Operational Research 2
Assignement

I - a) $f(x,y) = e^x \cos y$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$\begin{cases} \dfrac{df}{dx} = e^x \cos y \\ \\ \dfrac{df}{dy} = e^x \cdot -\sin y \end{cases}$$

- $\lim\limits_{h \to 0} \left[\dfrac{f_x(0+h,0) - f(0,0)}{h}\right] = \dfrac{e^h - 1}{h} = 0$

- $\lim\limits_{h \to 0} \left[\dfrac{f_y(0+h,0) - f(0,0)}{h}\right] = \dfrac{-0 + 0}{h} = 0$

- $f_x(0,0) = f_y(0,0) = 0$

b/ $g(x,y,z) = \sqrt{x^2+y^2+z^2}$, Pt $(2,2,2)$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-1/2}$$

I)' b/ $g(x,y,z) = \sqrt{x^2+y^2+z^2}$

Pt $(2,2,2)$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{1/2}$$

$$= \frac{1}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-1/2} \cdot \cancel{2}x$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial g}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \; ; \; \frac{\partial g}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

- the direction of maximum increase of g is given by $+\nabla g(x,y)$

$$\nabla g(x,y) = \left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

at Pt $(2,2,2)$, $\nabla g(2,2,2) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

since $\frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{2}{\sqrt{12}} = \frac{1}{\sqrt{3}}$

or we can also say $\nabla g(2,2,2) = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$

$$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

I)

c) $f(x,y,z) = 3x^2y - 4xy$

$Pt = (1,2)$

$$u = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$D_u f(x_0,y_0) = \nabla f(x_0,y_0) \cdot u$$

$$\nabla f(x,y) = \langle 6xy - 4y \, , \, 3x^2 - 4x \rangle$$

$$\nabla f(1,2) = \langle 4 \, , -1 \rangle$$

$$\|u\| = \sqrt{(\sqrt{3}/2)^2 + (-1/2)^2}$$

$$= \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

u is normalized already

$$D_{\vec{u}} f(1,2) = \langle 4, -1 \rangle \cdot \langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle$$

$$= 2\sqrt{3} + \frac{1}{2}$$

II) a) $f(x,y) = x^3 + y^3$ pt $(0,0)$

$\nabla f(x,y) = \langle 3x^2, 3y^2 \rangle$

$\nabla f(0,0) = 0$ (critical point) $(0,0)$

$\begin{pmatrix} 3x^2 \\ 3y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = y = 0$

- $\dfrac{\partial^2}{\partial x} = 6x$ $\qquad \dfrac{\partial^2}{\partial x \partial y} = \dfrac{\partial(3x^2)}{\partial y} = 0$

- $\dfrac{\partial^2}{\partial y} = 6y$

Discriminant of $f(x,y)$: $D = \dfrac{\partial^2 f(x,y)}{\partial x^2} \cdot \dfrac{\partial^2 f(x,y)}{\partial y^2} - \left(\dfrac{\partial^2 f(x,y)}{\partial x \cdot \partial y} \right)^2$

at $(0,0)$ $D = 0 \cdot 0 - (0)^2 = 0$

Donc there is <u>no local max or min</u>

- $D = 0 \Rightarrow$ "inconclusive."

II)

b) $g(x,y) = y^2 - x^2 + \frac{x^4}{2}$

$\nabla g(x,y) = 0$

$\nabla g(x,y) = \langle -2x + 2x^3 ,\ 2y \rangle$

$x = 0$

$y = 0$

First

$$\begin{cases} -2x + 2x^3 = 0 \\ 2y = 0 \end{cases}$$

$\boxed{1\ (0,0)}$

$x = 0$

$y = 0$

$$\begin{cases} x(x^2 - 1) = 0 \\ 2y = 0 \end{cases}$$

$x^2 - 1 = 0$ or $x = 1$ $\quad y = 0$

$x = -1$ $\quad y = 0$

$\Rightarrow$ 2 $\boxed{(1,0)}$; 3 $\boxed{(-1,0)}$

$$D = \frac{\partial^2 f(x,y)}{\partial x^2} \cdot \frac{\partial^2 f(x,y)}{\partial y^2} - \left(\frac{\partial^2}{\partial x y} f(x,y)\right)^2$$

II) b) $\frac{\partial^2 f(x,y)}{\partial^2 x} = \frac{\partial(-2x + 2x^3)}{\partial x}$

