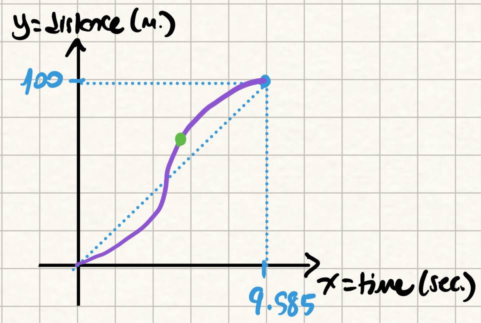
Defining Average and Instantaneous Rates of Change at a Point

06.06.2025

Newton, Leibniz, and Usain Bolt

* what is the instantenous rate of change of something? → DIFFERENTIAL CALCULUS
* How fast is Usain Bolt running at this instant? (not on average!)



$$\text{avg. speed} = \frac{\Delta \text{distance}}{\Delta \text{time}} = \frac{\Delta y}{\Delta x} = \frac{(\text{rise})}{(\text{run})}$$

$$= \frac{100}{9.585} \approx 10.4 \text{ m/s}$$

$$\approx 37.58 \text{ km/h}$$

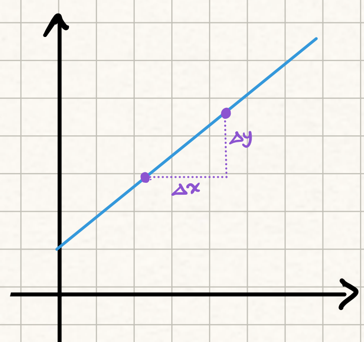
* To find the speed of Bolt at the ●, we need to find the slope of the tangent line to that point, which is $\frac{\Delta y}{\Delta x}$. But we need the smallest Δx possible. Otherwise it's not instantenous speed, it's still avg speed with a smaller interval. How do we find the smallest Δx? LIMITS!

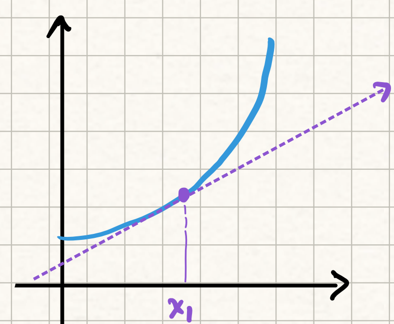
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

→ infinitely small change in y

→ infinitely small change in x

Derivative as a concept





$\text{Slope} = \frac{\Delta y}{\Delta x}$ = Instantenous rate of change = always same on the same line!

* "What is the slope of the tangent line to this point? Because that slope would tell us the instantenous rate of change at that point."

"DERIVATIVE"

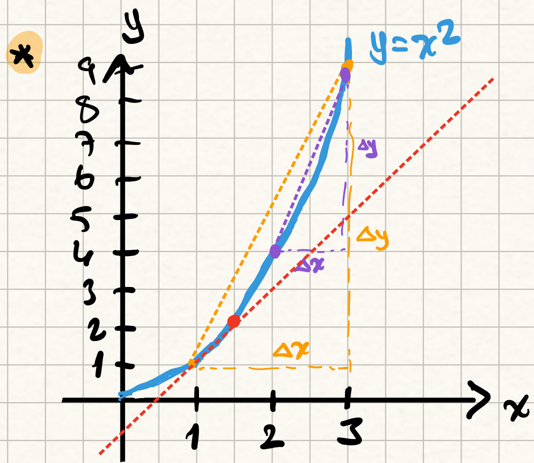
* The derivative quantifies the sensitivity to change of a function's output with respect to its input.
* The derivative of a function of a single variable at a chosen value, when it exists, is the slope of the tangent line to the graph of the function at that point. Formula of which is $\frac{\Delta y}{\Delta x}$.

* derivative $= \frac{dy}{dx} = \frac{d}{dx} f(x)$ (Leibniz's notation) (very useful when dealing with integral calculus, differential equations, multivariable calc.)

