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A PROOF OF SYRACUSE CONJECTURE

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Sequences, number theory, limits, parity, series

Abstract:

In this document, we study the syracuse sequences of all integers from 1 to infinity. Thus we obtain sequences whose limit at infinity are compared to the values (4,2 and 1) in order to verify the conjecture

INTRODUCTION

Of all the currently unsolved mathematical problems, which one has the most basic statement? This may well be the Syracuse conjecture: accessible to all in its statement, it has challenged researchers for decades. The 3n + 1 problem is posed in these terms: let us start from any positive integer, and apply the following transformation to it repeatedly (we speak of a trajectory): if this number is even, we divide it by 2, if the number is odd, we multiply it by three then we add 1, so we get another number. Is it true that sooner or later we will end up with 1? All calculations made to date confirm this prediction.

In this paper we are going to prove the Syracuse conjecture is false.

I. DEFINITIONS

1. Series:

A series is a sum of 2^n integers.

There are three types of series:

- **Heterogeneous series (m):**
  It is a sum of even and odd numbers. It is also an alternation of odd and even numbers.

  $m = \sum_{i=0}^{2^{n-1}} (ai + b)$ where a is an odd number and $b \in \mathrm{N} - \{0\}$

  Example:

  $m = \sum_{i=0}^{2^3-1} (3i + 7)$= 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28

- **Even series (p or t):**
  It's a sum of only even numbers.

  $t = \sum_{i=0}^{2^n-1} (ai + b)$ where a and b are even numbers.

  Example:

  $t = \sum_{i=0}^{2^3-1} (2i + 4)$ = 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18

- **Odd series (r):**
  It is a sum of only odd number

  $r = \sum_{i=0}^{2^3-1} (ai + b)$ where a is an even number and b an odd number.

  Example:

  $r = \sum_{i=0}^{2^n-1} (2i + 5)$ = 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19

2. Line:

A line is a sum of 2^p series where $p \geq 0$

There are four types of line:

- **Even line (P or T):**

  It is a sum of 2^p even series

  $$P = \sum_{p=1}^{2^P} \left(\sum_{i=0}^{2^n - 1} \left(a_p i + b_p \right) \right) \quad \text{or} \quad T = \sum_{t=1}^{2^P} \left(\sum_{i=0}^{2^n - 1} \left(a_t i + b_t \right) \right)$$

- **Odd line (R):**

  It is a sum of 2^p odd series.

  $$R = \sum_{r=1}^{2^P} \left(\sum_{i=0}^{2^n - 1} \left(a_r i + b_r \right) \right)$$

- **Homogeneous line (H):**

  It's a sum of 2^{p-1} even series and 2^{p-1} odd series.

  $$H = T + R$$

- **Heterogeneous line (M):**

  It's a sum of 2^p heterogeneous series.

  $$M = \sum_{m=1}^{2^P} \left(\sum_{i=0}^{2^n - 1} \left(a_m i + b_m \right) \right)$$

II. FUNCTIONS:

1. The separation function H:

The separation function H also called the to-homogeneous function is a sum of two functions: the left separation function H_l and the right separation function H_r

If $m=\sum_{i=0}^{2^n-1}(U_i)=\sum_{i=0}^{2^n-1}(ai+b)$ is the heterogeneous series, $H(m)$ gives two series: on odd series and another even series. The results are called homogeneous series.

$$H(m)=H_l(m)+H_r(m)$$

$$H_l(m)=\sum_{i=0}^{2^{n-1}-1}(U_{2i})=\sum_{i=0}^{2^{n-1}-1}(2ai+b)$$

$$H_r(m)=\sum_{i=0}^{2^{n-1}-1}(U_{2i+1})=\sum_{i=0}^{2^{n-1}-1}(a(2i+1)+b)=\sum_{i=0}^{2^{n-1}-1}(2ai+a+b)$$

So $H(m)=\sum_{i=0}^{2^{n-1}-1}(2ai+b)+\sum_{i=0}^{2^{n-1}-1}(2ai+a+b)$

NB:
If $H_l(m)$ l is odd then m is said to be odd-left or even-right heterogeneous series.
If $H_r(m)$ is odd then m is said to be odd-right or even-left heterogeneous series.

Odd-left and odd-right heterogeneous line?

If we apply the separation function to a heterogeneous line, we find an odd line and an even line. So if the odd series comes from the left-separation function applied to the heterogeneous series then this last one is said odd-left heterogeneous, else it's said odd-right. If the even series comes from the left-separation function applied to the heterogeneous series, then this last one is said to be odd-right.

