The Derivative of a Function

P ooled Samples

If you had to give a blood test to 3000 people to test for the presence of Hepatitis C, how would you do it? Would you test each person individually, which would require 3000 tests? That could be expensive, especially if each test were to cost $100. How can you use less tests? One way is to pool the samples. What sample size would you use? If you pooled the blood of 100 people and the test came out neg-ative, then one test was used instead of 100. But if the test came out posi-tive, then you would need to test each one, which now requires $1 + 100 = 101$ tests. Maybe the sample size to use is 200. Could it be 300? And the likelihood of a positive test must play a role as well. Sounds like a very hard problem. But with the discussion in this chapter and the Chapter Project to guide you, you can find the best sample size to use so cost is least.

4.1 The Definition of a Derivative

PREPARING FOR THIS SECTION *Before getting started, review the following:*

> Average Rate of Change (Section 1.3, pp. xx–xx)

> Secant Line (Section 1.3, pp. xx–xx)

> Factoring (Section 0.3, pp. xx–xx)

> Point–Slope Form of a Line (Section 0.8, pp. xx–xx)

> Difference Quotient (Section 1.2, pp. xx–xx)

OBJECTIVES
1. Find an equation of the tangent line to the graph of a function
2. Find the derivative of a function at a number c
3. Find the derivative of a function using the difference quotient
4. Find the instantaneous rate of change of a function
5. Find marginal cost and marginal revenue

The Tangent Problem

The geometry question that motivated the development of calculus was "What is the slope of the tangent line to the graph of a function $y = f(x)$ at a point P on its graph?" See Figure 1.
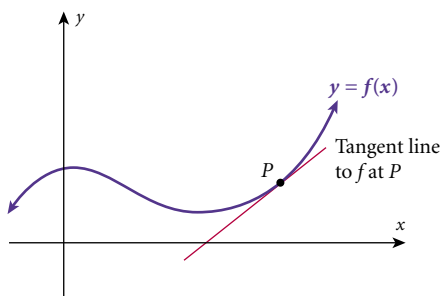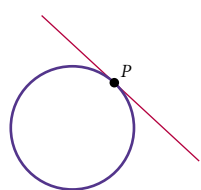
FIGURE 1

FIGURE 2

We first need to define what we mean by a *tangent* line. In high school geometry, the tangent line to a circle is defined as the line that intersects the graph in exactly one point. Look at Figure 2. Notice that the tangent line just touches the graph of the circle.

This definition, however, does not work in general. Look at Figure 3. The lines L_1 and L_2 only intersect the graph in one point P, but neither touches the graph at P. Additionally, the tangent line L_T shown in Figure 4 touches the graph of f at P, but also intersects the graph elsewhere. So how should we define the tangent line to the graph of f at a point P?

FIGURE 3
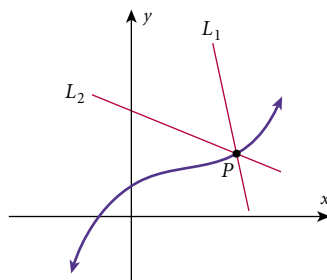
FIGURE 4
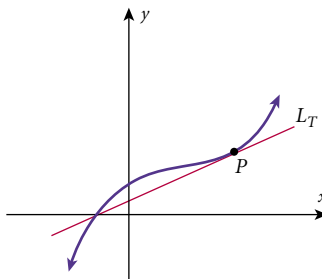
The tangent line L_T to the graph of a function $y = f(x)$ at a point P necessarily contains the point P. To find an equation for L_T using the point–slope form of the equation of a line, it remains to find the slope $m_{\tan}$ of the tangent line.

Suppose that the coordinates of the point P are $(c, f(c))$. Locate another point $Q = (x, f(x))$ on the graph of f. The line containing P and Q is a secant line. (Refer to Section 1.3.) The slope m_{sec} of the secant line is

$$m_{sec} = \frac{f(x) - f(c)}{x - c}$$
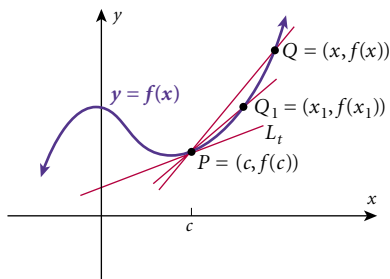
FIGURE 5

Now look at Figure 5.

As we move along the graph of f from Q toward P, we obtain a succession of secant lines. The closer we get to P, the closer the secant line is to the tangent line. The limiting position of these secant lines is the tangent line. Therefore, the limiting value of the slopes of these secant lines equals the slope of the tangent line. But, as we move from Q toward P, the values of x get closer to c. Therefore,

$$m_{\tan} = \lim_{x \to c} m_{sec} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The **tangent line** to the graph of a function $y = f(x)$ at a point $P = (c, f(c))$ on its graph is defined as the line containing the point P whose slope is

$$m_{\tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{1}$$

provided that this limit exists.

If $m_{\tan}$ exists, an equation of the tangent line is

$$y - f(c) = m_{\tan}(x - c) \tag{2}$$

1 **EXAMPLE 1** **Finding an Equation of the Tangent Line**

Find an equation of the tangent line to the graph of $f(x) = \dfrac{x^2}{4}$ at the point $\left(1, \dfrac{1}{4}\right)$. Graph the function and the tangent line.

SOLUTION The tangent line contains the point $\left(1, \dfrac{1}{4}\right)$. The slope of the tangent line to the graph of $f(x) = \dfrac{x^2}{4}$ at $\left(1, \dfrac{1}{4}\right)$ is

$$
\begin{aligned}
m_{\text{tan}} &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\dfrac{x^2}{4} - \dfrac{1}{4}}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x - 1)}(x + 1)}{4\cancel{(x - 1)}} \\
&= \lim_{x \to 1} \frac{x + 1}{4} = \frac{1}{2}
\end{aligned}
$$

An equation of the tangent line is

$$
y - \frac{1}{4} = \frac{1}{2}(x - 1) \qquad y - f(c) = m_{\text{tan}}(x - c)
$$

$$
y = \frac{1}{2}x - \frac{1}{4}
$$

Figure 6 shows the graph of $y = \dfrac{x^2}{4}$ and the tangent line at $\left(1, \dfrac{1}{4}\right)$. ▶

FIGURE 6
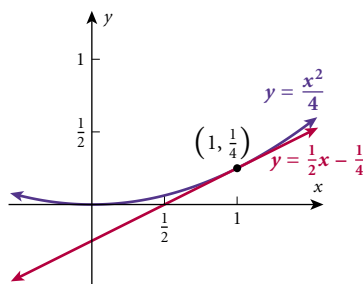
NOW WORK PROBLEM 3.

The limit in formula (1) has an important generalization: it is called the *derivative of f at c.*

The Derivative of a Function at a Number *c.*

Let $y = f(x)$ denote a function f. If c is a number in the domain of f, the **derivative of f at c,** denoted by $f'(c)$, read "f prime of c," is defined as

$$
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \tag{3}
$$

provided that this limit exists.

The steps for finding the derivative of a function are listed below:

> **Steps For Finding the Derivative of a Function at c**
>
> **STEP 1** Find $f(c)$.
> **STEP 2** Subtract $f(c)$ from $f(x)$ to get $f(x) - f(c)$ and form the quotient
>
> $$\boxed{\dfrac{f(x) - f(c)}{x - c}}$$
>
> **STEP 3** Find the limit (if it exists) of the quotient found in Step 2 as $x \to c$:
>
> $$\boxed{f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}}$$

2 **EXAMPLE 2**   **Finding the Derivative of a Function at a Number**

Find the derivative of $f(x) = 2x^2 - 5x$ at 2. That is, find $f'(2)$.

SOLUTION   **Step 1:** $f(2) = 2(4) - 5(2) = -2$

Step 2: $\dfrac{f(x) - f(2)}{x - 2} = \dfrac{(2x^2 - 5x) - (-2)}{x - 2} = \dfrac{2x^2 - 5x + 2}{x - 2} = \dfrac{(2x - 1)(x - 2)}{x - 2}$

Step 3: The derivative of f at 2 is

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{(2x - 1)(x - 2)}{x - 2} = 3$$

NOW WORK PROBLEM 13.

Example 2 provides a way of finding the derivative at 2 analytically. Graphing utilities have built-in procedures to approximate the derivative of a function at any number c. Consult your owner's manual for the appropriate keystrokes.
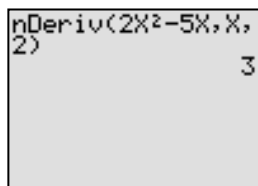
EXAMPLE 3   **Finding the Derivative of a Function Using a Graphing Utility**

Use a graphing utility to find the derivative of $f(x) = 2x^2 - 5x$ at 2. That is, find $f'(2)$.

SOLUTION   Figure 7 shows the solution using a TI-83 graphing calculator.

FIGURE 7


```
nDeriv(2X²-5X,X,
2)
                3
```

So $f'(2) = 3$.

NOW WORK PROBLEM 45.

EXAMPLE 4   **Finding the Derivative of a Function at c**

Find the derivative of $f(x) = x^2$ at c. That is, find $f'(c)$.

SOLUTION   Since $f(c) = c^2$, we have

$$\frac{f(x) - f(c)}{x - c} = \frac{x^2 - c^2}{x - c} = \frac{(x + c)(x - c)}{x - c}$$

The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{(x + c)\cancel{(x - c)}}{\cancel{x - c}} = 2c$$ ▶

As Example 4 illustrates, the derivative of $f(x) = x^2$ exists and equals $2c$ for any number c. In other words, the derivative is itself a function and, using x for the independent variable, we can write $f'(x) = 2x$. The function f' is called the **derivative function of f** or the **derivative of f.** We also say that f is **differentiable.** The instruction "differentiate f" means "find the derivative of f".

It is usually easier to find the derivative function by using another form. We derive this alternate form as follows:

Formula (3) for the derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Formula (3)}$$

Let $h = x - c$. Then $x = c + h$ and

$$\frac{f(x) - f(c)}{x - c} = \frac{f(c + h) - f(c)}{h}$$

Since $h = x - c$, then, as $x \to c$, it follows that $h \to 0$. As a result,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \qquad \textbf{(4)}$$

Now replace c by x in (4). This gives us the following formula for finding the derivative of f at any number x.

> **Formula for the Derivative of a Function $y = f(x)$ at x**
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad \textbf{(5)}$$
>
> That is, the derivative of the function f is the limit of its difference quotient.

3   **EXAMPLE 5**   **Using the Difference Quotient to Find a Derivative**

(a) Use formula (5) to find the derivative of $f(x) = x^2 + 2x$.
(b) Find $f'(0)$, $f'(-1)$, $f'(3)$.

SOLUTION   **(a)** First, we find the difference quotient of $f(x) = x^2 + 2x$.

$$\frac{f(x + h) - f(x)}{h} = \frac{[(x + h)^2 + 2(x + h)] - [x^2 + 2x]}{h}$$

$$= \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h}$$

$$= \frac{2xh + h^2 + 2h}{h} \qquad \text{Simplify.}$$

$$= \frac{h(2x + h + 2)}{h} \qquad \text{Factor out } h.$$

$$= 2x + h + 2 \qquad \text{Cancel the } h\text{'s.}$$

The derivative of f is the limit of the difference quotient as $h \to 0$. That is,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0}(2x + h + 2) = 2x + 2$$

(b) Since,

$$f'(x) = 2x + 2$$

we have

$$f'(0) = 2 \cdot 0 + 2 = 2$$
$$f'(-1) = 2(-1) + 2 = 0$$
$$f'(3) = 2(3) + 2 = 8 \qquad \blacktriangleright$$

NOW WORK PROBLEM 29.

Instantaneous Rate of Change

In Chapter 1 we defined the average rate of change of a function f from c to x as

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(c)}{x - c}$$

The limit as x approaches c of the average rate of change of f, based on formula (3), is the derivative of f at c. As a result, we call the derivative of f at c the **instantaneous rate of change of f with respect to x at c.** That is,

$$\left(\begin{array}{c}\text{Instantaneous rate of}\\\text{change of } f \text{ with respect to } x \text{ at } c\end{array}\right) = f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad (6)$$

4   **EXAMPLE 6**   **Finding the Instantaneous Rate of Change**

During a month-long advertising campaign, the total sales S of a magazine were given by the fraction

$$S(x) = 5x^2 + 100x + 10{,}000$$

where x represents the number of days of the campaign, $0 \le x \le 30$.

(a) What is the average rate of change of sales from $x = 10$ to $x = 20$ days?

(b) What is the instantaneous rate of change of sales when $x = 10$ days?

SOLUTION **(a)** Since $S(10) = 11{,}500$ and $S(20) = 14{,}000$, the average rate of change of sales from $x = 10$ to $x = 20$ is

$$\frac{\Delta S}{\Delta x} = \frac{S(20) - S(10)}{20 - 10} = \frac{14{,}000 - 11{,}500}{10} = 250 \text{ magazines per day}$$

(b) The instantaneous rate of change of sales when $x = 10$ is the derivative of S at 10.

$$S'(10) = \lim_{x \to 10} \frac{S(x) - S(10)}{x - 10} = \lim_{x \to 10} \frac{[(5x^2 + 100x + 10{,}000) - 11500]}{x - 10}$$

$$= \lim_{x \to 10} \frac{5(x^2 + 20x - 300)}{x - 10}$$

$$= 5 \lim_{x \to 0} \frac{(x + 30)(x - 10)}{x - 10} = 5 \lim_{x \to 10} (x + 30) = 5 \cdot 40 = 200$$

The instantaneous rate of change of S at 10 is 200 magazines per day. ▶

We interpret the results of Example 6 as follows: The fact that the average rate of sales from $x = 10$ to $x = 20$ is $\Delta S / \Delta x = 250$ magazines per day indicates that on the 10th day of the campaign, we can expect to average 250 magazines per day of additional sales if we continue the campaign for 10 more days. The fact that $S'(10) = 200$ magazines per day indicates that on the 10th day of the campaign, one more day of advertising will result in additional sales of approximately 200 magazines per day.

NOW WORK PROBLEM 39.

Find marginal cost and   **5**   **Application to Economics: Marginal Analysis**
marginal revenue

Economics is one of the many fields in which calculus has been used to great advantage. Economists have a special name for the application of derivatives to problems in economics—it is **marginal analysis.** Whenever the term *marginal* appears in a discussion, involving cost functions or revenue functions, it signals the presence of derivatives in the background.

