X is free.*

Proof (sketch). For the "only if" direction, check that the morphism $X \to A$ corresponding to the universal bundle $\underline{X} \to \underline{A}$ is a G-principal bundle isomorphic to $X \to X/G$. □

Example 5.2. Let $X = \mathrm{pt}$. Then the quotient stack $[X/G]$ is commonly named "BG".

Remark 5.3. The category of points of $[X/G]$, i. e. $[X/G](\mathrm{pt})$, is equivalent to the *action groupoid* (*weak quotient*) $X//G$. This groupoid has as objects the elements of X, and the set of morphisms $x \to x'$ is the set of group elements g such that $g \cdot x = x'$. The action groupoid is the "right replacement" for the ill-behaved space of orbits. A very accessible exposition is in *Higher-Dimensional Algebra VII: Groupoidification* by John Baez, Alexander Hoffnung and Christopher Walker: `http://arxiv.org/abs/0908.4305`.

Remark 5.4. In some cases, the category of coherent sheaves on $[X/G]$ can simply be described as the category of G-equivariant sheaves on X (sheaves equipped with a specified G-linearization). If S is a smooth projective surface over $\mathbb{C}$, the celebrated Bridgeland–King–Reid–Haiman equivalence states that the bounded derived category of coherent sheaves on $S^{[n]}$ (the Hilbert scheme of n points on S) and the bounded derived category of $\mathfrak{S}_n$-equivariant sheaves on S^n are equivalent. This statement can thus be interpreted as saying that the Hilbert scheme $S^{[n]}$ and the quotient stack $[S^n/\mathfrak{S}_n]$ are *derived equivalent*.