Tensor categories

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Tensor categories are categories equipped with a tensor operation between objects and between morphisms. They arise in abstract category theory, representation theory, quantum field theory and certain other subjects and provide an abstract setting for studying the general notions of the dual of an object and the trace of a morphism. Thanks to a rich graphical calculus, working with commutative diagrams in tensor categories is very enjoyable.

These are informal notes prepared for the February 2014 meeting of the *Kleine Bayerische AG* at TU München. The notes summarize the first chapter of the article *Tannakian Categories* (LNM 900, 1982; 2012) by P. Deligne and J. S. Milne.

Contents

1 Basics **2**

2 Structure in tensor categories **4**

3 Tensor functors **5**

4 Rigid tensor categories **6**

5 Abelian tensor categories **8**

6 The Tannaka reconstruction theorem **11**

1 Basics

Example 1.1. The category Mod_R of modules over a commutative ring R is the archetypical example of a tensor category.

Definition 1.2. A *tensor category* $(\mathcal{C}, \otimes)$ (i. e. a monoidal category with symmetric braiding) consists of

- a category $\mathcal{C}$,
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the tensor operation),
- an object $1 \in \mathcal{C}$ (the unit object),
- natural isomorphisms $X \otimes (Y \otimes Z) \xrightarrow{\phi_{XYZ}} (X \otimes Y) \otimes Z$ (the associator),
- natural isomorphisms $X \otimes Y \xrightarrow{\psi_{XY}} Y \otimes X$ (the braiding),
- natural isomorphisms $1 \otimes X \xrightarrow{\lambda_X} X$ and $X \otimes 1 \xrightarrow{\rho_X} X$ (the unitors),

such that

- the braiding is symmetric: $\psi_{YX} \circ \psi_{XY} = \mathrm{id}_{X \otimes Y}$ and
- the following coherence conditions are satisfied: the pentagon axiom (displayed); the hexagon axiom (relating the associator with the braiding); and the triangle axiom (relating the associator with the unitors).

$$\begin{tikzcd}
 & & X \otimes (Y \otimes (Z \otimes T)) \arrow[dll, "\mathrm{id}_X \otimes \phi_{Y,Z,T}"'] \arrow[drr, "\phi_{X,Y,Z \otimes T}"] & & \\
X \otimes ((Y \otimes Z) \otimes T \arrow[dr, "\phi_{X,Y \otimes Z,T}"'] & & & & (X \otimes Y) \otimes (Z \otimes T) \arrow[dl, "\phi_{X \otimes Y,Z,T}"] \\
 & (X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi_{X,Y,Z} \otimes \mathrm{id}_T"'] & & ((X \otimes Y) \otimes Z) \otimes T &
\end{tikzcd}$$

The definitions mimics the definition of a monoid, only "one level up". Demanding that the tensor operation is associative and commutative on the nose (e. g. $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$) would be *evil* in the technical sense, i. e. not invariant under equivalence of categories. This is the reason why we demand (natural) isomorphisms, i. e. natural transformations between easily inferred fuctors. The isomorphism classes of a tensor category form a (possibly large) commutative monoid.

Example 1.3. In Mod_R, ϕ_{XYZ} is given by

$$x \otimes (y \otimes z) \longmapsto (x \otimes y) \otimes z.$$

If $-1 \neq 1 \in R$, introducing a sign here will cause the pentagon diagram to fail to commute by a sign.

Example 1.4. Let $\mathcal{C}$ be a category with finite products. Then $(\mathcal{C}, \times)$ is a tensor category, with $\phi, \psi, \lambda, \rho$ given by the universal property of the product. In particular, $(\mathrm{Set}, \times)$ is a tensor category.

Example 1.5. Let $\mathcal{C}$ be a tensor category. Then $\mathcal{C}^{\mathrm{op}}$ is in a natural way a tensor category, by using the inverses of the given natural isomorphisms.

Remark 1.6. In a general tensor category, there is no natural morphism $X \to X \otimes X$.

