Test of Hypothesis (1)

❖ Statistical Inference

Statistical inference is the process of analyzing the result and making conclusions from data subject to random variation. It is also called inferential statistics. Hypothesis testing and confidence intervals are the applications of the statistical inference. Statistical inference is a method of making decisions about the parameters of a population, based on random sampling.

❖ Statistical Hypothesis

A statistical hypothesis is some statement or assertion about a population or equivalently about the probability distribution characterizing a population which we want to verify on the basis of information available from a sample.

Example: A few examples of statistical hypothesis that relate to our real life are as follows:

- A physiscian may hypothesize that the recommended drug is effective in 90 percent cases.
- A nutritionist claims that at most 75 percent of the per school children in a certain country have protein deficient diets.
- An administrator of business farm claims that the average work efficiency of his workers is at least 90 percent.
- A sewing machine company claims that their new machine is superior to the one availabe in the market.
- The court assumes that the indicated person is innocent.

❖ Null Hypothesis

According to Prof. R. A. Fisher, the hypothesis which we are going to test for possible rejection under the assumption that it is true is called the null hypothesis. Usually it is denoted by H_0.

Example: If $X \sim N(\mu, \sigma^2)$, where μ and σ^2 are parameters. Let us consider μ has a specified value μ_0(say). Then the null hypothesis is $H_0: \mu = \mu_0$

Some examples of null hypothesis relate to real life are as follows:

- There is no difference in the incidence of malnutrition between vaccinated and non-vaccinated.
- Males do not smoke more than females.
- There is no association between level of education and knowledge of child nutrition among women.
- Two teaching methods A and B are equally effective.

❖ Alternative Hypothesis

Any hypothesis which is complementary to the null hypothesis is called an alternative hypothesis. Usually it is denoted by H_1.

Example: If we want to test the null hypothesis that the population has a specified mean μ_0 (say) i.e. $H_0 : \mu = \mu_0$ then the alternative hypothesis could be $H_1 : \mu \neq \mu_0$ *or* $H_1 : \mu > \mu_0$ *or* $H_1 : \mu < \mu_0$.
A few more examples of alternative hypothesis corresponding to null hypothesis to relate real life are as follows:

- There has been a difference in the incidence of malnutrition between vaccinated and non-vaccinated.
- Males smoke more than females do.
- There is association between level of education and knowledge of child nutrition among women.
- Two teaching methods A and B are different.

❖ **Simple Hypothesis**

If the hypothesis specifies the population completely then it is termed as a simple hypothesis.

Example: If $x_1, x_2, \dots \dots, x_n$ is a random sample of size n from a normal population with mean μ and variance σ^2, then the hypothesis $H_0 : \mu = \mu_0,\ \sigma^2 = \sigma_0^2$ is a simple hypothesis.

❖ **Composite Hypothesis**

If the hypothesis does not specify the population completely then it is termed as a composite hypothesis.

Example: If $x_1, x_2, \dots \dots, x_n$ is a random sample of size n from a normal population with mean μ and variance σ^2 then each of the following hypothesis is a composite hypothesis.

→	$\mu = \mu_0$	→	$\sigma^2 = \sigma_0^2$
→	$\mu < \mu_0,\ \sigma^2 = \sigma_0^2$	→	$\mu > \mu_0,\ \sigma^2 = \sigma_0^2$
→	$\mu = \mu_0,\ \sigma^2 < \sigma_0^2$	→	$\mu = \mu_0,\ \sigma^2 > \sigma_0^2$
→	$\mu < \mu_0,\ \sigma^2 > \sigma_0^2$	→	$\mu > \mu_0,\ \sigma^2 < \sigma_0^2$

Note: A hypothesis which does not specify completely r parameters of a population is termed as a composite hypothesis with r degrees of freedom.

❖ **Parametric Hypothesis**

When the hypothesis concerning the parameters of the distribution, provided the form of the distribution is called parametric hypothesis.

Example: If $X \sim N(\mu, \sigma^2)$ then $H_0 : \mu = \mu_0$ is a parametric hypothesis.

❖ **Non-parametric Hypothesis**

While the hypothesis regarding the form of the distribution with specified or unspecified parameters is called non-parametric hypothesis.

Example: $H_0 : F_X(x) = F_Y(y)$ is a non-parametric hypothesis, where $F_X(x)$ and $F_Y(y)$ are the distribution function of two population.

