Bifurcation in parameter dependent systems

Numerical Methods for Systems Biology WS 12/13

Jonas Ibn-Salem

10.01.13

Freie Universität Berlin

Overview

1. Introduction: Fixed Point Analysis
2. Bifurcation
   Example: Logistic growth with harvesting
3. Hopf Bifurcation
4. Numerical Bifurcation Analysis: Path following

Introduction: Fixed Point Analysis

Given the system of differential equations:

$$y' = f(y)$$

Definition

A *fixed point* y^* is defined by $f(y^*) = 0$.

- Solve the equation $f(y) = 0$
- Analyse eigenvalues of the Jacobian at fixed points.

Bifurcation

Now: System with *controle parameter* μ.

$$y' = f(y, \mu)$$

How does μ influence the fixed points?

Bifurcation

Now: System with *controle parameter* μ.

$$y' = f(y, \mu)$$

How does μ influence the fixed points?

- Change of FP stability
- Change of FP number and location.

Bifurcation

Now: System with *controle parameter* μ.

$$y' = f(y, \mu)$$

How does μ influence the fixed points?

- Change of FP stability
- Change of FP number and location.

Definition

Bifurcation is the changing of the character of an equilibrium point and/or the creation of extra ones by alteration of a control parameter.

The value of μ where bifurcation occurs is called a *bifurcation point*.

Example: Logistic growth with harvesting

Growth of a (fish) population under harvesting (fishing):

$$y' = \frac{1}{10} y(10 - y) - \mu$$

Solving $f(y, \mu) = 0$ for any parameter μ.

Example: Logistic growth with harvesting

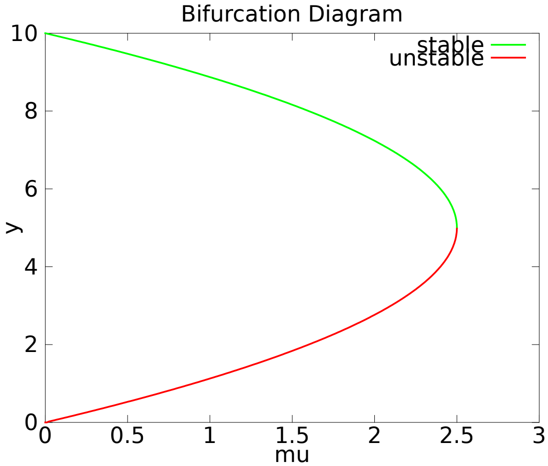
Growth of a (fish) population under harvesting (fishing):

$$y' = \frac{1}{10} y(10 - y) - \mu$$

Solving $f(y, \mu) = 0$ for any parameter μ.

Fixed points at
$y_{1/2} = 5 \pm \sqrt{25 - 10\mu}$

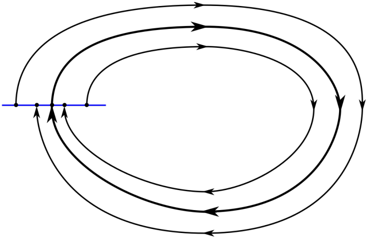
Stability:
$\lambda = \frac{df}{dy} = -\frac{2}{10} y + 1$



Hopf Bifurcation

Definition

A *Hopf Bifurcation* is the appearance or disappearance of a periodic orbit (limit cycle) through a local change in the stability properties of a fixed point.



It appears when a pair of complex conjugate eigenvalues around the fixed point crosses the imaginary axis of the complex plane.

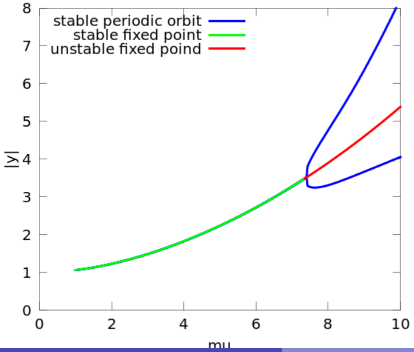
Hopf Bifurcation example

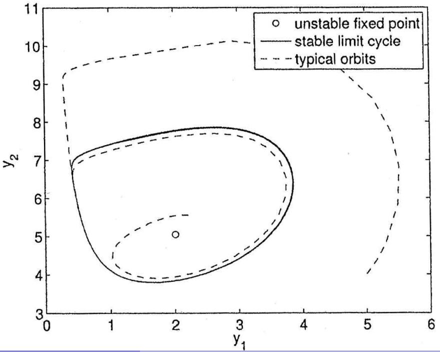
chlorine dioxide-iodine-malonic acid reaction

iodine: $y_1' = \mu - y_1 - \frac{4y_1y_2}{y_1^2+1}$

chlorine dioxine: $y_2' = y_1(1 - \frac{y_2}{y_1^2+1})$

- Fixed points at $(y_1, y_2) = (\frac{\mu}{5}, \frac{1}{25}(\mu^2 + 25))$
- Bifurcation point at $\mu \approx 7.3$

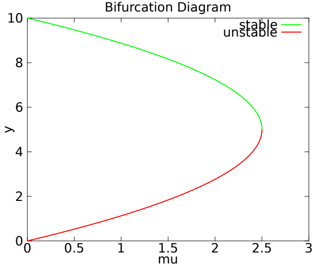




Numerical Bifurcation Analysis

How to draw the bifurcation diagram and find bifurcation points?

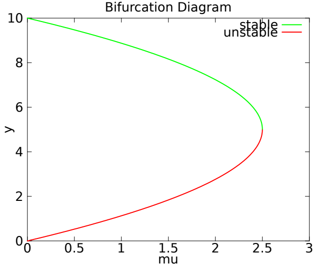
- Fixed point analysis for equidistant parameters μ is inefficient.



