COLLATZ-SYRACUSE CONJECTURE INVALIDATION

INTRODUCTION

Of all the currently unsolved mathematical problems, which one has the most basic statement? This may well be the Syracuse conjecture: accessible to all in its statement, it has challenged researchers for decades.
The 3n + 1 problem is posed in these terms: let us start from any positive integer, and apply the following transformation to it repeatedly (we speak of a trajectory): if this number is even, we divide it by 2, if the number is odd, we multiply it by three then we add 1, so we get another number. Is it true that sooner or later we will end up with 1? All calculations made to date confirm this prediction.
The suite is written as follows.

$$U_{n+1} = \begin{cases} \dfrac{U_n}{2} & if \quad U_n \ is \ \ even \\ 3U_n + 1 & if \ U_n \ is \ \ odd \end{cases}$$

In this paper we are going to prove the Syracuse conjecture is false.

I. DEFINITIONS :

0. Series :

A series q is a sum of 2^n integers.
There are four types of series :

- Heterogeneous series (m) :
  It is a sum of even and odd numbers. It is also an alternation of odd and even numbers.

$$m = \sum_{i=0}^{2^n-1}(ai+b)$$

Where a is odd number and $b \in \mathrm{N}^*$

Example :

$$m = \sum_{i=0}^{2^4}(3i+5) = 5+8+11+14+.....+53$$

- Even series (p or t) :
  It's a sum of only even numbers.

$$p = \sum_{i=0}^{2^n-1}(ai+b)$$

Where a and b are even numbers.

Example :

$$p = \sum_{i=0}^{2^4}(2i+8) = 8+10+12+14+.....+40$$

- Odd series r :
  It is a sum of only odd numbers.

$$r = \sum_{i=0}^{2^n-1}(ai+b)$$

Where a is even and b odd

Example :

$$m = \sum_{i=0}^{2^4}(4i+5) = 5+9+13+17+.....+69$$

- Homogeneous series h :

  It is a sum of one even series t and one odd series r

$h = t + r$

1. **Line :**

A line is a sum of 2^p series where $p \geq 0$

The generic name of any line is Q.

There are four types of lines : homogeneous line (H), odd line (R), even line (P) or (T) and heterogeneous line (M).

- An homogeneous line H is a sum of one even line T and one odd line R.

$$H = T + R = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_k i + b_k)) + \sum_{l=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_l i + b_l))$$

Where $T = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_k i + b_k)) = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1} t_k)$ and

$$R = \sum_{l=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_l i + b_l)) = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1} r_k)$$

- An even line P or T is a sum of 2^p even series .

$$P = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_k i + b_k)) = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1} p_k)$$

- An odd line is a sum of 2^p odd series (r_k).

$$R = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_k i + b_k)) = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1} r_k)$$

- An heterogeneous line is a sum of 2^p heterogeneous series (m_k).

$$M=\sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_k i+b_k))=\sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}m_k)$$

2. The Left function :

The left function G is applied to any line Q whose number of series is 2^p.

It results on the 2^{p-1} first series from the left side of Q.

Example :

If $$Q=\sum_{i=0}^{2^n-1}(3i+8)+\sum_{i=0}^{2^n-1}(i+9)+\sum_{i=0}^{2^n-1}(7i+26)+\sum_{i=0}^{2^n-1}(23i+4)$$

$$G(Q)=\sum_{i=0}^{2^n-1}(3i+8)+\sum_{i=0}^{2^n-1}(i+9)$$

3. The right function :

The right function D of any line Q is defined as follow

$$D(Q)=Q-G(Q)$$

4. The down function B :

If $$Q=\sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}(ai+b))$$

The down of Q is $$B(Q)=\sum_{k=1}^{2^p}(\sum_{i=0}^{2^{n-1}-1}(ai+b))$$

II. FUNCTIONS

1. **THE SEPARATION FUNCTION** H **:**

The separation function H also called the to-homogeneous function is a sum of two functions :

H_r the right-separation function and H_l the left-separation function.

If $m = \sum_{i=0}^{2^n-1}(U_i) = \sum_{i=0}^{2^n-1}(ai+b)$ is an heterogeneous series, $H(m)$ gives two series : one odd series and another even series. The results are also both called homogeneous series.

$$H = H(m) = H_l(m) + H_r(m)$$

$$H_l(m) = H_l(\sum_{i=0}^{2^n-1}(ai+b)) = \sum_{i=0}^{2^{n-1}-1}(U_{2i}) = \sum_{i=0}^{2^{n-1}-1}(2ai+b)$$

$$H_r(m) = H_r(\sum_{i=0}^{2^n-1}(ai+b)) = \sum_{i=0}^{2^{n-1}-1}(U_{2i+1}) = \sum_{i=0}^{2^{n-1}-1}(2ai+a+b)$$

So $H = \sum_{i=0}^{2^{n-1}-1}(2ai+b) + \sum_{i=0}^{2^{n-1}-1}(2ai+a+b)$

NB :

If $H_l(m)$ is odd then m is said to be odd-left or even-right heterogeneous series.

If $H_r(m)$ is odd then m is said to be odd-right or even-left heterogeneous series.

<u>**Odd-left and odd-right heterogeneous line**</u> **? :**

If we apply the separation function to an heterogeneous line, we find an odd line and an even line. So if the odd series comes from the left-separation function applied to the hrterogeneous series then this last one is said odd-left heterogeneous, else it’s said odd-right. If the even series comes from the left-separation function applied to the heterogeneous series, then this last one is said to be odd-right.

If $M = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^n-1}(a_k i + b_k))$

It can be written as follows

$$M = \sum_{x=1}^{X}\sum_{i=0}^{2^n-1}(a_x i + b_x) + \sum_{y=1}^{Y}\sum_{i=0}^{2^n-1}(a_y i + b_y)$$

Where $X + Y = 2^p$

$\overset{x}{M} = \sum_{x=1}^{X}\sum_{i=0}^{2^n-1}(a_x i + b_x)$ is the sum of odd-left series . It's also called the odd-left heterogeneous line

$\overset{y}{M} = \sum_{y=1}^{Y}\sum_{i=0}^{2^n-1}(a_y i + b_y)$ is the sum of odd-right series or the odd-right heterogeneous line

$M = \overset{x}{M}$ + $\overset{y}{M}$

2. The to-Even function E :

The to-Even function E receives in entry an odd series r then results in an even series p .

If $r = \sum_{i=0}^{2^n-1}(U_i) = \sum_{i=0}^{2^n-1}(ai + b)$ is an odd series, we have :

$$E(r) = \sum_{i=0}^{2^n-1}(3U_i + 1) = \sum_{i=0}^{2^n-1}(3ai + 3b + 1)$$

3. **The to-heterogeneous function H_e:**

The to-heterogeneous function H_e transforms an even series p into an heterogeneous series m.

$$p = \sum_{i=0}^{2^n-1}(U_i) = \sum_{i=0}^{2^n-1}(ai+b)$$ is an even series.

$$H_e(p) = \sum_{i=0}^{2^n-1}(\frac{U_i}{2}) = \sum_{i=0}^{2^n-1}(\frac{a}{2}i+\frac{b}{2})$$

III. Pyramid and blocs :

0. **Bloc B :**

A bloc B is a succession of three lines : it 's composed by one heterogeneous line M_0 followed by an homogeneous line H_0 then an even line P_0 that such $H(M_0) = H_0$, $E(H_0) = P_0$ and $H_e(P_0) = M_1$.

So $B_0 = (M_0, H_0, P_0)$

1. **Pyramid S :**

A pyramid S_n is a succession of n blocs (B_p) that such $H_e(P_p) = M_{p+1}$

The pyramid S_n which begin with the heterogeneous line M_0 is $S_n(M_0) = (B_0, B_1, B_2, ..., B_{n-1})$.

