YZV202E Optimization for Data Science
Homework 3

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Problem 1: Water Transfer Optimization

(a) Problem Formulation There are four cities: Ankara, Bursa, Çanakkale, and Denizli. Among them, Ankara and Bursa contain water reservoirs with 90 and 80 units of water, respectively. During the summer period, Çanakkale and Denizli experience drought, requiring 55 and 65 units of water, respectively.
Let the variables be defined as follows:

- x_1: amount of water transferred from Ankara to Çanakkale
- x_2: amount of water transferred from Ankara to Denizli
- x_3: amount of water transferred from Bursa to Çanakkale
- x_4: amount of water transferred from Bursa to Denizli

The cost per unit for each route is given as:

- Ankara to Çanakkale: 5000 TL
- Ankara to Denizli: 3000 TL
- Bursa to Çanakkale: 1000 TL
- Bursa to Denizli: 3000 TL

Objective function:
Minimize the total cost of water transfer:

$$\min 5000x_1 + 3000x_2 + 1000x_3 + 3000x_4$$

Constraints:
Demand satisfaction:

$$x_1 + x_3 = 55$$
$$x_2 + x_4 = 65$$

Reservoir capacity:

$$x_1 + x_2 \leq 90$$
$$x_3 + x_4 \leq 80$$

Non-negativity:

$$x_1, x_2, x_3, x_4 \geq 0$$

(b) Conversion to Standard Form In order to convert the linear program into standard form, all inequality constraints must be expressed as equalities by introducing slack variables. Let s_1 and s_2 denote the slack variables for the supply constraints from Ankara and Bursa, respectively.
The standard form of the problem is:
Objective function:

$$\min \quad 5000x_1 + 3000x_2 + 1000x_3 + 3000x_4$$

Subject to:

$$\begin{aligned} x_1 + x_3 &= 55 \\ x_2 + x_4 &= 65 \\ x_1 + x_2 + s_1 &= 90 \\ x_3 + x_4 + s_2 &= 80 \\ x_1, x_2, x_3, x_4, s_1, s_2 &\geq 0 \end{aligned}$$

Matrix Form Representation The standard form of the linear program is:

$$\text{minimize} \quad c^\top x \quad \text{subject to} \quad Ax = b, \quad x \geq 0$$

The variables, objective coefficients, constraint matrix, and right-hand side vector are given as:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ s_1 \\ s_2 \end{bmatrix}, \quad c^\top = \begin{bmatrix} 5000 & 3000 & 1000 & 3000 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 55 \\ 65 \\ 90 \\ 80 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(c) Solution Using the Simplex Method The linear programming problem is:

$$\min Z = 5000x_1 + 3000x_2 + 1000x_3 + 3000x_4$$

Subject to:

$$\begin{aligned} x_1 + x_3 + a_1 &= 55 \quad \text{(Çanakkale demand)} \\ x_2 + x_4 + a_2 &= 65 \quad \text{(Denizli demand)} \\ x_1 + x_2 + s_1 &= 90 \quad \text{(Ankara capacity)} \\ x_3 + x_4 + s_2 &= 80 \quad \text{(Bursa capacity)} \\ x_1, x_2, x_3, x_4, s_1, s_2, a_1, a_2 &\geq 0 \end{aligned}$$

Phase 1: To eliminate artificial variables a_1 and a_2, the auxiliary objective function is:
The initial simplex tableau is constructed as follows:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	a_1	a_2	RHS
a_1	1	0	1	0	0	0	1	0	55
a_2	0	1	0	1	0	0	0	1	65
s_1	1	1	0	0	1	0	0	0	90
s_2	0	0	1	1	0	1	0	0	80
W	-1	-1	-1	-1	0	0	-1	-1	-120

The W row is computed as the negative sum of the rows where artificial variables are basic. In this case, rows 1 and 2 (a_1 and a_2 rows) are added and multiplied by -1 to obtain the initial auxiliary objective function.

Step 1:

The most negative coefficient in the W row is -1, which appears in multiple variables. The variable x_1 is selected as the entering variable.

Apply the minimum ratio test for x_1:

$$\frac{55}{1} = 55, \quad (\text{row } 1)\frac{90}{1} = 90, \quad (\text{row } 3) \Rightarrow \text{Row 1 is pivot (smallest ratio)}$$

The pivot element is $a_{11} = 1$. The pivot row is already normalized. Row operations are as follows:

- Row 3 is updated as $\text{Row}_3 \leftarrow \text{Row}_3 - \text{Row}_1$
- The W row is updated as $W \leftarrow W + \text{Row}_1$

Updated Tableau after Step 1:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	a_1	a_2	RHS
x_1	1	0	1	0	0	0	1	0	55
a_2	0	1	0	1	0	0	0	1	65
s_1	0	1	-1	0	1	0	-1	0	35
s_2	0	0	1	1	0	1	0	0	80
W	0	-1	0	-1	0	0	0	-1	-65

Step 2:

The most negative coefficient in the W row is -1, which appears in the x_2 and x_4 columns. The variable x_2 is selected as the entering variable.