❖ Test of Significance

Test of significance is a statistical procedure to arrive at a conclusion or decision on the basis of samples and to test whether the formulated hypothesis can be accepted or rejected in probability sense. The aim of test of significance is to reject the null hypothesis.

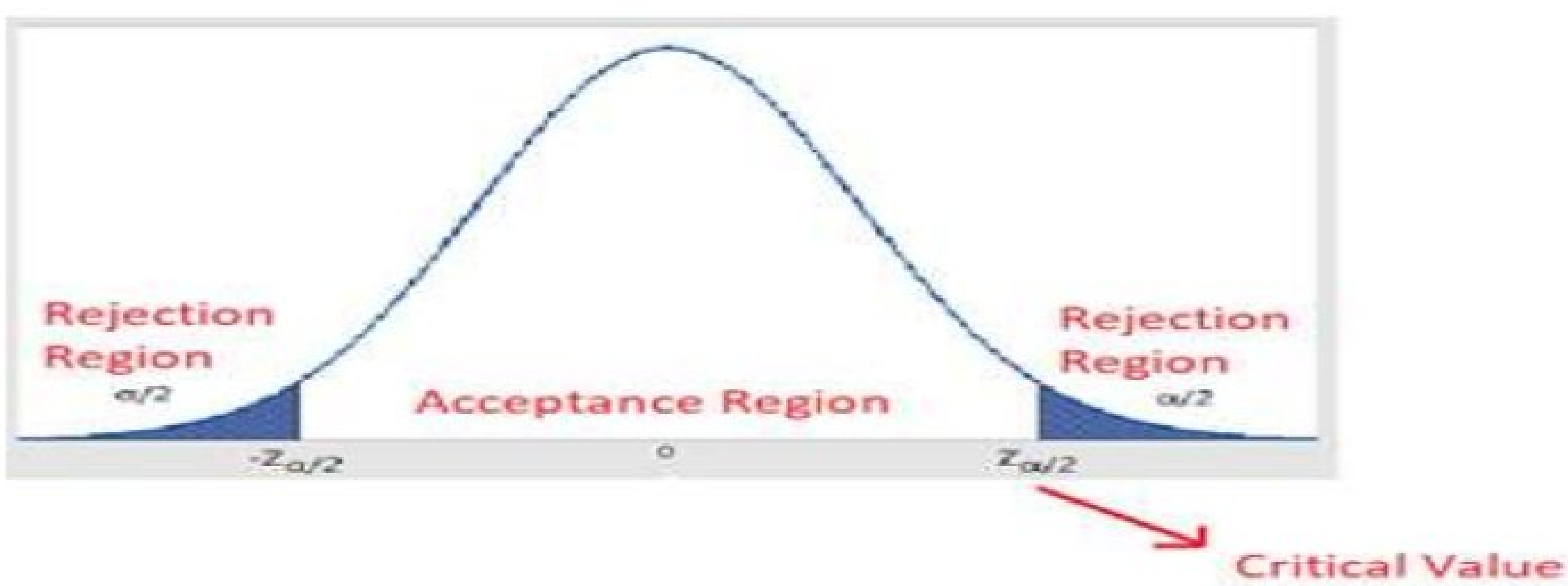
❖ Critical Region Or Rejection Region

Let $x_1, x_2, \dots \dots, x_n$ be a sample point designated by X in a n-dimensional sample space. If X falls in the region for which we reject H_0 when it is true then the region is called critical region. It is usually denoted by w.

In other wards, if the value of a test statistic falls in the region for which we reject H_0 when it is true then the region is known as critical region or rejection region. Usually it is denoted by w.

❖ Acceptance Region

If the value of a test statistic falls in the region for which we accept H_0 when it is true then the region is called acceptance region. It is denoted by w'.