* $y = f(x) \Rightarrow$ derivative $= f'(x)$ (Lagrange's notation) ("f prime")

"The slope of $f(x)$ for an x-value"

Secant lines & average rate of change



* Avg. rate of change over [1,3]

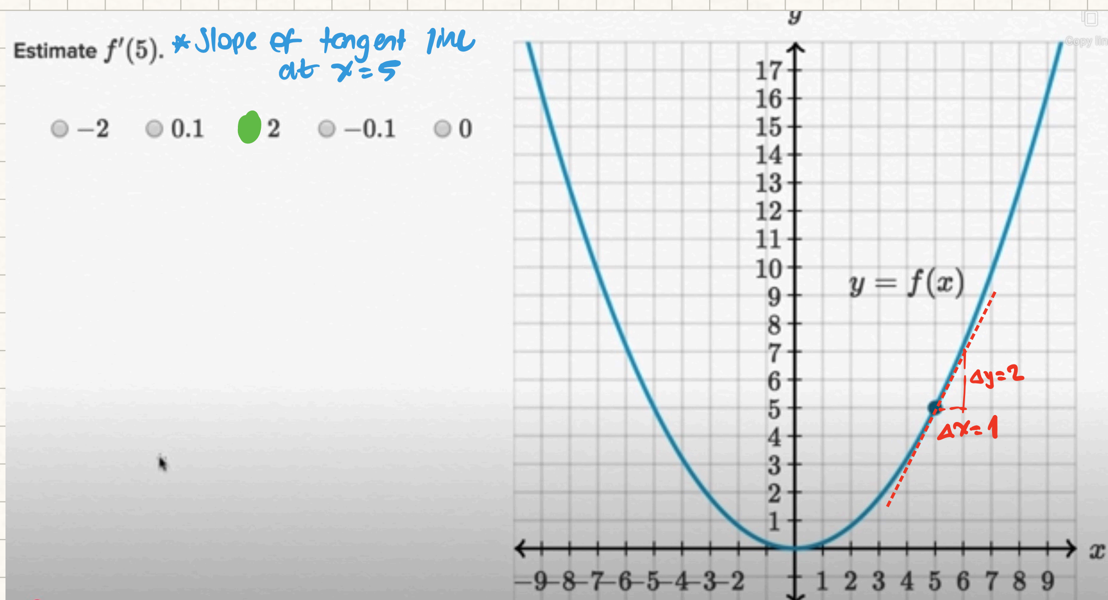
$= \frac{\Delta y}{\Delta x} = \frac{9-1}{3-1} = 4$ (slope of the secant line!)

* Avg. rate of change over [2,3]

$\frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5$

* Avg. rate of change at $x = 1.5$?
Not avg. anymore! It's instantaneous rate of change at $x=2$ and it's the slope of the tangent line.

Derivative as slope of curve



* Lagrange's notation: $f'(x)$ (Pronounced "f prime". Meaning "the slope of $f(x)$ for an x-value". Simple notation.

* Leibniz's notation: $\frac{dy}{dx}$ or $\frac{d}{dx} f(x)$ ("the derivative of $f(x)$ with respect to x."
Sounds more complex, but very useful when dealing with integral calculus, differential equations and multivariable calculus.

The derivative and tangent line equations

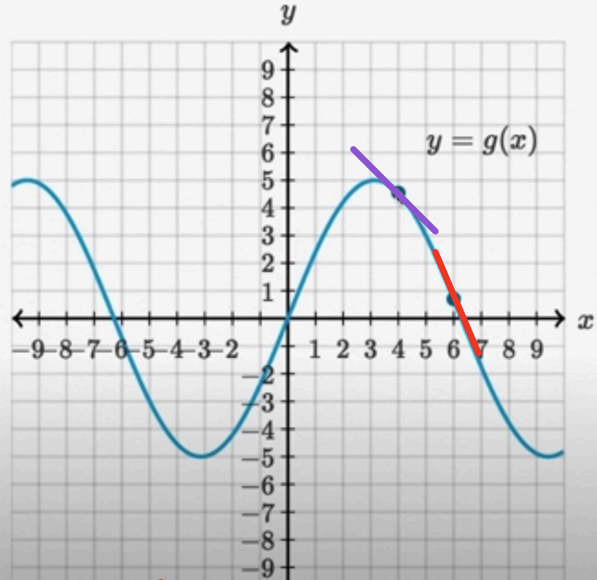
* Compare.

