Bifurcation in parameter dependent systems

Numerical Methods for Systems Biology WS 12/13

Jonas Ibn-Salem

10.01.13

Freie Universität Berlin

Overview

Introduction: Fixed Point Analysis

Given the system of differential equations:

$$y' = f(y)$$

Definition

A *fixed point* y^* is defined by $f(y^*) = 0$.

- Solve the equation $f(y) = 0$
- Analyse eigenvalues of the Jacobian at fixed points.

Now: System with *controle parameter* μ.

$$y' = f(y, \mu)$$

How does μ influence the number, location and stability of fixed points?

Bifurcation

Definition

Bifurcation is the changing of the character of an equalibrium point and/or the creation of extra ones by alteration of a control parameter.

Example: Logistig growth with harvesting

Growth of a population:

$$y' = \frac{1}{10} y(10 - y) - \mu$$

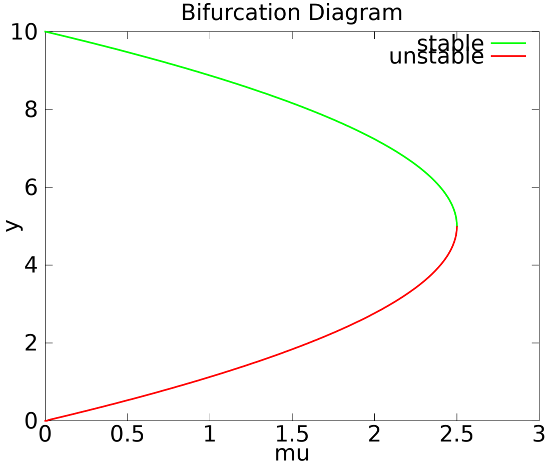
Solving $f(y, \mu) = 0$ for any parameter μ.

Example: Logistig growth with harvesting

Growth of a population:

$$y' = \frac{1}{10} y(10 - y) - \mu$$

Solving $f(y, \mu) = 0$ for any parameter μ.



Fixed points at
$y_{1/2} = 5 \pm \sqrt{25 - 10\mu}$

Stability:
$\frac{df}{dy} = -\frac{2}{10} y + 1$

Hopf Bifurcation

Definition

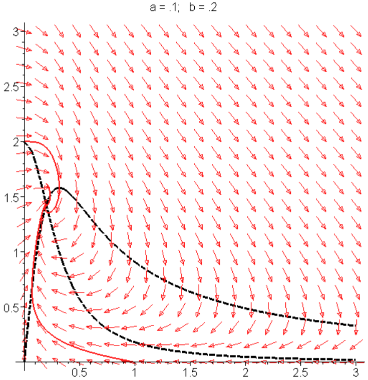
A *Hopf Bifurcation* is the appearance or disappearance of a periodic orbit through a local change in the stability properties of a fixed point.

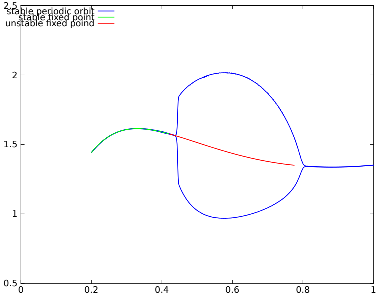
Appears when a pair of complex conjugate eigenvalues around the fixed point crosses the imaginary axis of the complex plane.

$$x' = -x + ay + x^2y \tag{1}$$

$$y' = b - ay - x^2y \tag{2}$$

Hopf Bifurcation example:





Numerical Bifurcation Analysis

How to draw the bifurcation diagram and find bifurcation points?

Fixed point analysis for equidistant parameters μ is inefficient.

Idea

Follow the fixed point around the fold bifurcation curve.

