GeoLing14: Determinants and inverse matrices.

Contents:

- Determinants and inverse matrices.

Recommended exercises: Leling 11, Leling 12.

EXERCISES

1. Compute the determinant of the following matrices:

$$(a)\quad A=\begin{pmatrix}1&2\\3&4\end{pmatrix}\qquad (b)\quad A=\begin{pmatrix}0&3\\-3&0\end{pmatrix}\qquad (c)\quad A=\begin{pmatrix}1&0\\3&6\end{pmatrix}$$

$$(d)\quad A=\begin{pmatrix}1&2&1\\3&6&0\\3&5&0\end{pmatrix}\qquad (e)\quad A=\begin{pmatrix}1&0&0\\0&6&0\\0&0&-2\end{pmatrix}\qquad (f)\quad A=\begin{pmatrix}1&0&0\\3&6&0\\3&5&4\end{pmatrix}$$

$$(g)\quad A=\begin{pmatrix}1&-2&-1\\0&6&0\\0&0&4\end{pmatrix}\qquad (h)\quad A=\begin{pmatrix}1&-2&-1\\0&6&0\\1&4&-1\end{pmatrix}$$

$$(i)\quad A=\begin{pmatrix}1&1&0&-1\\1&0&0&0\\1&1&0&-1\\1&1&0&-1\end{pmatrix}\qquad (l)\quad A=\begin{pmatrix}3&9&3&-10\\3&0&3&0\\3&7&3&-1\\3&11&3&-1\end{pmatrix}$$

$$(m)\quad \begin{pmatrix}-1&-1&-1&0&0&0\\1&0&0&-1&-1&0\\0&1&0&1&0&-1\\0&1&0&1&0&-1\\0&0&1&0&1&1\end{pmatrix}$$

2. Solve the following systems using determinants, i.e. Cramer's rule:

$$(a)\begin{cases}\alpha+\beta=1\\\alpha-\beta=0\end{cases}\qquad (b)\begin{cases}m-3n=0\\n=m-1\end{cases}$$

$$(c)\begin{cases}x+y+z=1\\x+y-z=0\\x=1\end{cases}\qquad (d)\begin{cases}x+y+z=1\\x+2y+z=1\\x+y+3z=0\end{cases}$$

$$(e)\begin{cases}x+y+z+w=1\\x+y-z-3w=0\\x+5w=1\\x-y=0\end{cases}$$

3. Find a vector $\overrightarrow{x}$ orthogonal to $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{R}^3$ where:

   (a) $\overrightarrow{v} = (1,2,3)\,,\ \overrightarrow{w} = (1,0,3)\,,$

   (b) $\overrightarrow{v} = (5,3,7)\,,\ \overrightarrow{w} = (1,0,0)\,.$

4. Solve the following system in terms of x_1 and x_2:

$$\begin{cases} x_1 + 4x_2 + x_3 + x_4 = 0 \\ 8x_1 + 1x_2 + 8x_3 + 3x_4 = 0 \end{cases}$$

   Is it possible to write the solutions using x_2 and x_4 instead?

5. Solve the following system in terms of x_1 and x_2:

$$\begin{cases} 5x_2 + x_3 + x_4 = 0 \\ 8x_1 + 1x_2 + 8x_3 + 3x_4 = 0 \end{cases}$$

   Is it possible to write the solutions using x_2 and x_4 instead?

6. Find the inverse of A:

$$(a)\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad (b)\quad A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \qquad (c)\quad A = \begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix}$$

$$(d)\quad A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 0 \\ 3 & 5 & 0 \end{pmatrix} \qquad (e)\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad (f)\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 6 & 0 \\ 3 & 5 & 4 \end{pmatrix}$$

7. The determinant of a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is given by $det(A) = aei + bfg + cdh - ceg - afh - bdi$.

8. Let $\overrightarrow{v} \times \overrightarrow{w}$ be the cross product of two vectors in space. Prove $(\overrightarrow{v} \times \overrightarrow{w}) \times \overrightarrow{v} = \overrightarrow{v}^2 \overrightarrow{w} - (\overrightarrow{v} \cdot \overrightarrow{w})\overrightarrow{v}$ (Hint: use that $\overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{v} \times (\overrightarrow{w} + m\overrightarrow{v})$) and notice $(\overrightarrow{v} \times \overrightarrow{w}) \times \overrightarrow{v}$ is perpendicular to $\overrightarrow{v}$).

9. A vector $\overrightarrow{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in space induces a vector function $f(\overrightarrow{X}) = \overrightarrow{v} \times \overrightarrow{X}$. If $\overrightarrow{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $f(\overrightarrow{X}) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

10. Find all numbers x such that the matrix $\begin{pmatrix} 12-x & 4 \\ 8 & 8-x \end{pmatrix}$ be singular, i.e. not invertible.

11. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ two given points in the plane $\mathbb{R}^2$. Prove that the equation of the line determined by P and Q is $det(\begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}) = 0$.

12. Show that $\begin{vmatrix} y+z & x+z & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$

13. Let M be an $(a+b \times a+b)$-block matrix, i.e. $M = \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)$ with A $a \times a$, B $b \times b$, C $a \times b$ and 0 is the zero $b \times a$ matrix. Prove $det(M) = det(A)det(B)$.

14. Calculate the characteristic polynomial fo the following matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 7777 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 7 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & -5 \\ 2 & 6 & -10 \\ 1 & 2 & -3 \end{pmatrix} \quad \begin{pmatrix} 5 & -1 \\ 9 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$