Tensor categories

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1 Basics

Example 1.1. The category Mod_R of modules over a commutative ring R is the archetypical example of a tensor category.

Definition 1.2. A *tensor category* $(\mathcal{C}, \otimes)$ (i. e. a monoidal category with symmetric braiding) consists of

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the tensor operation),
- an object $1 \in \mathcal{C}$ (the unit object),
- natural isomorphisms $X \otimes (Y \otimes Z) \xrightarrow{\phi_{XYZ}} (X \otimes Y) \otimes Z$ (the associator),
- natural isomorphisms $X \otimes Y \xrightarrow{\psi_{XY}} Y \otimes X$ (the braiding),
- natural isomorphisms $1 \otimes X \xrightarrow{\lambda_X} X$ and $X \otimes 1 \xrightarrow{\rho_X} X$ (the unitors),

such that

- the braiding is symmetric: $\psi_{YX} \circ \psi_{XY} = \mathrm{id}_{X \otimes Y}$ and
- the following coherence conditions are satisfied:

$$XXX$$

The definitions mimics the definition of a monoid, only "one level up". Demanding that the tensor operation is associative and commutative on the nose (e. g. $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$) would be *evil* in the technical sense, i. e. not invariant under equivalence of categories. The isomorphism classes of a tensor category form a commutative monoid.

Example 1.3. In Mod_R, ϕ_{XYZ} is given by

$$x \otimes (y \otimes z) \longmapsto (x \otimes y) \otimes z.$$

If $-1 \neq 1 \in R$, introducing a sign here will cause the pentagon diagram to fail to commute by a sign.

Example 1.4. Let $\mathcal{C}$ be a category with finite products. Then $(\mathcal{C}, \times)$ is a tensor category, with $\phi, \psi, \lambda, \rho$ given by the universal property of the product. In particular, $(\mathrm{Set}, \times)$ is a tensor category.

Example 1.5. Let $\mathcal{C}$ be a tensor category. Then $\mathcal{C}^{\mathrm{op}}$ is in a natural way a tensor category, by using the inverses of the given natural isomorphisms.

Remark 1.6. In a general tensor category, there is no natural morphism $X \to X \otimes X$.

The coherence conditions in the definition are needed for the following reason: In the category of modules, we are used to dropping all parentheses when dealing with iterated tensor products. This is justified because between any two given groupings, e. g.

$$X \otimes ((Y \otimes Z) \otimes (T \otimes U)) \quad \text{and} \quad ((X \otimes Y) \otimes (Z \otimes T)) \otimes U,$$

we have a *canonical* isomorphism. A classical theorem of Mac Lane guarantees that the stated coherence conditions suffice to render any "reasonable" diagram commutative:

Theorem 1.7 (Mac Lane)**.** *Any (formal) diagram in a tensor category built by*

$$\otimes, \mathrm{id}, \phi, \psi, \lambda, \rho, \phi^{-1}, \psi^{-1}, \lambda^{-1}, \rho^{-1}$$

(in which both sides have the same permutation) commutes.

The two caveats are the following: Firstly, in a given tensor category, there may hold certain *identities* between objects for no general abstract reason. For example, for some totally unrelated objects X, Y, A, B, it might hold that $X \otimes Y = A \otimes B$. Using those identities we can form diagrams which we do *not* expect to commute. Mac Lane's coherence theorem does not make any statement about those diagrams.

To understand the restriction about permutations, consider the diagram

$$\psi_{XX} \stackrel{?}{=} \mathrm{id}_{X \otimes X}.$$

We do not expect this diagram to be commutative; the permutations associated to both sides are not equal: $(1, 2) \neq \mathrm{id}$.

