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# A PROOF OF SYRACUSE CONJECTURE

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Sequences, number theory, parity, series

Abstract:

In this document, we study the syracuse sequences of all integers from 1 to infinity. Thus we obtain sequences whose limit at infinity are compared to the values (4,2 and 1) in order to verify the conjecture

## INTRODUCTION

Of all the currently unsolved mathematical problems, which one has the most basic statement? This may well be the Syracuse conjecture: accessible to all in its statement, it has challenged researchers for decades. The  $3n + 1$  problem is posed in these terms: let us start from any positive integer, and apply the following transformation to it repeatedly (we speak of a trajectory): if this number is even, we divide it by 2, if the number is odd, we multiply it by three then we add 1, so we get another number. Is it true that sooner or later we will end up with 1? All calculations made to date confirm this prediction.

In this paper we are going to prove the Syracuse conjecture is true.

# I. DEFINITIONS

## 1. Series:

A series is a sum of  $2^n$  integers.

There are three types of series:

- **Heterogeneous series (m):**

It is a sum of even and odd numbers. It is also an alternation of odd and even numbers.

$$m = \sum_{i=0}^{2^n-1} (ai + b) \quad \text{where } a \text{ is an odd number and } b \in \mathbb{N} - \{0\}$$

Example:

$$m = \sum_{i=0}^{2^3-1} (3i + 7) = 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28$$

- **Even series (p or t):**

It's a sum of only even numbers.

$$t = \sum_{i=0}^{2^n-1} (ai + b) \quad \text{where } a \text{ and } b \text{ are even numbers.}$$

Example:

$$t = \sum_{i=0}^{2^3-1} (2i + 4) = 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18$$

- **Odd series (r):**

It is a sum of only odd number

$$r = \sum_{i=0}^{2^n-1} (ai + b) \quad \text{where } a \text{ is an even number and } b \text{ an odd number.}$$

Example:

$$r = \sum_{i=0}^{2^n-1} (2i + 5) = 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$$

## 2. Line:

A line is a sum of  $2^p$  series where  $p \geq 0$

There are four types of line:

- **Even line (P or T):**

It is a sum of  $2^p$  even series

$$P = \sum_{p=1}^{2^p} \left( \sum_{i=0}^{2^n-1} (a_p i + b_p) \right) \quad \text{or} \quad T = \sum_{t=1}^{2^p} \left( \sum_{i=0}^{2^n-1} (a_t i + b_t) \right)$$

- **Odd line (R):**

It is a sum of  $2^p$  odd series.

$$R = \sum_{r=1}^{2^p} \left( \sum_{i=0}^{2^n-1} (a_r i + b_r) \right)$$

- **Homogeneous line (H):**

It's a sum of  $2^{p-1}$  even series and  $2^{p-1}$  odd series.

$$H = T + R$$

- **Heterogeneous line (M):**

It's a sum of  $2^p$  heterogeneous series.

$$M = \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^n-1} (a_m i + b_m) \right)$$

## II. FUNCTIONS:

### 1. The separation function $H$ :

The separation function  $H$  also called the to-homogeneous function is a sum of two functions: the left separation function  $H_l$  and the right separation function  $H_r$

If  $m = \sum_{i=0}^{2^n-1} (U_i) = \sum_{i=0}^{2^n-1} (ai + b)$  is the heterogeneous series,  $H(m)$  gives two series: on odd series and another even series. The results are called homogeneous series.

$$H(m) = H_l(m) + H_r(m)$$

$$H_l(m) = \sum_{i=0}^{2^{n-1}-1} (U_{2i}) = \sum_{i=0}^{2^{n-1}-1} (2ai + b)$$

$$H_r(m) = \sum_{i=0}^{2^{n-1}-1} (U_{2i+1}) = \sum_{i=0}^{2^{n-1}-1} (a(2i+1) + b) = \sum_{i=0}^{2^{n-1}-1} (2ai + a + b)$$

$$\text{So } H(m) = \sum_{i=0}^{2^{n-1}-1} (2ai + b) + \sum_{i=0}^{2^{n-1}-1} (2ai + a + b)$$

**NB:**

If  $H_l(m)$  is odd then  $m$  is said to be odd-left or even-right heterogeneous series.

