# A QUICKSTART GUIDE TO DERIVED FUNCTORS

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## 1. References

## 2. Abelian categories

- Definition
- Examples
- Counter examples
- Diagram chases
- Uses: category of coefficient objects, source and target of functors to derive
- Exact functors: classical and finite limit definition, in footnote: what, if one doesn't start with *short* exact sequences?

## 3. Injectives and projectives

For the theory of derived functors, injective and projective objects are very important. We'll discuss why below.

**Definition 3.1.** An object $I$ of an abelian category is *injective* if and only if, for any monomorphism $X \hookrightarrow Y$ and any morphism $X \to I$, there exists a lift of that morphism to $Y$.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, hook] \arrow[d] & Y \arrow[dl, dashed] \\ & I & \end{tikzcd}$$

Dually, an object $P$ is *projective* if and only if, for any epimorphism $X \twoheadrightarrow Y$ and any morphism $P \to X$, there exists a co-lift of that morphism to $Y$.

$$\begin{tikzcd} X \arrow[r, two heads] & Y \arrow[r] & 0 \\ & P \arrow[u] \arrow[ul, dashed] & \end{tikzcd}$$

Note that no uniqueness of the lifts is required.

**Example 3.2.** In the category of vector spaces over some field, assuming the axiom of choice, any object $I$ is injective: Simply take a basis of $X$, extend it to a basis of $Y$, and define $Y \to I$ by zero on the extended zero. This gives further indication that categories of vector spaces are not very interesting from a homological point of view.

**Example 3.3.** *Baer's criterion* states that in the category of abelian groups, assuming the axiom of choice, a group $G$ is injective if and only if it is *divisible*, i. e. if for any $x \in G$ and $n \geq 1$ there exists an element $y \in G$ such that $x = ny$.[1] For example, the groups $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Injective objects are generally regarded as huge, unwieldy objects. They are important for the theory, but no practical calculations are made with them. Projective objects, on the other hand, are often much more accessible. This is of course a purely social statement, since the injective objects in an abelian category $\mathcal{A}$ are precisely the projective objects in $\mathcal{A}^{\mathrm{op}}$.

**Example 3.4.** In the category of modules over some ring, any finite free module $P$ is projective: Choose a basis of $P$ and pick preimages under $X \twoheadrightarrow Y$ of the images of the basis vectors in $Y$. Use these preimages to define the co-lift $P \to X$. More generally, assuming the axiom of choice, any (not necessarily finite) free module is projective. The precise characterization is that a module is projective if and only if it is a direct summand of a free module.[2]

**Example 3.5.** Assuming the axiom of choice, any vector space is free and thus projective.

**Example 3.6.** An easy example of a projective module which is not free is the $\mathbb{Z}/(6)$-module $\mathbb{Z}/(2)$. It is projective because there is the direct sum decomposition $\mathbb{Z}/(6) \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(3)$. It is not free because its number of elements is not a multiple of 6.

**Example 3.7.** Let $\mathfrak{m} = (3, 1 + \sqrt{-5})$ be the famous example of an ideal of $R = \mathbb{Z}[\sqrt{-5}]$ which is locally, but not globally principal. Then $\mathfrak{m}$ is projective but not free.

**Example 3.8.** By the *Serre–Swan theorem*, vector bundles provide a systematic source of projective modules. Let $M$ be a module over a ring $R$. Then the induced quasicoherent module $M^{\sim}$ on $\operatorname{Spec} A$ is a vector bundle (i. e. a finite free sheaf of modules) if and only if $M$ is projective and finitely generated. It is a trivial bundle if and only if $M$ is finite free. There is an analogous relation in the smooth setting: The category of smooth vector bundles on a compact manifold $M$ is, by taking global sections, equivalent to the category of finitely generated projective modules over $\mathcal{C}^{\infty}(M)$.

