A QUICKSTART GUIDE TO DERIVED FUNCTORS

INGO BLECHSCHMIDT

ABSTRACT. These notes give an informal exposition of the basic theory on derived functors. They are aimed at students who have seen Ext or Tor appearing once or twice and want to know more about those derived functors. We conclude with a short outlook on the modern formalism of derived categories. No prior knowledge of homological algebra is assumed. However, one should be familiar with exact sequences and chain complexes, as for instance provided by a first course on algebraic topology.

IN A NUTSHELL

Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories (for instance, categories of modules). Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathcal{A}$. Then the induced sequence $0 \to FX \to FY \to FZ$ is only exact at the first two terms, the morphism $FY \to FZ$ may fail to be an epimorphism. It is therefore a natural question how to extend this sequence on the right to obtain an exact sequence. We could, of course, simply tack the cokernel of $FY \to FZ$ at the end; but there is a better way, given by the *right-derived functors* of F: There is a long exact sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1F(Y) \to R^1F(Z) \to R^2F(X) \to \cdots$$

depending functorially on the given short exact sequence. Note that this way, only the map $F(Z) \to R^1F(X)$, but not the object $R^1F(X)$, depends on Y and Z. To construct $R^nF(X)$, we pick an *injective resolution* $0 \to X \to I^\bullet$ of X and set $R^nF(X) := H^n(F(I^\bullet))$.

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References

Standard textbooks on homological algebra include:

- Weibel. *An Introduction to Homological Algebra.*
- Gelfand, Manin. *Methods of Homological Algebra.*

Beware of mathematical typos in the latter. For a comprehensive reference, turn to the *Stacks Project*, and don't forget to check *MathOverflow* in case of questions.

1. Abelian categories

Example 1.1. The prototypical example of an abelian category is the category of abelian groups.

Example 1.2. More generally, the category of modules over some ring is an abelian category.

Definition 1.3. An *abelian category* $\mathcal{A}$ is a category together with abelian group structures on the hom-sets $\mathrm{Hom}_\mathcal{A}(X, Y)$ such that composition is bilinear and:

(1) There exists a *zero object*, and for any pair of objects X and Y, there exists a *biproduct* $X \oplus Y$ (simultaneously a coproduct and a product).
(2) For any morphism $f : X \to Y$, there exists a *kernel* $\ker(f) \hookrightarrow X$ and a *cokernel* $Y \twoheadrightarrow \mathrm{cok}(f)$.
(3) An appropriate formulation of the homomorphism theorem holds.

Of course, since one cannot naively talk about elements of arbitrary categories, the axioms have to be formulated in a purely categorical way with universal properties. This is entirely possible, but for the purposes of these notes not necessary, since in any concrete abelian category which will appear here, there will be an obvious notion of kernel and cokernel.

Non-example 1.4. The category of sets, the category of topological spaces and the category of all (not necessarily commutative) groups are not abelian categories.

Non-example 1.5. The category of free abelian groups is not abelian. For instance, the linear map $\mathbb{Z} \to \mathbb{Z}$, $x \mapsto 2x$ doesn't have a cokernel in this category.

Non-example 1.6. The category of Hilbert spaces and the category of Banach spaces are not abelian. One can check than in these categories, the categorical image of a continuous linear map is calculated as the *closure* of the set-theoretical image. Therefore the homomorphism theorem fails: For instance, let $\iota : U \hookrightarrow X$ be the inclusion of a linear subspace which is not closed. Then the induced map $U/\ker(\iota) \to \mathrm{im}(\iota)$, i. e. $U \to \overline{U}$, is not an isomorphism.

Example 1.7. Let $\mathcal{A}$ be an abelian category. Then the category of (cochain) complexes in $\mathcal{A}$, $\mathrm{Kom}(\mathcal{A})$, is an abelian category with componentwise addition of morphisms and componentwise kernels and cokernels.

Example 1.8. Let X be a topological space. Then the category of sheaves of abelian groups on X, $\mathrm{AbSh}(X)$, is an abelian category. The kernel of a morphism of such sheaves is calculated as its naive presheaf kernel, its cokernel is the sheafification of the naive presheaf cokernel.

Abelian categories are a natural setting for talking about exact sequences. The usual lemmas on exact sequences, for instance the *five lemma* and the lemma on the existence of a connecting homomorphism, hold in any abelian category. In fact, there is the following metatheorem, which is very practical for working with abelian categories.

Theorem 1.9. *For the purposes of performing diagram chases in an abelian category $\mathcal{A}$, we may pretend that $\mathcal{A}$ is the category of modules over some ring. In particular, we may freely employ naive element-based proofs to verify statements about $\mathcal{A}$ – even though there need not be a notion of elements for objects of $\mathcal{A}$.*

This can be made precise in several ways; check the *Freyd–Mitchell embedding theorem* and (the first page of) Bergman's article *A note on abelian categories – translating element-chasing proofs, and exact embedding in abelian groups*. As a consequence, nothing is lost if the reader chooses to restrict all following abelian categories to module categories.

The main use of abelian categories in homological algebra is as *categories of coefficient objects* (for taking cohomology in, for example) and as source and target categories of functors to be derived. To this end, the following definition is crucial.

Definition 1.10. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is *additive* if and only if the induced maps $\mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ are homomorphisms of groups.

An additive functor F is *exact* if and only if, for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, the induced sequence $0 \to FX \to FY \to FZ \to 0$ is exact in $\mathcal{B}$.

It is *left-exact* if at least $0 \to FX \to FY \to FZ$ is always still exact and *right-exact* if at least $FX \to FY \to FZ \to 0$ is exact.

Example 1.11. Let T be an A-module. The functor $\mathrm{Mod}(A) \to \mathrm{Ab}$, $M \mapsto M \otimes_A T$ is right-exact. It is exact if and only if T is flat over A.

Example 1.12. Let T be an object of an arbitrary abelian category $\mathcal{A}$. The functor $\mathcal{A} \to \mathrm{Ab}$, $X \mapsto \mathrm{Hom}_{\mathcal{A}}(T, X)$ is left-exact. (It is exact if and only if T is a *projective object*, see below.) The contravariant Hom functor $X \mapsto \mathrm{Hom}_{\mathcal{A}}(X, T)$, regarded as a covariant functor $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$, is left-exact as well. (It is exact if and only if T is *injective*.)