$g'(4) > g'(6)$

○ $<$

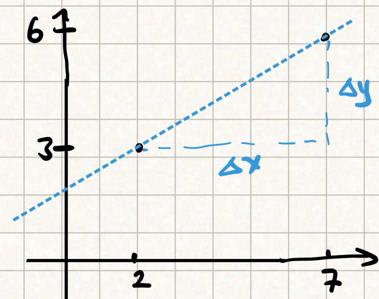
● $>$

red line is steeper but in negative direction

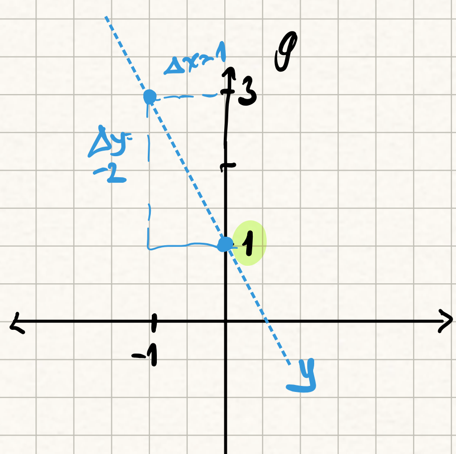


* The tangent line to the graph of f at the point $(2,3)$ passes through $(7,6)$. Find $f'(2)$.

$$= \frac{6-3}{7-2} = \frac{3}{5}$$



* $g(-1)=3$ $g'(-1)=-2$ ⇒ what's the equation of the tangent line to the raph of g at $x=-1$?



* $y-y_1 = m(x-x_1)$

$y-3 = -2(x-(-1))$

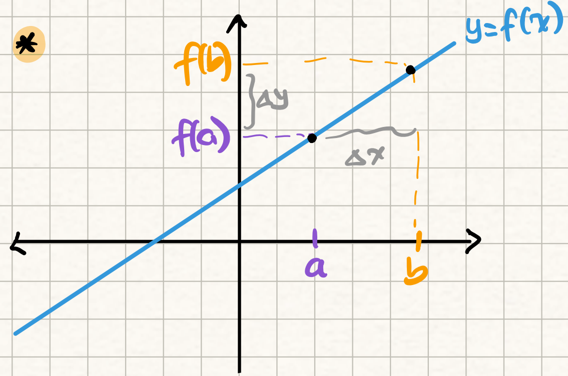
$y-3 = -2x+2$

$y = -2x+3$

Defining the Derivative of a Function and Using Derivative Notation

10.06.2025

Formal definition of the derivative as a limit

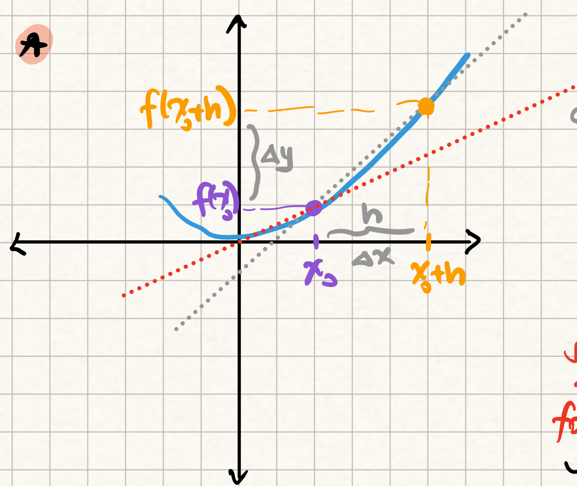


$f(x) = mx + b$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

* The slope at every point on a straight line is constant. Therefore, the derivative of the linear function $f(x) = mx + b$ with respect to x is equal to m for all x in its domain.

