Determinants

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \operatorname{Det}(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$\operatorname{Det}(A) = 0 \Rightarrow A$: singular

$\operatorname{Det}(A) \neq 0 \Rightarrow A$: invertible

$$* \begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 6 \\ 0 & -2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \Rightarrow |A| = +1 \underbrace{\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix}}_{\text{Det of other rows and columns}} - 0 \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} + 6 \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}$$

$$B = \begin{bmatrix} -2 & 3 & 1 & 4 \\ 1 & 1 & 7 & 2 \\ 0 & 0 & -1 & 2 \\ -3 & 2 & 1 & 1 \end{bmatrix} \Rightarrow |B| = 0 \cancel{\begin{vmatrix} 3 & 1 & 4 \\ 1 & 7 & 2 \\ 2 & 1 & 1 \end{vmatrix}} - 0 \cancel{\begin{vmatrix} -2 & 1 & 4 \\ 1 & 7 & 2 \\ -3 & 1 & 1 \end{vmatrix}}$$

$$+ (-1) \begin{vmatrix} -2 & 3 & 4 \\ 1 & 1 & 2 \\ -3 & 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 & 1 \\ 1 & 1 & 7 \\ -3 & 2 & 1 \end{vmatrix}$$

we choose this row because of zeros. They will cancel

$$= -1\left(-1\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 4 \\ -3 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix}\right) - 2\left(-1 \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 1 \\ -3 & 1 \end{vmatrix} - 7 \begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix}\right)$$

$$= -1(5 + 10 - 10) - 2(-1 + 1 - 35) = -5 + 70 = 65$$

Rule of Sarrus

(only for 3x3 matrices)

$$\begin{vmatrix} -2 & 3 & 1 \\ 1 & 1 & 7 \\ 0 & 0 & -1 \end{vmatrix} = \begin{bmatrix} -2 & 3 & 1 & -2 & 3 \\ 1 & 1 & 7 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$-2 \cdot 1 \cdot (-1)$ $-\ 3 \cdot 1 \cdot (-1)$

$+\ 3 \cdot 7 \cdot 0$ $-\ (-2) \cdot 7 \cdot 0$

$+\ 1 \cdot 1 \cdot 0$ $-\ 1 \cdot 1 \cdot 0$

$= 2 + 0 + 0 + 3 = 5$

* Rule of Sarrus for 3x3 matrices: $|A| = a \cdot e \cdot i + bfg + cdh - afh - bdi - ceg$

Quiz ① $\operatorname{Det}\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix}\right) = ?$

$1\cdot(-2)\cdot(-2) + 0 \dots -2\cdot 0 \dots - (-2)(-2)\,1 - 0 \dots - 1\cdot 3\cdot 1$

$= 4 - 4 - 3 = -3$

② $\operatorname{Det}\left(\begin{bmatrix} -3 & 1 & 5 & 2 \\ 0 & 0 & -1 & 1 \\ 2 & -2 & 3 & 0 \\ 1 & 4 & 0 & -4 \end{bmatrix}\right) = ?$

$= -(-1)\begin{vmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 4 & -4 \end{vmatrix} + 1 \begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$

$= 1\left(-24 + 0 + 16 + 4 + 8 - 0\right) + 1\left(0 + 3 + 40 + 10 - 0 + 36\right)$

$= +4 + 89 = 93$

③ Use the determinant to say whether $F = \begin{bmatrix} 1 & 5 & 0 & -1 \\ 3 & -2 & -1 & 2 \\ -1 & 1 & 0 & 3 \\ 1 & 3 & 2 & -2 \end{bmatrix}$ is invertible

$\operatorname{Det}(F) = -1\cdot\begin{vmatrix} 5 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & 2 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & -1 \\ 3 & -1 & 2 \\ 1 & 2 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 & 0 \\ 3 & -2 & -1 \\ 1 & 3 & 2 \end{vmatrix}$

$= -1\left(10 + 0 + 4 - 3 - 0 - 20\right) - 1\left(2 + 0 - 6 - 1 - 0 - 4\right) - 3\left(-4 - 5 - 30 + 3\right)$

$= 9 + 9 + 108 = 126$

Cramer's rule for solving systems

$$\left.\begin{aligned} 9x + 10y &= 34 \\ -6x - 5y &= -26 \end{aligned}\right\} \Rightarrow x = \frac{D_x}{D} \qquad y = \frac{D_y}{D}$$