If  $H_r(m)$  is odd then  $m$  is said to be odd-right or even-left heterogeneous series.

### Odd-left and odd-right heterogeneous line?

If we apply the separation function to a heterogeneous line, we find an odd line and an even line. So if the odd series comes from the left-separation function applied to the heterogeneous series then this last one is said odd-left heterogeneous, else it's said odd-right. If the even series comes from the left-separation function applied to the heterogeneous series, then this last one is said to be odd-right.

If  $M = \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^n-1} (a_m i + b_m) \right)$  is an heterogeneous line, it can be written as follow:

$$M = \sum_{x=1}^X \left( \sum_{i=0}^{2^n-1} (a_x i + b_x) \right) + \sum_{y=1}^Y \left( \sum_{i=0}^{2^n-1} (a_y i + b_y) \right)$$

Where  $X + Y = 2^p$

$M^x = \sum_{x=1}^X \left( \sum_{i=0}^{2^n-1} (a_x i + b_x) \right)$  is called the odd-left or even-right heterogeneous line

$M^y = \sum_{y=1}^Y \left( \sum_{i=0}^{2^n-1} (a_y i + b_y) \right)$  is called the odd-right or even-left heterogeneous line

$$M = M^x + M^y$$

## 2. The To-Even function $E$ :

The to-Even function  $E$  receives in entry an odd series  $r$  or an even series  $t$  then results in an even series  $p$ .

If  $r = \sum_{i=0}^{2^n-1} (U_i) = \sum_{i=0}^{2^n-1} (ai + b)$  is an odd series, we have :

$$E(r) = E \left( \sum_{i=0}^{2^n-1} (3U_i + 1) \right) = \sum_{i=0}^{2^n-1} (3ai + 3b + 1)$$

**NB:** The even series doesn't change if we pass them to the function.

## 3. The To-Heterogeneous function $H_e$ :

The to-heterogeneous function  $H_e$  transforms an even series  $p$  into an heterogeneous series  $m$ .

Given  $p = \sum_{i=0}^{2^n-1} (U_i) = \sum_{i=0}^{2^n-1} (ai + b)$  an even series.

$$H_e(p) = H_e \left( \sum_{i=0}^{2^n-1} (U_i) \right) = \sum_{i=0}^{2^n-1} \left( \frac{U_i}{2} \right) = \sum_{i=0}^{2^n-1} \left( \frac{a}{2} i + \frac{b}{2} \right)$$

## III. Bloc and Pyramid:

## 1. Bloc:

A bloc  $B_p$  is a succession of three lines : it's composed by one heterogeneous line  $M_p$  followed by an homogeneous line  $H_p$  then an even line  $P_p$  that such  $H_p = H(M_p) = T_p + R_p$  ,  $P_p = E(H_p) = E(R_p) + T_p$  and  $M_{p+1} = H_e(P_p)$ .

A bloc  $B_p$  is characterized by its index  $p$

## 2. Pyramid:

A pyramid  $S_{n-1}$  is a succession of  $n$  blocs ( $B_p$ ) that such  $H_e(P_p) = M_{p+1}$  .

The pyramid  $S_{n-1}$  which begin with the heterogeneous line  $M_0$  is  $S_{n-1}(M_0) = (B_0, B_1, B_2, \dots, B_{n-1})$ . This pyramid ends with  $P_{n-1}$  .

## 3. Construction of the pyramid $S_1 \left( \sum_{i=0}^{2^n-1} (i+1) \right)$

$M_0 = \sum_{i=0}^{2^n-1} (i+1)$  is the first heterogeneous line.

