Geometric Series

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Introduction to geometric series

Year	\$ in account (5% interest)
1	1,000
2	1,000 + 1,000 (1.05)
3	1,000 + 1,000 (1.05) + 1,000 (1.05)^2
⋮	⋮
n	1,000 + 1,000 (1.05) + ... + 1,000 (1.05)$^{n-1}$

} Geometric series

* A sequence is an order list of numbers. A series is sum of sequences.
  * Geometric sequence: $\{2, 6, 18, 54\}$ (finite)
  * Geometric series : $2+6+18+54$ (finite)

* Geometric sequence: (infinite)

$$\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \dots\right\} = \{a_n\}_{n=1}^{\infty} \text{ with } a_n = 1\left(\frac{1}{2}\right)^{n-1}$$

$$\left\{\left(\frac{1}{2}\right)^0, \left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3 \dots\right\}$$

* Geometric series (infinite)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \dots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

Finite geometric series formula

* a = first term n = # of terms (finite)

  r = common ratio S_n = sum of first n terms

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \dots + ar^{n-1}$$

$$\Rightarrow -rS_n = -\cancel{ar} - \cancel{ar^2} - \dots - ar^n$$

$$+ \quad \overline{}$$

$$S_n - rS_n = a - ar^n$$

$$\Rightarrow S_n(1-r) = a(1-r^n)$$

$$* \quad S_n = \frac{a \cdot (1-r^n)}{1-r}$$

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* A series is an addition of infinitely many terms, one after another.
* In geometric series, the ratio of consecutive terms is constant. $S_n = \frac{a(1-r^n)}{1-r}$ where "n" is the number of terms, "a" is the first term, and "r" is the ratio.

Examples:

* Find the sum of first 50 terms : $1 + \frac{10}{11} + \frac{100}{121} + \dots$

* $a = 1$
$n = 50$
$r = 10/11$

$$\Rightarrow S_{50} = \frac{1 \cdot \left(1 - \left(\frac{10}{11}\right)^{5}\right)}{1 - \frac{10}{11}} = \frac{1 - \left(\frac{10}{11}\right)^{5}}{\frac{1}{11}} = 11\left(1 - \left(\frac{10}{11}\right)^{50}\right)$$

* $1 - 0.99 + 0.99^{2} - 0.99^{3} + \dots\dots - 0.99^{79} = ?$ $\quad n = 80 \; (79+1)$

$\cdot(-0.99) \quad \cdot(-0.99)$

$$S_{80} = \frac{1\left(1 - \left((-0.99)^{80}\right)\right)}{1 - (-0.99)} = \frac{1 - 0.99^{80}}{1.99}$$

* $a_1 = 10$

$a_i = a_{i-1} \cdot \frac{9}{10}$ $\Rightarrow$ Find the sum of first 30 terms :

$$S_{30} = \frac{10\left(1 - \left(\frac{9}{10}\right)^{30}\right)}{1 - \frac{9}{10}} = 100\left(1 - \left(\frac{9}{10}\right)^{30}\right)$$

Real-world example: Swing

* A monkey is swinging from a tree. On the first swing, she passes through an arc of 24 m. With each swing, she passes through an arc $\frac{1}{2}$ the length of the previous swing.

1) **Which expression gives the total length the monkey swings in her first n swings?**

$$S_n = \frac{24\left(1 - \left(\frac{1}{2}\right)^{n}\right)}{1/2} = \frac{24(0.5)^{n}}{0.5}$$

$S_{25} = 48$

2) **What is the total distance the monkey has traveled when she completes her 25^{th} swing?**
Round your final answer to the nearest meter.

Real-world example: Hike

* Sloan went on a 4 day hiking trip. Each day, she walked 20% more than the distance that she walked the day before. She walked a total of 27 kilometers.

What is the distance Sloan walked in the 1^{st} day of the trip?
Round your final answer to the nearest kilometer.

$$27 = \frac{a\left(1 - 1.2^{4}\right)}{1 - 1.2}$$

$$27 = 5.368\,a$$

$$a \approx 5$$

Geometric Series with Summation Notation

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Summation notation

* $\sum_{i=1}^{10} i$: Sum of all i's where i starts at 1 and goes to 10.

* $1+2+3+\ldots 100 = \sum_{i=1}^{100} i$

* $\sum_{i=0}^{50} \pi i^2 = \pi \cdot 0^2 + \pi 1^2 + \pi 2^2 + \ldots + \pi 50^2$

Geometric series with sigma notation

* $S_n = a + a \cdot r + a \cdot r^2 + \ldots + ar^n$

* $\sum_{k=0}^{n} a \cdot r^k$

* $1 + (-3) + 9 + (-27) + 81 + \ldots = \sum_{k=0}^{\infty} (-3)^k$

Example :

* $\sum_{k=0}^{99} 2(3^k) = ?$

$a = 2$
$r = 3$
$n = 100$

"because starts at zero!"

$$S_{100} = \frac{2(1-3^{100})}{1-3}$$

$$= 3^{100} - 1$$

* $\sum_{i=1}^{10} i$: Sum of all i's where i starts at 1 and goes to 10.