Theorem 1.8 (Joyal, Street)**.** *If the graphical depictions of given morphisms of a tensor category are "the same" (in 4D space), the morphisms are equal.*

Example 1.9. XXX

Theorem 1.10 (Mac Lane)**.** *Any tensor category may be* strictified*, i. e. is equivalent as a tensor category to a* strict *tensor category: a category in which ϕ, λ, ρ (but not ψ) are identities.*

2 Structure in tensor categories

Definition 2.1. An *internal Hom* between objects X, Y of a tensor category $\mathcal{C}$ consists of

- an object $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{C}$ and
- a morphism $\underline{\mathrm{Hom}}(X, Y) \otimes X \xrightarrow{\mathrm{ev}} Y$ (evaluation morphism)

such that this pair is terminal among such pairs, i. e. such that for any object $T \in \mathcal{C}$ and a [fake] evaluation morphism $T \otimes X \xrightarrow{\widetilde{\mathrm{ev}}} Y$ there exists an unique morphism $f : T \to \underline{\mathrm{Hom}}(X, Y)$ such that the following diagram commutes:

$$\mathrm{ev} \circ (f \otimes \mathrm{id}_X) = \widetilde{\mathrm{ev}}.$$

Example 2.2. In $(\mathrm{Set}, \times)$, the internal Homs are given by the usual Hom sets. The evaluation morphism is given by $(f, x) \mapsto f(x)$.

Example 2.3. In Mod_R, the internal Homs are given by the Hom sets equipped with the usual module structure.

Remark 2.4. If an internal Hom $\underline{\mathrm{Hom}}(X, Y)$ exists for all objects $Y \in \mathcal{C}$ (and if appropriate choice principles are available), the internal Hom can be made into a functor $\underline{\mathrm{Hom}}(X, \_) : \mathcal{C} \to \mathcal{C}$ which is right adjoint to taking tensor product with X:

$$\_ \otimes X \dashv \underline{\mathrm{Hom}}(X, \_)$$

Remark 2.5. The relation with the usual Hom (which is only a set) is the following:

$$\mathrm{Hom}(1, \underline{\mathrm{Hom}}(X, Y)) \cong \mathrm{Hom}(1 \otimes X, Y) \cong \mathrm{Hom}(X, Y).$$

Remark 2.6. In $(\mathrm{Set}, \amalg)$, internal Homs do not exist in general: This is because in general, $\_ \amalg X$ does not preserve colimits and so cannot be a left adjoint.

Definition 2.7.

1. A *dual* of an object X is an internal Hom $X^\vee := \underline{\mathrm{Hom}}(X, 1)$.
2. The *dual* of a morphism $f : X \to Y$ is the unique morphism $f^t : Y^\vee \to X^\vee$ rendering the diagram commutative (if $X^\vee$ and $Y^\vee$ exist).

Example 2.8. In Mod_R, $f^t : \theta \mapsto \theta \circ f$.

Proposition 2.9. *In any tensor category, the set* $\mathrm{End}(1) = \mathrm{Hom}(1, 1)$ *is a commutative monoid with respect to composition of morphisms.*

Proof. The results holds even if there would be no braiding: On $\mathrm{End}(1)$, the tensor product induces a second binary operation. By the coherence conditions, this operation commutes with the operation given by composition, so by the famous Eckmann–Hilton theorem, both operations coincide and are commutative. ☐

Example 2.10. In Mod_R, $\mathrm{End}(1) \cong R$.

Example 2.11. In $(\mathrm{Set}, \prod)$, $\mathrm{End}(1) = \{\mathrm{id}\}$.

3 Tensor functors

Definition 3.1. A *tensor functor* $F : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ consists of

1. a functor $F : \mathcal{C} \to \mathcal{C}'$,
2. natural isomorphisms $FX \otimes' FY \xrightarrow{c_{XY}} F(X \otimes Y)$ and
3. an isomorphism $1' \xrightarrow{e} F1$

such that the following coherence conditions are satisfied:

$$XXX$$

Example 3.2. The forgetful functor $\mathrm{Rep}_k(G) \to \mathrm{Vect}_k$ of the category of finite-dimensional k-linear representations of a group (or group scheme) G is a tensor functor.

Example 3.3. Extension of scalars defines a tensor functor $\mathrm{Mod}_R \to \mathrm{Mod}_S$.

Example 3.4. A quantum field theory determines a tensor functor $\mathrm{Cob}_d \to \mathrm{Vect}_k$.