- **Determination of the first homogeneous line  $H_0$**

$$H_0 = H(M_0) = H \left( \sum_{i=0}^{2^n-1} (i+1) \right) = H_l \left( \sum_{i=0}^{2^n-1} (i+1) \right) + H_r \left( \sum_{i=0}^{2^n-1} (i+1) \right)$$

$$H_l \left( \sum_{i=0}^{2^n-1} (i+1) \right) = \sum_{i=0}^{2^{n-1}-1} (2i+1)$$

$$H_r \left( \sum_{i=0}^{2^n-1} (i+1) \right) = \sum_{i=0}^{2^{n-1}-1} (2i+2)$$

$$H_0 = \sum_{i=0}^{2^{n-1}-1} (2i+1) + \sum_{i=0}^{2^{n-1}-1} (2i+2)$$

$$\text{Where } T_0 = \sum_{i=0}^{2^{n-1}-1} (2i+2) \quad \text{and} \quad R_0 = \sum_{i=0}^{2^{n-1}-1} (2i+1)$$

$$H_0 = T_0 + R_0$$

- **Determination of the first even line  $P_0$**

$$P_0 = E(H_0) = E(T_0 + R_0) = T_0 + E(R_0)$$

**NB:** If we apply the to-Even function to an even line, it doesn't change (here we have  $T_0$  doesn't change)

$$E(R_0) = E\left(\sum_{i=0}^{2^{n-1}-1} (2i+1)\right) = \sum_{i=0}^{2^{n-1}-1} (3 \times (2i+1) + 1) = \sum_{i=0}^{2^{n-1}-1} (6i+4)$$

$$P_0 = \sum_{i=0}^{2^{n-1}-1} (2i+2) + \sum_{i=0}^{2^{n-1}-1} (6i+4)$$

So  $B_0 = (M_0, H_0, P_0)$  is the first bloc.

- **Determination of the second heterogeneous line  $M_1$**

$$M_1 = H_e(P_0) = H_e\left(\sum_{i=0}^{2^{n-1}-1} (2i+2) + \sum_{i=0}^{2^{n-1}-1} (6i+4)\right) = H_e\left(\sum_{i=0}^{2^{n-1}-1} (2i+2)\right) + H_e\left(\sum_{i=0}^{2^{n-1}-1} (6i+4)\right)$$

$$M_1 = \sum_{i=0}^{2^{n-1}-1} (i+1) + \sum_{i=0}^{2^{n-1}-1} (3i+2)$$

- **Determination of the second homogeneous line  $H_1$**

$$H_1 = H(M_1) = H\left(\sum_{i=0}^{2^{n-1}-1} (i+1) + \sum_{i=0}^{2^{n-1}-1} (3i+2)\right) = H\left(\sum_{i=0}^{2^{n-1}-1} (i+1)\right) + H\left(\sum_{i=0}^{2^{n-1}-1} (3i+2)\right)$$

$$H_1 = H_l \left( \sum_{i=0}^{2^{n-1}-1} (i+1) \right) + H_r \left( \sum_{i=0}^{2^{n-1}-1} (i+1) \right) + H_l \left( \sum_{i=0}^{2^{n-1}-1} (3i+2) \right) + H_r \left( \sum_{i=0}^{2^{n-1}-1} (3i+2) \right)$$

$$H_1 = \sum_{i=0}^{2^{n-2}-1} (2i+1) + \sum_{i=0}^{2^{n-2}-1} (2i+2) + \sum_{i=0}^{2^{n-2}-1} (6i+2) + \sum_{i=0}^{2^{n-2}-1} (6i+5)$$

$$\text{With } T_1 = \sum_{i=0}^{2^{n-2}-1} (2i+2) + \sum_{i=0}^{2^{n-2}-1} (6i+2) \quad \text{and} \quad R_1 = \sum_{i=0}^{2^{n-2}-1} (2i+1) + \sum_{i=0}^{2^{n-2}-1} (6i+5)$$