* Geometric series with sigma notation: $\sum_{k=0}^{n} a \cdot r^k$

Real-world example: Social Media

A new social media site boasts that its user base has increased 47% each month for the past year. The number of users on January 1st of last year was $50,000$. Which expression below gives the total number of *new* users (in thousands) that were added through month n of the past year, where $1 \le n \le 12$?

[redacted]

- $50(1.47)^n$
- ✓ $(0.47)(50)(1+(1.47)+(1.47)^2+\cdots+(1.47)^{n-1})$
- $50(1+0.47+(0.47)^2+\cdots+(0.47)^n)$
- $50(0.53+(0.53)^2+(0.53)^3+\cdots+(0.53)^n)$

m	s	n	t
1	50	$50(0.47)$	$50(1.47)$
2	$50(1.47)$	$50(1.47)(0.47)$	$50(1.47)^2$
3	$50(1.47)^2$	$50(1.47^2)(0.47)$	$50(1.47)^3$
⋮			
n	$50(1.47)^{n-1}$	$50(1.47)^{n-1}\cdot(0.47)$	$50(1.47)^n$

$$= 50(0.47) + 50(0.47)(1.47) + 50(0.47)(1.47)^2 + \cdots + 50(0.47)(1.47)^{n-1}$$

$$= 50(0.47)\left(1+(1.47)+(1.47)^2+\cdots+(1.47)^{n-1}\right)$$

Real-world problem: Mortgage

* 200k mortgage loan in 360 months with %0.5 monthly interest. How much should be the monthly payment?

$$\Big(\big(\underbrace{(200k(1.005)-x)}_{\text{remaining debt after 1 month}}1.005-x\big)1.005-x\Big)\ldots = 0$$

L = Loan amount
i = monthly interest
n = # of months
P = monthly payment

$$\underbrace{(((}_{n \text{ parenthesis}}L(1+i)-P)(1+i)-P)\ldots)$$

$$n=1 \Rightarrow L(1+i)-P=0 \Rightarrow P=L(1+i) \Rightarrow L=\frac{P}{i+1}$$

$$n=2 \Rightarrow (L(1+i)-P)(i+1)-P=0 \Rightarrow L=\frac{P}{i+1}+\frac{P}{(i+1)^2}$$

$$\Rightarrow L=\sum_{k=1}^{n} P\cdot\frac{1}{(i+1)^k} \Rightarrow 200{,}000=\frac{P\cdot\left(1-\left(\frac{1}{1.005}\right)^{360}\right)}{1-\frac{1}{1.005}} \Rightarrow P\approx 1193\$$$

The Binomial Theorem

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Introduction

* $(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\vdots$

$(a+b)^{20} =$ takes too long to calculate manually!

* The binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

* $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

$$\Rightarrow (a+b)^4 = \sum_{k=0}^{4} \binom{4}{k} a^{4-k} b^k$$

$$= \underbrace{\binom{4}{0}}_{1} a^4 b^{\underbrace{0}_{1}} + \underbrace{\binom{4}{1}}_{4} a^3 b^1 + \underbrace{\binom{4}{2}}_{6} a^2 b^2 + \underbrace{\binom{4}{3}}_{4} a^1 b^3 + \underbrace{\binom{4}{4}}_{1} a^{\underbrace{0}_{1}} b^4$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Pascal's triangle and binomial expansion

*

$$1 \quad \rightarrow 0^{th}$$

$$1 + 1 \quad \rightarrow 1^{st}$$

$$1 + 2 + 1 \quad \rightarrow 2^{nd}$$

$$1 + 3 + 3 + 1 \quad \rightarrow 3^{rd}$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1 \quad \rightarrow 4^{th}$$

$$\vdots$$

$$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^3$$

* The binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

Expanding binomials

$\ast \ (3y^2+6x^3)^5 = \dots + a\,x^6y^6 + \dots \Rightarrow a=?$

$\Rightarrow \binom{5}{2}(6x^3)^2(3y^2)^3 \dots$

$\Rightarrow \dots \dfrac{5\cdot 4\cdot \cancel{3}\cdot \cancel{2!}}{\cancel{3}\cdot 2\cdot 1\cdot \cancel{2!}} \cdot \underbrace{36x^6 \cdot 27y^6}_{972} \Rightarrow \dots 9{,}720\,x^6y^6 \dots$

Quiz

$\ast$ Expand $(z-k)^6$ using the binomial theorem:

$= z^6 - \binom{6}{1}z^5k + \binom{6}{2}z^4k^2 - \binom{6}{3}z^3k^3 + \binom{6}{4}z^2k^4 - \binom{6}{5}zk^5 + k^6$

$= z^6 - 6z^5k + 15z^4k^2 - 20z^3k^3 + 15z^2k^4 - 6zk^5 + k^6$

$\dfrac{6\cdot 5\cdot \cancel{4!}}{\cancel{4!}\cdot 2} \qquad \dfrac{6\cdot 5\cdot 4\cdot \cancel{3!}}{3\cdot 2\cdot \cancel{3!}}$

Expanding binomials w/o Pascal's triangle

$\ast \ (x+y)^7 = ?$

① we know it will have 8 terms, so we add indices.

