A Brief Introduction to the Linear Algebra - Systems of Linear Equations

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1 Introduction

1.1 Purpose and Scope

The publication is intended for the Bachelor's Technical or Natural Sciences Students. The publication aims to provide the basic theoretical knowledge and the different methods of how the system of linear equations (hereafter, SLE) can be solved. Linking the theory with practical examples after is foreseen as a valuable step to put the gained knowledge into the practice. At the end of the publication, the database where the main terms can be searched is provided. The publication has been created based on the "Mathematics 1" Course that was chaired by the author (P. Szabó) on the Faculty of Aeronautics (Academic year 2017/2018 and 2018/2019). The scope of the course is 6 hours of theory and 6 hours of practice.

1.2 Prerequisite to use the publication

As a prerequisite to use this publication, it is recommended to be familiar with the following terms: *real numbers*, *constant*, *variable*, *algebraic expression*, *basic algebraic operations* (*sum* and *multiplication* of real numbers) and editing the algebraic expressions and also with the terms from informatics and computer science such as *algorithm*, *method*, *algorithm complexity* and differences between method and algorithm.

1.3 The Field of Real Numbers

In this publication, authors are using the set of the real numbers $\mathcal{R}$ with arithmetic operations: Addition ($+$) and Multiplication ($\times$). This algebraic structure is a $field$ with 2 binary operations indexed as $(\mathcal{R}, +, \times)$.

Properties	Addition +	Multiplication ·
Closure	$a+b$ is a unique real number	$a \cdot b$ is a unique real number
Commutative	$a+b=b+a$	$a \cdot b = b \cdot a$
Associative	$(a+b)+c=$ $a+(b+c)$	$(ab)c=$ $a(bc)$
Identity	$0 \Rightarrow a+0=a$	$1 \Rightarrow a \cdot 1 = a$
Inverse	$a \Rightarrow a+(-a)=0$ $\Rightarrow (-a)+a=0$	$a \neq 0 \Rightarrow a \cdot \frac{1}{a} = 1$ $\Rightarrow \frac{1}{a} \cdot a = 1$
Distributive *or Linking Property*	$a \cdot (b+c) = a \cdot b + a \cdot c$	
Transitivity	$a=b \,\&\, b=c \Rightarrow a=c$ $a>b \,\&\, b>c \Rightarrow a>c$ $a<b \,\&\, b<c \Rightarrow a<c$	
Note: $ab=a(b)=(a)b$ *are alternate representations of* $a \cdot b$		

Figure 1: Main characteristics of $(\mathcal{R}, +, \times)$

With respect to defined operations: For this algebraic structure the following rules, laws apply - **Commutative, Associative and Distributive Law**. Field of real numbers! In the field is a Null element ($a + 0 = a$), Unit element ($a \times 1 = a$) and there are Inverse Elements ($a - a = 0$, $a \times \frac{1}{a} = 1$). The set of real numbers is closed for operations $(+, \times)(\forall a, b \in R,\ a + b \in R, a \times b \in R)$.

The main characteristics of $(\mathcal{R}, +, \times)$ are depicted in Fig. 1. More detailed information about this algebraic structure is possible to find in [3], [4], [5]. Note: in further chapters, we will be using ab instead of $a \times b$ and $3 \cdot 5$ instead of 3×5. The main terms or keywords are distinguished with **bold** or *italic*.

1.4 Linear Algebra and System of Linear Equations (SLE)

Linear algebra together with mathematical analysis and analytic geometry belong to the main mathematical disciplines. The scope of this publication is only linear algebra and is concerned with:

- system of linear equations (SLE):

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{gathered} \tag{1}
$$

- eigenvalue problem, special system of linear equations:

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2 \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = \lambda x_m
\end{gathered} \tag{2}
$$

Solvability and resolution of polynomials belong to the another part of algebra. The equations of type:

$$
f(x) = a_1x^n + a_2x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \tag{3}
$$

is a **polynomial of the** n^{th} **degree** . Real constants $a_1, a_2, \ldots, a_n$ are the coefficients of polynomial. Variable $z \in R$ is a **root** of polynomial (3) when $f(z) = 0$. Equation (3) is not linear, because the equation contains not only linear operations, higher powers of the variable x.

To summarize, the linear algebra is aiming to find a solution of the system of linear equations.

A table of constants

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
$$

is defining a **matrix** of the SLE of type $m \times n$. The matrix of type $m \times 1$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is a

right-side vector of SLE. If $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ we say that the SLE is homogenous, otherwise it is heterogeneous. The matrix of type $n \times 1, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a vector with the **unknown entries**. If we substitute the vector with the unknown entries into the eq. (1) and we get the equalities in all cases, then we say that this vector is a resolution of SLE.

In (2) we are searching for the vector with the unknown entries x (that is a solution of the SLE and is called **eigenvector of SLE**) and value λ (that is called **eigenvalue**).

In general, linear algebra is concerned with the algebraic structures and algebraic operations that are determining the entry conditions for the resolution of (3), (1), (2). The main topics of linear algebra are matrices, vectors, linear equations, and the algebraic structure known as a **vector space**. In this publication, authors only cover the examples of equation type (1) over the field of real numbers $(\mathcal{R}, +, \times)$.

1.5 Publication Topics

This publication covers the following topics:

- Elementary row operations
- Matrices and Vectors
- Operations with Matrices and Vectors
- Regular and Inverse matrices
- Determinants, Cramer's rule
- Gaussian Elimination Method
- Gauss-Jordan Elimination Method
- Linear Dependence and Independency of Vectors
- The rank of the matrix and Frobenius theorem
- Linear transformations
- Vector Spaces
- Computer Algebra
- Solutions of tasks
- Appendix I: Linear algebra – Questions
- Appendix II: Linear algebra – Classification
- Appendix III: The solution of equations in Bottleneck Algebra
- Conclusion
- References
- Register of terms

1.6 Reference Documents

For the development of this publication, authors used several books as a source, i.e. books from Slovak author [8] and also from foreign authors [5], [6], [4]. All used sources are listed in References.

More information on linear algebra applications can be found in other publications, i.e.

- Linear algebra application for **Computer science** [1]
- Linear algebra application for **Operation research** (management science) [7]
- The Handbook of Essential Mathematics [3]

In addition to linear algebra, other algebraic structures were created as a result of calculating different exercises. Some structures are e.g. **Max-Min Algebra**, **Bottleneck Algebra** and **Fuzzy Algebra**. This is over the $(\mathcal{R}, \max, \min)$, hence, the operations of addition and multiplication are substituted by operations of maximum and minimum. What concerns the solution of equations (3), (1), (2), they are solved in the same way as in linear algebra but new/different methods are applied. These methods are included in [2], the eq. (1) is solved in Bottleneck algebra.

1.7 Digital Infrastructure - Remote Cooperation

This sub-section has been developed and embedded in this publication to briefly describe the process and the technologies used. At the time of this publication development, both authors were located in the different European cities; the geographical distance between the authors did not allow the regular personal meetings where the development progress could be reviewed, discussed and approved.

Therefore, to facilitate communication and also to maintain traceability, authors were using 2 different technologies:

1.7.1 Cloud Computing in '*.tuke.sk' network - VPN Connection required

In the '*tuke.sk' network, authors were using the virtual PC (Microsoft Remote Desktop) with the IP address: 147.232.205.124. This option was since the virtual PC was offering the following program tools needed for the publication development:

- To ensure the correctness of the examples results, the authors used the program **MATLAB v. R2019b** with the academic license TAH and open-source system **SageMath v.8.8**.

- The publication was written in open-source document processor *LyX* ver. 2.2.3 that is based on the typesetting system *LaTeX*.

- For reference management, the authors used a reference management software *JabRef* that uses format *BiBTeX*.

1.7.2 Cloud Computing in '*ipower.sk' network - without VPN Connection

The second technology was via a specific author's network. This technology was used to be able to access the information securely but without a VPN connection. The authors were using MyCloud storage that allows having all files in one central place. The files were uploaded, accessed and shared from whichever spot with the established internet connection. In addition to that, this technology also offered the 'mobile, on-the-go access', i.e. the application allowing to upload, access and share the files (also photos, music, etc.) via the mobile device.

2 Elementary Matrix Row Operations

This Chapter provides an algorithm how to solve a **system of linear equations** (SLE). Such system can contain several linear equations; each linear equation contains **variables** that can be multiplied by a constant. The **solutions of the SLE** are the determined values of the variables for which all equations are always true, see [1].

[1] Solve a system of linear equations using row operations

$$x + 3y + 6z = 25 \tag{4}$$
$$2x + 7y + 14z = 58 \tag{5}$$
$$2y + 5z = 19 \tag{6}$$

Note: x,y, z are the variables of the SLE. The objective is to find the values x,y, z such as the relations (1), (2), (3) are valid. The SLE equations above are also called linear **algebraic expressions**.
Solution: The operations that lead to the **equivalent system of equations** are matrix row operations.

We will use the following matrix **row operations** to work out the solution:

I) Interchange two rows of SLE.
II) Multiply each element in a row by a nonzero number.
III) Multiply a row by a nonzero number and add the result to another row.

By applying the above row (linear) operations to the SLE, we create a new system of linear equations. This new created system is equivalent to the previous/initial one, i.e. both systems have the same solution or the same set of the solutions.

Solution of example [1]:

The example will be solved step by step by applying the operations I)-III).

At the first step, we try to eliminate the variable x from the second row. Therefore, we multiply the first row by the constant (-2) and then we add this result to the second row. As a result of this step, we receive the following SLE:

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ 2y + 5z &= 19 \end{aligned}$$

Now we try to eliminate the variable y from the last row. Therefore, we multiply the second row with the constant (-2) and then, the result we add to the third row. In other words, we replace the 3rd equation by (-2) times of the 2nd row. At this step, we receive the following SLE:

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3 \end{aligned}$$

Now, we can see that the variable $z = 3$. With this information we can calculate the variable y from the second row that is equal to 2 ($y + 6 = 8$ and so $y = 2$). Having these 2 variable, at the first row we can calculate the variable x that is equal to 1 ($x + 6 + 18 = 25$ and so $x = 1$).

By calculating all 3 variables: $(x, y, z) = (1, 2, 3)$ we have the solution of SLE. The SLE in [1] has only *one solution*, i.e. the **unique solution.**

[2] Solve a system of linear equations using row operations

$$\begin{aligned} 5x + 10y - 7z &= -2 \\ 2x + 4y - 3z &= -1 \\ 3x + 6y + 5z &= 9 \end{aligned}$$

Solution: At the first step, we change the first and the second row. Then, we multiply the first row by the constant (-5) and the second row by the constant 2. Consequently, we add the first row to the second row and the first row we multiply by the constant (-1/5). At this step, we receive the following SLE:

$$\begin{aligned} 2x + 4y - 3z &= -1 \\ z &= 1 \\ 3x + 6y + 5z &= 9 \end{aligned}$$

At the second step, we multiply the first row by the constant (-3) and the last row by the constant 2. We add the first row to the last row. Consequently, the first row we multiply by the constant (-1/3) and we receive the following system:

$$\begin{aligned} 2x + 4y - 3z &= -1 \\ z &= 1 \\ z &= 21 \end{aligned}$$

Now we see that according the last row, the variable $z = 1$ while according the second row, $z = 21$. This is not possible, hence, the SLE in [2] has **no solution**.

[3] Solve a system of linear equations using row operations

$$\begin{aligned} 3x - y - 5z &= 9 \\ y - 10z &= 0 \\ -2x + y &= -6 \end{aligned}$$

Solution:

At the first step, we multiply the last row by the constant 3 and the first row we multiply by 2. Consequently, we add the first row to the last one. We receive the following system:

$$\begin{aligned} 3x - y - 5z &= 9 \\ y - 10z &= 0 \\ y - 10z &= 0 \end{aligned}$$

The last row says that $y = 10z$. When we substitute y to the first row, we receive $x = 3 + 5z$. Apparently z can equal to any number. Therefore, we put $z = t$. Since $z = t$ where t is arbitrary, the variable z is called a **free variable**, x and y are basic variables. Now, we replace z by t so we receive $y = 10t$, $x = 3 + 5t$. SLE solution is $(x, y, z)=$ $(3 + 5t, 10t, t)$ where t can have any value. SLE in [3] has an **infinite set of solutions**.