Formal form of derivative



$$\text{slope of secant} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0}$$

$$= \frac{f(x_0+h) - f(x)}{h}$$

$$\underbrace{\text{slope of the tangent line for } x = x_0}_{= f'(x)} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Alternate form of derivative

* If we don't need a generalized equation, we only want to find the derivative for one specific point:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 1: Derivative as a limit

* $f(x) = \ln x \Rightarrow f'(e) = ?$

$$\lim_{h \to 0} \frac{f(e+h) - f(e)}{h} = \lim_{h \to 0} \frac{\ln(e+h) - \ln(e)}{h}$$

---

* The slope at every point on a straight line is constant. Therefore, the derivative of the linear function $f(x) = mx + b$ with respect to x is equal to m for all x in its domain.
* Formal form of the derivative: $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$
* If we don't need a generalized equation, we only want to find the derivative for one specific x: $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Example 2: Derivative from limit expression

The alternate form of the derivative of the function f at a number a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

With the Alternate Form of the Derivative as an aid, make sense of the following limit expression by identifying the function f and the number a.

$$f'(\underbrace{5}_{a}) = \lim_{x \to \underbrace{5}_{a}} \frac{\overbrace{x^3}^{f(x)} - \overbrace{125}^{f(a) = f(5)}}{x - \underbrace{5}_{a}} \Rightarrow$$

is the derivative of the function $f(x) =$ x^3

at the number $a =$ 5

Derivative of x^2 at $x=3$

$$f(x) = x^2 \Rightarrow f'(3) = ?$$

$$\lim_{\Delta x \to 0} \frac{(3+\Delta x)^2 - (\Delta x)^2}{(3+\Delta x) - 3} = \lim_{\Delta x \to 0} \frac{\cancel{9} + 6\Delta x + (\Delta x)^2 - \cancel{9}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{6\Delta x}{\Delta x} + \frac{(\Delta x)^2}{\Delta x} \right) = \lim_{\Delta x \to 0} 6 + \underbrace{\Delta x}_{\text{"0"}} = 6$$

Derivative of x^2 at any point

$$f(x) = x^2 \Rightarrow f'(x) = \frac{d}{dx} f(x) = ?$$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{x + \Delta x - x} = \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{2x\cancel{\Delta x}}{\cancel{\Delta x}} + \frac{(\Delta x)^{\cancel{2}}}{\cancel{\Delta x}} \right) = \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x$$

Estimating Derivatives of a Function at a Point

10.06.2025

A

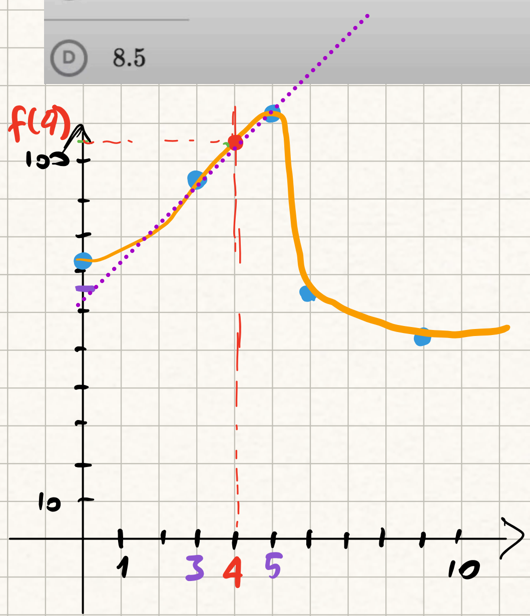
This table gives select values of the differentiable function f.

x	0	3	5	6	9
$f(x)$	72	95	112	77	54

What is the best estimate for $f'(4)$ we can make based on this table?

Choose 1 answer:

- (A) 17
- (B) -2
- (C) 103.5
- (D) 8.5



* we don't know the function's formula, our orange line is an estimation.

The closest points to 4, of which we know the outputs, are 3 and 5.