The coherence conditions in the definition are needed for the following reason: In the category of modules, we are used to dropping all parentheses when dealing with iterated tensor products. This is justified only because between any two given groupings, e. g.

$$X \otimes ((Y \otimes Z) \otimes (T \otimes U)) \quad \text{and} \quad ((X \otimes Y) \otimes (Z \otimes T)) \otimes U,$$

we have *exactly one* canonical isomorphism. A classical theorem of Mac Lane guarantees that the stated coherence conditions suffice to render any "reasonable" diagram commutative:

Theorem 1.7 (Mac Lane)**.** *Any (formal) diagram in a tensor category built by*

$$\otimes, \mathrm{id}, \phi, \psi, \lambda, \rho, \phi^{-1}, \psi^{-1}, \lambda^{-1}, \rho^{-1}$$

(in which both sides have the same permutation) commutes.

The two caveats are the following: Firstly, in a given tensor category, there may hold certain *identities* between objects for no general abstract reason. For example, for some totally unrelated objects X, Y, A, B, it might hold that $X \otimes Y = A \otimes B$. Using those identities we can form diagrams which we do *not* expect to commute. Mac Lane's coherence theorem does not make any statement about those diagrams.

To understand the restriction about permutations, consider the diagram

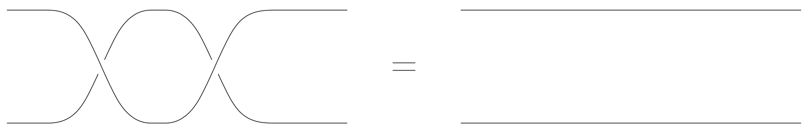
$$\begin{tikzcd} X \otimes X \arrow[r, bend left, "\psi_{XX}"] \arrow[r, bend right, "\mathrm{id}_{X\otimes X}"'] & X \otimes X. \end{tikzcd}$$

We do not expect this diagram to be commutative; the permutations associated to both sides are not equal: $(1, 2) \neq \mathrm{id}$.

Theorem 1.8 (Joyal, Street)**.** *If the graphical depictions of given morphisms of a tensor category are "the same" (in 4D space), the morphisms are equal.*

See *A survey of graphical languages for monoidal categories* by P. Selinger (LNP 813, arXiv:0908.3347) for a precise formulation and pointers to the relevant literature.

Example 1.9. In four-dimensional space, a double braiding can be unknotted (while holding the endpoints fixed):



Theorem 1.10 (Mac Lane)**.** *Any tensor category can be* strictified*, i. e. is equivalent as a tensor category to a* strict *tensor category: a category in which ϕ, λ, ρ (but not ψ) are identities.*

The strictification of a tensor category should *not* be thought of as some kind of quotienting process, i. e. one does not obtain the strictification by identifying certain morphisms. Instead, the objects of the strictification are "cliques" of possible groupings.

Also, skeleta of tensor categories are usually *not* strict: For instance, consider a skeleton of the category of sets with its unique object N isomorphic to set of natural numbers. Then the objects $N \times (N \times N)$ and $(N \times N) \times N$ are equal. But the unique morphism $N \times (N \times N) \to (N \times N) \times N$ compatible with the projections is not equal to the identity.

2 Structure in tensor categories

Definition 2.1. An *internal Hom* between objects X, Y of a tensor category $\mathcal{C}$ consists of

- an object $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{C}$ and
- a morphism $\underline{\mathrm{Hom}}(X, Y) \otimes X \xrightarrow{\mathrm{ev}} Y$ (evaluation morphism)

such that this pair is terminal among such pairs, i. e. such that for any object $T \in \mathcal{C}$ and a [fake] evaluation morphism $T \otimes X \xrightarrow{\widetilde{\mathrm{ev}}} Y$ there exists an unique morphism $f : T \to \underline{\mathrm{Hom}}(X, Y)$ such that the following diagram commutes:

$$\begin{tikzcd} T \otimes X \arrow[d, "f \otimes \mathrm{id}_X"'] \arrow[dr, "\widetilde{\mathrm{ev}}"] & \\ \underline{\mathrm{Hom}}(X, Y) \otimes X \arrow[r, "\mathrm{ev}"'] & Y \end{tikzcd}$$

Example 2.2. In $(\mathrm{Set}, \times)$, the internal Homs are given by the usual Hom sets. The evaluation morphism is given by $(f, x) \mapsto f(x)$.