Conclusion: Based on the examples explained above, the following can be concluded:
- SLE can be resolved by the row operations I) II) III) (linear operations).
- The new system we create by using the row operations is equivalent to the previous system.
- SLE can have **one solution**, **infinitely many solutions** or **no solution**.

2.1 Examples: Solving the System of Linear Equations using Matrix Row Operations (with solutions)

[4] Solve a system of linear equations using row operations

$$\begin{aligned} -41x + 15y &= 168 \\ 109x - 40y &= -447 \\ -3x + y &= 12 \\ 2x + z &= -1 \end{aligned}$$

[5] Solve a system of linear equations using row operations

$$\begin{aligned} 3x - y + 4z &= 6 \\ y + 8z &= 0 \\ -2x + y &= -4 \end{aligned}$$

[6] Solve a system of linear equations using row operations

$$\begin{aligned} x+2y+6z &= 5 \\ 3x+2y+6z &= 7 \\ -4x+5y+15z &= -7 \end{aligned}$$

[7] Solve a system of linear equations using row operations

$$\begin{aligned} x+2y+3z &= 5 \\ 3x+2y+z &= 7 \\ -4x+5y+z &= -7 \\ x+3z &= 5 \end{aligned}$$

[8] Solve a system of linear equations using row operations

$$\begin{aligned} x+z &= 2 \\ z+w &= 0 \\ 2y+z-w &= 4 \\ y-4z-5w &= 2 \end{aligned}$$

[9] Solve a system of linear equations using row operations

$$\begin{aligned} x+y+z &= 2 \\ z+w &= 0 \\ 2x+2y+z-w &= 4 \\ x+y-4z-5w &= 2 \end{aligned}$$

[10] Solve a system of linear equations using row operations

$$\begin{aligned} x+3y+3z &= 3 \\ 3x+2y+z &= 9 \\ -4x+z &= -9 \end{aligned}$$

3 Matrices and Vectors

3.1 The Augmented Matrix and the System of Linear Equations

The system of the following linear equations

$$\begin{aligned} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19 \end{aligned}$$

can also be defined by the matrix:

$$\begin{pmatrix} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{pmatrix}$$

This matrix is called an **augmented matrix of linear equations.** The matrix contains **coefficients of variables** and the **right side of the system**.

The first row (R1) of the matrix is the first equation from above and the first column is the column of variable x. Using this matrix, we can also solve the SLE, so that the individual adjustment steps (row operations) are applied to this matrix.

$$\begin{pmatrix} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{pmatrix} \begin{matrix} \\ -2R1 \\ \\ \end{matrix} \Rightarrow \begin{pmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{pmatrix} \begin{matrix} \\ \\ -2R2 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

From the last row in the final matrix, we see that $z = 3$, so we can gradually calculate the other variables y and x as follows:

$1 \cdot y + 2 \cdot 3 = 8 \rightarrow y = 2$

$1 \cdot x + 3 \cdot 2 + 6 \cdot 3 = 25 \rightarrow x = 1$

3.2 Examples: Solving the System of Linear Equations using the Augmented Matrix (with solutions)

[11] Solve a system of linear equations using augmented matrix

$$\begin{aligned} -41x + 15y &= 168 \\ 109x - 40y &= -447 \\ -3x + y &= 12 \\ 2x + z &= -1 \end{aligned}$$

[12] Solve a system of linear equations using augmented matrix

$$
\begin{aligned}
3x - y + 4z &= 6 \\
y + 8z &= 0 \\
-2x + y &= -4
\end{aligned}
$$

[13] Solve a system of linear equations using augmented matrix

$$
\begin{aligned}
x + 2y + 6z &= 5 \\
3x + 2y + 6z &= 7 \\
-4x + 5y + 15z &= -7
\end{aligned}
$$

[14] Solve a system of linear equations using augmented matrix

$$
\begin{aligned}
x + 2y + 3z &= 5 \\
3x + 2y + z &= 7 \\
-4x + 5y + z &= -7 \\
x + 3z &= 5
\end{aligned}
$$

[15] Solve a system of linear equations using augmented matrix

$$
\begin{aligned}
x + z &= 2 \\
z + w &= 0 \\
2y + z - w &= 4 \\
y - 4z - 5w &= 2
\end{aligned}
$$

4 Operations with Matrices and Vectors

This Chapter is presenting several definitions of operations with matrices/vectors that will enable users to resolve the different examples with matrices/vectors.

Definition 1:

Matrix A (below)

$$\mathbf{A} = (\mathbf{a_{ij}}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is an $m \times n$ real matrix if $a_{ij} \in \boldsymbol{\mathcal{R}}$ for all $i = 1, \cdots, m$, $j = 1, \cdots, n$. The set of all real matrices of type $m \times n$ is denoted as $\mathcal{R}^{m,n}$.

When $m = n$, we say that the matrix is **square**, otherwise, the matrix is **rectangular**.

The following matrix I is an **unit matrix** of type $n \times n$:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

The element a_{ij} of matrix I lies on the **diagonal of matrix** when $i = j$.

Matrices with only one column are **vectors**.

- Vector which has n rows is called **n-dimensional vector**. The set of **n-dimensional vectors** is denoted as $\mathcal{R}^n$.

Definition 2:

Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{m,n}$ and $\mathbf{B} = (\mathbf{b_{ij}}) \in \mathcal{R}^{m,n}$ then we can perform the following operations:

- $\mathbf{C} = (\mathbf{c_{ij}}) = \mathbf{A} + \mathbf{B}$ **is a sum of matrices** A and B, where $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$.
- $\mathbf{D} = (\mathbf{d_{ij}}) = \mathbf{A} - \mathbf{B}$ **is a difference of matrices** A and B where $d_{ij} = a_{ij} - b_{ij}$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$.
- $\mathbf{E} = (\mathbf{e_{ij}}) = \mathrm{r}\mathbf{A}$ is **the product of matrix A with a constant r** where $e_{ij} = ra_{ij}$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$.
- $\mathbf{A^T} = (\mathbf{a_{ji}}) \in \mathcal{R}^{m,n}$ is **a transposed matrix** of the matrix**A**.

Definition 3:

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathcal{R}^n$ be the column vectors.

Under the **scalar (dot) product of the two vectors** we understand the following number:

$$\mathbf{x^T y} = (x_1, x_{2,} \cdots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n. \tag{7}$$

Definition 4:

Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{m,n}$ and $\mathbf{B} = (\mathbf{b_{ij}}) \in \mathcal{R}^{m,n}$. Then we say that $\mathbf{C} = (\mathbf{c_{ij}}) = \mathbf{AB}$ is the **product of matrices** A and B when c_{ij} is the scalar product of the i-th row of the Matrix A and the j-th column of the Matrix B for $i, j = 1, \cdots, m$,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \tag{8}$$

Note: The squares of matrices are created by multiplication of matrices, e.g. $A^2 = AA$. For each square matrix A of type $n\ x\ n$ is valid that $A^0 = I$ where I is the Unit matrix of type $n\ x\ n$.

$$\text{I} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

Definition 5:

Let $\boldsymbol{m}$ be a set of linear equations of $\boldsymbol{n}$ unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{9}$$

Then the Matrix A (below)

$$\mathbf{A} = (\mathbf{a_{ij}}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is the **matrix of the system of linear equations**.

Vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a **vector of unknown** and vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is the **vector of the right sides**.

The system of $\boldsymbol{m}$ linear equations of $\boldsymbol{n}$ unknown values can be written in matrix form such as

$$\mathbf{Ax} = \mathbf{b} \tag{10}$$

4.1 Examples: Solving the System of Linear Equations using the Matrix Operations

[16] Calculate Ax.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

[17] Calculate Ax and Ay, Ax-Ay.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & -5 \\ 0 & 1 & -10 \\ -2 & 1 & 0 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} 8 \\ 10 \\ 1 \end{pmatrix}$$

[18] Calculate the sum and the difference of A, B; the matrix $\mathbf{A}^T$ and the product $\mathbf{AB}^T$.

$$\mathbf{A} = \begin{pmatrix} 5 & 7 & 2 \\ 0 & -3 & 5 \\ 4 & 8 & -1 \\ 3 & 9 & 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 4 & -2 \\ 1 & 1 & -1 \\ -3 & 5 & 0 \\ -5 & 0 & 4 \end{pmatrix}$$

[19] Calculate the sum and the difference of A, B; the matrix $\mathbf{A}^T$ and the product $\mathbf{AB}^T$.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -3 & 5 \\ 4 & -1 & -1 \\ 3 & 0 & 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 & -4 & 3 \\ 6 & 1 & -1 \\ -3 & 4 & 2 \\ -5 & \text{-1} & 4 \end{pmatrix}$$

[20] Calculate the matrix $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 0 & -3 & 5 \\ 4 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix}$$

[21] Calculate the matrix $\mathbf{C} = 3\mathbf{A} - \mathbf{2A^2}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -1 & 0 \\ 2 & 1 & -1 & 1 \\ 3 & 0 & 1 & 1 \end{pmatrix}$$

[22] Calculate the matrix $\mathbf{C} = 2\mathbf{A^T A} + \mathbf{A^2}$, where

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 2 & 1 \\ 0 & -3 & -1 & 2 \\ 1 & 3 & -1 & 1 \\ 7 & 0 & 4 & 1 \end{pmatrix}$$

[23] Calculate the matrix $\mathbf{C} = 2\mathbf{A}\mathbf{A}^{\mathbf{T}}\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 0 \end{pmatrix}$$

[24] Calculate the matrix $\mathbf{C} = (2\mathbf{A} - \mathbf{B})^{\mathbf{2}}$, where

$$\mathbf{A} = \begin{pmatrix} 5 & 7 & 2 \\ 0 & -3 & 5 \\ 4 & 8 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 5 & 0 \\ -5 & 0 & 4 \end{pmatrix}$$

[25] Calculate the matrix $\mathbf{C} = \mathbf{A}^{\mathbf{T}}\mathbf{A} + \mathbf{A}^{\mathbf{2}} - \mathbf{B}^{\mathbf{3}}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5 Regular and Inverse Matrices

This Chapter will present several definitions to allow users to resolve different examples with regular and inverse matrices.

Definition 6:

Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$ and $\mathbf{b} = (\mathbf{b_i}) \in \mathcal{R}^n$. Then the solution of the SLE $\mathbf{Ax} = \mathbf{b}$ can be denoted as

$$\mathbf{S}(\mathbf{A}, \mathbf{b}) = \{\mathbf{x} \in \mathcal{R}^n; \mathbf{Ax} = \mathbf{b}\}$$

Definition 7:

Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$ and $\mathbf{b} = (\mathbf{b_{ij}}) \in \mathcal{R}^n$**.** The matrix A is called **regular** when $|\mathbf{S}(\mathbf{A}, \mathbf{b})| = 1$.

Definition 8:

Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$ be a regular matrix. The matrix $\mathbf{A^{-1}} \in \mathcal{R}^{n,n}$ is an **inverse matrix** to $\mathbf{A}$, when $\mathbf{A^{-1}A} = \mathbf{AA^{-1}} = \mathbf{I}$.

Note: When a matrix A is a matrix of SLE and is regular, then the SLE has unique solution. The solution of SLE $\mathbf{Ax} = \mathbf{b}$ we can solve in a way that we multiply the equation with the inverse matrix, as follows:

$$\mathbf{A^{-1}Ax} = \mathbf{Ix} = \mathbf{x} = \mathbf{A^{-1}b}. \tag{11}$$

5.1 Calculation the inverse matrix using row operations

The algorithm for calculating the inverse matrix using row operations is simple. To matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{\mathbf{n,n}}$ we assign a special matrix $(\mathbf{A}, \mathbf{I}) \in \mathcal{R}^{\mathbf{n,2n}}$. Consequently, we apply row operations for matrix A until we get the math type $(\mathbf{I}, \mathbf{B}) \in \mathcal{R}^{n,2n}$.

When matrix$\mathbf{A}$ was regular at the beginning, then the matrix $\mathbf{B}$ will be the inverse to matrix $\mathbf{A}$.

We will demonstrate this algorithm on the example below.

