Mean and Variance of Discrete Random Variables

Expected Value of Discrete Random Variable

Suppose you and I play a betting game: we flip a coin and if it lands heads, I give you a dollar, and if it lands tails, you give me a dollar.

On average, how much am I expected to win or lose?

$$\text{expected winnings} = \underbrace{(-1)\left(\frac{1}{2}\right)}_{\text{win -\$1 half the time}} + \underbrace{(1)\left(\frac{1}{2}\right)}_{\text{win \$1 half the time}} = 0$$

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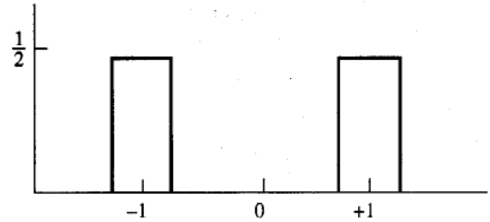
Figure 4.4. Relative frequency of winnings



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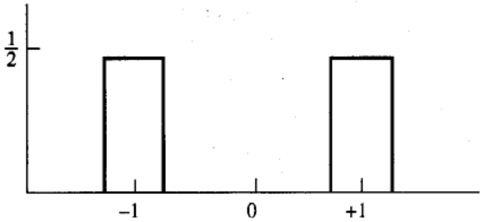

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Definition of Expected Value of a Discrete Random Variable

Definition

The **expected value of a discrete random variable** X with probability distribution $p(x)$ is given by

$$\mathrm{E}(X) \triangleq \mu = \sum_x x\, p_X(x) \qquad (\star)$$

where the sum is over all values of x for which $p_X(x) > 0$.

Note that in order for $(\star)$ to exist, the sum must converge absolutely; that is

$$\sum_x |x|\, p_X(x) < \infty \qquad (\star\star)$$

If $(\star\star)$ does not hold, we say the expected value of X does not exist.

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Expected value of functions of random variables

The expected value of $g(X)$ where g is any real-valued function is naturally

$$\mathrm{E}(g(X)) = \sum_x g(x)p(x)$$

Example

Consider the Bernoulli random variable

$$X = \begin{cases} 0, & \text{w.p. } 1/2 \\ 1, & \text{w.p. } 1/2 \end{cases}$$

Compute $\mathrm{E}\left(X^2 - 1\right)$.

$$\mathrm{E}\left(X^2 - 1\right) = (0^2 - 1)\left(\frac{1}{2}\right) + ((1)^2 - 1)\left(\frac{1}{2}\right) = \frac{-1}{2}$$

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Birthday Problem Revisited

65 people participated in the birthday game a few weeks back.

I claimed that if no two birthdays matched, then I would pay everyone 30 monopoly dollars, but otherwise each person would pay me one monopoly dollar.

Therefore either I would lose 65*30=1950 monopoly dollars, or I would win 65 monopoly dollars.

My expected earnings should be somewhere between -1950 and +65. Let's compute it.

My total earnings are represented by the following random variable

$$X = \begin{cases} 30, & \text{w.p. } p \\ -1950, & \text{w.p. } 1-p \end{cases}$$

where

$$p \approx \frac{365(364)(363)\cdots(301)}{365^{65}} \approx .9977$$

Therefore

$$\mathrm{E}(X) = 30(.9977) - 1950(.0023) = 25.446$$

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Definition (variance)

The **variance** of a random variable X with expected value μ is given by

$$\text{var}(X) \triangleq \sigma^2 = \mathrm{E}\left[(X-\mu)^2\right]$$

Definition

The **standard deviation** of a random variable X is, σ, the square root of the variance, i.e.

$$\text{sd}(X) \triangleq \sigma = \sqrt{\mathrm{E}\left[(X-\mu)^2\right]} = \sqrt{\text{var}(X)}$$