❖ One Tailed Test

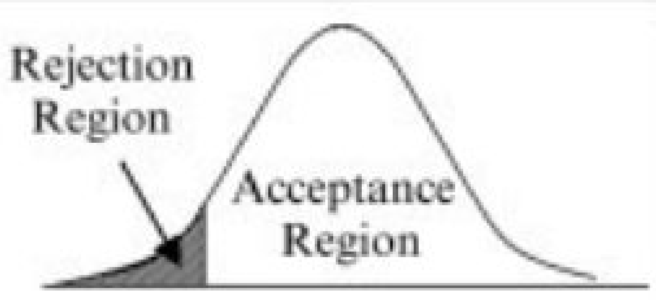
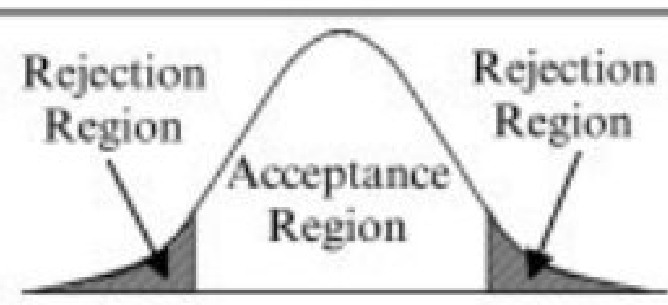
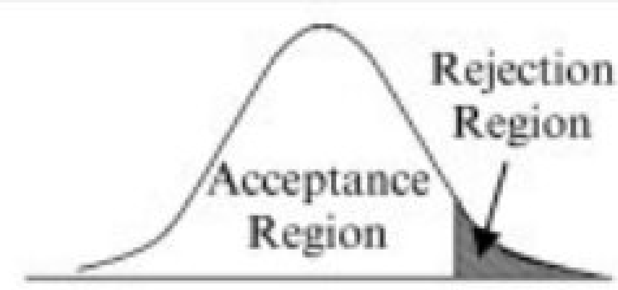
A test of any statistical hypothesis where the alternative hypothesis is one tailed (right or left tailed) is called a one tailed test. In such a case the critical region is given by the portion of the area lying in the first or last tails of the probability curve of the test statistic.

Example: A test for testing the mean of a population $H_0 : \mu = \mu_0$ against the alternative hypothesis $H_1 : \mu > \mu_0$ $(Right\ tailed)$ or $H_1 : \mu < \mu_0$ $(left\ tailed)$ is a one tailed test. In the right tailed test $H_1 : \mu > \mu_0$, the critical region lies entirely in the right tail of the sampling distribution of $\bar{x}$, while for the left tail test $H_1 : \mu < \mu_0$, the critical region is entirely in the left tail of the distribution.

❖ Two Tailed Test:

A test of statistical hypothesis where the alternative hypothesis is two tailed as called two tailed test. In such a case the critical region is given by the portion of the area lying in both the tails of the probability curve of the test statistic.

Example: Suppose that there are two population brands of bulbs, one manufactured by standard process (with mean life μ_1) and the other manufactured by some new technique (with mean life μ_2). If we want to test of the bulbs differ significantly then our null hypothesis is $H_0 : \mu_1 = \mu_2$ and the alternative will be $H_1 : \mu_1 \neq \mu_2$ this giving us a two tailed test.

One-Tailed Test (Left Tail)	Two-Tailed Test	One-Tailed Test (Right Tail)
$H_0 : \mu_X = \mu_0$ $H_1 : \mu_X < \mu_0$	$H_0 : \mu_X = \mu_0$ $H_1 : \mu_X \neq \mu_0$	$H_0 : \mu_X = \mu_0$ $H_1 : \mu_X > \mu_0$
		

❖ Type I Error

If the null hypothesis is actually true but on the basis of sample we reject H_0 (accepting H_1), this type of error is known as type I error.

The probability of type I error is denoted by α . Thus

$$\begin{aligned}\alpha &= \text{Probability of type I error} \\ &= \text{Probability of rejecting } H_0 \text{ when } H_0 \text{ is true}\end{aligned}$$

Symbolically, $P(x \in w \mid H_0) = \alpha \quad ; \quad where \;\; x = (x_1, x_2, \cdots, x_n)$

Example: In a murder case suppose that

H_0 : *The man is not guilty*
H_1 : *The man is guilty*

On the basis of sample if we reject the H_0 but it is true this type of error is known as type I error.

❖ Type II Error

If the null hypothesis is false but on the basis of sample we accept H_0 (H_1 is true), this type of error is known as type II error.

The probability of type II error is denoted by β . Thus

$$\begin{aligned}\beta &= \text{Probability of type II error} \\ &= \text{Probability of accepting } H_0 \text{ when } H_0 \text{ is false}\end{aligned}$$

Symbolically, $P(x \in \overline{w} \mid H_1) = \beta \quad ; \quad where \;\; x = (x_1, x_2, \cdots, x_n)$

Example: In a murder case suppose that

H_0 : *The man is not guilty*
H_1 : *The man is guilty*

On the basis of sample if we accept the H_0 but it is false this type of error is known as type II error.

Question 1: Which kinds of error is more serius and why?