- **Determination of the second even line  $P_1$**

$$P_1 = E(H_1) = T_1 + E(R_1)$$

$$P_1 = \sum_{i=0}^{2^{n-2}-1} (2i+2) + \sum_{i=0}^{2^{n-2}-1} (6i+2) + E \left( \sum_{i=0}^{2^{n-2}-1} (2i+1) + \sum_{i=0}^{2^{n-2}-1} (6i+5) \right)$$

$$P_1 = \sum_{i=0}^{2^{n-2}-1} (2i+2) + \sum_{i=0}^{2^{n-2}-1} (6i+2) + E \left( \sum_{i=0}^{2^{n-2}-1} (2i+1) \right) + E \left( \sum_{i=0}^{2^{n-2}-1} (6i+5) \right)$$

$$P_1 = \sum_{i=0}^{2^{n-2}-1} (2i+2) + \sum_{i=0}^{2^{n-2}-1} (6i+2) + \sum_{i=0}^{2^{n-2}-1} (6i+4) + \sum_{i=0}^{2^{n-2}-1} (18i+16)$$

So  $B_1 = (M_1, H_1, P_1)$  is the second bloc of the pyramid.

We just give the determination of the pyramid

$$S_1 \left( \sum_{i=0}^{2^n-1} (i+1) \right) = ((M_0, H_0, P_0), (M_1, H_1, P_1))$$

$$\text{We can see that: } M_p = \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_m i + b_m) \right)$$

$$H_p = \sum_{h=1}^{2^{p+1}} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_h i + b_h) \right)$$

$$P_p = \sum_{s=1}^{2^{p+1}} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_s i + b_s) \right)$$



**NB :**  $S_{\infty} \left( \sum_{i=0}^{2^n-1} (i+1) \right)$  is the infinite pyramid and its last lines  $(M_{\infty}, H_{\infty}, P_{\infty})$  will give us the result we are looking for.

#### IV. CALCULATIONS USING $H_p$ :

If we consider that all numbers from 1 to infinity are going to reach the loop (4,2,1) where  $p$  tend to  $n-1$  and  $n$  to infinity:

$T_p$  will be equal to:  $T_p = (2^{n-1}) \times 2$

$R_p$  will be equal to  $(2^{n-1}) \times 1$

The difference between  $T_p$  and  $R_p$  will be equal to :

$$T_p - R_p = (2^{n-1}) \times (2-1)$$

$$T_p - R_p = (2^{n-1})$$

Since we have  $2^{n-1}$  integers in this sum, the average is :

$$1 = \frac{T_p - R_p}{2^{n-1}}$$

a)

Given an heterogeneous line  $M_p$

$$M_p = \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_m i + b_m) \right)$$

- Finding the homogeneous line  $H_p$

$$H_p = H(M_p) = H_l(M_p) + H_r(M_p)$$

$$H_p = H_l \left( \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_m i + b_m) \right) \right) + H_r \left( \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_m i + b_m) \right) \right)$$

$$H_p = \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (2a_m i + b_m) \right) + \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (2a_m i + a_m + b_m) \right)$$

$$\text{If } H_p = \sum_{i=0}^{2^{n-p}-1} (A_p i + B_p) \quad \text{then} \quad A_p = \sum_{m=1}^{2^p} (2a_m) + \sum_{m=1}^{2^p} (2a_m)$$

$$A_p = \sum_{m=1}^{2^p} (4a_m)$$

**NB:** After the separation  $H_p$  can also be written as follow :

$$H_p = \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_i i + b_i) \right) + \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_r i + b_r) \right)$$

Where we distinguish odd and even series.

$$\text{This means : } \sum_{t=1}^{2^p} (a_t) = \sum_{r=1}^{2^p} (a_r) = \sum_{m=1}^{2^p} (2a_m)$$

b)

Given  $D(H_p)$  the absolute values of the differences between even suites and odd suites which come from the same heterogeneous suite in the line  $H_p$ .

