OLS Gradient Descent

November 18, 2018

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
```

0.0.1 Utility Objects

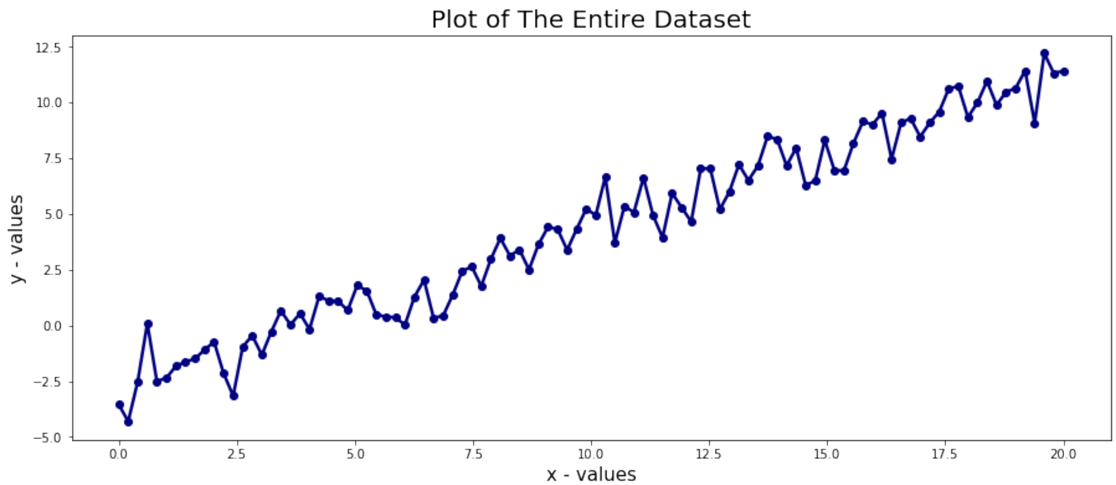
```
In [2]: ACTUAL_COLOR = "navy"
        PREDICTED_COLOR = "firebrick"

        def plot_actual_predicted(X, y_actual, y_pred):
            plt.figure(figsize=(15, 6))
            plt.scatter(
                X, y_actual, marker="o", label="Actual Curve", color=ACTUAL_COLOR, s=30
            )
            plt.plot(X, y_pred, "g-", lw=2.5, label="Fitted Curve", color=PREDICTED_COLOR)
            plt.title("Plot of Actual Points vs. Fitted Curve", fontsize=20)
            plt.xlabel("x - values", fontsize=15)
            plt.ylabel("y - values", fontsize=15)
            plt.legend(loc="upper left")
            plt.show()
```

1 Load Data

```
In [3]: data = np.loadtxt(open("linear_regression_test.csv", "rb"), delimiter=",")
        X = data[:, 0]
        y = data[:, 1]
```

2 Visualization

```
In [4]: plt.figure(figsize=(15, 6))
        plt.plot(X, y, lw=2.5, marker='o', color=ACTUAL_COLOR)
        plt.title("Plot of The Entire Dataset", fontsize=20)
        plt.xlabel("x - values", fontsize=15)
        plt.ylabel("y - values", fontsize=15)
        plt.show()
```



3 Ordinary Linear Regression

Let the linear regression function be:

$$y_i = \beta_0 + \beta_1 x_i \quad ---------\ (1)$$

The estimated parameters are the intercept and slope (beta 0 & 1 respectively). A unit vector (vector of all 1) should be stacked along the X values in order to compute beta 0 along side beta 1.

$$\mathbf{X} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{bmatrix}, \qquad \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad ---------\ (2)$$

```
In [5]: X_stack = np.stack([np.ones(X.shape[0]), X], axis=-1)
```

The estimated parameters are:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y} \quad ---------\ (3)$$

```
In [6]: def estimate_beta(X_stack, y):
            return np.dot(np.dot(np.linalg.inv(np.dot(X_stack.T, X_stack)), X_stack.T), y)
```

Having estimated the weights (betas), y can be predicted by using the equation below:

$$\hat{\boldsymbol{y}} = \boldsymbol{X}\hat{\boldsymbol{\beta}} \;=\; \hat{\beta_0} + \hat{\beta_1} x_i \quad ---------\ (4)$$