Solution:

I think that type I error is more serious than type II error

To support my idea an example are given below

$$H_0 : \textit{The man is not guilty}$$
$$H_1 : \textit{The man is guilty}$$

Suppose a judge is to give verdict an a criminal case. The accused may be guilty or not guilty. The judgement should be such that if the accused be guilty he should be punished while if he is not guilty he should be acequitted. If a guilty man be acquitted then an error of type II is commited whereas if an innocent may be convicted an error of type I is commited.

From the above example it is clear that type I error is more serious than type II error because in type I error an innocent man be convicted though he is not guilty.

❖ **Level of Significance**

The probability of type I error is known as the level of significance of the test. It is also called the size of the critical region. Usually it is denoted by α . Mathematically it can be written as

$P(x \in w \mid H_0) = \alpha$ or we can say

$$\alpha = \text{Probability of type I error}$$
$$= \text{Probability of rejecting } H_0 \text{ when } H_0 \text{ is true}$$

❖ **Power of the Test**

The power of the test is the probability of rejecting the null hypothesis when in fact it is false and should be rejected. It is the probability of correctly decision. It can be expressed by $1-\beta$. Mathematically it can be written as $P(x \in w \mid H_1) = 1-\beta$.

Example 16·1. *Given the frequency function :*

$$f(x, \theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta$$
$$= 0, \text{ elsewhere}$$

and that you are testing the null hypothesis $H_0 : \theta = 1$ against $H_1 : \theta = 2$, by means of a single observed value of x. What would be the sizes of the type I and type II errors, if you choose the interval (i) $0{\cdot}5 \le x$, (ii) $1 \le x \le 1{\cdot}5$ as the critical regions ? Also obtain the power function of the test.

[Gauhati Univ, B.Sc. 1993; Calcutta Univ. B.Sc. (Maths Hons.), 1987]

Solution. Here we want to test

$H_0 : \theta = 1$, against $H_1 : \theta = 2$.

(*i*) Here $W = \{x : 0{\cdot}5 \le x\} = \{x : x \ge 0{\cdot}5\}$

and $\bar{W} = \{x : x \le 0{\cdot}5\}$

$$\alpha = P\{x \in W \mid H_0\} = P\{x \ge 0{\cdot}5 \mid \theta = 1\}$$
$$= P\{0{\cdot}5 \le x \le \theta \mid \theta = 1\} = P\{0{\cdot}5 \le x \le 1 \mid \theta = 1\}$$
$$= \int_{0.5}^{1} [f(x,\theta)]_{\theta=1}\, dx = \int_{0.5}^{1} 1.\, dx = 0{\cdot}5$$

Similarly,

$$\beta = P\{x \in \bar{W} \mid H_1\} = P\{x \le 0{\cdot}5 \mid \theta = 2\}$$
$$= \int_0^{0.5} [f(x,\theta)]_{\theta=2}\, dx = \int_0^{0.5} \tfrac{1}{2}\, dx = 0{\cdot}25$$

Thus the sizes of type I and type II errors are respectively

$\alpha = 0{\cdot}5$ and $\beta = 0{\cdot}25$

and power function of the test $= 1 - \beta = 0{\cdot}75$

(*ii*) $W = \{x : 1 \le x \le 1{\cdot}5\}$

$$\alpha = P\{x \in W \mid \theta = 1\} = \int_1^{1.5} [f(x,\theta)]_{\theta=1}\, dx = 0,$$

since under $H_0 : \theta = 1, f(x,\theta) = 0$, for $1 \le x \le 1{\cdot}5$.

$$\beta = P\{x \in \bar{W} \mid \theta = 2\} = 1 - P\{x \in W \mid \theta = 2\}$$

$$= 1 - \int_1^{1.5} \left[f(x,\theta) \right]_{\theta=2} dx = 1 - \left| \frac{x}{2} \right|_1^{1.5} = 0.75$$

$\therefore$ Power Function $= 1 - \beta = 1 - 0.75 = 0.25$

Example 16·2. *If $x \geq 1$, is the critical region for testing $H_0 : \theta = 2$ against the alternative $\theta = 1$, on the basis of the single observation from the population,*

$$f(x, \theta) = \theta \exp(-\theta x),\ 0 \leq x < \infty,$$

obtain the values of type I and type II errors.