[26] Calculate the inverse matrix to matrix A when it exists.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix}$$

Solution: We will assign matrix $(\mathbf{A}, \mathbf{I})$ to matrix $\mathbf{A}$ and apply row operations:

$$(\mathbf{A}, \mathbf{I}) = \begin{pmatrix} 1 & 3 & 6 & 1 & 0 & 0 \\ 2 & 7 & 14 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ -2R1 \\ \\ \end{matrix} \Rightarrow \begin{pmatrix} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{pmatrix} -2R2 \quad \Rightarrow \begin{pmatrix} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{pmatrix} -2R3$$

$$\begin{pmatrix} 1 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -10 & 5 & -2 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{pmatrix} -6R3-3R2 \quad \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 7 & -3 & 0 \\ 0 & 1 & 0 & -10 & 5 & -2 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{pmatrix} = (\mathbf{I}, \mathbf{A^{-1}})$$

therefore

$$\mathbf{A^{-1}} = \begin{pmatrix} 7 & -3 & 0 \\ -10 & 5 & -2 \\ 4 & -2 & 1 \end{pmatrix}$$

To test the correctness of the result, we can do by using the inverse matrix definition$\mathbf{A^{-1}A} = \mathbf{AA^{-1}}=\mathbf{I}$.

When we have both, the $\mathbf{A^{-1}}$ matrix and the vector $\mathbf{b} = \begin{pmatrix} 25 \\ 58 \\ 19 \end{pmatrix}$ available, we can find a single solution of the system $\mathbf{Ax} = \mathbf{b}$ as follows:

$$\mathbf{x} = \mathbf{A^{-1}b} = \begin{pmatrix} 7 & -3 & 0 \\ -10 & 5 & -2 \\ 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} 25 \\ 58 \\ 19 \end{pmatrix}.$$

5.2 Examples: Solving the System of Linear Equations using Inverse Matrices (with solutions)

[27] Calculate the inverse matrix to matrix A when it exists. Let $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, if there is $\mathbf{A^{-1}}$available, find the solution $\mathbf{Ax} = \mathbf{b}$ using $\mathbf{A^{-1}}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

[28] Calculate the inverse matrix to matrix A when it exists. Let $\mathbf{b} = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$, if there is $\mathbf{A^{-1}}$available, find the solution $\mathbf{Ax} = \mathbf{b}$ using $\mathbf{A^{-1}}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

[29] Calculate the inverse matrix to matrix $\mathbf{A}$ when it exists. Let $\mathbf{b} = \begin{pmatrix} 7 \\ -2 \\ 3 \end{pmatrix}$, if there is $\mathbf{A^{-1}}$ available, find the solution $\mathbf{Ax} = \mathbf{b}$ using $\mathbf{A^{-1}}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{pmatrix}$$

[30] Calculate the inverse matrix to matrix $\mathbf{A}$ when it exists. Let $\mathbf{b} = \begin{pmatrix} 9 \\ 11 \\ 17 \end{pmatrix}$, if there is $\mathbf{A^{-1}}$available, find the solution $\mathbf{Ax} = \mathbf{b}$ using $\mathbf{A^{-1}}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 8 \end{pmatrix}$$

6 Determinants, Cramer's Rule

6.1 Determinants

Before we start any calculations using determinanst, we should first to define what a determinant is. Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$. The **determinant** is a number that we can assign to the matrix **A**; determinant is denoted as **det A** or **|A|.**

The **determinant of the second degree** of matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{2,2}$ is calculated as:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The **determinant of the third degree** of matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{3,3}$ is calculated as:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

This calculation can be realized using the so-called **Sarrus' Rule**. We add the first two rows of the matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{3,3}$ to the same matrix; the positive components of the determinant are three elements that lie on the diagonals from the right and the negative components lie on the diagonal from the left.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

The **determinant of the n-th degree** of matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$ is calculated as:

Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$. The value $\mathbf{D}_{ij}$ is a **subdeterminant** of element a_{ij} of matrix **A**. In fact, it is the determinant obtained from the matrix A when we delete the i-th row and the j-th column.The value

$$\mathbf{A}_{ij} = (-1)^{i+j}\mathbf{D}_{ij}$$

is called an **algebraic complement** of element a_{ij} of matrix **A**.

Determinant of matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$ is a number

$$\mathbf{detA} = a_{i1}A_{11} + a_{i2}A_{12} + \cdots + a_{in}A_{nn} = \sum_{j=1}^{n} a_{ij}A_{ij} \tag{12}$$

where i is an arbitrary row index of matrix **A.**

[31] Calculate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

Solution:

This example is resolved by definition of the second degree determinant as follows:

$$\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = a_{11}a_{12} - a_{12}a_{21} = 2 \cdot 3 - 1 \cdot (-1) = 6 + 1 = 7$$

[32] Calculate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Solution:

This example is calculated by applying the Sarrus' Rule, i,e. we add the first rows of the matrix $\mathbf{A}$ to the same matrix and calculate the individual sums:

$$\mathbf{det A} = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 3 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \\ -1 & 3 & 0 \end{vmatrix} = 2 \cdot 3 \cdot 1 + (-1) \cdot 2 \cdot 3 + 1 \cdot 1 \cdot 0 - 3 \cdot 3 \cdot 1 - 0 \cdot 2 \cdot 2 - 1 \cdot 1 \cdot (-1) = -8$$

[33] Calculate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 0 & 1 \end{pmatrix}$$

Solution:

In accordance with (12), we write the development of the determinants. It will be done according to the second row because this row contains only one non-zero element.

$$\mathbf{det A} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 0 & 1 \end{vmatrix} = (-1)^{2+1} A_{21} = - \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 0 & 1 \end{vmatrix} = -(1 \cdot 3 \cdot 1 + 2 \cdot 0 \cdot 1 + 3 \cdot 1 \cdot 4 - 1 \cdot 3 \cdot 3 - 4 \cdot 0 \cdot 1 - 1 \cdot 2 \cdot 1) = -4$$

[34] Calculate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

[35] Calculate the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 4 & 1 & 1 \\ 3 & 2 & 0 & 2 \\ 1 & 3 & 2 & 3 \end{pmatrix}$$

6.2 Regular matrices and determinants

Definition 8: Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$. Square matrix $\mathbf{A}$ is **regular** if and only $\mathbf{detA} \neq 0$. If $\mathbf{detA} = 0$, then $\mathbf{A}$ is **singular**.

Note: Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$. We can say that the condition $\mathbf{detA} \neq 0$ is equivalent to the condition $|\mathbf{S}(\mathbf{A}, \mathbf{b})| = 1$ for some $\mathbf{b} = (\mathbf{b_i}) \in \mathcal{R}^n$.

By using a determinant, we can also calculate:

- The SLE when the system $\mathbf{Ax} = \mathbf{b}$ has only one, unique solution ($\mathbf{detA} \neq 0$).

- The inverse matrix to the matrix A under the condition that the matrix is regular ($\mathbf{detA} = |A| \neq 0$)

$$A^{-1} = \left(\frac{A_{ij}}{|A|} \right) \tag{13}$$

Due to the complexity of this algorithm which is O(n!), the calculations with the big initial matrices is not possible.

[36] Use the determinant to calculate the inverse matrix to the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

Solution:

By the definition of the determinant:

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} = a_{11}a_{12} - a_{12}a_{21} = 2x3 - 1x(-1) = 6 + 1 = 7$$

$|\mathbf{A}| \neq 0$ therefore matrix $\mathbf{A}$ is regular and there is an inverse matrix to $\mathbf{A}$

Now we calculate the matrix according to the relationship (13)

$$\frac{1}{|\mathbf{A}|} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}^{-1}$$

Test of correctness: $\mathbf{A^{-1}A} = \frac{\mathbf{1}}{\mathbf{7}}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \frac{\mathbf{1}}{\mathbf{7}}\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$. We are testing similarly $\mathbf{AA^{-1}} = \mathbf{I}$.

[37] Use the determinant to calculate the inverse matrix to the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

[38] Use the determinant to calculate the inverse matrix to the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 0 & 1 \end{pmatrix}$$

[39] Use the determinant to calculate the inverse matrix to the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 4 & 1 & 1 \\ 3 & 2 & 0 & 2 \\ 1 & 3 & 2 & 3 \end{pmatrix}$$

[40] Use the determinant to calculate the inverse matrix to the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix}$$

6.3 Cramer's rule.

By determinants, we can calculate the SLE when the system has the only solution, i.e. $\mathbf{detA} \neq 0$.

Let have a given set of $\boldsymbol{n}$ equations of $\boldsymbol{n}$ unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{m2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{14}$$

Let

$$D = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

and let D_i be a determinant of a matrix created from the matrix $\mathbf{A}$ by replacing the i-th column with vector $\mathbf{b}$, e.g.

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Then, we can say that the only solution of this system is the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathcal{R}^n$, where

$$x_i = \frac{D_i}{D} \tag{15}$$

for each $i = 1, 2, \cdots, n$.

We can apply Cramer's Rule for small dimension tasks ($n = 2, 3, 4, 5$) and only when a square matrix of SLR $\mathbf{A}$ is regular ($\mathbf{detA} \neq \mathbf{0}$). For larger dimension tasks (n ≥ 8) we can not apply this rule because the computational complexity of the rule is large ($n!$).

[41] Solve a system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 + 4x_3 &= 2 \\ 3x_1 + 2x_2 + x_3 &= 1 \end{aligned}$$

Solution:

We calculate the matrix determinant

$\mathbf{detA} = D = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{vmatrix} = 10 \neq 0$ therefore, we can apply Cramer's rule to solve the SLE.

According to this rule, we calculate the determinants D_1, D_2, D_3.

$$D_1 = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 21 - 31 = -10$$

$$D_2 = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 2 & 4 \\ 3 & 1 & 1 \end{vmatrix} = 40 - 16 = 24$$

$$D_3 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 27 - 35 = -8$$

By relationship (15) $x_1 = \frac{D_1}{D} = \frac{-10}{10} = -1$, $x_2 = \frac{D_2}{D} = \frac{24}{10} = \frac{12}{5}$ and $x_3 = \frac{D_3}{D} = \frac{-8}{10} = -\frac{4}{5}$. The unique solution of the system is $\mathbf{x} = (-1, \frac{12}{5}, -\frac{4}{5})^T$.

6.4 Examples: Solving the System of Linear Equations using Cramer's Rule (with solutions)

[42] Solve a system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= -5 \\ x_1 + 2x_2 - x_3 &= 0 \\ -3x_1 + x_2 + 3x_3 &= 7 \end{aligned}$$

[43] Solve a system of linear equations

$$\begin{aligned} -2x_1 + x_2 + x_3 &= 3 \\ 4x_1 + 3x_2 - 2x_3 &= 4 \\ -x_1 + x_2 + 5x_3 &= 16 \end{aligned}$$

[44] Solve a system of linear equations

$$\begin{aligned} x_1 + x_2 + 4x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= 7 \\ 3x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

[45] Solve a system of linear equations

$$\begin{aligned} x_1 + x_2 + 2x_3 - 4x_4 &= 2 \\ 2x_1 + x_2 + x_3 - x_4 &= 3 \\ x_1 + 2x_2 + x_3 + x_4 &= 3 \\ -x_1 + x_2 + 2x_3 + x_4 &= 0 \end{aligned}$$

7 Gaussian Elimination Method

This chapter is providing one of the methods that can be used for the SLE resolution.

Let the matrix $A = (a_{ij}) \in R^{m,n}$ be a matrix we will use in the next example. A or (A, b) is a matrix (augmented matrix) of system of linear equations $Ax = b$. We will use the following **row and column operations**:

Ie) Interchange two rows/columns of a matrix

IIe) Multiply a row/column by a nonzero number.

IIIe) Replace a row/column by a multiple of another row/column added to it.

Note: In Chapter 2, we described the row operations I) II) III). The difference between those from Chapter 2 and these Ie) IIe) IIIe) provided here is that, the modifications in this chapter will be also used for the columns.

It is possible to demonstrate that by using the elementary modifications, we can transform matrix A to matrix $B = (b_{ij})$. For matrix B applies that $b_{kk} \neq 0$, and $b_{lj} = 0$ for $l \geq k$ and $j \geq k$. In this case we say that the matrix B is adjusted to the **upper triangular form** (B_2) or adjusted to the **trapezoidal form** (B_1 or B_3). Matrices $B_1$and $B_3$are trapezoidal matrices while matrix B_2 is triangular matrix.

$$\mathbf{B}_1 = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2m} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{mm} & \cdots & b_{mn} \end{pmatrix}$$

$$\mathbf{B}_2 = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{nn} \\ 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B}_3 = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2k} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{kk} & \cdots & b_{kn} \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

7.1 Gaussian Elimination Method

We assume that the input matrix **A** is nonzero matrix, i.e. it has at least one nonzero element a_{rs}. By interchanging rows and columns we can assure that the element $a_{11} \neq 0$. If $a_{i1} \neq 0$, then we multiply the first row with $(-\frac{ai1}{a11}\ )$ and the result is added to the i^{th} row. With this, we also reset the element a_{i1} to position 0 ($a_{i1} = 0$).We will gradually clear the entire first column. In the next step, we delete the first row and column of matrix A and repeat the procedure until the matrix has a nonzero element. The result of this procedure is a matrix B ($B = B_1$ or $B = B_2$ or $B = B_3$). The element that we use to reset the columns is called a **key element** or pivot entry. This method is called the **Gaussian Elimination Method**. The **computational complexity** of the method is $O(mn^2)$ (the approximate number of steps to be solved of system,m is the number of rows, n is the number of columns of the input matrix).

