Vector bundles on affine schemes

Lemma. *Let $\mathfrak{a}$ be a finitely generated idempotent ideal in a local ring A. Then $\mathfrak{a} = (0)$ or $\mathfrak{a} = (1)$.*

Proof. Consider $\mathfrak{a}$ as a finitely generated A-module. Then, by Nakayama's lemma, there exists an element $x \in A$ such that $x \equiv 1$ modulo $\mathfrak{a}$ and $x\mathfrak{a} = 0$. Since A is a local ring, x is invertible or $1 - x$ is invertible.[1] In the first case it follows that $\mathfrak{a} = (0)$, in the second that $\mathfrak{a} = (1)$. □

Lemma. *Let A be a local ring. Let P be an idempotent matrix over A. Then P is equivalent to a diagonal matrix with entries 1 and 0.*

Proof. Since P is idempotent, so are its ideals $(\Lambda^i P)$ of i-minors:

$$(\Lambda^i P) = (\Lambda^i (P \circ P)) = (\Lambda^i P \circ \Lambda^i P) \subseteq (\Lambda^i P) \cdot (\Lambda^i P) \subseteq (\Lambda^i P).$$

By the previous lemma, they are therefore each equal to (0) or (1). Since they form a descending chain, there exists a stage r such that $(\Lambda^r P) = (1)$ and $(\Lambda^{r+1} P) = (0)$. Therefore all $(r+1)$-minors of P are zero, and – since A is a local ring – there exists at least one invertible r-minor. Thus P can be made into a diagonal matrix of the desired form by applying row and column transformations. □

Remark. We can even show that P is *similar* to a diagonal matrix with entries 1 and 0: By the lemma, image and kernel of P are finite free. Combining bases of these subspaces, we obtain a basis of the full space; expressing P with respect to this basis, we obtain a diagonal matrix of the desired form.

Proposition. *An A-module M is finitely generated and projective if and only if there exists a partition $1 = \sum_i f_i \in A$ such that the localized modules $M[f_i^{-1}]$ are each finite free over $A[f_i^{-1}]$.*

Proof. Let M be a finitely generated and projective A-module. Then there exists a linear surjection $p : A^n \to M$ with a section $s : M \to A^n$. The composition $P := s \circ p$ is idempotent and M is isomorphic to $A^n / \ker(P)$. Interpreting the previous lemma in the little Zariski topos of $\operatorname{Spec} A$ (or alternatively just observing that the lemma invoked the defining axiom of local rings only a finite number of times), we see that there exists a partition of unity such that P is, over each of the localized rings, equivalent to a diagonal matrix with entries 1 and 0. Since localization is exact, the module $A^n / \ker(P)$ is therefore finite locally free.

Conversely, let M be a finite locally free A-module. Then M is locally finitely generated and therefore also globally finitely generated. Fix a linear surjection $A^n \to M$. Its kernel is finitely generated, since localization is exact and M is locally finitely presented.[2] Thus the kernel is also globally finitely generated. This shows that M is finitely presented.

To verify that M is projective, consider an arbitrary linear surjection $X \to Y$. We have to show that the postcomposition map $\operatorname{Hom}_A(M, X) \to \operatorname{Hom}_A(M, Y)$ is too surjective. Since M is locally projective and since $\operatorname{Hom}_A(M, \cdot)$ commutes with localization (this is because M is finitely presented),[3] this map is locally surjective and therefore surjective. □

Corollary. *Let M be an A-module. The induced quasicoherent sheaf of modules $M^\sim$ on $\operatorname{Spec} A$ is a vector bundle if and only if M is finitely generated and projective.*

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[1] Recall that a ring is *local* if and only if any invertible finite sum of elements contains an invertible summand. That is: $1 \neq 0$ and if $x + y$ is invertible, then x is invertible or y is invertible.

[2] Recall that a module N is finitely presented if and only if there exists an exact sequence $A^n \to A^m \to N \to 0$. So the definition only demands that the kernel of the given surjection $A^m \to N$ is finitely generated. However, one can show that even the kernel of *any* surjection $A^p \to N$ is finitely generated. See for instance Prop. 3.6 in Pete Clark's notes on commutative algebra.

[3] Write $M \cong A^m / \operatorname{im}(K)$ with $K = (k_{ij}) \in A^{m \times n}$. Then $\operatorname{Hom}_A(M, N) \cong \{x \in N^m \mid \sum_i k_{ij} x_i = 0,\ j = 1, \ldots, n\}$. With this explicit description one can verify that $\operatorname{Hom}_A(M, \cdot)$ commutes with localization.