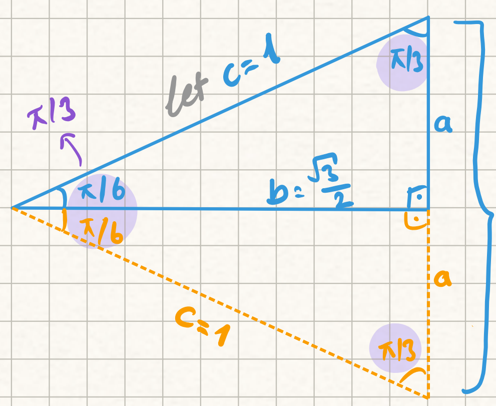
Special Trigonometric Values in the First Quadrant

08.05.2025

Cosine, sine, and tangent of $\pi/6$ and $\pi/3$

* Let's find the cos, sin, tan for $(\underbrace{\pi/3}_{60^\circ}, \underbrace{\pi/6}_{30^\circ})$

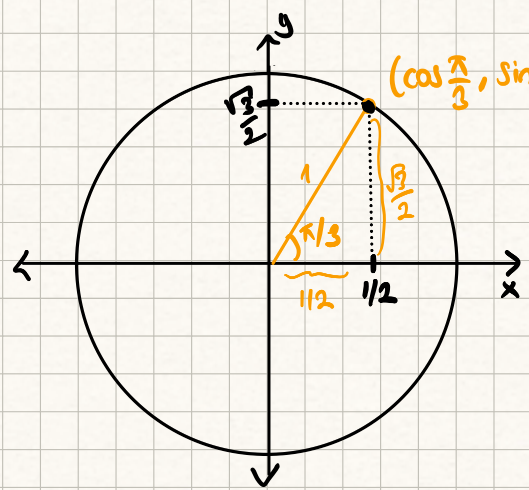


$2a = c \Rightarrow a = \frac{1}{2}$

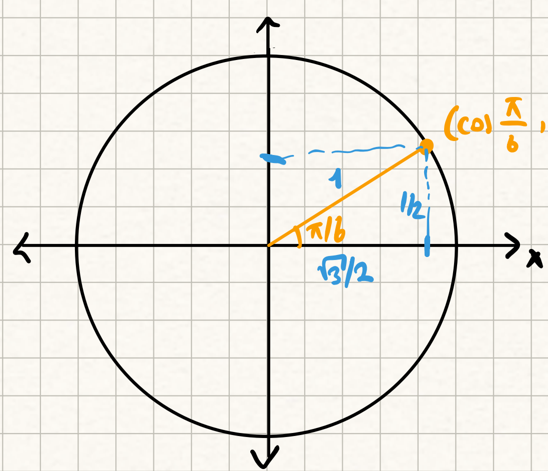
$a^2 + b^2 = c^2$

$(\frac{1}{2})^2 + b^2 = 1^2$

$b = \frac{\sqrt{3}}{2}$



$(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ $\quad \tan\frac{\pi}{3} = \frac{\sin\pi/3}{\cos\pi/3} = \sqrt{3}$

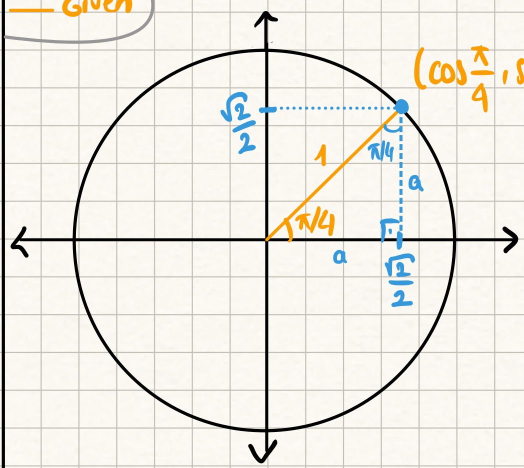


$(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2}) \Rightarrow \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

* The (x,y) coordinate on the unit circle corresponding to an angle of θ degrees gives us $(\cos\theta, \sin\theta)$

— Given

Trigonometric values of $\pi/4$



$\left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\tan\frac{\pi}{4} = 1$