A useful property of exact functors is that they *commute with cohomology*. More precisely, if $K^\bullet$ is a chain complex in an abelian category $\mathcal{A}$, and $F : \mathcal{A} \to \mathcal{B}$ is an exact functor, then there is a natural isomorphism $H^n(F(K^\bullet)) \cong F(H^n(K^\bullet))$. This is because exact functors preserve kernels, cokernels, and images.

Lemma 1.13. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. Let $0 \to X \to Y \to Z$ be an exact sequence in $\mathcal{A}$ (note the missing zero at the right end). Then the induced sequence $0 \to FX \to FY \to FZ$ is exact. The dual statement for right-exact functors holds as well.*

Lemma 1.14. *An additive functor between abelian categories is left-exact if and only if it preserves finite limits. It is right-exact if and only if it preserves finite colimits.*[1]

2. Injectives and projectives

For the theory of derived functors, injective and projective objects are very important. We'll discuss why below.

Definition 2.1. An object I of an abelian category is *injective* if and only if, for any monomorphism $X \hookrightarrow Y$ and any morphism $X \to I$, there exists a lift of that morphism to Y.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook] \arrow[d] & Y \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

Dually, an object P is *projective* if and only if, for any epimorphism $X \twoheadrightarrow Y$ and any morphism $P \to X$, there exists a co-lift of that morphism to Y.

$$\begin{tikzcd} X \arrow[r, two heads] & Y \arrow[r] & 0 \\ & P \arrow[u] \arrow[ul, dashed] & \end{tikzcd}$$

Note that no uniqueness of the lifts is required.

Example 2.2. In the category of vector spaces over some field, assuming the axiom of choice, any object I is injective: Simply take a basis of X, extend it to a basis of Y, and define $Y \to I$ on this basis. This gives further indication that categories of vector spaces are not very interesting from a homological point of view.

Example 2.3. *Baer's criterion* states that in the category of abelian groups, assuming the axiom of choice, a group G is injective if and only if it is *divisible*, i. e. if for any $x \in G$ and $n \geq 1$ there exists an element $y \in G$ such that $x = ny$.[2] For example, the groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Injective objects are generally regarded as huge, unwieldy objects. They are important for the theory, but no practical calculations are made with them. Projective objects, on the other hand, are often much more accessible. This is of course a purely social statement, since the injective objects in an abelian category $\mathcal{A}$ are precisely the projective objects in $\mathcal{A}^{\mathrm{op}}$.

Example 2.4. In the category of modules over some ring, any finite free module P is projective: Choose a basis of P and pick preimages under $X \twoheadrightarrow Y$ of the images of the basis vectors in Y. Use these preimages to define the co-lift $P \to X$. More generally, assuming the axiom of choice, any (not necessarily finite) free module is

[1] With this lemma, left-exactness of $\mathrm{Hom}_{\mathcal{A}}(T, \_)$ and of $\mathrm{Hom}_{\mathcal{A}}(\_, T)$ follows from general abstract nonsense, since the set-valued Hom functors of any category preserve limits and since limits in Ab are calculated just as in Set.

[2] The "only if" direction is easy: Let $x \in G$ and $n \geq 1$. Then consider the injective map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. The sought element y can be obtained as the image of 1 under a lift of the map $\mathbb{Z} \to G$, $1 \mapsto x$.

projective. The precise characterization is that a module is projective if and only if it is a direct summand of a free module.[3]

Example 2.5. Assuming the axiom of choice, any vector space is free and thus projective.

Example 2.6. An easy example of a projective module which is not free is the $\mathbb{Z}/(6)$-module $\mathbb{Z}/(2)$. It is projective because there is the direct sum decomposition $\mathbb{Z}/(6) \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$. It is not free because its number of elements is not a multiple of 6.

Example 2.7. Let $\mathfrak{m} = (3, 1 + \sqrt{-5})$ be the famous example of an ideal of $R = \mathbb{Z}[\sqrt{-5}]$ which is locally, but not globally principal. Considered as an R-module, it is projective but not free.

Example 2.8. By the *Serre–Swan theorem*, vector bundles provide a systematic source of projective modules. Let M be a module over a ring R. Then the induced quasicoherent module $M^\sim$ on $\operatorname{Spec} R$ is a vector bundle (i. e. a finite free sheaf of modules) if and only if M is projective and finitely generated.[4] It is a trivial bundle if and only if M is finite free. There is an analogous relation in the smooth setting: The category of smooth vector bundles on a compact manifold M is, by taking global sections, equivalent to the category of finitely generated projective modules over $\mathcal{C}^\infty(M)$.

The reason why injective and projective objects are important in homological algebra is that they *have great exactness properties*. The following lemma makes one of this properties precise. Injective and projective objects thus form the building blocks by which other objects are *resolved by* – see the next section.

Lemma 2.9. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $X^\bullet$ be a bounded below complex of injective objects or a bounded above complex of projective objects. If $X^\bullet$ is acyclic, then $F(X^\bullet)$ is acyclic as well (!).*

Note that the statement is totally false without the injectivity or projectivity assumption.

Proof. One can show that that such a complex is homotopy equivalent to the zero complex. Any additive functor preserves homotopy equivalences.[5] Therefore the image complex is too homotopy equivalent to zero and in particular acyclic.

[3]The relationship to other properties of R-modules is as follows:

$$\begin{tikzcd}
\text{free} \arrow[r, Rightarrow, bend left] & \text{projective} \arrow[l, Rightarrow, dashed, bend left, "R \text{ local or } R \text{ PID}"] \arrow[r, Rightarrow, bend left] & \text{flat} \arrow[l, Rightarrow, dashed, bend left, "R \text{ perfect}"] \arrow[r, Rightarrow, bend left] & \text{torsionfree} \arrow[l, Rightarrow, dashed, bend left, "R \text{ Dedekind domain}"]
\end{tikzcd}$$

[4]This corresponds to the fact that a module is projective and finitely generated if and only if there is a partition $1 = \sum_i f_i \in R$ such that the localized modules $M[f_i^{-1}]$ are finite free $R[f_i^{-1}]$-modules. See for instance Theorem 7.22 in Pete Clark's notes on commutative algebra.