Example 2.3. In Mod_R, the internal Homs are given by the Hom sets equipped with the usual module structure.

Remark 2.4. If an internal Hom $\underline{\mathrm{Hom}}(X, Y)$ exists for all objects $Y \in \mathcal{C}$ (and if appropriate choice principles are available), the internal Hom can be made into a functor $\underline{\mathrm{Hom}}(X, \_) : \mathcal{C} \to \mathcal{C}$. This functor is right adjoint to taking tensor product with X:

$$\_ \otimes X \;\dashv\; \underline{\mathrm{Hom}}(X, \_)$$

Remark 2.5. The relation with the usual Hom (which is only a set) is the following:

$$\mathrm{Hom}(1, \underline{\mathrm{Hom}}(X, Y)) \cong \mathrm{Hom}(1 \otimes X, Y) \cong \mathrm{Hom}(X, Y).$$

Remark 2.6. In $(\mathrm{Set}, \amalg)$, internal Homs do not exist in general. This is related to the fact that $\_ \amalg X$ does not preserve colimits and so cannot be a left adjoint.

Definition 2.7. 1. A *dual* of an object X is an internal Hom $X^\vee := \underline{\mathrm{Hom}}(X, 1)$.

2. The *dual* of a morphism $f : X \to Y$ is the unique morphism $f^t : Y^\vee \to X^\vee$ rendering the diagram

$$\begin{tikzcd} Y^\vee \otimes X \arrow[r, "f^t \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_{Y^\vee} \otimes f"'] & X^\vee \otimes X \arrow[d, "\mathrm{ev}_X"] \\ Y^\vee \otimes Y \arrow[r, "\mathrm{ev}_Y"'] & 1 \end{tikzcd}$$

commutative (if $X^\vee$ and $Y^\vee$ exist).

Example 2.8. In Mod_R, $f^t : \theta \mapsto \theta \circ f$.

If all objects possess a dual (and appropriate choice principles are available), the rule $X \mapsto X^\vee$ can be made into a functor $C \to \mathcal{C}^{\mathrm{op}}$.

Proposition 2.9. *In any tensor category, the set* $\mathrm{End}(1) = \mathrm{Hom}(1, 1)$ *is a commutative monoid with respect to composition of morphisms.*

Proof. The result holds even if there were no braiding: On $\mathrm{End}(1)$, the tensor product induces a second binary operation. By the coherence conditions, this operation commutes with the operation given by composition, so by the famous Eckmann–Hilton argument, both operations coincide and are commutative. ☐

Example 2.10. In Mod_R, $\mathrm{End}(1) \cong R$.

Example 2.11. In $(\mathrm{Set}, \times)$, $\mathrm{End}(1) = \{\mathrm{id}\}$.

3 Tensor functors

Definition 3.1. A *tensor functor* $F : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ consists of

1. a functor $F : \mathcal{C} \to \mathcal{C}'$,
2. natural isomorphisms $FX \otimes' FY \xrightarrow{c_{XY}} F(X \otimes Y)$ and
3. an isomorphism $1' \xrightarrow{e} F1$

such that the following coherence condition

$$\begin{tikzcd}
FX \otimes' (FY \otimes' FZ) \arrow[r, "\mathrm{id}\otimes c"] \arrow[d, "\phi'"'] & FX \otimes' F(Y \otimes Z) \arrow[r, "c"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F\phi"] \\
(FX \otimes' FY) \otimes' FZ \arrow[r, "c\otimes\mathrm{id}"'] & F(X \otimes Y) \otimes' Z \arrow[r, "c"'] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

and similar ones relating c with the braidings and with the unitors are satisfied.

Example 3.2. The forgetful functor $\mathrm{Rep}_k(G) \to \mathrm{Vect}_k$ of the category of finite-dimensional k-linear representations of a group (or group scheme) G is a tensor functor.

