# A QUICKSTART GUIDE TO DERIVED FUNCTORS

INGO BLECHSCHMIDT

## 1. References

## 2. Abelian categories

- Definition
- Examples
- Counter examples
- Diagram chases
- Uses: category of coefficient objects, source and target of functors to derive
- Exact functors: classical and finite limit definition, in footnote: what, if one doesn't start with *short* exact sequences?

## 3. Injectives and projectives

- Definition
- Examples
- Injectives are scary
- Uses: resolutions, lex[bounded acycl. inj.] acycl.

## 4. Resolutions

- Definition
- Existence
- Morphisms
- Uses: good replacements
- Motto: $X = I^0 - I^1 + \dots$
- Construction of free resolutions
- Koszul resolutions

## 5. Derived functors

**Definition 5.1.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then $\mathbb{R}F(X)$, the *total right-derived functor of $F$* evaluated at an object $X$ of $\mathcal{A}$, is the complex

$$\mathbb{R}F(X): \quad \cdots \to 0 \to F(I^0) \to F(I^1) \to \cdots,$$

where $0 \to X \to I^\bullet$ is an injective resolution of $X$. The *$n$-th right-derived functor of $F$* evaluated at $X$ is the $n$-th cohomology of this complex: $R^nF(X) := H^n(\mathbb{R}F(X)) = H^n(F(I^\bullet))$.

So, to calculate $R^nF(X)$, we take an injective resolution of $X$, apply $F$ to this complex (be sure to strip the leading "$F(X)$"), and then take cohomology. Note that, even though the complex $I^\bullet$ is not acyclic only at position zero, the complex $F(I^\bullet)$ may have cohomology in arbitrary high degrees (!). There is an analogous definition for the left-derived functors of right-exact functors, using projective instead of injective resolutions.

Since we made the arbitrary choice of an injective resolution, we have to discuss well-definedness of course. If $0 \to X \to J^\bullet$ is a second injective resolution, there is a homotopy equivalence $I^\bullet \simeq J^\bullet$ by Lemma **??**. This equivalence is preserved by the functor $F$, so the complexes $F(I^\bullet)$ and $F(J^\bullet)$ are homotopy equivalent as well. In particular, they are quasi-isomorphic and have isomorphic cohomology.

Summing up, the complex $\mathbb{R}F(X)$ is well-defined up to quasi-isomorphism and $R^nF(X)$ is well-defined up to isomorphism. For reasons to be stated below, we don't want to stress that with our definition, $\mathbb{R}F(X)$ is actually even well-defined up to homotopy equivalence.

Suppose we fix injective resolutions for every object of $\mathcal{A}$. Then the rule $X \mapsto R^nF(X)$ can be made into an additive functor $R^nF : \mathcal{A} \to \mathcal{B}$, since morphisms $X \to Y$ between objects induce morphisms between their associated injective resolutions (which are unique up to homotopy) and therefore well-defined morphisms $R^nF(X) \to R^nF(Y)$.

Different choices of resolutions will induce naturally isomorphic right-derived functors.

**Example 5.2.** We have $R^0F(X) \cong F(X)$, naturally in $X$. To verify this, pick an injective resolution $0 \to X \to I^\bullet$ and apply $F$ to the exact sequence $0 \to X \to I^0 \to I^1$.

It is in this sense that the $R^nF$ are "derived" functors of $F$; $R^0F$ coincides with $F$ and the higher derived functors bear some relation with $F$. But note that $R^{n+1}F$ is *not* the first derived functor of $R^nF$ (indeed, in general the higher derived functors are neither left- nor right-exact, so we cannot derive $R^nF$) and that there is no Leibniz rule to be found.

**Example 5.3.** Assume $F$ to even be exact. Then the higher derived functors of $F$ vanish, i. e. $R^nF(X) = 0$ for all objects $X$ and $n > 0$. To verify this, use Lemma **??**.

This suggests the following motto: *Derived functors measure the failure of a functor to be exact.*

**Example 5.4.** Consider the dualization functor $F : \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$, $M \mapsto \mathrm{Hom}(M, \mathbb{Z})$. Since $F$ is an instance of a Hom functor, it is left-exact. To ensure that the contravariance doesn't cause unnecessary difficulties, we spell out precisely what this means: If $0 \to M \to N \to P \to 0$ is a short exact sequence in $\mathrm{Ab}^{\mathrm{op}}$ (this really means that we have a short exact sequence $0 \leftarrow M \leftarrow N \leftarrow P \leftarrow 0$ in Ab), then $0 \to F(M) \to F(N) \to F(P)$ is exact in Ab. To compute $\mathbb{R}F(\mathbb{Z}/(2))$, we use the projective (even free) resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0,$$

which corresponds to an injective resolution in $\mathrm{Ab}^{\mathrm{op}}$. The total derived functor is therefore the complex

$$0 \longrightarrow F(\mathbb{Z}) \xrightarrow{2} F(\mathbb{Z}) \longrightarrow 0,$$

which can be simplified using $F(\mathbb{Z}) \cong \mathbb{Z}$ to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0.$$

Thus we obtain $R^0F(\mathbb{Z}) \cong F(\mathbb{Z}) = 0$ and $R^1F(\mathbb{Z}) \cong \mathbb{Z}/(2)$. Note that the torsion did not vanish, but instead moved to $R^1$.