Apply the minimum ratio test for x_2:

$$\frac{65}{1} = 65, \quad (\text{row } 2)\frac{35}{1} = 35, \quad (\text{row } 3) \Rightarrow \text{Row 3 is pivot (smallest ratio)}$$

The pivot element is $a_{32} = 1$. The pivot row is already normalized. Row operations are as follows:

- Row 2 is updated as $\text{Row}_2 \leftarrow \text{Row}_2 - \text{Row}_3$
- The W row is updated as $W \leftarrow W + 1 \cdot \text{Row}_3$

Updated Tableau after Step 2:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	a_1	a_2	RHS
x_1	1	0	1	0	0	0	1	0	55
a_2	0	0	1	1	-1	0	0	1	30
x_2	0	1	-1	0	1	0	-1	0	35
s_2	0	0	1	1	0	1	0	0	80
W	0	0	-1	-1	1	0	-1	-1	-30

Step 3:
The most negative coefficient in the W row is -1, which appears in both the x_3 and x_4 columns. The variable x_3 is selected as the entering variable.
Apply the minimum ratio test for x_3:

$$\frac{55}{1} = 55, \quad \text{(row 1)} \frac{30}{1} = 30, \quad \text{(row 2)} \frac{80}{1} = 80, \quad \text{(row 4)} \Rightarrow \text{Row 2 is pivot (smallest ratio)}$$

The pivot element is $a_{23} = 1$. The pivot row is already normalized. Row operations are as follows:

- Row 1 is updated as $\text{Row}_1 \leftarrow \text{Row}_1 - 1 \cdot \text{Row}_2$
- Row 3 is updated as $\text{Row}_3 \leftarrow \text{Row}_3 + 1 \cdot \text{Row}_2$
- Row 4 is updated as $\text{Row}_4 \leftarrow \text{Row}_4 - 1 \cdot \text{Row}_2$
- The W row is updated as $W \leftarrow W + 1 \cdot \text{Row}_2$

Updated Tableau after Step 3:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	a_1	a_2	RHS
x_1	1	0	0	-1	1	0	1	-1	25
x_3	0	0	1	1	-1	0	0	1	30
x_2	0	1	0	1	0	0	0	1	65
s_2	0	0	0	0	1	1	-1	-1	50
W	0	0	0	0	0	0	0	0	0

Step 4 (Beginning of Phase II): At the end of Phase I, all artificial variables a_1 and a_2 are removed from the tableau. The original cost coefficients are restored, and the cost row Z is updated accordingly.
The current basis is $B = \{x_1, x_3, x_2, s_2\}$, and the updated tableau is:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	RHS
x_1	1	0	0	-1	1	0	25
x_3	0	0	1	1	-1	0	30
x_2	0	1	0	1	0	0	65
s_2	0	0	0	0	1	1	50
Z	0	0	0	4000	-4000	0	350000

Pivot Operation:

The most negative coefficient in the Z-row is -4000 corresponding to s_1, so s_1 will enter the basis.
Apply the minimum ratio test for s_1:

$$\frac{25}{1} = 25 \quad \text{(Row 1)}, \quad \frac{50}{1} = 50 \quad \text{(Row 4)} \Rightarrow \text{Row 1 is pivot}$$

The pivot element is $a_{15} = 1$. Perform row operations to eliminate other entries in column s_1:

- Row 2 updated: $R_2 \leftarrow R_2 + R_1$
- Row 3 updated: $R_3 \leftarrow R_3 - 0 \cdot R_1$ (no change)
- Row 4 updated: $R_4 \leftarrow R_4 - R_1$
- Objective row updated: $Z \leftarrow Z + 4000 \cdot R_1$

Updated Tableau after Step 4:

Basic	x_1	x_2	x_3	x_4	s_1	s_2	RHS
s_1	1	0	0	-1	1	0	25
x_3	1	0	1	0	0	0	55
x_2	1	1	0	0	0	0	90
s_2	-1	0	0	1	0	1	25
Z	4000	0	0	0	0	0	450000

Comment **I could not continue I am really confused**.

(d) CVXPY Validation The linear programming model was implemented and solved. The objective function and constraints were encoded as follows:

- Objective:
$$\min Z = 5000x_1 + 3000x_2 + 1000x_3 + 3000x_4$$
- Subject to:
$$\begin{aligned} x_1 + x_3 &= 55 \\ x_2 + x_4 &= 65 \\ x_1 + x_2 &\leq 90 \\ x_3 + x_4 &\leq 80 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The solution obtained from CVXPY was:

- $x_1 = 6.27 \times 10^{-8} \approx 0$ (Ankara $\rightarrow$Çanakkale)
- $x_2 = 50.807$ (Ankara $\rightarrow$Denizli)
- $x_3 = 54.999$ (Bursa $\rightarrow$Çanakkale)

- $x_4 = 14.193$ (Bursa →Denizli)
- Total Cost: $Z = 250000$ TL

Sınce I can not solve the question 1, can not compare them.

