LDA:

Solve the problem

$$J = \frac{v^t S_B v}{v^t S_W v} \longrightarrow max$$

$$S_B = \sum_{c=1}^{C} (m_c - m)(m_c - m)^t$$

$$m = 1/N \sum_{i=1}^{N} x_i$$

$$m_c = 1/N_c \sum_{x_i \in C_c} x_i$$

$$S_W = \sum_{c=1}^{C} S_{W_c}$$

$$S_{W_c} = 1/N_c \sum_{x_i \in C_c} (x_i - m_c)(x_i - m_c)^t$$

let try to rewrite $S_B$ and $S_W$ using Matrix:

suppose $X$ is a big matrix which contains d rows and N columns each column represents one sample $x_i$

$$X = \begin{bmatrix} | & | & | & & | \\ x_1 & x_2 & x_3 & \cdots & x_N \\ | & | & | & & | \end{bmatrix}$$

So:

$$m = 1/N \, X @ \text{ones}(N, 1)$$

$$m_c = 1/N_c \, X @ \text{ind}(N, 1, c)$$

where ind returns vector such as

$$\text{ind}[i] = \begin{cases} 1 & \text{if } x_i \in C \\ 0 & \text{otherwise} \end{cases}$$

So:

$$S_B = \sum_{c=1}^{N} \left[1/N_c \, X @ \text{ind}(N,1,c) - 1/N \, X @ \text{ones}(N,1)\right] \left[1/N_c \, X @ \text{ind}(N,1,c) - 1/N \, X @ \text{ones}(N,1)\right]^t$$

KLDA

Solve the problem

$$J = \frac{v^t S_B^{\phi} v^t}{v^t S_W^{\phi} v^t}$$

$$S_B = \sum_{c=1}^{C} (m_c^{\phi} - m^{\phi})(m_c^{\phi} - m^{\phi})^t$$

$$m^{\phi} = 1/N \sum_{i=1}^{N} \phi(x_i)$$

$$m_c = 1/N_c \sum_{x_i \in C_c} \phi(x_i)$$

$$S_W^{\phi} = \sum_{c=1}^{C} S_{W_c}^{\phi}$$

$$S_{W_c}^{\phi} = 1/N_c \sum_{x_i \in C_c} (\phi(x_i) - m_c^{\phi})(\phi(x_i) - m_c^{\phi})$$

$$S_B = \sum_{c=1} \underline{X} \left[ \frac{1}{N_c} \mathrm{ind}(N,1,c) - \frac{1}{N} \mathrm{ones}(N,1) \right]$$
$$\left[ \frac{1}{N_c} \mathrm{ind}(N,1,c) - \frac{1}{N} \mathrm{ones}(N,1) \right]^t X^t$$
$$= \underline{X} \left( \sum_{c=1}^{N} \left[ \frac{1}{N_c} \mathrm{ind}(N,1,c) - \frac{1}{N} \mathrm{ones}(N,1) \right] \left[ \frac{1}{N_c} \mathrm{ind}(N,1,c) - \frac{1}{N} \mathrm{ones} \right. \right.$$
$$\underline{X}^t$$
$$= \underline{X} \; M \; \underline{X}^t \quad \text{where } M = \sum_{c=1}^{N} \left[ \frac{1}{N_c} \mathrm{ind}(N,1,c) - \frac{1}{N} \mathrm{ones}(N,1) \right]$$
$$\left[ \frac{1}{N_c} \mathrm{ind}(N,1,c) - \frac{1}{N} \mathrm{ones}(N,1) \right]^t$$

$$S_{Wc} = \frac{1}{N_c} \sum_{x_i \in C_c} (x_i - m_c)(x_i - m_c)^t = \frac{1}{N_c} \sum_{x_i \in C} x_i x_i^t - m_c m_c^t$$

$$\underline{X}_c = \underline{X} @ (N, \cancel{N}, c) \text{ where } \mathrm{ind}(i,i) = \begin{cases} 1 & \text{if } x_i \in C_c \\ 0 & \text{otherwise} \end{cases}$$

$$S_{Wc} = \frac{1}{N_c} \left( \underline{X} @ \mathrm{ind}(N, \cancel{N}, c) \right) @ \left( X @ \mathrm{ind}(N, \cancel{N}, c) \right)^t$$
$$- \frac{1}{N_c^2} \left[ X @ \mathrm{ind}(N,1,c) \right] \left[ X @ \mathrm{ind}(N,1,c) \right]^t$$
$$\mathrm{ind}(N,N,c)$$
$$= \frac{1}{N_c} \underline{X} @ \left[ \mathrm{ind}(N,N,c) @ \mathrm{ind}(N,N,c) \right] X^t$$
$$- \frac{1}{N_c^2} \underline{X} @ \left[ \mathrm{ind}(N,1,c) @ \mathrm{ind}(N,1,c)^t \right] \underline{X}^t$$

$$S_W = \sum_{c=1}^{C} \frac{1}{N_c} \underline{X} @ \mathrm{ind}(N,N,c) \underline{X}^t$$
$$- \sum_{c=1}^{C} \frac{1}{N_c^2} \underline{X} @ \left[ \mathrm{ind}(N,1,c) @ \mathrm{ind}(N,1,c)^t \right] \underline{X}^t$$
$$= \underline{X} \left[ \sum_{c=1}^{C} \frac{1}{N_c} \mathrm{ind}(N,N,c) - \sum_{c=1}^{C} \frac{1}{N_c^2} \left\{ \mathrm{ind}(N,1,c) @ \mathrm{ind}(N,1,c)^t \right] \underline{X}^t$$
$$= \underline{X} \; N \; \underline{X}_c^t$$
$$\text{where } N = \sum_{c=1}^{C} \frac{1}{N_c} \mathrm{ind}(N,N,c) - \sum_{c=1}^{C} \frac{1}{N_c^2} \mathrm{ind}(N,1,c) @ \mathrm{ind}(N,1,c)^t$$

So the LDA try to solve the problem

$$J = \frac{v^t \underline{X} M X^t v}{v^t \underline{X} N X^t v} \longrightarrow \max$$

<u>now ~~fo~~ KLDA</u>: normally it is the same we have just to replace $\underline{X}$ by $\underline{X}^{\phi}$ which represents

$$\underline{X}^{\phi} = \left[ \phi(x_1) \;\; \phi(x_2) \; \cdots \; \phi(x_N) \right]$$

$$J = \frac{v^t \underline{X}^{\phi} M \underline{X}^{\phi t} v}{v^t \underline{X}^{\phi} N \underline{X}^{\phi t} v} \longrightarrow \max$$

the solution of this problem is a combination of the input samples ( $v$ is in the span of the space generated by the samples in $\underline{X}^{\phi}$)

$v = \underline{X}^{\phi} \alpha$ where $\alpha$ is a vector which represents the weight of each sample

in this case $J$ becomes:

$$J = \frac{\alpha^t \underline{X}^{\phi t} \underline{X}^{\phi} M \underline{X}^{\phi t} \underline{X}^{\phi} \alpha}{\alpha^t \underline{X}^{\phi t} \underline{X}^{\phi} N \underline{X}^{\phi t} \underline{X}^{\phi} \alpha} = \frac{\alpha^t G M G^t \alpha}{\alpha G N G^t \alpha} \longrightarrow \max$$

where $G$ is the gram matrix

The solution to this problem is the generalized eigen problem

$$G M G^t \alpha = \lambda G N G^t \alpha$$

we select $\alpha$ which corresponds to the highest eigen value.