The Hilbert scheme

1. The symmetric product

<u>Def:</u> Let X be a geometric object (top. space, manifold, ...). Then
$$X^{(n)} := X^n / S_n$$
is its <u>n-th symmetric product</u>.

<u>Ex:</u> $(\mathbb{A}^1)^{(n)} \cong \mathbb{A}^n$ (as manifolds, if $k = \mathbb{C}$; for any ring, if read as schemes)
$$[(x_1, \dots, x_n)] \longmapsto (e_i(x_1, \dots, x_n))_{i=1}^n$$

<u>Ex:</u> $(\mathbb{P}^1)^{(n)} \cong \check{\mathbb{P}}^n \cong \mathbb{P}^n$
$[(x_1, \dots, x_n)] \longmapsto$ the hyperplane spanned by the $\varphi(x_i)$, $i = 1, \dots, n$,
where $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ is the rational normal curve

<u>Prop:</u> $\sum_n \chi(X^{(n)}) q^n = (1-q)^{-\chi(X)}$.

<u>Proof:</u>
$$\sum_n \chi(X^{(n)}) q^n = \sum_n \frac{1}{n!} \sum_\sigma \chi(\underbrace{(X^n)^\sigma}_{\cong X^{\sum v_i}}) q^n$$

$\chi(Y/G) = \frac{1}{|G|} \sum_g \chi(Y^g)$

let σ be of conj. type
$n = v_1 a_1 + \dots + v_m a_m$, $v_i \geq 1$, $a_1 > \dots > a_m \geq 1$, $m \geq 0$

$$= \sum_{m \geq 0} \sum_{v_1, \dots, v_m \geq 1} \sum_{a_1 > \dots > a_m \geq 1} \frac{1}{(\sum v_i a_i)!} \chi(X)^{\sum v_i} q^{\sum v_i a_i} \cdot \frac{(\sum v_i a_i)!}{a_1^{v_1} \cdots a_m^{v_m} v_1! \cdots v_m!}$$

$$= \sum_{m \geq 0} \sum_{v_1, \dots, v_m \geq 1} \sum_{a_1 > \dots > a_m} \prod_{i=1}^m \frac{1}{a_i^{v_i} \cdot v_i!} \chi(X)^{v_i} q^{v_i a_i}$$

sum is really over all finite multisets of $\mathbb{N}^+$

$$= \underbrace{\sum_{w_1 \geq 0} \sum_{w_2 \geq 0} \cdots}_{\substack{\text{sum over all sequences} \\ (w_1, w_2, \dots) \text{ which are} \\ \text{eventually zero}}} \prod_{i=1}^\infty \underbrace{\frac{1}{i^{w_i} \cdot w_i!} \chi(X)^{w_i} q^{w_i \cdot i}}_{= \frac{1}{w_i!} \left(\chi(X) \frac{q^i}{i}\right)^{w_i}}$$

$$= \prod_{i=1}^\infty e^{\chi(X) q^i / i} = e^{\chi(X) \cdot \sum q^i / i} = e^{-\chi(X) \ln(1-q)} = (1-q)^{-\chi(X)}.$$

:) easy to define

:(in general not smooth, e.g. for $\dim X > 1$

:(not a fine moduli space

2. The Hilbert scheme of points

Def: Let X be a geometric object (manifold, scheme, ...). Then

$$X^{[n]} := \{\text{closed subschemes } Z \hookrightarrow X \mid \dim Z = 0,\ \operatorname{len} Z = n\}$$

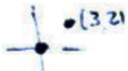
← contains unordered collections of n distinct points & nonreduced subschemes

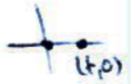
$$\overset{Z \mapsto I(Z)}{\underset{V(I) \leftarrow I}{\cong}} \{ I \hookrightarrow \mathcal{O}_X \mid \text{Hilbert polynomial of } \mathcal{O}_X/I = n \}$$

Ex: $(\mathbb{A}^m)^{[n]} \cong \{ I \in k[x_1,\dots,x_m] \text{ ideal} \mid \dim_k \underbrace{k[x_1,\dots,x_m]/I}_{\text{ring of functions on } V(I)} = n \}$.

↑ visualize as $V(I)$

Ex:

①  $= V((x,y) \cap (x-3, y-2)) = V(x^2-3x,\ xy-2x,\ xy-3y,\ y^2-2y) \in (\mathbb{A}^2)^{[2]}$

②  $= V((x,y) \cap (x-t, y)) = V(x^2-tx,\ xy,\ xy-ty,\ y^2) \underset{t \neq 0}{=} V(x^2-tx,\ y) \in (\mathbb{A}^2)^{[2]}$ (cobasis: $1, x$)

③  $=$ limit of ② as $t \to 0$ $= V(x^2, y) \in (\mathbb{A}^2)^{[2]}$ (cobasis: $1, x$)

④  $= \lim_{t \to 0} V((x,y) \cap (x - t x_0,\ y - t y_0)) = \{ f \in k[x,y] \mid f(0,0) = 0,\ Df(0,0)\binom{x_0}{y_0} = 0 \} \in (\mathbb{A}^2)^{[2]}$

⑤  $= V(x^3,\ y + \alpha x + \beta x^2) \in (\mathbb{A}^2)^{[3]}$

$= \lim_{t \to 0}$ 

⑥  $= V(x^2, xy, y^2) = \lim$  $\in (\mathbb{A}^2)^{[3]}$

Ex: $X^{[1]} \cong X$.