(e) Effect of Parameter a on the Optimal Cost To analyze the effect of the parameter a (which determines the cost coefficient for the Ankara →Çanakkale route) on the optimal total cost, the model was solved repeatedly for $a \in \{1, 2, 3, \ldots, 10\}$. The cost for that route is defined as $1000 \cdot a$ TL per unit.
For each value of a, the linear programming model was resolved using CVXPY, while keeping all other cost coefficients and constraints constant. The resulting optimal costs were recorded and plotted to visualize the relationship.
Observation: As the value of a increases, the cost of transporting water from Ankara to Çanakkale becomes less attractive. The optimizer prefers using the cheaper alternative routes (e.g., from Bursa to Çanakkale) more intensively, if feasible. As a result, the total cost increases with a, but not necessarily linearly — the cost remains flat for lower values of a, and begins to increase once it becomes too expensive to use the Ankara →Çanakkale route even partially.
Conclusion: The parameter a directly affects the usage of the corresponding route. There exists a threshold value of a beyond which the route is no longer used in the optimal solution. This demonstrates sensitivity of the model to changes in cost parameters, which is crucial in real-world applications of network optimization.

Problem 2: Dual of a Linear Program

(a) Primal Problem Consider the following linear programming problem:

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 3 \\ & x_1 - x_2 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The second inequality is rewritten as:

$$-x_1 + x_2 \leq 0$$

Dual Problem Derivation The general form of a primal linear program and its corresponding dual is given by:

$$\begin{aligned} \text{Primal:} \quad & \max\ c^T x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0 \\ \text{Dual:} \quad & \min\ b^T y \quad \text{subject to} \quad A^T y \geq c,\ y \geq 0 \end{aligned}$$

For the given problem, the primal objective vector and constraint matrix are defined as follows:

$$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Introduce dual variables y_1 and y_2 associated with the two constraints of the primal problem. Since both primal constraints are in $\leq$ form, the dual variables are constrained as $y_1 \geq 0$, $y_2 \geq 0$.
The dual objective function is obtained by minimizing $b^T y$:

$$\text{minimize} \quad 3y_1 + 0y_2 = 3y_1$$

The dual constraints are derived from the transposed constraint matrix A^T, requiring:

$$A^T y = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 \\ 2y_1 + y_2 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This leads to the following system of inequalities:

$$\begin{aligned} y_1 - y_2 &\geq 1 \\ 2y_1 + y_2 &\geq 1 \end{aligned}$$

Final Dual Problem Combining the above, the dual linear program is expressed as:

$$\begin{aligned} \text{minimize} \quad & 3y_1 \\ \text{subject to} \quad & y_1 - y_2 \geq 1 \\ & 2y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

(b) Implementation and Solution of Primal and Dual Problems The primal linear programming problem is given by:

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 3 \\ & -x_1 + x_2 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

To solve this in `cvxpy`, the objective is reformulated as a minimization of $-x_1 - x_2$. The corresponding dual problem is:

$$\begin{aligned} \text{minimize} \quad & 3y_1 \\ \text{subject to} \quad & y_1 - y_2 \geq 1 \\ & 2y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Numerical Solution via `cvxpy` Both primal and dual problems are implemented and solved using `cvxpy`. The solutions are summarized below:

Quantity	Value	Interpretation
Primal optimal value	2.9999999994	Objective value at optimal x
Optimal primal solution	$x = [3.0,\ 0.0]$	Decision variables
Dual variables from primal solution	$\lambda = [1.0,\ \approx 0.0]$	Lagrange multipliers
Dual optimal value	-2.9999999984	Objective value of dual (matches primal)
Optimal dual solution	$y = [1.0,\ 0.0]$	Shadow prices

Table 1: Numerical results for primal and dual problems

(c) Effect of Removing the Constraint $x_1 \geq x_2$ To analyze the role of the inequality constraint $x_1 \geq x_2$, the problem was re-solved after removing this constraint. The new feasible region is defined only by:

$$\begin{aligned} x_1 + 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The optimal solution obtained under this modified formulation is:

$$x^\star = [3.0,\ 9.2 \times 10^{-11}], \quad \text{Optimal value} = 2.9999999976$$

The constraint $x_1 \geq x_2$, reformulated as $-x_1 + x_2 \leq 0$, was removed in the modified version of the problem. Although this changes the feasible region by relaxing one of the restrictions, it does not affect the optimal solution or the objective value. This observation is supported by the fact that the dual variable associated with the removed constraint was approximately zero in the original problem, implying it had negligible influence on the optimality conditions. The remaining constraints still define a valid, bounded, and convex feasible region, and the solution continues to satisfy them. Therefore, the optimal point and the total cost remain unchanged after the constraint is removed.