This pyramid ends with P_{n-1}

1. **Construction of the pyramid** $S_1(\sum_{i=0}^{2^n-1}(i+1))$ **:**

$M_0 = \sum_{i=0}^{2^n-1}(i+1)$ is the first heterogeneous line

- **Determination of** H_0 **the first homogeneous line :**

$$H_0 = H(M_0) = H(\sum_{i=0}^{2^n-1}(i+1)) = H_l(\sum_{i=0}^{2^n-1}(i+1)) + H_r(\sum_{i=0}^{2^n-1}(i+1))$$

$$H_0 = \sum_{i=0}^{2^{n-1}-1}(2i+1) + \sum_{i=0}^{2^{n-1}-1}(2i+2)$$

With $T_0 = \sum_{i=0}^{2^{n-1}-1}(2i+2)$ and $R_0 = \sum_{i=0}^{2^{n-1}-1}(2i+1)$

- **Determination of the even line** P_0

$$P_0 = T_0 + E(R_0) = \sum_{i=0}^{2^{n-1}-1}(2i+2) + E(\sum_{i=0}^{2^{n-1}-1}(2i+1))$$

$$P_0 = \sum_{i=0}^{2^{n-1}-1}(2i+2) + \sum_{i=0}^{2^{n-1}-1}(6i+4)$$

Where $W_0 = \sum_{i=0}^{2^{n-1}-1}(6i+4)$

NB : if An even line comes from the separation function, it's noticed by T
If an even line comes from the to-even function, it's noticed by P

If an even line comes from an odd line, it's noticed by W

- **Determination of the second heterogeneous line** M_1 **:**

$$M_1 = H_e(P_0) = H_e(T_0 + W_0) = H_e(\sum_{i=0}^{2^{n-1}-1}(2i+2) + \sum_{i=0}^{2^{n-1}-1}(6i+4))$$

$$M_1 = \frac{1}{2}((\sum_{i=0}^{2^{n-1}-1}(2i+2) + \sum_{i=0}^{2^{n-1}-1}(6i+4)) = \sum_{i=0}^{2^{n-1}-1}(i+1) + \sum_{i=0}^{2^{n-1}-1}(3i+2)$$

$$M_1 = \sum_{i=0}^{2^{n-1}-1}(i+1) + \sum_{i=0}^{2^{n-1}-1}(3i+2)$$

- **Determination of the second homogeneous line** H_1

$$H_1 = H\ (M_1) = H(\sum_{i=0}^{2^{n-1}-1}(i+1) + \sum_{i=0}^{2^{n-1}-1}(3i+2))$$

$$H_1 = H(\sum_{i=0}^{2^{n-1}-1}(i+1)) + H(\sum_{i=0}^{2^{n-1}-1}(3i+2))$$

$$H_1 = H_l(\sum_{i=0}^{2^{n-1}-1}(i+1)) + H_r(\sum_{i=0}^{2^{n-1}-1}(i+1)) + H_l(\sum_{i=0}^{2^{n-1}-1}(3i+2)) + H_r(\sum_{i=0}^{2^{n-1}-1}(3i+2))$$

$$H_1 = \sum_{i=0}^{2^{n-2}-1}(2i+1) + \sum_{i=0}^{2^{n-2}-1}(2i+2) + \sum_{i=0}^{2^{n-2}-1}(6+2) + \sum_{i=0}^{2^{n-2}-1}(6+5)$$

Where $R_1 = \sum_{i=0}^{2^{n-2}-1}(2i+1) + \sum_{i=0}^{2^{n-2}-1}(6i+5)$ And $T_1 = \sum_{i=0}^{2^{n-2}-1}(6i+2) + \sum_{i=0}^{2^{n-2}-1}(2i+2)$

- **Determination of the even line** P_1 **:**

$$P_1 = T_1 + E(R_1)$$

$$E(R_1) = E(\sum_{i=0}^{2^{n-2}-1}(2i+1) + \sum_{i=0}^{2^{n-2}-1}(6i+5)) = \sum_{i=0}^{2^{n-2}-1}(3(2i+1)+1) + \sum_{i=0}^{2^{n-2}-1}(3(6i+5)+1)$$

$$E(R_1) = \sum_{i=0}^{2^{n-2}-1}(6i+4) + \sum_{i=0}^{2^{n-2}-1}(18i+16)$$

So $P_1 = \sum_{i=0}^{2^{n-2}-1}(2i+2) + \sum_{i=0}^{2^{n-2}-1}(6i+4)$ + $\sum_{i=0}^{2^{n-2}-1}(6i+2) + \sum_{i=0}^{2^{n-2}-1}(18i+16)$

We just give the determination of $S_1(M_0) = ((M_0, H_0, P_0), (M_1, H_1, P_1))$

NB : we can see that $S_\infty(M_0)$ is the infinite pyramid of M_0 and its last lines $(M_\infty, H_\infty, P_\infty)$ will give us the result we are looking for.

IV. DOING SOME CALCULATIONS USING H_p :

So if we consider that all numbers from 1 to infinity are going to the hailston (4,2,1) where p and n tend to infinity then :

T_p must be equal to $2^{n-1} \times 2$

R_p must be equal to $2^{n-1} \times 1$

And H_p must be equal to $2^{n-1} \times 1 + 2^{n-1} \times 2$

where p and n tend to infinity.

Let's consider an heterogeneous line M_p

$$M_p = \sum_{k=1}^{2^p}\left(\sum_{i=0}^{2^{n-p}-1}(a_k i + b_k)\right)$$

- Finding the homogeneous line H_p

$$H_p = H(M_p) = H_l(M_p) + H_r(M_p)$$

$$H_l(M_p) = \sum_{k=1}^{2^p}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k)\right)$$

$$H_r(M_p) = \sum_{k=1}^{2^p}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k + a_k)\right)$$

$$H_l(M_p) + H_r(M_p) = T_p + R_p$$

$$H_p = \sum_{k=1}^{2^p}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k)\right) + \sum_{k=1}^{2^p}\left(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k + a_k)\right)$$

If $H_p = \sum_{i=0}^{2^{n-p-1}-1}(A_p i + B_p)$ then $A_p = \sum_{k=1}^{2^p} 2a_k + \sum_{k=1}^{2^p} 2a_k = \sum_{k=1}^{2^p} 4a_k$

$$A_p = \sum_{k=1}^{2^p} 4a_k$$

Let's pose $D(H_p)$ as the absolute values of the differences between even suites and odd suites which come from the same heterogeneous suite in the line H_p

So if $M_p = \sum_{i=0}^{2^{n-p}-1}(a_1 i + b_1) + \sum_{i=0}^{2^{n-p}-1}(a_2 i + b_2) + \sum_{i=0}^{2^{n-p}-1}(a_3 i + b_3) + \ldots + \sum_{i=0}^{2^{n-p}-1}(a_p i + b_p)$

And

$$H_p = H_r\left(\sum_{i=0}^{2^{n-p}-1}(a_1 i + b_1)\right) + H_l\left(\sum_{i=0}^{2^{n-p}-1}(a_1 i + b_1)\right)$$

$$+H_r(\sum_{i=0}^{2^{n-p}-1}(a_2 i+b_2))+H_l(\sum_{i=0}^{2^{n-p}-1}(a_2 i+b_2))$$

$$+H_r(\sum_{i=0}^{2^{n-p}-1}(a_3 i+b_3))+H_l(\sum_{i=0}^{2^{n-p}-1}(a_3 i+b_3))$$

$$+\ldots..$$

$$+H_r(\sum_{i=0}^{2^{n-p}-1}(a_p i+b_p))+H_l(\sum_{i=0}^{2^{n-p}-1}(a_p i+b_p))$$

$$D(H_p) \;=\left|H_r(a_1 i+b_1)-H_l(a_1 i+b_1)\right|$$

$$+\left|H_r(a_2 i+b_2)+H_l(a_2 i+b_2)\right|$$

$$+\left|H_r(a_3 i+b_3)+H_l(a_3 i+b_3)\right|$$

$$+\ldots..$$

$$+\left|H_r(a_p i+b_p)+H_l(a_p i+b_p)\right|$$

Since $H_r(ai+b)=2ai+a+b$ and $H_l(ai+b)=2ai+b$

Then $H_r(ai+b)\geq H_l(ai+b)$

As a result

$$D(H_p)=H_r(M_p)-H_l(M_p) = \sum_{k=1}^{2^p}(2a_k i+b_k+a_k)-\sum_{k=1}^{2^p}(2a_k i+b_k)$$

$$D(H_p)=\sum_{k=1}^{2^p}(a_k)$$

$$D(H_p) \;=\; \frac{A_p}{4} \quad \text{-------> } (1)$$

- Finding the even line P_p

$$T_p = \sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_t i+b_t))$$

And $$R_p = \sum_{r=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_r i+b_r))$$

Where $H_p = \sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t))$ + $\sum_{r=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_r i + b_r))$

$$P_p = T_p + E(R_p)$$

= $\sum_{r=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(3a_r i + 3b_r + 1))$

So $W_p = \sum_{r=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(3a_r i + 3b_r + 1))$

$$P_p = \sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t)) + \sum_{r=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(3a_r i + 3b_r + 1))$$