[5]Additive functors preserve homotopies between morphisms of complexes, since they carry the defining relation "$f - g = dh + hd$" into an equation of similar kind.

The already interesting special case of three-term complexes, i. e. short exact sequences, can be proved by a simpler argument: If $0 \to A \to B \to C \to 0$ is a short exact sequence with A injective or C projective, one can construct a retraction of $A \to B$ respectively a section of $B \to C$. Therefore the sequence splits. The claim follows since additive functors preserve biproducts.[6] □

3. Resolutions

Definition 3.1. An *injective resolution* of an object X in an abelian category is an exact sequence of the form $0 \to X \to I^0 \to I^1 \to \cdots$, where the objects I^n are all injective. A short notation is $0 \to X \to I^\bullet$. Dually, a *projective resolution* is an exact sequence $P^\bullet \to X \to 0$ with the P^{-n}, $n \geq 0$, projective.

Example 3.2. Let M be an A-module. Picking some set of generators, we obtain a surjective map $A^{(I)} \twoheadrightarrow M$. If the chosen family is linearly independent, this map is an isomorphism and $0 \to A^{(I)} \twoheadrightarrow M \to 0$ is a projective resolution of M. (Recall that free modules are projective, assuming the axiom of choice.) But in general, the *module of relations* K_0, the kernel of $A^{(I)} \twoheadrightarrow M$, will be nontrivial. Picking generators for K_0, we obtain a surjective map $A^{(J)} \twoheadrightarrow K_0$ which we can compose with the inclusion $K_0 \hookrightarrow A^{(I)}$. In this way, the top row in the diagram

$$\begin{tikzcd}
A^{(J)} \arrow[rr] \arrow[dr, two heads] & & A^{(I)} \arrow[r, two heads] & M \arrow[r] & 0 \\
& K_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

is exact. If the chosen generators for K_0 are linearly independent, we can tack the zero module at the front and obtain in this manner a projective resolution of M. But in general, there will be nontrivial relations between the chosen generators – *second syzygies* – and even higher syzygies. Therefore we have to repeat this process ad infinitum.

$$\begin{tikzcd}
\cdots \arrow[rr] & & A^{(K)} \arrow[rr] \arrow[dr, two heads] & & A^{(J)} \arrow[rr] \arrow[dr, two heads] & & A^{(I)} \arrow[r, two heads] & M \arrow[r] & 0 \\
& \cdots \arrow[ur, hook] & & K_1 \arrow[ur, hook] & & K_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

Example 3.3. A projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}/(2)$, obtained by the method just sketched, is $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$.

Example 3.4. *Koszul resolutions* are an important source of resolutions in algebraic geometry. Let $f_1, \ldots, f_r$ be a *regular sequence* of elements of some ring A.[7] Denote

[6] Formally, a biproduct of two objects X and Y consists of morphisms $\iota_X : X \to X \oplus Y$, $\iota_Y : Y \to X \oplus Y$, $\pi_X : X \oplus Y \to X$, $\pi_Y : X \oplus Y \to Y$ such that $(X \oplus Y, \iota_X, \iota_Y)$ is a categorical coproduct of X and Y and such that $(X \oplus Y, \pi_X, \pi_Y)$ is a product of X and Y. This can also be characterized using the group structure on the hom-sets: The morphisms should satisfy the five relations $\pi_X \circ \iota_X = \mathrm{id}_X$, $\pi_Y \circ \iota_Y = \mathrm{id}_Y$, $\pi_Y \circ \iota_X = 0$, $\pi_X \circ \iota_Y = 0$, $\iota_X \circ \pi_X + \iota_Y \circ \pi_Y = \mathrm{id}_{X \oplus Y}$. Additive functors carry these equations into equations of a similar kind.

[7] This means that f_1 is regular in A, f_2 is regular in $A/(f_1)$, f_3 is regular in $A/(f_1, f_2)$ and so on. A ring element s is *regular* if and only if multiplication by s is injective. If A is a ring of functions, for

by $\Lambda^n M$ the n-th exterior power of M. Then $\Lambda^\bullet A^r \to A/(f_1, \ldots, f_r) \to 0$ is a free resolution. The differential sends $\vec{v}_1 \wedge \cdots \wedge \vec{v}_n$ to $\sum_{i=0}^r (-1)^i (\vec{v} \cdot \vec{f})\, \vec{v}_1 \wedge \cdots \wedge \widehat{\vec{v}_i} \wedge \cdots \wedge \vec{v}_n$ (scalar product). Switching toposes, Koszul resolutions can be used to construct locally free resolutions of structure sheaves of closed subschemes.

We can think of a projective resolution $P^\bullet \to X \to 0$ as providing us with successively improving approximations: A zeroth approximation of X is P^0. But this disregards possible relations between the generators (we are employing a module-theoretic metaphor here), so a better approximation is "$P^0 - P^{-1}$". If there are relations between the relations, we subtract too much in this expression; an even better approximation is "$P^0 - P^{-1} + P^{-2}$". Taking all the higher corrections into account, we obtain the symbolic identity "$X = P^0 - P^{-1} \pm \cdots$".[8]

Definition 3.5. An abelian category is said to have *enough injectives* if and only if for any object X there exists a monomorphism $X \hookrightarrow I$ into an injective object, i. e. if any object can be embedded into an injective object. Dually, an abelian category has *enough projectives* if and only if any object is a quotient of a projective object.

Lemma 3.6. *Let $\mathcal{A}$ be an abelian category with enough injectives or enough projectives. Then any object possesses an injective respectively a projective resolution.*

Proof. The method sketched in Example 3.2 depended only on the fact that any module admits a surjection from a free module. This fact can be substituted by the hypothesis on the existence of enough projectives. The statement about injective resolutions is the formal dual. $\square$

Remark 3.7. Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. We say that its *homological dimension* $\leq n$ if and only if any object possesses a projective (equivalently, injective) resolution of length n, i. e. comprising objects P^0 to P^{-n}.[9] The homological dimension is one way to measure the homological complexity of a category. For instance, the category of vector spaces over a field has homological dimension zero, since any object is projective (and injective).

