26. Directional Derivatives & The Gradient

Given a multivariable function $z = f(x, y)$ and a point on the *xy*-plane $P_0 = (x_0, y_0)$ at which f is differentiable (*i.e.* it is smooth with no discontinuities, folds or corners), there are infinitely many directions (relative to the *xy*-plane) in which to sketch a tangent line to f at P_0. A **directional derivative** is the slope of a tangent line to f at P_0 in which a *unit* direction vector $\mathbf{u} = \langle u_1, u_2 \rangle$ has been specified, and is given by the formula

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

The right side of the equation can be viewed as the result of a dot product:

$$D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle u_1, u_2 \rangle.$$

The vector-valued function $\langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$ is called the **gradient** of f at $x = x_0$ and $y = y_0$, and is written $\nabla f(x_0, y_0)$. Thus, the directional derivative of f at P_0 in the direction of **u** is written in the shortened form

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}.$$

♦ • ♦ • ♦ • ♦ • ♦

Example 26.1: Find $\nabla f(x, y)$, where $f(x, y) = x^2 y + 2xy^3$.

Solution: Since $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$, we have $\nabla f(x, y) = \langle 2xy + 2y^3, x^2 + 6xy^2 \rangle$.

♦ • ♦ • ♦ • ♦ • ♦

Example 26.2: Find the slope of the tangent line of $f(x, y) = x^2 y + 2xy^3$ at $x_0 = -1$, $y_0 = 2$ in the direction of $\mathbf{u} = \langle 4,3 \rangle$.

Solution: From the previous example, $\nabla f(x, y) = \langle 2xy + 2y^3, x^2 + 6xy^2 \rangle$. When evaluated at $x_0 = -1$ and $y_0 = 2$, we have

$$\nabla f(-1,2) = \langle 2(-1)(2) + 2(2)^3, (-1)^2 + 6(-1)(2)^2 \rangle = \langle 12, -23 \rangle.$$

The direction **u** is not a unit vector. Since $|\mathbf{u}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$, the unit vector in the direction of **u** is $\left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$. Thus,

$$D_{\mathbf{u}} f(-1,2) = \langle 12, -23 \rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle = 12\left(\frac{4}{5}\right) - 23\left(\frac{3}{5}\right) = -\frac{21}{5}.$$

♦ • ♦ • ♦ • ♦ • ♦

Example 26.3: Find the slope of the tangent line of $g(x,y) = \frac{x}{y^2}$ at $x_0 = 3$ and $y_0 = 5$, in the direction of the origin.

Solution: The vector from (3,5) to (0,0) is given by $\langle 0-3, 0-5\rangle = \langle -3,-5\rangle$. Its magnitude is $\sqrt{(-3)^2+(-5)^2} = \sqrt{34}$. Thus, the unit direction vector is

$$\mathbf{u} = \left\langle -\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}} \right\rangle.$$

The gradient of g is

$$\nabla g(x,y) = \left\langle \frac{1}{y^2}, -\frac{2x}{y^3} \right\rangle.$$

Therefore,

$$\nabla g(3,5) = \left\langle \frac{1}{(5)^2}, -\frac{2(3)}{(5)^3} \right\rangle = \left\langle \frac{1}{25}, -\frac{6}{125} \right\rangle.$$

The slope of the tangent line of g at $x_0 = 3$ and $y_0 = 5$ in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}} g(3,5) &= \left\langle \frac{1}{25}, -\frac{6}{125} \right\rangle \cdot \left\langle -\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}} \right\rangle \\ &= \left(\frac{1}{25}\right)\left(-\frac{3}{\sqrt{34}}\right) + \left(-\frac{6}{125}\right)\left(-\frac{5}{\sqrt{34}}\right) \\ &= -\frac{15}{125\sqrt{34}} + \frac{30}{125\sqrt{34}} = \frac{15}{125\sqrt{34}} \approx 0.0206. \end{aligned}$$

♦ • ♦ • ♦ • ♦ • ♦

Example 26.4: Find the slope of the tangent line of $h(x,y) = \sqrt{1+x^2+y^2}$ where $P_0 = (1,2)$ and the direction is given by a ray from P_0 oriented at $\theta = \frac{\pi}{6}$ radians, relative to the positive x-direction.