Yet

$$H_p = \sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t)) + \sum_{r=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_r i + b_r))$$

$$\begin{aligned}
H_p = {} & \sum_{i=0}^{2^{n-p-1}-1}(a_{t1} i + b_{t1}) + \sum_{i=0}^{2^{n-p-1}-1}(a_{r1} i + b_{r1}) \\
& + \sum_{i=0}^{2^{n-p-1}-1}(a_{t2} i + b_{t12}) + \sum_{i=0}^{2^{n-p-1}-1}(a_{r2} i + b_{r2}) \\
& + \sum_{i=0}^{2^{n-p-1}-1}(a_{t3} i + b_{t3}) + \sum_{i=0}^{2^{n-p-1}-1}(a_{r3} i + b_{r3}) \\
& + \ldots\ldots. \\
& + \sum_{i=0}^{2^{n-p-1}-1}(a_{tp} i + b_{tp}) + \sum_{i=0}^{2^{n-p-1}-1}(a_{rp} i + b_{rp})
\end{aligned}$$

Where each couple of series come from the same heterogeneous series

$$
\begin{aligned}
P_p = &\sum_{i=0}^{2^{n-p-1}-1}(a_{t1}i + b_{t1}) + E(\sum_{i=0}^{2^{n-p-1}-1}(a_{r1}i + b_{r1})) \\
&+ \sum_{i=0}^{2^{n-p-1}-1}(a_{t2}i + b_{t12}) + E(\sum_{i=0}^{2^{n-p-1}-1}(a_{r2}i + b_{r2})) \\
&+ \sum_{i=0}^{2^{n-p-1}-1}(a_{t3}i + b_{t3}) + E(\sum_{i=0}^{2^{n-p-1}-1}(a_{r3}i + b_{r3})) \\
&+ \dots\dots. \\
&+ \sum_{i=0}^{2^{n-p-1}-1}(a_{tp}i + b_{tp}) + E(\sum_{i=0}^{2^{n-p-1}-1}(a_{rp}i + b_{rp}))
\end{aligned}
$$

Let's consider $D(P)$ as the absolute values of the diferences beteween even suites which come from odd suites and even suites which come from heterogeneous suites on which we applied the E function (to-Even function)

So

$$
\begin{aligned}
D(P_p) = &\left|E(a_{t1}i + b_{t1}) - E(a_{r1}i + b_{r1})\right| \\
&+ \left|E(a_{t2}i + b_{t2}) - E(a_{r2}i + b_{r2})\right| \\
&+ \left|E(a_{t3}i + b_{t3}) - E(a_{r3}i + b_{r3})\right| \\
&+ \dots\dots. \\
&+ \left|E(a_{tp}i + b_{tp}) - E(a_{rp}i + b_{rp})\right|
\end{aligned}
$$

Let's pose $m = \sum_i (a_i + b)$ an heterogeneous series

Where $ai + b = U_m$ is the suite of the series m

$h = H(m) = H_r(\sum_i (m)) + H_l(\sum_i (m))$ is the homogeneous series from m

There are two cases to find the even series p which comes from h

- First case : $H_r(\sum_i (m))$ is the odd series

  $$p = E(H_r(\sum_i (m))) + H_l(\sum_i (m))$$

- second case : $H_l(\sum_i (m))$ is the odd series

  $$p = H_r(\sum_i (m)) + E(H_l(\sum_i (m)))$$

Whatever which is the odd series, we have $D(p) = \left|E(H_r(U_m)) - E(H_l(U_m))\right|$

Yet $H_r(U_m) \geq H_l(U_m)$

So , $E(H_r(U_m)) \geq E(H_l(U_m))$

As a result, $D(p) = E(H_r(U_m)) - E(H_l(U_m))$

Then, $D(P_p) = E(H_r(U_1)) - E(H_l(U_1))$

$+ E(H_r(U_2)) - E(H_l(U_2))$

$+ E(H_r(U_3)) - E(H_l(U_3))$

+........

$+ E(H_r(U_p)) - E(H_l(U_p))$

Where $M_p = \sum_{i=0}^{2^{n-p-1}-1} (U_1) + \sum_{i=0}^{2^{n-p-1}-1} (U_2) + \sum_{i=0}^{2^{n-p-1}-1} (U_3) + ... + \sum_{i=0}^{2^{n-p-1}-1} (U_p)$

$$H_r(M_p) = \sum_{k=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k + a_k)\Big)$$

$$H_l(M_p) = E\Big(\sum_{k=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k)\Big)\Big)$$

$$D(P_p) = E\Big(\sum_{k=1}^{2^p}(2a_k i + b_k + a_k)\Big) - E\Big(\sum_{k=1}^{2^p}(2a_k i + b_k)\Big)$$

$$D(P_p) = \sum_{k=1}^{2^p}(6a_k i + 3b_k + 3a_k + 1) - \sum_{k=1}^{2^p}(6a_k i + 3b_k + 1)$$

$$D(P_p) = \sum_{k=1}^{2^p}(3a_k) = \frac{3}{4}\sum_{k=1}^{2^p}(4a_k) = 3\frac{A_p}{4}$$

$$D(P_p) = \Big(3\frac{A_p}{4}\Big) \quad \text{------->} \quad (2)$$

Where $A_p = \sum_{k=1}^{2^p} 4a_k$

NB : let's proove that $\sum_{t=1}^{2^p} a_t = \sum_{r=1}^{2^p} a_r = \sum_{k=1}^{2^p} 2a_k$

$$T_p + R_p = \sum_{t=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t)\Big) + \sum_{r=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(a_r i + b_r)\Big)$$

$$T_p + R_p = H_l(M_p) + H_r(M_p)$$

Then , $\sum_{t=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t)\Big) + \sum_{r=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(a_r i + b_r)\Big) = H_l(M_p) + H_r(M_p)$

Consequently,

$$\sum_{t=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t)\Big) + \sum_{r=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(a_r i + b_r)\Big) = \sum_{k=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k)\Big) + \sum_{k=1}^{2^p}\Big(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i + b_k + a_k)\Big)$$

As a result, $\sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_t i)) + \sum_{r=1}^{2^p}\sum_{i=0}^{2^{n-p-1}-1}(a_r i) = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i) + \sum_{k=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i)$

We can see $\sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_t i)) = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i)$

And $\sum_{r=1}^{2^p}\sum_{i=0}^{2^{n-p-1}-1}(a_r i) = \sum_{k=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(2a_k i)$

As a result, $\sum_{t=1}^{2^p} a_t = \sum_{r=1}^{2^p} a_r = \sum_{k=1}^{2^p} 2a_k$

- Finding M_{p+1}

$$M_{p+1} = H_e(P_p) = H_e(\sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_t i + b_t)) + \sum_{r=1}^{2^p}\sum_{i=0}^{2^{n-p-1}-1}(3a_r i + 3b_r + 1))$$

$$M_{p+1} = \sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(\frac{a_t}{2} i + \frac{b_t}{2})) + \sum_{r=1}^{2^p}\sum_{i=0}^{2^{n-p-1}-1}(3\frac{a_r}{2}\ i + 3\frac{b_r}{2}\ + \frac{1}{2})$$

- Finding H_{p+1} :

Before finding H_{p+1} we must seperate odd-left and odd-right lines in M_{p+1}

$\overset{x}{M}_{p+1} = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-1}-1}(\frac{a_e}{2} i + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-1}-1}(3\frac{a_g}{2}\ i + 3\frac{b_g}{2}\ + \frac{1}{2})$ is the odd-left line

$\overset{y}{M}_{p+1} = \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-1}-1}(\frac{a_f}{2} i + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-1}-1}(3\frac{a_h}{2}\ i + 3\frac{b_h}{2}\ + \frac{1}{2})$ is the odd-right line

Where $E + F = 2^p$ and $G + H = 2^p$

$$H_{p+1} = H(M_{p+1}) = H\ (\overset{x}{M}_{p+1}) + H\ (\overset{y}{M}_{p+1})$$

$$H(\overset{x}{M}_{p+1}) = H_l(\overset{x}{M}_{p+1}) + H_r(\overset{x}{M}_{p+1})$$

$$H_l(\overset{x}{M}_{p+1}) = \sum_{e=1}^{E}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})\Big) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2})$$

$$H_r(\overset{x}{M}_{p+1}) = \sum_{e=1}^{E}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{a_e}{2} + \frac{b_e}{2})\Big) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{a_g}{2} + 3\frac{b_g}{2} + \frac{1}{2})$$

$$H(\overset{y}{M}_{p+1}) = H_l(\overset{y}{M}_{p+1}) + H_r(\overset{y}{M}_{p+1})$$

$$H_l(\overset{y}{M}_{p+1}) = \sum_{f=1}^{F}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2})\Big) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{b_h}{2} + \frac{1}{2})$$

$$H_r(\overset{y}{M}_{p+1}) = \sum_{f=1}^{F}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})\Big) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} + \frac{1}{2})$$