$$\text{So if } M_p = \sum_{i=0}^{2^{n-p}-1} (a_1 i + b_1) + \sum_{i=0}^{2^{n-p}-1} (a_2 i + b_2) + \dots + \sum_{i=0}^{2^{n-p}-1} (a_p i + b_p)$$

$$\begin{aligned} H_p &= H_l \left( \sum_{i=0}^{2^{n-p}-1} (a_1 i + b_1) \right) + H_r \left( \sum_{i=0}^{2^{n-p}-1} (a_1 i + b_1) \right) \\ &+ H_l \left( \sum_{i=0}^{2^{n-p}-1} (a_2 i + b_2) \right) + H_r \left( \sum_{i=0}^{2^{n-p}-1} (a_1 i + b_1) \right) \\ &+ \dots \\ &+ H_l \left( \sum_{i=0}^{2^{n-p}-1} (a_p i + b_p) \right) + H_r \left( \sum_{i=0}^{2^{n-p}-1} (a_p i + b_p) \right) \end{aligned}$$

When we distinguish odd and even series, we have:

$$\begin{aligned}
 H_p &= \sum_{i=0}^{2^{n-p-1}-1} (a_{r1}i + b_{r1}) + \sum_{i=0}^{2^{n-p-1}-1} (a_{t1}i + b_{t1}) \\
 &+ \sum_{i=0}^{2^{n-p-1}-1} (a_{r2}i + b_{r2}) + \sum_{i=0}^{2^{n-p-1}-1} (a_{t2}i + b_{t2}) \\
 &+ \dots \\
 &+ \sum_{i=0}^{2^{n-p-1}-1} (a_{rp}i + b_{rp}) + \sum_{i=0}^{2^{n-p-1}-1} (a_{tp}i + b_{tp})
 \end{aligned}$$

$$\begin{aligned}
 \text{So : } D(H_p) &= \left| \sum_{i=0}^{2^{n-p-1}-1} (a_{r1}i + b_{r1}) - \sum_{i=0}^{2^{n-p-1}-1} (a_{t1}i + b_{t1}) \right| \\
 &+ \left| \sum_{i=0}^{2^{n-p-1}-1} (a_{r2}i + b_{r2}) - \sum_{i=0}^{2^{n-p-1}-1} (a_{t2}i + b_{t2}) \right| \\
 &+ \dots \\
 &+ \left| \sum_{i=0}^{2^{n-p-1}-1} (a_{rp}i + b_{rp}) - \sum_{i=0}^{2^{n-p-1}-1} (a_{tp}i + b_{tp}) \right|
 \end{aligned}$$

$$\text{Since } H_r \left( \sum_{i=0}^{2^{n-p}-1} (ai + b) \right) = \sum_{i=0}^{2^{n-p}-1} (2ai + a + b) \text{ and}$$

$$H_l \left( \sum_{i=0}^{2^{n-p}-1} (ai + b) \right) = \sum_{i=0}^{2^{n-p}-1} (2ai + b)$$

$$\Rightarrow H_r \geq H_l$$

$$\text{As a result : } D(H_p) = H_r(M_p) - H_l(M_p)$$

$$\begin{aligned}
D(H_p) &= H_r \left( \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_m i + b_m) \right) \right) - H_l \left( \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_m i + b_m) \right) \right) \\
D(H_p) &= \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (2a_m i + a_m + b_m) \right) - \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (2a_m i + b_m) \right) \\
D(H_p) &= \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (a_m) \right) = \frac{1}{4} \sum_{m=1}^{2^p} \left( \sum_{i=0}^{2^{n-p}-1} (4a_m) \right) = \frac{1}{4} \sum_{i=0}^{2^{n-p}-1} (A_p) \\
D(H_p) &= \frac{1}{4} \times \frac{2^n}{2^{p+1}} A_p \quad \rightarrow \rightarrow (1)
\end{aligned}$$

- Finding the even line  $P_p$

$$\begin{aligned}
T_p &= \sum_{t=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_t i + b_t) \right) \quad \text{and} \quad R_p = \sum_{r=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_r i + b_r) \right) \\
P_p &= T_p + E(R_p) = \sum_{t=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_t i + b_t) \right) + E \left( \sum_{r=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_r i + b_r) \right) \right) \\
P_p &= \sum_{t=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_t i + b_t) \right) + \sum_{r=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (3a_r i + 3b_r + 1) \right)
\end{aligned}$$

