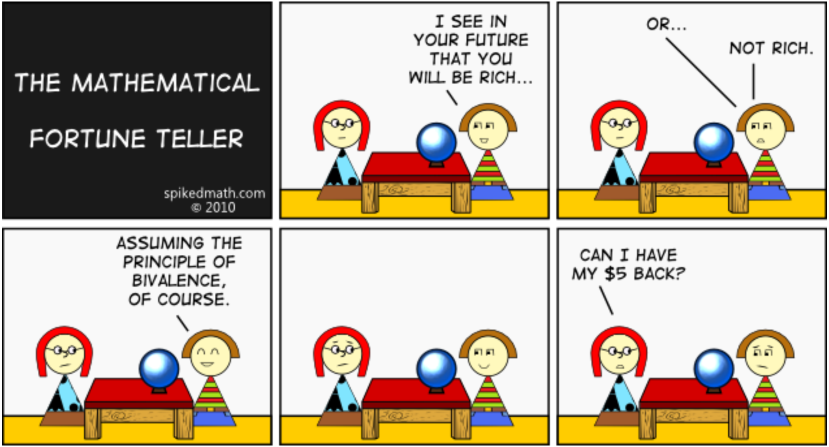
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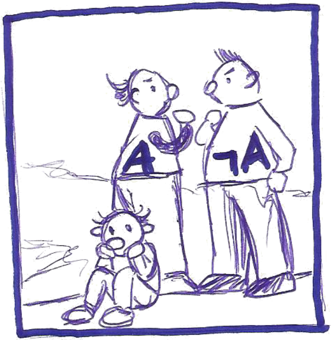
I SEE IN YOUR FUTURE THAT YOU WILL BE RICH...
OR...
NOT RICH.
ASSUMING THE PRINCIPLE OF BIVALENCE, OF COURSE.
CAN I HAVE MY $5 BACK?

# Double-negation translation and CPS transformation




Ingo Blechschmidt

June 3rd, 2015 at KU Leuven

# Outline

# Non-constructive proofs

**Theorem.** There exist **irrational** numbers $x, y$ such that $x^y$ is rational.

**Proof.** Either $\sqrt{2}^{\sqrt{2}}$ is rational or not.

In the first case, we are done.

In the second case, take $x := \sqrt{2}^{\sqrt{2}}$ and $y := \sqrt{2}$. Then $x^y = 2$ is rational.

# The law of excluded middle

"For any formula $A$, we may deduce $A \lor \neg A$."

Classical logic $=$
intuitionistic logic $+$ law of excluded middle.

**Classical interpretation**

| | |
|---|---|
| $\bot$ | There is a contradiction. |
| $A \land B$ | $A$ and $B$ are true. |
| $A \lor B$ | $A$ is true or $B$ is true. |
| $A \Rightarrow B$ | If $A$ holds, then also $B$. |
| $\forall x{:}X.\ A(x)$ | For all $x : X$ it holds that $A(x)$. |
| $\exists x{:}X.\ A(x)$ | There is an $x : X$ such that $A(x)$. |

# The law of excluded middle

"For any formula $A$, we may deduce $A \vee \neg A$."

Classical logic $=$
intuitionistic logic $+$ law of excluded middle.

## Constructive interpretation

| | |
|---|---|
| $\bot$ | There is a contradiction. |
| $A \wedge B$ | We have evidence for $A$ and for $B$. |
| $A \vee B$ | We have evidence for $A$ or for $B$. |
| $A \Rightarrow B$ | We can transform evidence for $A$ into one for $B$. |
| $\forall x{:}X.\, A(x)$ | Given $x : X$, we can construct evidence for $A(x)$. |
| $\exists x{:}X.\, A(x)$ | We have an $x : X$ together with evidence for $A(x)$. |

# Negated statements

"$\neg A$" is syntactic sugar for $(A \Rightarrow \bot)$
and means: There can't be any evidence for $A$.

**Constructive interpretation**

| | |
|---|---|
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# Doubly-negated statements

"$\neg\neg A$" means: There can't be any evidence for $\neg A$.

Trivially, we have $A \implies \neg\neg A$.
We can't deduce $\neg\neg A \implies A$.

**Constructive interpretation**

| | |
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**Where is the key?**

$\neg\neg(\exists x.\ \text{the key is at position } x)$

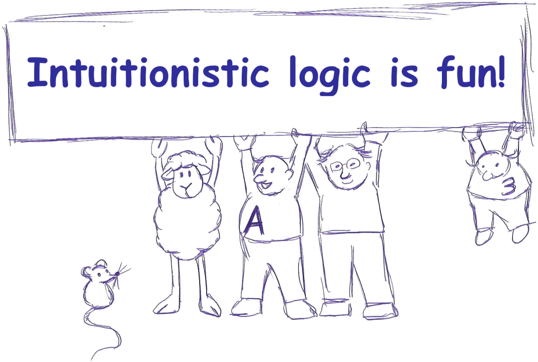
*versus*

$\exists x.\ \text{the key is at position } x$

# Applications

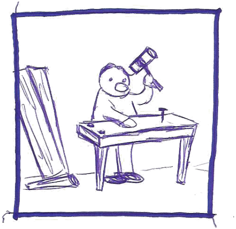
Intuitionistic logic …

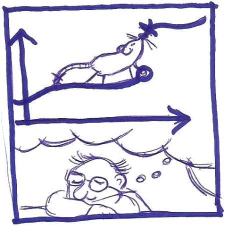
- can guide to more elegant proofs,
- is good for the mental hygiene, and
- allows to make finer distictions.