So $T_{p+1} = H_r(\overset{x}{M}_{p+1}) + H_l(\overset{y}{M}_{p+1})$ because all series in these two line are even

$$T_{p+1} = \sum_{e=1}^{E}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{a_e}{2} + \frac{b_e}{2})\Big) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{a_g}{2} + 3\frac{b_g}{2} + \frac{1}{2}) + \sum_{f=1}^{F}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2})\Big)$$

$$+ \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{b_h}{2} + \frac{1}{2})$$

and $R_{p+1} = H_l(\overset{x}{M}_{p+1}) + H_r(\overset{y}{M}_{p+1})$ because all series in these two lines are odd

$$R_{p+1} = \sum_{e=1}^{E}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})\Big) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2}) + \sum_{f=1}^{F}\Big(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})\Big)$$

$$+ \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} + \frac{1}{2})$$

$$H_{p+1} = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2}) + \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{a_e}{2} + \frac{b_e}{2})) +$$

$$\sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{a_g}{2} + 3\frac{b_g}{2} + \frac{1}{2}) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{b_h}{2} + \frac{1}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} + \frac{1}{2})$$

if $H_{p+1} = \sum_{i=0}^{2^{n-p-2}-1}(A_{p+1} i + B_{p+1})$ then

$$A_{p+1} = \sum_{e=1}^{E} a_e + \sum_{g=1}^{G} 3a_g + \sum_{e=1}^{E} a_e + \sum_{g=1}^{G} 3a_g + \sum_{f=1}^{F} a_f + \sum_{h=1}^{H} 3a_h + \sum_{f=1}^{F} a_f + \sum_{h=1}^{H} 3a_h$$

$$A_{p+1} = \sum_{e=1}^{E} a_e + \sum_{f=1}^{F} a_f + \sum_{g=1}^{G} 3a_g + \sum_{h=1}^{H} 3a_h + \sum_{e=1}^{E} a_e + \sum_{f=1}^{F} a_f + \sum_{g=1}^{G} 3a_g + \sum_{h=1}^{H} 3a_h$$

$$A_{p+1} = \sum_{t=1}^{2^p} a_t + \sum_{r=1}^{2^p} 3a_r + \sum_{t=1}^{2^p} a_t + \sum_{r=1}^{2^p} 3a_r = \sum_{t=1}^{2^p} 2a_t + \sum_{r=1}^{2^p} 6a_r$$

$$A_{p+1} = \sum_{t=1}^{2^p} a_t + 3\sum_{r=1}^{2^p} a_r + \sum_{t=1}^{2^p} a_t + 3\sum_{r=1}^{2^p} a_r = 2\sum_{t=1}^{2^p} a_t + 6\sum_{r=1}^{2^p} a_r$$

$$A_{p+1} = 2\sum_{k=1}^{2^p} 2a_k + 6\sum_{k=1}^{2^p} 2a_k = \sum_{k=1}^{2^p} 16a_k$$

$$A_{p+1} = \sum_{k=1}^{2^p} 16a_k = 4A_p$$

$$A_{p+1} = 4A_p$$

Let's find $D(H_{p+1})$

$$H_r(M_{p+1}) = H_r(\overset{x}{M}_{p+1}) + H_r(\overset{y}{M}_{p+1})$$

$$H_r(M_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{a_e}{2} + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{a_g}{2} + 3\frac{b_g}{2} + \frac{1}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} + \frac{1}{2})$$

$$H_l(M_{p+1}) = H_l(\overset{x}{M}_{p+1}) + H_l(\overset{y}{M}_{p+1})$$

$$H_l(M_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2}) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2}))$$

$$+ \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{b_h}{2} + \frac{1}{2})$$

$$D(H_{p+1}) = \sum_{e=1}^{E}(a_e i + \frac{a_e}{2} + \frac{b_e}{2}) + \sum_{g=1}^{G}(3a_g i + 3\frac{a_g}{2} + 3\frac{b_g}{2} + \frac{1}{2})$$

$$+ \sum_{f=1}^{F}(a_f i + \frac{a_f}{2} + \frac{b_f}{2}) + \sum_{h=1}^{H}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} + \frac{1}{2})$$

$$- (\sum_{e=1}^{E}(a_e i + \frac{b_e}{2}) + \sum_{g=1}^{G}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2}) + \sum_{f=1}^{F}(a_f i + \frac{b_f}{2})$$

$$+ \sum_{h=1}^{H}(3a_h i + 3\frac{b_h}{2} + \frac{1}{2}))$$

$$D(H_{p+1}) = \sum_{e=1}^{E}(\frac{a_e}{2}) + \sum_{g=1}^{G}(3\frac{a_g}{2}) + \sum_{f=1}^{F}(\frac{a_f}{2}) + \sum_{h=1}^{H}(3\frac{a_h}{2})$$

According to the separation of M_{p+1}, $D(H_{p+1}) = \sum_{t=1}^{2^p}(\frac{a_t}{2}) + \sum_{r=1}^{2^p}(3\frac{a_r}{2})$

$$D(H_{p+1}) = \frac{1}{2}\sum_{t=1}^{2^p}(a_t) + \frac{3}{2}\sum_{r=1}^{2^p}(a_r) = \frac{1}{2}\sum_{t=1}^{2^p}(2a_k) + \frac{3}{2}\sum_{r=1}^{2^p}(2a_k) =$$

$$D(H_{p+1}) = A_p \quad \text{------->} \quad (3)$$

- Finding P_{p+1}

$$P_{p+1} = T_{p+1} + E(R_{p+1})$$

$$E(R_{p+1}) = E(H_l(\overset{x}{M}_{p+1}) + H_r(\overset{y}{M}_{p+1}))$$

$$E(R_{p+1}) = E(H_l(\overset{x}{M}_{p+1})) + E(H_r(\overset{y}{M}_{p+1}))$$

$$E(H_l(\overset{x}{M}_{p+1})) = E(\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2}))$$

$$E(H_l(\overset{x}{M}_{p+1})) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{b_g}{2} + \frac{5}{2})$$

$$E(H_r(\overset{y}{M}_{p+1})) = E(\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} + \frac{1}{2}))$$

$$E(H_r(\overset{y}{M}_{p+1})) = \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + \frac{3a_f}{2} + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{a_h}{2} + 9\frac{b_h}{2} + \frac{5}{2})$$

$$W_{p+1} = E(R_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{b_g}{2} + \frac{5}{2})$$

$$+\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i+\frac{3a_f}{2}+3\frac{b_f}{2}+1))+\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i+9\frac{a_h}{2}+9\frac{b_h}{2}\ \ +\frac{5}{2})$$

So

$$P_{p+1}=\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i+\frac{a_e}{2}+\frac{b_e}{2}))+\sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i+3\frac{a_g}{2}+3\frac{b_g}{2}\ \ +\frac{1}{2})+\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i+\frac{b_f}{2}))$$

$$+\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i+3\frac{b_h}{2}\ \ +\frac{1}{2})$$

$$+\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i+3\frac{b_e}{2}+1))+\sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i+9\frac{b_g}{2}\ \ +\frac{5}{2})$$

$$+\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i+\frac{3a_f}{2}+3\frac{b_f}{2}+1))+\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i+9\frac{a_h}{2}+9\frac{b_h}{2}\ \ +\frac{5}{2})$$