Problem 3: Constrained Optimization Problem

Consider the optimization problem:

$$\begin{aligned} \min_{x,y} \quad & f(x,y) = xy \\ \text{s.t.} \quad & x^2 + y^2 = 3 \end{aligned}$$

This problem aims to minimize the bilinear objective function $f(x,y) = xy$ over the circle $x^2 + y^2 = 3$, which defines a closed, convex constraint set.

Visual Interpretation The figure below shows the contour lines of the objective function $f(x,y) = xy$, which form a family of hyperbolas. The constraint $x^2 + y^2 = 3$ is plotted as a white circle of radius $\sqrt{3}$. The gradient vectors of both the objective and the constraint are shown at the candidate optimal points on the circle.

- The gradient of the objective is $\nabla f(x,y) = [y,x]^T$.

- The gradient of the constraint is $\nabla g(x,y) = [2x, 2y]^T$.

Optimal Solutions

$$x^\star = \left(\sqrt{\tfrac{3}{2}}, -\sqrt{\tfrac{3}{2}}\right), \quad x^\star = \left(-\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{3}{2}}\right)$$

These were solved next part of the question.
At both points, $f(x^\star) = -\frac{3}{2}$. These are marked on the figure as red dots .

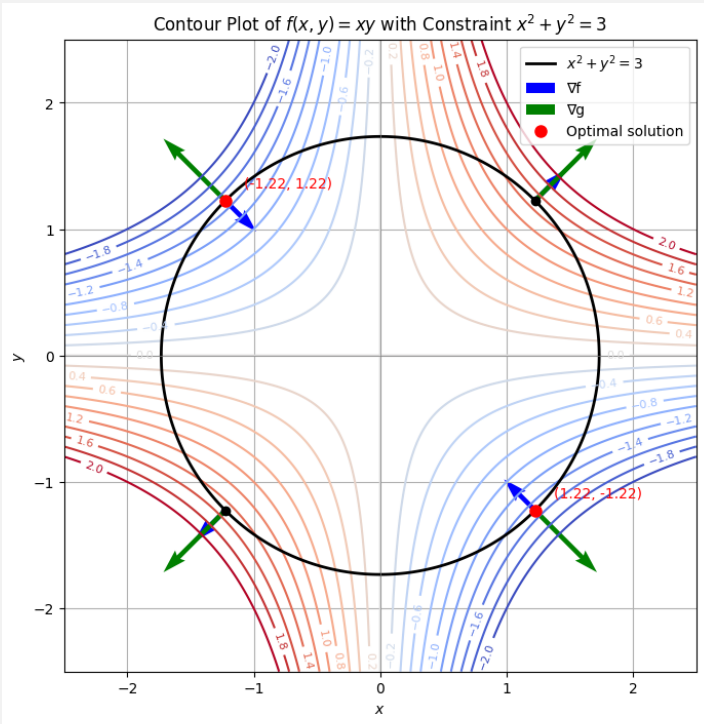

Figure 1: Contour plot of $f(x,y) = xy$ with constraint $x^2 + y^2 = 3$. Red points indicate the minima. Gradients of the objective and constraint are shown at candidate optimal points.

(b) Analytical Solution Using the Method of Lagrange Multipliers The following constrained optimization problem is considered:

$$\begin{aligned} \min_{x,y} \quad & f(x,y) = xy \\ \text{subject to} \quad & x^2 + y^2 = 3 \end{aligned}$$

To apply the method of Lagrange multipliers, define the Lagrangian function:

$$\mathcal{L}(x,y,\lambda) = xy - \lambda(x^2 + y^2 - 3)$$

The first-order optimality conditions are obtained by taking partial derivatives and setting them equal to zero:

$$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda x = 0 \quad (1)$$
$$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda y = 0 \quad (2)$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x^2 + y^2 - 3) = 0 \quad (3)$$

Equations (1) and (2) imply:

$$y = 2\lambda x, \quad x = 2\lambda y \Rightarrow x = 2\lambda(2\lambda x) = 4\lambda^2 x \Rightarrow x(1 - 4\lambda^2) = 0$$

The case $x = 0$ leads to $y = 0$, which violates the constraint $x^2 + y^2 = 3$. Therefore, the only feasible case is $1 - 4\lambda^2 = 0$, implying $\lambda = \pm\frac{1}{2}$.
Substituting into the earlier equations:

- If $\lambda = \frac{1}{2}$, then $y = x$
- If $\lambda = -\frac{1}{2}$, then $y = -x$