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instance $A = k[x, y]$, regularity of the sequence is intuitively a requirement on the equations "$f_1 = 0, \ldots, f_n = 0$" to be independent.

[8] The arithmetic operations with objects should be conceived in a purely figurative way. But in fact, there is a way to make these calculations completely rigorous. Namely, we can attach to any abelian category $\mathcal{A}$ a very nice invariant, its *K-theory group* $K(\mathcal{A})$ (also called *Grothendieck group*). It is the abelian group freely generated by the objects of $\mathcal{A}$ modulo the relation $X = X' + X''$ for any short exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$. One can then check that, if $P^\bullet \to X \to 0$ is a bounded resolution (projective or otherwise), the identity $X = \sum_n (-1)^n P^n$ holds in $K(\mathcal{A})$.

Here are four tangential remarks on the K-theory. (1) The map $(\mathcal{A}, \oplus) \to (K(\mathcal{A}), +)$ is the *universal additive invariant* of $\mathcal{A}$. (2) For a bounded complex $X^\bullet$, one can define its *Euler characteristic* as $\chi(X^\bullet) := \sum_n (-1)^n X^n \in K(\mathcal{A})$. Denoting by $H^\bullet(X^\bullet)$ its associated cohomology complex (with zero differentials), one can check that $\chi(X^\bullet) = \chi(H^\bullet(X^\bullet))$. (3) The K-theory of the category of vector spaces over some field is zero, by the *Eilenberg–Mazur swindle*. The K-theory of the category of finite-dimensional vector spaces is $\mathbb{Z}$, by associating to a vector space its dimension. (4) The K-theory group of the category of coherent sheaves of modules on a scheme carries important information about the intersection theory of the scheme.

[9] The equivalence is not entirely trivial. One can use the *dimension shifting trick* and the characterization of injective objects as those objects I such that $\operatorname{Ext}^1(T, I) = 0$ for all objects T; dually, an object P is projective if and only if $\operatorname{Ext}^1(P, T) = 0$ for alle objects T.

More generally, there is an intriguing relation between the homological dimension of the category of modules over a ring and its Krull dimension: If the ring is a *regular ring*, they coincide. For instance, the homological dimension of the category of $k[x_1, \ldots, x_n]$-modules, where k is a field, is n. Relatedly, the category of coherent sheaves on a smooth projective variety of dimension n has homological dimension n.

The following lemma investigates the functorial properties of injective and projective resolutions.

Lemma 3.8. *Let $0 \to X \to I^\bullet$ be a resolution by arbitrary objects in an abelian category. Let $0 \to Y \to J^\bullet$ be a resolution by injective objects. Let $f : X \to Y$ be a morphism. Then there exists a lift of f to the resolutions, i. e. a morphism $I^\bullet \to J^\bullet$ of complexes compatible with the maps $X \to I^0$ and $Y \to J^0$. Furthermore, this lift is unique up to homotopy.*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & I^0 \arrow[r] \arrow[d, dashed] & I^2 \arrow[r] \arrow[d, dashed] & \cdots \\
0 \arrow[r] & Y \arrow[r] & J^0 \arrow[r] & J^2 \arrow[r] & \cdots
\end{tikzcd}$$

Proof. We obtain a morphism $I^0 \to J^0$ by lifting the morphism $X \to Y \to J^0$ along the monomorphism $X \to I^0$. This is possible by the injectivity of J^0. The induction step is a bit more complicated. □

Corollary 3.9. *Any two injective resolutions of an object are homotopy equivalent.*

Proof. Let $0 \to X \to I^\bullet$ and $0 \to X \to J^\bullet$ be injective resolutions. By the previous lemma, the identity $\mathrm{id}_X : X \to X$ lifts to morphisms $I^\bullet \to J^\bullet$ and $J^\bullet \to I^\bullet$. The composition of these lifts is a lift of id_X to $I^\bullet \to I^\bullet$ respectively $J^\bullet \to J^\bullet$. By the uniqueness statement, these compositions are homotopic to the identity, since $\mathrm{id}_{I^\bullet}$ respectively $\mathrm{id}_{J^\bullet}$ are trivially lifts of id_X. □

Injective resolutions allow us to replace badly behaved objects by (complexes of) injective objects, which, thanks to their great exactness properties, mix much better with additive functors. This is the reason why injective (and dually, projective) resolutions are important. No information about morphisms is lost in this process, since morphisms between objects lift to morphisms between resolutions.

For future reference, we want to precisely state the relation between an object X, considered as a complex $X[0]$ concentrated in degree zero, and an associated injective resolution $I^\bullet$. Namely, the augmentation $X \to I^0$ gives rise to a morphism $X[0] \to I^\bullet$ of complexes which is a *quasi-isomorphism* by the following definition.

Definition 3.10. A morphism of complexes $K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if the induced morphisms $H^n(K^\bullet) \to H^n(L^\bullet)$ in cohomology are isomorphisms for all n.[10]

Any homotopy equivalence is a quasi-isomorphism, but the converse is totally false.

[10]Note that if $K^\bullet \to L^\bullet$ is a quasi-isomorphism, there is usually no quasi-isomorphism in the opposite direction. For instance, there is no quasi-isomorphism $(\mathbb{Z}/(2))[0] \to P^\bullet$ when $P^\bullet \to \mathbb{Z}/(2) \to 0$ is

4. Derived functors

Definition 4.1. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then $\mathbb{R}F(X)$, the *total right-derived functor of F* evaluated at an object X of $\mathcal{A}$, is the complex

$$\mathbb{R}F(X): \quad \cdots \to 0 \to F(I^0) \to F(I^1) \to \cdots,$$

where $0 \to X \to I^\bullet$ is an injective resolution of X. The *n-th right-derived functor of F* evaluated at X is the n-th cohomology of this complex: $R^nF(X) := H^n(\mathbb{R}F(X)) = H^n(F(I^\bullet))$.