By Analogy to $D(P),\ D(P_{p+1})=E(H_r(M_{p+1}))-E(H_l(M_{p+1}))$

$$H_r(M_{p+1})=H_r(\overset{x}{M}_{p+1})+H_r(\overset{y}{M}_{p+1})$$

$$E(H_r(M_{p+1}))=E(\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i+\frac{a_e}{2}+\frac{b_e}{2}))+\sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i+3\frac{a_g}{2}+3\frac{b_g}{2}\ \ +\frac{1}{2})$$

$$+\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i+\frac{a_f}{2}+\frac{b_f}{2}))+\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i+3\frac{a_h}{2}+3\frac{b_h}{2}\ \ +\frac{1}{2}))$$

$$E(H_r(M_{p+1}))=\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i+3\frac{a_e}{2}+3\frac{b_e}{2}+1))+$$

$$\sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i+9\frac{a_g}{2}+9\frac{b_g}{2}\ \ +\frac{5}{2})$$

$$+\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i+3\frac{a_f}{2}+3\frac{b_f}{2}+1))+\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i+9\frac{a_h}{2}+9\frac{b_h}{2}\ \ +\frac{5}{2})$$

$$E(H_l(M_{p+1})) = E(H_l(\overset{x}{M}_{p+1}) + H_l(\overset{y}{M}_{p+1}))$$

$$E(H_l(M_{p+1})) = E(\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{b_h}{2} + \frac{1}{2}))$$

$$E(H_l(M_{p+1})) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{b_g}{2} + \frac{5}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{b_h}{2} + \frac{5}{2})$$

$$D(P_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{a_e}{2} + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{a_g}{2} + 9\frac{b_g}{2} + \frac{5}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + 3\frac{a_f}{2} + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{a_h}{2} + 9\frac{b_h}{2} + \frac{5}{2})$$

$$- (\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{b_g}{2} + \frac{5}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{b_h}{2} + \frac{5}{2}))$$

$$D(P_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3\frac{a_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9\frac{a_g}{2}) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3\frac{a_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9\frac{a_h}{2})$$

$$D(P_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3\frac{a_e}{2})) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3\frac{a_f}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9\frac{a_g}{2}) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9\frac{a_h}{2})$$

According to the separation of M_{p+1}, $D(P_{p+1}) = \sum_{t=1}^{2^p}(\sum_{i=0}^{2^{n-p-2}-1}(3\frac{a_t}{2})) + \sum_{r=1}^{2^p}\sum_{i=0}^{2^{n-p-2}-1}(9\frac{a_r}{2})$

$$D(P_{p+1}) = \frac{3}{2}\sum_{i=0}^{2^{n-p-2}-1}(\sum_{k=1}^{2^p}(a_t)) + \frac{9}{2}\sum_{i=0}^{2^{n-p-2}-1}\sum_{k=1}^{2^p}(a_r)$$

$$D(P_{p+1}) = \frac{3}{2}\sum_{i=0}^{2^{n-p-2}-1}(\sum_{k=1}^{2^p}(2a_k)) + \frac{9}{2}\sum_{i=0}^{2^{n-p-2}-1}\sum_{k=1}^{2^p}(2a_k)$$

$$D(P_{p+1}) = \sum_{i=0}^{2^{n-p-2}-1}(\sum_{k=1}^{2^p}(12a_k))$$

$$D(P_{p+1}) = 3\sum_{i=0}^{2^{n-p-2}-1}(\sum_{k=1}^{2^p}(4a_k)) = 3\sum_{i=0}^{2^{n-p-2}-1}(A_p)$$

$$D(P_{p+1}) = 3\times\frac{2^n}{2^{p+2}}A_p$$

$$E(T_{p+1}) = E(H_r(\overset{x}{M}_{p+1}) + H_l(\overset{y}{M}_{p+1})) = E(H_r(\overset{x}{M}_{p+1})) + E(H_l(\overset{y}{M}_{p+1}))$$

$$E(H_r(\overset{x}{M}_{p+1})) = E(\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{a_e}{2} + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{a_g}{2} + 3\frac{b_g}{2} + \frac{1}{2}))$$

$$E(H_r(\overset{x}{M}_{p+1})) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{a_e}{2} + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{a_g}{2} + 9\frac{b_g}{2} + \frac{5}{2})$$

$$E(H_l(\overset{y}{M}_{p+1}) = E(\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{b_h}{2} + \frac{1}{2}))$$

$$E(H_l(\overset{y}{M}_{p+1}) = \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{b_h}{2} + \frac{5}{2})$$

$$E(T_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{a_e}{2} + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{a_g}{2} + 9\frac{b_g}{2} + \frac{5}{2})$$
$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{b_h}{2} + \frac{5}{2})$$

$$W_{p+1} - E(T_{p+1}) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{b_g}{2} + \frac{5}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + \frac{3a_f}{2} + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{a_h}{2} + 9\frac{b_h}{2} + \frac{5}{2})$$

$$-(\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{a_e}{2} + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{a_g}{2} + 9\frac{b_g}{2} + \frac{5}{2})$$

$$+ \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + 3\frac{b_f}{2} + 1)) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(9a_h i + 9\frac{b_h}{2} + \frac{5}{2}))$$

$$W_{p+1} - E(T_{p+1}) =$$

$$\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1} + 3\frac{a_e}{2}) + \sum_{g=1}^{G}(\sum_{i=0}^{2^{n-p-2}-1} + 9\frac{a_g}{2}) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1} 3\frac{a_f}{2}) + \sum_{h=1}^{H}(\sum_{i=0}^{2^{n-p-2}-1} 9\frac{a_h}{2})$$

$$\left|W_{p+1} - E(T_{p+1})\right| =$$

$$\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1} + 3\frac{a_e}{2}) + \sum_{g=1}^{G}(\sum_{i=0}^{2^{n-p-2}-1} - 9\frac{a_g}{2}) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1} 3\frac{a_f}{2}) + \sum_{h=1}^{H}(\sum_{i=0}^{2^{n-p-2}-1} 9\frac{a_h}{2})$$

####### false from here

- Finding W_{p+1}

$$W_{p+1} = E(R_{p+1}) = E(H_l(\overset{x}{M}_{p+1})) + E(H_r(\overset{y}{M}_{p+1}))$$

$$E(H_l(\overset{x}{M}_{p+1})) = E\ (\sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} + \frac{1}{2}))$$

$$E(H_l(\overset{x}{M}_{p+1})) = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(3a_e i + 3\frac{b_e}{2} + 1)) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(9a_g i + 9\frac{b_g}{2} + \frac{5}{2})$$

$$E(H_r(\overset{y}{M}_{p+1})) = E\ (\sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} \ + \frac{1}{2}))$$

$$E(H_r(\overset{y}{M}_{p+1})) = \sum_{f=1}^{F}\sum_{i=0}^{2^{n-p-2}-1}(3a_f i + 3\frac{a_f}{2} + 3\frac{b_f}{2} + 1) + \sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-1}-1}(9a_h i + 9\frac{a_h}{2} + 9\frac{b_h}{2} \ + \frac{5}{2}))$$

- **General form of** H_p

$$H_p = \sum_{i=0}^{2^{n-p-1}}(2i) + \sum_{k=1}^{2^p-1}(\sum_{i=0}^{2^{n-p-1}-1}(a_k i + b_k)) + \sum_{s=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_s i + b_s))$$

$$H_{p+1} = \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} \ + \frac{1}{2}) + \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{a_e}{2} + \frac{b_e}{2})) +$$
$$\sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{a_g}{2} + 3\frac{b_g}{2} \ + \frac{1}{2}) + \sum_{i=0}^{2^{n-p-2}}(2i) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2})) +$$
$$\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{b_h}{2} \ + \frac{1}{2}) + \sum_{i=0}^{2^{n-p-2}}(2i+1) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})) +$$
$$\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i + 3\frac{a_h}{2} + 3\frac{b_h}{2} \ + \frac{1}{2})$$

$$H_{p+1} = \ \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-2}-1}(a_e i + \frac{b_e}{2})) + \sum_{i=0}^{2^{n-p-2}}(2i) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{b_f}{2})) + \sum_{e=1}^{E}(\sum_{i=0}^{2^{n-p-1}-1}(a_e i + \frac{a_e}{2} + \frac{b_e}{2}))$$
$$\sum_{i=0}^{2^{n-p-2}-1}(2i+1) + \sum_{f=1}^{F}(\sum_{i=0}^{2^{n-p-2}-1}(a_f i + \frac{a_f}{2} + \frac{b_f}{2})) + \sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i + 3\frac{b_g}{2} \ + \frac{1}{2}) +$$

$$\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i+3\frac{b_h}{2}\ \ +\frac{1}{2})+\sum_{g=1}^{G}\sum_{i=0}^{2^{n-p-2}-1}(3a_g i+3\frac{a_g}{2}+3\frac{b_g}{2}\ \ +\frac{1}{2})+$$

$$\sum_{h=1}^{H}\sum_{i=0}^{2^{n-p-2}-1}(3a_h i+3\frac{a_h}{2}+3\frac{b_h}{2}\ \ +\frac{1}{2})$$

