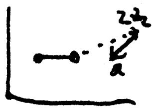
of the fixed points. For stable fixed points, $\mathbf{y}$ should tend towards $\mathbf{y}^*$; for unstable fixed points, $\mathbf{y}$ should move away from $\mathbf{y}^*$.

It should be noted that, although the methods described in this chapter are applicable to continuous dynamical systems (systems of differential equations) many are equally applicable to discrete, iterative systems, or maps (see Sections 13.3 and 14.4). The main differences between a dynamical system based on differential equations and one based on a map is that, for a map defined by $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, fixed points $\mathbf{x}^*$ satisfy $\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*)$ rather than equation (14.1.2), and the requirement for stability is that the eigenvalues of the Jacobian evaluated at $\mathbf{x}^*$ have modulus less than 1 (i.e., $|\lambda| < 1$).

14.2 Path-following and bifurcation analysis

Having found the fixed points of a dynamical system and determined their stability, it is often desirable to carry out a bifurcation analysis with respect to some parameter μ, as described in Section 13.1. When done numerically this is known as **path-following**. A naive approach would be simply to repeat the procedure of the previous section for a series of equally spaced values of μ. An exhaustive search of phase space for fixed points is usually prohibitively expensive, but this could be avoided because the fixed points already found for the preceding value of μ will be close to the fixed points for the current value of μ, and hence would be good candidates to try as starting values for Newton's method. The problem with this approach is that it breaks down just where we are most interested in its results: at a bifurcation. Consider for example the fold bifurcation shown in Figure 13.1.3. Starting with a small value of μ and working upwards will not work because, for small values of μ there are no fixed points to work with. Starting with a large value of μ and trying to follow one of the two fixed points as μ decreases will work initially. However, as μ approaches the bifurcation point, the method will simply 'fall' off the end of the fold bifurcation into the region where there are no fixed points, without finding the other branch of the bifurcation diagram.

Path-following proceeds by treating the parameter μ as an additional dependent variable in phase space. Suppose that two nearby points on the bifurcation curve $\mathbf{z}_1 = (\mathbf{y}_1, \mu_1)$ and $\mathbf{z}_2 = (\mathbf{y}_2, \mu_2)$ have been found by the naive approach described above (see Figure 14.2.1). We now seek a third point $\mathbf{z}_3 = (\mathbf{y}_3, \mu_3)$ as the solution to the equation



$$\mathbf{f}(\mathbf{y}, \mu) = \mathbf{0}. \tag{14.2.1}$$

We construct an initial approximation $\mathbf{z}_a$ (to feed into Newton's method) to this point simply by linearly extrapolating from the two known points $\mathbf{z}_a = 2\mathbf{z}_2 - \mathbf{z}_1$ (see Figure 14.2.1). By treating μ as a free variable, we have

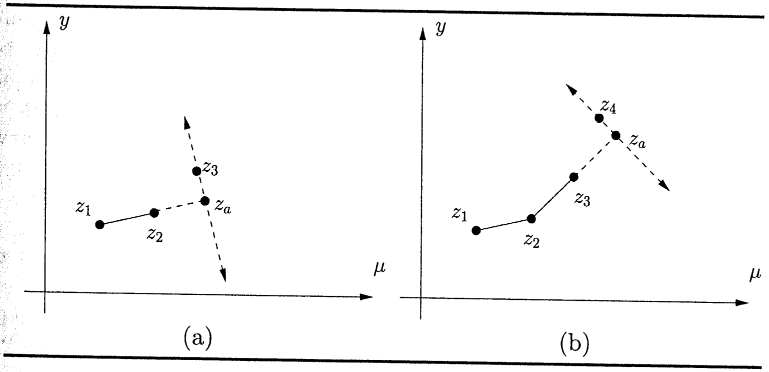


FIGURE 14.2.1: Numerical method for path-following: (a) two existing points $\mathbf{z}_1$ and $\mathbf{z}_2$ on the bifurcation diagram are used to generate a third point $\mathbf{z}_3$; (b) the process is repeated to generate $\mathbf{z}_4$, and so on.

effectively introduced an extra unknown into the system of equations, so we need an extra equation for the problem to remain well defined. Remember that the goal is to 'follow' the fixed point around the fold bifurcation, rather than to find the fixed point for pre-determined values of μ. Hence, we obtain an extra equation by constraining the third point in the bifurcation curve to lie on the line that: (i) passes through the initial approximation to the point; (ii) is senkrecht perpendicular to the line through the first two points (see Figure 14.2.1). This extra equation can be expressed in vector notation as extra equation

$$(\mathbf{z}_3 - \mathbf{z}_a).(\mathbf{z}_a - \mathbf{z}_2) = 0, \tag{14.2.2}$$

where the points $\mathbf{z}_2$ and $\mathbf{z}_a$ are known constants. (In higher dimensions, (14.2.2) is the equation of a hyperplane perpendicular to the line through the first two points.) Combining equations (14.2.1) and (14.2.2) gives $N+1$ equations in $N+1$ unknowns ($\mathbf{z}_3 = (y_1, \ldots, y_n, \mu)$). Solving these (via Newton's method with an initial approximation $\mathbf{z}_a$) gives the point $\mathbf{z}_3$. This process can now be repeated using the points $\mathbf{z}_2$ and $\mathbf{z}_3$ to construct a new point $\mathbf{z}_4$, and so on. The stability of each fixed point found can be determined by looking at the eigenvalues of the Jacobian.

Note that the distance between the initial points $\mathbf{z}_1$ and $\mathbf{z}_2$ is the approximate arc length distance between consecutive computed points on the bifurcation curve. These initial points should therefore be chosen to be sufficiently close that a reasonably smooth curve is produced, but not so close that an infeasibly large number of points are required to give the desired portion of the curve. ?

A MATLAB function implementing this procedure is shown below, and will be used in the following examples