So, to calculate $R^nF(X)$, we take an injective resolution of X, apply F to this complex (be sure to strip the leading "$F(X)$"), and then take cohomology. Note that, even though the complex $I^\bullet$ is not acyclic only at position zero, the complex $F(I^\bullet)$ may have cohomology in arbitrary high degrees (!). There is an analogous definition for the left-derived functors of right-exact functors, using projective instead of injective resolutions.

Since we made the arbitrary choice of an injective resolution, we have to discuss well-definedness of course. If $0 \to X \to J^\bullet$ is a second injective resolution, there is a homotopy equivalence $I^\bullet \simeq J^\bullet$ by Corollary 3.9. This equivalence is preserved by the functor F, so the complexes $F(I^\bullet)$ and $F(J^\bullet)$ are homotopy equivalent as well. In particular, they are quasi-isomorphic and have isomorphic cohomology.

Summing up, the complex $\mathbb{R}F(X)$ is well-defined up to quasi-isomorphism and $R^nF(X)$ is well-defined up to isomorphism. For reasons to be stated below, we don't want to stress that with our definition, $\mathbb{R}F(X)$ is actually even well-defined up to homotopy equivalence.

Suppose we fix an injective resolution for every object of $\mathcal{A}$. Then the rule $X \mapsto R^nF(X)$ can be made into an additive functor $R^nF : \mathcal{A} \to \mathcal{B}$, since a morphism $X \to Y$ between objects induces, by Lemma 3.8, a morphism between their associated injective resolutions (unique up to homotopy) and therefore a well-defined morphism $R^nF(X) \to R^nF(Y)$.

Different choices of resolutions lead to naturally isomorphic derived functors.

Example 4.2. We have $R^0F(X) \cong F(X)$, naturally in X. To verify this, pick an injective resolution $0 \to X \to I^\bullet$ and apply F to the exact sequence $0 \to X \to I^0 \to I^1$.

It is in this sense that the R^nF are "derived" functors of F; R^0F coincides with F and the higher derived functors bear some relation with F. But note that $R^{n+1}F$ is *not* the first derived functor of R^nF (indeed, in general the higher derived functors are neither left- nor right-exact, so we cannot derive R^nF) and that there is no Leibniz rule to be found.

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the projective resolution of Example 3.3. Accordingly, we say that two complexes $K^\bullet$ and $L^\bullet$ are *quasi-isomorphic* not when there exists a quasi-isomorphism between them, but when there exists a *zigzag* of quasi-isomorphisms $K^\bullet \to Z_0^\bullet \leftarrow Z_1^\bullet \to Z_2^\bullet \leftarrow \cdots \to Z_n^\bullet \leftarrow Y^\bullet$. This complication is a source of technical difficulties. Note that to say that $K^\bullet$ and $L^\bullet$ are quasi-isomorphic is slightly stronger than to say that they have isomorphic cohomology. In the latter case, there is no guarantee that the isomorphisms are induced by a (zigzag of) morphisms of complexes.

Example 4.3. Assume F to even be exact. Then the higher derived functors of F vanish, i. e. $R^nF(X) = 0$ for all objects X and $n > 0$. To verify this, use that exact functors commute with cohomology.

This suggests the following motto: *Derived functors measure the failure of a functor to be exact.*

Example 4.4. Consider the dualization functor $F : \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$, $M \mapsto \mathrm{Hom}(M, \mathbb{Z})$. Since F is an instance of a Hom functor, it is left-exact. To ensure that the contravariance doesn't cause unnecessary difficulties, we spell out precisely what this means: If $0 \to M \to N \to P \to 0$ is a short exact sequence in $\mathrm{Ab}^{\mathrm{op}}$ (this really means that we have a short exact sequence $0 \leftarrow M \leftarrow N \leftarrow P \leftarrow 0$ in Ab), then $0 \to F(M) \to F(N) \to F(P)$ is exact in Ab. To compute $\mathbb{R}F(\mathbb{Z}/(2))$, we use the projective (even free) resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0,$$

which corresponds to an injective resolution in $\mathrm{Ab}^{\mathrm{op}}$. The total derived functor is therefore the complex

$$0 \longrightarrow F(\mathbb{Z}) \xrightarrow{2} F(\mathbb{Z}) \longrightarrow 0,$$

which can be simplified using $F(\mathbb{Z}) \cong \mathbb{Z}$ to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0.$$

Thus we obtain $R^0F(\mathbb{Z}) \cong F(\mathbb{Z}) = 0$ and $R^1F(\mathbb{Z}) \cong \mathbb{Z}/(2)$. Note that the torsion did not vanish, but instead moved to R^1.

Many important functors are derived functors. For instance,

- the right derived functors of $\mathrm{Hom}(X, \_) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ are the *Ext functors* $\mathrm{Ext}^n(X, \_)$,
- the left derived functors of $M \otimes_A \_ : \mathrm{Mod}(A) \to \mathrm{Ab}$ are the *Tor functors* $\mathrm{Tor}_n(M, \_)$,
- the right derived functors of the global sections functor $\Gamma : \mathrm{AbSh}(X) \to \mathrm{Ab}$ calculate *sheaf cohomology*, and
- the right derived functors of the functor $\mathrm{Mod}(G) \to \mathrm{Ab}$ which associates to a G-module its subset of G-invariant elements calculate *group cohomology*.

The raison d'être for derived functors is given by the following lemma.

Lemma 4.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then, for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, there is an induced long exact sequence*

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1F(Y) \to R^1F(Z) \to R^2F(X) \to \cdots,$$

depending functorially on the short exact sequence.

Proof. Use the *horseshoe lemma* to obtain a short exact sequence relating injective resolutions of X, Y, and Z. Then perform the usual diagram chase to construct the connecting morphisms. $\square$

Note that the lemma shows very visibly that F is exact if and only if $R^1F = 0$.