[46] Using the Gaussian Elimination Method, adjust the matrix A to the triangular or trapezoidal form:

$$\mathbf{A} = \begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix}$$

Solution: The element $a_{11} = 2 \neq 0$, so we could start with reset of the first column but then we would have to multiply the first row by $\frac{-1}{2}$ and the results would be the fractions. Easier step is to interchange the first and second matrix rows.

$$\mathbf{A} = \begin{array}{c} \begin{pmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix} \end{array} \begin{array}{l} \\ -2R_1 \\ \\ -4R_1 \end{array}$$

Now we reset (eliminate) the first element of the second and the forth row. This is done in a way that we multiply the first row by -2 and add it to the second row; consequently, we multiply the first row by -4 and add it to the fourth row, $-2R_1$ and $-4R_1$ so we get:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{pmatrix}$$

Now we see that modified matrix, on the position (2, 2), has the element 0 and on the position (3, 2), has the element 1. In the modified matrix $a_{22} = 0$, hence, we can replace the second and third rows:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & 0 & 12 & -3 \end{pmatrix} \begin{array}{l} \\ \\ \\ -R_2 \end{array}$$

At the next step, we can subtract the second row from the forth and we will get:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 1 & 9 & -4 \end{pmatrix} \begin{array}{l} \\ \\ \\ -R_3 \end{array}$$

In the last step, we subtract the third row from the forth and we will get

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The result is a trapezoidal matrix of form B_3.

7.2 Examples: Solving the System of Linear Equations using Gaussian Elimination Method (with solutions)

In the following examples, adjust the matrix A to the triangular or trapezoidal form (use the Gaussian Elimination Method):

[47] :

$$\mathbf{A} = \begin{pmatrix} 25 & 2 & 3 & 4 & 2 \\ 75 & 6 & 2 & 11 & 3 \\ 75 & 6 & 3 & 4 & 8 \\ 25 & 4 & 5 & 1 & 4 \end{pmatrix}$$

[48] :

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ -1 & 3 & 0 \\ 2 & 1 & 5 \end{pmatrix}$$

[49] :

$$\mathbf{A} = \begin{pmatrix} 3 & 6 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

[50] :

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 3 & 0 & 2 \end{pmatrix}$$

8 Gauss-Jordan Elimination Method

This chapter is providing another method for the SLE resolution. It is a modification of the Gaussian Elimination Method, so some authors also call it Gauss-Jordan Elimination Method. To explain how we can use this method for calculations, we use the matrix A defined in the previous chapter, i.e. the matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{m,n}$. We can transform matrix $\mathbf{A}$, by row and column operations, into the matrix $\mathbf{B}$ of triangular or trapezoidal form $\mathbf{B}_1$, $\mathbf{B}_2$ or $\mathbf{B}_3$.

$$\mathbf{B}_1 = \begin{pmatrix} b_{11} & 0 & \cdots & 0 & b_{1m+1} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & 0 & b_{2m+1} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{nm} & b_{mm+1} & \cdots & b_{mn} \end{pmatrix}$$

$$\mathbf{B}_2 = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{nn} \\ 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B}_3 = \begin{pmatrix} b_{11} & 0 & \cdots & 0 & b_{1k+1} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & 0 & b_{2k+1} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_{kk} & b_{kk+1} & \cdots & b_{kn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We do the modification of the matrix $\mathbf{A}$ similarly as we did in the case of the Gaussian Elimination Method. The difference between these 2 methods is that in this case, we null also the elements above the diagonal once we get the upper triangular form; while in case of Gaussian Elimination Method we null only the elements under the diagonal. This method is called the **Gauss-Jordan Elimination Method**. We used this method in chapter 5 to calculate the inverse matrices. Method complexity (number of steps to solve the problem (1)) is $O(mn^2)$.

[51] Adjust the matrix A using the Gauss-Jordan Elimination Method.

$$\mathbf{A} = \begin{pmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{pmatrix}$$

Solution:

We will apply the same steps as in the example [47] and using the Gaussian Elimination Method, we modify the input matrix to the upper triangular matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, we will multiply the second row by $(+2)$ and add it to the first one to eliminate the element at position (1,2) (we want $a_{12} = 0$). We will get the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 2 & 4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

At the last step, we will add the third row to the first one and also to the second one. The resulting matrix is as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 11 & 0 \\ 0 & 1 & 0 & 12 & -3 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The output matrix has the **Jordan normal form** also known as **Jordan canonical form.**

8.1 Examples: Solve the System of Linear Equations using Gauss-Jordan Elimination Method

In the following examples, adjust the matrix A to the Jordan normal form (use the Gauss-Jordan Elimination Method):

[52] :

$$\mathbf{A} = \begin{pmatrix} 25 & 2 & 3 & 4 & 2 \\ 75 & 6 & 2 & 11 & 3 \\ 75 & 6 & 3 & 4 & 8 \\ 25 & 4 & 5 & 1 & 4 \end{pmatrix}$$

[53] :

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ -1 & 3 & 0 \\ 2 & 1 & 5 \end{pmatrix}$$

[54] :

$$\mathbf{A} = \begin{pmatrix} 3 & 6 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

[55] :

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 3 & 0 & 2 \end{pmatrix}$$

9 A Linear Dependence of the Vectors, Rank of Matrices

Definition 7: Let $a_1, \ldots, a_n \in R^n$ be *n-dimensional vectors*. Let $r_1, \ldots, r_n \in R$ be the real numbers and $(r_1, \ldots, r_n) \neq (0, \ldots, 0)$. Then the vector

$$x = r_1 a_1 + r_2 a_2 + \ldots + r_n a_n \tag{16}$$

is a **linear combination** of $a_1, \ldots, a_n$ vectors.

Definition 9: Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{m,n}$ be a defined matrix. Rank of a matrix **A**, denoted by **rank(A)**, h(A) - a number of nonzero rows of the output matrix **B** ($\mathbf{B} = \mathbf{B_1}$ or $\mathbf{B} = \mathbf{B_2}$ or $\mathbf{B} = \mathbf{B_3}$) obtained from matrix **A** after applying the elementary row operations. The matrix can be in triangular form or trapezoidal form.

Definition 10: Let $a_1^T, \ldots, a_m^T \in R^n$ be n-dimensional vectors and $a_i^T = (a_{i1}, a_{i2}, \ldots, a_{in})$ for each $i = 1, 2, \ldots, m$. Then vectors $a_1^T, \ldots, a_m^T$ are **linearly independent** when the matrix $A = (a_{ij})$ has the rank $r(A) = m$. If the rank $r(A) < m$, then vectors are **linearly dependent**.

Note: The lines of matrix **A** form row vectors. In the next chapter, a new algebraic structure, i.e.vector space will be defined. This structure will be defined by linearly independent vectors.

[56] Are the vectors a=(2,0,2,1), b=(-3,2,-2,5), c=(-1,2,0,6), d=(-4,5,-2,11) linearly dependent or independent?

Solution:

The vectors a, b, c, d form the following matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 2 & 1 \\ -3 & 2 & -2 & 5 \\ -1 & 2 & 0 & 6 \\ -4 & 5 & -2 & 11 \end{pmatrix} \in \mathcal{R}^{4,4}$$

The vectors will be linearly independent if the matrix can be converted into a matrix of *triangular form* by using matrix row operations, i.e. when the rank of the matrix will be $r(A) = 4$ (Note: row operations do not affect the rank of the matrix, so the resulting matrix will have the same rank as the input matrix A). Row operations: interchanging the first and the third rows and then the second and the third row.

$$\begin{pmatrix} -1 & 2 & 0 & 6 \\ 2 & 0 & 2 & 1 \\ -3 & 2 & -2 & 5 \\ -4 & 5 & -2 & 11 \end{pmatrix} \begin{matrix} \\ +2R_1 \\ -3R_1 \\ -4R_1 \end{matrix} \rightarrow \begin{pmatrix} -1 & 2 & 0 & 6 \\ 0 & 4 & 2 & 13 \\ 0 & -4 & -2 & -13 \\ 0 & -3 & -2 & -13 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 2 & 6 \\ 0 & 2 & 4 & 13 \\ 0 & -2 & -4 & -13 \\ 0 & -2 & -3 & -13 \end{pmatrix} \begin{matrix} \\ \\ +R_2 \\ +R_2 \end{matrix} \rightarrow \begin{pmatrix} -1 & 0 & 2 & 6 \\ 0 & 2 & 4 & 13 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = B$$

The output matrix B has *trapezoidal form* ($r(A) = r(B) = 3 < 4$), hence, the vectors a, b, c, d are not linearly independent, they are linearly dependent.

[57] In this example, assess if the vectors a,b,c linearly dependent or independent: a=(1,0,0,0), b=(2,1,0,1), c=(3,2,1,1).

Solution:

The vectors a, b, c form the following matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{pmatrix} \in \mathcal{R}^{3,4}$$

Firstly, we find a rank $r(A)$. By interchanging the rows, the matrix can be transformed into the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{pmatrix} \begin{matrix} \\ -2R_1 \\ -3R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ -2R_2 \end{matrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The output matrix has 3 nonzero rows ($r(A) = 3$) and a number of the input vectors is 3, therefore, the vectors a, b, c are linearly independent.

9.1 Examples: A linear dependence/independence of the vectors

[58] Assess if the vectors u, v, w, z are linearly dependent or independent:

$u = (1, 5, 3, -6)\,, v = (2, 4, 1, 0)\,, w = (1, 1, 2, 4)\,, z = (2, 8, 2, 10)$.

[59] Find α for which the following vectors will be linearly independent:

$a = (2, 1, 3, -1)\,, b = (3, -1, 2, 0)\,, c = (1, 3, 4, \alpha)\,, d = (4, \alpha, 1, 1)$.

[60] Find α for which the following vectors will be linearly independent:

$u = (1, 2\alpha + 3, 1, 0)\,, v = (0, 1, 1, 0)\,, w = (1, 1, -1, 2)\,, z = (1, -\alpha, 0, 3)$.

10 Frobenius Theorem

Theorem 2: Let $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$ be a defined matrix. Square matrix **A** is **regular** if and only **h(A)=n.** If **h(A)<n**, then the matrix **A** is **singular** (not regular).

Theorem 3 (Frobenius Theorem): Let have a defined SLE with $\boldsymbol{m}$ linear equations and $\boldsymbol{n}$ unknown elements.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\ldots\ldots\ldots\ldots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

The matrix of this system is following:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$\mathbf{Ax} = \mathbf{b}$ is the **matrix form** of the system and **(A,b)** is the **augmented matrix** of the system. Then Ax = b can be resolved if h(A) = h(A, b), i.e. if the rank of the matrix of SLE is equal to the rank of the augmented matrix.

- If **h(A)=h(A,b)=n,** then we say that $\mathbf{Ax} = \mathbf{b}$ has exactly **one solution**,
- If **h(A)=h(A,b)<n,** then we say that $\mathbf{Ax} = \mathbf{b}$ has **infinite many solution**.
- If **h(A)≠h(A,b),** then we say that $\mathbf{Ax} = \mathbf{b}$ has **no solution**.

Note: Frobenius Theorem provides the algorithm for a general resolution of the SLE. The complexity of SLE resolution is O(mn^2). The general algorithm includes 3 steps which are calculating the rank of the matrix A, augmented matrix (A, b) and application of Frobenius Theorem. By applying these 3 steps we can resolve the SLE.

10.1 Examples: Frobenius Theorem and Rank of Matrix

In the tasks [61]-[65], find the rank of the matrix A.

