The Chain Rule : Introduction

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Chain Rule

* $$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\, g'(x) = \frac{d[f(g(x))]}{dg(x)} \cdot \frac{d[g(x)]}{dx}$$

* $f(x) = \sin^2(x) \Rightarrow$

$$f'(x) = \underbrace{\frac{d}{d\sin(x)}\left[\sin^2(x)\right]} \cdot \underbrace{\frac{d}{dx}\left[\sin(x)\right]}$$

$$= \underline{2\sin(x)} \cdot \underline{\cos(x)}$$

* We take the derivative of the composite function with respect to the inner function. Then we multiply that with the derivative of the inner function with respect to x.

Common chain rule misunderstandings

* $$\underbrace{\frac{d}{dx}\left[\ln(\sin(x))\right]}_{\text{requires chain rule}} \neq \underbrace{\frac{d}{dx}\left[\ln(x)\sin(x)\right]}_{\text{requires product rule}}$$

* $$\frac{d}{dx}\left[\ln(\sin(x))\right] = \underbrace{\frac{1}{\sin(x)}}_{\text{b/c } \frac{d}{dx}\ln(x) = \frac{1}{x}} \cdot \underbrace{\cos(x)}_{\text{don't forget this step}}$$

* $$\frac{d}{dx}\left[\ln(\sin(x))\right] \neq \frac{1}{\cos(x)}$$

This is taking the derivative of the composite func. with respect to the derivative of the inner function which is not what we want!

* Chain Rule: We take the derivative of the composite function with respect to inner function and multiply that with the derivative of the inner function with respect to x.

$$\frac{d}{dx}\left[f(g(x))\right] = \frac{d[f(g(x))]}{dg(x)} \cdot \frac{d[g(x)]}{dx}$$

Identifying composite functions

(!) If you're not familiar w/ composite functions, see the Precalculus folder.

* $f(x) = 1 + x$, $g(x) = \cos(x)$
$\Rightarrow f \circ g(x) = f(g(x)) = 1 + \cos(x)$ $\left(x \to \boxed{g} \to \boxed{f} \to f(g(x)) \right)$
$\Rightarrow g \circ f(x) = g(f(x)) = \cos(1+x)$ $\left(x \to \boxed{f} \to \boxed{g} \to g(f(x)) \right)$

* Let $g(x) = \cos(\sin(x) + 1)$
$u(x) = \sin(x)$, $v(x) = x + 1 \Rightarrow g(x) = v(u(x))$

Example: Derivative of $\cos^3(x)$

* $f(x) = \cos^3(x) = (\cos(x))^3$
$\Rightarrow f'(x) = 3(\cos(x))^2(-\sin(x))$
$= -3\cos^2(x)\sin(x)$

Example 2:

* $f(x) = \sqrt{3x^2 - x} \Rightarrow f'(x) = ?$

$$= \frac{d(\sqrt{3x^2 - x})}{d_{3x^2-x}} \cdot \frac{d[3x^2 - x]}{dx}$$

$$= \frac{1}{2}(3x^2 - x)^{-1/2} \cdot (6x - 1) = \frac{6x - 1}{2\sqrt{3x^2 - x}}$$

Example 3:

* $f(x) = \ln(\sqrt{x}) \Rightarrow f'(x) = ?$

$$= \frac{d[\ln(\sqrt{x})]}{d\sqrt{x}} \cdot \frac{d[\sqrt{x}]}{dx} = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x}$$

The Chain Rule : Further Practice

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Chain rule with table

The following table lists the values of functions f and g, and of their derivatives, f' and g', for the x-values -2 and 4.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-2	5	-1	1	6
4	-4	-2	0	8

Let function F be defined as $F(x) = f\big(g(x)\big)$.