Ex: $X^{[2]} \cong \mathrm{Bl}_\Delta(X \times X)/S_2$.

Ex. $X^{[n]} \cong X^{(n)}$ if X is a curve.

Thm: (Fogarty) Let X be a smooth connected quasiprojective surface. Then $X^{[n]}$ is connected and smooth of dimension $2n$.

Rem: The Hilbert scheme of K3 surfaces provide one of the few examples for Hyperkähler manifolds.

:) smooth in important cases

:) fine moduli space

↖ universal family: $\Xi = \{(Z, x) \mid x \in Z\} \subseteq X^{[n]} \times X$

$$\begin{tikzcd} F \subseteq S \times X \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Xi \subseteq X^{[n]} \times X \arrow[d] \\ S \arrow[r, dashed, "\exists!"'] & X^{[n]} \end{tikzcd}$$

3. The relation to the symmetric power

There is the Hilbert–Chow morphism

$$\varphi : X^{[n]} \longrightarrow X^{(n)}$$
$$Z \longmapsto \operatorname{supp}(Z)$$

<u>Prop.</u>: ① If X is a curve, φ is an isomorphism.

② In any case φ restricts to an isomorphism on $\{\{x_1, \dots, x_n\} \in X^{(n)} \mid \text{p.w. distinct}\}$.

③ If X is a surface, φ is a crepant resolution of $X^{(n)}$.

$\varphi^* \omega_{X^{(n)}} \cong \omega_{X^{[n]}}$

④ ~~$X^{[2]} \cong$~~

<u>Rem.</u>: $X^{[n]}$ is <u>derived equivalent</u> to $X^n // S_n$. (in some cases)

4. An explicit model of $(\mathbb{A}^2)^{[n]}$

$$(\mathbb{A}^2)^{[n]} \cong \{(M, N, u) \mid M, N \in k^{n \times n},\ u \in k^n,\ MN = NM,\ \& \ \{f(M,N)u \mid f \in k[x,y]\} = k^n\} / GL_n$$

free action

$Z \longmapsto$ representation of

$$\mathcal{O}_Z \xrightarrow{\cdot x} \mathcal{O}_Z,\quad \mathcal{O}_Z \xrightarrow{\cdot y} \mathcal{O}_Z,\quad [1] \in \mathcal{O}_Z$$

with respect to any basis of $\mathcal{O}_Z = k[x,y]/I(Z)$

$$V(\{f \mid f(M,N)u = 0\}) \longleftarrow [(M, N, u)]$$

<u>Rem.</u>: ① This is one way to show that $(\mathbb{A}^2)^{[n]}$ is smooth.

② $\varphi : (\mathbb{A}^2)^{[n]} \longrightarrow (\mathbb{A}^2)^{(n)}$

$[(M,N,u)] \longmapsto \{(\lambda_1, \mu_1), \dots, (\lambda_n, \mu_n)\}$, if M, N are simultaneously similar to $\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, $\begin{pmatrix} \mu_1 & & * \\ & \ddots & \\ & & \mu_n \end{pmatrix}$

③ Generalizes to $(\mathbb{A}^m)^{[n]}$, $m \geq 0$.

e.g. for $Z = V(x^2, y - \alpha x)$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

5. Local properties

Prop.: $T_Z X^{[n]} \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}(Z), \mathcal{O}_Z) \cong \Gamma(Z, N_{Z|X})$.

(cf. Okounkov, ex. sheet 2, ex. 5)

Cor. Let X be a smooth projective surface.
Then $X^{[n]}$ is smooth of dimension $2n$. (cf. O, ex. 4)

Proof. We need to show that $\dim T_Z X^{[n]} = 2n$ for all $Z \in X^{[n]}$.

- $T_Z X^{[n]} \cong \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$, by considering

$$0 \to \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \xrightarrow{0} \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \to \dots$$

$$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \cong H^1(X, \mathcal{O}_Z) \cong H^1(Z, \mathcal{O}_Z) = 0 \quad (\dim Z = 0)$$

- $\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \cong \Gamma(Z, \mathcal{O}_Z)$ is of dimension n.
- $\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \cong \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes \omega_X)^\vee \cong \Gamma(Z, \underbrace{\omega_X|_Z}_{\cong \mathcal{O}_X|_Z})$ is of dim. n.
- $\chi(\mathcal{O}_Z, \mathcal{O}_Z) \overset{HRR}{=} \int_X \mathrm{ch}(\mathcal{O}_Z)\, \mathrm{ch}(\mathcal{O}_Z)^\vee\, \mathrm{td}(X) = 0$

$$\chi(\mathcal{O}_Z, \mathcal{O}_Z) = n - \dim \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) + n$$