- Finding  $M_{p+1}$

$$\begin{aligned}
M_{p+1} &= H_e(P_p) = H_e \left( \sum_{t=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (a_t i + b_t) \right) + \sum_{r=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} (3a_r i + 3b_r + 1) \right) \right) \\
M_{p+1} &= \sum_{t=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} \left( \frac{a_t}{2} i + \frac{b_t}{2} \right) \right) + \sum_{r=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-1}-1} \left( \frac{3}{2} a_r i + \frac{3}{2} b_r + \frac{1}{2} \right) \right)
\end{aligned}$$

- Finding  $H_{p+1}$

Before finding  $H_{p+1}$  we must separate odd-left and odd-right lines in  $M_{p+1}$

$\overset{x}{M}_{p+1} = \sum_{e=1}^E \left( \sum_{i=0}^{2^{n-p-1}-1} \left( \frac{a_e}{2} i + \frac{b_e}{2} \right) \right) + \sum_{g=1}^G \left( \sum_{i=0}^{2^{n-p-1}-1} \left( \frac{3}{2} a_g i + \frac{3}{2} b_g + \frac{1}{2} \right) \right)$  is the odd-left  
line from  $M_{p+1}$

$\overset{y}{M}_{p+1} = \sum_{f=1}^F \left( \sum_{i=0}^{2^{n-p-1}-1} \left( \frac{a_f}{2} i + \frac{b_f}{2} \right) \right) + \sum_{h=1}^H \left( \sum_{i=0}^{2^{n-p-1}-1} \left( \frac{3}{2} a_h i + \frac{3}{2} b_h + \frac{1}{2} \right) \right)$  is the odd-  
right line from  $M_{p+1}$

Where  $E + F = G + H = 2^p$

$$H_{p+1} = H(M_{p+1}) = H(\overset{x}{M}_{p+1} + \overset{y}{M}_{p+1}) = H(\overset{x}{M}_{p+1}) + H(\overset{y}{M}_{p+1})$$

$$H_{p+1} = H_l(\overset{x}{M}_{p+1}) + H_l(\overset{y}{M}_{p+1}) + H_r(\overset{x}{M}_{p+1}) + H_r(\overset{y}{M}_{p+1})$$

$$H_{p+1} = \sum_{e=1}^E \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_e i + \frac{b_e}{2} \right) \right) + \sum_{g=1}^G \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_g i + \frac{3}{2} b_g + \frac{1}{2} \right) \right)$$

$$+ \sum_{f=1}^F \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_f i + \frac{b_f}{2} \right) \right) + \sum_{h=1}^H \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_h i + \frac{3}{2} b_h + \frac{1}{2} \right) \right)$$

$$+ \sum_{e=1}^E \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_e i + \frac{a_e}{2} + \frac{b_e}{2} \right) \right) + \sum_{g=1}^G \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_g i + 3\frac{a_g}{2} + \frac{3}{2} b_g + \frac{1}{2} \right) \right)$$

$$+ \sum_{f=1}^F \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_f i + \frac{a_f}{2} + \frac{b_f}{2} \right) \right) + \sum_{h=1}^H \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_h i + 3\frac{a_h}{2} + \frac{3}{2} b_h + \frac{1}{2} \right) \right)$$

Let's find  $D(H_{p+1})$  by Analogy to  $D(H_p)$

$$D(H_{p+1}) = H_r(M_{p+1}) - H_l(M_{p+1})$$

$$D(H_{p+1}) = \sum_{e=1}^E \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_e i + \frac{a_e}{2} + \frac{b_e}{2} \right) \right) + \sum_{g=1}^G \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_g i + 3\frac{a_g}{2} + \frac{3}{2} b_g + \frac{1}{2} \right) \right)$$

