

# Fibered categories

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Fibered and indexed categories formalize objects and morphisms which can be pulled back along morphisms of a base category. One reason they are important is that stacks are fibered categories with certain extra properties.

These are informal notes prepared for the October 2014 meeting of the *Kleine Bayerische AG* at the University of Regensburg. The notes summarize part of chapter 3 of Angelo Vistoli's *Notes on Grothendieck topologies, fibered categories and descent theory*.

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## 1 Indexed categories

**Definition 1.1.** Let  $\mathcal{C}$  be a category. A  $\mathcal{C}$ -indexed category is a pseudofunctor  $F : \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$ .

The notion of a *pseudofunctor* is a slight weakening of the notion of an ordinary functor. It is best explained by looking at an example. By  $\text{Cat}$  we mean the (2-)category of categories, functors, and natural transformations. We ignore all set-theoretical issues.

**Example 1.2.** Let  $\mathcal{C}$  be a category of spaces (topological spaces, manifolds, schemes, schemes over a fixed scheme, ...). Then there is a  $\mathcal{C}$ -indexed category of vector bundles:

$$\begin{aligned} X &\longmapsto \text{VB}(X) = \text{category of vector bundles on } X \\ (f : X \rightarrow Y) &\longmapsto (f^* : \text{VB}(Y) \rightarrow \text{VB}(X)) = \text{pullback along } f \end{aligned}$$

Pullback of vector bundles is not associative on the nose; for maps  $X \xrightarrow{f} Y \xrightarrow{g} Z$  it is not the case that  $(g \circ f)^* = f^* \circ g^*$  as functors  $\text{VB}(Z) \rightarrow \text{VB}(X)$ . In fact, recall from general category theory that it does not make sense to compare functors on equality, since this involves comparing objects on equality. A pseudofunctor does not need to satisfy the functor axioms on the nose, it suffices that they are satisfied up to coherent natural isomorphisms. We give the precise definition below.

**Example 1.3.** Let  $\mathcal{C}$  be a category of schemes. Then there is a  $\mathcal{C}$ -indexed category of quasicoherent sheaves of modules, defined by  $X \mapsto \text{QCoh}(X)$ .

**Example 1.4.** There is a Ring-indexed category of modules, given by  $R \mapsto \text{Mod}(R)$  (the category of  $R$ -modules) and the restriction of scalars functors.

**Example 1.5.** Let  $\mathcal{C}$  be a category with pullbacks. There is a canonical  $\mathcal{C}$ -indexed category  $\mathbb{C}$ , the *self-indexing* of  $\mathcal{C}$ :

$$\begin{aligned} X &\longmapsto \mathcal{C}/X \\ (f : X \rightarrow Y) &\longmapsto (f^* : \mathcal{C}/Y \rightarrow \mathcal{C}/X) \end{aligned}$$

As usual,  $\mathcal{C}/X$  is the slice category of morphisms to  $X$ ; its objects are morphisms  $T \rightarrow X$ , where  $T \in \mathcal{C}$  is arbitrary, and its morphisms are commutative triangles. The functor  $f^*$  is defined by sending an object  $(T \rightarrow Y)$  to some chosen pullback  $(T \times_Y X \rightarrow X)$ .

*Remark 1.6.* A morphism  $T \xrightarrow{\pi} X$  in the category of sets can be thought of as an  $X$ -indexed family of sets, namely the fibers  $\pi^{-1}[\{x\}]$ , where  $x$  ranges over  $X$ . By analogy, we visualize a morphism  $T \rightarrow X$  in an arbitrary category  $\mathcal{C}$  as an  $X$ -indexed family of objects in  $\mathcal{C}$ . Thus  $\mathcal{C}/X$  can be thought of as the category of  $X$ -indexed families of objects in  $\mathcal{C}$ , and the functors  $f^*$  *reindex families*.