Definition 3.5. A *morphism of tensor functors* $\eta : (F, c, e) \to (\tilde{F}, \tilde{c}, \tilde{e})$ consists of a natural transformation $\eta : F \to \tilde{F}$ which is compatible with the coherence isomorphisms:

$$XXX$$

4 Rigid tensor categories

Definition 4.1. A tensor category $\mathcal{C}$ is *rigid* iff

- all internal Homs exist,
- the natural morphisms

  $$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \longrightarrow \underline{\mathrm{Hom}}(X_1 \otimes Y_1, X_2 \otimes Y_2)$$

  are isomorphisms and
- all objects $X \in \mathcal{C}$ are *reflexive* (i. e. the natural map $X \to X^{\vee\vee}$ is an isomorphism).

Example 4.2. The category $\mathrm{Vect}_k^{\mathrm{fd}}$ of finite-dimensional is rigid.

Example 4.3. More generally, the category $\mathrm{Mod}_R^{\mathrm{fin.\,free}}$ of finitely free R-modules is rigid.

Example 4.4. Changing toposes, the category of locally free $\mathcal{O}_{\mathrm{Spec}\,R}$ modules is rigid. (This category is equivalent to category of finitely generated projective R-modules, by the tilde construction.)

Example 4.5. The category $\mathrm{Rep}_k(G)$ is rigid.

Remark 4.6. Let $\mathcal{C}$ be a rigid category. Then the functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}, X \mapsto X^\vee$ is an equivalence of tensor categories.

In a rigid tensor category, we can define the notion of traces:

Definition 4.7. 1. The *trace* of an endomorphism $f : X \to X$ in a rigid tensor category is the following element of End(1):

$$f \in \mathrm{Hom}(X, X) \cong \mathrm{Hom}(1, \underline{\mathrm{Hom}}(X, X)) \cong \mathrm{Hom}(1, X^\vee \otimes X) \to \mathrm{Hom}(1, 1) \ni: \mathrm{tr} f.$$

2. The *rank* (or *Euler characteristic*) of an object X is the trace of the identity morphism on X.

Trace and rank satisfy the relations you expect:

Lemma 4.8. *1.* $\mathrm{tr}(f \otimes f') = \mathrm{tr}(f) \circ \mathrm{tr}(f'), \quad \mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f).$

2. $\mathrm{rk}(X \otimes X') = \mathrm{rk}(X) \circ \mathrm{rk}(X'), \quad \mathrm{rk}(1) = \mathrm{id}_1.$

Lemma 4.9. *In a rigid tensor category, a pair $(Y, Y \otimes X \xrightarrow{\mathrm{ev}} 1)$ is a dual of X iff there exists a morphism $1 \xrightarrow{\varepsilon} X \otimes Y$ such that the following triangle identities hold:*

$$XXX$$

Proof. Given a pair $(T, T \otimes X \xrightarrow{\widetilde{\mathrm{ev}}} 1)$, construct $f : T \to Y$ as follows:

$$XXX$$

Then the relevant diagram commutes, since

$$XXX.$$

Use the second triangle identity to show uniqueness.

For the converse direction, construct ε by dualizing ev. □

Remark 4.10. Rigidity is only necessary for the converse direction: A pair $(Y, Y \otimes X \xrightarrow{\mathrm{ev}} 1)$ which satisfies the condition stated in lemma is always a dual—and in fact, because the condition is symmetric, the coevaluation morphism exhibits X as a dual of Y. In particular, X is reflexive.

Proposition 4.11. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a tensor functor, with $\mathcal{C}$ rigid. Let $X, Y \in \mathcal{C}$. Then an internal Hom $\underline{\mathrm{Hom}}(FX, FY)$ exists in $\mathcal{C}'$ and the natural morphism*

$$F(\underline{\mathrm{Hom}}(X, Y)) \longrightarrow \underline{\mathrm{Hom}}(FX, FY)$$

is an isomorphism.

Proof. It is enough to show that in $\mathcal{C}'$ a dual of FX exists and that the natural morphism

$$F(X^\vee) \longrightarrow (FX)^\vee$$

is an isomorphism. This is obvious by the lemma, as the given condition is preserved by F. □

Corollary 4.12. $\mathrm{tr}F(f) = F(\mathrm{tr}f), \quad \mathrm{rk}FX = F(\mathrm{rk}X)$.