$a^2 + a^2 = 1^2$

$2a^2 = 1$

$a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

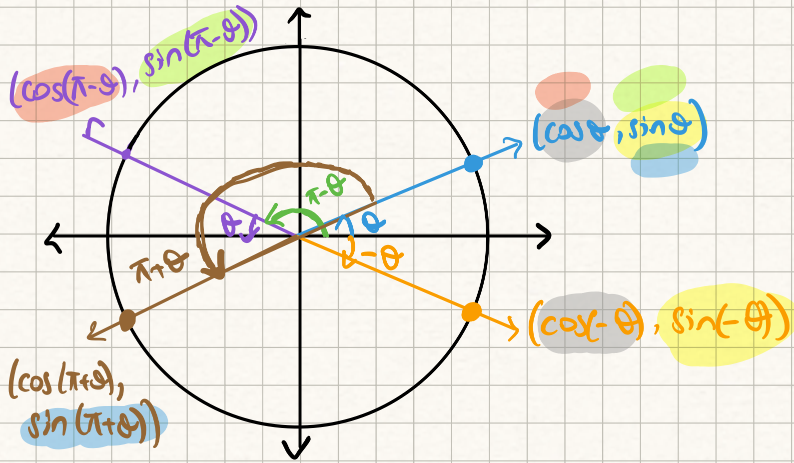
* $\vec{S} = [0 \ 1 \ 2 \ 3 \ 4]$ $\quad \vec{C} = [4 \ 3 \ 2 \ 1 \ 0]$ $\quad \vec{\theta} = [0 \ 30 \ 45 \ 60 \ 90]$

$\sin(\theta_i) = \frac{\sqrt{\vec{S}_i}}{2}$ $\quad \cos(\theta_i) = \frac{\sqrt{\vec{C}_i}}{2}$

Trigonometric Identities on the Unit Circle

08.05.2025

Sine & Cosine Identities: Symmetry



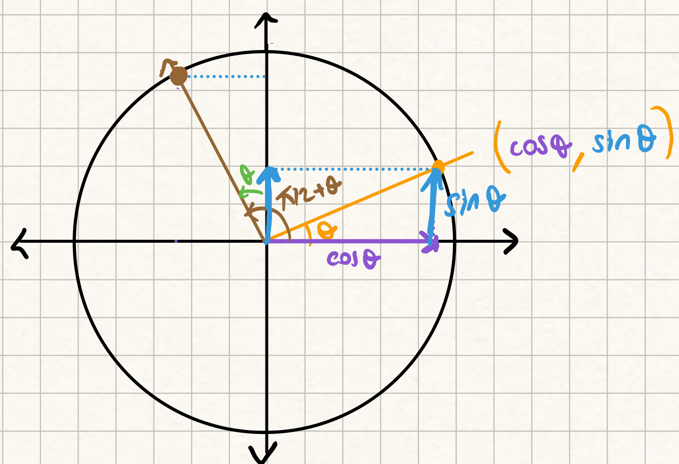
- $\cos\theta = \cos(-\theta)$
- $\sin(-\theta) = -\sin\theta$
- $\sin(\theta) = \sin(\pi-\theta)$
- $\cos(\theta) = -\cos(\pi-\theta)$
- $\sin(\theta+\pi) = -\sin(\theta)$

(... many more...)

Tangent Identities: Symmetry:

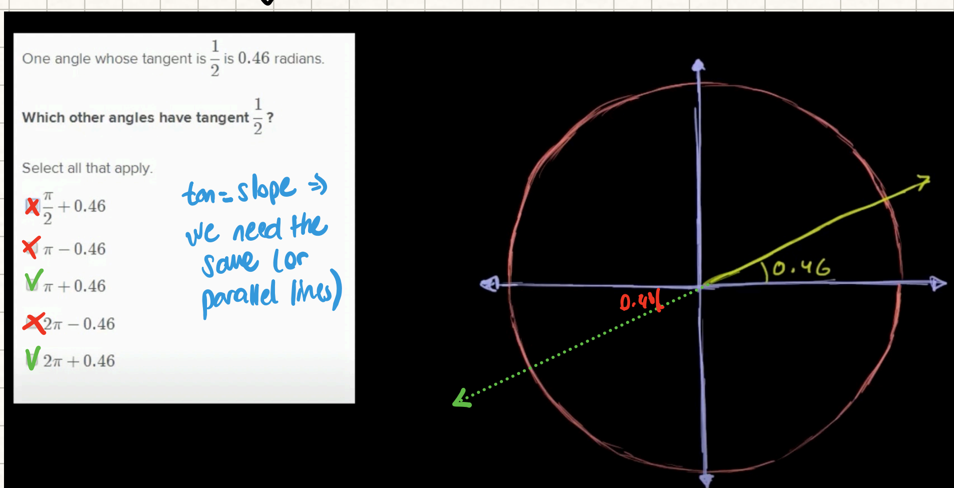
- $\tan(\pi+\theta) = \tan\theta$
- $\tan(\pi-\theta) = -\tan\theta$
- $-\tan\theta = \tan(-\theta)$

