AP CALCULUS BC

01 LIMITS AND CONTINUITY

* A limit is the value that a function (or sequence) approaches as the argument (or index) approaches some value.
* When a limit doesn't approach the same value from both sides, then the limit doesn't exist: $\lim_{x\to c} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$
* Just because a function is undefined for some x-value doesn't mean there's no limit. On the other hand, just because a function is defined for some x-value doesn't mean that limit exists.
* Let $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$,
    * $\lim_{x\to c} (f(x)+g(x)) = L+M$
    * $\lim_{x\to c} (f(x)-g(x)) = L-M$
    * $\lim_{x\to c} (f(x)\cdot g(x)) = L\cdot M$
    * $\lim_{x\to c} (k\cdot f(x)) = k\cdot L$
    * $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$
    * $\lim_{x\to c} \left(f(x)^{\frac{r}{s}}\right) = L^{\frac{r}{s}}$
* When evaluating the limits of combined functions, we must verify whether the left-hand and right-hand limits are equal. If they are, the overall limit exist, even if the individual limit of one of the component functions does not.
* $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ if and only if:

    ① $\lim_{x\to a} g(x) = L$ (exists)

    ② AND $f(x)$ is continuous at L.

* Selecting procedures for determining limits:

Calculating $\lim_{x \to a} f(x)$

A. Direct substitution

Try to evaluate the function directly.

$$f(a)$$

$$f(a) = \frac{b}{0}$$

where ***b*** is not zero

B. Asymptote (probably)

example:

$$\lim_{x \to 1} \frac{1}{x-1}$$

Inspect with a graph or table to learn more about the function at ***x=a***.

$$f(a) = b$$

where ***b*** is a real number

C. Limit found (probably)

example:

$$\lim_{x \to 3} x^2 = (3)^2 = 9$$

$$f(a) = \frac{0}{0}$$

D. Indeterminate form

example:

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

Try rewriting the limit in an equivalent form.

E. Factoring

example:

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

can be reduced to

$$\lim_{x \to -1} \frac{x-2}{x-3}$$

by factoring and cancelling.

F. Conjugates

example:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

can be rewritten as

$$\lim_{x \to 4} \frac{1}{\sqrt{x} + 2}$$

using conjugates and cancelling.

G. Trig identities

example:

$$\lim_{x \to 0} \frac{\sin(x)}{\sin(2x)}$$

can be rewritten as

$$\lim_{x \to 0} \frac{1}{2\cos(x)}$$

using a trig identity.

Try evaluating the limit in its new form.

H. Approximation

When all else fails, graphs and tables can help approximate limits.

* If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$

* f is discontinuous if we have to pick up our pencil while graphing it. There are 3 types of discontinuity:
   1. Removable (Point) Discontinuity: Two-sided limit exists, which means one-sided limits exist and are equal to each other. However, two-sided limit is not equal to the function's value at that point.
   2. Jump Discontinuity : Two-sided limit does not exist because one-sided limits are not equal to each other, even though they exist.
   3. Asymptotic Discontinuity: Two-sided limit does not exist because one-sided limits are unbounded, therefore they don't exist.
* f is continuous at $x = c$ $\iff$ $\lim_{x \to c} f(x) = f(c)$

* f is continuous over (a,b) $\iff$ f is continuous over every point in the interval
* f is continuous over $[a,b]$ $\iff$ f is continuous over (a,b) AND $\lim_{x \to a^+} f(x) = f(a)$ AND $\lim_{x \to b^-} f(x) = f(b)$
* f is continuous on all real numbers $\iff$ it has no types of discontinuity (removable, jump, or asymptotic).
* When the one-sided limits are unbounded to the same direction, we can say that limit is going to infinity in that direction.
* A function can not cross its vertical asymptote, but it can cross its horizontal asymptote (even multiple times).
* Functions with horizontal asymptotes have one-sided limits for x approaches ∞ and $-\infty$. These limits are finite real numbers.
* The intermediate value theorem states that for any function f that is continuous over the interval $[a,b]$, the function will take any value between $f(a)$ and $f(b)$ over the interval. More formally, for any value L between $f(a)$ and $f(b)$, there's a value c in $[a,b]$ for which $f(c) = L$.