$f(3) = 95 \qquad f(5) = 112$

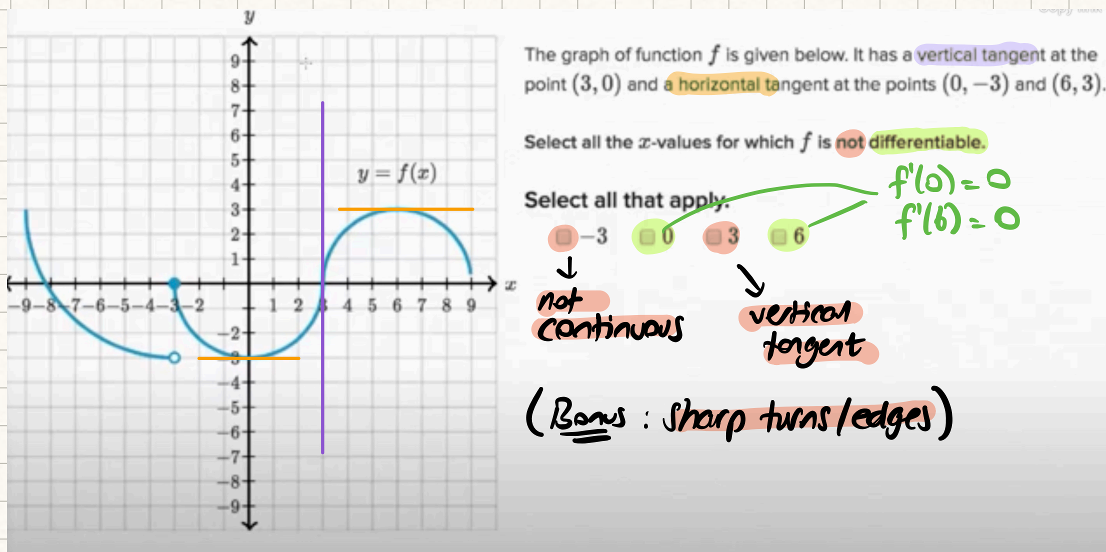
slope of the closest secant line $= \dfrac{f(5) - f(3)}{5-3} = 8.5$

Determining When Derivatives Do and Don't Exist — 10.06.2025

Differentiability and continuity

* If f not continuous at $x=c$, then f is not differentiable at $x=c$.
* If there's jump or removable discont. the slopes of the tangent lines will approach to different values as $x \to c^-$ and $x \to c^+$. If $f(c)$ is undefined, then $\lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c}$ won't even work.
* If f is continuous at $x=c$, it doesn't necessarily mean that it's differentiable at $x=c$.
* $f(x) = |x-c| \Rightarrow$ the slopes of the tangent lines will approach to different values as $x \to c^-$ and $x \to c^+$

Example: Graphical

* 

* If f is not continuous at $x=c$, then f is not differentiable at $x=c$.
* If f is continuous at $x=c$, it doesn't necessarily mean that it's differentiable at $x=c$. The slopes of the tangent lines might approach to different values as $x \to c^-$ and $x \to c^+$.

Example : Algebraic

* $f(x) = \begin{cases} x^2 & , x < 3 \\ 6x-9 & , x \geq 3 \end{cases}$

Is $f(x)$ $\underbrace{\text{continuous}}_{\text{YES}}$ and $\underbrace{\text{differentiable}}_{\text{YES}}$ at $x = 3$?

* $\lim_{x \to c^-} f(3) = \lim_{x \to c^+} f(3) = f(3)$

$9 = 9 = 9$

* $\lim_{x \to 3} \frac{f(x) - f(3)}{x-3}$ 9

$\lim_{x \to 3^-} \frac{f(x)-9}{x-3} = \frac{x^2-9}{x-3} = \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}}$

$= x+3 = \underline{\underline{6}}$

$\lim_{x \to 3^+} \frac{f(x)-9}{x-3} = \frac{6x-9-9}{x-3} = \frac{6(x-3)}{x-3} = \underline{\underline{6}}$

Example 2: Algebraic

* Is $f(x) = \begin{cases} x-1 & , x < 1 \\ (x-1)^2 & , x \geq 1 \end{cases}$ continuous / differentiable at $x = 1$?