$F'(4) =$

$$F'(4) = f'(g(4)) \cdot g'(4)$$
$$= f'(-2) \cdot (8)$$
$$= 1 \cdot 8 = 8$$

Derivative of a^x (for any positive base a)

$$\frac{d}{dx}[a^x] = ?$$

we know that $a^{\log_a x} = x$, which means $e^{\log_e x} = x = e^{\ln x}$

$$\Rightarrow \frac{d}{dx}\left[\left(e^{\ln(a)}\right)^x\right] = ?$$

$$= \frac{d}{dx}\left[e^{\ln(a)\cdot x}\right] = \frac{d\left[e^{\ln(a)\cdot x}\right]}{d\,\ln(a)\cdot x} \cdot \frac{d\left[\overset{\text{a constant!}}{\ln(a)}\cdot x\right]}{dx}$$

$$= e^{\ln(a)\cdot x} \cdot \ln(a) = a^x \cdot \ln(a) = \frac{d}{dx}[a^x]$$

$$\frac{d}{dx}[8 \cdot 3^x] = 8 \cdot \ln(3) \cdot 3^x$$

Derivative of $\log_a x$ (for any positive base $a \neq 1$)

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln(x)}{\underbrace{\ln(a)}_{\text{constant!}}}\right] = \frac{d}{dx}\left[\frac{1}{\ln(a)} \cdot \ln(x)\right]$$

$$= \frac{1}{\ln(a)} \cdot \frac{1}{x} = \frac{1}{\ln(a) \cdot x} = \log_a(x)$$

$$\frac{d}{dx}[a^x] = a^x \cdot \ln(a)$$

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{x \cdot \ln(a)}$$

Example:

$$* \ \frac{d}{dx}\left[7^{x^2-x}\right] = ?$$

$$= \frac{d\left[7^{(x^2-x)}\right]}{d(x^2-x)} \cdot \frac{d\left[x^2-x\right]}{dx} = 7^{x^2-x} \cdot \ln(7) \cdot (2x-1)$$

Example:

$$* \ \frac{d}{dx}\left[\log_4(x^2+x)\right] = ?$$

$$= \frac{d\left[\log_4(x^2+x)\right]}{d\left[x^2+x\right]} \cdot \frac{d\left[x^2+x\right]}{dx} = \frac{2x+1}{\ln(4)\cdot(x^2+x)}$$

Example:

$$* \ \frac{d}{dx}\left[\sec\left(\frac{3\pi}{2}-x\right)\right] = ? \qquad * \ \dots \text{ at } x = \frac{\pi}{4}$$

$$= \frac{d\left[\sec\left(\frac{3\pi}{2}-x\right)\right]}{d\left[\frac{3\pi}{2}-x\right]} \cdot \frac{d\left[\frac{3\pi}{2}-x\right]}{dx} = \sec\left(\frac{3\pi}{2}-x\right)\cdot\tan\left(\frac{3\pi}{2}-x\right)\cdot(-1)$$

$$* \ (-1)\cdot\sec\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)\cdot\tan\left(\frac{3\pi}{2}-\frac{\pi}{4}\right) = \sqrt{2}$$

Example

$$* \ \frac{d}{dx}\left[\sqrt[4]{x^3+4x^2+7}\right] = \frac{d\left[\sqrt[4]{x^3+4x^2+7}\right]}{d(x^3+4x^2+7)} \cdot \frac{d\left[x^3+4x^2+7\right]}{dx}$$

$$= \frac{1}{4}\cdot(x^3+4x^2+7)^{-3/4}\cdot(3x^2+8x)$$

Implicit Differentiation

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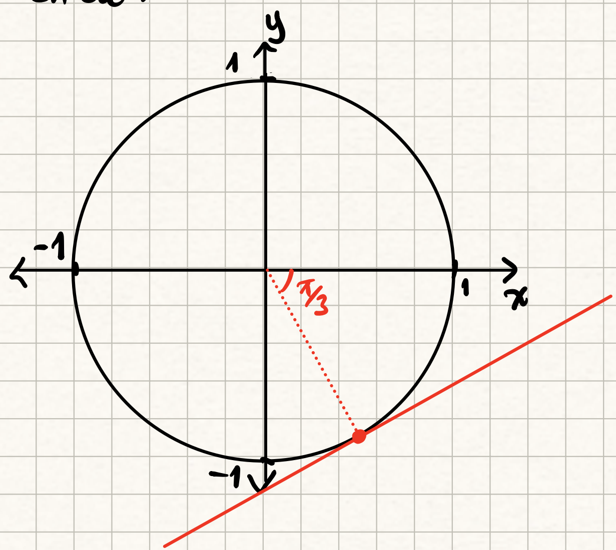
* Let $x^2+y^2=1$. When we graph this relationship (this is not a function, b/c no one-to-one mapping), we get the unit circle:



* Now imagine we want to find the slope of the tangent line that passes through $(\cos(\frac{-\pi}{3}), \sin(\frac{-\pi}{3}))$. For this, we will use implicit differentiation.