D_y → Determinant with respect to y

D → Determinant of the coefficients

$$\begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} -9 & 10 \\ -6 & -5 \end{bmatrix} \quad \begin{bmatrix} 34 \\ -26 \end{bmatrix}$$

$$D_x = \begin{vmatrix} 34 & 10 \\ -26 & -5 \end{vmatrix} = 90 \qquad D_y = \begin{vmatrix} -9 & 34 \\ -6 & -26 \end{vmatrix} = -30 \qquad D = \begin{vmatrix} 9 & 10 \\ -6 & -5 \end{vmatrix} = 15$$

$$x = \frac{90}{15} = 6 \qquad y = \frac{-30}{15} = -2$$

Ex:

$$\begin{aligned} 3x - 2y + 7z &= -41 \\ -2x + y - 5z &= 26 \\ x + 5y - 4z &= 57 \end{aligned} \Rightarrow (x, y, z) = ?$$

$$D_x = \begin{vmatrix} -41 & -2 & 7 \\ 26 & 1 & -5 \\ 57 & 5 & -4 \end{vmatrix} = 12 \qquad D_z = \begin{vmatrix} 3 & -2 & -41 \\ -2 & 1 & 26 \\ 1 & 5 & 57 \end{vmatrix} = -48$$

$$D_y = \begin{vmatrix} 3 & -41 & 7 \\ -2 & 26 & -5 \\ 1 & 57 & -4 \end{vmatrix} = 96 \qquad D = \begin{vmatrix} 3 & -2 & 7 \\ -2 & 1 & -5 \\ 1 & 5 & -4 \end{vmatrix} = 12$$

$$\Rightarrow (x, y, z) = \left(\frac{12}{12}, \frac{96}{12}, \frac{-48}{12}\right) = (1, 8, -4)$$

* Cramer's Rule : $a_1x + b_1y = d_1$, $a_2x + b_2y = d_2$ $\Rightarrow$

$x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, with $D \neq 0$, where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_x = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$$

Modifying determinants

* Scalar multipl: $\text{Det}(k \cdot A) = k^2 \cdot |A|$
* $\text{Det}\left(\begin{bmatrix} k.a & k.b \\ c & d \end{bmatrix}\right) = k.\ \text{Det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ (If only one row)
* Sum of Rows: $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ → one row is same → one row is the sum

  $|A| + |B| = |C|$

* Swapped Row Rule: For each row swap, multiply determinant by -1.

  Therefore if we have identical rows, Det. is zero.

* Row operations don't effect the determinant.

Quiz ① Find the Det(C), using only the Det(A) and Det(B).

$A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 1 \\ -2 & 4 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 \\ -1 & 5 \end{bmatrix}$

* $A_{1,c} = B_{1,c} = C_{1,c}$ and $A_{2,c} + B_{2,c} = C_{2,c} \Rightarrow$

  $|C| = |A| + |B| = -3 - 6 = -9$ ✓

Upper and lower triangular matrices

* Upper Triangle Matrix: All the values below main diagonal are zero.
* Lower " " " " " " above " " " " .
* Det. for UTM or LTM is the product of the values in main diagon.

Ex (UTF)

$R_3 - R_1 \to R_3,\ R_1 + R_4 \to R_4 \qquad \tfrac{3}{2}R_2 + R_3 \to R_3$

$A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 2 & 2 \\ -1 & 0 & 1 & -2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 4 & -1 \end{vmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -5/2 & 1 \\ 0 & 0 & 4 & -1 \end{vmatrix}$

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* Determinant Modification Rules:

  ① Multiplying a row of the matrix by a scalar requires that we multiply the determinant by the same scalar

  ② When two rows of a matrix are swapped, the determinant must be multiplied by -1.

$$\xrightarrow{\frac{8}{5}R_3+R_4\to R_4}\quad \Rightarrow |A| = \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -5/2 & 1 \\ 0 & 0 & 0 & 3/5 \end{vmatrix} = -3$$

Ex: (LTF)

$$|A| = \begin{vmatrix} 1 & 0 & 3 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 2 & 2 \\ -1 & 0 & 1 & -2 \end{vmatrix} \xRightarrow{R_3+R_4\to R_3,\ R_4/2+R_1\to R_1} |A| = \begin{vmatrix} 1/2 & 0 & 7/2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ -1 & 0 & 1 & -2 \end{vmatrix}$$

$$\xRightarrow{\frac{2}{7}\cdot R_1+R_2\to R_2} |A| = \begin{vmatrix} 1/2 & 0 & 7/2 & 0 \\ 1/7 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ -1 & 0 & 1 & -2 \end{vmatrix} \xRightarrow{-7/6\cdot R_3+R_1\to R_1} |A| = \begin{vmatrix} 1/2 & 7/2 & 0 & 0 \\ 1/7 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ -1 & 0 & 1 & -2 \end{vmatrix}$$

$$\xRightarrow{-7/4\cdot R_2+R_1\to R_1} |A| = \begin{vmatrix} +1/4 & 0 & 0 & 0 \\ 1/7 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ -1 & 0 & 1 & -2 \end{vmatrix} = -3$$