Solution: The direction vector is given by $\mathbf{u} = \left\langle \cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right) \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$. It is a unit vector. The gradient of h is

$$\nabla h(x,y) = \left\langle \frac{x}{\sqrt{1+x^2+y^2}}, \frac{y}{\sqrt{1+x^2+y^2}} \right\rangle,$$

So upon substitution,

$$\nabla h(1,2) = \left\langle \frac{(1)}{\sqrt{1+(1)^2+(2)^2}}, \frac{(2)}{\sqrt{1+(1)^2+(2)^2}} \right\rangle = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle.$$

The directional derivative of h at (1,2) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}}h(1,2) &= \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle \\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{6}}\right) \\ &= \frac{\sqrt{3}+2}{2\sqrt{6}} \approx 0.762. \end{aligned}$$

♦ • ♦ • ♦ • ♦ • ♦

Directional derivatives can be extended into higher dimensions.

Example 26.5: Find the slope of the tangent line of $f(x,y,z) = xy^2z^3$ at $x_0 = 2, y_0 = 1$ and $z_0 = 3$ in the direction of $\langle 2,4,-5 \rangle$.

Solution: The gradient of f is

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle.$$

At (2,1,3), we have

$$\nabla f(2,1,3) = \langle 27, 108, 54 \rangle.$$

The unit direction vector is $\mathbf{u} = \left\langle \frac{2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, -\frac{5}{\sqrt{45}} \right\rangle$. The slope of the tangent line of f at (2,1,3) in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}}f(2,1,3) &= \nabla f(2,1,3) \cdot \mathbf{u} \\ &= \langle 27, 108, 54 \rangle \cdot \left\langle \frac{2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, -\frac{5}{\sqrt{45}} \right\rangle \\ &= \frac{54}{\sqrt{45}} + \frac{432}{\sqrt{45}} - \frac{270}{\sqrt{45}} \approx 32.2. \end{aligned}$$

♦ • ♦ • ♦ • ♦ • ♦

Using the cosine form of the formula for the dot product of two vectors, $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, we can rewrite $D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$ as

$$D_{\mathbf{u}} f(x_0, y_0) = |\nabla f(x_0, y_0)||\mathbf{u}| \cos\theta.$$

Since $\mathbf{u}$ is a unit vector, then $|\mathbf{u}| = 1$, so that

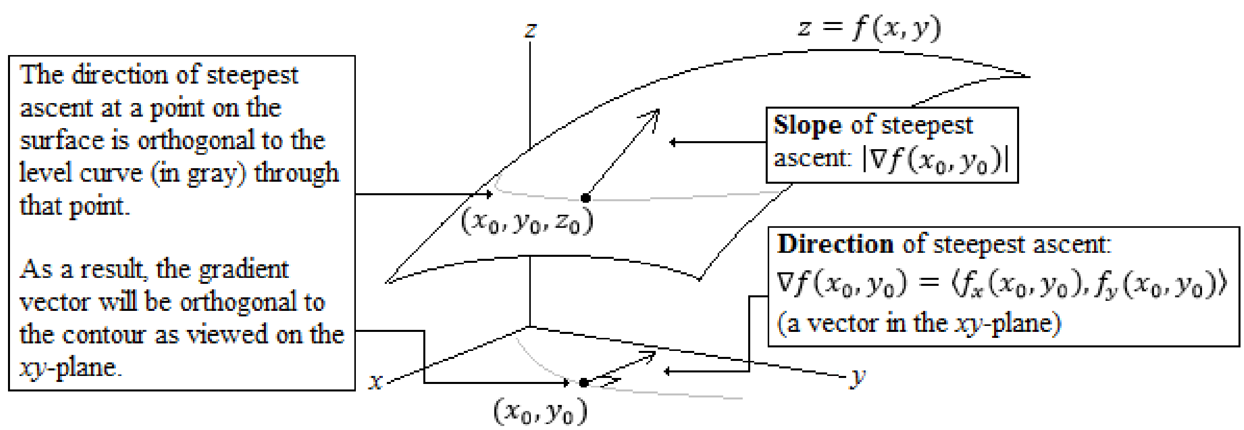
$$|\nabla f(x_0, y_0)||\mathbf{u}| \cos\theta = |\nabla f(x_0, y_0)| \cos\theta,$$

where θ is the angle between the gradient vector at (x_0, y_0), and the direction vector $\mathbf{u}$. From this, we can infer that $|\nabla f(x_0, y_0)| \cos\theta$ is maximized when $\nabla f(x_0, y_0)$ and $\mathbf{u}$ are parallel, or when $\theta = 0$ (so that $\cos\theta = 1$). This leads to a significant result in directional derivatives.

