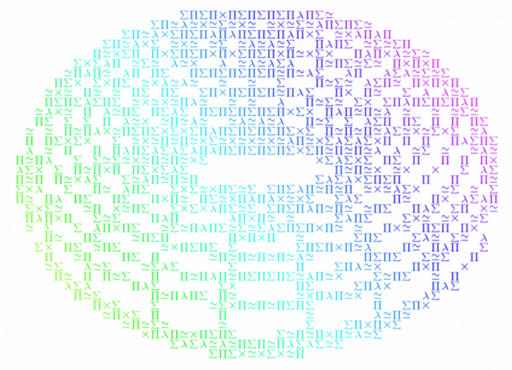
Homotopy type theory



Ingo Blechschmidt

November 25th, 2014

Outline

What are foundations?

- Foundations set the logical context for doing maths.
- Their details don't matter in everyday work (mostly).
- But their main concepts do.

http://collabcubed.com/2012/10/24/high-trestle-trail-bridge-rdg/

What are foundations?

- Foundations set the logical context for doing maths.
- Their details don't matter in everyday work (mostly).
- But their main concepts do.

- Classical foundations are *set-based* (ZF, ZFC, ...):
  Everything is a set.
- $0 := \emptyset, \quad 1 := \{0\}, \quad 2 := \{0, 1\}, \quad \ldots$
- $(x, y) := \{\{x\}, \{x, y\}\}$ (Kuratowski pairing)
- $(x, y, z) := (x, (y, z))$
- maps: (X, Y, R) with $R \subseteq X \times Y$ such that ...

What’s wrong with set-based foundations?

Set-based foundations …

- do not reflect typed mathematical practice,
- do not respect equivalence of structures,
- require complex encoding of “higher-level” subjects, complicating interactive proof environments.

What is homotopy type theory?

- Homotopy type theory is a new foundational theory.
- Basic notions have a homotopy-theoretic flavour.
- One can start doing “real mathematics” right away, without complex encodings.
- Initiated by Voevodsky in 2005.

Some participants of the IAS special year

What is homotopy type theory?

Homotopy type theory …

- is elegant,
- reflects mathematical practice,
- contains wondrous new concepts,
- ensures that everything respects equivalences,
- simplifies the plumbing of homotopy theory,
- allows for accessible computer formalization.

What is homotopy type theory?

Homotopy type theory …

- is elegant,
- reflects mathematical practice,
- contains wondrous new concepts,
- ensures that everything respects equivalences,
- simplifies the plumbing of homotopy theory,
- allows for accessible computer formalization.

What are values and types?

- In type theory, there are **values** and **types**.
- Every value is of exactly one type.
- Types may depend on values.

$$7 : \mathbb{N}$$
$$(3, 5) : \mathbb{N} \times \mathbb{N}$$
$$\mathsf{succ} : \mathbb{N} \to \mathbb{N}$$
$$\text{zero vector} : \mathbb{R}^n \quad (n : \mathbb{N})$$

What are values and types?

- In type theory, there are **values** and **types**.
- Every value is of exactly one type.
- Types may depend on values.

$$
\begin{aligned}
7 &: \mathbb{N} \\
(3,5) &: \mathbb{N} \times \mathbb{N} \\
\mathsf{succ} &: \mathbb{N} \to \mathbb{N} \\
\text{zero vector} &: \mathbb{R}^n \quad (n : \mathbb{N})
\end{aligned}
$$

Let $B(x)$ be a type family depending on $x : A$.

- $\sum_{x:A} B(x) = \text{“}\{(a,b) \mid a : A, b : B(a)\}\text{”}$
- $\prod_{x:A} B(x) = \text{“}\{f : A \to ?? \mid f(a) : B(a) \text{ for all } a : A\}\text{”}$

What is the dependent equality type?

In set theory, for a set X and elements $x, y \in X$:

- "$x = y$" is a **proposition**.
- Set theory is **layered above** predicate logic.

In type theory, for a type X and values $x, y : X$:

- There is the **equality type** $\mathrm{Id}_X(x, y)$ or $(x =_X y)$.
- To verify that "$x = y$", exhibit a value of $(x = y)$.
- Have $\mathsf{refl}_x : (x = x)$.
- Identity types may contain zero or **many** values!

Intuition: $(x = y)$ is the type of **proofs** that "$x = y$".

What is the dependent equality type?

In set theory, for a set X and elements $x, y \in X$:

- "$x = y$" is a **proposition**.
- Set theory is **layered above** predicate logic.

In type theory, for a type X and values $x, y : X$:

- There is the **equality type** $\mathrm{Id}_X(x, y)$ or $(x =_X y)$.
- To verify that "$x = y$", exhibit a value of $(x = y)$.
- Have $\mathsf{refl}_x : (x = x)$.
- Identity types may contain zero or **many** values!

Intuition: $(x = y)$ is the type of **proofs** that "$x = y$".