Example 3.3. Extension of scalars defines a tensor functor $\mathrm{Mod}_R \to \mathrm{Mod}_S$.

Example 3.4. A quantum field theory determines a tensor functor $\mathrm{Cob}_d \to \mathrm{Vect}_k$.

Definition 3.5. A *morphism of tensor functors* $\eta : (F, c, e) \to (\widetilde{F}, \tilde{c}, \tilde{e})$ consists of a natural transformation $\eta : F \to \widetilde{F}$ which is compatible with the coherence isomorphisms:

$$\begin{tikzcd}
FX \otimes' FY \arrow[r, "c"] \arrow[d, "\eta_X\otimes\eta_Y"'] & F(X \otimes Y) \arrow[d, "\eta_{(X\otimes Y)}"] \\
\widetilde{F}X \otimes' \widetilde{F}Y \arrow[r, "\tilde{c}"'] & \widetilde{F}(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
1' \arrow[r, "e"] \arrow[d, "\mathrm{id}_1"'] & F1 \arrow[d, "\eta_1"] \\
1' \arrow[r, "\tilde{e}"'] & \widetilde{F}1
\end{tikzcd}$$

4 Rigid tensor categories

Definition 4.1. A tensor category $\mathcal{C}$ is *rigid* iff

- all internal Homs exist,
- the natural morphisms

$$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \longrightarrow \underline{\mathrm{Hom}}(X_1 \otimes Y_1, X_2 \otimes Y_2)$$

  are isomorphisms and
- all objects $X \in \mathcal{C}$ are *reflexive* (i. e. the natural map $X \to X^{\vee\vee}$ is an isomorphism).

Example 4.2. The category $\mathrm{Vect}_k^{\mathrm{fd}}$ of finite-dimensional is rigid.

Example 4.3. More generally, the category $\mathrm{Mod}_R^{\mathrm{fin.\,free}}$ of finitely free R-modules is rigid.

Example 4.4. Changing toposes, the category of locally free $\mathcal{O}_{\mathrm{Spec}\,R}$-modules is rigid. (This category is equivalent to category of finitely generated projective R-modules, by the tilde construction.)

Example 4.5. The category $\mathrm{Rep}_k(G)$ is rigid.

Remark 4.6. In a rigid tensor category, internal Homs can be calculated using the tensor product:

$$\underline{\mathrm{Hom}}(X, Y) \cong X^\vee \otimes Y.$$

Remark 4.7. Let $\mathcal{C}$ be a rigid category. Then the functor $\mathcal{C} \to \mathcal{C}^{\mathrm{op}}, X \mapsto X^\vee$ is an equivalence of tensor categories.

In a rigid tensor category, we can define the notion of traces:

Definition 4.8. 1. The *trace* of an endomorphism $f : X \to X$ in a rigid tensor category is the following element of $\mathrm{End}(1)$:

$$f \in \mathrm{Hom}(X, X) \cong \mathrm{Hom}(1, \underline{\mathrm{Hom}}(X, X)) \cong \mathrm{Hom}(1, X^\vee \otimes X) \to \mathrm{Hom}(1, 1) \ni: \mathrm{tr}\, f.$$

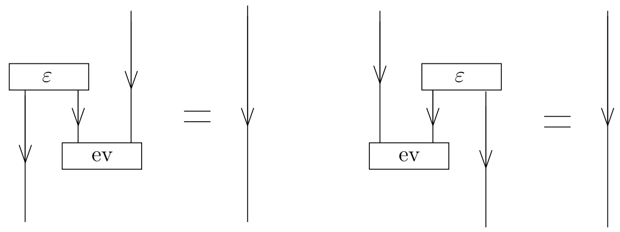
2. The *rank* (or *Euler characteristic*) of an object X is the trace of the identity morphism on X.