* $x^2+y^2=1 \Rightarrow \frac{d}{dx}[x^2+y^2] = \frac{d}{dx}[1]$

$$\Rightarrow \frac{d}{dx}[x^2] + \underbrace{\frac{d}{dx}[y^2]} = 0$$

* At this step, we assume that y does change with respect to x. So it's not a constant. So we calculate $\frac{d}{dx}[(y(x))^2]$

$$\Rightarrow (2x) + \frac{d[y^2]}{dy} \cdot \frac{d[y]}{dx} = 0$$

$$\Rightarrow (2x) + (2y) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

* $(x,y) \Rightarrow (\cos(\frac{-\pi}{3}), \sin(\frac{-\pi}{3}))$

$$\Rightarrow \frac{dy}{dx} = \frac{-\cos(\frac{-\pi}{3})}{\sin(\frac{-\pi}{3})} = \frac{-1/2}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

* In implicit differentiation, we differentiate each side of an equation with two variables by treating one of the variables as a function of the other, not as a constant. This calls for using the chain rule.

Example :

$$(x-y)^2 = x + y - 1 \Rightarrow \frac{dy}{dx} = ?$$

$$\frac{d}{dx}\left[(x-y)^2\right] = \frac{d}{dx}\left[x+y-1\right]$$

$$\Rightarrow 2(x-y)\left(1-\frac{d}{dx}\right) = 1 + \frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{2y-2x+1}{2y-2x-1}$$

Example :

$$x^2 + (y-x)^3 = 28 \Rightarrow \text{slope of the tangent line at } x=1\,?$$

$$x=1 \Rightarrow 1+(y-1)^3 = 28$$
$$y = 4$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[(y-x)^3\right] = \frac{d}{dx}\left[28\right]$$

$$\Rightarrow (2x) + 3(y-x)^2\left(\frac{dy}{dx}-1\right) = 0$$

$$(x,y) = (1,4) \Rightarrow 2(1) + 3(4-1)^2\left(\frac{dy}{dx}-1\right) = 0$$

$$\Rightarrow 2 + 27\left(\frac{dy}{dx}-1\right) = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{-2}{27} + 1 = \frac{25}{27}$$

Quiz

$$x^3y - 2x^2 + y^4 = 8 \Rightarrow \frac{dy}{dx} = ?$$

$$\frac{d}{dx}\left[\underbrace{x^3y}_{\text{product rule}}\right] - \frac{d}{dx}\left[2x^2\right] + \frac{d}{dx}\left[y^4\right] = 0$$

$$3x^2y + x^3\underbrace{\frac{dy}{dx}}_{a} - 4x + 4y^3\left(\underbrace{\frac{dy}{dx}}_{a}\right) = 0$$

$$3x^2y + x^3a - 4x + 4y^3a = 0 \quad \Rightarrow x^3a + 4y^3a = -3x^2y + 4x$$

$$a = \frac{-3x^2y+4x}{x^3+4y^3}$$

"Showing explicit and implicit diff. give same result"

* $x\sqrt{y} = 1 \Rightarrow \sqrt{y} = \frac{1}{x} \Rightarrow y = \frac{1}{x^2} = x^{-2}$

$\Rightarrow y' = -2x^{-3}$

$x\sqrt{y} = 1 \Rightarrow \frac{d}{dx}[x\sqrt{y}] = \frac{d}{dx}[1]$

$= \frac{d}{dx}[x] \cdot \sqrt{y} + x \frac{d}{dx}[\sqrt{y}] = 0$

$\Rightarrow \sqrt{y} + x \cdot \frac{1}{2} y^{-1/2} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \sqrt{y} + \frac{x}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\sqrt{y} \cdot \frac{2\sqrt{y}}{x} = \frac{-2y}{x}$

* $y = \frac{1}{x^2} \Rightarrow \frac{-2y}{x} = \frac{-2\left(\frac{1}{x^2}\right)}{x} = -2x^{-3}$