**Example 1.7.** Any presheaf  $F : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$  (that is, an ordinary functor) gives rise to a  $\mathcal{C}$ -indexed category by postcomposing with the embedding  $\text{Set} \rightarrow \text{Cat}$ , which associates to any set its induced discrete category. In particular, any object  $A \in \mathcal{C}$  gives rise to a *representable*  $\mathcal{C}$ -indexed category  $\underline{A}$  given by  $T \mapsto \text{Hom}_{\mathcal{C}}(T, A)$ .

Stacks are special kinds of indexed categories, so indexed categories should be geometric in some sense. Where is the geometry? Where are the topological spaces, the points?

Let  $\mathcal{C}$  be a category of spaces and  $F$  a  $\mathcal{C}$ -indexed category. We can infuse geometric content by imagining, for any space  $A \in \mathcal{C}$ , the category  $F(A)$  to be *the category of maps*  $A \rightarrow F$ . So even though  $F$  does not define topological structure directly, we can *probe* it by test spaces. This is the general idea of Grothendieck's *functor*

of *points* philosophy (see <http://ncatlab.org/nlab/show/motivation+for+sheaves,+cohomology+and+higher+stacks> for a beautiful exposition).

How good is this idea? We expect to be able to precompose morphisms  $f : A \rightarrow F$  by morphisms  $g : B \rightarrow A$  in  $\mathcal{C}$  to obtain morphisms  $f \circ g : B \rightarrow F$ . This is indeed possible:  $f \circ g$  can be realized as  $g^*(f)$ .

But the resulting notion fails in general to be local: If  $A = \bigcup_i U_i$  is covered by open subsets, we expect to be able to define maps  $A \rightarrow F$  by glueing compatible maps  $U_i \rightarrow F$ . This is not possible with an arbitrary indexed category – the notion of a topology does not enter the definition in any way. A *stack* is an indexed category where this kind of pathology does not arise.

**Example 1.8.** The *points* of the indexed category  $\mathbf{VB}$  of vector bundles (example 1.2) are by definition (and by analogy with the classical situation) the maps  $\text{pt} \rightarrow \mathbf{VB}$ . So for any natural number  $n$ , there is a point of  $\mathbf{VB}$  (corresponding to a vector space of dimension  $n$ ). These points have lots of automorphisms, namely the invertible  $(n \times n)$ -matrices.

*Remark 1.9.* In speaking of morphisms from a test space  $A$  to an indexed category  $F$  we commit a type error. This can be fixed by replacing  $A$  with its induced indexed category  $\underline{A}$ . A 2-categorical Yoneda lemma then gives a canonical equivalence  $\text{Hom}(\underline{A}, F) \simeq F(A)$ .

We are now ready to appreciate a precise definition of a pseudofunctor.

**Definition 1.10.** Let  $\mathcal{C}$  be a category. A *pseudofunctor*  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$  consists of

- a category  $F(X)$  for each object  $X$  of  $\mathcal{C}$ ,
- a functor  $F(f) =: f^* : F(Y) \rightarrow F(X)$  for each morphism  $X \xrightarrow{f} Y$  in  $\mathcal{C}$ ,
- a natural isomorphism  $\eta_X : (\text{id}_X)^* \Rightarrow \text{Id}_{F(X)}$  for each object  $X$  in  $\mathcal{C}$ , and
- a natural isomorphism  $\alpha_{f,g} : f^* \circ g^* \Rightarrow (g \circ f)^*$  for each diagram  $X \xrightarrow{f} Y \xrightarrow{g} Z$  in  $\mathcal{C}$

such that the following *coherence conditions* hold:

- For any  $X \xrightarrow{f} Y$  in  $\mathcal{C}$ ,  $\alpha_{\text{id}_X, f} = \eta_X f^* : \text{id}_X^* \circ f^* \Rightarrow f^*$ .
- For any  $X \xrightarrow{f} Y$  in  $\mathcal{C}$ ,  $\alpha_{f, \text{id}_Y} = f^* \eta_Y : f^* \circ \text{id}_Y^* \Rightarrow f^*$ .
- For any  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$  in  $\mathcal{C}$ , the diagram commutes.

