Bifurcation in parameter dependent systems

Numerical Methods for Systems Biology WS 12/13

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Overview

Introduction: Fixed Point Analysis

Given the system of differential equations:

$$y' = f(y)$$

Definition

A *fixed point* y^* is defined by $f(y^*) = 0$.

- Solve the equation $f(y) = 0$
- Analyse eigenvalues of the Jacobian at fixed points.

Bifurcation

Now: System with *controle parameter* μ.

$$y' = f(y, \mu)$$

How does μ influence the fixed points?

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- Change in FP stability
- Change in FP number and location.

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Definition

Bifurcation is the changing of the character of an equalibrium point and/or the creation of extra ones by alteration of a control parameter.

The value of μ where bifurcation occurs is called a *bifurcation point*.

Example: Logistig growth with harvesting

Growth of a (fish) population under harvesting (fishing):

$$y' = \frac{1}{10} y(10 - y) - \mu$$

Solving $f(y, \mu) = 0$ for any parameter μ.

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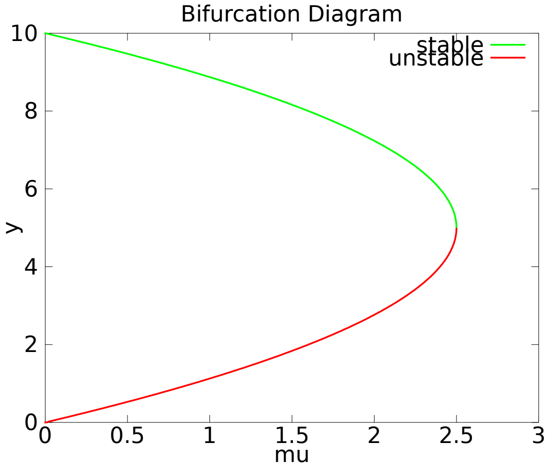
Fixed points at
$y_{1/2} = 5 \pm \sqrt{25 - 10\mu}$

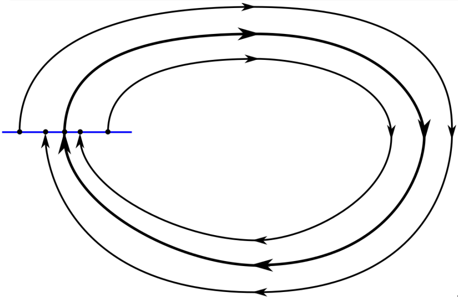
Stability:
$\lambda = \frac{df}{dy} = -\frac{2}{10} y + 1$



Hopf Bifurcation

Definition

A *Hopf Bifurcation* is the appearance or disappearance of a periodic orbit (limit cycle) through a local change in the stability properties of a fixed point.



Appears when a pair of complex conjugate eigenvalues around the fixed point crosses the imaginary axis of the complex plane.

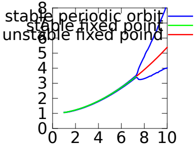
Hopf Bifurcation example

chlorine dioxide-iodine-malonic acid reaction

iodine: $y_1' = \mu - y_1 - \frac{4y_1y_2}{y_1^2+1}$

chlorine dioxine: $y_2' = y_1(1 - \frac{y_2}{y_1^2+1})$

- Fixed points at $(y_1, y_2) = (\frac{\mu}{5}, \frac{1}{25}(\mu^2 + 25))$
- Bifurcation point at $\mu \approx 7.3$



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Numerical Bifurcation Analysis

How to draw the bifurcation diagram and find bifurcation points?

Fixed point analysis for equidistant parameters μ is inefficient.

Idea

Follow the fixed point around the fold bifurcation curve.

Treating μ as an aditional dependent variable in phase space and solve

$$f(y, \mu) = 0$$

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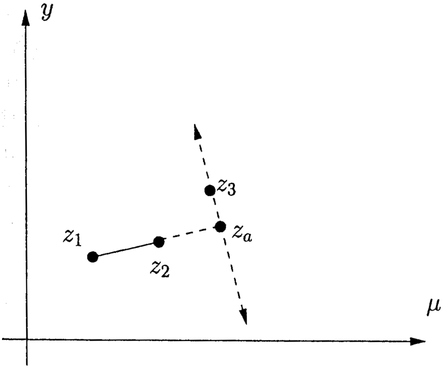
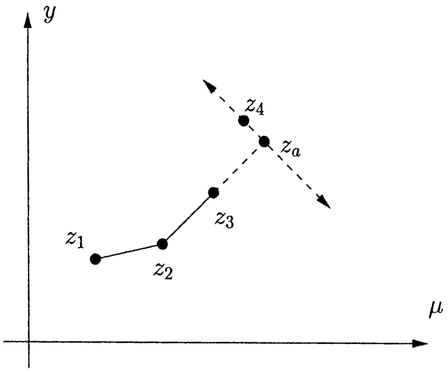
Numerical Bifurcation Analysis

- Given two nearby points $z_1 = (y_1, \mu_1)$ and $z_2 = (y_2, \mu_2)$
- Initial approximation $z_a = 2z_2 - z_1$ as starting point for Newton's method.
- Additional equation: $(z_3 - z_a) \cdot (z_a - z_2) = 0$



(a)



(b)

Example: Path following

Bifurcation Diagram for $y' = \mu y - y^3$

- Fixing $\mu = -1$ yield
  $z_1 = (0, -1)$ and
  $z_2 = (0, -1 + \delta\mu)$ for small
  approximate distance $\delta\mu$

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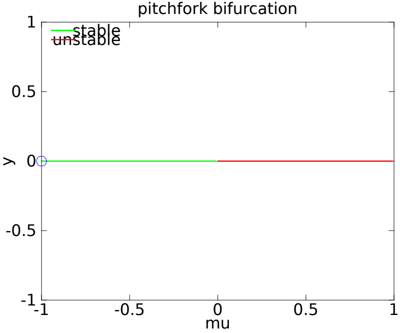

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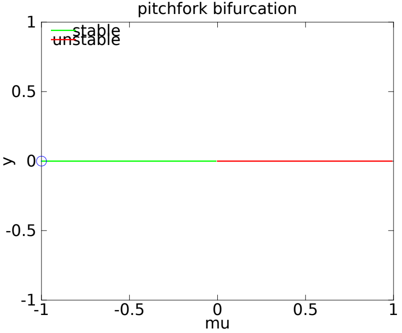

- Other fixed points for $\mu = 1$ yield $z_1 = (1, 1)$ and $z_2 = (1, 1 + \delta\mu)$

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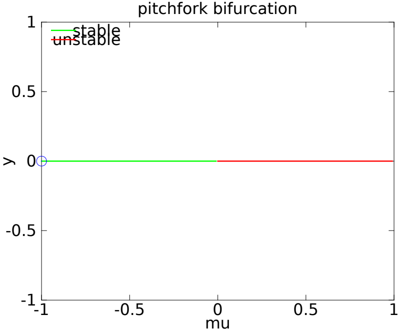

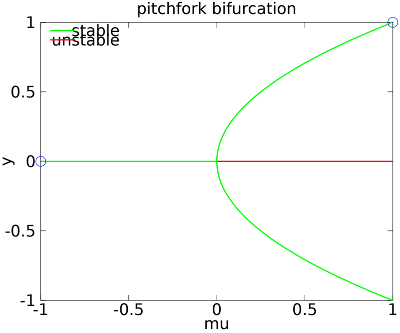

Summary

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Thank you!