1 2 3 4 5 6 7 8

② we know x will go from x^7 to x^0, any y will go from y^0 to y^7

1	2	3	4	5	6	7	8
x^7	x^6y	x^5y^2	x^4y^3	x^3y^4	x^2y^5	xy^6	y^7

③ we calculate the first coefficient. Formula for the rest is:

$$\frac{(\text{exponent of the previous term})(\text{coefficient of the previous term})}{\text{index of the previous term}}$$

We calculate the first half of the coefs b/c we know the rest will be symmetrical

$= \underset{1^7}{\_\_}x^7 + \underset{\frac{7\cdot 1}{1}}{7}x^6y + \underset{\frac{6\cdot 7}{2}}{21}x^5y^2 + \underset{\frac{5\cdot 21}{3}}{35}x^4y^3 + \underline{35}x^3y^4 + \underline{21}x^2y^5 + \underline{7}xy^6 + y^7$

reflection of the first half of coefficients

Arithmetic Series

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Introduction

* $\{1,2,3 \ldots, n\}$

$\Rightarrow S_n = 1+2+3+\ldots+n$

$\Rightarrow S_n = n+(n-1)+(n-2)+\ldots 1$

$+$

$2 \cdot S_n = \underbrace{(n+1)+(n+1)+\ldots(n+1)}_{n \text{ times}}$

$S_n = \frac{n \cdot (n+1)}{2} = n \cdot \frac{n+1}{2}$

last term

first term

n: number of terms

$\frac{n+1}{2}$: average of the first and last terms.

$= \left(\begin{matrix}\text{number of terms}\end{matrix}\right)\left(\begin{matrix}\text{avg. of first and last terms}\end{matrix}\right)$

Arithmetic Series formula

* $S_n = n\left(\frac{a_1+a_2}{2}\right)$

* $11+20+29+\ldots+4052=?$

$n = \frac{4052-2}{9} = 450$

$a_1 = 11$

$a_{450} = 4052$

$S_{450} = 450 \; \frac{11+4052}{2} = 914,175$

* $10-1-12-\ldots-10,979=?$

$n = \frac{-10,979-21}{-11} = 1000$

$S_n = 1,000 \; \frac{-10,969}{2} = -5,484,500$

Example: Sigma Notation

* $\sum_{k=1}^{550}(2k+50) = ? = 52+54+56+\ldots+1,150$

$= 550 \; \frac{52+1,150}{2} = 330,550$

Example: Sum Expression

* $-50 \underbrace{+}_{6} (-44) + (-38) + \ldots + 2038 \underbrace{+}_{6} 2044 = ?$

$n = \frac{2044-(-50)}{6} + 1 = 350$

$S_{350} = 350 \; \frac{2044-50}{2} = 348,950$

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* An arithmetic series is a sum of terms in which each term is found by adding a constant number (a.k.a. the common difference) to the previous term.
* Arithmetic series formula: $S_n = n\left(\frac{a_1+a_n}{2}\right)$
* $n = \frac{a_n-a_1}{d}+1$, where d is the common difference

Example: Recursive formula

* $a_i = a_{i-1} + 11$, $a_1 = 4$ $\Rightarrow$ $S_{650} = ?$

$n = \frac{a_n - a_1}{d} + 1 \Rightarrow 649 = \frac{a_{650} - 4}{11} = 7{,}143$

$S_n = 650 \frac{4 + 7{,}143}{2} = 2{,}322{,}775$

Quiz

* $48 + 45 + 42 + \dots + (-81) + (-84) = ?$

$a_1 = 48$

$n = \frac{-84-48}{-3} + 1$

$= 45$

$S_n = 45 \cdot \frac{-84+48}{2} = -810$

* $\sum_{k=1}^{40} (5k - 87) = ?$

$a_1 = -82$

$a_{40} = 113$

$n = 40$

$\Rightarrow S_{40} = 40 \frac{113 - 82}{2} = 620$

* $a_1 = 92$, $a_i = a_{i-1} - 8$ $\Rightarrow$ $S_{28} = ?$

$n = 28$

$\Rightarrow \frac{a_{28} - 92}{-8} + 1 = 28$

$a_{28} = -124$

$S_{28} = 28 \frac{92 - 124}{2}$

$= -448$