```
In [7]: def estimate_y(X_stack, betas_hat):
            return np.dot(X_stack, betas_hat)
```

4 Linear Regression model with Gradient Descent

Gradient descent is an optimization technique for finding the minima (preferably the local minima) of a function (called cost function). While gradient ascent is the opposite.

There are different types of gradient descent, the most common are: - batch gradient descent - stochastic gradient descent - mini-batch gradient descent

In this case, let the cost function be MSE, hence I need to find the minimum MSE:

$$Mean\ Square\ Error\ (MSE) = \frac{1}{N}\sum_i (y_i - \hat{y})^2 \quad ---------\ (5)$$

```
In [8]: def mse(y, y_hat):
            return round(np.sum((y - y_hat)**2) / len(y))
```

Substituting equation (4) into (6):

$$MSE = \frac{1}{N}\sum_i (y_i - \ (\hat{\beta_0} + \hat{\beta_1} x_i))^2 \quad ---------\ (6)$$

Hence, I will derivation the 1st-iterative of MSE with respect to its parameters (beta 0 & 1).

$$\frac{\partial}{\partial \hat{\beta_0}}\ MSE \ = \frac{2}{N}\sum_i -\ (y_i - \ (\hat{\beta_0} + \hat{\beta_1} x_i)) \quad ---------\ (7a)$$

$$\frac{\partial}{\partial \hat{\beta_0}}\ MSE \ = \frac{-2}{N}\sum_i (y_i - \ \hat{y_i}) \quad ---------\ (7b) \quad [using\ equ\ (4)]$$

```
In [9]: def gradient_beta_0(y, y_hat):
            return -2/len(y) * np.sum(y - y_hat)
```

$$\frac{\partial}{\partial \hat{\beta_1}}\ MSE \ = \frac{2}{N}\sum_i -x_i\ (y_i - \ (\hat{\beta_0} + \hat{\beta_1} x_i)) \quad ---------\ (8a)$$

$$\frac{\partial}{\partial \hat{\beta_1}}\ MSE \ = \frac{-2}{N}\sum_i x_i\ (y_i - \ \hat{y_i}) \quad ---------\ (8b) \quad [using\ equ\ (4)]$$

```
In [10]: def gradient_beta_1(X, y, y_hat):
             return -2/len(y) * np.sum(X * (y - y_hat))
```

The objective is to find the best set of betas that will yield the minimal MSE. The initial betas can be set to a very small number:

$$\hat{\beta_0} = 0.01 \quad ---------\ (9)$$

$$\hat{\beta_1} = 0.01 \quad ---------\ (10)$$

Learning rate is a technique used to control the steps taken while descending. A high learning rate implies that it covers more in each step, which can be detrimental to achieving the local minima. Conversely, a low learning rate means declining down the slope gradually.

Set the learning rate to:

$$l_{rate} = 0.005 \quad - - - - - - - - - (11)$$

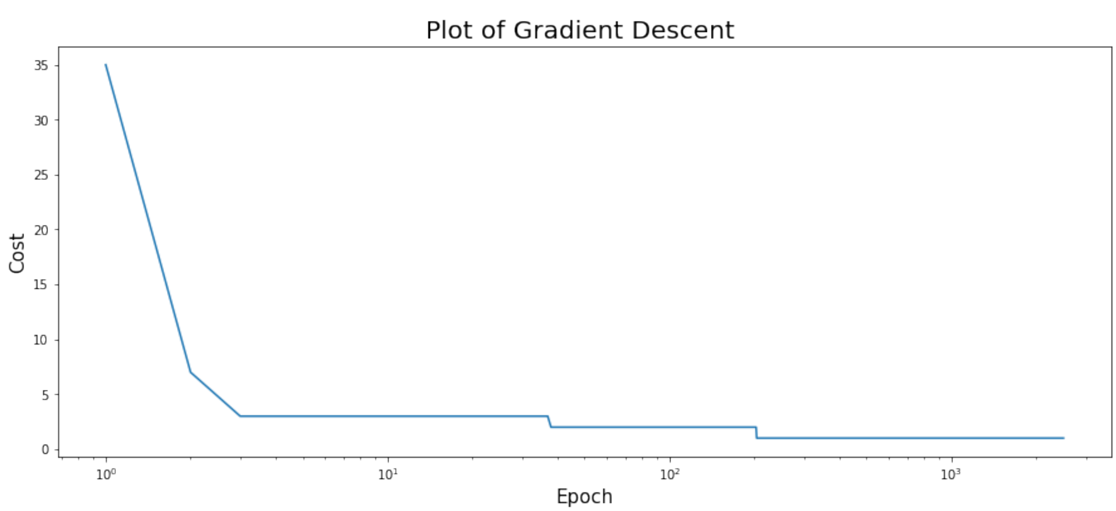
At each step of the gradient descent, we are required to do the following: - use the estimated betas to predict y - compute the cost function (MSE) - update the betas - repeat until the cost function is almost indifferece - or the total number of steps taken down the slope (called epoch). We will take this approach.

