Gift #6 – The Dot Product
Due Friday, September 20
(at the start of class)

This gift covers lesson 8 (from Monday, Sept. 16). To spread out your work you should complete exercises 1-4 by Wednesday. Feel free to work with others on them, but the write-ups must be your own. As with any assignment, please document any help you receive at the top of your submission.

1. **Short Answer/Computations:**

a) Compute the dot product $\vec{u}\cdot\vec{v}$ for $\vec{u} = 4\hat{\imath} + 10\hat{\jmath} - 3\hat{k}$ and $\vec{v} = 3\hat{\imath} - \hat{\jmath} - 2\hat{k}$.

b) Your answer to part a) should have been positive. What does this say about the orientation of the two vectors $\vec{u}$ and $\vec{v}$?

c) Compute the dot product $\vec{u}\cdot\vec{v}$ given that $\|\vec{u}\| = 8$, $\|\vec{v}\| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{3}$.

d) Find the angle between $\vec{u} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{v} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$.

2. For each pair of vectors below, determine if the vectors are parallel, orthogonal (i.e., perpendicular) or neither. Show your work.

a) $\vec{u} = -2\hat{\imath} + 3\hat{\jmath} - \hat{k},\ \vec{v} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$

b) $\vec{u} = \langle 2, 2, -2\rangle,\ \vec{u} = \langle 4, -4, -4\rangle$

c) $\vec{u} = \langle \cos(\theta), \sin(\theta), 1\rangle,\ \vec{v} = \langle \sin(2\theta), 2\sin^2(\theta), 2\sin(\theta)\rangle$

3. Prove the property $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$. (You may assume $\vec{u} = \langle u_1, u_2, u_3\rangle$ is a three-dimensional vector, although it's easy to see that the argument extends to vectors of any dimension.)

4. The proof (presented in class) of the equivalence of the analytic and geometric definitions of the dot product made good use of the property:

$$\vec{u}\cdot(\lambda\vec{v}) = \lambda(\vec{u}\cdot\vec{v}) \text{ and } (\lambda\vec{u})\cdot\vec{v} = \lambda(\vec{u}\cdot\vec{v}) \text{ for all scalars } \lambda.$$

Do parts a) and b) below to prove this property.

a) Using the geometric definition of the dot product, show that $\vec{u}\cdot(-\vec{v}) = -(\vec{u}\cdot\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$. (Hint: What happens to the angle when you multiply $\vec{v}$ by -1?)

b) Using the geometric definition of the dot product, show that for any positive scalar λ

$$\vec{u}\cdot(\lambda\vec{v}) = \lambda(\vec{u}\cdot\vec{v})$$
$$\text{and } (\lambda\vec{u})\cdot\vec{v} = \lambda(\vec{u}\cdot\vec{v}).$$

5. Assume $\vec{u}\cdot\vec{v} = 2$, $\|\vec{u}\| = 4$, and $\|\vec{v}\| = 3$. What is the value of $5\vec{u}\cdot(\vec{u} - 2\vec{v})$? (Hint: Use previous exercises and the distributive property: $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.)

6. **More Practice…**

a) Given nonzero vectors $\vec{u}$ and $\vec{v}$, prove that $\|\vec{v}\|\vec{u} + \|\vec{u}\|\vec{v}$ is perpendicular to $\|\vec{v}\|\vec{u} - \|\vec{u}\|\vec{v}$.

b) Prove the "Cauchy-Schwarz" Inequality: $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$ for any two vectors $\vec{u}$ and $\vec{v}$.

7. **True or False (and Explain)** If the statement is true, prove it; if the statement is false, provide a counterexample.

a) If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ and $\vec{u}$ is nonzero, then $\vec{v} = \vec{w}$.

b) If $\vec{u}$ and $\vec{v}$ are perpendicular to $\vec{w}$, then $\vec{u} + \vec{v}$ is perpendicular to $\vec{w}$, too.

8. a) Find the equation of the plane through the origin and perpendicular to the vector $\vec{v} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$.

b) Find the equation of the plane through the point (1, 1, -3) and perpendicular to the vector $\vec{v} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$.

c) Find the equation of the plane through the point (1, 1, -3) and parallel to the plane

$$3x - 2y + z = 6.$$