Basic properties of probability

Math 308

Definition: Let S be a sample space. A probability on S is a real valued function P,

$$P : \{\text{Events}\} \to \mathbb{R},$$

satisfying:

1. $P(A) \geq 0$ for any event A.
2. $P(S) = 1$.
3. If $A_1, A_2, \ldots$ are **mutually exclusive events** (m.e.e) then

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i),$$

where by **m.e.e.** we mean $A_i \cap A_j = \emptyset$ when $i \neq j$.

Basic properties of probability:

1. $P(\emptyset) = 0$.
2. Let $A_1, A_2, \ldots A_n$ be **m.e.e.**, then

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i).$$

3. $P(A') = 1 - P(A)$.
4. If $A \subset B$ then $P(A) \leq P(B)$.
   In particular, if B is S, we get $0 \leq P(A) \leq 1$ for any event A.
5. Let $B_1, B_2, \ldots$ be **m.e.e.** such that $S = \cup_{i=1}^{\infty} B_i$, that is, the B_i's form a **partition** of S. Then for any event A,

$$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i).$$

6. It follows from (5) that for any event B, since $S = B \cup B'$ is a **partition** of S, then

$$P(A) = P(A \cap B) + P(A \cap B').$$

In particular, since $A \cap B' = A \setminus B$ (draw the Venn diagram), we have

$$P(A \setminus B) = P(A) - P(A \cap B).$$

7. For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

8. (probability of a finite union)

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i_1<i_2} P(A_{i_1} \cap A_{i_2}) + \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots + (-1)^{n+1} P(\cap_{i=1}^{n} A_i).$$

In particular,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

9. Using the basic properties (and Venn diagrams) you can find formulas for probabilities of other operations on sets. For example, if A and B are events, then the probability that event A occur **or** B occur, but **not both** is

$$P\left((A \cup B) \setminus (A \cap B)\right) = P\left((A \setminus B) \cup (B \setminus A)\right) = P(A) + P(B) - 2P(A \cap B).$$

Note that the last equality follows from property (2) since $(A \setminus B)$ and $(B \setminus A)$ are **m.e.e.** (we also used property (6)).