> **Marginal Cost**
>
> Suppose $C = C(x)$ is the cost of producing x units. Then the derivative $C'(x)$ is called the **marginal cost.**

We interpret the marginal cost as follows. Since

$$C'(x) = \lim_{h \to 0} \frac{C(x + h) - C(x)}{h}$$

it follows for small values of h that

$$C'(x) \approx \frac{C(x + h) - C(x)}{h}$$

That is to say,

$$C'(x) \approx \frac{\text{cost of increasing production from } x \text{ to } x + h}{h}$$

In most practical situations x is very large. Because of this, many economists let $h = 1$, which is small compared to large x. Then, marginal cost may be interpreted as

$$C'(x) = C(x + 1) - C(x) = \text{cost of increasing production by one unit}$$

EXAMPLE 7   **Finding Marginal Cost**

Suppose that the cost in dollars for a weekly production of x tons of steel is given by the function:

$$C(x) = \frac{1}{10} x^2 + 5x + 1000$$

(a) Find the marginal cost.
(b) Find the cost and marginal cost when $x = 1000$ tons.
(c) Interpret $C'(1000)$.

SOLUTION   **(a)** The marginal cost is the derivative $C'(x)$. We use the difference quotient of $C(x)$ to find $C'(x)$.

$$C'(x) = \lim_{h \to 0} \frac{C(x + h) - C(x)}{h} = \lim_{h \to 0} \frac{\left\{ \left[\frac{1}{10}(x + h)^2 + 5(x + h) + 1000 \right] - \left[\frac{1}{10}x^2 + 5x + 1000 \right] \right\}}{h}$$

$$= \lim_{h \to 0} \frac{\left\{ \frac{1}{10}(x^2 + 2xh + h^2) + 5x + 5h - \frac{1}{10}x^2 - 5x \right\}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{5}xh + \frac{1}{10}h^2 + 5h \right)}{h} = \lim_{h \to 0} \left(\frac{1}{5}x + \frac{1}{10}h + 5 \right) = \frac{1}{5}x + 5$$

(b) We evaluate $C(x)$ and $C'(x)$ at $x = 1000$

The cost when $x = 1000$ tons is $C(1000) = \frac{1}{10}(1000)^2 + 5 \cdot 1000 + 1000 = \$106{,}000$

The marginal cost when $x = 1000$ tons is $C'(1000) = \frac{1}{5} \cdot 1000 + 5 = \$205/\text{ton}$

(c) $C'(1000) = \$205$ per ton means that the cost of producing one additional ton of steel after 1000 tons have been produced is approximately \$205.   ▶

Note that the average cost of producing one more ton of steel after the 1000th ton is

$$\frac{\Delta C}{\Delta x} = \frac{C(1001) - C(1000)}{1001 - 1000}$$

$$= \left(\frac{1}{10} \cdot 1001^2 + 5 \cdot 1001 + 1000 \right) - \left(\frac{1}{10} \cdot 1000^2 + 5 \cdot 1000 + 1000 \right)$$

$$= \$205.10/\text{ton}$$

We observe that the average cost differs from the marginal cost by only 0.1 dollar/ton, which is less than $\frac{1}{20}$th of 1%. Note, too, that the marginal cost is easier to compute than the actual average cost.

The money received by our hypothetical steel producer when he sells his product is the revenue. Specifically, let $R = R(x)$ be the total revenue received from selling x tons. Then the derivative $R'(x)$ is called the **marginal revenue.** For this example, marginal revenue, like marginal cost, is measured in dollars per ton. An approximate value for $R'(x)$ is obtained by noting again that

$$R'(x) \approx \frac{R(x + h) - R(x)}{h}$$

When x is large, then $h = 1$ is small by comparison, so that

$$R'(x) = R(x + 1) - R(x) = \text{revenue resulting from the sale of one additional unit}$$

This is the interpretation many economists give to marginal revenue.

EXAMPLE 8

Suppose that the revenue R for a weekly sale of x tons of steel is given by the formula

$$R = x^2 + 5x$$

(a) Find the marginal revenue.
(b) Find the revenue and marginal revenue when $x = 1000$ tons.
(c) Interpret $R'(1000)$.

SOLUTION   (a) The marginal revenue is the derivative $R'(x)$. We use the difference quotient of $R(x)$ to find $R'(x)$.

$$R'(x) = \lim_{h \to 0} \frac{R(x + h) - R(x)}{h} = \lim_{h \to 0} \frac{\{[(x + h)^2 + 5(x + h)] - [x^2 + 5x]\}}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 + 5x + 5h - x^2 - 5x)}{h}$$

$$= \lim_{h \to 0} \frac{(2xh + h^2 + 5h)}{h} = \lim_{h \to 0} (2x + h + 5) = 2x + 5$$

(b) The revenue when $x = 1000$ tons is

$$R(1000) = (1000)^2 + 5(1000) = \$1,005,000$$

The marginal revenue when $x = 1000$ tons is

$$R'(1000) = 2(1000) + 5 = \$2005/\text{ton}$$

(c) $R'(1000) = \$2005/\text{ton}$ means that the revenue due to selling one additional ton of steel after 1000 tons have been sold is approximately $2005. ▶

Note that the average revenue derived from selling one additional ton after 1000 tons have been sold is

$$\frac{\Delta R}{\Delta x} = \frac{R(1001) - R(1000)}{1001 - 1000} = 1{,}007{,}006 - 1{,}005{,}000 = \$2006/\text{ton}$$

Observe that the actual average revenue differs from the marginal revenue by only $1/ton, or 0.05%.

NOW WORK PROBLEM 63.

SUMMARY   The derivative of a function $y = f(x)$ at c is defined as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

The derivative $f'(x)$ of a function $y = f(x)$ is $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

In geometry, $f'(c)$ equals the slope of the tangent line to the graph of f at the point $(c, f(c))$.

In applications, if two variables are related by the function $y = f(x)$, then $f'(c)$ equals the instantaneous rate of change of f with respect to x at c.

In economics, the derivative of a cost function is the marginal cost and the derivative of a revenue function is the marginal revenue.

EXERCISE 4.1   **Answers to Odd-Numbered Problems Begin on Page AN-XX.**

In Problems 1–12, find the slope of the tangent line to the graph of f at the given point. What is an equation of the tangent line? Graph f and the tangent line.

1. $f(x) = 3x + 5$ at $(1, 8)$

2. $f(x) = -2x + 1$ at $(-1, 3)$

3. $f(x) = x^2 + 2$ at $(-1, 3)$

4. $f(x) = 3 - x^2$ at $(1, 2)$

5. $f(x) = 3x^2$ at $(2, 12)$

6. $f(x) = -4x^2$ at $(-2, -16)$

7. $f(x) = 2x^2 + x$ at $(1, 3)$

8. $f(x) = 3x^2 - x$ at $(0, 0)$

9. $f(x) = x^2 - 2x + 3$ at $(-1, 6)$

10. $f(x) = -2x^2 + x - 3$ at $(1, -4)$

11. $f(x) = x^3 + x$ at $(2, 10)$

12. $f(x) = x^3 - x^2$ at $(1, 0)$

In Problems 13–24, find the derivative of each function at the given number.

13. $f(x) = -4x + 5$ at 3

14. $f(x) = -4 + 3x$ at 1

15. $f(x) = x^2 - 3$ at 0

16. $f(x) = 2x^2 + 1$ at -1

17. $f(x) = 2x^2 + 3x$ at 1

18. $f(x) = 3x^2 - 4x$ at 2

19. $f(x) = x^3 + 4x$ at -1

20. $f(x) = 2x^3 - x^2$ at 2

21. $f(x) = x^3 + x^2 - 2x$ at 1

22. $f(x) = x^3 - 2x^2 + x$ at -1

23. $f(x) = \dfrac{1}{x}$ at 1

24. $f(x) = \dfrac{1}{x}$ at 1

In Problems 25–36, find the derivative of f using difference quotients.

25. $f(x) = 2x$

26. $f(x) = 3x$

27. $f(x) = 1 - 2x$

28. $f(x) = 5 - 3x$

29. $f(x) = x^2 + 2$

30. $f(x) = 2x^2 - 3$

31. $f(x) = 3x^2 - 2x + 1$

32. $f(x) = 2x^2 + x + 1$

33. $f(x) = x^3$

34. $f(x) = \dfrac{1}{x}$

35. $f(x) = mx + b$

36. $f(x) = ax^2 + bx + c$

In Problems 37–44, find

(a) The average rate of change as x changes from 1 to 3.
(b) The (instantaneous) rate of change at 1.

37. $f(x) = 3x + 4$ **38.** $f(x) = 2x - 6$ **39.** $f(x) = 3x^2 + 1$ **40.** $f(x) = 2x^2 + 1$

41. $f(x) = x^2 + 2x$ **42.** $f(x) = x^2 - 4x$ **43.** $f(x) = 2x^2 - x + 1$ **44.** $f(x) = 2x^2 + 3x - 2$

In Problems 45–54, find the derivative of each function at the given number using a graphing utility.

45. $f(x) = 3x^3 - 6x^2 + 2$ at -2 **46.** $f(x) = -5x^4 + 6x^2 - 10$ at 5

47. $f(x) = \dfrac{-x^3 + 1}{x^2 + 5x + 7}$ at 8 **48.** $f(x) = \dfrac{-5x^4 + 9x + 3}{x^2 + 5x^2 - 6}$ at -3

49. $f(x) = xe^x$ at 0 **50.** xe^x at 1 **51.** $f(x) = x^2 e^x$ at 1

52. $f(x) = x^2 e^x$ at 0 **53.** $f(x) = xe^{-x}$ at 1 **54.** $f(x) = x^2 e^{-x}$ at 2

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55. Does the tangent line to the graph of $y = x^2$ at $(1, 1)$ pass through the point $(2, 5)$?

56. Does the tangent line to the graph of $y = x^3$ at $(1, 1)$ pass through the point $(2, 5)$?

57. A dive bomber is flying from right to left along the graph of $y = x^2$. When a rocket bomb is released, it follows a path that approximately follows the tangent line. Where should the pilot release the bomb if the target is at $(1, 0)$?

58. Answer the question in Problem 57 if the plane is flying from right to left along the graph of $y = x^3$.

59. Ticket Sales The cumulative ticket sales for the 10 days preceding a popular concert is given by

$$S = 4x^2 + 50x + 5000$$

where x represents the 10 days leading up to the concert, $1 \le x \le 10$.

(a) What is the average rate of change in sales from day 1 to day 5?
(b) What is the average rate of change in sales from day 1 to day 10?
(c) What is the average rate of change in sales from day 5 to day 10?
(d) What is the instantaneous rate of change in sales on day 5?
(e) What is it on day 10?

60. Computer Sales The weekly revenue R, in dollars, due to selling x computers is

$$R(x) = -20x^2 + 1000x$$

(a) Find the average rate of change in revenue due to selling 5 additional computers after the 20th has been sold.
(b) Find the marginal revenue.
(c) Find the marginal revenue at $x = 20$.
(d) Interpret the answers found in (a) and (c).
(e) For what value of x is $R'(x) = 0$?

61. Supply and Demand Suppose $S(x) = 50x^2 - 50x$ is the supply function describing the number of crates of grapefruit a farmer is willing to supply to the market for x dollars per crate.

(a) How many crates is the farmer willing to supply for $10 per crate?
(b) How many crates is the farmer willing to supply for $13 per crate?
(c) Find the average rate of change in supply from $10 per crate to $13 per crate.
(d) Find the instantaneous rate of change in supply at $x = 10$.
(e) Interpret the answers found in (c) and (d).

62. Glucose Conversion In a metabolic experiment, the mass M of glucose decreases over time t according to the formula

$$M = 4.5 - 0.03t^2$$

(a) Find the average rate of change of the mass from $t = 0$ to $t = 2$.
(b) Find the instantaneous rate of change of mass at $t = 0$.
(c) Interpret the answers found in (a) and (b).

63. Cost and Revenue Functions For a certain production facility, the cost function is

$$C(x) = 2x + 5$$

and the revenue function is

$$R(x) = 8x - x^2$$

where x is the number of units produced (in thousands) and R and C are measured in millions of dollars. Find:

(a) The marginal revenue.
(b) The marginal cost.
(c) The break-even point(s) [the number(s) x for which $R(x) = C(x)$].
(d) The number x for which marginal revenue equals marginal cost.
(e) Graph $C(x)$ and $R(x)$ on the same set of axes.

64. Cost and Revenue Functions For a certain production facility, the cost function is

$$C(x) = x + 5$$

and the revenue function is

$$R(x) = 12x - 2x^2$$

where x is the number of units produced (in thousands) and R and C are measured in millions of dollars. Find:

(a) The marginal revenue.
(b) The marginal cost.
(c) The break-even point(s) [the number(s) x for which $R(x) = C(x)$].
(d) The number x for which marginal revenue equals marginal cost.
(e) Graph $C(x)$ and $R(x)$ on the same set of axes.

65. Demand Equation The price p per ton of cement when x tons of cement are demanded is given by the equation

$$p = -10x + 2000$$

dollars. Find:

(a) The revenue function $R = R(x)$ (*Hint: R = xp*, where p is the unit price.)
(b) The marginal revenue.
(c) The marginal revenue at $x = 100$ tons.
(d) The average rate of change in revenue from $x = 100$ to $x = 101$ tons.

66. Demand Equation The cost function and demand equation for a certain product are

$$C(x) = 50x + 40{,}000 \quad \text{and} \quad p = 100 - 0.01x$$

Find:

(a) The revenue function.
(b) The marginal revenue.
(c) The marginal cost.
(d) The break-even point(s).
(e) The number x for which marginal revenue equals marginal cost.

67. Demand Equation A certain item can be produced at a cost of \$10 per unit. The demand equation for this item is

$$p = 90 - 0.02x$$

where p is the price in dollars and x is the number of units. Find:

(a) The revenue function.
(b) The marginal revenue.
(c) The marginal cost.
(d) The break-even point(s).
(e) The number x for which marginal revenue equals marginal cost.

68. Instantaneous Rate of Change A circle of radius r has area $A = \pi r^2$ and circumference $C = 2\pi r$. If the radius changes from r to $(r + h)$, find the:

(a) Change in area.
(b) Change in circumference.
(c) Average rate of change of area with respect of the radius.
(d) Average rate of change of the circumference with respect to the radius.
(e) Instantaneous rate of change of area with respect to the radius.
(f) Instantaneous rate of change of the circumference with respect to the radius.