Many important functors are derived functors. For instance,

- the right derived functors of $\mathrm{Hom}(X, \_) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ are the *Ext functors* $\mathrm{Ext}^n(X, \_)$,
- the left derived functors of $M \otimes_A \_ : \mathrm{Mod}(A) \to \mathrm{Ab}$ are the *Tor functors* $\mathrm{Tor}_n(M, \_)$,
- the right derived functors of the global sections functor $\Gamma : \mathrm{AbSh}(X) \to \mathrm{Ab}$ calculate *sheaf cohomology*, and
- the right derived functors of the functor $\mathrm{Mod}(G) \to \mathrm{Ab}$ which associates to a $G$-module its subset of $G$-invariant elements calculate *group cohomology*.

The raison d'être for derived functors is given by the following lemma.

**Lemma 5.5.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then, for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, there is an induced long exact sequence*

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1F(Y) \to R^1F(Z) \to R^2F(X) \to \cdots,$$

*depending functorially on the short exact sequence.*

*Proof.* Use the *horseshoe lemma* to obtain a short exact sequence relating injective resolutions of $X$, $Y$, and $Z$. Then perform the usual diagram chase to construct the connecting morphisms. □

Note that the lemma shows very visibly that $F$ is exact if and only if $R^1F = 0$.

In practice, calculating the higher derived functors can be quite hard. There are two main techniques: One can use exact sequences (and spectral sequences) to draw some conclusions about the derived functors, or hope that there are more amenable resolutions than resolutions by unwieldy injectives available. To this end, the following lemma is crucial:

**Lemma 5.6.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Let $X$ be an object of $\mathcal{A}$. Let $0 \to X \to U^\bullet$ be a resolution of $X$ by $F$-*acyclic objects, *i. e. objects such that $R^{\geq 1}F(U^m) = 0$. Then $\mathbb{R}F(X)$ is quasi-isomorphic to $F(U^\bullet)$ and in particular $R^nF(X) \cong H^n(F(U^\bullet))$.*[1]

In other words, we may use resolutions by $F$-acyclic objects instead of injective objects to calculate the derived functors. For instance, we can use flat resolutions to calculate Tor and flabby resolutions to calculate sheaf cohomology.

[1] Note that $\mathbb{R}F(X)$ will in general *not* be homotopy equivalent to $F(U^\bullet)$. This is the reason why we identify $\mathbb{R}F(X)$ up to quasi-isomorphism and not up to homotopy equivalence.

Derived functors can be used to fix several defects in the original functors. The paramount defect of a non-exact functor is, of course, its failure to preserve exactness. This is fixed by the long exact sequence. But derived functors also fix the failure of certain identities. For instance, if $M$ is an arbitrary $A$-module, in general it does not hold that

$$M^{\vee\vee} := \mathrm{Hom}(\mathrm{Hom}(M, A), A) \cong M.$$

But, denoting by $(\_)^{\mathbb{V}}$ the total derived functor of the dualization functor, the complex $M^{\mathbb{V}\mathbb{V}}$ *is* quasi-isomorphic to $M$ (regarded as a complex concentrated in degree zero).[2] Similarly, in algebraic geometry, there is the *projection formula*

$$\pi_* \mathcal{E} \otimes \mathcal{F} \cong \pi_*(\mathcal{E} \otimes \pi^* \mathcal{G}),$$

valid in the case that $\mathcal{G}$ is a locally free sheaf. For the derived functors, the formula holds in complete generality, assuming only some finiteness conditions:

$$\mathbb{R}\pi_*(\mathcal{E}^\bullet) \otimes^{\mathbb{L}} \mathcal{F}^\bullet \simeq_{\mathrm{qis}} \mathbb{R}\pi_*(\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathbb{L}\pi^* \mathcal{G}^\bullet).$$

- somewhere: "complexes good, cohomology bad", example from homotopy theory

## 6. Example: Ext

- Lifting problems
- Extensions (give correspondence in footnote)

## 7. Example: Sheaf cohomology

- Resolution of $\underline{\mathbb{R}}$
- Comparison with singular cohomology
- Paucity of projective sheaves
- Example: section-wise surjectivity
- Uninteresting: $\mathcal{C}^\infty$-modules

## 8. Outlook

- Derived categories: definition, composition of derived functors, $D^b(X)$ $f^!$
- Model categories: $\mathrm{sAb} \simeq \mathrm{Kom}^{\leq 0}(\mathrm{Ab})$

[2] We are skating over some details here. In particular, we have not explained how to apply $\mathbb{R}F$ to a complex $K^\bullet$ (instead of a single object); this is done by finding a quasi-isomorphism $K^\bullet \to I^\bullet$ to a complex consisting of injectives and setting $\mathbb{R}F(K^\bullet) := F(I^\bullet)$. Also one has to impose some finiteness restrictions, for instance $A$ should be Noetherian and $M$ finitely generated.