Proposition 4.13. *In a rigid category, $\_ \otimes X$ is not only a left, but also a right adjoint:*

$$\underline{\mathrm{Hom}}(X, \_) \dashv \_ \otimes X \dashv \underline{\mathrm{Hom}}(X, \_).$$

Corollary 4.14. *In a rigid category, the tensor operation is continuous and cocontinuous.*

5 Abelian tensor categories

Definition 5.1. An *abelian tensor category* is a tensor category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is abelian and $\otimes$ is bi-additive.

Remark 5.2. In an abelian category, $\mathrm{End}(1)$ is a commutative ring.

Lemma 5.3. *Let $\mathcal{C}$ be an abelian tensor category. Let $X \in \mathcal{C}$.*

1. *Assume that the unit 1 is a simple object and that $\mathcal{C}$ is rigid. Then:*

$$X \ncong 0 \quad \Longrightarrow \quad X^\vee \otimes X \to 1 \textit{ is epic.}$$

2. *Assume that $1 \ncong 0 \in \mathcal{C}$. Then:*

$$X \ncong 0 \quad \Longleftarrow \quad X^\vee \otimes X \to 1 \textit{ is epic.}$$

Proof.
1. Because the unit is simple, the natural morphism $X^\vee \otimes X \to 1$ is either zero or epic. (This follows by considering the image of the morphism.) Under the correspondence
$$\mathrm{Hom}(X^\vee \otimes X, 1) \cong \mathrm{Hom}(X, X^{\vee\vee}),$$
the morphism corresponds to the natural morphism $X \to X^{\vee\vee}$. If this was zero, X were zero as well.
2. Assume $X \cong 0$. Then we have an epic morphism $0 \to 1$. By terminality of the zero object, this is an isomorphism.

□

Remark 5.4. The rigidity is necessary: In $\mathrm{Mod}_\mathbb{Z}$, the object $\mathbb{Z}/(2)$ is not isomorphic to the zero object, but the morphism $(\mathbb{Z}/(2))^\vee \otimes \mathbb{Z}/(2) \to \mathbb{Z}$ has zero codomain.

Proposition 5.5. *Let $\mathcal{C}$ be a rigid abelian tensor category with $1 \in \mathcal{C}$ simple. Let $\mathcal{C}'$ be an abelian tensor category with $1 \ncong 0$. Then any exact tensor functor $F : \mathcal{C} \to \mathcal{C}'$ is faithful.*

Proof. Let $Ff = 0$. If $f \neq 0$, then $\operatorname{im}(f) \ncong 0$. So by the lemma, the natural morphism $\operatorname{im}(f)^\vee \otimes \operatorname{im}(f) \to 1$ is epic. Because F is exact and preserves duals, the natural morphism $\operatorname{im}(Ff)^\vee \otimes \operatorname{im}(Ff) \to 1$ is epic as well. Again by the lemma, it follows that $\operatorname{im}(Ff) \ncong 0$. This is a contradiction. ☐

Remark 5.6. The simpleness of the unit is necessary: Let $\mathcal{C}$ and $\mathcal{D}$ be rigid abelian categories. Then the projection functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C}$ is exact, but in general not faithful.

Corollary 5.7. *Let k be a field and R be a ring with $1 \neq 0 \in R$. Let $\varphi : k \to R$ be a ring homomorphism. Then φ is injective.*

Proof. By the proposition, the functor $F : \mathrm{Vect}_k^{\mathrm{fd}} \to \mathrm{Mod}_R$ (extension of scalars) is faithful. Let $\varphi(x) = 0$. Then the image of the map $k \to k$, multiplication by x, is zero in Mod_R. So by faithfullness, the map is already zero on k, so $x = 0$. ☐

Proposition 5.8. *Let $\mathcal{C}$ be a rigid abelian tensor category. Let $U \hookrightarrow 1$ be a subobject. Then the unit object decomposes as the direct sum*

$$1 \cong U \oplus U^\perp,$$

where $U^\perp = \ker(1 \to U^\vee)$.

