The tangent bundle of the projective space

The aim of this short note is to calculate the tangent bundle of the projective space in the setting of synthetic differential geometry.

Axiom of microaffinity. For any function $f : \Delta \to \mathbb{R}$, where $\Delta = \{\varepsilon \in \mathbb{R} \,|\, \varepsilon^2 = 0\}$, there exists a unique pair (a, b) of real numbers such that

$$f(\varepsilon) = a + b\varepsilon$$

for all $\varepsilon \in \Delta$.

Example 1. For all $\varepsilon \in \Delta$, the number $1 + \varepsilon$ is invertible with inverse given by

$$\frac{1}{1+\varepsilon} = \frac{1-\varepsilon}{(1+\varepsilon)(1-\varepsilon)} = \frac{1-\varepsilon}{1-\varepsilon^2} = 1 - \varepsilon.$$

Definition 2. The *derivative* $f'(x)$ of a set-theoretical function $f : \mathbb{R} \to \mathbb{R}$ at a point x is the unique number b such that

$$f(x + \varepsilon) = f(x) + b\varepsilon$$

for all $\varepsilon \in \Delta$.

Example 3. Let $f(x) = x^3$. Then $f'(x) = 3x^2$, since

$$f(x + \varepsilon) = (x + \varepsilon)^3 = x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 = x^3 + 3x^2\varepsilon$$

for all $\varepsilon \in \Delta$.

Definition 4. The *projective space* $\mathbb{P}V$ associated to a vector space V is the set

$$\mathbb{P}V = \{\ell \subseteq V \,|\, \ell \text{ is a one-dimensional subspace of } V\}.$$

Definition 5. The *tangent bundle* TX of a set X is the set

$$TX = X^\Delta = \{\gamma : \Delta \to X\}$$

of all set-theoretical maps $\Delta \to X$.

Theorem 6. *Let V be a vector space. Then there is a canonical isomorphism (bijection)*

$$T(\mathbb{P}V) \cong \coprod_{\ell \in \mathbb{P}V} \mathrm{Hom}_{\mathbb{R}}(\ell, V/\ell).$$

The symbol "$\coprod$" for the disjoint union might seem dubious on first sight: Aren't the fibers of the tangent bundle supposed to have some cohesion and fit continuously together? In synthetic differential geometry, all sets have automatically a continuous/geometric flavour. The cohesion present in the index set over which the disjoint union is taken over, the set $\mathbb{P}V$, induces a kind of cohesion in the disjoint union.

A simpler example is given by the tangent bundle of $\mathbb{R}^n$: This admits the description

$$T\mathbb{R}^n \cong \coprod_{x \in \mathbb{R}^n} \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n.$$

The description using the disjoint union might feel weird while the description using the cartesian product seems right, but there is in fact a canonical bijection between these sets, sending an element $\langle x, v\rangle \in \coprod_{x \in \mathbb{R}^n} \mathbb{R}^n$ to $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$.

The point of synthetic differential geometry is, however, that one doesn't need to make this idea of "cohesion" explicit (unlike in the standard approach, where one defines topological spaces and manifolds to capture it). The sets themselves already exhibit the smooth behaviour we're interested in.

Proof of Theorem 6. Let $\gamma : \Delta \to \mathbb{P}V$ be a tangent vector with base point $\ell := \gamma(0) \in \mathbb{P}V$. There's a lift $\overline{\gamma} : \Delta \to V$ such that $\gamma(\varepsilon) = \operatorname{span}(\overline{\gamma}(\varepsilon))$ for all $\varepsilon \in \Delta$. We define a linear map $\alpha : \ell \to V/\ell$ by setting

$$x \longmapsto \alpha(x) = [x/\overline{\gamma}(0) \cdot \overline{\gamma}'(0)].$$

The expression "$x/\overline{\gamma}(0)$" should be read as follows: The vector x, being an element of ℓ, is some multiple λ of $\overline{\gamma}(0)$. The expression "$x/\overline{\gamma}(0)$" denotes this unique number λ. It can be checked that the vector $\alpha(x) \in V/\ell$ does not depend on the choice of the lifting $\overline{\gamma}$. The element $\langle \ell, \alpha\rangle$ is therefore a well-defined element of $\coprod_{\ell \in \mathbb{P}V} \operatorname{Hom}_{\mathbb{R}}(\ell, V/\ell)$.

Conversely, let an element $\langle \ell, \alpha\rangle \in \coprod_{\ell \in \mathbb{P}V} \operatorname{Hom}_{\mathbb{R}}(\ell, V/\ell)$ be given. We choose vectors $x_0 \in V$ and $z \in V$ such that $\ell = \operatorname{span}(x_0)$ and $\alpha(x_0) = [z]$ and define $\gamma : \Delta \to \mathbb{P}V$ by setting

$$\varepsilon \longmapsto \gamma(\varepsilon) = \operatorname{span}(x_0 + \varepsilon z).$$

The definition of $\gamma(\varepsilon)$ is invariant under scaling of x_0 and also under changing z to some other vector $z + \lambda x_0$ in its equivalence class:

$$\begin{aligned}
\operatorname{span}(x_0 + \varepsilon(z + \lambda x_0)) &= \operatorname{span}((1 + \varepsilon\lambda)x_0 + \varepsilon z) \\
&= \operatorname{span}(x_0 + \varepsilon/(1 + \varepsilon\lambda)z) \\
&= \operatorname{span}(x_0 + \varepsilon z),
\end{aligned}$$

since $\varepsilon/(1 + \varepsilon\lambda) = \varepsilon(1 - \varepsilon\lambda) = \varepsilon$. Therefore γ is a well-defined element of $T(\mathbb{P}V)$ which only depends on $\langle \ell, \alpha\rangle$ and not on the arbitrary choices of x_0 and z.

One can check that the two described constructions are mutually inverse. $\square$

At no point in the proof one has to check that certain maps are continuous or smooth. The proof can be easily adapted to verify that the tangent bundle of the Grassmannian of r-dimensional subspaces admits the description

$$T(\mathrm{Gr}_r V) \cong \coprod_{U \in \mathrm{Gr}_r V} \operatorname{Hom}_{\mathbb{R}}(U, V/U).$$