$$H_{p+1}=\sum_{i=0}^{2^{n-p-2}}(2i)+\sum_{k=1}^{2^p-1}(\sum_{i=0}^{2^{n-p-2}-1}(a_k i+\frac{b_k}{2}))+\sum_{k=1}^{2^p-1}(\sum_{i=0}^{2^{n-p-2}-1}(a_k i+\frac{a_k}{2}+\frac{b_k}{2}))+\sum_{i=0}^{2^{n-p-2}-1}(2i+1)+$$

$$\sum_{s=1}^{2^p}\sum_{i=0}^{2^{n-p-2}-1}(3a_s i+3\frac{b_s}{2}\ \ +\frac{1}{2})+\sum_{s=1}\sum_{i=0}^{2^{n-p-2}-1}(3a_z i+3\frac{a_s}{2}+3\frac{b_s}{2}\ \ +\frac{1}{2})$$

$$H_{p+1}=\sum_{i=0}^{2^{n-p-1}}(i)+\sum_{k=1}^{2^p-1}(\sum_{i=0}^{2^{n-p-2}-1}(2a_k i+\frac{a_k}{2}+b_k))+\sum_{s=1}^{2^p}\sum_{i=0}^{2^{n-p-2}-1}(6a_s i+3\frac{a_s}{2}\ \ +3b_s+1)$$

So

1) $$H_p=\sum_{i=0}^{2^{n-p-1}}(2i)+\sum_{k=1}^{2^p-1}(\sum_{i=0}^{2^{n-p-1}-1}(a_k i+b_k))+\sum_{s=1}^{2^p}(\sum_{i=0}^{2^{n-p-1}-1}(a_s i+b_s))$$

2) $$H_{p+1}=\sum_{i=0}^{2^{n-p-1}}(i)+\sum_{k=1}^{2^p-1}(\sum_{i=0}^{2^{n-p-2}-1}(2a_k i+\frac{a_k}{2}+b_k))+\sum_{s=1}^{2^p}\sum_{i=0}^{2^{n-p-2}-1}(6a_s i+3\frac{a_s}{2}\ \ +3b_s+1)$$

- **Devlopment of** H_p

$$\sum_{i=0}^{2^{n-p-1}}(2i)=2\sum_{i=0}^{2^{n-p-1}}(i)=2(2^{n-p-1})(2^{n-p-1}+1)\frac{1}{2}=\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}}$$

$$\sum_{k=1}^{2^p-1}(\sum_{i=0}^{2^{n-p-1}-1}(a_k i+b_k))=\sum_{i=0}^{2^{n-p-1}}(A_k i+B_k)$$

Where $A_k=\sum_{k=1}^{2^p-1}(a_k)$ and $B_k=\sum_{k=1}^{2^p-1}(b_k)$

$$\sum_{i=0}^{2^{n-p-1}-1}(A_k i+B_k)=A_k\sum_{i=0}^{2^{n-p-1}-1}(i)+B_k\sum_{i=0}^{2^{n-p-1}-1}(1)=\frac{A_k}{2}(\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}})+\frac{2^n}{2^{p+1}}B_k$$

$$\sum_{s=1}^{2^p}\left(\sum_{i=0}^{2^{n-p-1}-1}(a_s i+b_s)\right) = \sum_{i=0}^{2^{n-p-1}-1}(A_s i+B_s)$$

$$\sum_{i=0}^{2^{n-p-1}-1}(A_s i+B_s) = \frac{A_s}{2}\left(\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}}\right)+\frac{2^n}{2^{p+1}}B_s$$

So

$$H_p = \frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}} + \frac{A_k}{2}\left(\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}}\right)+\frac{2^n}{2^{p+1}}B_k + \frac{A_s}{2}\left(\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}}\right)+\frac{2^n}{2^{p+1}}B_s$$

- **Devlopment of** H_{p+1}

3) $$H_{p+1} = \sum_{i=0}^{2^{n-p-1}}(i)+\sum_{k=1}^{2^p-1}\left(\sum_{i=0}^{2^{n-p-2}-1}(2a_k i+\frac{a_k}{2}+b_k)\right) + \sum_{s=1}^{2^p}\sum_{i=0}^{2^{n-p-2}-1}(6a_s i+3\frac{a_s}{2}+3b_s+1)$$

$$\sum_{i=0}^{2^{n-p-1}}(i)=\frac{1}{2}\left(\frac{2^n}{2^{p+1}}\right)\left(\frac{2^n}{2^{p+1}}+1\right)$$

$$\sum_{k=1}^{2^p-1}\left(\sum_{i=0}^{2^{n-p-2}-1}(2a_k i+\frac{a_k}{2}+b_k)\right) = \sum_{i=0}^{2^{n-p-2}-1}(2A_k i+\frac{A_k}{2}+B_k)$$

$$= 2A_k\sum_{i=0}^{2^{n-p-2}-1}(i)+\left(\frac{A_k}{2}+B_k\right)\sum_{i=0}^{2^{n-p-2}-1}(1) = A_k\left(\frac{2^n}{2^{p+2}}-1\right)\left(\frac{2^n}{2^{p+2}}\right)+\left(\frac{A_k}{2}+B_k\right)\left(\frac{2^n}{2^{p+2}}\right)$$

$$\sum_{s=1}^{2^p}\sum_{i=0}^{2^{n-p-2}-1}(6a_s i+3\frac{a_s}{2}\ \ +3b_s+1) = \sum_{i=0}^{2^{n-p-2}-1}(6A_s i+\frac{3}{2}A_s+3B_s+2^p)$$

$$= 6A_s\sum_{i=0}^{2^{n-p-2}-1}(i)+(\frac{3}{2}A_s+3B_s+2^p)\sum_{i=0}^{2^{n-p-2}-1}(1) = 3A_s(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+(\frac{3}{2}A_s+3B_s+2^p)(\frac{2^n}{2^{p+2}})$$

So $H_{p+1}=\frac{1}{2}(\frac{2^n}{2^{p+1}})(\frac{2^n}{2^{p+1}}+1)$ + $A_k(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+(\frac{A_k}{2}+B_k)(\frac{2^n}{2^{p+2}})$ +

$3A_s(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+(\frac{3}{2}A_s+3B_s+2^p)(\frac{2^n}{2^{p+2}})$

$$H_{p+1}=\frac{2^n}{2^{p+2}}(\frac{2^n}{2^{p+1}}+1)+A_k(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+A_k\frac{2^n}{2^{p+3}}+B_k\frac{2^n}{2^{p+2}}$$

$$+3A_s(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+\frac{3}{2}A_s\ \ \frac{2^n}{2^{p+2}}+3B_s\frac{2^n}{2^{p+2}}+2^p\ \frac{2^n}{2^{p+2}}$$

$$H_{p+1}=\frac{2^n}{2^{p+2}}(\frac{2^n}{2^{p+1}}+1)+A_k((\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+\frac{2^n}{2^{p+3}})+A_s(3(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+\frac{3}{2}\ \ (\frac{2^n}{2^{p+2}}))$$

$$+B_k\frac{2^n}{2^{p+2}}+3B_s\frac{2^n}{2^{p+2}}$$

So

$H_p=\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}}$ + $\frac{A_k}{2}(\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}})+\frac{2^n}{2^{p+1}}B_k$ + $\frac{A_s}{2}(\frac{2^{2n}}{2^{2p+2}}+\frac{2^n}{2^{p+1}})+\frac{2^n}{2^{p+1}}B_s$

and

$$H_{p+1}=\frac{2^n}{2^{p+2}}(\frac{2^n}{2^{p+1}}+1)+A_k((\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+\frac{2^n}{2^{p+3}})+A_s(3(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}})+\frac{3}{2}\ \ (\frac{2^n}{2^{p+2}}))$$

$$+B_k \frac{2^n}{2^{p+2}} + 3B_s \frac{2^n}{2^{p+2}}$$

- Calculation of $H_{p+1} - H_p$

$$H_{p+1} - H_p = \frac{2^n}{2^{p+2}}(\frac{2^n}{2^{p+1}}+1) - (\frac{2^{2n}}{2^{2p+2}} + \frac{2^n}{2^{p+1}}) + A_k((\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}}) + \frac{2^n}{2^{p+3}} - (\frac{2^{2n}}{2^{2p+3}} + \frac{2^n}{2^{p+2}}))$$

$$+A_s(3(\frac{2^n}{2^{p+2}}-1)(\frac{2^n}{2^{p+2}}) + \frac{3}{2}\ (\frac{2^n}{2^{p+2}}) - (\frac{2^{2n}}{2^{2p+3}} + \frac{2^n}{2^{p+2}})) + B_k(\frac{2^n}{2^{p+2}} - \frac{2^n}{2^{p+1}}) + B_s(3\frac{2^n}{2^{p+2}} - \frac{2^n}{2^{p+1}})$$