If $M=\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^n-1}(a_m i+b_m)\right)$ is an heterogeneous line, it can be written as follow:

$$M=\sum_{x=1}^{X}\left(\sum_{i=0}^{2^n-1}(a_x i+b_x)\right)+\sum_{y=1}^{Y}\left(\sum_{i=0}^{2^n-1}(a_y i+b_y)\right)$$

Where $X+Y=2^p$

$\overset{x}{M}=\sum_{x=1}^{X}\left(\sum_{i=0}^{2^n-1}(a_x i+b_x)\right)$ is called the odd-left or even-right heterogeneous line

$\overset{y}{M}=\sum_{y=1}^{Y}\left(\sum_{i=0}^{2^n-1}(a_y i+b_y)\right)$ is called the odd-right or even-left heterogeneous line

$$M=\overset{x}{M}+\overset{y}{M}$$

2. The To-Even function E:

The to-Even function E receives in entry an odd series r or an even series t then results in an even series p .

If $r=\sum_{i=0}^{2^n-1}(U_i)=\sum_{i=0}^{2^n-1}(ai+b)$ is an odd series, we have :

$$E(r)=E\left(\sum_{i=0}^{2^n-1}(3U_i+1)\right)=\sum_{i=0}^{2^n-1}(3ai+3b+1)$$

NB: The even series doesn't change if we pass them to the function.

3. The To-Heterogeneous function H_e:

The to-heterogeneous function H_e transforms an even series p into an heterogeneous series m .

Given $p=\sum_{i=0}^{2^n-1}(U_i)=\sum_{i=0}^{2^n-1}(ai+b)$ an even series.

$$H_e(p)=H_e\left(\sum_{i=0}^{2^n-1}(U_i)\right)=\sum_{i=0}^{2^n-1}\left(\frac{U_i}{2}\right)=\sum_{i=0}^{2^n-1}\left(\frac{a}{2}i+\frac{b}{2}\right)$$

III. Bloc and Pyramid:

1. Bloc:

A bloc B_p is a succession of three lines : it's composed by one heterogeneous line M_p followed by an homogeneous line H_p then an even line P_p that such $H_p = H(M_p) = T_p + R_p$, $P_p = E(H_p) = E(R_p) + T_p$ and $M_{p+1} = H_e(P_p)$.

A bloc B_p is characterized by its index p

2. Pyramid:

A pyramid S_{n-1} is a succession of n blocs (B_p) that such $H_e(P_p) = M_{p+1}$.

The pyramid S_{n-1} which begin with the heterogeneous line M_0 is $S_{n-1}(M_0) = (B_0, B_1, B_2, \ldots, B_{n-1})$. This pyramid ends with P_{n-1} .

3. Construction of the pyramid $S_1\left(\sum_{i=0}^{2^n-1}(i+1)\right)$

$M_0 = \sum_{i=0}^{2^n-1}(i+1)$ is the first heterogeneous line.

- **Determination of the first homogeneous line H_0**

$$H_0 = H(M_0) = H\left(\sum_{i=0}^{2^n-1}(i+1)\right) = H_l\left(\sum_{i=0}^{2^n-1}(i+1)\right) + H_r\left(\sum_{i=0}^{2^n-1}(i+1)\right)$$

$$H_l\left(\sum_{i=0}^{2^n-1}(i+1)\right) = \sum_{i=0}^{2^{n-1}-1}(2i+1)$$

$$H_r\left(\sum_{i=0}^{2^n-1}(i+1)\right) = \sum_{i=0}^{2^{n-1}-1}(2i+2)$$

$$H_0 = \sum_{i=0}^{2^{n-1}-1}(2i+1) + \sum_{i=0}^{2^{n-1}-1}(2i+2)$$

Where $T_0 = \sum_{i=0}^{2^{n-1}-1}(2i+2)$ and $R_0 = \sum_{i=0}^{2^{n-1}-1}(2i+1)$

$$H_0 = T_0 + R_0$$

- **Determination of the first even line** P_0

$$P_0 = E(H_0) = E(T_0 + R_0) = T_0 + E(R_0)$$

NB: If we apply the to-Even function to an even line, it doesn't change (here we have T_0 doesn't change)

$$E(R_0) = E\left(\sum_{i=0}^{2^{n-1}-1}(2i+1)\right) = \sum_{i=0}^{2^{n-1}-1}(3\times(2i+1)+1) = \sum_{i=0}^{2^{n-1}-1}(6i+4)$$

$$P_0 = \sum_{i=0}^{2^{n-1}-1}(2i+2) + \sum_{i=0}^{2^{n-1}-1}(6i+4)$$

So $B_0 = (M_0, H_0, P_0)$ is the first bloc.