Intuition: $(x = y)$ is the type of **paths** $x \rightsquigarrow y$.

How are types like spaces?

homotopy theory	type theory
space X	type X
point $x \in X$	value $x : X$
path $x \rightsquigarrow y$	value of $(x = y)$
(continuous) map	value of $X \to Y$

- A **homotopy** between maps $f, g : X \to Y$ is a value of

$$(f \simeq g) :\equiv \prod_{x:X} (f(x) = g(x)).$$

- A space X is **contractible** iff

$$\mathsf{IsContr}(X) :\equiv \sum_{x:X} \prod_{y:X} (x = y).$$

How are types like spaces?

- "The type X is **contractible**":

$$\mathsf{IsContr}(X) :\equiv \sum_{x:X} \prod_{y:X} (x = y).$$

- "The type X is a **mere proposition**":

$$\mathsf{IsMereProp}(X) :\equiv \prod_{x,y:X} (x = y)$$

- "The type X is a **set** or **discrete space**":

$$\mathsf{IsSet}(X) :\equiv \prod_{x,y:X} \mathsf{IsMereProp}(x = y)$$

- For instance, $\mathbb{N}$ is a set.

How are types like spaces?

- Functions are automatically **continuous/functorial**:

$$(x = y) \longrightarrow (f(x) = f(y)).$$

- Type families $P : X \to \mathcal{U}$ automatically behave like **fibrations**, in that they have the path lifting property:

$$(x = y) \longrightarrow (P(x) \simeq P(y)).$$

How are constructions encoded?

- The **fiber** of a map $f : X \to Y$ over a point $y : Y$ is

$$\mathsf{fib}_f(y) :\equiv \sum_{x:X} (f(x) = y).$$

- The **path space** of X is

$$X^I :\equiv \sum_{x,y:X} (x = y).$$

- The **based loop space** of X at x is

$$\Omega^1(X, x) :\equiv (x = x).$$

- The **path fibration** of (X, x) is the map

$$\mathsf{snd} : \sum_{y:X} (x = y) \to X.$$

How are constructions encoded?

- The **fiber** of a map $f : X \to Y$ over a point $y : Y$ is

$$\mathsf{fib}_f(y) :\equiv \sum_{x:X} (f(x) = y).$$

- The **path space** of X is

$$X^I :\equiv \sum_{x,y:X} (x = y).$$

- The **based loop space** of X at x is

$$\Omega^1(X, x) :\equiv (x = x).$$

- The **path fibration** of (X, x) is the map

$$\mathsf{snd} : \sum_{y:X} (x = y) \to X.$$

What are higher inductive definitions?

The type $\mathbb{N}$ of natural numbers is **freely generated** by

- a point $0 : \mathbb{N}$ and
- a function $\mathsf{succ} : \mathbb{N} \to \mathbb{N}$.

This definition gives rise to an **induction principle**

$$\prod_{A:\mathbb{N}\to\mathcal{U}} \left(A(0) \times \left(\prod_{n:\mathbb{N}} A(n) \to A(\mathsf{succ}(n)) \right) \longrightarrow \prod_{n:\mathbb{N}} A(n) \right),$$

and a **recursion principle**

$$\prod_{X:\mathcal{U}} \left(X \times \left(\mathbb{N} \to (X \to X) \right) \longrightarrow (\mathbb{N} \to X) \right).$$

How to present famous spaces?

The **circle** S^1 is generated by

- a point $\mathsf{base} : S^1$ and
- a path $\mathsf{loop} : (\mathsf{base} = \mathsf{base})$.

The **sphere** S^2 is generated by

- a point $\mathsf{base} : S^2$ and
- a path $\mathsf{surf} : (\mathsf{refl}_{\mathsf{base}} = \mathsf{refl}_{\mathsf{base}})$.

The **torus** T^2 is generated by

- a point $b : T^2$,
- a path $p : (b = b)$,
- a path $q : (b = b)$, and
- a 2-path $t : (p \cdot q = q \cdot p)$.

How to present famous spaces?

The **suspension** ΣX of X is generated by

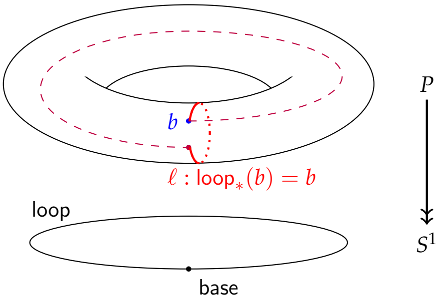
- a point $\mathsf{N} : \Sigma X$ and
- a point $\mathsf{S} : \Sigma X$ and
- a function $\mathsf{merid} : X \to (N = S)$.

The **cylinder** $\mathrm{Cyl}(X)$ of X is generated by

- a function $\mathsf{bot} : X \to \mathrm{Cyl}(X)$ and
- a function $\mathsf{top} : X \to \mathrm{Cyl}(X)$ and
- a function $\mathsf{seg} : \prod_{x:X}(\mathsf{bot}(x) = \mathsf{top}(x))$.