Given a function $z = f(x, y)$ and a point $P_0 = (x_0, y_0, z_0)$:

- The **direction of steepest ascent** at P_0 is given by $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$. In this case, it is permissible to state the direction as a non-unit vector.
- The **slope of steepest ascent** at P_0 is given by $|\nabla f(x_0, y_0)|$.
- The **direction of steepest descent** at P_0 is opposite the direction of steepest ascent, and is given by $-\nabla f(x_0, y_0) = \langle -f_x(x_0, y_0), -f_y(x_0, y_0) \rangle$.
- The **slope of steepest descent** at P_0 is $-|\nabla f(x_0, y_0)|$.

A path that follows the directions of steepest ascent is called a **gradient path** and is always orthogonal to the contours of the surface.



♦ • ♦ • ♦ • ♦ • ♦

Example 26.6: Let $f(x, y) = x^2 + 2xy^2$. State the direction(s) in which the slope of the tangent line at $x_0 = 2$ and $y_0 = 1$ is 0.

Solution: We have $\nabla f(x, y) = \langle 2x + 2y^2, 4xy\rangle$. Let $\mathbf{u} = \langle u_1, u_2\rangle$. We have

$$\begin{aligned} D_{\mathbf{u}} f(2,1) &= \nabla f(2,1) \cdot \mathbf{u} \\ &= \langle 6,8\rangle \cdot \langle u_1, u_2\rangle \\ &= 6u_1 + 8u_2. \end{aligned}$$

If the slope is to be 0, we set $6u_1 + 8u_2 = 0$. Thus, whenever $u_2 = -\frac{3}{4}u_1$, then the slope of the tangent line at $x_0 = 2$ and $y_0 = 1$ will be 0.

♦ • ♦ • ♦ • ♦ • ♦

Example 26.7: Find the direction of steepest ascent of $f(x, y) = x^2y + 2xy^3$ at $x_0 = -1$ and $y_0 = 2$, then find the slope of steepest ascent.

Solution: From an earlier example, we found that $\nabla f(x, y) = \langle 2xy + 2y^3, x^2 + 6xy^2\rangle$ and that $\nabla f(-1,2) = \langle 12, -23\rangle$. This is the *direction* of steepest ascent. The *slope* of steepest ascent is $|\langle 12, -23\rangle| = \sqrt{12^2 + (-23)^2} \approx 25.94$.

When finding a directional derivative where the direction is stated or to be determined, you *must* be sure that it is stated as a unit vector. However, when asked to find a direction of steepest ascent, it is permissible to leave it as a non-unit vector since you will likely be calculating the slope as well. While it is not incorrect to state the direction of steepest ascent as a unit vector, a common error is to then use that unit vector to find the slope, in which case the answer will be 1, which is likely incorrect.

♦ • ♦ • ♦ • ♦ • ♦

Example 26.8: Suppose the slope of the tangent line of $z = f(x, y)$ at $P_0 = (x_0, y_0)$ in the direction of $\langle 3,1\rangle$ is $\sqrt{10}$, and that the slope of the tangent line at the same point in the direction of $\langle 1,4\rangle$ is $\frac{18}{\sqrt{17}}$. What is the direction of steepest ascent of f at P_0, and what is the slope in this direction?