In practice, calculating the higher derived functors can be quite hard. There are two main techniques: One can use exact sequences (and spectral sequences) to draw some conclusions about the derived functors, or hope that there are more amenable resolutions than resolutions by unwieldy injectives available. To this end, the following lemma is crucial:

Lemma 4.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Let X be an object of $\mathcal{A}$. Let $0 \to X \to U^\bullet$ be a resolution of X by F-*acyclic objects*, i. e. objects such that $R^{\geq 1}F(U^m) = 0$. Then $\mathbb{R}F(X)$ is quasi-isomorphic to $F(U^\bullet)$ and in particular $R^nF(X) \cong H^n(F(U^\bullet))$.*[11]

In other words, we may use resolutions by F-acyclic objects instead of injective objects to calculate the derived functors. For instance, we can use *flat resolutions* to calculate Tor and *flabby resolutions* to calculate sheaf cohomology. The general philosophy of derived functors is therefore as follows: The complex $\mathbb{R}F(X)$ is obtained by replacing X with a (complex of) objects which are in some sense well-suited to F and applying F to those.

Derived functors can be used to fix several defects in the original functors. The paramount defect of a non-exact functor is, of course, its failure to preserve exactness. This is fixed by the long exact sequence. But derived functors also fix the failure of certain identities. For instance, if M is an arbitrary A-module, in general it does not hold that

$$M^{\vee\vee} := \mathrm{Hom}(\mathrm{Hom}(M, A), A) \cong M.^{12}$$

But, denoting by $(\_)^{\mathbb{V}}$ the total derived functor of the dualization functor, the complex $M^{\mathbb{V}\mathbb{V}}$ *is* quasi-isomorphic to M (regarded as a complex concentrated in degree zero).[13] Similarly, in algebraic geometry, there is the *projection formula*

$$\pi_*\mathcal{E} \otimes \mathcal{F} \cong \pi_*(\mathcal{E} \otimes \pi^*\mathcal{G}),$$

valid in the case that $\mathcal{G}$ is a locally free sheaf. For the derived functors, the formula holds in complete generality, assuming only some finiteness conditions:

$$\mathbb{R}\pi_*(\mathcal{E}^\bullet) \otimes^{\mathbb{L}} \mathcal{F}^\bullet \simeq_{\mathrm{qis}} \mathbb{R}\pi_*(\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathbb{L}\pi^*\mathcal{G}^\bullet).$$

- somewhere: “complexes good, cohomology bad”, example from homotopy theory

5. Example: Ext

Let $\mathcal{A}$ be an abelian category with enough injectives and projectives. Then one can show that the right-derived functors of the Hom functor coincide in the sense

[11] Note that $\mathbb{R}F(X)$ will in general *not* be homotopy equivalent to $F(U^\bullet)$. This is the reason why we identify $\mathbb{R}F(X)$ up to quasi-isomorphism and not up to homotopy equivalence.

[12] Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/(2)$. Then $M^\vee = 0$.

[13] We are skating over some details here. In particular, we have not explained how to apply $\mathbb{R}F$ to a *complex* $K^\bullet$ (instead of a single object); this is done by finding a quasi-isomorphism $K^\bullet \to I^\bullet$ to a complex consisting of injectives and setting $\mathbb{R}F(K^\bullet) := F(I^\bullet)$. Also one has to impose some finiteness conditions, for instance A should be Noetherian and M finitely generated.

that
$$R^n(\mathrm{Hom}_{\mathcal{A}}(X, \_))(Y) \cong R^n(\mathrm{Hom}_{\mathcal{A}}(\_, Y))(X).$$
We can thus define $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ as either object.

The elements of $\mathrm{Ext}^n(X, Y)$ have a concrete interpretation. Namely, they correspond to *n-extensions* of Y by X, that is exact sequences of the form
$$0 \longrightarrow Y \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow X \longrightarrow 0,$$
up to a certain equivalence relation.

Proposition 5.1. *There is a natural bijective correspondence between elements of* $\mathrm{Ext}^1(X, Y)$ *and equivalence classes of short exact sequences* $0 \to Y \to E \to X \to 0$.

Proof. Let $0 \to Y \to E \to X \to 0$ be a short exact sequence. Applying the functor $\mathrm{Hom}_{\mathcal{A}}(X, \_)$, we obtain the long exact sequence
$$0 \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, E) \to \mathrm{Hom}(X, X) \xrightarrow{\partial} \mathrm{Ext}^1(X, Y) \to \cdots.$$
The element of $\mathrm{Ext}^1(X, Y)$ corresponding to the short exact sequence is then $\partial(\mathrm{id}_X)$.

The converse direction is a bit more complicated.[14] □

Under this correspondence, the trivial short exact sequence $0 \to Y \to Y \oplus X \to X \to 0$ corresponds to the zero element in $\mathrm{Ext}^1(X, Y)$. Thus a short exact sequence splits if and only if its Ext class vanishes.

Here is a fun application of Ext^1. Let $i : U \hookrightarrow X$ be a subobject. Let $f : U \to Y$ be a morphism. Is there an extension of f to X, i. e. a morphism $\bar{f} : X \to Y$ such that $\bar{f} \circ i = f$? Such a question is of course trivial in the category of vector spaces over a field, since there we can just extend a basis of U to a basis of X and construct $\bar{f}$ on this basis. But in general abelian categories, the question is non-trivial.

We can give a precise answer to the question using the Ext functor. Applying the functor $\mathrm{Hom}(\_, Y)$ to the short exact sequence $0 \to U \to X \to X/U \to 0$, we obtain the long exact sequence
$$0 \to \mathrm{Hom}(X/U, Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(U, Y) \xrightarrow{\partial} \mathrm{Ext}^1(X/U, Y) \to \cdots.$$
The map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(U, Y)$ is given by precomposing with $i : U \to X$. Therefore the morphism f lifts to X if and only if it has a preimage under this map; by exactness, this is the case if and only if $\partial(f)$ vanishes in $\mathrm{Ext}^1(X/U, Y)$.