Numerical Bifurcation Analysis

How to draw the bifurcation diagram and find bifurcation points?

- Fixed point analysis for equidistant parameters μ is inefficient.



Idea

Follow the fixed point around the fold bifurcation curve.

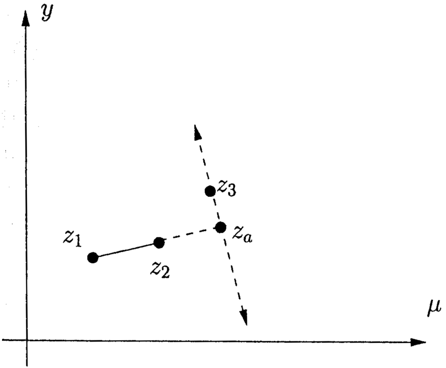
Treat μ as an additional dependent variable in phase space and solve

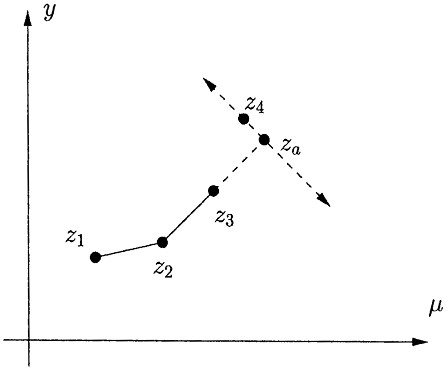
$$f(y, \mu) = 0$$

.

Numerical Bifurcation Analysis

- Given two nearby points $z_1 = (y_1, \mu_1)$ and $z_2 = (y_2, \mu_2)$
- Initial approximation $z_a = 2z_2 - z_1$
- Additional equation: $(z_3 - z_a) \cdot (z_a - z_2) = 0$
- Solving for z_3 via Newton's method with init. approx. z_a





Example: Path following

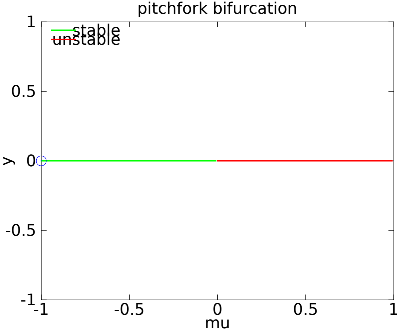
Bifurcation Diagram for $\quad y' = \mu y - y^3$

- Fixing $\mu = -1$ yield
  $z_1 = (0, -1)$ and
  $z_2 = (0, -1 + \delta\mu)$ for small
  approximate distance $\delta\mu$

Example: Path following

Bifurcation Diagram for $\quad y' = \mu y - y^3$

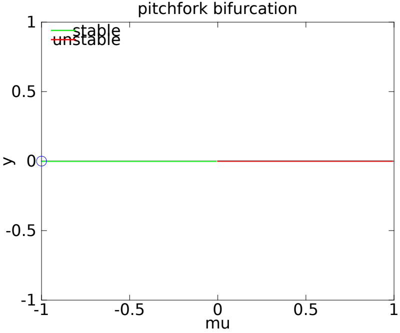
- Fixing $\mu = -1$ yield
  $z_1 = (0, -1)$ and
  $z_2 = (0, -1 + \delta\mu)$ for small
  approximate distance $\delta\mu$



Example: Path following

Bifurcation Diagram for $\quad y' = \mu y - y^3$

- Fixing $\mu = -1$ yield
  $z_1 = (0, -1)$ and
  $z_2 = (0, -1 + \delta\mu)$ for small approximate distance $\delta\mu$

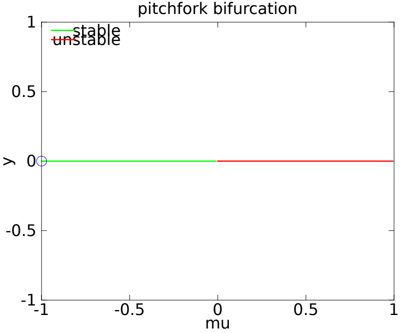


- Other fixed points for $\mu = 1$ yield $z_1 = (1, 1)$ and
  $z_2 = (1, 1 - \delta\mu)$

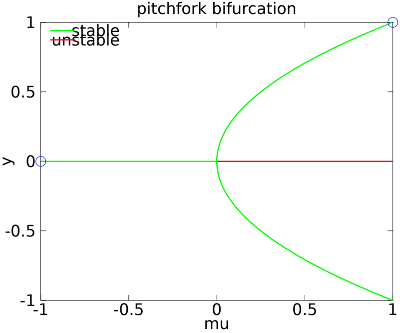
Example: Path following

Bifurcation Diagram for $\quad y' = \mu y - y^3$

- Fixing $\mu = -1$ yield $z_1 = (0, -1)$ and $z_2 = (0, -1 + \delta\mu)$ for small approximate distance $\delta\mu$



- Other fixed points for $\mu = 1$ yield $z_1 = (1, 1)$ and $z_2 = (1, 1 - \delta\mu)$



Summary

- Controle parameters can influence fixed points in ODE systems.
- At bifurcation points the stability, location and/or number of fixed points change.
- In a Hopf bifurcation a fixed point loses stability and a limit cycle occurs.
- Bifurcation can be analysed numerically using path following.

Sources:

- D.S. Jones et al.: *Differential Equations and Mathematical Biology.* CRC Press (2010)

Thank you!