[Poona Univ. M.C.A. 1993; Allahabad Univ. B.Sc., 1993; Delhi Univ. B.Sc (Stat. Hons.), 1988]

Solution. Here $W = \{x : x \geq 1\}$ and $\bar{W} = \{x : x < 1\}$.

and $H_0 : \theta = 2,\ H_1 : \theta = 1$

$$\begin{aligned}
\alpha &= \text{Size of Type I error} \\
&= P[x \in W \mid H_0] = P[x \geq 1 \mid \theta = 2] \\
&= \int_1^{\infty} \left[f(x,\theta) \right]_{\theta=2} dx \\
&= 2 \int_1^{\infty} e^{-2x}\, dx = 2 \left| \frac{e^{-2x}}{-2} \right|_1^{\infty} \\
&= e^{-2} = 1/e^2
\end{aligned}$$

$$\begin{aligned}
\beta &= \text{Size of type II error} \\
&= P[x \in \bar{W} \mid H_1] = P(x < 1 \mid \theta = 1) \\
&= \int_0^1 e^{-x}\, dx = \left| \frac{e^{-x}}{-1} \right|_0^1 \\
&= (1 - e^{-1}) = \frac{e-1}{e}
\end{aligned}$$

Example 16·3. *Let p be the probability that a coin will fall head in a single toss in order to test $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{3}{4}$. The coin is tossed 5 times and H_0 is rejected if more than 3 heads are obtained. Find the probability of type I error and power of the test.*

Solution. Here

$$H_0 : p = \frac{1}{2} \text{ and } H_1 : p = \frac{3}{4}.$$

If the r.v. X denotes the number of heads in n tosses of a coin then $X \sim B(n, p)$ so that

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{5}{x} p^x (1-p)^{5-x}, \qquad \ldots(*)$$

since $n = 5$, (given). The critical region is given by

$$W = \{x : x \geq 4\} \Rightarrow \overline{W} = \{x : x \leq 3\}$$

α = Probability of type I error

$$= P[X \geq 4 \mid H_0]$$

$$= P[X = 4 \mid p = \tfrac{1}{2}] + P[X = 5 \mid p = \tfrac{1}{2}]$$

$$= \binom{5}{4} (\tfrac{1}{2})^4 (\tfrac{1}{2})^{5-4} + \binom{5}{5} (\tfrac{1}{2})^5 \qquad \text{[From (*)]}$$

$$= 5 (\tfrac{1}{2})^5 + (\tfrac{1}{2})^5 = 6 (\tfrac{1}{2})^5$$

$$= \frac{3}{16}$$

β = Probability of Type II error

$$= P[x \in \overline{W} \mid H_1] = 1 - P\,[x \in W \mid H_1]$$

$$= 1 - [P(X = 4 \mid p = \tfrac{3}{4}) + P(X = 5 \mid p = \tfrac{3}{4})]$$

$$= 1 - \left[\binom{5}{4} (\tfrac{3}{4})^4 (\tfrac{1}{4}) + \binom{5}{5} (\tfrac{3}{4})^5\right]$$

$$= 1 - (\tfrac{3}{4})^4 \{ \tfrac{5}{4} + \tfrac{3}{4} \}$$

$$= 1 - \frac{81}{128} = \frac{47}{128}$$

$\therefore$ Power of the test is

$$1 - \beta = \frac{81}{128}$$

❖ Test of Significance

Test of significance is a statistical procedure to arrive at a conclusion or decision on the basis of samples and to test whether the formulated hypothesis can be accepted or rejected in probability sense. The aim of test of significance is to reject the null hypothesis.

❖ Steps in Solving Testing of Hypothesis Problem:

The major steps involved in the solution of a "testing of hypothesis" problem may be outlined as follows:

- Explicit knowledge of the nature of the population distribution and the parameter(s) of interest, i.e., the parameter(s) about which the hypothesis are set up.
- Setting up of the null hypothesis H_0 and the alternative hypothesis H_1 in terms of the range of the parameter values each one embodies.
- Choose the appropriate level of significance (α) depending on the reliability of the estimates and permissible risk.
- Choose the suitable test statistic and compute the test statistic under the null hypothesis.

- We compare the computed value of test statistic with the significant value (tabulated value) at the given level of significance α.
  - If the computed value of test statistic (in modulus value) is greater than the significant value (tabulated value) then we reject the null hypothesis at level of significance α and we say that it is significant.
  - If the computed value of test statistic is less than the tabulated value then we accept the null hypothesis at level of significance α and we say that it is not significant.

❖ The p-Value Approach to Hypothesis Testing:

The $p-value$ is the probability of obtaining a test statistic equal to or more extreme than the result obtained from the sample data, given that the null hypothesis H_0 is really true.