YES / NO

* $f(1) = (1-1)^2 = \underline{\underline{0}}$

$\lim_{x \to 1^-} f(x) = (1-1) = \underline{\underline{0}}$

$\lim_{x \to 1^+} f(x) = (1-1)^2 = \underline{\underline{0}}$

* $\lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{f(x) - 0}{x-1}$

$\lim_{x \to 1^-} \frac{(x-1) - 0}{(x-1) - 1} = \lim_{x \to 1^-} \frac{x-1}{x-1} = \underline{\underline{1}}$

$\lim_{x \to 1^+} \frac{(x-1)^2 - 0}{(x-1)^2 - 1} = \underline{\underline{0}}$

Applying the Power Rule

10.06.2025

Power Rule

* $f(x) = x^n \quad , n \neq 0$
  $f'(x) = n \cdot x^{n-1}$

* $f(x) = x^2 \Rightarrow f'(x) = 2x$
  $f(x) = x^3 \Rightarrow f'(x) = 3x^2$
  $f(x) = x^{-100} \Rightarrow f'(x) = -100\, x^{-101}$
  $f(x) = x^{2.571} \Rightarrow f'(x) = 2.571 \cdot x^{1.571}$

... with rewriting the expression

* $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right) = -1 \cdot x^{-2} = \frac{-1}{x^2}$

* $f(x) = \sqrt[3]{x} \Rightarrow f'(x) = f'\left(x^{1/3}\right) = \frac{1}{3} \cdot x^{-2/3}$

* $\frac{d}{dx}\left(\sqrt[3]{x^2}\right)$ at $x = 8$?

  $\frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3} x^{-1/3} \Rightarrow x = 8 \Rightarrow \frac{2}{3} \cdot 8^{(-1/3)}$

  $= \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{\sqrt[3]{8}}_{\,2}} = \frac{1}{3}$

* **Power Rule:** $f(x) = x^n, n \neq 0 \Rightarrow f'(x) = n \cdot x^{n-1}$

Derivative Rules: Introduction

10.06.2025

Basic derivative rules

* $\frac{d}{dx}[k] = 0$

* $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx} f(x) = k \cdot f'(x)$

* $\frac{d}{dx}[2 \cdot x^5] = 2 \cdot \frac{d}{dx}[x^5] = (2)(5)x^4 = 10x^4$

* $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

* $\frac{d}{dx}[x^3 + x^{-4}] = 3x^2 + (-4)x^{-5} = 3x^2 - 4x^{-5}$

* $\frac{d}{dx}[\underbrace{2x^2}_{4x} + \underbrace{3x}_{3} + \underbrace{4}_{0}] = 4x + 3$

Example: Table

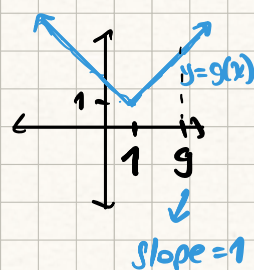
*

x	0	1	4	9	16
$f(x)$	2	1	-2	1	6
$f'(x)$	-3	-1	1	3	4

$g(x) = |x-1| + 1$

$h(x) = 3f(x) + 2g(x)$

$\frac{d}{dx} h(x)$ at $x = 9$ = ?



* $= 3 \cdot \frac{d}{dx}(f(x)) + 2 \frac{d}{dx}(g(x))$

$= 3 \cdot (3) + 2 \cdot (1) = 11$

* Constant Rule: $\frac{d}{dx}[k] = 0$
* Multiplication by a constant: $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}[f(x)]$
* Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Derivative Rules: Connecting with the Power Rule