$$\begin{aligned}
& + \sum_{f=1}^F \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_f i + \frac{a_f}{2} + \frac{b_f}{2} \right) \right) + \sum_{h=1}^H \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_h i + 3\frac{a_h}{2} + \frac{3}{2}b_h + \frac{1}{2} \right) \right) \\
& - \sum_{e=1}^E \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_e i + \frac{b_e}{2} \right) \right) - \sum_{g=1}^G \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_g i + \frac{3}{2}b_g + \frac{1}{2} \right) \right) \\
& - \sum_{f=1}^F \left( \sum_{i=0}^{2^{n-p-2}-1} \left( a_f i + \frac{b_f}{2} \right) \right) - \sum_{h=1}^H \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3a_h i + \frac{3}{2}b_h + \frac{1}{2} \right) \right)
\end{aligned}$$

$$\begin{aligned}
D(H_{p+1}) &= \sum_{e=1}^E \left( \sum_{i=0}^{2^{n-p-2}-1} \left( \frac{a_e}{2} \right) \right) + \sum_{g=1}^G \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3\frac{a_g}{2} \right) \right) \\
&+ \sum_{f=1}^F \left( \sum_{i=0}^{2^{n-p-2}-1} \left( \frac{a_f}{2} \right) \right) + \sum_{h=1}^H \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3\frac{a_h}{2} \right) \right)
\end{aligned}$$

According to the separation of  $M_{p+1}$

$$\begin{aligned}
D(H_{p+1}) &= \sum_{t=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-2}-1} \left( \frac{a_t}{2} \right) \right) + \sum_{r=1}^{2^p} \left( \sum_{i=0}^{2^{n-p-2}-1} \left( 3\frac{a_r}{2} \right) \right) \\
&= \sum_{i=0}^{2^{n-p-2}-1} \left( \frac{1}{2} \sum_{t=1}^{2^p} (a_t) \right) + \sum_{i=0}^{2^{n-p-2}-1} \left( \frac{3}{2} \sum_{r=1}^{2^p} (a_r) \right) \\
&= \sum_{i=0}^{2^{n-p-2}-1} \left( \frac{1}{2} \sum_{m=1}^{2^p} (2a_m) \right) + \sum_{i=0}^{2^{n-p-2}-1} \left( \frac{3}{2} \sum_{m=1}^{2^p} (2a_m) \right) \\
&= \sum_{i=0}^{2^{n-p-2}-1} \left( \sum_{m=1}^{2^p} (4a_m) \right) = \sum_{i=0}^{2^{n-p-2}-1} (A_p)
\end{aligned}$$

$$D(H_{p+1}) = \frac{2^n}{2^{p+2}} A_p \rightarrow \rightarrow (2)$$

c) Finding the general form of  $D(H_{p+1})$

from (1) and (2) we have:

$$D(H_{p+1}) = \frac{2^n}{2^{p+2}} A_p \quad \text{and} \quad D(H_p) = \frac{1}{4} \times \frac{2^n}{2^{p+1}} A_p$$

Then  $D(H_p)$  is a geometric suite where the first term is  $D(H_0)$  and the common ratio is 2.

- Let's find  $D(H_0)$

In the first bloc of de pyramid  $S_{n-1} \left( \sum_{i=0}^{2^{n-1}-1} (i+1) \right)$ ,

$$H_0 = \sum_{i=0}^{2^{n-1}-1} (2i+1) + \sum_{i=0}^{2^{n-1}-1} (2i+2)$$

$$D(H_0) = \sum_{i=0}^{2^{n-1}-1} (1) = 2^{n-1}$$

The general form is :  $D(H_p) = 2^{n-1} \times 2^p$

In the last line, the index  $p$  reach  $n-1$ , so

$$D(H_{n-1}) = 2^{n-1} \times 2^{n-1}$$

$$Avg(D(H_{n-1})) = Avg(D(H_0)) Avg(2^{n-1}) = 1$$

$$Avg(D(H_{n-1})) = 1$$

We know that:  $1 = \frac{T_p - R_p}{2^{n-1}}$

$$1 = \frac{|T_p - R_p|}{2^{n-1}}$$

We are expecting:  $1 = Avg(D(H_{n-1}))$

The result we found verify the equality:  $1 = Avg(D(H_{n-1}))$

It means the conjecture of Syracuse is then true.

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