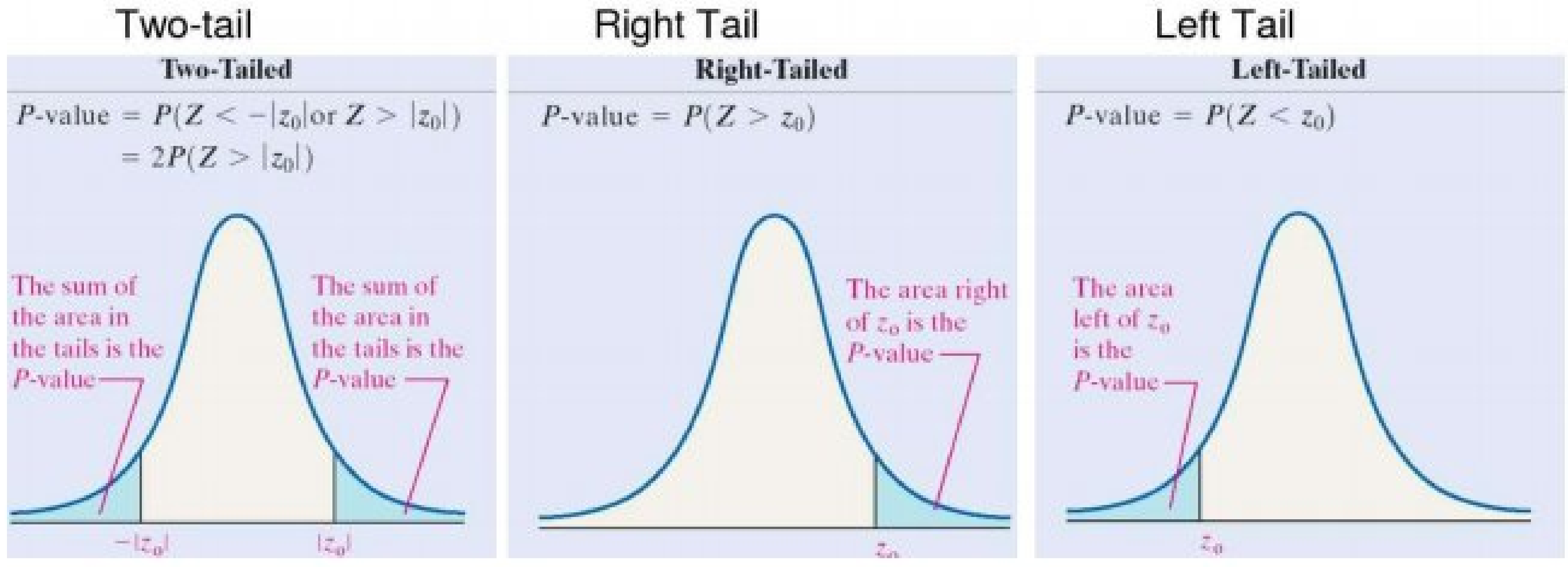
The $p-value$ is often referred to as the observed level of significance, which is the smallest level at which H_0 can be rejected for a given set of data. The decision rule for rejecting H_0 in the $p-value$ approach is:

- If the $p-value$ greater than or equal to α, the null hypothesis is not rejected.
- If the $p-value$ is smaller than α, the null hypothesis is rejected.

P-Value Approach

Assume that the null hypothesis is true.
The P-Value is the probability of observing a sample mean that is as or more extreme than the observed.
How to compute the P-Value for each type of test:
Step 1: Compute the test statistic $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma / \sqrt{n}}$

Two-tail | Right Tail | Left Tail



❖ Steps in Determining the $p-value$

- State the null hypothesis H_0
- State the alternative hypothesis H_1.
- Choose the level of significance α.
- Choose the sample size n.
- Determine the appropriate statistical technique and corresponding test statistic to use.
- Collect the data and compute the sample value of the appropriate test statistic.
- Calculate the $p-value$ based on the test statistic. This involves

→ Sketching the distribution under the null hypothesis H_0.

→ Placing the test statistic on the horizontal axis.

→ Shading in the appropriate area under the curve, on the basis of the alternative hypothesis H_1.

- Compare the $p-value$ to α.
- Make the statistical decision. If the $p-value$ is greater than or equal to α, the null hypothesis is not rejected. If the $p-value$ is smaller than α, the null hypothesis is rejected.
- Express the statistical decision in terms of the particular situation.