10.06.2025

Differentiating polynomials

* $f(x) = x^5 + 2x^3 - x^2$

$\frac{d}{dx} f(x) = 5x^4 + 6x^2 - 2x$

* at $x = 2$ = ? $= 5(2)^4 + 6(2)^2 - 2(2) = 100$

Differentiating integer powers

* $g(x) = \frac{2}{x^3} - \frac{1}{x^2} \Rightarrow g'(x) = ? \quad g'(2) = ?$

$g(x) = 2x^{-3} - x^{-2}$

$g'(x) = -6x^{-4} + 2x^{-3}$

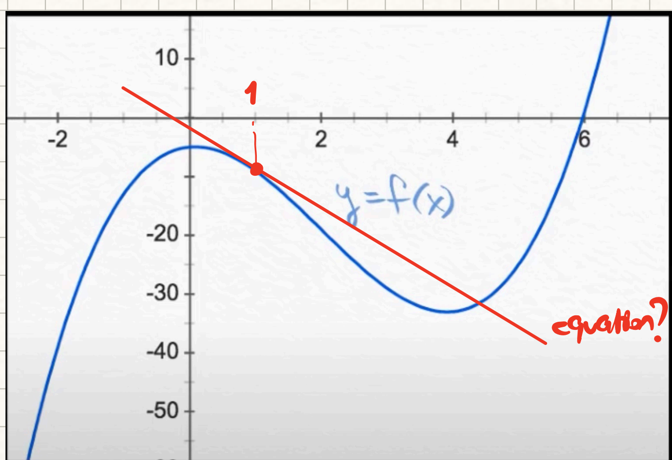
$g'(2) = -6(2)^{-4} + 2(2)^{-3}$

$= \frac{-6}{16} + \frac{2}{8}$

$= \frac{-6+4}{16} = \frac{-2}{16} = \frac{-1}{8}$

Tangents of polynomials

* $f(x) = x^3 - 6x^2 + x - 5$



* $f(1) = (1)^3 - 6(1)^2 + 1 - 5$

$= -9$

$f'(x) = 3x^2 - 12x + 1$

$f'(1) = 3(1) - 12(1) + 1$

$= -8$

$y = m \cdot x + b$

$-8 = -9(1) + b$

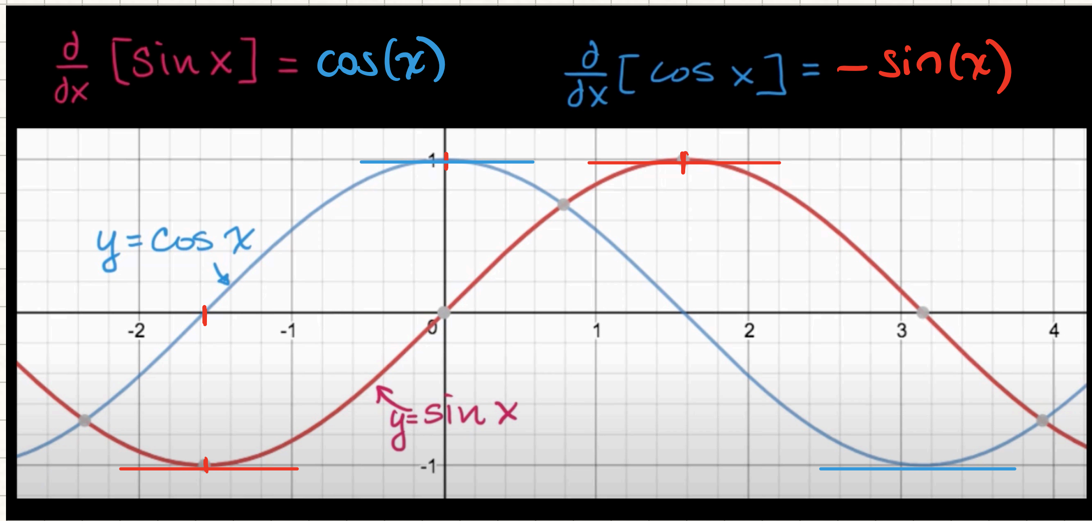
$-1 = b \Rightarrow y = -8x - 1$

Derivatives of $\cos(x)$, $\sin(x)$, e^x, and $\ln(x)$

10.06.2025

Derivatives of $\sin(x)$ and $\cos(x)$

*



Example:

* $g(x) = 7\sin(x) - 3\cos(x) - \left(\frac{\pi}{\sqrt[3]{x}}\right)^2 \Rightarrow g'(x) = ?$

$$g(x) = 7\sin(x) - 3\cos(x) - \left(\pi \cdot x^{-1/3}\right)^2$$

$$= 7\sin(x) - 3\cos(x) - \pi^2 \cdot x^{-2/3}$$

$$g'(x) = 7\cos(x) + 3\sin(x) + \frac{2\pi^2}{3} x^{-5/3}$$

Derivative of e^x

* $\frac{d}{dx}[e^x] = e^x$

Derivative of $\ln(x)$

* $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

---

* $\frac{d}{dx}\sin(x) = \cos(x)$ * $\frac{d}{dx}\cos(x) = \sin(x)$
* $\frac{d}{dx}[e^x] = e^x$ * $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