The reason why injective and projective objects are important in homological algebra is that they *have great exactness properties*. The following lemma makes one of this properties precise. Injective and projective objects thus form the building blocks by which other objects are *resolved by* – see the next section.

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[1]The "only if" direction is easy: Let $x \in G$ and $n \geq 1$. Then consider the injective map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. The sought element $y$ can be obtained as the image of 1 under a lift of the map $\mathbb{Z} \to G$, $1 \mapsto x$.

[2]The relationship to other properties of $R$-modules is as follows:

$$\begin{tikzcd} \text{free} \arrow[r, Rightarrow, bend left] & \text{projective} \arrow[l, Rightarrow, dashed, bend left, "R\text{ local or }R\text{ PID}"] \arrow[r, Rightarrow, bend left] & \text{flat} \arrow[l, Rightarrow, dashed, bend left, "R\text{ perfect}"] \arrow[r, Rightarrow, bend left] & \text{torsionfree} \arrow[l, Rightarrow, dashed, bend left, "R\text{ Dedekind domain}"] \end{tikzcd}$$

**Lemma 3.9.** *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Let $X^\bullet$ be a bounded below complex of injective objects or a bounded above complex of projective objects. If $X^\bullet$ is acyclic, then $F(X^\bullet)$ is acyclic as well (!).*

Note that the statement is totally false without the injectivity or projectivity assumption.

*Proof.* One can show that that such a complex is homotopy equivalent to the zero complex. Any additive functor preserves homotopy equivalences. Therefore the image complex is too homotopy equivalent to zero and in particular acyclic.

The already interesting special case of three-term complexes, i. e. short exact sequences, can be proved by a simpler argument: If $0 \to A \to B \to C \to 0$ is a short exact sequence with $A$ injective or $C$ projective, one can construct a retraction of $A \to B$ respectively a section of $B \to C$. Therefore the sequence splits. The claim follows since additive functors preserve biproducts. □

- Check statement about vector bundles

## 4. Resolutions

- Definition
- Existence
- Morphisms
- Uses: good replacements
- Motto: $X = I^0 - I^1 + \ldots$
- Construction of free resolutions
- Koszul resolutions
- Homological dimension

## 5. Derived functors

**Definition 5.1.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then $\mathbb{R}F(X)$, the *total right-derived functor of $F$* evaluated at an object $X$ of $\mathcal{A}$, is the complex

$$\mathbb{R}F(X): \quad \cdots \to 0 \to F(I^0) \to F(I^1) \to \cdots,$$

where $0 \to X \to I^\bullet$ is an injective resolution of $X$. The *$n$-th right-derived functor of $F$* evaluated at $X$ is the $n$-th cohomology of this complex: $R^nF(X) := H^n(\mathbb{R}F(X)) = H^n(F(I^\bullet))$.

So, to calculate $R^nF(X)$, we take an injective resolution of $X$, apply $F$ to this complex (be sure to strip the leading "$F(X)$"), and then take cohomology. Note that, even though the complex $I^\bullet$ is not acyclic only at position zero, the complex $F(I^\bullet)$ may have cohomology in arbitrary high degrees (!). There is an analogous definition for the left-derived functors of right-exact functors, using projective instead of injective resolutions.

Since we made the arbitrary choice of an injective resolution, we have to discuss well-definedness of course. If $0 \to X \to J^\bullet$ is a second injective resolution, there is a homotopy equivalence $I^\bullet \simeq J^\bullet$ by Lemma **??**. This equivalence is preserved by the functor $F$, so the complexes $F(I^\bullet)$ and $F(J^\bullet)$ are homotopy equivalent as well. In particular, they are quasi-isomorphic and have isomorphic cohomology.

Summing up, the complex $\mathbb{R}F(X)$ is well-defined up to quasi-isomorphism and $R^nF(X)$ is well-defined up to isomorphism. For reasons to be stated below, we don't want to stress that with our definition, $\mathbb{R}F(X)$ is actually even well-defined up to homotopy equivalence.