- Determination of the general form of H_p

Let us pose

$$U_p = H_{p+1} - H_p$$

So $H_{p+1} = U_p + H_p$

$$H_1 = U_0 + H_0$$
$$H_2 = U_1 + H_1 = U_1 + U_0 + H_0$$
$$H_3 = U_2 + H_2 = U_2 + U_1 + U_0 + H_0$$
......
$$H_p = \sum_{q=0}^{p-1}(U_q) + H_0$$

« « « « « «

Calculation of $\sum_{q=0}^{p-1}(U_q)$

$$\sum_{q=0}^{p-1}(U_q) = \sum_{q=0}^{p-1}\left(\frac{2^n}{2^{q+2}}(\frac{2^n}{2^{q+1}}+1)-(\frac{2^{2n}}{2^{2q+2}}+\frac{2^n}{2^{q+1}})+A_k((\frac{2^n}{2^{q+2}}-1)(\frac{2^n}{2^{q+2}})+\frac{2^n}{2^{q+3}}-(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}}))\right.$$

$$\left.+A_s(3(\frac{2^n}{2^{q+2}}-1)(\frac{2^n}{2^{q+2}})+\frac{3}{2}(\frac{2^n}{2^{q+2}})-(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}}))+B_k(\frac{2^n}{2^{q+2}}-\frac{2^n}{2^{q+1}})+B_s(3\frac{2^n}{2^{q+2}}-\frac{2^n}{2^{q+1}})\right)$$

1) $$\sum_{q=0}^{p-1}\frac{2^n}{2^{q+2}}(\frac{2^n}{2^{q+1}}+1) = \sum_{q=0}^{p-1}\frac{2^n}{4}+\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}} = \frac{2^n}{4}\sum_{q=0}^{p-1}(1)+\frac{2^{2n}}{2^3}\sum_{q=0}^{p-1}(\frac{1}{4})^q+\frac{2^n}{4}\sum_{q=0}^{p-1}(\frac{1}{2})^q$$

$$\sum_{q=0}^{p-1}\frac{2^n}{2^{q+2}}(\frac{2^n}{2^{q+1}}+1) = \frac{2^n p}{4}+(\frac{2^{2n}}{2^3})(\frac{1-\frac{1}{4^p}}{1-\frac{1}{4}})+\frac{2^n}{4}(\frac{1-\frac{1}{2^p}}{1-\frac{1}{2}})$$

$$= \frac{4}{3}(\frac{2^{2n}}{2^3})(1-\frac{1}{4^p})+\frac{2^n}{2}(1-\frac{1}{2^p})$$

2) $$\sum_{q=0}^{p-1}-(\frac{2^{2n}}{2^{2q+2}}+\frac{2^n}{2^{q+1}}) = -\frac{2^n}{4}\sum_{q=0}^{p-1}\frac{1}{4^q}-\frac{2^n}{2}\sum_{q=0}^{p-1}\frac{1}{2^q}$$

$$\sum_{q=0}^{p-1}-(\frac{2^{2n}}{2^{2q+2}}+\frac{2^n}{2^{q+1}}) = -\frac{2^n}{4}(\frac{1-\frac{1}{4^p}}{1-\frac{1}{4}})-\frac{2^n}{2}(\frac{1-\frac{1}{2^p}}{1-\frac{1}{2}})$$

$$\sum_{q=0}^{p-1}-(\frac{2^{2n}}{2^{2q+2}}+\frac{2^n}{2^{q+1}}) = -\frac{2^n}{3}(1-\frac{1}{4^p})-2^n(1-\frac{1}{2^p})$$

3) $$\sum_{q=0}^{p-1}A_k((\frac{2^n}{2^{q+2}}-1)(\frac{2^n}{2^{q+2}})+\frac{2^n}{2^{q+3}}-(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}})) = ?$$

$$A_k\sum_{q=0}^{p-1}(\frac{2^n}{2^{q+2}}-1)(\frac{2^n}{2^{q+2}}) = A_k\sum_{q=0}^{p-1}\frac{2^{2n}}{2^{2q+4}}-A_k\sum_{q=0}^{p-1}\frac{2^n}{2^{q+2}} = \frac{A_k 2^{2n}}{16}\sum_{q=0}^{p-1}\frac{1}{4^q}-\frac{A_k 2^n}{4}\sum_{q=0}^{p-1}\frac{1}{2^q}$$

$$= \frac{A_k 2^{2n}}{16}\left(\frac{1-\frac{1}{4^p}}{1-\frac{1}{4}}\right)-\frac{A_k 2^n}{4}\left(\frac{1-\frac{1}{2^p}}{1-\frac{1}{2}}\right) = \frac{4}{3}\left(\frac{A_k 2^{2n}}{16}\right)\left(1-\frac{1}{4^p}\right)-\frac{A_k 2^n}{2}\left(1-\frac{1}{2^p}\right)$$

$$A_k\sum_{q=0}^{p-1}\left(\left(\frac{2^n}{2^{q+2}}-1\right)\left(\frac{2^n}{2^{q+2}}\right)\right)=\frac{4}{3}\left(\frac{A_k 2^{2n}}{16}\right)\left(1-\frac{1}{4^p}\right)-\frac{A_k 2^n}{2}\left(1-\frac{1}{2^p}\right)$$

$$A_k\sum_{q=0}^{p-1}\left(\frac{2^n}{2^{q+3}}\right)=\frac{2^n A_k}{8}\left(\frac{1-\frac{1}{2^p}}{\frac{1}{2}}\right)=\frac{2^n A_k}{4}\left(1-\frac{1}{2^p}\right)$$

$$A_k\sum_{q=0}^{p-1}-\left(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}}\right)=-\frac{A_k 2^{2n}}{8}\left(\frac{1-\frac{1}{4^p}}{\frac{3}{4}}\right)-\frac{A_k 2^n}{4}\left(\frac{1-\frac{1}{2^p}}{\frac{1}{2}}\right)$$

$$A_k\sum_{q=0}^{p-1}-\left(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}}\right)=-\frac{A_k 2^{2n}}{6}\left(1-\frac{1}{4^p}\right)-\frac{A_k 2^n}{2}\left(1-\frac{1}{2^p}\right)$$

$$\sum_{q=0}^{p-1}A_k\left(\left(\frac{2^n}{2^{q+2}}-1\right)\left(\frac{2^n}{2^{q+2}}\right)+\frac{2^n}{2^{q+3}}-\left(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}}\right)\right)=\frac{4}{3}\left(\frac{A_k 2^{2n}}{16}\right)\left(1-\frac{1}{4^p}\right)-\frac{A_k 2^n}{2}\left(1-\frac{1}{2^p}\right)+$$
$$\frac{2^n A_k}{4}\left(1-\frac{1}{2^p}\right)-\frac{A_k 2^{2n}}{6}\left(1-\frac{1}{4^p}\right)-\frac{A_k 2^n}{2}\left(1-\frac{1}{2^p}\right)$$