69. Instantaneous Rate of Change The volume V of a right circular cylinder of height 3 feet and radius r feet is $V = V(r) = 3\pi r^2$. Find the instantaneous rate of change of the volume with respect to the radius r at $r = 3$.

70. Instantaneous Rate of Change The surface area S of a sphere of radius r feet is $S = S(r) = 4\pi r^2$. Find the instantaneous rate of change of the surface area with respect to the radius r at $r = 2$.

4.2   The Derivative of a Power Function; Sum and Difference Formulas

OBJECTIVES   **1**   Find the derivative of a power function
2   Find the derivative of a constant times a function
3   Find the derivative of a polynomial function

In the previous section, we found the derivative $f'(x)$ of a function $y = f(x)$ by using the difference quotient:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{1}$$

We use this form for the derivative to derive formulas for finding derivatives.

We begin by considering the constant function $f(x) = b$, where b is a real number. Since the graph of the constant function f is a horizontal line (see Figure 8), the tangent line to f at any point is also a horizontal line. Since the derivative equals the slope of the tangent line to the graph of a function f at a point, then the derivative of f should be 0.

Algebraically, the derivative is obtained by using formula (1). The difference quotient of $f(x) = b$ is

$$\frac{f(x + h) - f(x)}{h} = \frac{b - b}{h} = \frac{0}{h} = 0$$

The derivative of $f(x) = b$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} 0 = 0$$

FIGURE 8

Derivative of the Constant Function

For the constant function $f(x) = b$, the derivative is $f'(x) = 0$. In other words, the derivative of a constant is 0.

Besides the **prime notation** f', there are several other ways to denote the derivative of a function $y = f(x)$. The most common ones are

$$y' \qquad \text{and} \qquad \frac{dy}{dx}$$

The notation $\dfrac{dy}{dx}$, often referred to as the **Leibniz notation,** may also be written as

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx} f(x)$$

where $\dfrac{d}{dx} f(x)$ is an instruction to compute the derivative of the function f with respect to its independent variable x. A change in the symbol used for the independent variable does not affect the meaning. If $s = f(t)$ is a function of t, then $\dfrac{ds}{dt}$ is an instruction to differentiate f with respect to t.
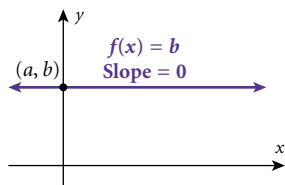
In terms of the Leibniz notation, if b is a constant, then

$$\frac{d}{dx}b = 0 \tag{2}$$

EXAMPLE 1 | **Finding the Derivative of _a_ Constant Function**

(a) If $f(x) = 5$, then $f'(x) = 0$.

(b) If $y = -1.7$, then $y' = 0$.

(c) If $y = \dfrac{2}{3}$, then $\dfrac{dy}{dx} = 0$.

(d) If $s = f(t) = \sqrt{5}$, then $\dfrac{ds}{dt} = f'(t) = 0$. ▶

In subsequent work with derivatives we shall use the prime notation or the Leibniz notation, or sometimes a mixture of the two, depending on which is more convenient.

NOW WORK PROBLEM (1)

Derivative of a Power Function

We now investigate the derivative of the power function $f(x) = x^n$, where n is a positive integer, to see if a pattern appears.

For $f(x) = x$, $n = 1$, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)-x}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$$

For $f(x) = x^2$, $n = 2$, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0}(2x+h) = 2x$$

For $f(x) = x^3$, $n = 3$, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}$$

$$= \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$$

In the Leibniz notation, these results take the form

$$\frac{d}{dx}x = 1 \qquad \frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}x^3 = 3x^2$$

This pattern suggests the following formula:

Derivative of $f(x) = x^n$

For the power function $f(x) = x^n$, n a positive integer, the derivative is $f'(x) = nx^{n-1}$. That is,

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$ (3)

Formula (3) may be stated in words as

The derivative with respect to x of x raised to the power n, where n is a positive integer, is n times x raised to power $n - 1$.

Problems 68 and 69 outline proofs of Formula (3).

1   **EXAMPLE 2   Finding the Derivative of a Power Function**

(a) If $f(x) = x^6$, then $f'(x) = 6x^{6-1} = 6x^5$

(b) $\dfrac{d}{dt}t^5 = 5t^4$ (c) $\dfrac{d}{dx}x = 1$ ▶

NOW WORK PROBLEM 3.

EXAMPLE 3   Finding the Derivative of a Power Function at a Number

Find $f'(4)$ if $f(x) = x^3$

SOLUTION   We use formula (3) $f'(x) = 3x^2$ Formula (3)
$f'(4) = 3(4)^2 = 48$ Substitute 4 for x.   ▶

Formula (3) allows us to compute some derivatives with ease. However, do not forget that a derivative is, in actuality, the limit of a difference quotient.

The next formula is used often.

Derivative of a Constant Times a Function

The derivative of a constant times a function equals the constant times the derivative of the function. That is, if C is a constant and f is a differentiable function, then

$$\boxed{\frac{d}{dx}[Cf(x)] = C\frac{d}{dx}f(x)}$$ (4)

Proof   We prove formula (4) as follows.

$$\frac{d}{dx}Cf(x) = \lim_{h \to 0}\frac{Cf(x+h) - Cf(x)}{h}$$

$$= \lim_{h \to 0}C\frac{f(x+h) - f(x)}{h}$$

$$= C\lim_{h \to 0}\frac{f(x+h) - f(x)}{h}$$

$$= C\frac{d}{dx}f(x) \qquad \blacktriangleright$$

The usefulness and versatility of this formula are often overlooked, especially when the constant appears in the denominator. Note that

$$\frac{d}{dx}\left[\frac{f(x)}{C}\right] = \frac{d}{dx}\left[\frac{1}{C}f(x)\right] = \frac{1}{C}\frac{d}{dx}[f(x)]$$

Always be on the lookout for constant factors *before* differentiating.

2   **EXAMPLE 4**   **Finding the Derivative of a Constant Times a Function**

(a) If $f(x) = 10x^3$, then

$$f'(x) = \frac{d}{dx}10x^3 = 10\frac{d}{dx}x^3 = 10 \cdot 3x^2 = 30x^2$$

(b) $\dfrac{d}{dx}\dfrac{x^5}{10} = \dfrac{1}{10}\dfrac{d}{dx}x^5 = \dfrac{1}{10}\cdot 5x^4 = \dfrac{1}{2}x^4$

(c) $\dfrac{d}{dt}6t = 6\dfrac{d}{dt}t = 6 \cdot 1 = 6$

(d) $\dfrac{d}{dx}\dfrac{2\sqrt{3}}{3}x^3 = \dfrac{2\sqrt{3}}{3}\dfrac{d}{dx}x^3 = \dfrac{2\sqrt{3}}{3}\cdot 3x^2 = 2\sqrt{3}\,x^2 \qquad \blacktriangleright$

NOW WORK PROBLEM 7.

Sum and Difference Formulas

> **Derivative of a Sum**
>
> The derivative of the sum of two differentiable functions equals the sum of their derivatives. That is,
>
> $$\boxed{\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)}\qquad (5)$$

A proof is given at the end of this section.

This formula for differentiating states that functions that are sums can be differentiated "term by term."

EXAMPLE 5   **Finding the Derivative of a Function**

Find the derivative of: $f(x) = x^2 + 4x$

SOLUTION   The function f is the sum of the two power functions x^2 and $4x$. We can differentiate term by term.

$$\frac{d}{dx} f(x) = \frac{d}{dx} (x^2 + 4x) = \frac{d}{dx} x^2 + \frac{d}{dx} (4x) = 2x + 4\frac{d}{dx} x = 2x + 4 \quad \blacktriangleright$$

$\qquad\qquad\qquad$ Formula (5) $\qquad\qquad$ Formulas (3) and (4) $\qquad \frac{d}{dx} x = 1$

> **Derivative of a Difference**
>
> The derivative of the difference of two differentiable functions equals the difference of their derivatives. That is,
>
> $$\boxed{\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)} \tag{6}$$

Formulas (5) and (6) extend to sums and differences of more than two functions. Since a polynomial function is a sum (or difference) of power functions, we can find the derivative of any polynomial function by using a combination of Formulas (3), (4), (5), and (6).

3   **EXAMPLE 6**   **Finding the Derivative of a Polynomial Function**

Find the derivative of: $f(x) = 6x^4 - 3x^2 + 10x - 8$

SOLUTION $$f'(x) = \frac{d}{dx} (6x^4 - 3x^2 + 10x - 8)$$

$$= \frac{d}{dx} (6x^4) - \frac{d}{dx} (3x^2) + \frac{d}{dx} (10x) - \frac{d}{dx} 8 \qquad \text{Use Formulas (5) and (6).}$$

$$= 6\frac{d}{dx} x^4 - 3\frac{d}{dx} x^2 + 10\frac{d}{dx} x - 0 \qquad \text{Use Formulas (2) and (4).}$$

$$= 24x^3 - 6x + 10 \qquad\qquad\qquad \text{Use Formula (3); Simplify.} \quad \blacktriangleright$$

NOW WORK PROBLEM 21.

EXAMPLE 7   **Finding the Derivative of a Polynomial Function**

If $f(x) = -\dfrac{x^4}{2} - 2x + 3$, find

(a) $f'(x)$ **(b)** $f'(-1)$

SOLUTION   **(a)** $f'(x) = -\dfrac{4x^3}{2} - 2 + 0 = -2x^3 - 2$

(b) $f'(-1) = -2(-1)^3 - 2 = 0$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \blacktriangleright$

Proof of the Sum Formula   We verify Formula (5) as follows. To compute

$$\frac{d}{dx}[f(x) + g(x)]$$

we need to find the limit of the difference quotient of $f(x) + g(x)$.

$$\frac{d}{dx}[f(x) + g(x)] = \lim_{h \to 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{[f(x + h) - f(x)] + [g(x + h) - g(x)]}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h} \right] + \lim_{h \to 0} \left[\frac{g(x + h) - g(x)}{h} \right]$$

$$= \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \qquad \blacktriangleright$$

Proof of the Difference Formula   The proof uses Formulas (4) and (5).

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x) + (-1)g(x)]$$

$$= \frac{d}{dx} f(x) + \frac{d}{dx}[(-1)g(x)]$$

$$= \frac{d}{dx} f(x) + (-1)\frac{d}{dx} g(x)$$

$$= \frac{d}{dx} f(x) - \frac{d}{dx} g(x) \qquad \blacktriangleright$$

EXAMPLE 8   **Analyzing a Cost Function**

The total daily cost C, in dollars, of producing dishwashers is

$$C(x) = 1000 + 72x - 0.06x^2 \qquad 0 \le x \le 60$$

where x represents the number of dishwashers produced.
(a) Find the total daily cost of producing 50 dishwashers.
(b) Determine the marginal cost function.
(c) Find $C'(50)$ and interpret its meaning.
(d) Use the marginal cost to estimate the cost of producing 51 dishwashers.
(e) Find the actual cost of producing 51 dishwashers. Compare the actual cost of making 51 dishwashers to the estimated cost of producing 51 dishwashers found in part (d).
(f) Determine the actual cost of manufacturing the 51st dishwashwer.
(g) The average cost function is defined as $\overline{C}(x) = \dfrac{C(x)}{x}, 0 < x \le 60$. Determine the average cost function for producing x dishwashers.
(h) Find the average cost of producing 51 dishwashers.

SOLUTION   **(a)** The total daily cost of producing 50 dishwashers is

$$C(50) = 1000 + 72(50) - 0.06(50)^2 = \$4450$$

(b) The marginal cost is

$$C'(x) = \frac{d}{dx}(72x - 0.06x^2) = 72 - 0.12x$$

(c) $C'(50) = 72 - 0.12(50) = \$66$

The marginal cost of producing 50 dishwashers may be interpreted as the cost to produce the 51st dishwasher.

(d) From part (a) the cost to produce 50 dishwashers is $4450. If the 51st costs $66, then the cost to produce 51 will be

$$\$4450 + \$66 = \$4516$$

(e) The actual cost to produce 51 dishwashers is

$$C(51) = \$1000 + 72(51) - 0.06(51)^2 = \$4515.90$$

There is a $0.10 difference between the actual cost and the cost obtained using the marginal cost.

(f) The actual cost of producing the 51st dishwasher is

$$C(51) - C(50) = \$4515.90 - \$4450 = \$65.90$$

(g) The average cost function is

$$\overline{C}(x) = \frac{c(x)}{x} = \frac{1000 + 72x - 0.06x^2}{x} = \frac{1000}{x} + 72 - 0.06x$$

(h) The average cost of producing 51 dishwashers is

$$\overline{C}(51) = \frac{1000}{51} + 72 - 0.06(51) = \$88.55$$

NOW WORK PROBLEM 65.

EXERCISE 4.2 Answers to Odd-Numbered Problems Begin on Page AN-XX.

In Problems 1–20, find the derivative of each function.

1. $f(x) = 4$

2. $f(x) = -2$

3. $f(x) = x^3$

4. $f(x) = x^4$

5. $f(x) = 6x^2$

6. $f(x) = -8x^3$

7. $f(t) = \frac{t^4}{4}$

8. $f(t) = \frac{t^3}{6}$

9. $f(x) = x^2 + x$

10. $f(x) = x^2 - x$

11. $f(x) = x^3 - x^2 + 1$

12. $f(x) = x^4 - x^3 + x$

13. $f(t) = 2t^2 - t + 4$

14. $f(t) = 3t^3 - t^2 + t$

15. $f(x) = \frac{1}{2}x^8 + 3x + \frac{2}{3}$

16. $f(x) = \frac{2}{3}x^6 - \frac{1}{2}x^4 + 2$

17. $f(x) = \frac{1}{3}(x^5 - 8)$

18. $f(x) = \frac{x^3 + 2}{5}$

19. $f(x) = ax^2 + bx + c$
a, b, c are constants

20. $f(x) = ax^3 + bx^2 + cx + d$
a, b, c, d are constants

In Problems 21–28, find the indicated derivative.

21. $\frac{d}{dx}(-6x^2 + x + 4)$

22. $\frac{d}{dx}(8x^3 - 6x^2 + 2x)$

23. $\frac{d}{dt}(-16t^2 + 80t)$

24. $\frac{d}{dt}(-16t^2 + 64t)$

25. $\frac{dA}{dr}$ if $A = \pi r^2$

26. $\frac{dC}{dr}$ if $C = 2\pi r$

27. $\frac{dV}{dr}$ if $V = \frac{4}{3}\pi r^3$

28. $\frac{dP}{dt}$ if $P = 0.2t$

In Problems 29–38, find the value of the derivative at the indicated number.