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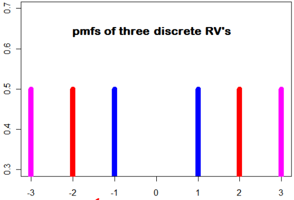
Some Distributions



$$X_1 = \begin{cases} -1, & \text{w.p. } 1/2 \\ 1, & \text{w.p. } 1/2 \end{cases} \quad X_2 = \begin{cases} -2, & \text{w.p. } 1/2 \\ 2, & \text{w.p. } 1/2 \end{cases} \quad X_3 = \begin{cases} -3, & \text{w.p. } 1/2 \\ 3, & \text{w.p. } 1/2 \end{cases}$$

Hence $\mathrm{E}(X_1) = \mathrm{E}(X_2) = \mathrm{E}(X_3) = 0$ and

$$\mathrm{var}(X_1) = (1-0)^2\left(\frac{1}{2}\right) + (-1-0)^2\left(\frac{1}{2}\right) = 1$$

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Therefore $\mathrm{sd}(X_1) = 1$, $\mathrm{sd}(X_2) = 2$, and $\mathrm{sd}(X_3) = 3$.

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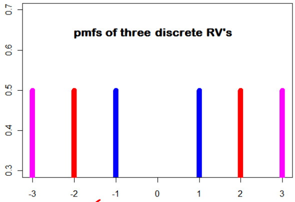

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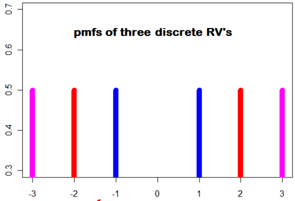

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Theorem 4.2

Theorem (mean and variance of $aX + b$)

For any random variable X (discrete or not) and constants a and b,

1. $\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$
2. $\mathrm{var}(aX + b) = a^2\,\mathrm{var}(X)$

It follows that if X has mean μ and standard deviation σ, then

$$Y = \frac{x - \mu}{\sigma}$$

has mean 0 and standard deviation 1. *This is the standardized form of X.*

Theorem

If X and Y are random variables, then $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$.

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Theorem 4.3

Theorem ($\text{var}(X) = \text{E}(X^2) - \mu^2$)

If X is a random variable with mean μ, then

$$\text{var}(X) = \text{E}(X^2) - \mu^2$$

Proof.

$$\begin{aligned}
\text{var}(X) &= \text{E}\left[(X-\mu)^2\right] \\
&= \text{E}(X^2 - 2X\mu + \mu^2) \\
&= E(X^2) - E(2X\mu) + E(\mu^2) \\
&= E(X^2) - 2\mu^2 + \mu^2 \\
&= E(X^2) - \mu^2
\end{aligned}$$

□

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$\square$

Chebyshev's inequality

There are many random variables X with a given mean μ and a given variance σ^2, but they all must satisfy the following inequality.

Theorem

Chebyshev's inequality Let X be a random variable with mean μ and variance σ^2. Then for any positive k,

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$$

Letting $k = 2$ in Chebyshev's inequality gives

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \geq 1 - \frac{1}{4} = \frac{3}{4}$$

That is, the interval from $\mu - 2\sigma$ to $\mu + 2\sigma$ must contain at least 3/4 of the probability mass.

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Exercise 4.17

4.17. You are to pay \$1.00 to play a game that consists of drawing one ticket at random from a box of numbered tickets. You win the amount (in dollars) of the number on the ticket you draw. The following two boxes of numbered tickets are available.

I. $\boxed{0, 1, 2}$ II. $\boxed{0, 0, 0, 1, 4}$

a. Find the expected value and variance of your net gain per play with box I.
b. Repeat part (a) for box II.
c. Given that you have decided to play, which box would you choose, and why?

Exercise 4.37

4.37. Four couples go to dinner together. The waiter seats the men randomly on one side of the table and the women randomly on the other side of the table. Find the expected value and variance of the number of couples who are seated across from each other.