Differentiating Inverse Functions

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* $f(x) = g^{-1}(x) \iff g'(x) = f(x)$

$\Rightarrow g(f(x)) = x$

$\Rightarrow \frac{d}{dx}\left[g(f(x))\right] = \frac{d}{dx}[x]$

$\Rightarrow g'(f(x)) \cdot f'(x) = 1$

$\Rightarrow f'(x) = \dfrac{1}{g'(f(x))}$

* $f(x) = e^x \Rightarrow f^{-1}(x) = \ln(x)$

$\Rightarrow f'(x) = e^x = \dfrac{1}{\frac{1}{e^x}}$

Example: From Equation

* Let $f(x) = \frac{1}{2}x^3 + 3x - 4$ and $h(x) = f^{-1}(x)$. Notice that $f(-2) = -14$. What is $h'(-14) = ?$

$$h'(-14) = \frac{1}{f'(\underbrace{h(-14)}_{-2})} = \frac{1}{f'(-2)} = \frac{1}{9}$$

$$f'(x) = \frac{3}{2}x^2 + 3 \Rightarrow f'(-2) = \frac{3}{2} \cdot 4 + 3 = 9$$

Example: From table

* Let g and h be **inverse** functions.

The following table lists a few values of g, h, and g'.

x	$g(x)$	$h(x)$	$g'(x)$
3	5	4	$-\frac{1}{4}$
4	3	1	$\frac{1}{2}$

$h'(3) =$

$$h'(3) = \frac{1}{g'(\underbrace{h(3)}_{4})} = \frac{1}{g'(4)} = \frac{1}{1/2} = 2$$

* $f^{-1}(x) = g(x) \Rightarrow f'(x) = \dfrac{1}{g'(f(x))}$

Differentiating Inverse Trigonometric Functions

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Derivative of inverse sine

* $y = \sin^{-1}(x) \Rightarrow y' = ?$

$y = \sin^{-1}(x) \Rightarrow x = \sin(y) = \frac{d}{dx}[x] = \frac{d}{dx}[\sin(y)]$

$\Rightarrow 1 = \cos(y) \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}}$ *

* $\sin^2(y) + \cos^2(y) = 1$

$\cos^2(y) = 1 - \sin^2(y)$

$\cos(y) = \sqrt{1-\sin^2(y)}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Derivative of inverse cosine

* $y = \cos^{-1}(x) \Rightarrow y' = ?$

$y = \cos^{-1}(x) \Rightarrow x = \cos(y) \Rightarrow \frac{d}{dx}[x] = \frac{d}{dx}[\cos(y)]$

$\Rightarrow 1 = -\sin(y) \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{\sin(y)}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-\cos^2(y)}} \Rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

Derivative of inverse tangent

* $y = \tan^{-1}(x) \rightarrow y' = ?$

$\Rightarrow x = \tan(y) \Rightarrow \frac{dx}{x}[x] = \frac{dx}{x}[\tan(y)]$

$\Rightarrow \frac{1}{\cos^2(y)} \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \cos^2(y) = \frac{\cos^2(y)}{\cos^2(y) + \sin^2(y)}$

$= \frac{1}{\frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}} = \frac{1}{1 + \tan^2(y)} = \frac{1}{1+x^2}$

* $\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$

* $\frac{d}{dx}[\arccos(x)] = \frac{-1}{\sqrt{1-x^2}}$

* $\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}$

Selecting Procedures For Calculating Derivatives

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Strategy for Single Rule

* "Do I see a product, quotient, or composition of functions?" tells us the right strategy.
  * we see product → Product Rule
  * we see quotient → Quotient Rule
  * We see composition → Chain Rule

Manipulating Functions Before Differentiation

* Sometimes it's better to manipulate the function:
* $\frac{d}{dx}\left[(x+5)(x-3)\right] = ?$

→ Product Rule:

$$\frac{d}{dx}\left[(x+5)\right]\cdot(x-3) + (x+5)\cdot\frac{d}{dx}\left[(x-3)\right]$$

$$= 1\cdot(x-3) + (x+5)\,1$$

$$= x-3+x+5$$

$$= 2x+2$$

→ Chain Rule:

$$\frac{d}{dx}\left[x^2+2x-15\right] = 2x+2$$

Strategy for multiple rules

* $\frac{d}{dx}\left[\underbrace{\sin\left((x^2+5)(\cos(x)\right)}_{\text{This is sine of sth.}}\right]$