[61]

$$\mathbf{A} = \begin{pmatrix} -1 & 9 & -1 & 0 \\ 4 & 4 & 2 & 4 \\ 9 & -1 & 5 & 8 \\ 4 & 7 & 11 & 6 \end{pmatrix}$$

[62]

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 & 2 \\ -2 & 1 & 0 & 1 \\ -1 & 1 & 3 & 1 \\ -1 & 2 & 9 & 4 \end{pmatrix}$$

[63]

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$$

[64]

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 4 \\ 1 & 7 & 8 & 8 \\ 4 & 7 & 11 & 6 \end{pmatrix}$$

[65]

$$\mathbf{A} = \begin{pmatrix} -1 & 9 & -1 \\ 4 & 4 & 2 \\ 9 & -1 & 5 \\ 4 & 7 & 11 \\ 2 & 5 & 1 \end{pmatrix}$$

In the tasks [66]-[70], solve the system of linear equations:

[66]

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 3 \\ 4x_1 + 3x_2 + 5x_3 &= 4 \\ 2x_1 + 3x_3 &= 2 \end{aligned}$$

[67]

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + 4x_3 &= 3 \\ 3x_1 + 2x_2 + x_3 &= 6 \end{aligned}$$

[68]

$$\begin{aligned} 2x_1 - x_2 + x_3 - x_4 &= 1 \\ 2x_1 - x_2 - 3x_4 &= 2 \\ 3x_1 - x_3 + x_4 &= -3 \\ 2x_1 + 2x_2 - 2x_3 + 5x_4 &= -6 \end{aligned}$$

[69]

$$\begin{aligned} 3x_1 + 4x_2 + 2x_3 + 5x_4 &= 3 \\ 7x_1 + 3x_2 + x_3 + 2x_4 &= 2 \\ -4x_1 + 3x_2 - 3x_3 + 2x_4 &= -5 \\ 6x_1 + 10x_2 + 9x_4 &= 1 \end{aligned}$$

[70]

$$\begin{aligned} x_1 + 6x_2 + 5x_3 + 5x_4 &= 6 \\ 3x_1 + 4x_2 + 2x_3 + x_4 &= 2 \\ 5x_1 + 2x_2 - x_3 - 3x_4 &= -2 \\ 6x_1 + 8x_2 + +4x_3 + 2x_4 &= 4 \end{aligned}$$

In the tasks [71]-[75], using Frobenius Theorem, assess whether the system of linear equations has a solution or not.

[71]

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 3 \\ 4x_1 - 3x_2 + 5x_3 &= -1 \\ 2x_1 + 3x_3 &= 2 \end{aligned}$$

[72]

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + 4x_3 &= 3 \\ 3x_1 + 2x_2 + x_3 &= 6 \end{aligned}$$

[73]

$$\begin{aligned}
2x_1 - x_2 + x_3 - x_4 &= 1 \\
2x_1 - x_2 - 3x_4 &= 2 \\
3x_1 - x_3 + x_4 &= -3 \\
2x_1 + 2x_2 - 2x_3 + 5x_4 &= -6
\end{aligned}$$

[74]

$$\begin{aligned}
3x_1 + 4x_2 + 2x_3 + 5x_4 &= 3 \\
7x_1 + 3x_2 + x_3 + 2x_4 &= 2 \\
-4x_1 + 3x_2 - 3x_3 + 2x_4 &= -5 \\
6x_1 + 10x_2 + 9x_4 &= 1
\end{aligned}$$

[75]

$$\begin{aligned}
x_1 + 6x_2 + 5x_3 + 5x_4 &= 6 \\
3x_1 + 4x_2 + 2x_3 + x_4 &= 2 \\
5x_1 + 2x_2 - x_3 - 3x_4 &= -2 \\
6x_1 + 8x_2 + +4x_3 + 2x_4 &= 4
\end{aligned}$$

11 Linear Transformations

11.1 Matrices Multiplication and Elementary Operations

Each matrix defines a determined linear transformation. The simplest linear transformations are the elementary row or column operations. This chapter shows that the elementary operations can be applied in a way that the initial matrix will be multiplied from the right or left side with another appropriate matrix. Consequently, the sequence of the elementary operations can be expressed by 1 matrix that is a multiplication of these 2 matrices.

The elementary modifications are as follows:

Ie) Interchange 2 rows or columns of a matrix;

IIe) Multiply a row or column by a nonzero constant;

IIIe) Replace the row by the sum of 2 rows;

11.2 Matrices of type 2 x 2

For a demonstration, we will consider the matrices of type 2×2, the unit matrix is $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If we apply the elementary modification Ie) to this matrix, the resulting matrix is $\mathbf{D_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

If we apply the elementary modification IIe) to this matrix, i.e. we multiply the first row by the constant $k \neq 0$, the resulting matrix is $\mathbf{D_2} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

If we mutiple the second row by $k \neq 0$ and we add it to the first row, the resulting matrix is $\mathbf{D_3} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

If we mutiple the first row by $k \neq 0$ and we add it to the second row, the resulting matrix is $\mathbf{D_4} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

Now, we will consider the general matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we apply a multiplication with the matrices $\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}, \mathbf{D_4}$ from the left side, we get:

$\mathbf{B_1} = \mathbf{D_1}\mathbf{A} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$ – Interchanging the first and the second row;

$\mathbf{B_2} = \mathbf{D_2}\mathbf{A} = \begin{pmatrix} ka & kb \\ c & d \end{pmatrix}$ – Multiplying the first row by the constant $k \neq 0$;

$\mathbf{B_3} = \mathbf{D_3}\mathbf{A} = \begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix}$ – Multiplying the second row by $k \neq 0$ and adding it to the first row;

$\mathbf{B_4} = \mathbf{D_4}\mathbf{A} = \begin{pmatrix} a & b \\ ka+c & kb+d \end{pmatrix}$ – Multiplying the first row by $k \neq 0$ and adding it to the second row.

We can also apply a multiplication with the matrices $\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}, \mathbf{D_4}$ from the right side. If the input matrix is multiplied by $\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}, \mathbf{D_4}$ from the right side, the given matrices are as follows:

$\mathbf{C_1} = \mathbf{A}\mathbf{D_1} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ – Interchanging the first and the second column;

$\mathbf{C_2} = \mathbf{AD_2} = \begin{pmatrix} ka & b \\ kc & d \end{pmatrix}$ – Multiplying the first column by the nonzero constant $k \neq 0$;

$\mathbf{C_3} = \mathbf{AD_3} = \begin{pmatrix} a & ka+b \\ c & kc+d \end{pmatrix}$ – Multiplying the first column by $k \neq 0$ and adding it to the second column;

$\mathbf{C_4} = \mathbf{AD_4} = \begin{pmatrix} a+kb & b \\ c+kd & d \end{pmatrix}$ – Multiplying the second column by $k \neq 0$ and adding it to the first row;

11.3 Matrices of type n x n

In general, these ideas can be applied for the real matrices $n \times n$. Instead of matrices $\mathbf{D_1}, \mathbf{D_2}, \mathbf{D_3}, \mathbf{D_4}$ we will consider the matrices $\mathbf{T_1}, \mathbf{T_2}, \mathbf{T_3}, \mathbf{T_4}$ of type $n \times n$.

$$\mathbf{T}_1 = \begin{array}{c} \begin{array}{ccccccc} & & i-th & & j-th & & \end{array} \\ \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i-th \\ \\ j-th \\ \\ \\ \end{array}$$

The matrix $\mathbf{T}_1$ was created from the unit matrix $n \times n$ after interchanging the $i^{th}$and j^{th} row. The matrix $\mathbf{T}_1$ is named **permutation matrix** because multiplying the matrix from the right/left side means to interchange the i^{th} and j^{th} row or column of the initial matrix.

$$\mathbf{T}_2 = \begin{array}{c} \begin{array}{ccccc} & & i-th & & \end{array} \\ \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i-th \\ \\ \\ \\ \end{array}$$

The matrix $\mathbf{T_2}$ was created from the unit matrix $n \times n$ after multiplying the i^{th} row with the constant $k \neq 0$.

$$\mathbf{T}_3 = \begin{array}{c} \begin{array}{ccccccc} & & i-th & & j-th & & \end{array} \\ \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \cdots & k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i-th \\ \\ j-th \\ \\ \\ \end{array}$$

The matrix $\mathbf{T_3}$ was created from the unit matrix $n \times n$ after multiplying the j^{th} row with $k \neq 0$ and adding it to the i^{th} row.

$$\mathbf{T}_4 = \begin{array}{cc} \begin{matrix} & & i-th & & j-th & & \end{matrix} & \\ \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} & \begin{matrix} \\ \\ i-th \\ \\ j-th \\ \\ \\ \end{matrix} \end{array}$$

The matrix $\mathbf{T_4}$ was created from the unit matrix $n \times n$ after multiplying the i^{th} row with $k \neq 0$ and adding it to the j^{th} row.

As a consequence of this consideration, it can be concluded that the Gaussian Elimination Method can be expressed as a multiplication of the special matrices.

11.4 Linear Transformations and Gaussian Elimination Method

[76] Find matrix form of system below and the matrices sequence that is created during the calculations using the Gaussian Elimination Method:

$$\begin{aligned} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19 \end{aligned}$$

Solution:

As the first step, we will create the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix}$$

Now we want to eliminate the element a_{21}. That is why we multiply the first row by (-2) and we add it to the second row. We can perform this operation by the following matrix:

$$\mathbf{T_4^1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consequently, we observe the following:

$$\mathbf{A_1} = \mathbf{T_4^1 A} = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

Now we want to eliminate the element a_{32}. That is why we multiply the second row by (-2) and we add it to the third row. We can perform this operation by the following matrix:

$$\mathbf{T_4^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

After multiplying $\mathbf{T_4^2}$ and$\mathbf{A_1}$, we get the following matrix:

$$\mathbf{A_2}{=}\mathbf{T_4^2}\mathbf{A_1} = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

A sequence of the matrices is $\mathbf{T_4^1}$ and $\mathbf{T_4^2}$. Now, when we multiply the $\mathbf{T_4^1}$ and $\mathbf{T_4^2}$, we receive the matrix $\mathbf{T}$:

$$\mathbf{T}{=}\mathbf{T_4^2}\mathbf{T_4^1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$$

We can say that the matrix $\mathbf{T}$ contains all elementary modifications of the input matrix $\mathbf{A}$ necessary for adjusting to the triangular or trapezoidal matrix. This type of matrix we call a **transformation matrix**.

When we multiply the matrix $\mathbf{T}$ and $\mathbf{A}$, we get the following output matrix:

$$\mathbf{TA} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Note: For each square matrix A of type $n \times n$ there exist the square matrices P (for row operations - lower triangular form) and square matrices Q (for column operations - upper triangular form). Both matrices are of type $n \times n$ and the output is matrix $\mathbf{PAQ}$ of type $n \times n$ that has an upper triangular or trapezoidal form (form after applying the Gaussian Elimination Method).

In the previous example, $\mathbf{P} = \mathbf{T}$ and $\mathbf{Q} = \mathbf{I}$ (the unit matrix because we did not need to perform any elementary column operations). This decomposition we call **PAQ expression of the Gaussian Elimination Method**. This expression is not single-valued because the optional permutation of the matrix rows and columns can give us the same result. Although the definition of a permutation is out of the scope of this publication, we will explain the permutation matrix on the example.

In the example [76] we are considering the initial matrix $\mathbf{B}{=} \begin{pmatrix} 1 & 6 & 3 \\ 2 & 14 & 7 \\ 0 & 5 & 2 \end{pmatrix}$ instead of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix}$. The matrix $\mathbf{B}$ was created from the matrix $\mathbf{A}$ after interchanging the second and the third column. If the matrix $\mathbf{B}$ is multiplied by the matrix

$$\mathbf{T_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

from the right side, then we get the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix} = \mathbf{BT_1} = \begin{pmatrix} 1 & 6 & 3 \\ 2 & 14 & 7 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The matrix $\mathbf{T}_1$ is **permutation matrix**. Permutation matrices are created from the unit matrix after interchanging 2 rows or columns. Multiplication of the initial matrix with a permutation matrix from the right side means interchanging 2 columns of the input matrix while the multiplication with the permutation matrix from the left side means interchanging 2 rows of the input matrix.