Trace and rank satisfy the relations you expect:

Lemma 4.9. *1.* $\mathrm{tr}(f \otimes f') = \mathrm{tr}(f) \circ \mathrm{tr}(f'), \quad \mathrm{tr}(f \circ g) = \mathrm{tr}(g \circ f).$

2. $\mathrm{rk}(X \otimes X') = \mathrm{rk}(X) \circ \mathrm{rk}(X'), \quad \mathrm{rk}(1) = \mathrm{id}_1.$

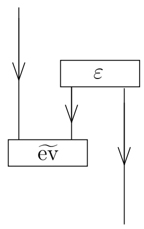
Lemma 4.10. *In a rigid tensor category, a pair* $(Y, Y \otimes X \xrightarrow{\mathrm{ev}} 1)$ *is a dual of* X *iff there exists a morphism* $1 \xrightarrow{\varepsilon} X \otimes Y$ *such that the following triangle identities hold:*



Suppressing coherence isomorphisms, these conditions can also be expressed as

$$(\mathrm{id}_X \otimes \mathrm{ev}) \circ (\varepsilon \otimes \mathrm{id}_X) = \mathrm{id}_X, \quad (\mathrm{ev} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_Y \otimes \varepsilon) = \mathrm{id}_Y.$$

Proof. Given a pair $(T, T \otimes X \xrightarrow{\widetilde{\mathrm{ev}}} 1)$, construct $f : T \to Y$ as follows:



Using the graphical calculus it's fun to check that with this definition of f, the diagram in the definition of the internal Hom commutes (try it!). Use the second triangle identity to show uniqueness.

For the converse direction, construct ε by dualizing ev. $\square$

Remark 4.11. Rigidity is only necessary for the converse direction: A pair $(Y, Y \otimes X \xrightarrow{\text{ev}} 1)$ which satisfies the condition stated in lemma is always a dual—and in fact, because the condition is symmetric, the coevaluation morphism exhibits X as a dual of Y. In particular, X is reflexive.

Example 4.12. Let M be a finitely free R-module with basis $(x_1, \ldots, x_n)$. Let $(\vartheta_1, \ldots, \vartheta_n)$ be the associated dual basis of $M^\vee$. Then ε is given by

$$\varepsilon : 1 \longrightarrow M \otimes M^\vee, \; 1 \longmapsto \sum_i x_i \otimes \vartheta_i.$$

The condition in the lemma expresses that for any $x \in M$ and any $\vartheta \in M^\vee$, it holds that

$$\begin{aligned} x &= \sum_i \theta_i(x)\, x_i, \\ \vartheta &= \sum_i \vartheta(x_i)\, \vartheta_i. \end{aligned}$$

Proposition 4.13. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a tensor functor, with $\mathcal{C}$ rigid. Let $X, Y \in \mathcal{C}$. Then an internal Hom $\underline{\mathrm{Hom}}(FX, FY)$ exists in $\mathcal{C}'$ and the natural morphism*

$$F(\underline{\mathrm{Hom}}(X, Y)) \longrightarrow \underline{\mathrm{Hom}}(FX, FY)$$

is an isomorphism.

Proof. It is enough to show that in $\mathcal{C}'$ a dual of FX exists and that the natural morphism

$$F(X^\vee) \longrightarrow (FX)^\vee$$

is an isomorphism. This is obvious by the lemma, as the given condition is preserved by F. □

Corollary 4.14. $\operatorname{tr} F(f) = F(\operatorname{tr} f), \quad \operatorname{rk} FX = F(\operatorname{rk} X).$

Proposition 4.15. *In a rigid category, $\_ \otimes X$ is not only a left, but also a right adjoint:*

$$\underline{\mathrm{Hom}}(X, \_) \;\dashv\; \_ \otimes X \;\dashv\; \underline{\mathrm{Hom}}(X, \_).$$

Corollary 4.16. *In a rigid category, the tensor operation is continuous and cocontinuous, i. e. commutes with arbitrary limits and colimits.*

5 Abelian tensor categories

Definition 5.1. An *abelian tensor category* is a tensor category $(\mathcal{C}, \otimes)$ such that $\mathcal{C}$ is abelian and $\otimes$ is bi-additive.

Remark 5.2. In an abelian category, $\mathrm{End}(1)$ is a commutative ring with respect to addition and composition of morphisms.