Quiz (1) Use UTF or LTF to find $\det\left(\begin{bmatrix} 4 & -2 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{bmatrix}\right)$

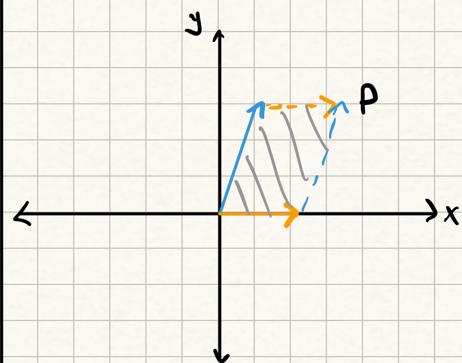
$$\xrightarrow{R_2+R_4\to R_2} |A| = \begin{vmatrix} 4 & -2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{vmatrix} \xRightarrow{-\frac{3}{2}R_3+R_2\to R_2} |A| = \begin{vmatrix} 4 & -2 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{vmatrix} \xRightarrow{-2R_2+R_1\to R_1} |A| = \begin{vmatrix} -6 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{vmatrix}$$

$$\Rightarrow |A| = (-6)\cdot(-1)\cdot 2\cdot(-1) = -12$$

- Upper Triangular Matrix: All entries below main diagonal are zeros.
- Lower Triangular Matrix: All entries above main diagonal are zeros.
- Determinant of a LTM or UTM is the product of the entries in the main diagonal.

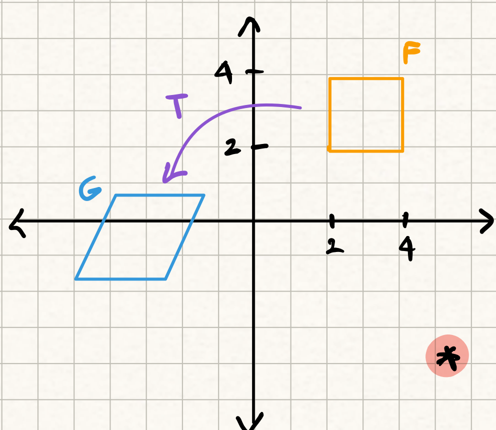
Using determinants to find area

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$



$$\begin{aligned} \text{Area}_P &= |\text{Det}(A)| \\ &= |0-6| \\ &= 6 \end{aligned}$$

Using determinant to find the area of the transformed image



$$T(\vec{x}) = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

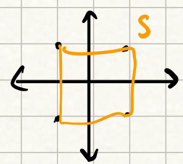
$$\begin{aligned} \text{Area}_G &= |\text{Area}_F \cdot \text{Det}(T)| \\ &= |(2\times 2)\cdot(-1-0)| \\ &= |4\cdot(-1)| = |-4| = 4 \end{aligned}$$

Quiz ① Find the area of P formed by (2,3) and (-1,4) vectors.

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \Rightarrow \text{Det}(A) = 11 \qquad \text{Area}_P = |11| = 11$$

- $\text{Area}_F = \text{Det}\,|A|$ (F is the figure created by the column vectors)
- $T: F \to G \Rightarrow \text{Area}_G = |\text{Area}_F \cdot \text{Det}(T)|$

(2) The square S has $(1,1), (-1,1), (-1,-1), (1,-1)$ as vertices. If $T(S) = F$ and $T(\vec{x}) = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \vec{x}$, then $Area_F = ?$

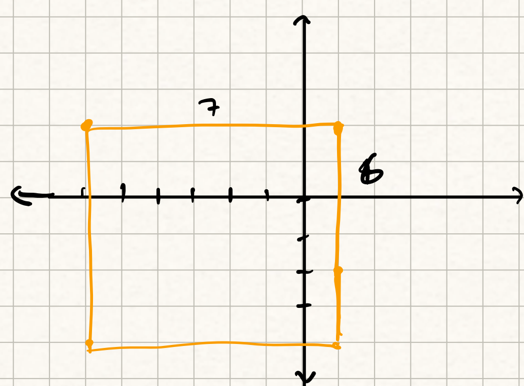


$Area_S = 4$

$Det(T) = -3 + 4 = 1$

$Area_F = |4 \cdot 1| = 4$

(3) The rectangle R has $(-6,2), (1,2), (1,-4), (-6,-4)$ as its corners. $T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix} \cdot \vec{x}$, and $T(R) = L \Rightarrow Area_L = ?$



$Area_R = 42$

$Det(T) = 8$

$Area_L = |42 \cdot 8|$

$= 336$