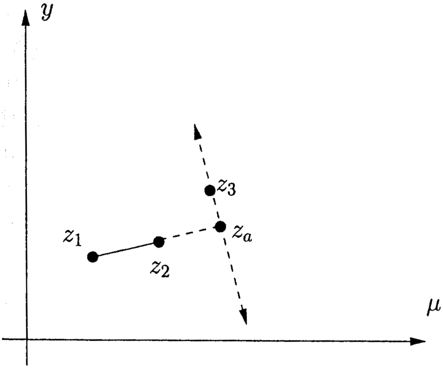
Treating μ as an aditional dependent variable in phase space and solve

$$f(y, \mu) = 0$$

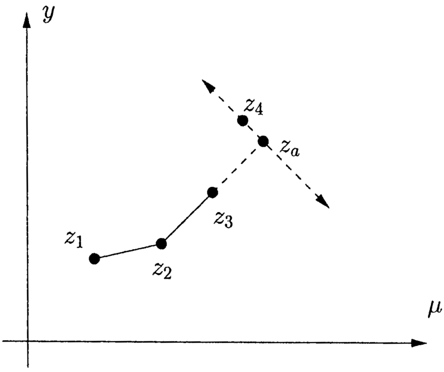
.

Numerical Bifurcation Analysis

- Given two nearby points $z_1 = (y_1, \mu_1)$ and $z_2 = (y_2, \mu_2)$
- Initial approximation $z_a = 2z_2 - z_1$ as starting point for Newton's method.
- Additional equation: $(z_3 - z_a) \cdot (z_a - z_2) = 0$



(a)



(b)

Example: Path following

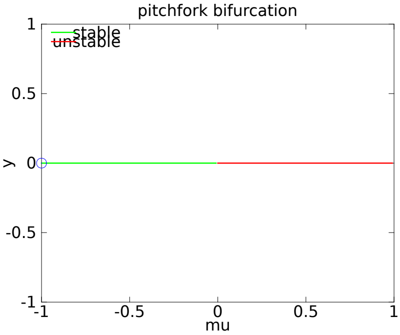
Bifurcation Diagram for $y' = \mu y - y^3$

- Fixing $\mu = -1$ yield
  $z_1 = (0, -1)$ and
  $z_2 = (0, -1 + \delta\mu)$ for small
  approximate distance $\delta\mu$

Example: Path following

Bifurcation Diagram for $\quad y' = \mu y - y^3$

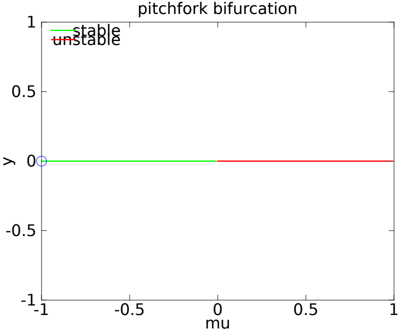
- Fixing $\mu = -1$ yield $z_1 = (0, -1)$ and $z_2 = (0, -1 + \delta\mu)$ for small approximate distance $\delta\mu$



Example: Path following

Bifurcation Diagram for $\quad y' = \mu y - y^3$

- Fixing $\mu = -1$ yield $z_1 = (0, -1)$ and $z_2 = (0, -1 + \delta\mu)$ for small approximate distance $\delta\mu$

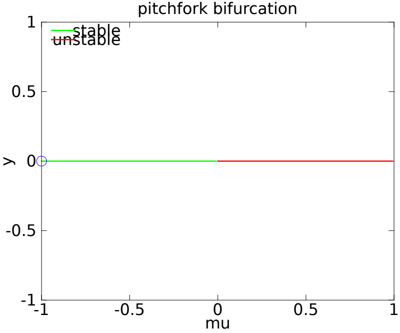


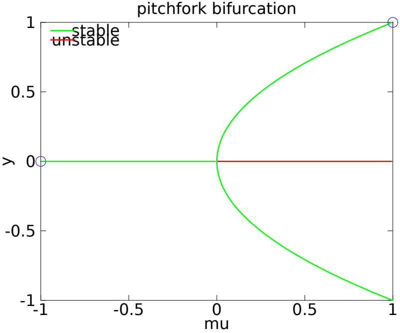
- Other fixed points for $\mu = 1$ yield $z_1 = (1, 1)$ and $z_2 = (1, 1 + \delta\mu)$

Example: Path following

Bifurcation Diagram for $\quad y' = \mu y - y^3$

- Fixing $\mu = -1$ yield $z_1 = (0, -1)$ and $z_2 = (0, -1 + \delta\mu)$ for small approximate distance $\delta\mu$
- Other fixed points for $\mu = 1$ yield $z_1 = (1, 1)$ and $z_2 = (1, 1 + \delta\mu)$





Summary

- ...

Thank you!