(t) $\lambda = 0$ is not an eigenvalue of A.

This theorem relates all of the major topics we have studied thus far.

Concept Review

- Eigenvector
- Eigenvalue
- Characteristic equation
- Characteristic polynomial
- Eigenspace
- Equivalence Theorem

Skills

- Find the eigenvalues of a matrix.
- Find bases for the eigenspaces of a matrix.

Exercise Set 5.1

In Exercises 1–2, confirm by multiplication that **x** is an eigenvector of A, and find the corresponding eigenvalue.

1. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Answer:

5

2. $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. Find the characteristic equations of the following matrices:

(a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & -7 \\ 1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answer:

(a) $\lambda^2 - 2\lambda - 3 = 0$

(b) $\lambda^2 - 8\lambda + 16 = 0$

(c) $\lambda^2 - 12 = 0$

(d) $\lambda^2 + 3 = 0$

(e) $\lambda^2 = 0$

(f) $\lambda^2 - 2\lambda + 1 = 0$

4. Find the eigenvalues of the matrices in Exercise 3

5. Find bases for the eigenspaces of the matrices in Exercise 3

Answer:

(a) Basis for eigenspace corresponding to $\lambda = 3$: $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$; basis for eigenspace corresponding to $\lambda = -1$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(b) Basis for eigenspace corresponding to $\lambda = 4$: $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$

(c) Basis for eigenspace corresponding to $\lambda = \sqrt{12}$: $\begin{bmatrix} \frac{3}{\sqrt{12}} \\ 1 \end{bmatrix}$; basis for eigenspace corresponding to $\lambda = -\sqrt{12}$: $\begin{bmatrix} -\frac{3}{\sqrt{12}} \\ 1 \end{bmatrix}$

(d) There are no eigenspaces.

(e) Basis for eigenspace corresponding to $\lambda = 0$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(f) Basis for eigenspace corresponding to $\lambda = 1$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

6. Find the characteristic equations of the following matrices:

(a) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

7. Find the eigenvalues of the matrices in Exercise 6.

Answer:

(a) 1, 2, 3

(b) $-\sqrt{2}, 0, \sqrt{2}$

(c) -8

(d) 2

(e) 2

(f) $-4, 3$

8. Find bases for the eigenspaces of the matrices in Exercise 6.

9. Find the characteristic equations of the following matrices:

(a) $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 10 & -9 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Answer:

(a) $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = 0$

(b) $\lambda^4 - 8\lambda^3 + 19\lambda^2 - 24\lambda + 48 = 0$

10. Find the eigenvalues of the matrices in Exercise 9.

11. Find bases for the eigenspaces of the matrices in Exercise 9.

Answer:

(a)
$$\lambda = 1: \text{basis} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \ \lambda = -2: \text{basis} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \ \lambda = -1: \text{basis} \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(b)
$$\lambda = 4: \text{basis} \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

12. By inspection, find the eigenvalues of the following matrices:

(a) $\begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -\frac{1}{3} & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

13. Find the eigenvalues of A^9 for

$$A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & \frac{1}{2} & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Answer:

$1, \ \left(\frac{1}{2}\right)^9 = \frac{1}{512}, \ 2^9 = 512$

14. Find the eigenvalues and bases for the eigenspaces of A^{25} for

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

15. Let *A* be a 2×2 matrix, and call a line through the origin of R^2 *invariant* under *A* if *A***x** lies on the line when **x** does. Find equations for all lines in R^2, if any, that are invariant under the given matrix.

(a) $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

Answer:

(a) $y = x$ and $y = 2x$

(b) No lines

(c) $y = 0$

16. Find $\det(A)$ given that *A* has $p(\lambda)$ as its characteristic polynomial.

(a) $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$

(b) $p(\lambda) = \lambda^4 - \lambda^3 + 7$

[*Hint:* See the proof of Theorem 5.1.5.]

17. Let *A* be an $n \times n$ matrix.

(a) Prove that the characteristic polynomial of *A* has degree *n*.

(b) Prove that the coefficient of λ^n in the characteristic polynomial is 1.