In the example [76], the Gaussian Elimination Method for the matrix $\mathbf{B}$ is represented by $\mathbf{P} = \mathbf{T_4^2 T_4^1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ this means that

$$\mathbf{PBQ} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 6 & 3 \\ 2 & 14 & 7 \\ 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

In case the input matrix is not square, then we have to apply different steps. If we consider that the matrix is augmented:

$$[\mathbf{A}, \mathbf{b}] = \begin{pmatrix} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{pmatrix}$$

In this case, we will add the zero row (the row with zero constants) into this augmented matrix to get a square matrix. Sometimes we might need to add the zero column.

$$\mathbf{C} = \begin{pmatrix} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we will add the 4^{th} column and the 4^{th} row to the transformation matrix T. This column and row contain the constants 0 and the constant 1 on the 4^{th} position.

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The result of the transformation is as follows:

$$\mathbf{TC} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In practical calculations, we can also apply the decomposition called **LU** and **LDU** decomposition of matrix **A**.

According to this decomposition, each matrix A can be written as a multiplication of the matrices with lower ($\mathbf{L}$) and upper ($\mathbf{U}$) triangular form, $\mathbf{A} = \mathbf{LU}$ or with lower ($\mathbf{L}$), diagonal ($\mathbf{D}$) and upper ($\mathbf{U}$) triangular form $\mathbf{A} = \mathbf{LDU}$. A more detailed description/study of this decomposition is out of the scope of this publication. Note: Our Computer Algebra systems (SageMath) have 14 different square matrix decomposition, see chapter 13: Computer Algebra.

11.5 Examples - PAQ expression of Gaussian Elimination Method

In the tasks [77]-[80], find the PAQ expression of Gaussian Elimination Method:

[77]

$$\mathbf{A} = \begin{pmatrix} 3 & \text{-}1 & 4 \\ 0 & 1 & 8 \\ -2 & 1 & 0 \end{pmatrix}$$

[78]

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 \\ 3 & 2 & 6 \\ -4 & 5 & 15 \end{pmatrix}$$

[79]

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & -4 & -5 \end{pmatrix}$$

[80]

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 1 & -1 \\ 1 & 1 & -4 & -5 \end{pmatrix}$$

12 Vector Spaces

12.1 Linear Vector Space

Let matrix $\mathbf{A} = (\mathbf{a_{ij}}) \in \mathcal{R}^{n,n}$ have the rank $h(\mathbf{A}) = n$. The columns of matrix $\mathbf{A}$ are linearly independent vectors (note: the same applies to the row because $h(\mathbf{A}) = h(\mathbf{A}^T) = n$). These column or row vectors create a **basis of linear vector space**. This basis is a set of vectors $\mathcal{B} = \{\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}\}$,where $\mathbf{a_1}, \mathbf{a_2} \cdots \mathbf{a_n} \in \mathcal{R}^n$ are **linear independent column vectors** of matrix $\mathbf{A}$. The amount of vectors in the basis is called **a dimension** of vector space. By the basis we can define a new algebraic structure where each vector $\mathbf{x} \in \mathcal{R}^n$ can be expressed by linear combination of basis vectors :

$$\mathbf{x} = r_1\mathbf{a}_1 + r_2\mathbf{a}_2 + \cdots + r_n\mathbf{a}_n$$

for constants $r_i \in \mathcal{R}$ for $i = 1, 2, \cdots, n$ that $(r_1, r_2, \cdots, r_n) \neq (0, \cdots, 0)$.

Such algebraic structure contains vectors (from $\mathcal{R}^n$) and scalars (from $\mathcal{R}$). There are two linear operations defined here: the sum of two vectors and the multiplication of the vector by scalar. The main entities of this structure are vectors, hence, we can say that a vector space is linear and can be depicted as $\mathbf{L}_n$. This structure has also other characteristics which are a part of the next section.

12.2 General Definition of Vector Space

In general, a vector space over the entity $\mathbf{F}$ is a set of the vectors $\mathbf{V}$ with the following operations:

a) **Sum of 2 vectors** $+ : \mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$ $((\forall \mathbf{u}, \mathbf{w} \in \mathbf{V}) \Rightarrow \mathbf{u} + \mathbf{w} \in \mathbf{V})$

b) **Multiplication of a vector by scalar**: $\cdot : \mathbf{F} \times \mathbf{V} \rightarrow \mathbf{V}$ $((\forall a \in \mathbf{F})\ (\forall \mathbf{w} \in \mathbf{V}) \Rightarrow a\mathbf{w} \in \mathbf{V})$

The elements of set $\mathbf{V}$ are the **vectors** and the elements of entity $\mathbf{F}$ are the **scalars**. Let's assume that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are optional vectors $\mathbf{V}$ and a, b are optional scalars.

For the vector spaces the following rules and axioms apply:

1) **Associativity**: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$. (**associative law** considering the operation).

2) **Commutativity**: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. (**commutative law** considering the operation).

3) Existence of **additive identity**: $(\exists \mathbf{0} \in \mathbf{V})(\forall \mathbf{v} \in \mathbf{V})$:$\mathbf{0} + \mathbf{v} = \mathbf{v}$.

4) Existence for **additive inverse**:$(\forall \mathbf{v} \in \mathbf{V})(\exists\text{-}\mathbf{v} \in \mathbf{V})$: $\mathbf{v}+(\text{-}\mathbf{v}) = \mathbf{0}$.

5) **Associativity of scalar mutiplication**:$(\forall a, b \in \mathbf{F})(\forall \mathbf{v} \in \mathbf{V})$: $a(b\mathbf{v}) = (ab)\mathbf{v}$.

6) **Scalar multiplication identity** :$(\exists 1 \in \mathbf{F})(\forall \mathbf{v} \in \mathbf{V})$:$1\mathbf{v} = \mathbf{v}$.

7) **Distributivity of scalar multiplication and sum of vectors**: $(\forall a \in \mathbf{F})(\forall \mathbf{u}, \mathbf{v} \in \mathbf{V})$: $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$. (**distributivity law** considering the operations)

8) **Distributivity of scalar mutiplication and sum in the entity**: $(\forall a, b \in \mathbf{F})(\forall \mathbf{v} \in \mathbf{V})$:$(a+b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$. (**distributivity law** considering the operations)

In the vector space each vector has its own dimension. The vector dimension $\mathbf{v} \in \mathbf{V}$ we depict as $|\mathbf{v}|$. It is a scalar of the entity $\mathbf{F}$. The vector $\mathbf{w} \in \mathbf{V}$ is an **unit vector** where $|\mathbf{w}| = 1$. The vector dimension has the specific characteristics; a function of the vector dimension is sometimes called a **norm of vector space**. In the linear vector space $\mathbf{L}_n$ over the field of real numbers $\mathcal{R}$,a dimension of vector $\mathbf{x} \in \mathbf{L}_n$ can be defined as:

$$|\mathbf{x}| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{x_1x_1 + x_2x_2 + \cdots + x_nx_n}. \tag{17}$$

This is so-called **Euclidean Norm** of linear vector space.

12.3 Vector Spaces E_2 and E_3

All vectors $\mathbf{x} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2$ create a vector space over the field of real numbers $\mathcal{R}$, that together with the norm (17) is called **2-dimensional Euclidean vector space** and depicted as E_2. The vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{R}^2$ and $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{R}^2$ create basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ of the vector space.

$$E_2 = \{\mathbf{x}; \mathbf{x} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2, r_1, r_2 \in \mathcal{R}\}$$

All vectors $\mathbf{x} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3$ create vector space over the field of real numbers $\mathcal{R}$, that together with the norm (17) is called **3-dimensional Euclidean vector space** and depicted as E_3. The vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{R}^3$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \mathcal{R}^3$ and $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathcal{R}^3$ create a basis $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the vector space.

$$E_3 = \{\mathbf{x}; \mathbf{x} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3, r_1, r_2, r_3 \in \mathcal{R}\}$$

The vectors $\mathbf{e}_i$, for $i = 1, 2, 3$ are **unit vectors** due to the application of the **Euclidean Norm**: (17) $|\mathbf{e}_i| = \sqrt{\mathbf{e}_i^T \mathbf{e}_i} = 1$.

12.4 Vectors Spaces in Physics

The vector space can be also used in Physics for a description of the forces and velocities. Before identifying vector spaces, we need to have a **defined fixed point** that will serve for further calculations. The forces **v** a **w** with a defined **dimension** and **direction** are affecting this point**.** These forces (or different **physical quantities**) are the elements of the vector space. In a plane, all these forces, can be depicted by narrows; the direction of these vectors is depicted by a narrow above the vector, such as $\overrightarrow{v}$. In case of 2 optional narrows $\overrightarrow{v}$ a $\overrightarrow{\mathbf{w}}$, a parallelogram created by these 2 narrows include another narrow which begins in the fixed point defined at the beginning. This is a definition of the **vectors sum** $\overrightarrow{v + w}$ for 2 optional vectors. In case there are 2 narrows on the same line, the sum of these 2 narrows will be another narrow on the same line. The dimension of the resulting narrow is dependent on the direction of 2 initial narrows - narrows can have the same direction or each narrow can have a different diirection. When defining the **multiplication of a vector by scalar**, the following vectors are created $\overrightarrow{2w}$ alebo $-\overrightarrow{1v}$. We can also demonstrate that such defined vectors create a vector space. If a defined fixed point is located in $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in E_2$, then the vector space will be equivalent with the spaceE_2.

13 Computer Algebra

The problems covered by this publication can be also resolved by the computer using the special programs for calculation of the linear algebra problems with the common name of Computer Algebra. To verify the results of the problems, we have used the program MATLAB v. R2019b. This program has the academical license TAH and open source system SageMath v. 8.8. The system SageMath uses the coding language Python (eventually the special programming libraries) for programming the algorithms and for the application of a linear algebra.

13.1 MATLAB and the resolution of the system Ax=b

The code from MATLAB R2019b system provided below can be used for finding the solutions of the equation $\mathbf{Ax} = \mathbf{b}$ in general:

```
% 30. task - solution of system of linear equations
 A=[[1,3,6];[2,7,14];[0,2,5]];
 b=[25,58,19]';
 x=A\b
```

Similarly, the same code we get from the SageMath (Python). However, here we need to define a type of input matrix - QQ means that elements of input matrix are rational numbers.

```
% 30. task - solution of system of linear equations
 A = matrix(QQ, [[1,3,6],[2,7,14],[0,2,5]])
 b = vector(QQ, [25,58,19])
 x = A.solve_right(b)
```

It is important to note that the code from above works only in case the SLE has 1 solution, in other words, when the input matrix **A** is regular. For remaining cases, i.e. when the input matrix is arbitrary, SLE has infinitely many solutions or no solution, then we used the **Frobenius Theorem:**

The **inputs of the algorithm** are: **Matrix A** and **Vector b**.

The **output of the algorithm** is as follows:

- **Vector x** - the system has only 1 exact solution (matrix A is regular);

- **Reduced echelon form matrix** - the system has infinitely many solutions; - a possibility to join a general solution to the matrix;

- **'System of linear equations has no solution'** - the system has no solution;

The following code is for testing the individual problems:

```
%% A general solution of a system of linear equations (SLE)
% Read an input matrix A
% 1.task - unique solution
% A=[[1,3,6];[2,7,14];[0,2,5]]
% 2. task - no solution
% A=[[5,10,-7];[2,4,-3];[3,6,5]]
```

```
% 3. task - infinitely many solution
A=[[3,-1,-5];[0,1,-10];[-2,1,0]]

% Read an input right side vector b
% 1. task - unique solution
%b=[25,58,19]'
% 2. task - no solution
% b=[-2,-1,9]'
% 3. task - infinitely many solution
b=[9,0,-6]'

% Determine the size of matrix A
[m,n]=size(A);
n=min(m,n);
% Apply the Frobenius theorem
% Compute kA - the rank of matrix A
% Compute kAu - the rank of augmented matrix Au
Au=[A,b];
kA=rank(A);
kAu=rank(Au);
if kA==kAu
  if kA==n
    % the matrix A is regular, SLE has a unique solution X (kA=KAu=n)
    S= 'The unique solution X of a system of linear equation:';
    disp(S);
    X = linsolve(A,b)
  else
    % the matrix A is irregular, SLE has infinitely many solution (kA=KAu<n)
    % in this case we need compute
    S= 'The system has infinitely many solution. Reduced row echelon form matrix:';
    disp(S);
    % Gauss - Jordan elimination, reduced row echelon form matrix
    X=rref(Au)
  end
else
  % the matrix A is irregular, SLE has no solution (kA<>kAu, Frobenius theorem)
  S='System of linear equation has no solution';
  disp(S);
end
```

MATLAB program allows to solve also different types of problems, e.g. the solution of SLE by using Cramer's Rule depicted in the code below:

```
%% 42. task - Cramer's rule
A=[[2,3,4];[1,2,-1];[-3,1,3]]
```

```
b=[-5,0,7]
[m,n]=size(A);
n=min(m,n);
rA=rank(A);
if rA==n
  d=det(A)
  for j=1:n
    B=A;
    for i=1:n
      B(i,j)=b(i);
    end
    di(j)=det(B);
  end
  disp(di)
  C=A\b'
 else
   disp('Matrix A is singular')
 end
```

Computation of the determinant

```
%% 35. task - the determinant of matrix
A=[[1,4,2,0];[2,4,1,1];[3,2,0,2];[1,3,2,3]];
[m,n]=size(A);
n=min(m,n);
rA=rank(A);
if rA==n
  d=det(A)
else
  disp('Matrix A is singular')
end
```

Computation of elementary operations - linear transformation

```
%% 74. task - PAQ decomposition
A=[[1,0,1,0];[0,0,1,1];[0,2,1,-1];[0,1,-4,-5]]
T1=[[1,0,0,0];[0,0,0,1];[0,0,1,0];[0,1,0,0]];
T2=[[1,0,0,0];[0,1,0,0];[0,-2,1,0];[0,0,0,1]];
T3=[[1,0,0,0];[0,1,0,0];[0,0,1/9,0];[0,0,0,1]];
T4=[[1,0,0,0];[0,1,0,0];[0,0,1,0];[0,0,-1,1]];
% Elementary row operations and matrices
T=T4*T3*T2*T1
C=T*A
```

There are also the problems with the analytical character that are more difficult to resolve by the Computer Algebra. Such examples are problems 74-75 of this book.