Of course, we can show $\mathrm{Cyl}(X) \simeq X \times I \simeq X$.

What is circle induction?



The **induction principle** of S^1 states: Given $P : S^1 \to \mathcal{U}$,

- a point $b : P(\mathsf{base})$, and
- a path $\ell : \mathsf{loop}_*(b) = b$

there is a function $f : \prod_{x:S^1} P(x)$ such that

- $f(\mathsf{base}) \equiv b$ and
- $f(\mathsf{loop}) = \ell$.

What is type truncation?

Let X be a type.

The **propositional truncation** $\|X\|_{-1}$ is generated by

- a function $X \to \|X\|_{-1}$ and
- for any $x, y : \|X\|_{-1}$, a path $x = y$.

The **0-truncation** $\|X\|_0$ is generated by

- a function $X \to \|X\|_0$ and
- for any $x, y : \|X\|_0$, $p, q : (x = y)$, a path $p = q$.

The **fundamental group** of (X, x_0) is

$$\pi_1(X, x_0) :\equiv \|\Omega^1(X, x_0)\|_0 :\equiv \|(x_0 = x_0)\|_0.$$

What is the univalence axiom?

An **equivalence** is a function $f : X \to Y$ such that

$$\mathsf{IsEquiv}(f) :\equiv \prod_{y:Y} \mathsf{IsContr}(\mathsf{fib}_f(y)).$$

Types X and Y are **equivalent** iff

$$(X \simeq Y) :\equiv \sum_{f:X\to Y} \mathsf{IsEquiv}(f).$$

The **univalence axiom** states: The canonical function

$$(X = Y) \longrightarrow (X \simeq Y)$$

is an equivalence, for all types X and Y.

What's the status of the axiom of choice?

- The following proposition is **just true**, but is not a faithful rendition of the axiom of choice:

$$\Big(\prod_{x:A}\sum_{y:B} R(x,y)\Big) \longrightarrow \sum_{f:A\to B}\prod_{x:A} R(x,f(x)).$$

- The real axiom of choice,

$$\Big(\prod_{x:A}\Big\|\sum_{y:B} R(x,y)\Big\|_{-1}\Big) \longrightarrow \Big\|\sum_{f:A\to B}\prod_{x:A} R(x,f(x))\Big\|_{-1},$$

can be added as an axiom, but is rarely needed.

What's the status of the axiom of choice?

- The following proposition is **just true**, but is not a faithful rendition of the axiom of choice:

$$\Big(\prod_{x:A}\sum_{y:B} R(x,y)\Big) \longrightarrow \sum_{f:A\to B}\prod_{x:A} R(x,f(x)).$$

- The real axiom of choice,

$$\Big(\prod_{x:A}\Big\|\sum_{y:B} R(x,y)\Big\|_{-1}\Big) \longrightarrow \Big\|\sum_{f:A\to B}\prod_{x:A} R(x,f(x))\Big\|_{-1},$$

  can be added as an axiom, but is rarely needed.
- The law of excluded middle is too rarely needed.

$$\mathsf{LEM} :\equiv \prod_{A:\mathcal{U}}\Big(\mathsf{IsMereProp}(A) \to A + \neg A\Big).$$

What are models of HoTT?

Conjecturally, HoTT can be interpreted in any **$(\infty, 1)$-topos**. Verified models include

- ∞Grpd, i. e. a model in simplicial sets, and
- $(\infty, 1)$-presheaf toposes over elegant Reedy categories.

Thus, any theorem proven in HoTT holds in the context of classical homotopy theory and in more general contexts.

What is homotopy type theory?

Homotopy type theory …

- is elegant,
- reflects mathematical practice,
- contains wondrous new concepts,
- ensures that everything respects equivalences,
- simplifies the plumbing of homotopy theory,
- allows for accessible computer formalization.

What is homotopy type theory?

Homotopy type theory …

- is elegant,
- reflects mathematical practice,
- contains wondrous new concepts,
- ensures that everything respects equivalences,
- simplifies the plumbing of homotopy theory,
- allows for accessible computer formalization.

References

- The textbook
  http://homotopytypetheory.org/book/
- Voevodsky on his motivations
  http://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations_files/2014_IAS.pdf
- Seminar slides
  http://www.math.ias.edu/~mshulman/hottseminar2012/01intro.pdf
  http://www.math.ias.edu/~mshulman/hottminicourse2012/04induction.pdf
  https://coq.inria.fr/files/coq5-slides-spitters.pdf
  https://www.andrew.cmu.edu/user/awodey/hott/CMUslides.pdf
- An application unrelated to homotopy theory
  http://www.cs.nott.ac.uk/~txa/talks/lyon14.pdf
- hott-amateurs mailing list