Suppose we fix injective resolutions for every object of $\mathcal{A}$. Then the rule $X \mapsto R^nF(X)$ can be made into an additive functor $R^nF : \mathcal{A} \to \mathcal{B}$, since morphisms $X \to Y$ between objects induce morphisms between their associated injective resolutions (which are unique up to homotopy) and therefore well-defined morphisms $R^nF(X) \to R^nF(Y)$.

Different choices of resolutions will induce naturally isomorphic right-derived functors.

**Example 5.2.** We have $R^0F(X) \cong F(X)$, naturally in $X$. To verify this, pick an injective resolution $0 \to X \to I^\bullet$ and apply $F$ to the exact sequence $0 \to X \to I^0 \to I^1$.

It is in this sense that the $R^nF$ are "derived" functors of $F$; $R^0F$ coincides with $F$ and the higher derived functors bear some relation with $F$. But note that $R^{n+1}F$ is *not* the first derived functor of $R^nF$ (indeed, in general the higher derived functors are neither left- nor right-exact, so we cannot derive $R^nF$) and that there is no Leibniz rule to be found.

**Example 5.3.** Assume $F$ to even be exact. Then the higher derived functors of $F$ vanish, i. e. $R^nF(X) = 0$ for all objects $X$ and $n > 0$. To verify this, use Lemma **??**.

This suggests the following motto: *Derived functors measure the failure of a functor to be exact.*

**Example 5.4.** Consider the dualization functor $F : \mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}$, $M \mapsto \mathrm{Hom}(M, \mathbb{Z})$. Since $F$ is an instance of a Hom functor, it is left-exact. To ensure that the contravariance doesn't cause unnecessary difficulties, we spell out precisely what this means: If $0 \to M \to N \to P \to 0$ is a short exact sequence in $\mathrm{Ab}^{\mathrm{op}}$ (this really means that we have a short exact sequence $0 \leftarrow M \leftarrow N \leftarrow P \leftarrow 0$ in Ab), then $0 \to F(M) \to F(N) \to F(P)$ is exact in Ab. To compute $\mathbb{R}F(\mathbb{Z}/(2))$, we use the projective (even free) resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0,$$

which corresponds to an injective resolution in $\mathrm{Ab}^{\mathrm{op}}$. The total derived functor is therefore the complex

$$0 \longrightarrow F(\mathbb{Z}) \xrightarrow{2} F(\mathbb{Z}) \longrightarrow 0,$$

which can be simplified using $F(\mathbb{Z}) \cong \mathbb{Z}$ to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0.$$

Thus we obtain $R^0F(\mathbb{Z}) \cong F(\mathbb{Z}) = 0$ and $R^1F(\mathbb{Z}) \cong \mathbb{Z}/(2)$. Note that the torsion did not vanish, but instead moved to $R^1$.

Many important functors are derived functors. For instance,

- the right derived functors of $\mathrm{Hom}(X, \_) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ are the *Ext functors* $\mathrm{Ext}^n(X, \_)$,
- the left derived functors of $M \otimes_A \_ : \mathrm{Mod}(A) \to \mathrm{Ab}$ are the *Tor functors* $\mathrm{Tor}_n(M, \_)$,
- the right derived functors of the global sections functor $\Gamma : \mathrm{AbSh}(X) \to \mathrm{Ab}$ calculate *sheaf cohomology*, and
- the right derived functors of the functor $\mathrm{Mod}(G) \to \mathrm{Ab}$ which associates to a $G$-module its subset of $G$-invariant elements calculate *group cohomology*.

The raison d'être for derived functors is given by the following lemma.

**Lemma 5.5.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Then, for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, there is an induced long exact sequence*

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1F(Y) \to R^1F(Z) \to R^2F(X) \to \cdots,$$

*depending functorially on the short exact sequence.*

*Proof.* Use the *horseshoe lemma* to obtain a short exact sequence relating injective resolutions of $X$, $Y$, and $Z$. Then perform the usual diagram chase to construct the connecting morphisms. □

Note that the lemma shows very visibly that $F$ is exact if and only if $R^1F = 0$.