14 Solutions

Elementary Row Operations

[1]: $X = (1, 2, 3)^T$ - system has a unique solution
[2]: no solution
[3]: $X = (3 + 5t, 10t, t)^T,\ t \in \boldsymbol{\mathcal{R}}$
Solution:

$$\begin{aligned} 3x - y - 5z &= 9 \\ y - 10z &= 0 \\ -2x + y &= -6 \end{aligned}$$

a) We assign the augmented matrix to the system

$$\begin{pmatrix} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{pmatrix}$$

b) We adjust the augmented matrix using the Gauss-Jordan elimination. We get a **reduced echelon form of A**.

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By matrix R we can define **a fundamental system of solutions**. Elements r_{11}, r_{22} are **pivot entries** and variables x, y, are **basic variables** (**pivot or basic variables**) of the system. Variable z is not a basic variable, but it is a **free variable**. Free variable is defined by a parameter, e.g. $t \in \boldsymbol{R}$, $z = t$. From the second equation of the matrix $\mathbf{R}$ we can express a variable $y = 10t$; from the first equation of the same matrix we can express a variable $x = 3 + 5t$. The main system of solutions or parametric solution of the SLE is vector $(x, y, z)^T = (3 + 5t, 10t, t)^T$ for $t \in \boldsymbol{R}$. In case the SLE has infinite solutions, it is necessary to find the main system of solutions.

Note: In MATLAB, while having matrix $\mathbf{A} = \begin{pmatrix} 3 & -1 & -5 \\ 0 & 1 & -10 \\ -2 & 1 & 0 \end{pmatrix}$ and vector of right sides $\mathbf{b} = \begin{pmatrix} 9 \\ 0 \\ -6 \end{pmatrix}$, the augmented matrix of the system will be generated by using the command $[\mathbf{A}, \mathbf{b}]$; The final matrix after applying Gauss-Jordan Elimination Method we get by using the command **R=ref(A)**.

[4]: no solution
[5]: $X = (2 - 4t, -8t, t)^T,\ t \in \boldsymbol{\mathcal{R}}$
[6]: no solution
[7]: $X = (2, 0, 1)^T$
[8]: $X = (2 + t, 2 + t, -t, t)^T,\ t \in \boldsymbol{\mathcal{R}}$
[9]: $X = (2 - u + t, u, -t, t)^T,\ t, u \in \boldsymbol{\mathcal{R}}$

Solution: In this example, we will resolve the SLE in a similar way as we resolved example [3].

$$\begin{aligned} x+y+z&=2\\ z+w&=0\\ 2x+2y+z-w&=4\\ x+y-4z-5w&=2 \end{aligned}$$

First, we add the augmented matrix to the input; consequently, this matrix will be adjusted by Gauss-Jordan Elimination Method. After this adjustment, we get the following matrix:

$$\mathbf{R}=\begin{pmatrix} 1 & 1 & 0 & -1 & 2\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The variables x, z will be basic ($r_{11}=1$,$r_{23}=1$,column indexes indicate which variables are basic and variablesy, w will be free variables. That is why we can allocate $w=t$, $y=u$. From the second equation has resulted that $z=-t$ and from the first one that $x=2-u+t$. Hence, the fundamental (parametric) system resolution is vector $X=(2-u+t, u, -t, t)^T,\ t, u \in \boldsymbol{\mathcal{R}}$.

[10]: $X=(1.2, 4.8, -4.2)^T$

Matrices and vectors

[11]: no solution
[12]: $X=(2-4t, -8t, t)^T,\ t \in \boldsymbol{\mathcal{R}}$
[13]: no solution
[14]: $X=(2, 0, 1)^T$
[15]: $X=(2+t, 2+t, -t, t)^T,\ t \in \boldsymbol{\mathcal{R}}$

Matrix operations

[16]: $\mathbf{Ax}=(25, 58, 19)^T$
[17]: $\mathbf{Ax}=\mathbf{Ay}=(9, 0, 6)^T$
[18]: $\mathbf{A}-\mathbf{B}=\begin{pmatrix} 3 & 3 & 4\\ -1 & -4 & 6\\ 7 & 3 & -1\\ 8 & 9 & 2 \end{pmatrix}, \mathbf{A^T}=\begin{pmatrix} 5 & 0 & 4 & 3\\ 7 & -3 & 8 & 9\\ 2 & 5 & -1 & 6 \end{pmatrix}, \mathbf{A^TB}=\begin{pmatrix} -17 & 40 & 2\\ -58 & 65 & 25\\ -18 & 8 & 15 \end{pmatrix}$
[19]: $\mathbf{A}-\mathbf{B}=\begin{pmatrix} 4 & 5 & -1\\ -5 & -4 & 6\\ 7 & -5 & -3\\ 8 & 1 & 2 \end{pmatrix}, \mathbf{A^T}=\begin{pmatrix} 3 & 1 & 4 & 3\\ 1 & -3 & -1 & 0\\ 2 & 5 & -1 & 6 \end{pmatrix}, \mathbf{A^TB}=\begin{pmatrix} -24 & 2 & 28\\ -16 & -11 & 4\\ 1 & -13 & 23 \end{pmatrix}$
[20]: $\mathbf{A^TA}=\begin{pmatrix} 34 & 7 & 5\\ 7 & 11 & -14\\ 5 & -14 & 31 \end{pmatrix}$

[21]: $\mathbf{C = 3A - 2A^2} = \begin{pmatrix} -13 & 3 & 6 & -5 \\ 4 & -25 & -11 & 2 \\ 0 & 7 & -13 & -1 \\ -7 & -8 & -9 & -7 \end{pmatrix}$

[22]: $\mathbf{C = 2A^TA + A^2} = \begin{pmatrix} 112 & -6 & 47 & 22 \\ 13 & 60 & 24 & -5 \\ 55 & 3 & 48 & 13 \\ 18 & 33 & 20 & 26 \end{pmatrix}$

[23]: $\mathbf{C = 2AA^TA} = \begin{pmatrix} 92 & 178 & 14 \\ 54 & 108 & 2 \\ 78 & 154 & 8 \end{pmatrix}$

[24]: $\mathbf{C} = (\mathbf{2A - B})^{\mathbf{2}} = \begin{pmatrix} 185 & 54 & 145 \\ 124 & 320 & -155 \\ 87 & -103 & 261 \end{pmatrix}$

[25]: $\mathbf{C = A^TA + A^2 - B^3} = \begin{pmatrix} 1 & -3 & 5 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{pmatrix}$

Inverse matrices

[26]: $\mathbf{A^{-1}} = \begin{pmatrix} -0.1429 & 0.2857 & 0.2857 \\ 0.5 & -0.5 & 0 \\ 0.0714 & 0.3571 & -0.1429 \end{pmatrix}$

[27]: $\mathbf{A^{-1}}$ does not exist

[28]: $\mathbf{A^{-1}} = \begin{pmatrix} -2 & 0 & 3 \\ 0 & 0.3333 & -0.6667 \\ 1 & 0 & -1 \end{pmatrix}$

[29]: $\mathbf{X} = (-5.6667, -1.6667, 6)^T$

[30]: $\mathbf{X} = (1, 2, 3)^T$

Determinants

[31]: $det\mathbf{A}$=7

[32]: $det\mathbf{A} = -8$

[33]: $det\mathbf{A}$=-4

[34]: $det\mathbf{A}$=-5

[35]: $det\mathbf{A}$=-18

[36]: $\mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

[37]: $\mathbf{A^{-1}} = \frac{1}{-8}\begin{pmatrix} 3 & 5 & -9 \\ 1 & -1 & -3 \\ -5 & -3 & 7 \end{pmatrix}$

[38]: $\mathbf{A^{-1}} = \frac{1}{-4}\begin{pmatrix} 0 & -4 & 0 & 0 \\ -3 & 3 & 1 & -1 \\ -10 & -10 & 2 & 2 \\ 9 & 7 & -3 & -1 \end{pmatrix}$

[39]: $\mathbf{A^{-1}} = \frac{1}{-18}\begin{pmatrix} -12 & 16 & -14 & 4 \\ 9 & -18 & 9 & 0 \\ -21 & 28 & -11 & -2 \\ 9 & -6 & 3 & -6 \end{pmatrix}$

[40]: $\mathbf{A^{-1}} = \begin{pmatrix} 7 & -3 & 0 \\ -10 & 5 & -2 \\ 4 & -2 & 1 \end{pmatrix}$

Cramer's Rule

[41]: $D = 10,\ D_1 = -10,\ D_2 = 24,\ D_3$=-8
[42]: $D = 42,\ D_1 = -112,\ D_2 = 42,\ D_3$=-28
[43]: $D = -45,\ D_1 = -45,\ D_2 = -90,\ D_3$=-135
[44]: $D = -12,\ D_1 = -12,\ D_2 = -48,\ D_3$=0
[45]: $D = 20,\ D_1 = 20,\ D_2 = 20,\ D_3$=0, $D_4 = 0$

Gaussian Elemination Method

[46]:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[47]:

$$\mathbf{A} = \begin{pmatrix} 25 & 2 & 3 & 4 & 2 \\ 0 & 2 & 2 & -3 & 2 \\ 0 & 0 & 3 & 4 & -1 \\ 0 & 0 & 0 & 25 & -16 \end{pmatrix}$$

[48]:

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 7 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

[49]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[50]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Gauss-Jordan Elimination Method

[51]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 11 & 0 \\ 0 & 1 & 0 & 12 & -3 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[52]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & -0.0751 \\ 0 & 1 & 0 & 0 & 2.4390 \\ 0 & 0 & 1 & 0 & -0.8537 \\ 0 & 0 & 0 & 1 & 0.3902 \end{pmatrix}$$

[53]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

[54]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[55]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Linear dependence and independence of vectors

[56]: $r(A) = 3 < n = 4$, vectors are linearly dependent.
[57]: $r(A) = 2 < n = 4$, vectors are linearly dependent.
[58]: $r(A) = 4 = n = 4$, vectors are linearly independent.
[59]: We get a matrix after the elementary row operations

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 & -1 \\ 0 & -5 & -5 & 1 \\ 0 & 0 & \alpha+3 & 2 \\ 0 & 0 & 0 & 2\alpha \end{pmatrix}$$

rank of the matrix $r(A) = 4$ if and only if $\alpha \neq 0$ and $\alpha \neq$ -3.
[60]: We get a matrix after the elementary row operations

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, the vectors are linearly independent for each α.