Lemma 5.3. *Let $\mathcal{C}$ be an abelian tensor category. Let $X \in \mathcal{C}$.*

1. *Assume that the unit* 1 *is a simple object and that* $\mathcal{C}$ *is rigid. Then:*

$$X \ncong 0 \quad \Longrightarrow \quad X^\vee \otimes X \to 1 \textit{ is epic.}$$

2. *Assume that* $1 \ncong 0 \in \mathcal{C}$*. Then:*

$$X \ncong 0 \quad \Longleftarrow \quad X^\vee \otimes X \to 1 \textit{ is epic.}$$

Proof. 1. Because the unit is simple, the natural morphism $X^\vee \otimes X \to 1$ is either zero or epic. (This follows by considering the image of the morphism.) Under the correspondence

$$\mathrm{Hom}(X^\vee \otimes X, 1) \cong \mathrm{Hom}(X, X^{\vee\vee}),$$

the morphism corresponds to the natural morphism $X \to X^{\vee\vee}$. If this was zero, X would be zero as well.

2. Assume $X \cong 0$. Then we have an epic morphism $0 \to 1$. By terminality of the zero object, this is an isomorphism. □

Remark 5.4. The rigidity is necessary: In $\mathrm{Mod}_\mathbb{Z}$, the object $\mathbb{Z}/(2)$ is not isomorphic to the zero object, but the morphism $(\mathbb{Z}/(2))^\vee \otimes \mathbb{Z}/(2) \to \mathbb{Z}$ has zero domain.

Proposition 5.5. *Let $\mathcal{C}$ be a rigid abelian tensor category with $1 \in \mathcal{C}$ simple. Let $\mathcal{C}'$ be an abelian tensor category with $1 \ncong 0$. Then any exact tensor functor $F : \mathcal{C} \to \mathcal{C}'$ is faithful.*

Proof. Let $Ff = 0$. If $f \neq 0$, then $\mathrm{im}(f) \ncong 0$. So by the lemma, the natural morphism $\mathrm{im}(f)^\vee \otimes \mathrm{im}(f) \to 1$ is epic. Because F is exact and preserves duals, the natural morphism $\mathrm{im}(Ff)^\vee \otimes \mathrm{im}(Ff) \to 1$ is epic as well. Again by the lemma, it follows that $\mathrm{im}(Ff) \ncong 0$. This is a contradiction. □

Remark 5.6. The simpleness of the unit is necessary: Let $\mathcal{C}$ and $\mathcal{D}$ be rigid abelian categories. Then the projection functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C}$ is exact, but in general not faithful.

The following corollary shows that the proposition can be interpreted as a *categorification* of a certain well-known lemma of commutative algebra:

Corollary 5.7. *Let k be a field and R be a ring with $1 \neq 0 \in R$. Let $\varphi : k \to R$ be a ring homomorphism. Then φ is injective.*

Proof. By the proposition, the functor $F : \mathrm{Vect}_k^{\mathrm{fd}} \to \mathrm{Mod}_R$ (extension of scalars) is faithful. Let $\varphi(x) = 0$. Then the image of the map $k \to k$ given by multiplication with x is zero in Mod_R. So by faithfulness, the map is already zero on k, so $x = 0$. □

Proposition 5.8. *Let $\mathcal{C}$ be a rigid abelian tensor category. Let $U \hookrightarrow 1$ be a subobject. Then the unit object decomposes as a direct sum*

$$1 \cong U \oplus U^{\perp},$$

where $U^{\perp} = \ker(1 \to U^{\vee})$.

Proof. The argument is given in six steps. Rigidity mainly enters in that tensoring is exact instead of merely right-exact.

1. Consider the cokernel V of $U \hookrightarrow 1$; we then have a short exact sequence

$$0 \longrightarrow U \longrightarrow 1 \longrightarrow V \longrightarrow 0.$$

2. By rigidity, tensoring is exact; so we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] & 1 \arrow[r] & V \arrow[r] & 0 \\ 0 \arrow[r] & U \otimes U \arrow[r] \arrow[u, hook] & U \arrow[r] \arrow[u, hook] & V \otimes U \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

with exact rows. Because the morphism $U \to V \otimes U \to V$ is zero, $V \otimes U$ is zero; and by exactness of the bottom row, $U \otimes U \cong U$.