6. Example: Tor

Let A be a ring (commutative, with unit). Tensoring with a fixed module defines a right-exact functor $\mathrm{Mod}(A) \to \mathrm{Mod}(A)$. Since one can show that
$$L^n(M \otimes_A \_)(N) \cong L^n(\_ \otimes_A N)(M),$$
we can use both to define $\mathrm{Tor}^A_n(M, N)$.

Here is a fun example from algebraic geometry. Let I and J be ideals of $A := k[x, y]$. These define closed subschemes $V(I)$ and $V(J)$ of $\mathbb{A}^2_k$, the vanishing locus

[14] Let $\alpha \in \mathrm{Ext}^1(X, Y)$. Embed Y into an injective object I. The long exact sequence induced by the short exact sequence $0 \to Y \to I \to I/Y \to 0$ shows that the map $\partial : \mathrm{Hom}(X, I/Y) \to \mathrm{Ext}^1(X, Y)$ is surjective. Therefore there exists a morphism $f : X \to I/Y$ such that $\partial(f) = \alpha$. The short exact sequence corresponding to α is then $0 \to Y \to E \to X \to 0$, where $E \to X$ is the pullback of $I \to I/Y$ along f.

of the polynomials in I respectively J. The ring of functions is A/I and A/J respectively. The scheme-theoretical intersection is given by $V(I+J)$ with ring of functions $A/(I+J) \cong A/I \otimes_A A/J$.

We can then define the *intersection multiplicity* of $V(I)$ and $V(J)$ at a point $x \in \mathbb{A}^2_k$ (corresponding to a prime ideal $\mathfrak{p}$) as the dimension of the stalk $(A/(I+J))_{\mathfrak{p}}$ as a k-vector space.

Example 6.1. Let $I = (y - x^2)$ and $J = (y)$ be ideals in $k[x,y]$ defining the parabola and the x-axis, respectively. The ring of functions of $V(I+J)$ is

$$k[x,y]/(I+J) = k[x,y]/(y-x^2,y) \cong k[x]/(x^2).$$

Since any elements not contained in the ideal $(x,y) = (x)$ of this ring are invertible, localizing this ring at (x,y) doesn't change it. Thus the intersection multiplicity of $V(I)$ and $V(J)$ at the the origin (corresponding to the prime ideal (x,y)) is 2, just as we would expect.

We can globalize the definition. Let $\mathcal{I}$ and $\mathcal{J}$ be sheaves of ideals on a scheme X over a field k of dimension 2, defining closed subschemes $V(\mathcal{I})$ and $V(\mathcal{J})$. Their scheme-theoretical intersection is $V(\mathcal{I}+\mathcal{J})$ with structure sheaf $\mathcal{O}_X/(\mathcal{I}+\mathcal{J}) \cong \mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}$.[15] If $V(\mathcal{I})$ and $V(\mathcal{J})$ are "in general position" and intersect in a finite number of points, we can define their *intersection product* as the number

$$V(\mathcal{I}) \cdot V(\mathcal{J}) := \sum_{x \in V(\mathcal{I}) \cap V(\mathcal{J})} \dim_k \left(\mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}\right)_x.$$

Since the tensor product sheaf $\mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}$ is supported precisely at the intersection points, this can also be stated more abstractly using sheaf cohomology as

$$V(\mathcal{I}) \cdot V(\mathcal{J}) = \dim_k H^0(X, \mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}) = \chi(\mathcal{O}_X/\mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}).$$

Since discrete points don't have higher cohomology, the *Euler characteristic* appearing in this formula, defined as $\chi(\mathcal{E}) := \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{E})$, is simply given by the first summand.

However, this definition can't be used to calculate non-proper intersections like $V(\mathcal{I}) \cdot V(\mathcal{I})$. In this case, the intersection number should be defined in a way that if we can move $V(\mathcal{I})$ to a "linearly equivalent divisor" $V(\mathcal{I}')$, then $V(\mathcal{I}) \cdot V(\mathcal{I}) := V(\mathcal{I}) \cdot V(\mathcal{I}')$. For instance, any two lines in $\mathbb{P}^2_k$ should have intersection product 1, even if we intersect a line with itself.

The correct definition is given by *Serre's intersection formula*

$$V(\mathcal{I}) \cdot V(\mathcal{J}) := \chi(\mathcal{O}_X/\mathcal{I} \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{J}) := \sum_{n=0}^{\infty} (-1)^n \chi(\mathrm{Tor}_n^{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}, \mathcal{O}_X/\mathcal{J})),$$

It is a refinement of the naive definition, which is the $n = 0$ term in the sum. In the case that $V(\mathcal{I})$ and $V(\mathcal{J})$ happen to be in general position, one can show that the higher Tor sheaves vanish. Therefore Serre's intersection formula recovers the naive definition in this case.

[15] More precisely, $\mathcal{O}_X/(\mathcal{I}+\mathcal{J})$ is the pushforward of the structure sheaf of $V(\mathcal{I}+\mathcal{J})$ along the closed immersion $V(\mathcal{I}+\mathcal{J}) \hookrightarrow X$. As is customary in this context, we will identify sheaves on closed subspaces with their pushforward to the ambient scheme.

For a very readable account of this topic, see Manin's *Lectures on the K-functor in algebraic geometry*.

7. Outlook

The *derived category* $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the *localization* of the category $\mathrm{Kom}(\mathcal{A})$ of complexes *at the class of quasi-isomorphisms*. Explicitly, its objects are simply the objects of $\mathrm{Kom}(\mathcal{A})$. For any morphism in $\mathrm{Kom}(\mathcal{A})$, there is a corresponding morphism in $D(\mathcal{A})$; but additionally, we adjoin a formal inverse for any quasi-isomorphism in $\mathrm{Kom}(\mathcal{A})$. Morphisms in $D(\mathcal{A})$ are therefore formal "zigzags" $K^\bullet \to Z_0^\bullet \leftarrow Z_1^\bullet \to Z_2^\bullet \leftarrow \cdots \to Z_n^\bullet \leftarrow Y^\bullet$ consisting of honest morphisms from $\mathrm{Kom}(\mathcal{A})$ and formal inverses to quasi-isomorphisms.