- **Determination of the second heterogeneous line** M_1

$$M_1 = H_e(P_0) = H_e\left(\sum_{i=0}^{2^{n-1}-1}(2i+2) + \sum_{i=0}^{2^{n-1}-1}(6i+4)\right) = H_e\left(\sum_{i=0}^{2^{n-1}-1}(2i+2)\right) + H_e\left(\sum_{i=0}^{2^{n-1}-1}(6i+4)\right)$$

$$M_1 = \sum_{i=0}^{2^{n-1}-1}(i+1) + \sum_{i=0}^{2^{n-1}-1}(3i+2)$$

- **Determination of the second homogeneous line** H_1

$$H_1 = H(M_1) = H\left(\sum_{i=0}^{2^{n-1}-1}(i+1) + \sum_{i=0}^{2^{n-1}-1}(3i+2)\right) = H\left(\sum_{i=0}^{2^{n-1}-1}(i+1)\right) + H\left(\sum_{i=0}^{2^{n-1}-1}(3i+2)\right)$$

$$H_1 = H_l\left(\sum_{i=0}^{2^{n-1}-1}(i+1)\right) + H_r\left(\sum_{i=0}^{2^{n-1}-1}(i+1)\right) + H_l\left(\sum_{i=0}^{2^{n-1}-1}(3i+2)\right) + H_r\left(\sum_{i=0}^{2^{n-1}-1}(3i+2)\right)$$

$$H_1 = \sum_{i=0}^{2^{n-2}-1}(2i+1) + \sum_{i=0}^{2^{n-2}-1}(2i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+5)$$

With $T_1 = \sum_{i=0}^{2^{n-2}-1}(2i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+2)$ and $R_1 = \sum_{i=0}^{2^{n-2}-1}(2i+1) + \sum_{i=0}^{2^{n-2}-1}(6i+5)$

- **Determination of the second even line** P_1

$$P_1 = E(H_1) = T_1 + E(R_1)$$

$$P_1 = \sum_{i=0}^{2^{n-2}-1}(2i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+2) + E\left(\sum_{i=0}^{2^{n-2}-1}(2i+1) + \sum_{i=0}^{2^{n-2}-1}(6i+5)\right)$$

$$P_1 = \sum_{i=0}^{2^{n-2}-1}(2i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+2) + E\left(\sum_{i=0}^{2^{n-2}-1}(2i+1)\right) + E\left(\sum_{i=0}^{2^{n-2}-1}(6i+5)\right)$$

$$P_1 = \sum_{i=0}^{2^{n-2}-1}(2i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+4) + \sum_{i=0}^{2^{n-2}-1}(18i+16)$$

So $B_1 = (M_1, H_1, P_1)$ is the second bloc of the pyramid.

We just give the determination of the pyramid $S_1\left(\sum_{i=0}^{2^n-1}(i+1)\right) = ((M_0, H_0, P_0), (M_1, H_1, P_1))$

We can see that: $M_p = \sum_{m=1}^{2^p}\left(\sum_{i=0}^{2^{n-p}-1}(a_m i + b_m)\right)$

$$H_p = \sum_{h=1}^{2^{p+1}}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_h i + b_h)\right)$$

$$P_p = \sum_{s=1}^{2^{p+1}}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_s i + b_s)\right)$$

NB : $S_\infty\left(\sum_{i=0}^{2^n-1}(i+1)\right)$ is the infinite pyramid and its last lines $(M_\infty, H_\infty, P_\infty)$ will give us the result we are looking for.

IV. CALCULATIONS USING H_p :

If we consider that all numbers from 1 to infinity are going to reach the loop (4,2,1) where and p tend to $n-1$ and n to infinity:

T_p will be defined by: $(2^{n-1})\times 2 \leq T_p \leq (2^{n-1})\times 4$
R_p will be equal to $(2^{n-1})\times 1$

The difference between T_p and R_p will be defined by :

$$(2^{n-1})\times(2-1)\leq T_p - R_p \leq (2^{n-1})\times(4-1)$$
$$(2^{n-1})\leq T_p - R_p \leq (2^{n-1})\times 3$$

Since we have 2^{n-1} integers in this sum, the average of the framing is :

$$1 \leq \frac{T_p - R_p}{2^{n-1}} \leq 3$$

a)