⇓

Chain Rule

* $\frac{d}{dx}\left[\underbrace{\sin(x^2+5)\;\cos(x)}_{\text{This is product of 2 things}}\right]$

⇓

Product Rule

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* When selecting a procedure for calculating derivatives, ask "Do I see a product, quotients, or composition of functions.
* Consider manipulating the function if it will make the calculation easier.
* When determining the order in which to apply differentiation rules to a complex expression, begin by identifying the outermost operation (based on the structure of the equation) and proceed inward, applying the appropriate rules step by step.

Applying the chain rule and product rule

* $\frac{d}{dx}\left[(x^2\sin x)^3\right] = ?$

CHAIN RULE FIRST

$$= \frac{d\left[(x^2\sin x)^3\right]}{d(x^2\sin x)} \cdot \frac{d\left[x^2\sin(x)\right]}{dx}$$

$$= 3(x^2 \cdot \sin x)^2 \cdot \left[(2x)(\sin x) + x^2(\cos x)\right]$$

$$= 3 \cdot x^4 \cdot \sin^2 x \cdot \left[2x\sin x + x^2\cos x\right]$$

$$= 6x^5\sin^3 x + 3x^6\sin^2 x\cos x$$

PRODUCT RULE FIRST

$$* = \frac{d}{dx}\left[x^6 \cdot \sin^3 x\right]$$

$$= 6x^5\sin^3 x + x^6 \cdot \frac{d}{dx}\left[\sin^3 x\right]$$

$$= 6x^5\sin^3 x + x^6 \cdot 3 \cdot \sin^2 x \cdot \cos x$$

Applying the chain rule twice

* $\frac{d}{dx}\left[\sin^3(x^2)\right] = ?$

$$= \frac{d\left[\sin^3(x^2)\right]}{d\sin(x^2)} \cdot \frac{d\left[\sin(x^2)\right]}{dx}$$

$$= 3 \cdot \sin^2(x^2) \cdot \frac{d\left[\sin x^2\right]}{dx^2} \cdot \frac{d\left[x^2\right]}{dx}$$

$$= 3 \cdot \sin^2(x^2) \cdot \cos(x^2) \cdot 2x$$

$$= 6x\sin^2(x^2)\cos(x^2)$$

Derivative of $e^{\cos(x)} \cdot \cos(e^x)$

* $\frac{d}{dx}\left[e^{\cos(x)} \cdot \cos(e^x) \right] = ?$

$$= \frac{d}{dx}\left[e^{\cos(x)}\right] \cdot \cos(e^x) + e^{\cos(x)} \cdot \frac{d}{dx}\left[\cos(e^x)\right]$$

$$= \frac{d\left[e^{\cos(x)}\right]}{d\cos(x)} \cdot \frac{d[\cos(x)]}{dx} \cdot \cos(e^x) + e^{\cos(x)} \cdot \frac{d[\cos(e^x)]}{d e^x} \cdot \frac{d[e^x]}{dx}$$

$$= e^{\cos(x)} \cdot -\sin(x) \cdot \cos(e^x) + e^{\cos(x)} \cdot -\sin(e^x) \cdot e^x$$

$$= -e^{\cos(x)} \left(\sin(x) \cdot \cos(e^x) + \sin(e^x) \cdot e^x\right)$$

Derivative of $\sin[\ln(x^2)]$

* $\frac{d}{dx}\left[\sin(\ln(x^2))\right]$

$$= \frac{d[\sin(\ln(x^2))]}{d\ln(x^2)} \cdot \frac{d[\ln(x^2)]}{dx}$$

$$= \cos(\ln x^2) \cdot \frac{d[\ln(x^2)]}{dx^2} \cdot \frac{d[x^2]}{dx}$$

$$= \cos(\ln x^2) \cdot \frac{1}{x^2} \cdot 2x$$

$$= \frac{2 \cdot \cos(\ln x^2)}{x}$$

Calculating Higher Order Derivatives

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Second derivatives

* $y = \frac{6}{x^2} \Rightarrow \frac{d^2}{dx^2}[y] = ? = \frac{d}{dx}\left[\frac{d}{dx}[y]\right]$

$$\Rightarrow \frac{d}{dx}\left[\frac{d}{dx}[6x^{-2}]\right] = \frac{d}{dx}\left[-12x^{-3}\right] = 36x^{-4}$$