Remark 7.1. Many important categories can be obtained as localizations. For instance, the category of complete metric spaces is the localization of the category of all metric spaces at the class of bilipschitz maps with dense image. The category of sheaves on a topological space X is the localization of the category of presheaves on X at the class of stalkwise isomorphisms. The category of germs of spaces is the localization of the category of pointed spaces at the class of maps which restrict to isomorphisms in neighbourhoods of the base points.

We already discussed the motto of replacing objects by resolutions. This motto can be rigorously formalized in derived categories: If $0 \to X \to I^\bullet$ is a resolution (injective or otherwise), the morphism $X[0] \to I^\bullet$ is a quasi-isomorphism and therefore an isomorphism in $D(\mathcal{A})$.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. Recall that the complex $\mathbb{R}F(X)$ is well-defined up to quasi-isomorphism. Considered as an object of $D(\mathcal{B})$, it is therefore well-defined up to isomorphism; thus $\mathbb{R}F$ can be understood as a functor $D(\mathcal{A}) \to D(\mathcal{B})$.[16] *Derived categories are thus the natural source and target of total derived functors.*

The main advantage of the derived category setting is that, unlike with the classical approach, we forget only so much information as strictly needed. In particular, we retain the information about the *complex* $\mathbb{R}F(X)$ as opposed to only the cohomology groups $R^nF(X)$. A concrete benefit of this is that there is an easy formula for the composition of derived functors.

Proposition 7.2. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left-derived functors. Assume that a certain homological compatibility assumption is satisfied. Then $\mathbb{R}G \circ \mathbb{R}F \cong \mathbb{R}(G \circ F)$.*

This formula generalizes to the composition of more than two functors. Explicitly, to calculate $\mathbb{R}(G \circ F)(X)$, we can first pick a suitable resolution $0 \to X \to I^\bullet$, apply F to obtain $\mathbb{R}F(X) = F(I^\bullet)$, pick a resolution of that, i. e. a quasi-isomorphism $F(I^\bullet) \to J^\bullet$, and finally apply G to obtain $\mathbb{R}(G \circ F)(X) \cong G(J^\bullet)$.[17]

Summarizing, the information contained in the *complex* $\mathbb{R}F(X)$ is sufficient to determine $\mathbb{R}(G \circ F)(X)$. In contrast, for the classical right-derived functors, there

[16] This is not quite correct. In general, $\mathbb{R}F$ is only well-defined as a functor $D^+(\mathcal{A}) \to D^+(\mathcal{B})$, where $D^+(\mathcal{A})$ is the localization of the category of *bounded below* chain complexes at the quasi-isomorphisms. We ignore such boundedness issues.

[17] We could also try to find a resolution $0 \to X \to I^\bullet$ such that the objects $F(I^n)$ are G-acyclic. Then it's not necessary to resolve $F(I^\bullet)$.

is only the *Grothendieck spectral sequence*

$$E_2^{pq} = R^p G(R^q F(X)) \Longrightarrow R^n(G \circ F)(X).$$

The objects contained in the E_2-page are of course not enough to uniquely determine the limit; one needs the differentials and the higher pages. Thus the cohomology of $\mathbb{R}F(X)$ doesn't suffice to uniquely determine the cohomology of $\mathbb{R}(G \circ F)(X)$.

★ *Complexes good, cohomology bad.* ★

Besides its use as a great technical tool for dealing with derived functors, derived categories are also interesting on their own. For a ring A or a scheme X, the derived category of modules on A and the derived category of sheaves of modules on X is a valuable invariant which appears to sit just right on the spectrum between "extremely valuable, but not computable" and "easily computable, but not very interesting".

There are some relations between rings and schemes which can only be expressed on a derived level. For example, the polynomial ring $k[x_1, \ldots, x_n]$ and the exterior algebra over k^n are not at all isomorphic. However, their associated derived categories *are* equivalent. This instance of *Koszul duality* has even practical implications, in that certain algorithms dealing with the polynomial ring (which is infinite dimensional over k) can be massively sped up by transporting them along the equivalence to work with the exterior algebra instead (which is finite dimensional over k).[18]

In geometry, there are are important examples of schemes which "should" be the same in some sense but are not isomorphic. In this case, their relation is sometimes captured by a *derived equivalence*, i. e. an equivalence between the associated derived categories of sheaves of modules. This is, for instance, the case for the orbifold $X^n /\!/ S_n$ of unordered n-tuples of points on a surface X and the Hilbert scheme $X^{[n]}$ of n points on X. Their derived categories of sheaves of modules are equivalent.

Remark 7.3. Despite their applications, derived categories do have several problems. One is that they "don't glue very well": Denote by $D(X)$ the derived category of sheaves on a scheme X. Let X be covered by open subsets U and V. Then it would be nice if $D(X)$ was a (2-)fiber product of $D(U)$ and $D(V)$ over $D(U \cap V)$. However, this is not the case. A way to fix these problems is to turn to *enriched derived categories*.

There are a number of fine expository articles on derived categories. The reader is warmly encouraged to study them.

- R. P. Thomas. *Derived categories for the working mathematician*. (Start here.)
- M. Haiman. *Notes on derived categories and derived functors*.
- B. Keller. *Derived categories and their uses*.
- B. Keller. *Introduction to abelian and derived categories*.
- L. Nicolaescu. *The derived categories of sheaves and the Poincaré–Verdier duality*. (With an emphasis on duality theory.)

[18] See articles by Mohamed Barakat for details.

Finally, we want to mention *model categories*. They are necessary to extend the theory to non-abelian situations. For instance, let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Then there is an exact sequence

$$f^*\Omega^1_{Y/Z} \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

relating several sheaves of Kähler differentials. Of course, noticing the missing zero at the front, you can hear this sequence screaming “derive me!”.

This phenomenon is already visible on the level of affine schemes: Let $A \to B \to C$ be a sequence of rings. Then there is an exact sequence

$$\Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0.$$

Unfortunately, the functor associating to an algebra its module of Kähler differentials can’t be derived with the techniques described in these notes, since its source category is not abelian. Check out A. Raksit, *Defining the cotangent complex*.

E-mail address: `iblech@web.de`