Rank of matrices, Frobenius Theorem

[61]:

$$\mathbf{A} = \begin{pmatrix} -1 & 9 & -1 & 0 \\ 4 & 4 & 2 & 4 \\ 9 & -1 & 5 & 8 \\ 4 & 7 & 11 & 6 \end{pmatrix}, r(A) = 3$$

[62]:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 & 2 \\ -2 & 1 & 0 & 1 \\ -1 & 1 & 3 & 1 \\ -1 & 2 & 9 & 4 \end{pmatrix}, r(A) = 3$$

[63]:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}, r(A) = 4$$

[64]:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & -1 & 1 & 4 \\ 1 & 7 & 8 & 8 \\ 4 & 7 & 11 & 6 \end{pmatrix}, r(A) = 3$$

[65]:

$$\mathbf{A} = \begin{pmatrix} -1 & 9 & -1 \\ 4 & 4 & 2 \\ 9 & -1 & 5 \\ 4 & 7 & 11 \\ 2 & 5 & 1 \end{pmatrix}, r(A) = 3$$

[66]: no solution
[67]: no solution
[68]: $X = (0, 2, 1.\bar{6}, 1.\bar{3})^T$
[69]: no solution
[70]: $(-0.8571 + u + 0.5714t, 1.1429 - u - 0.928t, u, t)^T \; t, u \in \boldsymbol{\mathcal{R}}$
[71]: $X = (-2, 1, 2)^T$
[72]: no solution

[73]: $X = (0, 2, 1.\bar{6}, 1.\bar{3})^T$
[74]: no solution
[75]: $(-0.8571 + u + 0.5714t, 1.1429 - u - 0.928t, u, t)^T \ t, u \in \boldsymbol{\mathcal{R}}$

PAQ Expression of Gaussian Elimination Method

[76]:

$$\mathbf{T^1_4} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{T^2_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\mathbf{P} = \mathbf{T} = \mathbf{T^1_4} * \mathbf{T^2_4} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}, \mathbf{Q} = \mathbf{I},$$

[77]:

$$\mathbf{T^1_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{2}{3} & 0 & 1 \end{pmatrix}, \mathbf{T^2_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \ \mathbf{T^3_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\mathbf{P} = \mathbf{T} = \mathbf{T^3_3} * \mathbf{T^2_2} * \mathbf{T^1_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 3 \end{pmatrix}, \mathbf{Q} = \mathbf{I},$$

note: $\mathbf{Q} = \mathbf{I}$ means that elementary operations with columns were not applied.
[78]:

$$\mathbf{T^1_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \mathbf{T^2_3} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathbf{T^3_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{13} \end{pmatrix}, \ \mathbf{T^4_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

$$\mathbf{P} = \mathbf{T} = \mathbf{T^3_2} * \mathbf{T^2_3} * \mathbf{T^1_4}, \mathbf{Q} = \mathbf{I}.$$

[79]:

$$\mathbf{T^1_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathbf{T^2_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\mathbf{T^3_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{T^4_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

$$\mathbf{P} = \mathbf{T} = \mathbf{T^4_3} * \mathbf{T^3_2} * \mathbf{T^2_3} * \mathbf{T^1_1}, \mathbf{Q} = \mathbf{I}.$$

[80]:

$$\mathbf{T_4^1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \mathbf{T_4^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{P} = \mathbf{T} = \mathbf{T_4^2} * \mathbf{T_4^1}, \mathbf{Q} = \mathbf{I}.$$

15 Appendix I: Linear Algebra - Questions

This chapter contains the list of questions the users/readers of this publication should be familiar with. The questions are related to the solution of the system of linear equations (SLE) and the vector spaces.

1.What is the solution of the SLE ?

2.How many solutions the SLE can have ?

3.What is a free variable in the solution of SLE ?

4.What are the matrix elementary rows operations ?

5.What is the matrix form of the SLE ?

6.What is the augmented matrix of the SLE ?

7.Which algorithms can be used for the solution of the SLE ?

8.Take 1 optional algorithm related to the SLE and describe it.

9.What is the Gaussian Elimination Method ?

10.What is the difference between the elementary rows operations and elementary adjustments of matrix ?

11.What is the computing complexity of the Gaussian Elimination Method ?

12.What is the determinant of the matrix.

13.What is the method to calculate the determinant of the square matrix of type $n \times n$?

14.What is the the computing complexity to solve SLE by the matrix determinant ?

15.Which matrices can be defined as regular ?

16.What is the linear combination of the vectors ?

17.Which vectors are linearly independent ?

18.What is the rank of the matrix ?

19.By the rank of matrix, define the regular matrices.

20.By the amount of the SLE solutions, define the regular matrices.

21.By the determinants, define the regular matrices.

22.How we can resolve the SLE by the inverse matrix ?

23.How we can resolve the inverse matrices by the elementary rows operations ?

24.How we can resolve the inverse matrices by the determinant ?

25.What is the algebraic complement of the matrix element ?

26.What is the transposed matrix ?

27.What is the relation between the rank of the matrix and the transposed matrix ?

28.What is the Cramer's Rule ?

29.In which cases we can apply the Cramer's Rule for the solution of the SLE ?

30.What is the Sarrus's Rule ?

31.What is the Frobenius Theorem with respect to the SLE ?

32.What is the basis of vector space ?

33.What is the dimension of vector space ?

34.How we can define the size of the vector in E_3?

16 Appendix II: Linear Algebra - Subject Classification

In this publication we were dealing with the resolution of the system of linear equations of type (1). According to the current **Mathematical Subject Classification** this is defined as a discipline called **15A06 Linear equations**.

The vector spaces defined in the previous chapter are a discipline **15A03 Vector spaces**, **linear dependence and rank**.

Despite above-mentioned 2 disciplines, the current research of linear algebra recognize also other disciplines, e.g.

15-XX **Linear and multilinear algebra; matrix theory**

15Axx Basic linear algebra
15A03 Vector spaces, linear dependence, rank
15A04 Linear transformations, semilinear transformations
15A06 Linear equations
15A09 Matrix inversion, generalized inverses
15A12 Conditioning of matrices [See also 65F35]
15A15 Determinants, permanents, other special matrix functions [See also 19B10, 19B14] 15A16 Matrix exponential and similar functions of matrices
15A18 Eigenvalues, singular values, and eigenvectors
15A21 Canonical forms, reductions, classification
15A22 Matrix pencils [See also 47A56]
15A23 Factorization of matrices
15A24 Matrix equations and identities
15A27 Commutativity
15A29 Inverse problems
15A30 Algebraic systems of matrices [See also 16S50, 20Gxx, 20Hxx]
15A39 Linear inequalities
15A42 Inequalities involving eigenvalues and eigenvectors
15A45 Miscellaneous inequalities involving matrices
15A54 Matrices over function rings in one or more variables
15A60 Norms of matrices, numerical range, applications of functional analysis to matrix theory [See also 65F35, 65J05]
15A63 Quadratic and bilinear forms, inner products [See mainly 11Exx]
15A66 Clifford algebras, spinors
15A69 Multilinear algebra, tensor products
15A72 Vector and tensor algebra, theory of invariants [See also 13A50, 14L24]
15A75 Exterior algebra, Grassmann algebras
15A78 Other algebras built from modules
15A80 Max-plus and related algebras
15A83 Matrix completion problems
15A86 Linear preserver problems
15A99 Miscellaneous topics 15Bxx Special matrices
15B05 Toeplitz, Cauchy, and related matrices
15B10 Orthogonal matrices

15B15 Fuzzy matrices
15B33 Matrices over special rings (quaternions, finite fields, etc.)
15B34 Boolean and Hadamard matrices
15B35 Sign pattern matrices
15B36 Matrices of integers [See also 11C20]
15B48 Positive matrices and their generalizations; cones of matrices
15B51 Stochastic matrices
15B52 Random matrices
15B57 Hermitian, skew-Hermitian, and related matrices
15B99 None of the above, but in this section

The provided list is not exhaustive one; it provides only the basic level of information. More details of the linear algebra are provided in the Slovak publication of the Professor Zlatos [8] or Professor Kuros [4].

The last chapter of this publication is dealing with the resolution of the equations in the so-called bottleneck algebra. Concurrently, it is the example of the current research in the Linear Algebra .

17 Appendix III: Solution of system of equations in Bottleneck Algebra

This chapter describes an example on the current research in the linear algebra in accordance with the classification above (Appendix II). The mentioned contribution is the outcome of the work of 3 authors. The solution of the system of equations

$$\mathbf{Ax} = \mathbf{b}$$

in bottleneck algebra is described in the the work (Butkovic et al. [2]).

Bottleneck algebra serves for mathematical modelling of the current, practical problems. This algebra can be defined as an algebra in which the operations $(\oplus, \otimes) =$ (maximum, minimum) are used instead of the basic algebraic operations of linear algebra. In the bottleneck algebra these operations are extended also for the vectors and matrices in similar way than in linear algebra. The difference between linear algebra and this algebraic structure is that the operations minimum and maximum are not group operations, meaning that the inverse elements to the individual ones are not existing. As a consequence, there is no possibility to apply the elementary row operations for the solution of the system of equations. Additionally, the important role in this structure are the elements $-\infty$ and ∞. The exact definition of the bottleneck algebra and more details can be found in the publication (Butkovic et al. [2]).

Note: Currently, for this algebra (i.e. with the operations $(\oplus, \otimes) =$ (maximum, minimum)) some authors are using the name *max-min* or *fuzzy algebra*. The work referred above (Butkovic et al. [1987]) explains when the system of equations

$$\mathbf{A} \otimes \mathbf{x} = \mathbf{b} \tag{18}$$

has the only one solution or, in other words, when the matrix A is **strong regular** in bottleneck algebra. In this algebra, the class of the strong regular matrices form trapezoidal matrices and matrices which can be converted to trapezoidal matrix by using the permutations of matrix rows and columns.

The work (Butkovic et al. [2]) includes also polynomial algorithm for the testing the matrix features. The real square matrix $\mathbf{A} = (a_{ij})$ of type $n \times n$ is **trapezoidal** (has trapezoidal form in Bottleneck algebra) when

$$a_{ii} > \max_{k=1,\cdots,i} \min_{l=k+1,\cdots,n} a_{kl} = \sum_{k=1}^{i}{}^{\oplus} \sum_{l=k+1}^{n}{}^{\otimes} a_{kl} \tag{19}$$

for each $i = 1, \cdots, n$.

Below we will describe the steps how to check when the input matrix $\mathbf{A} = (a_{ij})$ of the type $n \times n$ is a trapezoidal matrix. The right side of the inequality is possible to define for $i = 1, \cdots, n$ as follows:

$$P(1) = \min_{l=2,\cdots,n} a_{1l} \tag{20}$$

$$P(i) = \max \left\{ P(i-1), \min_{l=i+1,\cdots,n} a_{il} \right\} \tag{21}$$

therefore, there is a need to calculate (20) and then (21). The last step of this algorithm is comparing the $a_{ii} > P(i)$ for each $i = 1, \cdots, n$.

As for the solution of the (2) and (15) it has been discovered that also in the bottleneck algebra, the system can have **one solution**, **infinitely many solutions** or **no solution**. The regular matrices in the bottleneck algebra are described as follows:

The real square matrix $A = (a_{ij})$ of type $n \times n$ is **trapezoidal** if there is a such permutation π that for each $i = 1, \cdots, n$.

$$a_{i\pi(i)} > \max_{k=1,\cdots,i} \min_{l=k+1,\cdots,n} a_{k\pi(l)} \tag{22}$$

The system (18) has only one solution when the matrix A is trapezoidal. The testing whether the input matrix is trapezoidal or not is out of scope of this publication. However, in case of interest, there is a possibility to find the polynomial algorithm for testing of this feature in the above referred work.

The condition (22) applies for these bottleneck algebras that are defined over a **dense set**. A set M is dense, in regards to configuration of the elements "$<$", when $(\forall x, y \in M)(x < y)(\exists z \in M)(x < z < y)$. A dense set is e.g. the set of the real numbers of rational numbers. The set of the integers is not dense because there is no integer between 2 and 3.

18 Conclusion

This publication has been developed to provide the study material about linear algebra. The publication contains the theoretical as well as the practical part.

As for the theory, this publication provides the fundamental theoretical knowledge, information and different methods on how to resolve the system of equations; besides, the fundamental information about a vector space is provided. Despite the theoretical part, in order to support practicing a gained theory, this publication contains the database of the practical exercises with its solutions. Such exercises can also be resolved by using Computer Algebra. The simple tutorial on how to apply Computer Algebra in such cases is also a part of this publication. The elementary keywords and the main concepts of this problematic are provided in the Register.

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