XXX diagram

The coherence conditions are needed for the following reason: When dealing with vector bundles or similar objects which can be pulled back, we are used to identify iterated pullbacks with pullbacks along compositions (“ $f^* \circ g^* = (g \circ f)^*$ ”). This is justified only because between any two pullback expressions, e. g.

$$(f^* \circ (h \circ g)^* \circ p^* \circ q^*)(E) \quad \text{and} \quad ((g \circ f)^* \circ (q \circ p \circ h)^*)(E),$$

we have *exactly one* agreed-upon canonical isomorphism (of course, there are lots of other non-canonical isomorphisms as well). In a general indexed category, we can construct these canonical isomorphisms out of the given  $\eta_X$  and  $\alpha_{f,g}$ ; the coherence conditions ensure that – no matter in which way we do this – we always obtain the same canonical isomorphism.

*Remark 1.11.* Indexed categories are also important in categorical logic and topos theory. For instance, any geometric morphism  $\mathcal{F} \xrightarrow{f} \mathcal{E}$  of toposes induces an  $\mathcal{E}$ -indexed category  $\mathbb{F}$ , given by  $X \mapsto \mathcal{F}/f^*X$ . From the point of view of the internal language of  $\mathcal{E}$ , an  $\mathcal{E}$ -indexed category looks like an ordinary (large, but locally small) category. This is useful for doing *relative category theory*.

## 2 Fibered categories

Recall that an  $I$ -indexed family  $(X_i)_{i \in I}$  of sets can equivalently be given by

- a map  $I \rightarrow \text{Ob Set}$ ,  $i \mapsto X_i$ , or
- a map  $\coprod_{i \in I} X_i \xrightarrow{\pi} I$ . (Recover  $X_i$  as the fiber of  $i$  under  $\pi$ .)

Indexed categories are like the first approach, while fibered categories are like the second approach.

**Definition 2.1.** Let  $\mathcal{C}$  be a category. A *category fibered over  $\mathcal{C}$*  is a category  $\mathcal{F}$  together with a functor  $\mathcal{F} \xrightarrow{p} \mathcal{C}$  such that “pullbacks exist”: For any morphism  $Y \xrightarrow{f} X$  in  $\mathcal{C}$  and any object  $E$  in the fiber category  $\mathcal{F}(X)$  (the subcategory of  $\mathcal{F}$  consisting of those objects which map to  $X$  and those morphisms which map to  $\text{id}_X$  under  $p$ ), there should exist an object  $E'$  of  $\mathcal{F}(X)$  and a *cartesian* morphism  $E' \xrightarrow{\phi} E$  with  $p(\phi) = f$ .

$$\begin{array}{ccc} E' & \xrightarrow{\phi} & E \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

The notion of a cartesian morphism is modeled on the universal property of usual pullbacks: A morphism  $\phi : E' \rightarrow E$  with  $p(\phi) = f$  as in the definition is cartesian if and only if, for any morphism  $g : Z \rightarrow Y$ , for any object  $E''$  in  $\mathcal{F}(Z)$ , and any morphism  $\psi : E'' \rightarrow E$  such that  $p(\psi) = f \circ g$ , there exists a unique morphism  $\alpha : E'' \rightarrow E'$  such that  $p(\alpha) = g$  and  $\phi \circ \alpha = \psi$ .

XXX diagram

*Remark 2.2.* The definition of a fibered category as stated is *evil* in the technical sense that it requires certain equalities between objects (namely,  $p(E') = Y$ ). Since

important examples of categories which ought to be fibered categories actually fulfill these evil equalities, this defect is tolerable in practice. But for some purposes, a non-evil variant is necessary; see for instance <http://ncatlab.org/nlab/show/Grothendieck+fibration#StreetFibration>.

**Example 2.3.** The incarnation of example 1.2 as a fibered category is the category  $\mathbf{VB}$  of vector bundles on arbitrary base spaces. The projection functor  $\mathbf{VB} \rightarrow \mathcal{C}$  associates to any vector bundle its base space.