4) $$\sum_{q=0}^{p-1}A_s\left(3\left(\frac{2^n}{2^{q+2}}-1\right)\left(\frac{2^n}{2^{q+2}}\right)+\frac{3}{2}\left(\frac{2^n}{2^{q+2}}\right)-\left(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}}\right)\right)=?$$

$$\sum_{q=0}^{p-1}A_s\left(3\left(\frac{2^n}{2^{q+2}}-1\right)\left(\frac{2^n}{2^{q+2}}\right)\right)=\sum_{q=0}^{p-1}3A_s\frac{2^{2n}}{2^{2q+4}}-3A_s\frac{2^n}{2^{q+2}}=3A_s\frac{2^{2n}}{16}\left(\frac{1-\frac{1}{4^p}}{\frac{3}{4}}\right)-3A_s\frac{2^n}{4}\left(\frac{1-\frac{1}{2^p}}{\frac{1}{2}}\right)$$

$$\sum_{q=0}^{p-1}A_s\left(3\left(\frac{2^n}{2^{q+2}}-1\right)\left(\frac{2^n}{2^{q+2}}\right)\right)=A_s\frac{2^{2n}}{4}\left(1-\frac{1}{4^p}\right)-3A_s\frac{2^n}{2}\left(1-\frac{1}{2^p}\right)$$

$$\sum_{q=0}^{p-1}\frac{3}{2}A_s\frac{2^n}{2^{q+2}}=\frac{3}{8}A_s 2^n(\frac{1-\frac{1}{2^p}}{\frac{1}{2}})=\frac{3}{4}A_s 2^n(1-\frac{1}{2^p})$$

$$\sum_{q=0}^{p-1}-A_s(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}})=-A_s\frac{2^{2n}}{8}(\frac{1-\frac{1}{4^p}}{\frac{3}{4}})-A_s\frac{2^n}{4}(\frac{1-\frac{1}{2^p}}{\frac{1}{2}})=-A_s\frac{2^{2n}}{6}(1-\frac{1}{4^p})-A_s\frac{2^n}{2}(1-\frac{1}{2^p})$$

$$\sum_{q=0}^{p-1}A_s(3(\frac{2^n}{2^{q+2}}-1)(\frac{2^n}{2^{q+2}})+\frac{3}{2}(\frac{2^n}{2^{q+2}})-(\frac{2^{2n}}{2^{2q+3}}+\frac{2^n}{2^{q+2}}))=A_s\frac{2^{2n}}{4}(1-\frac{1}{4^p})-3A_s\frac{2^n}{2}(1-\frac{1}{2^p})+$$
$$\frac{3}{4}A_s 2^n(1-\frac{1}{2^p})+-A_s\frac{2^{2n}}{6}(1-\frac{1}{4^p})-A_s\frac{2^n}{2}(1-\frac{1}{2^p})$$

5) $$\sum_{q=0}^{p-1}B_k(\frac{2^n}{2^{q+2}}-\frac{2^n}{2^{q+1}})=B_k\frac{2^n}{4}(\frac{1-\frac{1}{2^p}}{\frac{1}{2}})+B_k\frac{2^n}{2}(\frac{1-\frac{1}{2^p}}{\frac{1}{2}})=B_k\frac{2^n}{2}(1-\frac{1}{2^p})+B_k 2^n(1-\frac{1}{2^p})$$

6) $$\sum_{q=0}^{p-1}B_s(3\frac{2^n}{2^{q+2}}-\frac{2^n}{2^{q+1}})=3B_s\frac{2^n}{4}(\frac{1-\frac{1}{2^p}}{\frac{1}{2}})+B_s\frac{2^n}{2}(\frac{1-\frac{1}{2^p}}{\frac{1}{2}})=3B_s\frac{2^n}{2}(1-\frac{1}{2^p})+B_s 2^n(1-\frac{1}{2^p})$$

So

$$\sum_{q=0}^{p-1}(U_q) = \frac{4}{3}(\frac{2^{2n}}{2^3})(1-\frac{1}{4^p})+\frac{2^n}{2}(1-\frac{1}{2^p})-\frac{2^n}{3}(1-\frac{1}{4^p})-2^n(1-\frac{1}{2^p}) + \frac{4}{3}(\frac{A_k 2^{2n}}{16})$$
$$(1-\frac{1}{4^p})-\frac{A_k 2^n}{2}(1-\frac{1}{2^p}) + \frac{2^n A_k}{4}(1-\frac{1}{2^p}) -\frac{A_k 2^{2n}}{6}(1-\frac{1}{4^p})-\frac{A_k 2^n}{2}(1-\frac{1}{2^p})+$$
$$A_s\frac{2^{2n}}{4}(1-\frac{1}{4^p})-3A_s\frac{2^n}{2}(1-\frac{1}{2^p})+$$
$$\frac{3}{4}A_s 2^n(1-\frac{1}{2^p})+-A_s\frac{2^{2n}}{6}(1-\frac{1}{4^p})-A_s\frac{2^n}{2}(1-\frac{1}{2^p})+B_k\frac{2^n}{2}(1-\frac{1}{2^p})+B_k 2^n(1-\frac{1}{2^p})+3B_s\frac{2^n}{2}(1-\frac{1}{2^p})$$
$$+B_s 2^n(1-\frac{1}{2^p})$$

$$\sum_{q=0}^{p-1}(U_q) =$$

We know $H_p = \sum_{q=0}^{p-1}(U_q)+H_0$

« « « « « « «

In order to find H_0 we must return to the pyramid

Where $H_0 = \sum_{i=0}^{2^{n-1}}(2i)+\sum_{i=0}^{2^{n-1}-1}(2i+1)$

$$H_0 = \frac{2^n}{2}(\frac{2^n}{2}+1)+(\frac{2^n}{2}-1)\frac{2^n}{2}+\frac{2^n}{2}$$

Now $H_p = \frac{4}{3}(\frac{2^{2n}}{2^3})(1-\frac{1}{4^p})+\frac{2^n}{2}(1-\frac{1}{2^p})-\frac{2^n}{3}(1-\frac{1}{4^p})-2^n(1-\frac{1}{2^p}) + \frac{4}{3}(\frac{A_k 2^{2n}}{16})$

$$(1-\frac{1}{4^p})-\frac{A_k 2^n}{2}(1-\frac{1}{2^p}) + \frac{2^n A_k}{4}(1-\frac{1}{2^p}) -\frac{A_k 2^{2n}}{6}(1-\frac{1}{4^p})-\frac{A_k 2^n}{2}(1-\frac{1}{2^p})+$$
$$A_s\frac{2^{2n}}{4}(1-\frac{1}{4^p})-3A_s\frac{2^n}{2}(1-\frac{1}{2^p})+\frac{3}{4}A_s 2^n(1-\frac{1}{2^p})+-A_s\frac{2^{2n}}{6}(1-\frac{1}{4^p})-A_s\frac{2^n}{2}(1-\frac{1}{2^p})+B_k\frac{2^n}{2}(1-\frac{1}{2^p})$$
$$+B_k 2^n(1-\frac{1}{2^p})+3B_s\frac{2^n}{2}(1-\frac{1}{2^p})+B_s 2^n(1-\frac{1}{2^p})+\frac{2^n}{2}(\frac{2^n}{2}+1)+(\frac{2^n}{2}-1)\frac{2^n}{2}+\frac{2^n}{2}$$

Now to reach the last line of the pyramid p must tend to n.

Let's calculate $\lim_{p\to n-1}(H_p)$

$$H_{n-1} = \lim_{p\to n-1}(H_p) = \lim_{p\to n-1}\Big(\frac{4}{3}\left(\frac{2^{2n}}{2^3}\right)\left(1-\frac{1}{4^{n-1}}\right) + \frac{2^n}{2}\left(1-\frac{1}{2^{n-1}}\right) - \frac{2^n}{3}\left(1-\frac{1}{4^{n-1}}\right) - 2^n\left(1-\frac{1}{2^{n-1}}\right) +$$

$$\frac{4}{3}\left(\frac{A_k 2^{2n}}{16}\right)\left(1-\frac{1}{4^{n-1}}\right) - \frac{A_k 2^n}{2}\left(1-\frac{1}{2^{n-1}}\right) + \frac{2^n A_k}{4}\left(1-\frac{1}{2^{n-1}}\right) - \frac{A_k 2^{2n}}{6}\left(1-\frac{1}{4^{n-1}}\right) - \frac{A_k 2^n}{2}\left(1-\frac{1}{2^{n-1}}\right) +$$

$$A_s\frac{2^{2n}}{4}\left(1-\frac{1}{4^{n-1}}\right) - 3A_s\frac{2^n}{2}\left(1-\frac{1}{2^{n-1}}\right) + \frac{3}{4}A_s 2^n\left(1-\frac{1}{2^{n-1}}\right) + -A_s\frac{2^{2n}}{6}\left(1-\frac{1}{4^{n-1}}\right) - A_s\frac{2^n}{2}\left(1-\frac{1}{2^{n-1}}\right) + B_k\frac{2^n}{2}\ (1-\frac{1}{2^{n-1}}$$

$$+ B_k 2^n\left(1-\frac{1}{2^{n-1}}\right) + 3B_s\frac{2^n}{2}\left(1-\frac{1}{2^{n-1}}\right) + B_s 2^n\left(1-\frac{1}{2^{n-1}}\right) + \frac{2^n}{2}\left(\frac{2^n}{2}+1\right) + \left(\frac{2^n}{2}-1\right)\frac{2^n}{2} + \frac{2^n}{2}$$

$$\Big)$$

« « « « « « « «