29. $f(x) = 4x^2$ at $x = -3$

30. $f(x) = -10x^3$ at $x = -2$

31. $f(x) = 2x^2 - x$ at $x = 4$

32. $f(x) = x^4 - 2x^2$ at $x = 2$

33. $f(t) = -\frac{1}{3}t^3 + 5t$ at $t = 3$

34. $f(t) = -\frac{1}{4}t^4 + \frac{1}{2}t^2 + 4$ at $t = 1$

35. $f(x) = \frac{1}{2}(x^6 - x^4)$ at $x = 1$

36. $f(x) = \frac{1}{3}(x^6 + x^3 + 1)$ at $x = -1$

37. $f(x) = ax^2 + bx + c$ at $x = -b/2a$
    a, b, c are constants

38. $f(x) = ax^3 + bx^2 + cx + d$ at $x = 0$
    a, b, c, d are constants

In Problems 39–48, find the value of $\dfrac{dy}{dx}$ at the indicated point.

39. $y = x^4$ at $(1, 1)$

40. $y = x^4$ at $(2, 16)$

41. $y = x^2 - 14$ at $(4, 2)$

42. $y = x^3 + 1$ at $(3, 28)$

43. $y = 3x^2 - x$ at $(-1, 4)$

44. $y = x^2 - 3x$ at $(-1, 4)$

45. $y = \frac{1}{2}x^2$ at $\left(1, \frac{1}{2}\right)$

46. $y = x^3 - x^2$ at $(1, 0)$

47. $y = 2 - 2x + x^3$ at $(2, 6)$

48. $y = 2x^2 - \frac{1}{2}x + 3$ at $(0, 3)$

In Problems 49–50, find the slope of the tangent line to the graph of the function at the indicated point. What is an equation of the tangent line?

49. $f(x) = x^3 + 3x - 1$ at $(0, -1)$

50. $f(x) = x^4 + 2x - 1$ at $(1, 2)$

In Problems 51–56, find those x, if any, at which $f'(x) = 0$.

51. $f(x) = 3x^2 - 12x + 4$

52. $f(x) = x^2 + 4x - 3$

53. $f(x) = x^3 - 3x + 2$

54. $f(x) = x^4 - 4x^3$

55. $f(x) = x^3 + x$

56. $f(x) = x^5 - 5x^4 + 1$

---

57. Find the point(s), if any, on the graph of the function $y = 9x^3$ at which the tangent line is parallel to the line $3x - y + 2 = 0$.

58. Find the points(s), if any, on the graph of the function $y = 4x^2$ at which the tangent line is parallel to the line $2x - y - 6 = 0$.

59. Two lines through the point $(1, -3)$ are tangent to the graph of the function $y = 2x^2 - 4x + 1$. Find the equations of these two lines.

60. Two lines through the point $(0, 2)$ are tangent to the graph of the function $y = 1 - x^2$. Find the equations of these two lines.

61. Marginal Cost The cost per day, $C(x)$, in dollars, of producing x pairs of eyeglasses is

$$C(x) = 0.2x^2 + 3x + 1000$$

(a) Find the average cost due to producing 10 additional pairs of eyeglasses after 100 have been produced.
(b) Find the marginal cost.
(c) Find the marginal cost at $x = 100$.
(d) Interpret $C'(100)$.

62. Toy Truck Sales At Dan's Toy Store, the revenue R, in dollars, derived from selling x electric trucks is

$$R(x) = -0.005x^2 + 20x$$

(a) What is the average rate of change in revenue due to selling 10 additional trucks after 1000 have been sold?
(b) What is the marginal revenue?
(c) What is the marginal revenue at $x = 1000$?
(d) Interpret $R'(1000)$.
(e) For what value of x is $R'(x) = 0$?

63. Medicine The French physician Poiseville discovered that the volume V of blood (in cubic centimeters) flowing through a clogged artery with radius R (in centimeters) can be modeled by

$$V(R) = kR^4$$

where k is a positive constant.

(a) Find the derivative $V'(R)$.
(b) Find the rate of change of volume for a radius of 0.3 cm.
(c) Find the rate of change of volume for a radius of 0.4 cm.
(d) If the radius of a clogged artery is increased from 0.3 cm to 0.4 cm, estimate the effect on the volume of blood flowing through the enlarged artery.

64. **Respiration Rate** A human being's respiration rate R (in breaths per minute) is given by

$$R = -10.35p + 0.59p^2$$

where p is the partial pressure of carbon dioxide in the lungs. Find the rate of change in respiration rate when $p = 50$.

65. **Analyzing a Cost Function** The total daily cost C of producing microwave ovens is

$$C(x) = 2000 + 50x - 0.05x^2, 0 \le x \le 50$$

where x represents the number of microwave ovens produced.

(a) Find the total daily cost of producing 40 microwave ovens.
(b) Determine the marginal cost function.
(c) Find $C'(40)$ and interpret its meaning.
(d) Use the marginal cost to estimate the cost of producing 41 microwave ovens.
(e) Find the actual cost of producing 41 microwave ovens. Compare the actual cost of making 51 microwave ovens to the estimated cost of producing 51 microwave ovens.
(f) Determine the actual cost of manufacturing the 51st microwave oven.
(g) The average cost function is defined as $\overline{C}(x) = \dfrac{C(x)}{x}$, $0 < x \le 50$. Determine the average cost function for producing x microwave ovens.
(h) Find the average cost of producing 51 microwave ovens.
(i) Compare your answers from parts (g), (h), and (j). Give explanations for the differences.

66. **Analyzing a Cost Function** The total daily cost C of producing small televisions is

$$C(x) = 1500 + 25x - 0.05x^2, 0 \le x \le 100$$

where x represents the number of televisions produced.

(a) Find the total daily cost of producing 70 televisions.
(b) Determine the marginal cost function.
(c) Find $C'(70)$ and interpret its meaning.
(d) Use the marginal cost to estimate the cost of producing 71 televisions.
(e) Find the actual cost of producing 71 televisions. Compare the actual cost of making 71 televisions to the estimated cost of producing 71 televisions.
(f) Determine the actual cost of manufacturing the 71st televisions.
(g) The average cost function is defined as $\overline{C}(x) = \dfrac{C(x)}{x}$, $0 < x \le 100$. Determine the average cost function for producing x televisions.
(h) Find the average cost of producing 71 televisions.
(i) Compare your answers from parts (g), (h), and (j). Give explanations for the differences.

65. **Price of Beans** The price per unit in dollars per cwt for beans from 1993 through 2002 can be modeled by the polynomial function $p(t) = 0.007t^3 - 0.63t^2 + 0.005t + 6.123$,

where t is in years, and $t = 0$ corresponds to 1993.

(a) Find the marginal price of beans for the year 1995.
(b) Find the marginal price for beans for the year 2002.
(c) How do you interpret the two marginal prices? What is the trend?

66. **Price of Beans** price per unit in dollars per cwt for beans from 1993 through 2002 can also be modeled by the polynomial function, $p(t) = -0.002t^4 + 0.044t^3 - 0.335t^2 + 0.750t + 5.543$, where t is in years and $t = 0$ corresponds to 1993.

(a) Find the marginal price for beans for the year 1995.
(b) Find the marginal price for beans for the year 2002.
(c) How do you interpret the two marginal prices? What is the trend?
(d) Explain why there might be two different functions that can model the price of beans.

67. **Instantaneous Rate of Change** The volume V of a sphere of radius r feet is $V = V(r) = \frac{4}{3}\pi r^3$. Find the instantaneous rate of change of the volume with respect to the radius r at $r = 2$.

68. **Instantaneous Rate of Change** The volume V of a cube of side x meters is $V = V(x) = x^3$. Find the instantaneous rate of change of the volume with respect to the side x at $x = 3$.

69. **Work Output** The relationship between the amount $A(t)$ of work output and the elapsed time t, $t \ge 0$, was found through empirical means to be

$$A(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$

where a_0, a_1, a_2, a_3 are constants. Find the instantaneous rate of change of work output at time t.

70. **Consumer Price Index** The consumer price index (CPI) of an economy is described by the function

$$I(t) = -0.2t^2 + 3t + 200 \qquad 0 \le t \le 10$$

where $t = 0$ corresponds to the year 2000.

(a) What was the average rate of increase in the CPI over the period from 2000 to 2003?
(b) At what rate was the CPI of the economy changing in 2003? in 2006?

71. Use the binomial theorem to prove formula (3)

[**Hint:** $(x + h)^n - x^n = x^n + nx^{n-1}h + \dfrac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n = nx^{n-1}h + h^2 \cdot$ (terms involving x and h). Now apply formula (1), page 000.]

72. Use the following factoring rule to prove formula (3)

$$f(x) = x^n - c^n = (x - c)(x^{n-1} + x^{n-2}c + x^{n-3}c^2 + \cdots + c^{n-1})$$

Now apply Formula (3), page 000, to find $f'(c)$.

4.3   Product and Quotient Formulas

OBJECTIVES   **1** Find the derivative of a product
2 Find the derivative of a quotient
3 Find the derivative of $f(x) = x^n$, n is a negative integer

The Derivative of a Product

In the previous section we learned that the derivative of the sum or the difference of two functions is simply the sum or the difference of their derivatives. The natural inclination at this point may be to assume that differentiating a product or quotient of two functions is as simple. But this is not the case, as illustrated for the case of a product of two functions. Consider

$$F(x) = f(x) \cdot g(x) = (3x^2 - 3)(2x^3 - x) \tag{1}$$

where $f(x) = 3x^2 - 3$, and $g(x) = 2x^3 - x$. The derivative of $f(x)$ is $f'(x) = 6x$ and the derivative of $g(x)$ is $g'(x) = 6x^2 - 1$. The product of these derivatives is

$$f'(x) \cdot g'(x) = 6x(6x^2 - 1) = 36x^3 - 6x \tag{2}$$

To see if this is equal to the derivative of the product, we first multiply the right side of (1) and then differentiate using the rules of differentiation of the previous section:

$$F(x) = (3x^2 - 3)(2x^3 - x) = 6x^5 - 9x^3 + 3x$$

so that

$$F'(x) = 30x^4 - 27x^2 + 3 \tag{3}$$

Since (2) and (3) are not equal, we conclude that the derivative of a product *is not* equal to the product of the derivatives.

The formula for finding the derivative of the product of two functions is given below:

Derivative of a Product

The derivative of the product of two differentiable functions equals the first function times the derivative of the second plus the second function times the derivative of the first. That is,

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \tag{4}$$

The following version of formula (4) may help you remember it.

$$\frac{d}{dx}(\text{first} \cdot \text{second}) = \text{first} \cdot \frac{d}{dx}\text{second} + \text{second} \cdot \frac{d}{dx}\text{first}$$

1   **EXAMPLE 1**   **Finding the Derivative of a Product**

Find the derivative of: $F(x) = (x^2 + 2x - 5)(x^3 - 1)$

SOLUTION   The function F is the product of the two functions $f(x) = x^2 + 2x - 5$ and $g(x) = x^3 - 1$ so that, by (1), we have

$$F'(x) = (x^2 + 2x - 5)\left[\frac{d}{dx}(x^3 - 1)\right] + (x^3 - 1)\left[\frac{d}{dx}(x^2 + 2x - 5)\right] \quad \text{Use formula (4).}$$

$$= (x^2 + 2x - 5)(3x^2) + (x^3 - 1)(2x + 2) \qquad\qquad \text{Differentiate.}$$

$$= 3x^4 + 6x^3 - 15x^2 + 2x^4 + 2x^3 - 2x - 2 \qquad\qquad \text{Simplify.}$$

$$= 5x^4 + 8x^3 - 15x^2 - 2x - 2 \qquad\qquad\qquad\qquad \text{Simplify.} \quad \blacktriangleright$$

Now that you know the formula for the derivative of a product, be careful not to use it unnecessarily. When one of the factors is a constant, you should use the formula for the derivative of a constant times a function. For example, it is easier to work

$$\frac{d}{dx}[5(x^2 + 1)] = 5\frac{d}{dx}(x^2 + 1) = (5)(2x) = 10x$$

than it is to work

$$\frac{d}{dx}[5(x^2 + 1)] = 5\left[\frac{d}{dx}(x^2 + 1)\right] + (x^2 + 1)\left(\frac{d}{dx}5\right)$$

$$= (5)(2x) + (x^2 + 1)(0) = 10x$$

NOW WORK PROBLEM 1.

The Derivative of a Quotient

As in the case with a product, the derivative of a quotient is *not* the quotient of the derivatives.

> **Derivative of a Quotient**
>
> The derivative of the quotient of two differentiable functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.
>
> $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2} \quad \text{where } g(x) \neq 0 \qquad (5)$$

You may want to memorize the following version of Formula (5):

$$\frac{d}{dx}\,\frac{\text{numerator}}{\text{denominator}}$$

$$= \frac{(\text{denominator})\dfrac{d}{dx}(\text{numerator}) - (\text{numerator})\dfrac{d}{dx}(\text{denominator})}{(\text{denominator})^2}$$

2 **EXAMPLE 2**   **Finding the Derivative of a Quotient**

Find the derivative of: $F(x) = \dfrac{x^2 + 1}{x - 3}$

SOLUTION Here, the function F is the quotient of $f(x) = x^2 + 1$ and $g(x) = x - 3$. We use Formula (5) to get

$$\frac{d}{dx}\left(\frac{x^2 + 1}{x - 3}\right) = \frac{(x - 3)\dfrac{d}{dx}(x^2 + 1) - (x^2 + 1)\dfrac{d}{dx}(x - 3)}{(x - 3)^2} \qquad \text{Use formula (5).}$$

$$= \frac{(x - 3)(2x) - (x^2 + 1)(1)}{(x - 3)^2} \qquad \text{Differentiate.}$$

$$= \frac{2x^2 - 6x - x^2 - 1}{(x - 3)^2} \qquad \text{Simplify.}$$

$$= \frac{x^2 - 6x - 1}{(x - 3)^2} \qquad \text{Simplify.} \qquad \blacktriangleright$$

NOW WORK PROBLEM 9.

We shall follow the practice of leaving our answers in factored form as shown in Example 2.