The Product Rule

10.06.2025

Product Rule

* $$\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)g(x) + f(x)\,g'(x)$$

* $$\frac{d}{dx}\left[x^2\sin(x)\right] = \underbrace{2x}_{\frac{d}{dx}(x^2)}\sin(x) + x^2\underbrace{\cos(x)}_{\frac{d}{dx}[\sin(x)]}$$

Differentiating products

* $$\frac{d}{dx}\left(e^x\cos(x)\right) = -e^x\sin(x) + e^x\cos(x)$$
$$= e^x\left(\cos(x) - \sin(x)\right)$$

Product rule with table

* The following table lists the values of functions f and h, and of their derivatives, f' and h', for $x = 3$.

x	$f(x)$	$h(x)$	$f'(x)$	$h'(x)$
3	6	0	6	4

Evaluate $\frac{d}{dx}[f(x)\cdot h(x)]$ at $x = 3$. $= f'(3)\cdot h(3) + f(3)\,h'(3)$
$= 6\cdot 0 + 6\cdot 4 = 24$

Product Rule with implicit and explicit

* $f(-1) = 3$, $f'(-1) = 5$, $g(x) = \frac{1}{x}$, $F(x) = f(x)\cdot g(x)$

$\Rightarrow F'(-1) = ?$

$= f'(-1)\cdot g(-1) + f(-1)\cdot g'(-1)$

$= (5)(-1) + (3)(-1)$

$= -5 - 3 = -8$

$g'(x) = \frac{d}{dx}(x^{-1})$
$= -x^{-2}$
$= \frac{-1}{x^2}$

* Product Rule: $\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)g(x) + f(x)g'(x)$

The Quotient Rule

10.06.2025

Quotient Rule

* $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

* $$f(x) = \frac{x^2}{\cos(x)} \Rightarrow f'(x) = ?$$

$$= \frac{2x\cdot\cos(x) + x^2\sin(x)}{\cos^2(x)}$$

... with table

*
- Let f be a function such that $f(-1) = 3$ and $f'(-1) = 5$.
- Let g be the function $g(x) = 2x^3$. $g'(x) = 6x^2$

Let F be a function defined as $F(x) = \dfrac{f(x)}{g(x)}$.

$g(-1) = -2$

$g'(-1) = 6$

$F'(-1) =$ ______

$$F(-1) = \frac{f'(-1)\cdot g(-1) - f(-1)\cdot g'(-1)}{(g(-1))^2} = \frac{(5)(-2) - (3)(6)}{(-2)^2}$$

$$= \frac{-10-18}{4} = \frac{-28}{4} = -7$$

Differentiating rational functions

* $$\frac{d}{dx}\left[\frac{5-3x}{x^2+3x}\right] = \frac{(-3)(x^2+3x) - (5-3x)(2x+3)}{(x^2+3x)^2}$$

$$= \frac{-3x^2 - \cancel{9x} - 10x - 15 + 6x + \cancel{9x}}{x^2(x+3)^2} = \frac{3x^2 - 4x - 15}{(x^2+3x)^2}$$

* Quotient Rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Finding the Derivatives of tan(x), cot(x), sec(x), and csc(x)

10.06.2025

Derivatives of tan(x) and cot(x)

* $$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$$
$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

* $$\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)}$$
$$= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

Derivatives of sec(x) and csc(x)

* $$\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] = \frac{0\cdot\cos(x) + \sin(x)\cdot 1}{\cos^2(x)}$$
$$= \frac{\sin(x)}{\cos^2(x)}$$

* $$\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right] = \frac{0\cdot\cos(x) - \cos(x)}{\sin^2(x)}$$
$$= \frac{-\cos(x)}{\sin^2(x)}$$

* $\frac{d}{dx}[\tan(x)] = \frac{1}{\cos^2(x)} = \sec^2(x)$
* $\frac{d}{dx}[\cot(x)] = \frac{-1}{\sin^2(x)} = -\csc^2(x)$
* $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
* $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$