Fibered categories

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Fibered and indexed categories formalize the notion of objects and morphisms which can be pulled back along morphisms of a base category. One reason they are important is that stacks are fibered categories with certain extra properties.

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1 Indexed categories

Definition 1.1. Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-indexed category* is a pseudofunctor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$.

The notion of a *pseudofunctor* is a slight weakening of the notion of an ordinary functor. It is best explained by looking at a example. By Cat we mean the (2-)category of categories, functors, and natural transformations. We ignore all set-theoretical issues.

Example 1.2. Let $\mathcal{C}$ be a category of spaces (topological spaces, manifolds, schemes, schemes over a fixed scheme, ...). Then there is a $\mathcal{C}$-indexed category of vector bundles:

$$\begin{array}{rcl} X & \longmapsto & \mathrm{VB}(X) = \text{category of vector bundles on } X \\ (f : X \to Y) & \longmapsto & (f^* : \mathrm{VB}(Y) \to \mathrm{VB}(X)) = \text{pullback along } f \end{array}$$

Pullback of vector bundles is not associative on the nose; for maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ it is not the case that $(g \circ f)^* = f^* \circ g^*$ as functors $\mathrm{VB}(Z) \to \mathrm{VB}(X)$. In fact, recall from general category theory that it does not make sense to compare functors on equality, since this involves comparing objects on equality. A pseudofunctor does not need to satisfy the functor axioms on the nose, it suffices that they are satisfied up to coherent natural isomorphisms. We give the precise definition below.

Example 1.3. Let $\mathcal{C}$ be a category of schemes. Then there is a $\mathcal{C}$-indexed category of quasicoherent sheaves of modules, defined by $X \mapsto \mathrm{QCoh}(X)$.

Example 1.4. There is a Ring-indexed category of modules, given by $R \mapsto \mathrm{Mod}(R)$ (the category of R-modules) and the restriction of scalars functors.

Example 1.5. Let $\mathcal{C}$ be a category with pullbacks. There is a canonical $\mathcal{C}$-indexed category, the *self-indexing of* $\mathcal{C}$:

$$\begin{array}{rcl} X & \longmapsto & \mathcal{C}/X \\ (f : X \to Y) & \longmapsto & (f^* : \mathcal{C}/Y \to \mathcal{C}/X) \end{array}$$

As usual, $\mathcal{C}/X$ is the slice category of morphisms to X; its objects are morphisms $T \to X$, where $X \in \mathcal{C}$ is arbitrary, and its morphisms are commutative triangles. The functor f^* is defined by sending an object $(T \to Y)$ to some chosen pullback $(T \times_Y X \to X)$.

Remark 1.6. A morphism $T \xrightarrow{\pi} X$ in the category of sets can be thought of as an X-indexed family of sets, namely the fibers $\pi^{-1}[\{x\}]$, where x ranges over X. By analogy, we visualize a morphism $T \to X$ in an arbitrary category $\mathcal{C}$ as an X-indexed family of objects in $\mathcal{C}$. Thus $\mathcal{C}/X$ can be thought of as the category of X-indexed families of objects in $\mathcal{C}$, and the functors f^* *reindex families.*

Example 1.7. Any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ (that is, an ordinary functor) gives rise to a $\mathcal{C}$-indexed category by postcomposing with the embedding $\mathrm{Set} \to \mathrm{Cat}$, which associates to any set its induced discrete category. In particular, any object $A \in \mathcal{C}$ gives rise to a *representable* $\mathcal{C}$-indexed category $\underline{A}$ given by $T \mapsto \underline{\mathrm{Hom}_{\mathcal{C}}}(T, A)$.

Stacks are special kinds of indexed categories, so indexed categories should be geometric in some sense. Where is it? Where are the topological spaces, the points? Let $\mathcal{C}$ be a category of spaces and F a $\mathcal{C}$-indexed category. We can infuse geometric content by imagining, for any space $A \in \mathcal{C}$, the category $F(A)$ to be *the category of maps* $A \to F$. So even though F does not define topological structure directly, we can *probe* it by test spaces. This is the general idea of Grothendieck's *functor of points* philosophy (see XXXnLab).

How good is this idea? We expect to be able to precompose morphisms $f : A \to F$ by morphisms $g : B \to A$ in $\mathcal{C}$ to obtain morphisms $f \circ g : B \to F$. This is indeed possible – $f \circ g$ can be realized as $g^*(f)$.

But the resulting notion fails in general to be local: If $A = \bigcup_i U_i$ is covered by open subsets, we expect to be able to define maps $A \to F$ by glueing compatible maps $U_i \to F$.

This is not possible with an arbitrary indexed category – the notion of a topology does not enter the definition in any way. A stack is a fibered category where this kind of pathology does not arise.

Example 1.8. The *points* of the fibered category VB of vector bundles (example 1.2) are by definition (and by analogy with the classical situation) the maps $\mathrm{pt} \to \mathrm{VB}$. So for any natural number n, there is a point of VB (corresponding to a vector space of dimension n). These points have lots of automorphisms, namely the invertible $(n \times n)$-matrices.

Remark 1.9. In speaking of morphisms from a test space A to a fibered category F we commit a type error. This can be fixed by replacing A with its induced fibered category $\underline{A}$. A 2-categorical Yoneda lemma then gives a canonical equivalence $\underline{\mathrm{Hom}}(\underline{A}, F) \simeq F(A)$.

We are now ready to appreciate a precise definition of a pseudofunctor.

Definition 1.10. Let $\mathcal{C}$ be a category. A *pseudofunctor* $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ consists of

- a category $F(X)$ for each object X of $\mathcal{C}$,
- a functor $F(f) =: f^* : F(Y) \to F(X)$ for each morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$,
- a natural isomorphism $\eta_X : (\mathrm{id}_X)^* \Rightarrow \mathrm{Id}_{F(X)}$ for each object X in $\mathcal{C}$, and
- a natural isomorphism $\alpha_{f,g} : f^* \circ g^* \Rightarrow (g \circ f)^*$ for each diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$

such that the following *coherence conditions* hold:

- For any $X \xrightarrow{f} Y$ in $\mathcal{C}$, $\alpha_{\mathrm{id}_X, f} = \eta_X f^* : \mathrm{id}_X^* \circ f^* \Rightarrow f^*$.
- For any $X \xrightarrow{f} Y$ in $\mathcal{C}$, $\alpha_{f, \mathrm{id}_Y} = f^* \eta_Y : f^* \circ \mathrm{id}_Y^* \Rightarrow f^*$.
- For any $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ in $\mathcal{C}$, the diagram commutes.
  XXX

The coherence conditions are needed for the following reason: XXX