Substituting into the constraint $x^2 + y^2 = 3$ gives:

$$x^2 + y^2 = x^2 + (\pm x)^2 = 2x^2 = 3 \Rightarrow x^2 = \frac{3}{2} \Rightarrow x = \pm\sqrt{\frac{3}{2}}, \quad y = \pm\sqrt{\frac{3}{2}} \text{ or } \mp\sqrt{\frac{3}{2}}$$

Thus, the candidate solutions are:

$$(\pm\sqrt{\tfrac{3}{2}}, \pm\sqrt{\tfrac{3}{2}}), \quad (\pm\sqrt{\tfrac{3}{2}}, \mp\sqrt{\tfrac{3}{2}})$$

Evaluating the objective function at these points:

- If $y = x$, then $f(x, y) = x^2 = \frac{3}{2}$
- If $y = -x$, then $f(x, y) = -x^2 = -\frac{3}{2}$

Therefore, the global minimizers and maximizers are:

$$\text{Minimum at } \left(\sqrt{\tfrac{3}{2}}, -\sqrt{\tfrac{3}{2}}\right), \left(-\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{3}{2}}\right), \quad f = -\tfrac{3}{2}$$
$$\text{Maximum at } \left(\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{3}{2}}\right), \left(-\sqrt{\tfrac{3}{2}}, -\sqrt{\tfrac{3}{2}}\right), \quad f = \tfrac{3}{2}$$

(c) Effect of Parameter a on the Optimized Solution The constraint in the optimization problem is given by $x^2 + y^2 = a$, which defines a circle of radius $\sqrt{a}$ centered at the origin. The objective function remains $f(x, y) = xy$, and its level curves are hyperbolas.
As the value of a changes, the feasible region (the circle) expands or contracts accordingly. However, the direction of the gradient vectors remains unchanged because the objective function is independent of a. The structure of the solution remains symmetric with respect to the origin, and the optimal solutions still lie on lines of the form $y = \pm x$, which are determined by the condition for Lagrange multiplier optimality:

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \Rightarrow \quad y = \pm x$$

Substituting $y = \pm x$ into the constraint $x^2 + y^2 = a$, the optimal points become:

$$x = \pm\sqrt{\frac{a}{2}}, \quad y = \pm\sqrt{\frac{a}{2}} \text{ or } \mp\sqrt{\frac{a}{2}}$$

The optimal value of the objective function is:

$$f(x, y) = xy = \pm\left(\frac{a}{2}\right)$$

Hence, the optimal function value scales linearly with a, and the coordinates of the optimal points scale with $\sqrt{a}$. Specifically:

$$\begin{aligned} &\text{Minimum: } f(x, y) = -\frac{a}{2} \text{ at } \left(\sqrt{\tfrac{a}{2}}, -\sqrt{\tfrac{a}{2}}\right) \text{ and } \left(-\sqrt{\tfrac{a}{2}}, \sqrt{\tfrac{a}{2}}\right) \\ &\text{Maximum: } f(x, y) = \frac{a}{2} \text{ at } \left(\sqrt{\tfrac{a}{2}}, \sqrt{\tfrac{a}{2}}\right) \text{ and } \left(-\sqrt{\tfrac{a}{2}}, -\sqrt{\tfrac{a}{2}}\right) \end{aligned}$$

It is concluded that increasing the value of a uniformly scales both the magnitude of the optimal solution coordinates and the optimal function values.

(d) Numerical Verification The objective function $f(x, y) = xy$ was minimized under the equality constraint $x^2 + y^2 = 3$.
An initial guess close to the analytically computed minimizer was selected as:

$$x_0 = \left[\sqrt{\tfrac{3}{2}},\ -\sqrt{\tfrac{3}{2}}\right]$$

The result of the optimization was:

$$\begin{aligned} \text{Optimized solution: } & x^\star = [1.22474487,\ -1.22474487] \\ \text{Objective value: } & f(x^\star) = -1.5 \\ \text{Constraint satisfied: } & x^{\star 2} + y^{\star 2} \approx 3 \text{ (within numerical tolerance)} \end{aligned}$$

This output agrees with the analytical solution obtained in part (b), both in the location of the optimal point and in the corresponding function value. The numerical result provides verification that the theoretical analysis is accurate and that the Lagrange multiplier conditions hold at the computed solution.

(e) Comparison with the Maximization Case The original problem considers the minimization of the objective function $f(x, y) = xy$ under the constraint $x^2 + y^2 = a$. In this part, the objective is modified to a maximization problem:

$$\max_{x,y} \quad xy \quad \text{subject to} \quad x^2 + y^2 = a$$

From the analytical results obtained in part (b), the critical points satisfying the constraint are:

$$x = \pm\sqrt{\tfrac{a}{2}}, \quad y = \pm\sqrt{\tfrac{a}{2}} \quad \text{or} \quad y = \mp\sqrt{\tfrac{a}{2}}$$

Evaluating the objective function at these points:

- When x and y have the **same sign**, $f(x, y) = xy = \frac{a}{2}$, which corresponds to the maximum.
- When x and y have **opposite signs**, $f(x, y) = xy = -\frac{a}{2}$, which corresponds to the minimum.