Given an heterogeneous line M_p

$$M_p = \sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p}-1}(a_m i + b_m)\right)$$

- Finding the homogeneous line H_p

$$H_p = H(M_p) = H_l(M_p) + H_r(M_p)$$

$$H_p = H_l\left(\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p}-1}(a_m i + b_m)\right)\right) + H_r\left(\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p}-1}(a_m i + b_m)\right)\right)$$

$$H_p = \sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_m i + b_m)\right) + \sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_m i + a_m + b_m)\right)$$

If $H_p = \sum_{i=0}^{2^{n-p-1}-1}(A_p i + B_p)$ then $A_p = \sum_{m=1}^{2^p}(2a_m) + \sum_{m=1}^{2^p}(2a_m)$

$$A_p = \sum_{m=1}^{2^p}(4a_m)$$

NB: After the separation H_p can also be written as follow :

$$H_p = \sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t)\right) + \sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_r i + b_r)\right)$$

Where we distinguish odd and even series.

This means : $\sum_{t=1}^{2^p}(a_t) = \sum_{r=1}^{2^p}(a_r) = \sum_{m=1}^{2^p}(2a_m)$

b)

Given $D(H_p)$ the absolute values of the differences between even suites and odd suites which come from the same heterogeneous suite in the line H_p.

So if $M_p = \sum_{i=0}^{2^{n-p}-1}(a_1 i + b_1) + \sum_{i=0}^{2^{n-p}-1}(a_2 i + b_2) + \ldots\ldots + \sum_{i=0}^{2^{n-p}-1}(a_p i + b_p)$

$$\begin{aligned}H_p &= H_l\left(\sum_{i=0}^{2^{n-p}-1}(a_1 i + b_1)\right) + H_r\left(\sum_{i=0}^{2^{n-p}-1}(a_1 i + b_1)\right)\\ &+ H_l\left(\sum_{i=0}^{2^{n-p}-1}(a_2 i + b_2)\right) + H_r\left(\sum_{i=0}^{2^{n-p}-1}(a_1 i + b_1)\right)\\ &+ \ldots\ldots\\ &+ H_l\left(\sum_{i=0}^{2^{n-p}-1}(a_p i + b_p)\right) + H_r\left(\sum_{i=0}^{2^{n-p}-1}(a_p i + b_p)\right)\end{aligned}$$

When we distinguish odd and even series, we have:

$$
\begin{aligned}
H_p = &\sum_{i=0}^{2^{n-p-1}-1}(a_{r1}i + b_{r1}) + \sum_{i=0}^{2^{n-p-1}-1}(a_{t1}i + b_{t1}) \\
&+ \sum_{i=0}^{2^{n-p-1}-1}(a_{r2}i + b_{r2}) + \sum_{i=0}^{2^{n-p-1}-1}(a_{t2}i + b_{t2}) \\
&+ \ldots\ldots \\
&+ \sum_{i=0}^{2^{n-p-1}-1}(a_{rp}i + b_{rp}) + \sum_{i=0}^{2^{n-p-1}-1}(a_{tp}i + b_{tp})
\end{aligned}
$$

So : $D(H_p) = \left|\sum_{i=0}^{2^{n-p-1}-1}(a_{r1}i + b_{r1}) - \sum_{i=0}^{2^{n-p-1}-1}(a_{t1}i + b_{t1})\right|$

$$
\begin{aligned}
&+ \left|\sum_{i=0}^{2^{n-p-1}-1}(a_{r2}i + b_{r2}) - \sum_{i=0}^{2^{n-p-1}-1}(a_{t2}i + b_{t2})\right| \\
&+ \ldots\ldots \\
&+ \left|\sum_{i=0}^{2^{n-p-1}-1}(a_{rp}i + b_{rp}) - \sum_{i=0}^{2^{n-p-1}-1}(a_{tp}i + b_{tp})\right|
\end{aligned}
$$

Since $H_r\left(\sum_{i=0}^{2^{n-p}-1}(ai + b)\right) = \sum_{i=0}^{2^{n-p}-1}(2ai + a + b)$ and

$H_l\left(\sum_{i=0}^{2^{n-p}-1}(ai + b)\right) = \sum_{i=0}^{2^{n-p}-1}(2ai + b)$