Solution: We don't know f, but we can treat the components in its gradient, $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$, as a pair of unknowns. In the direction of $\langle 3,1\rangle$, the slope of the tangent line is $\sqrt{10}$. Considering the unit direction vector $\mathbf{u} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle$, we have $D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = \sqrt{10}$. Thus, we initially have

$$\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle = \sqrt{10},$$

which gives

$$f_x(x_0,y_0)\frac{3}{\sqrt{10}}+f_y(x_0,y_0)\frac{1}{\sqrt{10}}=\sqrt{10}. \qquad (\mathbf{1})$$

In a similar way, we consider the unit direction vector in the direction of $\langle 1,4\rangle$, which is $\left\langle \frac{1}{\sqrt{17}},\frac{4}{\sqrt{17}}\right\rangle$. The slope in this direction is $\frac{18}{\sqrt{17}}$. We have

$$\langle f_x(x_0,y_0),f_y(x_0,y_0)\rangle\cdot\left\langle \frac{1}{\sqrt{17}},\frac{4}{\sqrt{17}}\right\rangle=\frac{18}{\sqrt{17}},$$

which gives

$$f_x(x_0,y_0)\frac{1}{\sqrt{17}}+f_y(x_0,y_0)\frac{4}{\sqrt{17}}=\frac{18}{\sqrt{17}}. \qquad (\mathbf{2})$$

Taking equations **(1)** and **(2)** together, we have a system of two unknowns in two equations:

$$\begin{gathered}f_x(x_0,y_0)\frac{3}{\sqrt{10}}+f_y(x_0,y_0)\frac{1}{\sqrt{10}}=\sqrt{10}\\ f_x(x_0,y_0)\frac{1}{\sqrt{17}}+f_y(x_0,y_0)\frac{4}{\sqrt{17}}=\frac{18}{\sqrt{17}}.\end{gathered}$$

The first equation is multiplied by $\sqrt{10}$, and the second by $\sqrt{17}$ to clear fractions:

$$\begin{gathered}f_x(x_0,y_0)(3)+f_y(x_0,y_0)(1)=10\\ f_x(x_0,y_0)(1)+f_y(x_0,y_0)(4)=18.\end{gathered}$$

The bottom equation is multiplied by –3:

$$\begin{gathered}f_x(x_0,y_0)(3)+f_y(x_0,y_0)(1)=10\\ f_x(x_0,y_0)(-3)+f_y(x_0,y_0)(-12)=-54.\end{gathered}$$

Adding the second equation to the first, we have $-11f_y\,(x_0,y_0)=-44$. Thus, $f_y(x_0,y_0)=4$. Substituting this into either of the equations **(1)** or **(2)**, we find that $f_x(x_0,y_0)=2$. Therefore, we now know $\nabla f(x_0,y_0)$, which is $\langle 2,4\rangle$. This is the direction of steepest ascent of f. The slope at P_0 in this direction is $\sqrt{2^2+4^2}=\sqrt{20}=2\sqrt{5}\approx 4.47$.

♦ • ♦ • ♦ • ♦ • ♦

Example 26.9: A plane tilts to the north at a 6% grade – that is, for every 100 feet one moves horizontally north, he or she will gain 6 feet vertically. Find the slope and the grade if someone walks to the northeast.

Solution: Assume the plane passes through the origin, assuming also that the *y*-axis is north and south, and the *x*-axis is east and west, in the usual map orientation. When $y = 100$, we have $z = 6$, so that another ordered triple on the plane is (0,100,6). Thus, we can write $z = \frac{6}{100}y = 0.06y$ as the equation of the plane. The gradient of f is $\nabla f(x, y) = \langle 0, 0.06\rangle$. Note that *x* is an independent variable but has no effect on the values of *z*. If it helps, write the plane as $z = 0x + 0.06y$.

Furthermore, at the origin, we still have $\nabla f(0,0) = \langle 0, 0.06\rangle$. Meanwhile, movement to the northeast can be modeled by the vector $\langle 1,1\rangle$, or as a unit vector, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

The slope at the origin in the direction of northeast is given by

$$\begin{aligned} D_{\mathbf{u}} f(0,0) &= \nabla f(0,0) \cdot \mathbf{u} \\ &= \langle 0, 0.06\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ &= \frac{0.06}{\sqrt{2}} \approx 0.0424. \end{aligned}$$

The grade can be inferred by the fact that 1 foot of movement in the northeast direction results in a rise of 0.0424 feet vertically. Thus, the grade is about 4.24%.

Note that a movement east or west would result in no change in *z*. The directional derivative in either direction (the positive or negative *x* direction) is 0. Let $\mathbf{u} = \langle 1,0\rangle$ or $\langle -1,0\rangle$ and verify that the directional derivative would be 0.

♦ • ♦ • ♦ • ♦ • ♦