Ex.:

$$\dim T_{V((x,y,z)^2)} (\mathbb{A}^3)^{[4]} \cong \mathrm{Hom}_{k[x,y,z]}(m^2/m^4, k[x,y,z]/m^2)$$

$$\cong \mathrm{Hom}_k(m^2, k[x,y,z]/m^2)$$

$$\cong \mathrm{Hom}_k(m^2/m^3, m/m^2) \text{ is of dimension } 6 \cdot 3 = 18 > 3 \cdot 4 = 12.$$

6. Invariants

Lemma: Let T be a complex torus acting on a complex manifold M with finitely many fixed points. Then $\chi(M) = |M^T|$. (see chromotopy.org) ← /torus-actions-maximal-tori-2

Proof: Let $t \in T$ be a topological generator: $T = \overline{\langle t \rangle}$.
Then: $M^T = M^t$ and $t_*: M \to M$ is homotopic to id_M (since T is connected).

Let $U \subseteq M$ be a disjoint union of coordinate patches, one for each fixed point of M.
Let $V := M \setminus M^T$.
Assuming that $t_*: M \to M$ maps U, V and $U \cap V$ into itself, we obtain

$$\underbrace{L(id_M)}_{=\chi(M)} = L(t_*) \overset{MV}{=} \underbrace{L(t_*|_U)}_{=|M^T|} + \underbrace{L(t_*|_V)}_{=0} - \underbrace{L(t_*|_{U\cap V})}_{=0}$$

U disj. union of $|M^T|$ contr. spaces

$t_*|_V$ has no fixed points; Lefschetz fixed pt. thm.

Prop. $\chi((\mathbb{A}^2)^{[n]}) = p(n) :=$ number of partitions of n. (cf. 0, ex. 4)

Proof: The torus $T := \mathbb{C}^\times \times \mathbb{C}^\times$ acts on $\mathbb{A}^2$.
This induces an action of T on $(\mathbb{A}^2)^{[n]}$: $\overset{(t_1,t_2)}{\underset{\shortparallel}{}} t \cdot V(I) := V(\{ f \mid f(t_1^{\pm1} x, t_2^{\pm1} y) \in I \})$
The fixed points are precisely the subschemes cut out by <u>monomial ideals</u>, i.e.: (cf. 0, ex. 1)

$$\{\text{partitions of } n\} \cong \{\text{fixed points}\}$$
$$\lambda \longmapsto V(I_\lambda), \text{ where } I_\lambda = (x^i y^j \mid (i,j) \notin \lambda)$$

Rem. The open subsets $U_\lambda := \{ Z \in (\mathbb{A}^2)^{[n]} \mid (x^i y^j \mid (i,j) \in \lambda) \text{ is a basis for } \mathcal{O}_Z \}$ cover $(\mathbb{A}^2)^{[n]}$. Each contains exactly one fixed point, namely $V(I_\lambda)$.

Rem: $V(I_\lambda) = \lim_{t \to 0} \text{"}t\lambda\text{"}$. (fun! Usually, Young Diagrams are not taken literally.)

Ex. ~~$n=2$.~~ $U_{(2)}$

Thm. (Göttsche)
$$\sum_{n=0}^{\infty} \sum_{i=0}^{4n} b_i(X^{[n]})\, p^{i-2n} q^n = \prod_{m=1}^{\infty} \prod_{j=0}^{4} (1 - (-1)^j p^{j-2} q^m)^{-(-1)^j b_j(X)} \quad (X \text{ surface})$$

Cor.: $\sum_n \chi(X^{[n]}) q^n = \prod_{m=1}^{\infty} (1-q^m)^{-\chi(X)}$

Thm. (BKRH) $D^b(X^{[n]}) \simeq D^b_{S_n} D^b(X^n)$.
Bridgeland–King–Reid–Haiman

Cor.: $\sum_n \chi((\mathbb{A}^2)^{[n]}) q^n = \prod_m \frac{1}{1-q^m} = \sum p(n) q^n.$

7. Creation and annihilation operators

We have the creation operator

$$\alpha_{-m} : H^*(X^{[n]}) \longrightarrow H^*(X^{[n+m]})$$

$$y \longmapsto PD(\pi_{2*}(\pi_1^*(y) \cap Z_{n,m})),$$

$$\{(Z,P,W) \in X^{[n]} \times X \times X^{[n+m]} \mid Z \subseteq W,\ g(W)-g(Z) = mP\} = Z_{n,m}$$

$$\begin{tikzcd} & Z_{n,m} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X^{[n]} & & X^{[n+m]} \end{tikzcd}$$

similarly annihilation operators α_m.

Thm: (Nakajima) $[\alpha_i, \alpha_j] = i\,\delta_{i+j,0}\ \mathrm{id}$ on $\bigoplus_{n \geq 0} H^*(X^{[n]}, \mathbb{Q})$.

Thm: $\bigoplus_{n \geq 0} H^*(\cancel{\mathrm{Hilb}}\ X^{[n]})$ (with some shifts) is a representation of the Heisenberg algebra associated to X.

See Manfred Lehn's Montreal lecture notes for more details and Haiman's notes on Macdonald polynomials and the geometry of Hilbert schemes for an accessible introduction to applications in combinatorics.