# Applications

- We can **mechanically extract algorithms** from intuitionistic proofs of existence statements.
- The **internal language of toposes** is intuitionistic.
- **Dream mathematics** only works intuitionistically.



# Topos power

Any finitely generated vector space does *not not* possess a basis.

$$\Downarrow$$

Any sheaf of modules of finite type on a reduced scheme is locally free on a dense open subset.

# Dream mathematics

### Synthetic differential geometry

Any map $\mathbb{R} \to \mathbb{R}$ is smooth. There are infinitesimal numbers $\varepsilon$ such that $\varepsilon^2 = 0$ and $\varepsilon \neq 0$.

### Synthetic domain theory

For any set $X$ there exists a map

$$\mathsf{fix} : (X \to X) \to X$$

such that $f(\mathsf{fix}(f)) = \mathsf{fix}(f)$ for any $f : X \to X$.

### Synthetic computability theory

There are only countably many subsets of $\mathbb{N}$.

# The doubly-negated LEM

Even intuitionistically "$\neg\neg(A \lor \neg A)$" holds.

**Proof**. Assume $\neg(A \lor \neg A)$, we want to show $\bot$.
If $A$, then $A \lor \neg A$, thus $\bot$. Therefore $\neg A$.
Since $\neg A$, we have $A \lor \neg A$, thus $\bot$.

# The ¬¬-translation

$$
\begin{aligned}
A^{\square} &:\equiv \neg\neg A \text{ for atomic formulas } A \\
(A \wedge B)^{\square} &:\equiv \neg\neg(A^{\square} \wedge B^{\square}) \\
(A \vee B)^{\square} &:\equiv \neg\neg(A^{\square} \vee B^{\square}) \\
(A \Rightarrow B)^{\square} &:\equiv \neg\neg(A^{\square} \Rightarrow B^{\square}) \\
(\forall x{:}X.\, A(x))^{\square} &:\equiv \neg\neg(\forall x{:}X.\, A^{\square}(x)) \\
(\exists x{:}X.\, A(x))^{\square} &:\equiv \neg\neg(\exists x{:}X.\, A^{\square}(x))
\end{aligned}
$$

**Theorem.** $A$ classically $\Longleftrightarrow$ $A^{\square}$ intuitionistically.

# A classical logic fairy tale

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$A$ intuitionistically $\iff$ we can defend $A$ in any dialog.

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$A$ classically $\iff$ we can defend $A^{\Box}$ in any dialog.

$\iff$ we can defend $A$ in any dialog with jumps back in time allowed.

# Curry–Howard correspondence

| logic | programming |
|---:|---|
| formula $A$ | type $A$ |
| intuitionistic proof $p : A$ | term $p : A$ |
| conjuction $A \wedge B$ | product type $(A, B)$ |
| disjunction $A \vee B$ | sum type Either $A$ $B$ |
| implication $A \Rightarrow B$ | function type $A \to B$ |

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| **$\neg\neg$-translation** | **CPS transformation** |

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| **$\neg\neg$-translation** | **CPS transformation** |
| $\neg\neg A$ | ?? |

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| $(A \Rightarrow \bot) \Rightarrow \bot$ | ?? |

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| conjuction $A \wedge B$ | product type $(A, B)$ |
| disjunction $A \vee B$ | sum type Either $A$ $B$ |
| implication $A \Rightarrow B$ | function type $A \rightarrow B$ |
| **$\neg\neg$-translation** | **CPS transformation** |
| $(A \Rightarrow \bot) \Rightarrow \bot$ | $(A \rightarrow r) \rightarrow r$ |

# Computational content of classical proofs

```haskell
type Cont r a = ((a -> r) -> r)

-- Decide an arbitrary statement a.
lem :: Cont r (Either a (a -> Cont r b))
lem k = k $ Right $ \x -> (\k' -> k (Left x))

-- Calculate the minimum of an infinite list
-- of natural numbers.
min :: [Nat] -> Cont r (Int, Int -> Cont r ())
min xs = ...
```

# Outlook

- CPS transformation = Yoneda embedding
- What about delimited continuations?
- Geometrical interpretation:

$$\mathrm{Sh}(X) \models A^{\Box} \quad \Longleftrightarrow \quad \mathrm{Sh}(X_{\neg\neg}) \models A$$

- Generalize from $\neg\neg$ to arbitrary **modal operators** (monads): Relevant axioms are
  1. $A \Rightarrow \Box A$
  2. $\Box\Box A \Rightarrow \Box A$
  3. $\Box(A \wedge B) \Leftrightarrow \Box A \wedge \Box B$

# Outlook

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github /iblech/talk-constructive-mathematics