$\Rightarrow H_r \geq H_l$

As a result : $D(H_p) = H_r(M_p) - H_l(M_p)$

$$D(H_p)=H_r\left(\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p}-1}(a_m i+b_m)\right)\right)-H_l\left(\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p}-1}(a_m i+b_m)\right)\right)$$

$$D(H_p)=\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_m i+a_m+b_m)\right)-\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_m i+b_m)\right)$$

$$D(H_p)=\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_m)\right)=\frac{1}{4}\sum_{m=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(4a_m)\right)=\frac{1}{4}\sum_{i=0}^{2^{n-p-1}-1}(A_p)$$

$$D(H_p)=\frac{1}{4}\times\frac{2^n}{2^{p+1}}A_p \qquad \rightarrow\rightarrow(1)$$

- Finding the even line P_p

$$T_p=\sum_{t=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_t i+b_t)\right) \quad \text{and} \quad R_p=\sum_{r=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_r i+b_r)\right)$$

$$P_p=T_p+E(R_p)=\sum_{t=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_t i+b_t)\right)+E\left(\sum_{r=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_r i+b_r)\right)\right)$$

$$P_p=\sum_{t=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_t i+b_t)\right)+\sum_{r=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(3a_r i+3b_r+1)\right)$$

- Finding M_{p+1}

$$M_{p+1}=H_e(P_p)=H_e\left(\sum_{t=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_t i+b_t)\right)+\sum_{r=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}(3a_r i+3b_r+1)\right)\right)$$

$$M_{p+1}=\sum_{t=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}\left(\frac{a_t}{2}i+\frac{b_t}{2}\right)\right)+\sum_{r=1}^{2^P}\left(\sum_{i=0}^{2^{n-p-1}-1}\left(\frac{3}{2}a_r i+\frac{3}{2}b_r+\frac{1}{2}\right)\right)$$

- Finding H_{p+1}

Before finding H_{p+1} we must separate odd-left and odd-right lines in M_{p+1}

$$\overset{x}{M}_{p+1} = \sum_{e=1}^{E}\left(\sum_{i=0}^{2^{n-p-1}-1}\left(\frac{a_e}{2}i+\frac{b_e}{2}\right)\right)+\sum_{g=1}^{G}\left(\sum_{i=0}^{2^{n-p-1}-1}\left(\frac{3}{2}a_g i+\frac{3}{2}b_g+\frac{1}{2}\right)\right)$$ is the odd-left line from M_{p+1}

$$\overset{y}{M}_{p+1} = \sum_{f=1}^{F}\left(\sum_{i=0}^{2^{n-p-1}-1}\left(\frac{a_f}{2}i+\frac{b_f}{2}\right)\right)+\sum_{h=1}^{H}\left(\sum_{i=0}^{2^{n-p-1}-1}\left(\frac{3}{2}a_h i+\frac{3}{2}b_h+\frac{1}{2}\right)\right)$$ is the odd-right line from M_{p+1}

Where $E+F=G+H=2^p$

$$H_{p+1} = H(M_{p+1}) = H\left(\overset{x}{M}_{p+1}+\overset{y}{M}_{p+1}\right) = H\left(\overset{x}{M}_{p+1}\right)+H\left(\overset{y}{M}_{p+1}\right)$$

$$H_{p+1} = H_l\left(\overset{x}{M}_{p+1}\right)+H_l\left(\overset{y}{M}_{p+1}\right)+H_r\left(\overset{x}{M}_{p+1}\right)+H_r\left(\overset{y}{M}_{p+1}\right)$$

$$\begin{aligned} H_{p+1} = {} & \sum_{e=1}^{E}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_e i+\frac{b_e}{2}\right)\right)+\sum_{g=1}^{G}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_g i+\frac{3}{2}b_g+\frac{1}{2}\right)\right) \\ & +\sum_{f=1}^{F}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_f i+\frac{b_f}{2}\right)\right)+\sum_{h=1}^{H}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_h i+\frac{3}{2}b_h+\frac{1}{2}\right)\right) \\ & +\sum_{e=1}^{E}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_e i+\frac{a_e}{2}+\frac{b_e}{2}\right)\right)+\sum_{g=1}^{G}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_g i+3\frac{a_g}{2}+\frac{3}{2}b_g+\frac{1}{2}\right)\right) \\ & +\sum_{f=1}^{F}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_f i+\frac{a_f}{2}+\frac{b_f}{2}\right)\right)+\sum_{h=1}^{H}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_h i+3\frac{a_h}{2}+\frac{3}{2}b_h+\frac{1}{2}\right)\right) \end{aligned}$$