18. Show that the characteristic equation of a 2×2 matrix *A* can be expressed as $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$, where $\mathrm{tr}(A)$ is the trace of *A*.

19. Use the result in Exercise 18 to show that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the solutions of the characteristic equation of *A* are

$$\lambda = \frac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\right]$$

Use this result to show that *A* has

(a) two distinct real eigenvalues if $(a-d)^2 + 4\mathrm{bc} > 0$.

(b) two repeated real eigenvalues if $(a-d)^2 + 4bc = 0$.

(c) complex conjugate eigenvalues if $(a-d)^2 + 4bc < 0$.

20. Let A be the matrix in Exercise 19. Show that if $b \neq 0$, then

$$\mathbf{x}_1 = \begin{bmatrix} -b \\ a - \lambda_1 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} -b \\ a - \lambda_2 \end{bmatrix}$$

are eigenvectors of A that correspond, respectively, to the eigenvalues

$$\lambda_1 = \frac{1}{2}\left[(a+d) + \sqrt{(a-d)^2 + 4bc}\right]$$

and

$$\lambda_2 = \frac{1}{2}\left[(a+d) - \sqrt{(a-d)^2 + 4bc}\right]$$

21. Use the result of Exercise 18 to prove that if $p(\lambda)$ is the characteristic polynomial of a 2×2 matrix A, then $p(A) = 0$.

22. Prove: If a, b, c, and d are integers such that $a + b = c + d$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has integer eigenvalues—namely, $\lambda_1 = a + b$ and $\lambda_2 = a - c$.

23. Prove: If λ is an eigenvalue of an invertible matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of A^{-1}, and $\mathbf{x}$ is a corresponding eigenvector.

24. Prove: If λ is an eigenvalue of A, $\mathbf{x}$ is a corresponding eigenvector, and s is a scalar, then $\lambda - s$ is an eigenvalue of $A - sI$, and $\mathbf{x}$ is a corresponding eigenvector.

25. Prove: If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, then $s\lambda$ is an eigenvalue of sA for every scalar s, and $\mathbf{x}$ is a corresponding eigenvector.

26. Find the eigenvalues and bases for the eigenspaces of

$$A = \begin{bmatrix} -2 & 2 & 3 \\ -2 & 3 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

and then use Exercises 23 and 24 to find the eigenvalues and bases for the eigenspaces of

(a) A^{-1}

(b) $A - 3I$

(c) $A + 2I$

27. (a) Prove that if A is a square matrix, then A and A^T have the same eigenvalues. [*Hint:* Look at the characteristic equation $\det(\lambda I - A) = 0$.]

(b) Show that A and A^T need not have the same eigenspaces. [*Hint:* Use the result in Exercise 20 to find a 2×2 matrix for which A and A^T have different eigenspaces.]

28. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. In each part, answer the question and explain your reasoning.

(a) What is the size of A?

(b) Is A invertible?

(c) How many eigenspaces does A have?

29. The eigenvectors that we have been studying are sometimes called ***right eigenvectors*** to distinguish them from ***left eigenvectors***, which are $n \times 1$ column matrices $\mathbf{x}$ that satisfy the equation $\mathbf{x}^T A = \mu \mathbf{x}^T$ for some scalar μ. What is the relationship, if any, between the right eigenvectors and corresponding eigenvalues λ of A and the left eigenvectors and corresponding eigenvalues μ of A?

True-False Exercises

In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) If A is a square matrix and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero scalar λ, then $\mathbf{x}$ is an eigenvector of A.

Answer:

False

(b) If λ is an eigenvalue of a matrix A, then the linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Answer:

False

(c) If the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2 + 1$, then A is invertible.

Answer:

True

(d) If λ is an eigenvalue of a matrix A, then the eigenspace of A corresponding to λ is the set of eigenvectors of A corresponding to λ.

Answer:

False

(e) If 0 is an eigenvalue of a matrix A, then A^2 is singular.

Answer:

True

(f) The eigenvalues of a matrix A are the same as the eigenvalues of the reduced row echelon form of A.

Answer:

False

(g) If 0 is an eigenvalue of a matrix A, then the set of columns of A is linearly independent.

Answer:

False