EXAMPLE 3   **Finding the Derivative of a Quotient**

Find the derivative of: $y = \dfrac{(1 - 3x)(2x + 1)}{3x - 2}$

SOLUTION We shall solve the problem in two ways.

Method 1   Use the formula for the derivative of a quotient right away.

$$y' = \frac{d}{dx}\,\frac{(1 - 3x)(2x + 1)}{3x - 2}$$

$$= \frac{(3x - 2)\dfrac{d}{dx}[(1 - 3x)(2x + 1)] - (1 - 3x)(2x + 1)\dfrac{d}{dx}(3x - 2)}{(3x - 2)^2} \qquad \text{Use Formula (5).}$$

$$= \frac{(3x - 2)[(1 - 3x)\dfrac{d}{dx}(2x + 1) + (2x + 1)\dfrac{d}{dx}(1 - 3x)] - (1 - 3x)(2x + 1)\dfrac{d}{dx}(3x - 2)}{(3x - 2)^2} \qquad \text{Differentiate.}$$

$$= \frac{(3x - 2)[(1 - 3x)(2) + (2x + 1)(-3)] - (1 - 3x)(2x + 1)(3)}{(3x - 2)^2}$$ Using Formula (4)
Differentiate.

$$= \frac{(3x - 2)[2 - 6x - 6x - 3] - (-6x^2 - x + 1)(3)}{(3x - 2)^2}$$ Simply.

$$= \frac{(3x - 2)(-12x - 1) - (-18x^2 - 3x + 3)}{(3x - 2)^2}$$ Simplify.

$$= \frac{-36x^2 + 21x - 2 + 18x^2 + 3x - 3}{(3x - 2)^2}$$ Simplify.

$$= \frac{-18x^2 + 24x - 1}{(3x - 2)^2}$$

Method 2   First, multiply the factors in the numerator and then apply the formula for the derivative of a quotient.

$$y = \frac{(1 - 3x)(2x + 1)}{3x - 2} = \frac{-6x^2 - x + 1}{3x - 2}$$

Now use Formula (5):

$$y' = \frac{d}{dx} \frac{-6x^2 - x + 1}{3x - 2}$$

$$= \frac{(3x - 2)\frac{d}{dx}(-6x^2 - x + 1) - (-6x^2 - x + 1)\frac{d}{dx}(3x - 2)}{(3x - 2)^2}$$ Formula (5).

$$= \frac{(3x - 2)(-12x - 1) - (-6x^2 - x + 1)(3)}{(3x - 2)^2}$$ Differentiate.

$$= \frac{-36x^2 + 21x + 2 + 18x^2 + 3x - 3}{(3x - 2)^2}$$ Simplify.

$$= \frac{-18x^2 + 24x - 1}{(3x - 2)^2}$$ Simplify.   ▶

As you can see from this example, looking at alternative methods may make the differentiation easier.

The Derivative of x^n, n a Negative Integer

Find the derivative of   **3**   In the previous section, we learned that the derivative of a power function $f(x) = x^n$,
$f(x) = x^n$, n a negative $n \geq 1$ an integer, is $f'(x) = nx^{n-1}$.
integer   The formula for the derivative of x raised to a negative integer exponent follows the
same form.

> The derivative of $f(x) = x^n$, where n is *any* integer, is n times x to the $n - 1$ power. Thus,

$$\boxed{\frac{d}{dx} x^n = nx^{n-1} \qquad \text{for any integer } n} \qquad (3)$$

EXAMPLE 4   **Using Formula (3)**

The proof is left as an exercise. See Problem 54.

(a) $\dfrac{d}{dx} x^{-3} = -3x^{-4} = \dfrac{-3}{x^4}$

(b) $\dfrac{d}{dx} \dfrac{4}{x^2} = \dfrac{d}{dx} 4x^{-2} = 4\dfrac{d}{dx} x^{-2} = 4(-2x^{-3}) = \dfrac{-8}{x^3}$

(c) $\dfrac{d}{dx}\left(x + \dfrac{2}{x}\right) = \dfrac{d}{dx}(x + 2x^{-1}) = \dfrac{d}{dx}x + 2\dfrac{d}{dx}x^{-1} = 1 + 2(-1)x^{-1} = 1 - \dfrac{2}{x^2}$

NOW WORK PROBLEM 17.

EXAMPLE 5 **Finding the Derivative of a Function**

Find the derivative of: $g(x) = \left(1 - \dfrac{1}{x^2}\right)(x + 1)$

SOLUTION Since $g(x)$ is the product of two simpler functions, we begin by applying the formula for the derivative of a product:

$$g'(x) = \left(1 - \dfrac{1}{x^2}\right)\dfrac{d}{dx}(x + 1) + (x + 1)\dfrac{d}{dx}\left(1 - \dfrac{1}{x^2}\right) \qquad \text{Derivative of a Product.}$$

$$= \left(1 - \dfrac{1}{x^2}\right)(1) + (x + 1)\dfrac{d}{dx}(1 - x^{-2}) \qquad \text{Differentiate; } \dfrac{1}{x^2} = x^{-2}.$$

$$= 1 - \dfrac{1}{x^2} + (x + 1)(2x^{-3}) \qquad \text{Differentiate.}$$

$$= 1 - \dfrac{1}{x^2} + \dfrac{2(x + 1)}{x^3} \qquad \text{Simplify.}$$

$$= 1 - \dfrac{1}{x^2} + \dfrac{2x}{x^3} + \dfrac{2}{x^3} \qquad \text{Simplify.}$$

$$= 1 + \dfrac{1}{x^2} + \dfrac{2}{x^3} \qquad \blacktriangleright$$

Alternatively, we could have solved Example 5 by multiplying the factors first. Then

$$g(x) = \left(1 - \dfrac{1}{x^2}\right)(x + 1) = x + 1 - \dfrac{1}{x} - \dfrac{1}{x^2}$$

so

$$g'(x) = \dfrac{d}{dx}\left(x + 1 - \dfrac{1}{x} - \dfrac{1}{x^2}\right) = \dfrac{d}{dx}x + \dfrac{d}{dx}(1) - \dfrac{d}{dx}\dfrac{1}{x} - \dfrac{d}{dx}\dfrac{1}{x^2}$$

$$= 1 + 0 - \dfrac{d}{dx}x^{-1} - \dfrac{d}{dx}x^{-2} = 1 - (-1)x^{-2} - (-2)x^{-3} = 1 + \dfrac{1}{x^2} + \dfrac{2}{x^3}$$

EXAMPLE 4 **Application**

The value $V(t)$, in dollars, of a car t years after its purchase is given by the equation

$$V(t) = \dfrac{8000}{t} + 5000 \qquad 1 \le t \le 5$$

Graph the function $V = V(t)$. Then find:

Art to come

(a) The average rate of change in value from $t = 1$ to $t = 4$.
(b) The instantaneous rate of change in value.
(c) The instantaneous rate of change in value after 1 year.
(d) The instantaneous rate of change in value after 3 years.
(e) Interpret the answers to (c) and (d).

SOLUTION The graph of $V = V(t)$ is given in Figure 9.

(a) The average rate of change in value from $t = 1$ to $t = 4$ is given by

$$\frac{V(4) - V(1)}{4 - 1} = \frac{7000 - 13{,}000}{3} = -2000$$

So the average rate of change in value from $t = 1$ to $t = 4$ is $-\$2000$ per year, that is, it is decreasing at the rate of $\$2000$ per year.

(b) The derivative $V'(t)$ of $V(t)$ equals the instantaneous rate of change in the value of the car.

$$V'(t) = \frac{d}{dt}\left(\frac{8000}{t} + 5000\right) = \frac{d}{dt}\frac{8000}{t} + \frac{d}{dt}(5000)$$

$$= \frac{d}{dt}8000t^{-1} + 0 = 8000(-1)t^{-2} = -\frac{8000}{t^2}$$

FIGURE 9

Notice that $V'(t) < 0$; we interpret this to mean that the value of the car is decreasing over time.

(c) After 1 year, $V'(1) = -\dfrac{8000}{1} = -\$8000/\text{year}$

(d) After 3 years, $V'(3) = -\dfrac{8000}{9} = -\$888.89/\text{year}$

(e) $V'(1) = -\$8000$ means that the value of the car after 1 year will decline by approximately $\$8000$ over the next year; $V'(3) = -\$888.89$ means that the value of the car after 3 years will decline by approximately $\$888.89$ over the next year. ▶

Figure 9 shows the graph of $V = V(t)$.

NOW WORK PROBLEM 41.
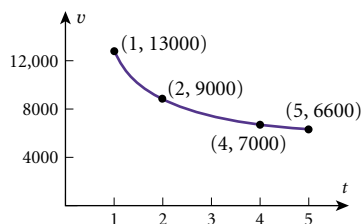
SUMMARY Each of the derivative formulas given so far can be written without reference to the independent variable of the function. If f and g are differentiable functions, we have the following formulas:

Derivative of a constant times a function	$(cf)' = cf'$
Derivative of a sum	$(f + g)' = f' + g'$
Derivative of a difference	$(f - g)' = f' - g'$
Derivative of a product	$(f \cdot g)' = f \cdot g' + g \cdot f'$
Derivative of a quotient	$\left(\dfrac{f}{g}\right)' = \dfrac{g \cdot f' - f \cdot g'}{g^2}$

In Problems 1–8, find the derivative of each function by using the formula for the derivative of a product.

1. $f(x) = (2x + 1)(4x - 3)$

2. $f(x) = (3x - 4)(2x + 5)$

3. $f(t) = (t^2 + 1)(t^2 - 4)$

4. $f(t) = (t^2 - 3)(t^2 + 4)$

5. $f(x) = (3x - 5)(2x^2 + 1)$

6. $f(x) = (3x^2 - 1)(4x + 1)$

7. $f(x) = (x^5 + 1)(3x^3 + 8)$

8. $f(x) = (x^6 - 2)(4x^2 + 1)$

In Problems 9–20, find the derivative of each function.

9. $f(x) = \dfrac{x}{x + 1}$

10. $f(x) = \dfrac{x + 4}{x^2}$

11. $f(x) = \dfrac{3x + 4}{2x - 1}$

12. $f(x) = \dfrac{3x - 5}{4x + 1}$

13. $f(x) = \dfrac{x^2}{x - 4}$

14. $f(x) = \dfrac{x}{x^2 - 4}$

15. $f(x) = \dfrac{2x + 1}{3x^2 + 4}$

16. $f(x) = \dfrac{2x^2 - 1}{5x + 2}$

17. $f(t) = \dfrac{-2}{t^2}$

18. $f(t) = \dfrac{4}{t^3}$

19. $f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

20. $f(x) = 1 - \dfrac{1}{x} + \dfrac{1}{x^2}$

In Problems 21–24, find the slope of the tangent line to the graph of the function f at the indicated point. What is an equation of the tangent line?

21. $f(x) = (x^3 - 2x + 2)(x + 1)$ at $(1, 2)$

22. $f(x) = (2x^2 - 5x + 1)(x - 3)$ at $(1, 4)$

23. $f(x) = \dfrac{x^3}{x + 1}$ at $\left(1, \frac{1}{2}\right)$

24. $f(x) = \dfrac{x^2}{x - 1}$ at $\left(-1, -\frac{1}{2}\right)$

In Problems 25–28, find those x, if any, at which f'(x) = 0.

25. $f(x) = (x^2 - 2)(2x - 1)$

26. $f(x) = (3x^2 - 3)(2x^3 - x)$

27. $f(x) = \dfrac{x^2}{x + 1}$

28. $f(x) = \dfrac{x^2 + 1}{x}$

In Problems 29–40, find y'.

29. $y = x^2(3x - 2)$

30. $y = (x^2 + 2)(x - 1)$

31. $y = (x^{-2} + 4)(4x^2 + 3)$

32. $y = (2x^{-1} + 3)(x^{-3} + x^{-2})$

33. $y = \dfrac{(2x + 3)(x - 4)}{3x + 5}$

34. $y = \dfrac{(3x - 2)(x^2 + 1)}{4x - 3}$

35. $y = \dfrac{3x + 1}{(x - 2)(x + 2)}$

36. $y = \dfrac{2x - 5}{(1 - x)(1 + x)}$

37. $y = \dfrac{(3x + 4)(2x - 3)}{(2x + 1)(3x - 2)}$

38. $y = \dfrac{(2 - 3x)(1 - x)}{(x + 2)(3x + 1)}$

39. $y = \dfrac{x^{-2} - x^{-1}}{x^{-2} + x^{-1}}$

40. $y = \dfrac{3x^{-4} - x^{-2}}{x^{-3} + x^{-1}}$

---

41. Value of a Car The value V of a luxury car after t years is

$$V(t) = \frac{10{,}000}{t} + 6000 \qquad 1 \le t \le 6$$

(a) What is the average rate of change in value from $t = 2$ to $t = 5$?

(b) What is the instantaneous rate of change in value?

(c) What is the instantaneous rate of change after 2 years?

(d) What is the instantaneous rate of change after 5 years?

(e) Interpret the answers found in (c) and (d).

42. Value of a Painting The value V of a famous painting t years after it is purchased is

$$V(t) = \frac{100t^2 + 50}{t} + 400 \qquad 1 \le t \le 5$$

(a) What is the average rate of change in value from $t = 1$ to $t = 3$?

(b) What is the instantaneous rate of change in value?

(c) What is the instantaneous rate of change after 1 year?

(d) What is the instantaneous rate of change after 3 years?

(e) Interpret the answers found in (c) and (d).

43. Demand Equation The demand equation for a certain commodity is

$$p = 10 + \frac{40}{x} \qquad 1 \le x \le 10$$

where p is the price in dollars when x units are demanded. Find:

(a) The revenue function.
(b) The marginal revenue.
(c) The marginal revenue for $x = 4$.
(d) The marginal revenue for $x = 6$.

44. Cost Function The cost of fuel in operating a luxury yacht is given by the equation

$$C(s) = \frac{-3s^2 + 1200}{s}$$

where s is the speed of the yacht. Find the rate at which the cost is changing when $s = 10$.

45. Price–Demand Function The price–demand function for calculators is given by

$$D(p) = \frac{100,000}{p^2 + 10p + 50} \qquad 5 \le p \le 20$$

where D is the quantity demanded per week and p is the unit price in dollars.

(a) Find $D'(p)$, the rate of change of demand with respect to price.
(b) Find $D'(5)$, $D'(10)$, and $D'(15)$
(c) Interpret the results found in part (b).