3. For any subobject $T \hookrightarrow X$ the following chain of equivalences holds:

$$T \otimes U \cong 0 \iff T \otimes U \hookrightarrow T \text{ is zero} \iff T \twoheadrightarrow U^{\vee} \otimes T \to U^{\vee} \otimes X \text{ is zero.}$$

The first "$\Leftarrow$" is because $T \otimes U \hookrightarrow T$ is a monomorphism (by exactness of tensoring with T) and the second "$\Leftrightarrow$" is by the isomorphisms

$$\underline{\mathrm{Hom}}(T \otimes U, T) \cong U^{\vee} \otimes T^{\vee} \otimes T \cong \underline{\mathrm{Hom}}(T, U^{\vee} \otimes T).$$

So the largest such subobject $T \hookrightarrow X$ is given by

$$T = \ker(X \to U^{\vee} \otimes X) \cong U^{\perp} \otimes X.$$

(The isomorphism is by exactness of tensoring with X.)

4. Applying this observation to $X = V$, it follows that $U^{\perp} \otimes V \cong V$, because even $V \otimes U \cong 0$ holds.

Applying it to $X = U$, it follows that $U^{\perp} \otimes U \cong 0$: Let $T \hookrightarrow U$ with $T \otimes U \cong 0$. By exactness of tensoring with T, the sequence

$$0 \longrightarrow T \otimes U \longrightarrow T \longrightarrow T \otimes V \longrightarrow 0$$

is exact. Since $T \otimes U \cong 0$, we have

$$T \cong T \otimes V \hookrightarrow U \otimes V \cong 0.$$

The "$\hookrightarrow$" is—again—by exactness of tensoring (with V).

5. By exactness of tensoring with $U^\perp$, the sequence

$$0 \longrightarrow U^\perp \otimes U \longrightarrow U^\perp \longrightarrow U^\perp \otimes V \longrightarrow 0$$

is exact. By the previous step, this shows $U^\perp \cong V$.

6. By applying the five lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & U \oplus U^\perp \arrow[r] \arrow[d] & U^\perp \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & U \arrow[r] & 1 \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

it follows that $1 \cong U \oplus U^\perp$. ☐

Remark 5.9. Rigidity is necessary: In $\mathrm{Mod}_{\mathbb{Z}}$, the unit object is not simple but admits no non-trivial decompositions either.

Corollary 5.10. *In a rigid abelian category, the unit object is simple iff* $\mathrm{End}(1)$ *is a field.*

Proof. For the "only if" direction, let $f \in \mathrm{End}(1)$. As the unit object is simple, f is either zero or an isomorpism (i. e. invertible in the ring $\mathrm{End}(1)$). This follows by considering kernel and image of f.

For the "if" direction, let $U \hookrightarrow 1$. By the lemma, there exists a projection operator $P : 1 \to 1$ with $\mathrm{im}(P) = U$. If P is zero in $\mathrm{End}(1)$, $U = 0$; if P is invertible, $U = 1$ (as subojects of 1). ☐

6 The Tannaka reconstruction theorem

Theorem 6.1. *Let $\mathcal{C}$ be a rigid abelian tensor category. Let $k := \mathrm{End}(1)$ be a field. Let $\omega : \mathcal{C} \to \mathrm{Vect}_k^{\mathrm{fd}}$ be an exact faithful k-linear tensor functor. Then:*

1. *The functor $\underline{\mathrm{Aut}}^{\otimes}(\omega) : \mathrm{Alg}_k \to \mathrm{Set}$, given by*

$$R \longmapsto \textit{set of tensor automorphisms of } (\_ \otimes_k R) \circ \omega,$$

is the functor of points of an affine group scheme G.

2. *A certain functor $\mathcal{C} \to \mathrm{Rep}_k(G)$ induced by ω is an equivalence of tensor categories.*