$= -2 + 6x^2$

$\frac{\partial^2 f(x,y)}{\partial^2 y} = \frac{\partial(2y)}{\partial y} = 2$

$\frac{\partial^2 f(x,y)}{x\, y} = 0$

- D at $(0,0) \Rightarrow -2 \cdot 2 - 0^2$

$= -4 < 0$

$\rightarrow$ saddle point $(0,0)$

- D at $(+1,0)$ / at $(-1,0)$ $\Rightarrow 4 \times 2 - 0 = 8 > 0$

f_{xx} at $(1,0) = 4 > 0$
at $(-1,0) = 4 > 0$

so $(1,0)$, $(-1,0)$ Both are Local minimum.

c) $h(x,y) = x^2 - 2xy + y^2 + 6$

$= (x-y)^2 + 6$

$\nabla h(x,y) = \langle 2x - 2y, 2y - 2x \rangle$

$\nabla f(x,y) = 0$

$\Rightarrow x = y \Rightarrow$ 0,0 (infinit critical points)

1,1

2,2

- $\frac{\partial h(x,y)}{\partial x^2} = 2$ $\quad$ $\frac{\partial h(x,y)}{\partial xy} = 2$

- $\frac{\partial h(x,y)}{\partial y^2} = 2$ $\quad$ $D = 2 \times 2 - 4 = 0$

inconclusive

but giving $h(x,y) = \underbrace{(x-y)^2}_{\geq 0} + \underbrace{6}_{\geq 0} \geq 0$

the minim $((x-y)^2 + 6)$ for $x = y$

$= 6$

donc $\boxed{\text{le minimum of } h(x,y) = 6}$

III - a) max $g(x,y) = \frac{1}{x} + \frac{1}{y}$

st. $x \cdot y = 9$ $\Longrightarrow$ $f(x,y) = 9$

Using Langrangian $x, y > 0$

$L(x, y, \lambda) = 0$

$$\nabla g(x,y) - \lambda \nabla f(x,y) = 0$$

$$\nabla g = \left\langle -\frac{1}{x^2}, -\frac{1}{y^2} \right\rangle$$

$$\nabla f = \langle y, x \rangle$$

$$\nabla g(x,y) = \lambda \nabla f(x,y)$$

$$\begin{cases} -\frac{1}{x^2} = \lambda y \\ -\frac{1}{y^2} = \lambda x \end{cases} \Rightarrow \begin{cases} \lambda y + \frac{1}{x^2} = 0 & (1) \\ \lambda x + \frac{1}{y^2} = 0 & (2) \\ xy - 9 = 0 & (3) \\ x, y > 0 & (4) \end{cases}$$

(1) $\times x$ $\quad \lambda yx + \frac{1}{x} = 0$

(2) $\times y$ $\quad \lambda xy + \frac{1}{y} = 0$

$$\begin{cases} \lambda x y - \dfrac{1}{x} = 0 & x \neq 0 \\ \lambda x y - \dfrac{1}{y} = 0 & y \neq 0 \\ x y = 9 \end{cases}$$

$$9\lambda = \frac{1}{x} \qquad 9\lambda = \frac{1}{y}$$

$$\Rightarrow \boxed{x = y} \qquad x^2 y = 9$$

$$x^2 = 9$$

$$\begin{cases} x = 3 \quad \text{ou} \quad x = -3 \\ y = 3 \quad \text{ou} \quad y = -3 \end{cases}$$

$$\boxed{(3,3)} \; ; \; \boxed{(-3,-3)}$$

$$\begin{cases} g(3,3) = 2/3 & \longleftarrow \underline{\text{max}} \\ g(-3,-3) = -2/3 & \leftarrow \underline{\text{min}} \end{cases}$$

$$\rightarrow \text{Max}(g(x,y)) \text{ at } (3,3) = \underline{\underline{2/3}}$$

III b/ $\min f(x,y) = x^2 - y^2$

$h(x,y) = x^2 + y^2 = 100$

$\nabla f = \langle 2x, -2y \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$L(x, y, \lambda) = 0 \quad f(x,y) - \lambda g(x,y) = 0$

$\nabla f = \lambda \nabla g$

$2x = \lambda 2x$
$2y = -\lambda 2y$

$$\begin{cases} 2x(1-\lambda) = 0 \\ 2y(1+\lambda) = 0 \\ x^2 + y^2 - 100 = 0 \rightarrow x, y \neq 0 \text{ at the same time} \\ x, y \geq 0 \end{cases}$$

$x, y \neq 0$ (both can not be zero at the same time)

Thus

$x = 0$ and $y = \pm 10$

or

$y = 0$ and $x = \pm 10$

$$\left\{\begin{array}{l} f(0,-10) = -100 \\ f(0,10) = -100 \end{array}\right\}$$ Minimum of $f(x,y)$ at $(0,-10)$ $(0,10)$

$$\left\{\begin{array}{l} f(-10,0) = 100 \\ f(10,0) = 100 \end{array}\right.$$