46. Rising Object The height, in kilometers, that a balloon will rise in t hours is given by the formula

$$s = \frac{t^2}{2 + t}$$

Find the rate at which the balloon is rising after (a) 10 minutes, (b) 20 minutes.

47. Population Growth A population of 1000 bacteria is introduced into a culture and grows in number according to the formula

$$P(t) = 1000\left(1 + \frac{4t}{100 + t^2}\right)$$

where t is measured in hours. Find the rate at which the population is growing when

(a) $t = 1$ (b) $t = 2$ (c) $t = 3$ (d) $t = 4$

48. Drug Concentration The concentration of a certain drug in a patient's bloodstream t hours after injection is given by

$$C(t) = \frac{0.4t}{2t^2 + 1}$$

Find the rate at which the concentration of the drug is changing with respect to time. At what rate is the concentration changing

(a) 10 minutes after the injection?
(b) 30 minutes after the injection?
(c) 1 hour after the injection?
(d) 3 hours after the injection?

49. Intensity of Illumination The intensity of illumination I on a surface is inversely proportional to the square of the distance r from the surface to the source of light. If the intensity is 1000 units when the distance is 1 meter, find the rate of change of the intensity with respect to the distance when the distance is 10 meters.

50. Cost Function The cost, C, in thousands of dollars, for removal of pollution from a certain lake is

$$C(x) = \frac{5x}{110 - x}$$

where x is the percent of pollutant removed. Find:

(a) $C'(x)$, the rate of change of cost with respect to the amount of pollutant removed.
(b) Compute $C'(10)$, $C'(20)$, $C'(70)$, $C'(90)$.

51. Cost Function An airplane crosses the Atlantic Ocean (3000 miles) with an airspeed of 500 miles per hour. The cost C (in dollars) per person is

$$C(x) = 100 + \frac{x}{10} + \frac{36,000}{x}$$

where x is the ground speed (airspeed $\pm$ wind). Find:

(a) The marginal cost.
(b) The marginal cost at a ground speed of 500 mph.
(c) The marginal cost at a ground speed of 550 mph.
(d) The marginal cost at a ground speed of 450 mph.

52. Average Cost Function If C is the total cost function then $\overline{C}(x) = \frac{C(x)}{x}$ is defined as the **average cost function,** that is, the cost per unit produced. Typically, the graph of the average cost function has a U-shape. This is so since we expect higher average costs because of plant inefficiency at low output levels and also at high output levels near plant capacity. Suppose a company estimates that the total cost of producing x units of a certain product is given by

$$C(x) = 400 + 0.02x + 0.0001x^2$$

Then the average cost is given by

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{400}{x} + 0.02 + 0.0001x$$

(a) Find the marginal average cost $\overline{C}'(x)$.
(b) Find the marginal average cost at $x = 200$, 300, and 400.
(c) Interpret your results.

53. **Satisfaction and Reward** The relationship between satisfaction S and total reward r has been found to be

$$S(r) = \frac{ar}{g - r}$$

where $g \geq 0$ is the predetermined goal level and $a > 0$ is the perceived justice per unit of reward. Show that the instantaneous rate of change of satisfaction with respect to reward is inversely proportional to the square of the difference between the personal goal of the individual and the amount of reward received.

54. Prove Formula (3).

Hint: If $n < 0$, then $-n > 0$. Now use the fact that

$$\frac{d}{dx} x^n = \frac{d}{dx} \frac{1}{x^{-n}}$$

and use the quotient formula.

4.4 The Power Rule

OBJECTIVES   **1** Find derivatives using the Power Rule
                 2 Find derivatives using the Power Rule and other formulas

When a function is of the form $y = [g(x)]^n$, n an integer, the formula used to find the derivative y' is called the *Power Rule*. Let's see if we can guess this formula by finding the derivative of $y = [g(x)]^n$ when $n = 2$, $n = 3$, and $n = 4$.

If $n = 2$,

$$\frac{d}{dx} [g(x)]^2 = \frac{d}{dx} [g(x)g(x)] = g'(x)g(x) + g(x)g'(x) = 2g(x)g'(x)$$

<div align="center">Product formula</div>

If $n = 3$,

$$\frac{d}{dx} [g(x)]^3 = \frac{d}{dx} \{[g(x)]^2 g(x)\} = [g(x)]^2 g'(x) + g(x)\left\{\frac{d}{dx} [g(x)]^2\right\}$$

$$= [g(x)]^2 g'(x) + g(x)[2g(x)g'(x)] = 3[g(x)]^2 g'(x)$$

If $n = 4$,

$$\frac{d}{dx} [g(x)]^4 = \frac{d}{dx} \{[g(x)]^3 g(x)\} = [g(x)]^3 g'(x) + g(x)\left\{\frac{d}{dx} [g(x)]^3\right\}$$

$$= [g(x)]^3 g'(x) + g(x)\{3[g(x)]^2 g'(x)\} = 4[g(x)]^3 g'(x)$$

Let's summarize what we've found:

$$\frac{d}{dx} [g(x)]^2 = 2g(x)g'(x)$$

$$\frac{d}{dx} [g(x)]^3 = 3[g(x)]^2 g'(x)$$

$$\frac{d}{dx} [g(x)]^4 = 4[g(x)]^3 g'(x)$$

These results suggest the following formula:

> **The Power Rule**
>
> If g is a differentiable function and n is any integer, then
>
> $$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x) \qquad (1)$$

Note the similarity between the Power Rule and the formula for the derivative of a power function:

$$\frac{d}{dx}x^n = nx^{n-1}$$

The main difference between these formulas is the factor $g'(x)$. Be sure to remember to include $g'(x)$ when using formula (1).

1   **EXAMPLE 1**   **Using the Power Rule to Find a Derivative**

Find the derivative of the function: $f(x) = (x^2 + 1)^3$

SOLUTION   We could, of course, expand the right-hand side and proceed according to techniques discussed earlier. However, the usefulness of the Power Rule is that it enables us to find derivatives of functions like this without resorting to tedious (and sometimes impossible) computation.

The function $f(x) = (x^2 + 1)^3$ is the function $g(x) = x^2 + 1$ raised to the power 3. Using the Power Rule,

$$\frac{d}{dx}f(x) = \frac{d}{dx}(x^2 + 1)^3 = 3(x^2 + 1)^2 \frac{d}{dx}(x^2 + 1)$$

$$\uparrow$$
Use the Power Rule

$$= 3(x^2 + 1)^2(2x) = 6x(x^2 + 1)^2 \qquad \blacktriangleright$$

NOW WORK PROBLEM 1.

EXAMPLE 2   **Using the Power Rule**

Find $f'(x)$

(a) $f(x) = \dfrac{1}{(x^3 + 4)^5}$ **(b)** $f(x) = \dfrac{1}{(x^2 + 4)^3}$

SOLUTION   **(a)** We write $f(x)$ as $f(x) = (x^3 + 4)^{-5}$. Then we use the Power Rule:

$$f'(x) = \frac{d}{dx}(x^3 + 4)^{-5} = -5(x^3 + 4)^{-6}\frac{d}{dx}(x^3 + 4)$$

$$\uparrow$$
Use the Power Rule

$$= -5(x^3 + 4)^{-6}(3x^2) = \frac{-15x^2}{(x^3 + 4)^6}$$

(b) $\dfrac{d}{dx} \dfrac{1}{(x^2 + 4)^3} = \dfrac{d}{dx}(x^2 + 4)^{-3} = \underset{\uparrow}{-3(x^2 + 4)^{-4}} \dfrac{d}{dx}(x^2 + 4)$

<div align="center">Use the Power Rule</div>

$$= -3(x^2 + 4)^{-4} \cdot 2x = \dfrac{-6x}{(x^2 + 4)^4} \qquad \blacktriangleright$$

Often, we must use at least one other formula along with the Power Rule to differentiate a function. Here are two examples.

2 | **EXAMPLE 3** | **Using the Power Rule with Other Formulas**

Find the derivative of the function: $f(x) = x(x^2 + 1)^3$

SOLUTION   The function f is the product of x and $(x^2 + 1)^3$. We begin by using the formula for the derivative of a product. That is,

$$f'(x) = x\dfrac{d}{dx}(x^2 + 1)^3 + (x^2 + 1)^3 \dfrac{d}{dx}x \qquad \text{Product formula.}$$

We continue by using the Power Rule:

$$f'(x) = x\left[3(x^2 + 1)^2 \dfrac{d}{dx}(x^2 + 1)\right] + (x^2 + 1)^3 \cdot 1 \qquad \text{Power Rule; } \dfrac{d}{dx}x = 1.$$

$$= (x)(3)(x^2 + 1)^2(2x) + (x^2 + 1)^3 \qquad \text{Differentiate.}$$

$$= (x^2 + 1)^2(6x^2) + (x^2 + 1)^2(x^2 + 1) \qquad \text{Simplify.}$$

$$= (x^2 + 1)^2[6x^2 + (x^2 + 1)] \qquad \text{Factor.}$$

$$= (x^2 + 1)^2(7x^2 + 1) \qquad \text{Simplify.} \qquad \blacktriangleright$$

NOW WORK PROBLEM 7.

EXAMPLE 4 | **Using the Power Rule with Other Formulas**

Find the derivative of the function: $f(x) = \left(\dfrac{3x + 2}{4x^2 - 5}\right)^5$

SOLUTION   Here, f is the quotient $\dfrac{3x + 2}{4x^2 - 5}$ raised to the power 5. We begin by using the Power Rule and then use the formula for the derivative of a quotient:

$$f'(x) = (5)\left(\dfrac{3x + 2}{4x^2 - 5}\right)^4\left[\dfrac{d}{dx}\left(\dfrac{3x + 2}{4x^2 - 5}\right)\right] \qquad \text{Power Rule.}$$

$$= (5)\left(\dfrac{3x + 2}{4x^2 - 5}\right)^4\left[\dfrac{(4x^2 - 5)\dfrac{d}{dx}(3x + 2) - (3x + 2)\dfrac{d}{dx}(4x^2 - 5)}{(4x^2 - 5)^2}\right] \qquad \begin{array}{l}\text{Quotient}\\\text{Formula.}\end{array}$$

$$= (5)\left(\dfrac{3x + 2}{4x^2 - 5}\right)^4\left[\dfrac{(4x^2 - 5)(3) - (3x + 2)(8x)}{(4x^2 - 5)^2}\right] \qquad \text{Differentiate.}$$

$$= \dfrac{5(3x + 2)^4(-12x^2 - 16x - 15)}{(4x^2 - 5)^6} \qquad \text{Simplify.} \qquad \blacktriangleright$$

NOW WORK PROBLEM 19.

Application

The revenue $R = R(x)$ derived from selling x units of a product at a price p per unit is

$$R = xp$$

where $p = d(x)$ is the demand equation, namely, the equation that gives the price p when the number x of units demanded is known. The marginal revenue is then the derivative of R with respect to x:

$$R'(x) = \frac{d}{dx}(xp) = p + x\frac{dp}{dx} \qquad (2)$$

It is sometimes easier to find the marginal revenue by using formula (2) instead of differentiating the revenue function directly.

EXAMPLE 5   **Finding the Marginal Revenue**

Suppose the price p per ton when x tons of polished aluminum are demanded is given by the equation

$$p = \frac{2000}{x + 20} - 10 \quad 0 < x < 90$$

Find:

(a) The rate of change of price with respect to x.
(b) The revenue function.
(c) The marginal revenue.
(d) The marginal revenue at $x = 20$ and $x = 80$.

SOLUTION   **(a)** The rate of change of price with respect to x is the derivative $\dfrac{dp}{dx}$.

$$\frac{dp}{dx} = \frac{d}{dx}\left(\frac{2000}{x + 20} - 10\right) = \frac{d}{dx}\,2000(x + 20)^{-1} - \frac{d}{dx}\,10$$

$$\underset{\underset{\text{Power Rule}}{\uparrow}}{=} -2000\,(x + 20)^{-2}\frac{d}{dx}(x + 20) - 0 = \frac{-2000}{(x + 20)^2}$$

(b) The revenue function is

$$R(x) = xp = x\left[\frac{2000}{x + 20} - 10\right] = \frac{2000x}{x + 20} - 10x$$

(c) Using formula (2), the marginal revenue is

$$R'(x) = p + x\frac{dp}{dx} \qquad \text{Formula (2)}$$

$$= \left[\frac{2000}{x + 20} - 10\right] + x\left(\frac{-2000}{(x + 20)^2}\right) \qquad \text{Use result from (a).}$$

$$= \frac{2000}{x + 20} - 10 - \frac{2000x}{(x + 20)^2} \qquad \text{Simplify.}$$

(d) Using the result from part (c), we find

$$R'(20) = \frac{2000}{40} - 10 - \frac{2000(20)}{(40)^2} = \$15/\text{ton}$$

$$R'(80) = \frac{2000}{100} - 10 - \frac{2000(80)}{(100)^2} = -\$6/\text{ton}$$

▶

NOW WORK PROBLEM 31.

EXERCISE 4.4 Answers to Odd-Numbered Problems Begin on Page AN-XX.

In Problems 1–28, find using the derivative of each function Power Rule.

1. $f(x) = (2x - 3)^4$
2. $f(x) = (5x + 4)^3$
3. $f(x) = (x^2 + 4)^3$
4. $f(x) = (x^2 - 1)^4$

5. $f(x) = (3x^2 + 4)^2$
6. $f(x) = (9x^2 + 1)^2$
7. $f(x) = x(x + 1)^3$
8. $f(x) = x(x - 4)^2$

9. $f(x) = 4x^2(2x + 1)^4$
10. $f(x) = 3x^2(x^2 + 1)^3$
11. $f(x) = [x(x - 1)]^3$
12. $f(x) = [x(x + 4)]^4$

13. $f(x) = (3x - 1)^{-2}$
14. $f(x) = (2x + 3)^{-3}$
15. $f(x) = \dfrac{4}{x^2 + 4}$
16. $f(x) = \dfrac{3}{x^2 - 9}$

17. $f(x) = \dfrac{-4}{(x^2 - 9)^3}$
18. $f(x) = \dfrac{-2}{(x^2 + 2)^4}$
19. $f(x) = \left(\dfrac{x}{x + 1}\right)^3$
20. $f(x) = \left(\dfrac{x^2}{x + 5}\right)^4$

21. $f(x) = \dfrac{(2x + 1)^4}{3x^2}$
22. $f(x) = \dfrac{(3x + 4)^3}{9x}$
23. $f(x) = \dfrac{(x^2 + 1)^3}{x}$
24. $f(x) = \dfrac{(3x^2 + 4)^2}{2x}$

25. $f(x) = \left(x + \dfrac{1}{x}\right)^3$
26. $f(x) = \left(x - \dfrac{1}{x}\right)^4$
27. $f(x) = \dfrac{3x^2}{(x^2 + 1)^2}$
28. $g(x) = \dfrac{2x^3}{(x^2 - 4)^2}$

---

29. Car Depreciation A certain car depreciates according to the formula

$$V(t) = \frac{29000}{1 + 0.4t + 0.1t^2}$$

where V is the value of the car at time t in years. The derivative $V'(t)$ gives the rate at which the car depreciates. Find the rate at which the car is depreciating:

(a) 1 year after purchase. (b) 2 years after purchase.
(c) 3 years after purchase. (d) 4 years after purchase.