Let's find $D(H_{p+1})$ by Analogy to $D(H_p)$

$$D(H_{p+1}) = H_r(M_{p+1}) - H_l(M_{p+1})$$

$$D(H_{p+1}) = \sum_{e=1}^{E}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_e i+\frac{a_e}{2}+\frac{b_e}{2}\right)\right)+\sum_{g=1}^{G}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_g i+3\frac{a_g}{2}+\frac{3}{2}b_g+\frac{1}{2}\right)\right)$$

$$+\sum_{f=1}^{F}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_f i+\frac{a_f}{2}+\frac{b_f}{2}\right)\right)+\sum_{h=1}^{H}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_h i+3\frac{a_h}{2}+\frac{3}{2}b_h+\frac{1}{2}\right)\right)$$
$$-\sum_{e=1}^{E}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_e i+\frac{b_e}{2}\right)\right)-\sum_{g=1}^{G}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_g i+\frac{3}{2}b_g+\frac{1}{2}\right)\right)$$
$$-\sum_{f=1}^{F}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(a_f i+\frac{b_f}{2}\right)\right)-\sum_{h=1}^{H}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3a_h i+\frac{3}{2}b_h+\frac{1}{2}\right)\right)$$

$$D(H_{p+1})=\sum_{e=1}^{E}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(\frac{a_e}{2}\right)\right)+\sum_{g=1}^{G}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3\frac{a_g}{2}\right)\right)$$
$$+\sum_{f=1}^{F}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(\frac{a_f}{2}\right)\right)+\sum_{h=1}^{H}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3\frac{a_h}{2}\right)\right)$$

According to the separation of M_{p+1}

$$D(H_{p+1})=\sum_{t=1}^{2^p}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(\frac{a_t}{2}\right)\right)+\sum_{r=1}^{2^p}\left(\sum_{i=0}^{2^{n-p-2}-1}\left(3\frac{a_r}{2}\right)\right)$$
$$=\sum_{i=0}^{2^{n-p-2}-1}\left(\frac{1}{2}\sum_{t=1}^{2^p}(a_t)\right)+\sum_{i=0}^{2^{n-p-2}-1}\left(\frac{3}{2}\sum_{r=1}^{2^p}(a_r)\right)$$
$$=\sum_{i=0}^{2^{n-p-2}-1}\left(\frac{1}{2}\sum_{m=1}^{2^p}(2a_m)\right)+\sum_{i=0}^{2^{n-p-2}-1}\left(\frac{3}{2}\sum_{m=1}^{2^p}(2a_m)\right)$$
$$=\sum_{i=0}^{2^{n-p-2}-1}\left(\sum_{m=1}^{2^p}(4a_m)\right)=\sum_{i=0}^{2^{n-p-2}-1}(A_p)$$

$$D(H_{p+1})=\frac{2^n}{2^{p+2}}A_p \quad \rightarrow\rightarrow (2)$$

c) Finding the general form of $D(H_{p+1})$

from (1) and (2) we have:

$D(H_{p+1})=\frac{2^n}{2^{p+2}}A_p$ and $D(H_p)=\frac{1}{4}\times\frac{2^n}{2^{p+1}}A_p$

Then $D(H_p)$ is a geometric suite where the first term is $D(H_0)$ and the common ratio is 2.

- Let's find $D(H_0)$

  In the first bloc of de pyramid $S_{n-1}\left(\sum_{i=0}^{2^n-1}(i+1)\right)$,

$$H_0 = \sum_{i=0}^{2^{n-1}-1}(2i+1) + \sum_{i=0}^{2^{n-1}-1}(2i+2)$$

$$D(H_0) = \sum_{i=0}^{2^{n-1}-1}(1) = 2^{n-1}$$

$$Avg(D(H_0)) = \frac{2^{n-1}}{2^{n-1}} = 1$$

The general form is : $D(H_p) = 2^p$

In the last line, the index p reach $n-1$, so

$D(H_{n-1}) = 2^{n-1}$

$Avg(D(H_{n-1})) = 1$

$\lim_{n\to\infty} Avg(D(H_{n-1})) = \lim_{n\to\infty} 1 = 1$

We know that: $1 \le \frac{T_p - R_p}{2^{n-1}} \le 3$

$$1 \le \frac{|T_p - R_p|}{2^{n-1}} \le 3$$

We are expecting: $1 \le Avg(D(H_{n-1})) \le 3$

This value of the limit verify the framing $1 \le Avg(D(H_{n-1})) \le 3$

$$1 \le 1 \le 3 \text{ true}$$

It means that the Syracuse conjecture is true.

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