CC 112: Discrete Mathematics, Problem Set VI
Session 2017-18

1. A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of *n* pesos if the order in which the coins and bills are paid matters.

2. (a) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and *n* as their last term, where *n* is a positive integer. That is, sequences $a_1, a_2, ..., a_k$, where $a_1 = 1, a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, ..., k-1$.
   (b) What are the initial conditions?
   (c) How many sequences of the type described in (a) are there when n is an integer with $n \geq 2$?

3. (a) Find a recurrence relation for the number of bit strings of length *n* that contain three consecutive 0s.
   (b) What are the initial conditions?
   (c) How many bit strings of length seven contain three consecutive 0s?

4. (a) Find a recurrence relation for the number of bit strings of length n that contain the string 01.
   (b) What are the initial conditions?
   (c) How many bit strings of length seven contain the string 01?

5. (a) Find a recurrence relation for the number of ways to climb *n* stairs if the person climbing the stairs can take one, two, or three stairs at a time.
   (b) What are the initial conditions?
   (c) In many ways can this person climb a flight of eight stairs?

6. A string that contains only 0s, 1s, and 2s is called a **ternary string**.
   (a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive symbols that are the same.
   (b) What are the initial conditions?
   (c) How many ternary strings of length six contain consecutive symbols that are the same?

7. Let *S(m, n)* denote the number of onto functions from a set with *m* elements to a set with *n* elements. Show that *S(m, n)* satisfies the recurrence relation

   $S(m, n) = n^m - \sum_{k=1}^{n-1} C(n, k) S(m, k)$

   whenever $m \geq n$ and $n > 1$, with the initial condition $S(m, 1) = 1$.

8. In theTower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.
   (a) Find a recurrence relation for the number of moves required to solve the puzzle for *n* disks with this added restriction.
   (b) Solve this recurrence relation to find a formula for the number of moves required to solve the puzzle for *n* disks.
   (c) How many different arrangements are there of the *n* disks on three pegs so that no disk is on top of a smaller disk?
   (d) Show that every allowable arrangement of the *n* disks occurs in the solution of this variation of the puzzle.

9. Solve the simultaneous recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + 2b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

10. A binary tree is called full if every internal vertex has either two children or no children. Let H_n denote the number of full binary trees with $n + 1$ leaves. Derive a recurrence equation for H_n with initial conditions.

11. Consider an $n \times n$ grid, consisting of n^2 square cells. Suppose you want to travel from the lower left corner to the upper right corner, where you are allowed to move exactly one cell at a time, either to the right or to the top. Then derive a formula for the total number of valid paths possible, satisfying the above constraints.

12. Draw all the non-isomorphic simple graphs on n vertices for $n = 1, 2, 3$ and 4.

13. Prove or disprove the following: in every simple graph G of 5 vertices, either $K_3 \subseteq G$ or $K_3 \subseteq \overline{G}$.

14. Give an example of a simple connected undirected **non-Hamilton** graph with n nodes and exactly $\frac{(n-1)(n-2)}{2} + 1$ edges; you should give a generic graph for a general n and not for a specific value of n.

15. Prove or disprove the following: if in a simple graph G, every $G - v_i$ is disconnected then it implies that G is also disconnected.

16. Draw the simple graph G, whose incidence matrix B is such that:

$$BB^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

Discuss the idea used in your algorithm for computing G.

17. A tree is a connected graph without any cycles. Prove that a tree with n nodes has $n - 1$ edges.

18. Prove of disprove the following: If $H \subseteq G$, then $\overline{H} \subseteq \overline{G}$.

19. Show that if G is a simple graph with 6 vertices then either $K_3 \subseteq G$ or $K_3 \subseteq \overline{G}$. You should give a generalized argument, and not with respect to a "specific" G with 6 vertices.

20. The degree sequence of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. What is the degree sequence of K_n and $K_{m,n}$?

21. A sequence $d_1, d_2, \ldots d_n$ is called graphic if it is the degree sequence of a simple graph. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

    (a) $5, 4, 3, 2, 1, 0$
    (b) $6, 5, 4, 3, 2, 1$
    (c) $2, 2, 2, 2, 2, 2$

22. Show that a sequence $d_1, d_2, \ldots d_n$ of nonnegative integers in nonincreasing order is a graphic sequence if and only if the sequence obtained by reordering the terms of the sequence $d_2 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ so that the terms are in nonincreasing order is a graphic sequence.

23. A simple graph is called regular if every vertex of this graph has the same degree. A regular graph is called n-regular if every vertex in this graph has degree n. For which values of n are these graphs regular?

    (a) K_n
    (b) C_n
    (c) W_n
    (d) Q_n

24. The complementary graph $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.

(a) K_n
(b) $K_{m,n}$
(c) C_n
(d) Q_n

25. Can you find a simple graph with n vertices with $n \geq 3$ that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n-1)/2$?.

Fleury's algorithm, published in 1883, constructs Euler circuits by first choosing an arbitrary vertex of a connected multigraph, and then forming a circuit by choosing edges successively. Once an edge is chosen, it is removed. Edges are chosen successively so that each edge begins where the last edge ends, and so that this edge is not a cut edge unless there is no alternative.

26. Prove that Fleury's algorithm always produces an Euler circuit.
27. Give a variant of Fleury's algorithm to produce Euler paths.

An **edge coloring** of a graph is an assignment of colors to edges so that edges incident with a common vertex are assigned different colors.

The **edge chromatic number** of a graph is the smallest number of colors that can be used in an edge coloring of the graph. The edge chromatic number of a graph G is denoted by $\chi'(G)$.

28. Show that the edge chromatic number of a graph must be at least as large as the maximum degree of a vertex of the graph.
29. Show that if G is a graph with n vertices, then no more than $n/2$ edges can be colored the same in an edge coloring of G.
30. Find the edge chromatic number of K_n when n is a positive integer.

ALGORITHM 1: First, list the vertices $v_1, v_2, v_3, \ldots, v_n$ in order of decreasing degree so that $deg(v_1) \geq deg(v_2) \geq \cdots \geq deg(v_n)$. Assign color 1 to v_1 and to the next vertex in the list not adjacent to v_1 (if one exists), and successively to each vertex in the list not adjacent to a vertex already assigned color 1. Then assign color 2 to the first vertex in the list not already colored. Successively assign color 2 to vertices in the list that have not already been colored and are not adjacent to vertices assigned color 2. If uncolored vertices remain, assign color 3 to the first vertex in the list not yet colored, and use color 3 to successively color those vertices not already colored and not adjacent to vertices assigned color 3. Continue this process until all vertices are colored.

31. Show that the coloring produced by this algorithm may use more colors than are necessary to color a graph.