Proof.
1. Consider the cokernel V of $U \hookrightarrow 1$; we then have a short exact sequence
$$0 \longrightarrow U \longrightarrow 1 \longrightarrow V \longrightarrow 0.$$
2. By rigidity, tensoring is exact; so we obtain the diagram
$$XXX$$
with exact rows. Because the morphism $U \to V \otimes U \to V$ is zero, $V \otimes U$ is zero; and by exactness of the bottom row, $U \otimes U$ is zero.
3. For any subobject $T \hookrightarrow X$ the following chain of equivalences holds:
$$T \otimes U \cong 0 \iff T \otimes U \hookrightarrow T \text{ is zero} \iff T \twoheadrightarrow U^\vee \otimes T \to U^\vee \otimes X \text{ is zero.}$$
The first "$\Leftarrow$" is because $T \otimes U \hookrightarrow T$ is a monomorphism (by exactness of tensoring with T) and the second "$\Leftrightarrow$" is by the isomorphisms
$$\underline{\mathrm{Hom}}(T \otimes U, T) \cong U^\vee \otimes T^\vee \otimes T \cong \underline{\mathrm{Hom}}(T, U^\vee \otimes T).$$
So the largest such subobject $T \hookrightarrow X$ is given by
$$T = \ker(X \to U^\vee \otimes X) \cong U^\perp \otimes X.$$
(The isomorphism is by exactness of tensoring with X.)

4. Applying this observation to $X = V$, it follows that $U^\perp \otimes V \cong V$.

   Applying it to $X = U$, it follows that $U^\perp \otimes U \cong 0$: Let $T \hookrightarrow U$ with $T \otimes U \cong 0$. By exactness of tensoring with T, the sequence

$$0 \longrightarrow T \otimes U \longrightarrow T \longrightarrow T \otimes V \longrightarrow 0$$

   is exact. Since $T \otimes U \cong 0$, we have

$$T \cong T \otimes V \hookrightarrow U \otimes V \cong 0.$$

   The "$\hookrightarrow$" is—again—by exactness of tensoring (with V).

5. By exactness of tensoring with $U^\perp$, the sequence

$$0 \longrightarrow U^\perp \otimes U \longrightarrow U^\perp \longrightarrow U^\perp \otimes V \longrightarrow 0$$

   is exact. By the previous observation, this shows $U^\perp \cong V$.

6. By applying the five lemma to the diagram

$$XXX,$$

   it follows that $1 \cong U \oplus U^\perp$. □

Remark 5.9. Rigidity is necessary: In $\mathrm{Mod}_\mathbb{Z}$, the unit object is not simple but admits to non-trivial decompositions.

Corollary 5.10. *In a rigid abelian category, the unit object is simple iff* $\mathrm{End}(1)$ *is a field.*

Proof. For the "only if" direction, let $f \in \mathrm{End}(1)$. As the unit object is simple, f is either zero or an isomorpism (i. e. invertible in the ring $\mathrm{End}(1)$). This follows by considering kernel and image of f.

For the "if" direction, let $U \hookrightarrow 1$. By the lemma, there exists a projection operator $P : 1 \to 1$ with $\mathrm{im}(P) = U$. If P is zero in $\mathrm{End}(1)$, $U = 0$; if P is invertible, $U = 1$ (as subojects of 1). □

6 The Tannaka reconstruction theorem

Theorem 6.1. *Let $\mathcal{C}$ be a rigid abelian tensor category. Let $k := \mathrm{End}(1)$ be a field. Let $\omega : \mathcal{C} \to \mathrm{Vect}_k^{fd}$ be an exact faithful k-linear tensor functor. Then:*

1. *The functor $\underline{\mathrm{Aut}}^\otimes(\omega) : \mathrm{Alg}_k \to \mathrm{Set}$, given by*

$$R \longmapsto \textit{set of tensor automorphisms of } (\_ \otimes_k R) \cdot \omega,$$

   is the functor of points of an affine group scheme G.

2. *A certain functor $\mathcal{C} \to \mathrm{Rep}_k(G)$ induced by ω is an equivalence of tensor categories.*