30. Demand Function The demand function for a certain calculator is given by

$$d(x) = \frac{100}{0.02x^2 + 1} \qquad 0 \le x \le 20$$

where x (measured in units of a thousand) is the quantity demanded per week and $d(x)$ is the unit price in dollars.

(a) Find $d'(x)$.
(b) Find $d'(10)$, $d'(15)$, and $d'(20)$ and interpret your results.
(c) Find the revenue function.
(d) Find the marginal revenue.

31. Demand Equation The price p per pound when x pounds of a certain commodity are demanded is

$$p = \frac{10{,}000}{5x + 100} - 5 \qquad 0 < x < 90$$

Find:

(a) The rate of change of price with respect to x.
(b) The revenue function.
(c) The marginal revenue.
(d) The marginal revenue at $x = 10$ and at $x = 40$.
(e) Interpret the answer to (d).

32. Revenue Function The weekly revenue R in dollars resulting from the sale of x typewriters is

$$R(x) = \frac{100x^5}{(x^2 + 1)^2} \qquad 0 \le x \le 100$$

Find:

(a) The marginal revenue.
(b) The marginal revenue at $x = 40$.
(c) The marginal revenue at $x = 60$.
(d) Interpret the answers to (b) and (c).

33. **Amino Acids** A protein disintegrates into amino acids according to the formula

$$M = \frac{28}{t + 2}$$

where M, the mass of the protein, is measured in grams and t is time measured in hours.

(a) Find the average rate of change in mass from $t = 0$ to $t = 2$ hours.
(b) Find $M'(0)$.
(c) Interpret the answers to (a) and (b).

4.5   The Derivatives of the Exponential and Logarithmic Functions; the Chain Rule

PREPARING FOR THIS SECTION   *Before getting started, review the following:*

> The Exponential Function (Section 2.3, pp. xx–xx)
> Change of Base Formula (Section 2.5, pp. xx–xx)
> The Logarithmic Function (Section 2.4, pp. xx–xx)

OBJECTIVES
1. Find the derivative of functions involving e^x
2. Find a derivative using the Chain Rule
3. Find the derivative of functions involving $\ln x$
4. Find the derivative of functions including $\log a^x$ and a^x

Up to now, our discussion of finding derivatives has been focused on polynomial functions (derivative of a sum or difference), rational functions (derivative of a quotient), and these functions raised to an integer power (the Power Rule). In this section we present the formulas for finding the derivative of the exponential function and the logarithm function.

The Derivative of $f(x) = e^x$

We begin the discussion of the derivative of $f(x) = e^x$ by considering the function

$$f(x) = a^x \qquad a > 0, \qquad a \neq 1$$

To find the derivative of $f(x) = a^x$, we use the formula for finding the derivative of f at x using the difference quotient, namely:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

For $f(x) = a^x$, we have

$$f'(x) = \frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \left[a^x \left(\frac{a^h - 1}{h} \right) \right] = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Factor out a^x

Suppose we seek $f'(0)$. Assuming the limit on the right exists and equals some number, it follows (since $a^0 = 1$) that the derivative of $f(x) = a^x$ at 0 is

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$$

This limit equals the slope of the tangent line to the graph of $f(x) = a^x$ at the point $(0, 1)$. The value of this limit depends upon the choice of a. Observe in Figure 10 that the slope of the tangent line to the graph of $f(x) = 2^x$ at $(0, 1)$ is less than 1, and that the slope of the tangent line to the graph of $f(x) = 3^x$ at $(0, 1)$ is greater than 1.

FIGURE 10
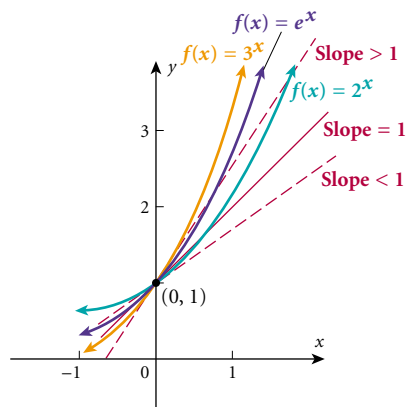
From this, we conclude there is a number a, $2 < a < 3$, for which the slope of the tangent line to the graph of $f(x) = a^x$ at $(0, 1)$ is exactly 1. The function $f(x) = a^x$ for which $f'(0) = 1$ is the function $f(x) = e^x$, whose base is the number e, we introduced in Chapter 2. A further property of the number e is that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Using this result, we find that

$$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x(1) = e^x$$

Derivative of $f(x) = e^x$

The derivative of the exponential function $f(x) = e^x$ is e^x. That is,

$$\boxed{\frac{d}{dx} e^x = e^x} \tag{1}$$

The simple nature of formula (1) is one of the reasons the exponential function $f(x) = e^x$ appears so frequently in applications.

1   EXAMPLE 1   Finding the Derivative of Functions Involving e^x

Find the derivative of each function:

(a) $f(x) = x^2 + e^x$ **(b)** $f(x) = xe^x$ **(c)** $f(x) = \dfrac{e^x}{x}$

SOLUTION   (a) Use the sum formula. Then

$$f'(x) = \frac{d}{dx}(x^2 + e^x) = \frac{d}{dx} x^2 + \frac{d}{dx} e^x = 2x + e^x$$

(b) Use the formula for the derivative of a product: Then

$$f'(x) = \frac{d}{dx}(xe^x) = x\frac{d}{dx}e^x + e^x\frac{d}{dx}x = xe^x + e^x(1) = e^x(x + 1)$$

(c) Use the formula for the derivative of a quotient. Then

$$f'(x) = \underset{\underset{\text{Quotient formula}}{\uparrow}}{\frac{d}{dx}\frac{e^x}{x}} = \underset{\underset{\text{Differentiate}}{\uparrow}}{\frac{x\dfrac{d}{dx}e^x - e^x\dfrac{d}{dx}x}{x^2}} = \underset{\underset{\text{Factor}}{\uparrow}}{\frac{xe^x - e^x \cdot 1}{x^2}} = \frac{(x - 1)e^x}{x^2}$$

▶

NOW WORK PROBLEM 3.

To find the derivative of other functions involving e^x and to find the derivative of the logarithmic function requires a formula called *the Chain Rule.*

The Chain Rule

The Power Rule is a special case of a more general, and more powerful formula, called the *Chain Rule.* This formula enables us to find the derivative of a *composite function.*
    Consider the function $y = (2x + 3)^2$. If we write $y = f(u) = u^2$ and $u = g(x) = 2x + 3$, then, by a substitution process, we can obtain the original function, namely, $y = f(u) = f(g(x)) = (2x + 3)^2$. This process is called **composition** and the function $y = (2x + 3)^2$ is called the **composite function** of $y = f(u) = u^2$ and $u = g(x) = 2x + 3$.

EXAMPLE 2   **Finding a Composite Function**

Find the composite function of

$$y = f(u) = \sqrt{u} \qquad \text{and} \qquad u = g(x) = x^2 + 4$$

SOLUTION   The composite function is

$$y = f(u) = \sqrt{u} = \sqrt{g(x)} = \sqrt{x^2 + 4}$$

▶

The Chain Rule will require that we find the components of a composite function.

EXAMPLE 3   **Decomposing a Composite Function**

(a) If $y = (5x + 1)^3$, then $y = u^3$ and $u = 5x + 1$.
(b) If $y = (x^2 + 1)^{-2}$, then $y = u^{-2}$ and $u = x^2 + 1$.

(c) If $y = \dfrac{5}{(2x + 3)^3}$, then $y = \dfrac{5}{u^3}$ and $u = 2x + 3$.

▶

In the above examples, the composite function was "broken up" into simpler functions. The Chain Rule provides a way to use these simpler functions to find the derivative of the composite function.

> **The Chain Rule**
>
> Suppose f and g are differentiable functions. If $y = f(u)$ and $u = g(x)$, then, after substitution, y is a function of x. The Chain Rule states that the derivative of y with

respect to x is the derivative of y with respect to u times the derivative of u with respect to x. That is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ (2)

2 **EXAMPLE 4** **Finding a Derivative Using the Chain Rule**

Use the Chain Rule to find the derivative of: $y = (5x + 1)^3$

SOLUTION We break up y into simpler functions: If $y = (5x + 1)^3$, then $y = u^3$ and $u = 5x + 1$. To find $\frac{dy}{dx}$, we first find $\frac{dy}{du}$ and $\frac{du}{dx}$:

$$\frac{dy}{du} = \frac{d}{du}(u^3) = 3u^3 \quad \text{and} \quad \frac{du}{dx} = \frac{d}{dx}(5x + 1) = 5$$

By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot 5 = 15u^2 = \underset{\underset{u = 5x + 1}{\uparrow}}{15(5x + 1)^2}$$ ▶

Notice that when using the Chain Rule, we must substitute for u in the expression for $\frac{dy}{du}$ so that we obtain a function of x.

NOW WORK PROBLEM 9.

EXAMPLE 5 **Finding a Derivative Using The Chain Rule**

Find the derivative of: $y = e^{x^2}$

SOLUTION We break up y into simpler functions. If $y = e^{x^2}$, then $y = e^u$ and $u = x^2$. Now use the Chain Rule to find $y' = \frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot 2x = \underset{\underset{u = x^2}{\uparrow}}{2xe^{x^2}}$$ ▶

The result of Example 5 can be generalized.

Derivative of $y = e^{g(x)}$

The derivative of a composite function $y = e^{g(x)}$, where g is a differentiable function, is

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} \frac{d}{dx} g(x)$$ (3)

The proof is left as an exercise. See Problem 67.

EXAMPLE 6   **Finding the Derivative of Functions of the Form $e^{g(x)}$**

Find the derivative of each function:

(a) $f(x) = 4e^{5x}$ **(b)** $f(x) = e^{x^2+1}$

SOLUTION   **(a)** Use Formula (3) with $g(x) = 5x$. Then

$$f'(x) = \frac{d}{dx}(4e^{5x}) = 4\frac{d}{dx}e^{5x} = 4 \cdot e^{5x}\frac{d}{dx}(5x) = 4e^{5x}(5) = 20e^{5x}$$

$$\uparrow$$
$$\text{Formula (3)}$$

(b) Use Formula (3) with $g(x) = x^2 + 1$. Then

$$f'(x) = \frac{d}{dx}e^{x^2+1} = e^{x^2+1} = \frac{d}{dx}(x^2 + 1) = e^{x^2+1}(2x) = 2xe^{x^2+1}$$

$$\uparrow$$
$$\text{Formula (3)}$$

NOW WORK PROBLEM 23.

EXAMPLE 7   **Finding the Derivative of Functions Involving e^x**

Find the derivative of each function:

(a) $f(x) = xe^{x^2}$ **(b)** $f(x) = \dfrac{x}{e^x}$ **(c)** $f(x) = (e^x)^2$

SOLUTION   **(a)** The function f is the product of two simpler functions, so we start with the product formula.

$$f'(x) = \frac{d}{dx}(xe^{x^2}) = x\frac{d}{dx}e^{x^2} + e^{x^2}\frac{d}{dx}x = x \cdot e^{x^2} \cdot \frac{d}{dx}x^2 + e^{x^2} \cdot 1 = xe^{x^2} \cdot 2x + e^{x^2} = e^{x^2}(2x^2 + 1)$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow$$
$$\text{Product Formula} \qquad \text{Formula (3);}\ \frac{d}{dx}x = 1 \qquad\qquad \text{Factor}\ \blacktriangleright$$

(b) We could use the quotient formula, but it is easier to rewrite f in the form $f(x) = xe^{-x}$ and use the product formula

$$f'(x) = \frac{d}{dx}xe^{-x} = x\frac{d}{dx}e^{-x} + e^{-x}\frac{d}{dx}x = x \cdot e^{-x}\frac{d}{dx}(-x) + e^{-x} \cdot 1 = xe^{-x}(-1) + e^{-x} = e^{-x}(1 - x)$$

$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad\qquad\qquad\qquad \uparrow$$
$$\text{Product Formula} \qquad\qquad \text{Formula (3)} \qquad\qquad\qquad\qquad \text{Factor}\ \blacktriangleright$$

(c) Here the function is e^x raised to the power 2. We first apply a Law of Exponents and write $f(x) = (e^x)^2 = e^{2x}$.
Then we can use Formula (3).

$$f'(x) = \frac{d}{dx}e^{2x} = e^{2x}\frac{d}{dx}(2x) = e^{2x} \cdot 2 = 2e^{2x} \qquad \blacktriangleright$$

CAUTION: Notice the difference between e^{x^2} and $(e^x)^2$. In the first, one e is raised to the power x^2; in the second, the parentheses tell us e^x is raised to the power 2. $\blacktriangleright$

NOW WORK PROBLEM 29.

The Derivative of $f(x) = \ln x$

To find the derivative of $f(x) = \ln x$, we observe that if $y = \ln x$, then $e^y = x$. That is,

$$e^{\ln x} = x$$

If we differentiate both sides with respect to x, we obtain

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x$$

$$e^{\ln x} \frac{d}{dx} \ln x = 1 \qquad \text{Apply Formula (3) on the left.}$$

$$\frac{d}{dx} \ln x = \frac{1}{e^{\ln x}} \qquad \text{Solve for } \frac{d}{dx} \ln x.$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad e^{\ln x} = x.$$

We have proved the following formula:

> **Derivative of $f(x) = \ln x$**
>
> If $f(x) = \ln x$, then $f'(x) = \dfrac{1}{x}$. That is
>
> $$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$
> (4)

---

3   **EXAMPLE 8**   **Finding the Derivative of Functions Involving ln x**

Find the derivative of each function.