In practice, calculating the higher derived functors can be quite hard. There are two main techniques: One can use exact sequences (and spectral sequences) to draw some conclusions about the derived functors, or hope that there are more amenable resolutions than resolutions by unwieldy injectives available. To this end, the following lemma is crucial:

**Lemma 5.6.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. Let $X$ be an object of $\mathcal{A}$. Let $0 \to X \to U^\bullet$ be a resolution of $X$ by $F$-*acyclic objects*, i. e. objects such that $R^{\geq 1}F(U^m) = 0$. Then $\mathbb{R}F(X)$ is quasi-isomorphic to $F(U^\bullet)$ and in particular $R^nF(X) \cong H^n(F(U^\bullet))$.*[3]

In other words, we may use resolutions by $F$-acyclic objects instead of injective objects to calculate the derived functors. For instance, we can use flat resolutions to calculate Tor and flabby resolutions to calculate sheaf cohomology.

Derived functors can be used to fix several defects in the original functors. The paramount defect of a non-exact functor is, of course, its failure to preserve exactness. This is fixed by the long exact sequence. But derived functors also fix the failure of certain identities. For instance, if $M$ is an arbitrary $A$-module, in general it does not hold that

$$M^{\vee\vee} := \mathrm{Hom}(\mathrm{Hom}(M, A), A) \cong M.$$

But, denoting by $(\_)^{\mathbb{V}}$ the total derived functor of the dualization functor, the complex $M^{\mathbb{V}\mathbb{V}}$ *is* quasi-isomorphic to $M$ (regarded as a complex concentrated in

[3]Note that $\mathbb{R}F(X)$ will in general *not* be homotopy equivalent to $F(U^\bullet)$. This is the reason why we identify $\mathbb{R}F(X)$ up to quasi-isomorphism and not up to homotopy equivalence.

degree zero).[4] Similarly, in algebraic geometry, there is the *projection formula*

$$\pi_*\mathcal{E} \otimes \mathcal{F} \cong \pi_*(\mathcal{E} \otimes \pi^*\mathcal{G}),$$

valid in the case that $\mathcal{G}$ is a locally free sheaf. For the derived functors, the formula holds in complete generality, assuming only some finiteness conditions:

$$\mathbb{R}\pi_*(\mathcal{E}^\bullet) \otimes^{\mathbb{L}} \mathcal{F}^\bullet \simeq_{\text{qis}} \mathbb{R}\pi_*(\mathcal{E}^\bullet \otimes^{\mathbb{L}} \mathbb{L}\pi^*\mathcal{G}^\bullet).$$

- somewhere: "complexes good, cohomology bad", example from homotopy theory

## 6. Example: Ext

- Lifting problems
- Extensions (give correspondence in footnote)

## 7. Example: Sheaf cohomology

- Resolution of $\underline{\mathbb{R}}$
- Comparison with singular cohomology
- Paucity of projective sheaves
- Example: section-wise surjectivity
- Uninteresting: $\mathcal{C}^\infty$-modules

## 8. Outlook

- Unbounded resolutions
- Derived categories: definition, composition of derived functors, $D^b(X)$ $f^!$
- Model categories: sAb $\simeq \mathrm{Kom}^{\leq 0}(\mathrm{Ab})$

[4]We are skating over some details here. In particular, we have not explained how to apply $\mathbb{R}F$ to a complex $K^\bullet$ (instead of a single object); this is done by finding a quasi-isomorphism $K^\bullet \to I^\bullet$ to a complex consisting of injectives and setting $\mathbb{R}F(K^\bullet) := F(I^\bullet)$. Also one has to impose some finiteness restrictions, for instance $A$ should be Noetherian and $M$ finitely generated.