**Definition 2.4.** An *indexed functor*  $\mathcal{F} \rightarrow \mathcal{G}$  between fibered categories over  $\mathcal{C}$  is an ordinary functor  $\mathcal{F} \rightarrow \mathcal{G}$  which commutes with the projection functors to  $\mathcal{C}$  and which carries cartesian morphisms to cartesian morphisms.

### 3 Equivalence between indexed and fibered categories

**Proposition 3.1.** *There is a pseudoequivalence between the 2-category of indexed categories, indexed functors, and indexed natural transformations; and the 2-category of fibered categories, fibered functors, and fibered natural transformations.*

**From fibered categories to indexed categories.** If suitable choice principles are available, any fibered category  $\mathcal{F} \rightarrow \mathcal{C}$  defines a  $\mathcal{C}$ -indexed category  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$  by setting  $F(X) := \mathcal{F}(X)$ . For morphisms  $f : Y \rightarrow X$  in  $\mathcal{C}$ , the pullback functors  $f^* : F(X) \rightarrow F(Y)$  are defined by associating to any object  $E$  a chosen pullback  $E'$  as in the definition of a fibered category. The action of  $f^*$  on morphisms is given by the universal property of cartesian morphisms.

One can check that this construction gives indeed rise to a pseudofunctor. In fact, the proof is very similar to the proof that the self-indexing  $X \mapsto \mathcal{C}/X$  is a pseudofunctor.

**From indexed categories to fibered categories.** In the opposite direction, any  $\mathcal{C}$ -indexed category  $F : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Cat}$  induces a fibered category  $\int F \rightarrow \mathcal{C}$  via the *Grothendieck construction*. Mimicking the case of families of sets, the objects of  $\int F$  are pairs  $(X \in \mathcal{C}, E \in F(X))$ . A morphism  $(X, E) \rightarrow (Y, F)$  is a pair  $(f : X \rightarrow Y, \alpha : E \rightarrow f^*(F))$ . The projection functor  $\int F \rightarrow \mathcal{C}$  sends an object  $(X, E)$  to  $X$  and a morphism  $(f, \alpha)$  to  $f$ .

One can check that this category over  $\mathcal{C}$  is indeed a fibered category. The cartesian morphisms  $\phi : (Y, E') \rightarrow (X, E)$  required by the definition are the morphisms  $(f : Y \rightarrow X, \text{id} : f^*(E) \rightarrow f^*(E))$ .

**Example 3.2.** Let  $\mathcal{C}$  be a category with pullbacks. Then  $\int \mathbb{C}$  (see example 1.5) is equivalent to the *category of morphisms in  $\mathcal{C}$* . This category has arbitrary morphisms  $Y \rightarrow X$  as objects and commutative squares as morphisms. The equivalence is given by sending an object  $(X \in \mathcal{C}, (Y \xrightarrow{f} X))$  to  $f$ .

**Example 3.3.** Let  $\mathcal{C}$  be an arbitrary category and  $A$  a fixed object of  $\mathcal{C}$ . Then  $\int \underline{A}$  (see example 1.7) is equivalent to the slice category  $\mathcal{C}/A$ , by sending an object  $(X \in \mathcal{C}, X \xrightarrow{f} A)$  to  $f$ .

## 4 Special types of indexed and fibered categories

**Definition 4.1.** A *category fibered in groupoids (sets) [equivalence relations]* is a fibered category  $\mathcal{F}$  such that the fiber categories  $\mathcal{F}(X)$  are all groupoids (discrete categories) [categories equivalent to discrete categories].

Equivalently, the associated pseudofunctor  $\mathcal{C}^{\text{op}} \rightarrow \text{Cat}$  factors over  $\text{Grpd} \hookrightarrow \text{Cat}$  ( $\text{Set} \hookrightarrow \text{Cat}$ ) [ $\text{EqRel} \hookrightarrow \text{Cat}$ ].