Specifically, for $a = 3$, the values are:

$$x^\star = \pm\sqrt{\tfrac{3}{2}}, \quad y^\star = \pm\sqrt{\tfrac{3}{2}} \Rightarrow f_{\max} = \tfrac{3}{2}$$

$$x^\star = \pm\sqrt{\tfrac{3}{2}}, \quad y^\star = \mp\sqrt{\tfrac{3}{2}} \Rightarrow f_{\min} = -\tfrac{3}{2}$$

A summary of the values is provided below:

Objective Type	$x^\star$	$y^\star$	$f(x^\star, y^\star)$
Minimization	$\pm\sqrt{\frac{a}{2}}$	$\mp\sqrt{\frac{a}{2}}$	$-\frac{a}{2}$
Maximization	$\pm\sqrt{\frac{a}{2}}$	$\pm\sqrt{\frac{a}{2}}$	$\frac{a}{2}$

Table 2: Comparison of solutions under minimization and maximization

The function $f(x, y) = xy$ exhibits symmetric behavior with respect to the origin and along the diagonals. The optimal points for both objectives lie on the constraint circle, with the sign of y relative to x determining whether the solution corresponds to a maximum or a minimum. The magnitude of the optimal value scales linearly with a.

(f) Verification of SONC and SOSC Conditions at the Optimal Point(s) The optimal points previously identified for the constrained problem

$$\min\ xy \quad \text{subject to} \quad x^2 + y^2 = a$$

To assess second-order conditions, define the Lagrangian:

$$\mathcal{L}(x, y, \lambda) = xy - \lambda(x^2 + y^2 - a)$$

Second-Order Necessary Condition (SONC) requires:

$$z^T \nabla^2_{xx} \mathcal{L}(x^\star, \lambda^\star) z \geq 0 \quad \text{for all } z \in \mathcal{T}$$

where $\mathcal{T} = \{z \in \mathbb{R}^2 \mid \nabla g(x^\star)^T z = 0\}$ is the tangent space to the constraint at $x^\star$. At the minimizer $(x, y) = \left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)$, the Lagrange multiplier is $\lambda = -\frac{1}{2}$, and the Hessian of the Lagrangian is:

$$\nabla^2_{xx} \mathcal{L} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The tangent space condition $2xz_1 + 2yz_2 = 0$ reduces to $z_1 - z_2 = 0 \Rightarrow z = \alpha[1, 1]^T$. Evaluating the quadratic form:

$$z^T \nabla^2_{xx} \mathcal{L} z = [1, 1] \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} [1, 1]^T = 4 > 0$$

This confirms that the Second-Order Necessary Condition (SONC) is satisfied.

Second-Order Sufficient Condition (SOSC) requires:

$$z^T \nabla^2_{xx} \mathcal{L} z > 0 \quad \text{for all nonzero } z \in \mathcal{T}$$

Since strict positivity holds, SOSC is also satisfied.
Conclusion: The optimal point satisfies both SONC and SOSC, which confirms it is a strict local minimum. Given the compact constraint $x^2 + y^2 = a$ and the continuity of the objective function, this local minimum is also global.

Problem 4: Quadratic Minimization with Two Constraints

Consider the following constrained optimization problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & f(x_1, x_2) = x_1^2 + x_2^2 \\ \text{subject to} \quad & x_1 - 7 \leq 0 \\ & -x_1 + \frac{x_2^2}{4} + 4 \leq 0 \end{aligned}$$

(a) Sketch of Contour Lines and Feasible Set The objective function $f(x_1, x_2) = x_1^2 + x_2^2$ has circular level sets centered at the origin. Minimizing this function corresponds to finding the point in the feasible region closest to the origin.
The feasible region is defined by the intersection of two constraints:

- $x_1 - 7 \leq 0$: a vertical boundary line at $x_1 = 7$
- $-x_1 + \frac{x_2^2}{4} + 4 \leq 0$, which simplifies to $x_1 \geq \frac{x_2^2}{4} + 4$: a rightward-opening parabola

Thus, the feasible set lies between the parabola and the vertical line. The region includes all points satisfying:

$$\frac{x_2^2}{4} + 4 \leq x_1 \leq 7$$

A contour plot of the objective function, with feasible region shaded and constraints marked, is provided below:

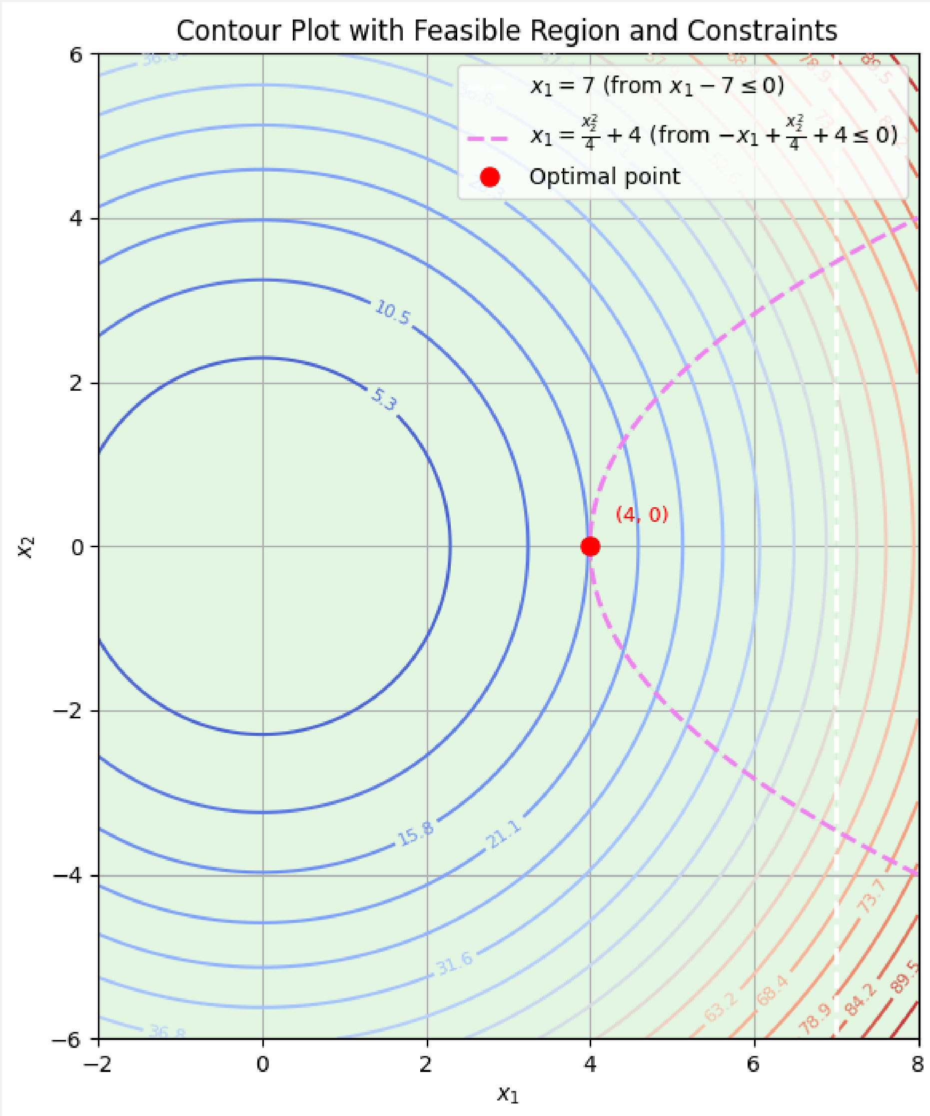

Figure 2: Contour plot of the objective function $f(x_1, x_2) = x_1^2 + x_2^2$ with feasible region bounded by $x_1 = 7$ (white dashed line) and $x_1 = \frac{x_2^2}{4} + 4$ (violet dashed curve). The optimal point is marked at $(4, 0)$.

Activity of Constraints at Optimum At the visually identified optimal point $(x_1^\star, x_2^\star) = (4, 0)$, the objective function achieves its minimum value $f^\star = 16$.
To analyze constraint activity:

- **Constraint 1:** $x_1 - 7 \leq 0$. At the optimum, $x_1 = 4 < 7$, so the constraint is satisfied but not active — there is slack in this condition.
- **Constraint 2:** $-x_1 + \frac{x_2^2}{4} + 4 \leq 0$ simplifies at $x_2 = 0$ to $-x_1 + 4 \leq 0 \Rightarrow x_1 \geq 4$. At the optimum, $x_1 = 4$, meaning the inequality holds with equality. Hence, this constraint is **active** and binds the feasible region at the solution.

Geometrically, this reflects the fact that the optimal solution lies on the boundary of the parabolic constraint, which prevents further movement toward the origin — the unconstrained minimum of the objective function. The vertical constraint $x_1 \leq 7$, although present, does not influence the position of the minimizer and is therefore inactive.