```
In [11]: beta_0 = 0.01
         beta_1 = 0.01
         l_rate = 0.005
         epoch = 2500
         costs = [None] * epoch

         for i in range(epoch):
             y_hat = estimate_y(X_stack, np.array([beta_0, beta_1]))
             costs[i] = mse(y, y_hat)
             beta_0 -= l_rate * gradient_beta_0(y, y_hat)
             beta_1 -= l_rate * gradient_beta_1(X, y, y_hat)
```

```
In [12]: print(f'The optimal betas are {round(beta_0, 5)} and {round(beta_1,5)} respectively.')
```

```
The optimal betas are -2.83248 and 0.7128 respectively.
```

```
In [13]: plt.figure(figsize=(15, 6))
         plt.semilogx(list(range(1,  epoch+1)), costs)
         plt.title('Plot of Gradient Descent ', fontsize=20)
         plt.xlabel('Epoch', fontsize=15)
         plt.ylabel('Cost',fontsize=15)
         plt.show()
```

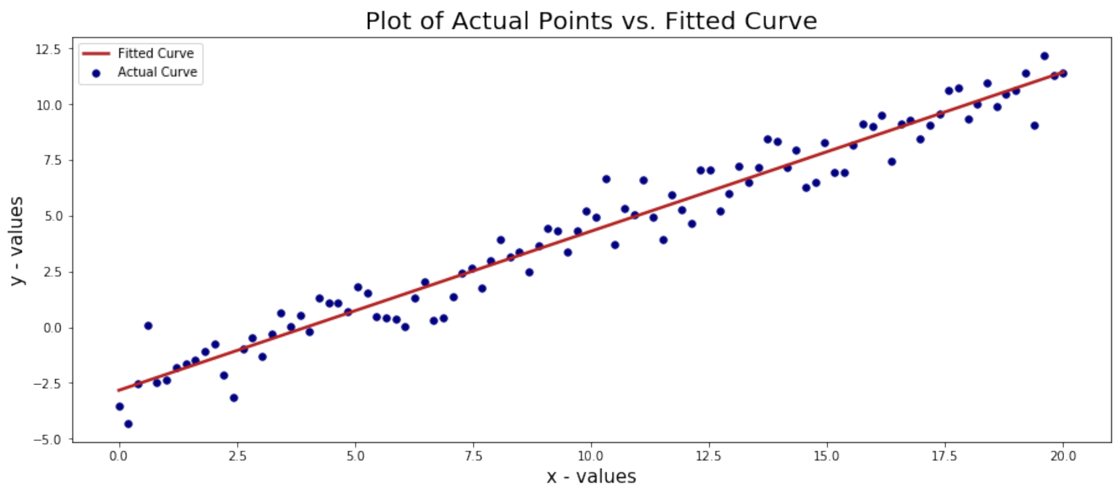


5 Estimating y with the optimal betas

```
In [14]: y_pred_desc = estimate_y(X_stack, np.array([beta_0, beta_1]))
```

6 Visualize the fitted model against the data

```
In [15]: plot_actual_predicted(X, y, y_pred_desc)
```



7 R-squared value

$$R^2 = 1 - \frac{\sum_i (y_i - \hat{y})^2}{\sum_i (y_i - \bar{y})^2} \quad - - - - - - - - - (12)$$

```
In [16]: def r_squared(y, y_hat):
             numerator = np.sum((y - y_hat)**2)
             denominator = np.sum((y - np.average(y))**2)
             return round(1 - numerator / denominator, 6)
```

```
In [17]: print('The R-squared is {}'.format(r_squared(y, y_pred_desc)))
```

```
The R-squared is 0.958823
```