**Proposition 4.2.** A functor  $\mathcal{F} \xrightarrow{p} \mathcal{C}$  is ...

- a *category fibered in groupoids* if and only if any morphism in  $\mathcal{F}$  is cartesian and for any morphism  $Y \xrightarrow{f} X$  in  $\mathcal{C}$  and any object  $E$  in  $\mathcal{F}(X)$ , there exists a morphism  $E' \xrightarrow{\phi} E$  in  $\mathcal{F}$  such that  $p(\phi) = f$ .
- a *category fibered in sets* if and only if for any morphism  $Y \xrightarrow{f} X$  in  $\mathcal{C}$  and any object  $E$  in  $\mathcal{F}(X)$ , there exists a unique morphism  $E' \xrightarrow{\phi} E$  in  $\mathcal{F}$  such that  $p(\phi) = f$ .

## 5 An important example: quotient stacks

For this example, we fix a category  $\mathcal{C}$  of spaces. We will be so sketchy that it does not matter which category  $\mathcal{C}$  actually is. The source for this section is the article *Algebraic stacks* of Tomás Gómez.

Let  $G$  be a group acting on a space  $X$ . Assume that the action is *free*, i.e. for any  $g \in G$  and any  $x \in X$ ,  $g \cdot x = x$  implies  $g = 1$ . Then recall that there is a well-defined projection  $X \rightarrow X/G$ ; that this map is a  $G$ -principal bundle; and that this map is the *universal*  $G$ -principal bundle equipped with a  $G$ -equivariant map to  $X$ : For any  $G$ -principal bundle  $P \rightarrow Y$  equipped with a  $G$ -equivariant map  $P \rightarrow X$ , there is a unique map  $Y \rightarrow X/G$  such that the diagram

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X/G \end{array}$$

is a pullback diagram. Thus the space  $X/G$  has the pleasant property that maps  $Y \rightarrow X/G$  correspond to  $G$ -principal bundles  $P \rightarrow Y$  equipped with  $G$ -equivariant maps  $P \rightarrow X$ . (If such a bundle is given, construct  $Y \rightarrow X/G$  by selecting, for any  $y \in Y$ , a preimage under  $P \rightarrow Y$ , and applying  $P \rightarrow X \rightarrow X/G$  to this preimage. One can check that the resulting element in  $X/G$  is independent of the choice.)

Unfortunately, if the action of  $G$  on  $X$  is not free,  $X/G$  fails to exist as an object of  $\mathcal{C}$  or fails to have this universal property. But there is always a quotient *stack*  $[X/G]$  (at least an indexed category) with this property, tautologically defined by sending a test

object  $Y$  of  $\mathcal{C}$  to the category of  $G$ -principal bundles  $P \rightarrow Y$  equipped with  $G$ -equivariant maps  $Y \rightarrow X$ .

Indeed, one can canonically define a map  $\underline{X} \rightarrow [X/G]$  and check that is the universal bundle.

**Proposition 5.1.** *The stack  $[X/G]$  is representable, i. e. of the form  $\underline{A}$  for some space  $A$  in  $\mathcal{C}$ , if and only if the action of  $G$  on  $X$  is free.*

*Proof (sketch).* For the “only if” direction, check that the morphism  $X \rightarrow A$  corresponding to the universal bundle  $\underline{X} \rightarrow \underline{A}$  is a  $G$ -principal bundle isomorphic to  $X \rightarrow X/G$ .  $\square$

**Example 5.2.** Let  $X = \text{pt}$ . Then the quotient stack  $[X/G]$  is commonly named “ $BG$ ”.

*Remark 5.3.* The category of points of  $[X/G]$ , i. e.  $[X/G](\text{pt})$ , is equivalent to the *action groupoid (weak quotient)*  $X//G$ . This groupoid has as objects the elements of  $X$ , and the set of morphisms  $x \rightarrow x'$  is the set of group elements  $g$  such that  $g \cdot x = x'$ . The action groupoid is the “right replacement” for the ill-behaved space of orbits. A very accessible exposition is in *Higher-Dimensional Algebra VII: Groupoidification* by John Baez, Alexander Hoffnung and Christopher Walker: <http://arxiv.org/abs/0908.4305>.

*Remark 5.4.* I believe that in some cases, the category of coherent sheaves on  $[X/G]$  can simply be described as the category of  $G$ -equivariant sheaves on  $X$ .