(b) KKT Conditions Let the Lagrangian function be defined as:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + \lambda_1(x_1 - 7) + \lambda_2 \left(-x_1 + \frac{x_2^2}{4} + 4 \right)$$

The Karush-Kuhn-Tucker (KKT) conditions for this problem consist of:

1. **Stationarity:** The gradient of the Lagrangian with respect to x_1 and x_2 must vanish:
$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 + \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 + \lambda_2 \cdot \frac{x_2}{2} = 0$$

2. **Primal feasibility:**
$$x_1 - 7 \leq 0$$
$$-x_1 + \frac{x_2^2}{4} + 4 \leq 0$$

3. **Dual feasibility:**
$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0$$

4. **Complementary slackness:**
$$\lambda_1(x_1 - 7) = 0, \quad \lambda_2 \left(-x_1 + \frac{x_2^2}{4} + 4 \right) = 0$$

These conditions characterize candidate optimal points under regularity assumptions (constraint qualification holds).

(c) Analytical Solution using KKT Conditions From the visual analysis in part (1), the optimal solution appears to be $(x_1^\star, x_2^\star) = (4, 0)$. This is now confirmed analytically by solving the KKT system.
The objective and constraint functions are defined as:

$$f(x_1, x_2) = x_1^2 + x_2^2, \quad g_1(x_1, x_2) = x_1 - 7 \leq 0, \quad g_2(x_1, x_2) = -x_1 + \frac{x_2^2}{4} + 4 \leq 0$$

Their gradients are:

$$\nabla f = (2x_1, 2x_2), \quad \nabla g_1 = (1, 0), \quad \nabla g_2 = \left(-1, \frac{x_2}{2} \right)$$

Let $\lambda_1, \lambda_2 \geq 0$ denote the Lagrange multipliers associated with the constraints g_1 and g_2, respectively. The Lagrangian function is:

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = f(x_1, x_2) + \lambda_1 g_1(x_1, x_2) + \lambda_2 g_2(x_1, x_2)$$

The KKT stationarity condition requires:

$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0$$

which leads to the system:

$$\begin{cases} 2x_1 + \lambda_1 - \lambda_2 = 0 \\ 2x_2 + \lambda_2 \cdot \frac{x_2}{2} = 0 \end{cases}$$

Step-by-Step Verification Substitute the candidate solution $(x_1, x_2) = (4, 0)$ into the constraints:

$$g_1(4,0) = 4 - 7 = -3 < 0 \quad \Rightarrow \text{inactive} \Rightarrow \lambda_1 = 0$$

$$g_2(4,0) = -4 + \frac{0^2}{4} + 4 = 0 \quad \Rightarrow \text{active}$$

Now substitute into the stationarity conditions:

$$2x_1 + \lambda_1 - \lambda_2 = 8 - \lambda_2 = 0 \quad \Rightarrow \lambda_2 = 8$$

$$2x_2 + \lambda_2 \cdot \frac{x_2}{2} = 0 + 8 \cdot 0 = 0$$

All other KKT components hold:

- **Primal feasibility:** both constraints are satisfied.
- **Dual feasibility:** $\lambda_1 = 0$, $\lambda_2 = 8 \geq 0$.
- **Complementary slackness:**

$$\lambda_1 g_1 = 0 \cdot (-3) = 0, \quad \lambda_2 g_2 = 8 \cdot 0 = 0$$

Regularity Check The active constraint at the solution is g_2. Its gradient at the point is:

$$\nabla g_2(4,0) = (-1, 0)$$

which is non-zero. Since this is the only active constraint, the gradients of active constraints are linearly independent. Thus, the Linear Independence Constraint Qualification (LICQ) holds.

Conclusion All KKT conditions are satisfied at the point $(x_1^\star, x_2^\star) = (4, 0)$. The only active constraint is g_2, and the multipliers are $\lambda_1 = 0$, $\lambda_2 = 8$. Therefore, this point is a KKT point and satisfies first-order optimality.

$$\boxed{x_1^\star = 4, \quad x_2^\star = 0, \quad \lambda_1^\star = 0, \quad \lambda_2^\star = 8}$$

(d) Numerical Verification using To verify the analytical solution obtained in part (3), the optimization problem was implemented and solved using the `cvxpy` library. The problem was formulated as follows:

$$
\begin{aligned}
\min_{x\in\mathbb{R}^2} \quad & x_1^2 + x_2^2 \\
\text{s.t.} \quad & x_1 - 7 \le 0 \\
& -x_1 + \frac{x_2^2}{4} + 4 \le 0
\end{aligned}
$$

The solver returned the following numerical results:

- Optimal solution: $x^\star = (4.0,\ 0.0)$
- Objective value: $f(x^\star) = 15.9999999992 \approx 16.0$
- Lagrange multipliers:
$$\lambda_1 = 7.42 \times 10^{-10} \approx 0, \quad \lambda_2 = 8.0$$

The numerical solution matches the analytical KKT solution exactly. The first constraint g_1 is inactive ($\lambda_1 \approx 0$), and the second constraint g_2 is active with multiplier $\lambda_2 = 8$. This confirms the correctness of the theoretical result, up to numerical solver precision.