Second derivatives: Implicit equations

* $y^2 - x^2 = 4 \Rightarrow \frac{d^2y}{dx^2} = ?$

* $\frac{d}{dx}[y^2] - \frac{d}{dx}[x^2] = \frac{d}{dx}[4]$

$$\Rightarrow 2y\frac{dy}{dx} - 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

* $\frac{d}{dx}\left[x \cdot y^{-1}\right] = \frac{d}{dx}[x]\, y^{-1} + (x)\frac{d}{dx}[y^{-1}]$

$$= y^{-1} + x(-y^{-2})\left(\frac{dy}{dx}\right) = \frac{1}{y} - x \cdot \frac{1}{y^2} \cdot \frac{x}{y} = \frac{1}{y} - \frac{x^2}{y^3}$$

Second derivatives (implicit equations): Evaluate derivative

* 6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

$$\frac{d}{dx}\left[\frac{y}{3y^2 - x}\right] = \frac{\frac{d}{dx}[y](3y^2 - x) - y\frac{d}{dx}[3y^2 - x]}{(3y^2 - x)^2}$$

$$= \frac{\frac{y}{\cancel{(3y^2-x)}} \cdot \cancel{(3y^2 - x)} - y \cdot \left((6y) \cdot \left(\frac{y}{3y^2 - x}\right) - 1\right)}{(3y^2 - x)^2}$$

For $(x, y) = (-1, 1)$

$$= \frac{1 - \left(6 \cdot \frac{1}{4} - 1\right)}{(4)^2} = \frac{\frac{1}{2}}{16} = \frac{1}{32}$$

* The second derivative of a function is simply the derivative of the function's derivative.

* Notation: $f''(x)$ OR $\frac{d^2}{dx^2}[f(x)]$

Further Practice Connecting Derivatives and Limits

Disguised derivatives

* $\lim_{h\to 0} \frac{5\log(2+h) - 5\log(2)}{h} = ?$

$= 5 \cdot \underbrace{\lim_{h\to 0} \frac{\log(2+h) - \log(2)}{h}}$ $\qquad f(2) = \log(2)$

$= 5 \cdot f'(2) = 5 \cdot \frac{1}{\ln(10)\cdot 2} =$

Quiz

* $\lim_{h\to 0} \frac{2\sqrt[3]{8+h} - 2\sqrt[3]{8}}{h} = ?$

$\Rightarrow f'(8) = ?$ for $f(x) = 2\sqrt[3]{x}$

$\frac{d}{dx}\left[2\sqrt[3]{x}\right] = 2 \cdot \frac{1}{3} \cdot x^{-\frac{2}{3}}$

$f'(8) = 2 \cdot \frac{1}{3} \cdot \frac{1}{\sqrt[3]{8^2}} = \frac{1}{6}$

* $\lim_{h\to 0} \frac{3(2+h)^4 - 3(2)^4}{h} = ?$ $\quad = \quad f(x) = 3x^4$

$f'(x) = 12x^3$

$f'(2) = 96$

* $\lim_{h\to 0} \frac{3\ln(e+h) - 3\ln(e)}{h} = ?$ $\quad = \quad f(x) = 3\ln(x) \to f'(e) = ?$

$f'(x) = \frac{3}{x} \Rightarrow f'(e) = \frac{3}{e}$

* $\lim_{h\to 0} \frac{5\arcsin\left(\frac{2}{3}+h\right) - 5\arcsin\left(\frac{2}{3}\right)}{h} = ?$ $\Leftrightarrow f(x) = 5 \cdot \arcsin(x) \Rightarrow$

$f'\left(\frac{2}{3}\right) = ?$

$f'(x) = 5 \cdot \frac{1}{\sqrt{1-x^2}} \Rightarrow f'\left(\frac{2}{3}\right) = 5 \cdot \frac{1}{\sqrt{1-\frac{4}{9}}} = \frac{5 \cdot 3}{\sqrt{5}} = \frac{15}{\sqrt{5}} = \frac{15\sqrt{5}}{5} = 3$