(a) $f(x) = x^2 + \ln x$ **(b)** $f(x) = x \ln x$

SOLUTION   **(a)** Use the sum formula. Then $f'(x) = \dfrac{d}{dx}(x^2 + \ln x) = \dfrac{d}{dx} x^2 + \dfrac{d}{dx} \ln x = 2x + \dfrac{1}{x}$

(b) Use the product formula. Then $f'(x) = \dfrac{d}{dx}(x \ln x) = x\dfrac{d}{dx} \ln x + \ln x \dfrac{d}{dx} x$

$$= (x)\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x \qquad \blacktriangleright$$

NOW WORK PROBLEM 35.

To differentiate the natural logarithm of a function $g(x)$, namely, $\ln g(x)$, use the following formula.

> **Derivative of $\ln g(x)$**
>
> The formula for finding the derivative of the composite function $f(x) = \ln g(x)$, where g is a differentiable function, is
>
> $$\boxed{\frac{d}{dx} \ln g(x) = \frac{\dfrac{d}{dx} g(x)}{g(x)}}$$
> (5)

The proof uses the Chain Rule and is left as an exercise. See Problem 68.

EXAMPLE 9   **Finding the Derivative of Functions Involving In x**

Finding the derivative of each function.

(a) $f(x) = \ln(x^2 + 1)$ **(b)** $f(x) = (\ln x)^2$

SOLUTION   **(a)** The function $f(x) = \ln(x^2 + 1)$ is of the form $f(x) = \ln g(x)$. We use Formula (4) with $g(x) = x^2 + 1$. Then,

$$f'(x) = \frac{d}{dx} \ln(x^2 + 1) = \frac{\dfrac{d}{dx}(x^2 + 1)}{x^2 + 1} = \frac{2x}{x^2 + 1}$$

(b) The function $f(x)$ is $\ln x$ raised to the power 2. We use the Power Rule. Then

$$f'(x) = 2 \ln x \left(\frac{d}{dx} \ln x \right) = \frac{2 \ln x}{x}$$ ▶

NOW WORK PROBLEM 45.

The Derivative of $f(x) = \log_a x$ and $f(x) = a^x$

Find the derivative of functions including log a^x and a^x **4**   To find the derivative of the logarithm function $f(x) = \log_a x$ for any base a, we use the Change-of-Base Formula: Then

$$f(x) = \log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}$$

Since $\ln a$ is a constant, we have

$$f'(x) = \frac{d}{dx} \log_a x = \frac{d}{dx} \frac{\ln x}{\ln a} = \frac{1}{\ln a} \frac{d}{dx} \ln x = \frac{1}{\ln a} \frac{1}{x} = \frac{1}{x \ln a}$$

We have the formula

> **Derivative of $f(x) = \log_a x$**
>
> If $f(x) = \log_a x$, then $f'(x) = \dfrac{1}{x \ln a}$. That is,
>
> $$\boxed{\frac{d}{dx} \log_a x = \frac{1}{x \ln a}}$$ (6)

EXAMPLE 10   **Finding the Derivative of $\log_2 x$**

Find the derivative of: $f(x) = \log_2 x$

SOLUTION   Using Formula (6), we have

$$f'(x) = \frac{d}{dx} \log_2 x = \frac{1}{x \ln 2}$$ ▶

NOW WORK PROBLEM 47.

To find the derivative of $f(x) = a^x$, where $a > 0$, $a \neq 1$, is any real constant, we use the definition of a logarithm and the change-of-base formula. If $y = a^x$, we have

$$x = \log_a y$$

$$x = \frac{\ln y}{\ln a}$$

$$x = \frac{\ln a^x}{\ln a}$$

Now, we differentiate both sides with respect to x:

$$\frac{d}{dx} x = \frac{d}{dx} \frac{\ln a^x}{\ln a}$$

$$1 = \frac{1}{\ln a} \frac{d}{dx} \ln a^x \qquad \text{$\ln a$ is a constant.}$$

$$1 = \frac{1}{\ln a} \frac{\frac{d}{dx} a^x}{a^x} \qquad \text{Use Formula (5).}$$

$$1 = \frac{\frac{d}{dx} a^x}{a^x \ln a} \qquad \text{Simplify.}$$

$$\frac{d}{dx} a^x = a^x \ln a \qquad \text{Solve for } \frac{d}{dx} a^x.$$

We have derived the formula:

> **Derivative of $f(x) = a^x$**
>
> The derivative of $f(x) = a^x$, $a > 0$, $a \neq 1$, is $f'(x) = a^x \ln a$. That is,
>
> $$\boxed{\frac{d}{dx} a^x = a^x \ln a} \qquad (7)$$

EXAMPLE 11   **Finding the Derivative of 2^x**

Find the derivative of: $f(x) = 2^x$

SOLUTION   Using Formula (7), we have

$$f'(x) = \frac{d}{dx} 2^x = 2^x \ln 2$$

▶

NOW WORK PROBLEM 51.

EXAMPLE 12   **Maximizing Profit**

At a Notre Dame football weekend, the demand for game-day t-shirts is given by

$$p = 30 - 5 \ln\left(\frac{x}{100} + 1\right)$$

where p is the price of the shirt in dollars and x is the number of shirts demanded.

(a) At what price can 1000 t-shirts be sold?

(b) At what price can 5000 t-shirts be sold?

(c) Find the marginal demand for 1000 t-shirts and interpret the answer.

(d) Find the marginal demand for 5000 t-shirts and interpret the answer.

(e) Find the revenue function $R = R(x)$.

(f) Find the marginal revenue from selling 1000 t-shirts and interpret the answer.

(g) Find the marginal revenue from selling 5000 t-shirts and interpret the answer.

(h) If each t-shirt costs $4, find the profit function $P = P(x)$.

(i) What is the profit if 1000 t-shirts are sold?

(j) What is the profit if 5000 t-shirts are sold?

(k) Use the TABLE feature of a graphing utility to find the quantity x (to the nearest hundred) that maximizes profit.

(l) What price should be charged for a t-shirt to maximize profit?

SOLUTION   (a) For $x = 1000$, the price p is

$$p = 30 - 5 \ln\left(\frac{1000}{100} + 1\right) = \$18.01$$

(b) For $x = 5000$, the price p is

$$p = 30 - 5 \ln\left(\frac{5000}{100} + 1\right) = \$10.34$$

(c) The marginal demand for x shirts is

$$p'(x) = \frac{dp}{dx} = \frac{d}{dx}\left[30 - 5\ln\left(\frac{x}{100} + 1\right)\right] = -5\,\frac{\frac{1}{100}}{\frac{x}{100} + 1} = \frac{-5}{x + 100}$$

For $x = 1000$,

$$p'(1000) = \frac{-5}{1000 + 100} = -\$0.0045$$

This means that another t-shirt will be demanded if the price is reduced by $0.0045.

(d) For $x = 5000$,

$$p'(5000) = \frac{-5}{5000 + 100} = -\$0.00098$$

This means that another t-shirt will be demanded if the price is reduced by $0.00098.

(e) The revenue function $R = R(x)$ is

$$R = xp = x\left[30 - 5\ln\left(\frac{x}{100} + 1\right)\right]$$

(f) The marginal revenue is

$$R'(x) = \frac{d}{dx}\left[xp(x)\right] = xp'(x) + p(x)$$

$$= x \cdot \frac{-5}{x + 100} + 30 - 5\ln\left(\frac{x}{100} + 1\right)$$

$$= \frac{-5x}{x + 100} + 30 - 5\ln\left(\frac{x}{100} + 1\right)$$

If $x = 1000$,

$$R'(1000) = \frac{-5000}{5100} + 30 - 5 \ln(11) = \$17.03$$

The revenue received for selling the 1001st t-shirt is $17.03

(g) If $x = 5000$

$$R'(5000) = \frac{-25000}{5100} + 30 - 5 \ln(51) = \$5.44$$

The revenue received for selling the 5001st t-shirt is $5.44.

(h) The cost C for x t-shirts is $C = -4x$, so the product function P is

$$P = P(x) = R(x) - C(x) = x\left[30 - 5 \ln\left(\frac{x}{100} + 1\right)\right] - 4x$$

$$= 26x - 5x \ln\left(\frac{x}{100} + 1\right)$$

(i) If $x = 1000$, the profit is

$$P(1000) = 26(1000) - 5(1000) \ln\left(\frac{1000}{100} + 1\right) = \$14,010.52$$

(j) If $x = 5000$, the profit is

FIGURE 11

$$P(5000) = 26(5000) - 5(5000) \ln\left(\frac{5000}{100} + 1\right) = \$31,704.36$$

X	Y1
6500	32836
6600	32845
6700	**32846**
6800	32840
6900	32827
7000	32806
7100	32778
X=6700	

(k) See Figure 11. For $x = 6700$ t-shirts, the profit is largest. ($32,846)

(l) If $x = 6700$, the price p is

$$p(6700) = 30 - 5 \ln\left(\frac{6700}{100} + 1\right) = \$8.90$$

SUMMARY

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x) \qquad \frac{d}{dx} a^x = a^x \ln a$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad \frac{d}{dx} \ln g(x) = \frac{g'(x)}{g(x)} \qquad \frac{d}{dx} \log_a x = \frac{1}{x \log_a x}$$

EXERCISE 4.5   Answers to Odd-Numbered Problems Begin on Page AN-XX.

In Problems 1–8, find the derivative of each function.

1. $f(x) = x^3 - e^x$

2. $f(x) = 2e^x - x$

3. $f(x) = x^2 e^x$

4. $f(x) = x^3 e^x$

5. $f(x) = \dfrac{e^x}{x^2}$

6. $f(x) = \dfrac{5x}{e^x}$

7. $f(x) = \dfrac{4x^2}{e^x}$

8. $f(x) = \dfrac{3x^3}{e^x}$

In Problems 9–20, find $\dfrac{dy}{dx}$ using the Chain Rule.

9. $y = u^5, \quad u = x^3 + 1$

10. $y = u^3, \quad u = 2x + 5$

11. $y = \dfrac{u}{u+1}, \quad u = x^2 + 1$

12. $y = \dfrac{u-1}{u}, \quad u = x^2 - 1$

13. $y = (u+1)^2, \quad u = \dfrac{1}{x}$

14. $y = (u^2 - 1)^3, \quad u = \dfrac{1}{x+2}$

15. $y = (u^3 - 1)^5, \quad u = x^{-2}$

16. $y = (u^2 + 4)^4, \quad u = x^{-2}$

17. $y = u^3, \quad u = e^x$

18. $y = 4u^2, \quad u = e^x$

19. $y = e^u, \quad u = x^3$

20. $y = e^u, \quad u = \dfrac{1}{x}$

21. Find the derivative y' of $y = (x^3 + 1)^2$ by:

   (a) Using the Chain Rule.
   (b) Using the Power Rule.
   (c) Expanding and then differentiating.

22. Follow the directions in Problem 21 for the function $y = (x^2 - 2)^3$.

In Problems 23–54, find the derivative of each function.

23. $f(x) = e^{5x}$

24. $f(x) = e^{-3x}$

25. $f(x) = 8e^{-x^2}$

26. $f(x) = -e^{3x^2}$

27. $f(x) = x^2 e^{x^2}$

28. $f(x) = x^3 e^{x^2}$

29. $f(x) = 5(e^x)^3$

30. $f(x) = 4(e^x)^4$

31. $f(x) = \dfrac{x^2}{e^x}$

32. $f(x) = \dfrac{8x}{e^{-x}}$

33. $f(x) = \dfrac{(e^x)^2}{x}$

34. $f(x) = \dfrac{e^{-2x}}{x^2}$

35. $f(x) = x^2 - 3\ln x$

36. $f(x) = 5\ln x - 2x$

37. $f(x) = x^2 \ln x$

38. $f(x) = x^3 \ln x$

39. $f(x) = 3\ln(5x)$

40. $f(x) = -2\ln(3x)$

41. $f(x) = x\ln(x^2 + 1)$

42. $f(x) = x^2 \ln(x^2 + 1)$

43. $f(x) = x + 8\ln(3x)$

44. $f(x) = 3\ln(2x) - 5x$

45. $f(x) = 8(\ln x)^3$

46. $f(x) = 2(\ln x)^4$

47. $f(x) = \log_3 x$

48. $f(x) = x + \log_4 x$

49. $f(x) = x^2 \log_2 x$

50. $f(x) = x^3 \log_3 x$

51. $f(x) = 3^x$

52. $f(x) = x + 4^x$

53. $f(x) = x^2 \cdot 2^x$

54. $f(x) = x^3 \cdot 3^x$

In Problems 55–62, find an equation of the tangent line to the graph of each function at the given point.

55. $f(x) = e^{70}$ at $(0,1)$

56. $f(x) = e^{4x}$ at $(0,1)$

57. $f(x) = \ln x$ at $(1,0)$

58. $f(x) = \ln(3x)$ at $(1,0)$

59. $f(x) = e^{3x-2}$ at $\left(\frac{2}{3},1\right)$

60. $f(x) = e^{-x^2}$ at $\left(1,\frac{1}{e}\right)$

61. $f(x) = x\ln x$ at $(1,0)$

62. $f(x) = \ln x^2$ at $(1,0)$

63. Find the equation of the tangent line to $y = e^x$ that is parallel to the line $y = x$.

64. Find the equation of the tangent line to $y = e^{3x}$ that is perpendicular to the line $y = -\frac{1}{2}x$.

65. Weber–Fechner Law When a certain drug is administered, the reaction R to the dose x is given by the **Weber–Fechner law:**

$$R = 5.5\ln x + 10$$

(a) Find the reaction rate for a dose of 5 units.
(b) Find the reaction rate for a dose of 10 units.
(c) Interpret the results of parts (a) and (b).

66. Marginal Cost The cost (in dollars) of producing x units (measured in thousands) of a certain product is found to be

$$C(x) = 20 + \ln(x+1)$$

Find the marginal cost.

67. Atmospheric Pressure The atmospheric pressure at a height of x meters above sea level is $P(x) = 10^4 e^{-0.00012x}$ kilograms per square meter. What is the rate of change of the pressure with respect to the height at $x = 500$ meters? At $x = 700$ meters?

68. Revenue Revenue sales analysis of a new toy by Toys Inc. indicates that the relationship between the unit price p and the monthly sales x of its new toy is given by the equation

$$p = 10e^{-0.04x}$$

Find

(a) The revenue function $R = R(x)$.
(b) The marginal revenue R when $x = 200$.