Sine and cosine identities: periodicity



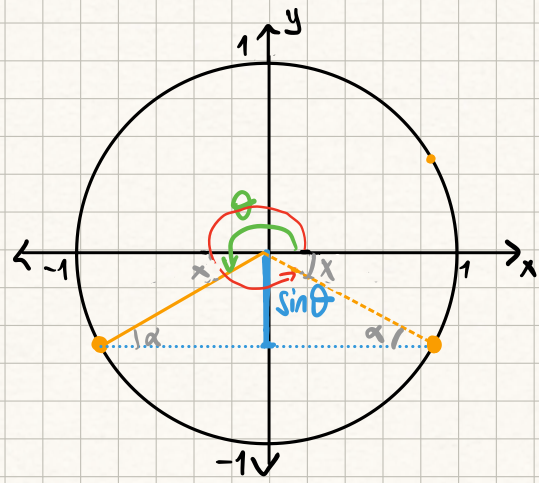
- $\cos\theta = \sin(\theta+\frac{\pi}{2})$

Tangent Identities: Periodicity:



! I'm not copying these information to this summary section because no need to memorize / study them. I should be able to derive these conclusions when I need to.

Quiz



which two of the following $= \sin\theta$?

① $\sin\left(\frac{\pi}{2} - \theta\right)$

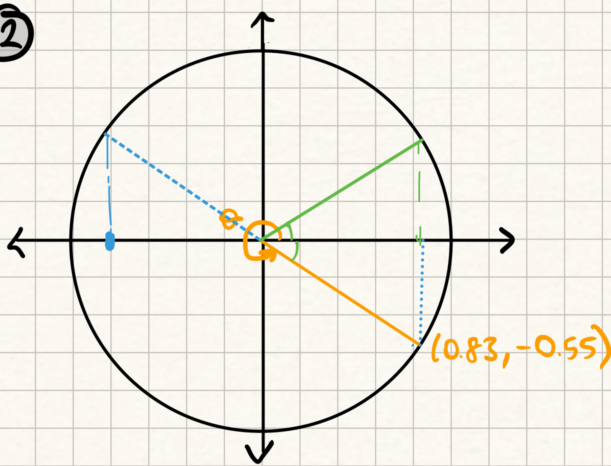
② $\sin(2\pi - \theta)$

③ $\sin(2\pi + \theta)$

④ $\sin(\pi - \theta)$

$-x = ?$ $\quad x = \theta - \pi \Rightarrow -x = \pi - \theta$

②



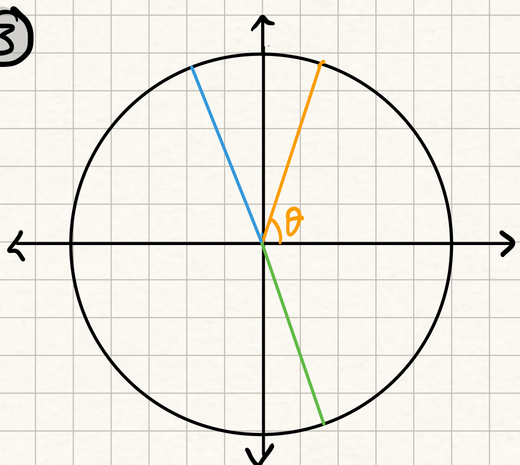
① $\cos(\pi + \theta) = ?$

-0.83

② $\cos(2\pi - \theta) = ?$

0.83

③



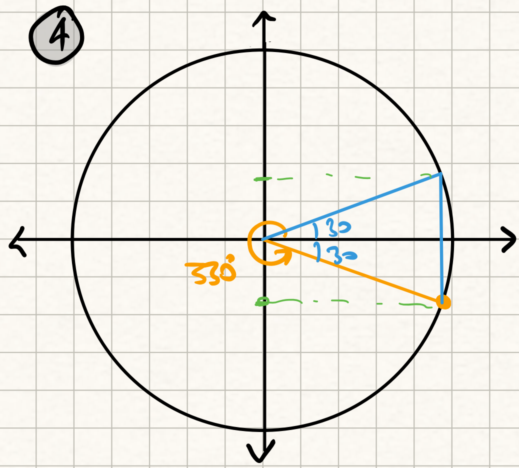
$\tan\theta = \dfrac{0.98}{0.2} = 4.9 \Rightarrow$

① $\tan(\pi - \theta) = ?$ $\quad -4.9$

② $\tan(2\pi - \theta) = ?$

-4.9

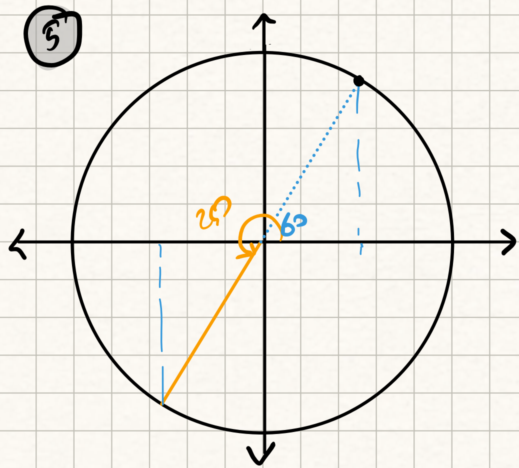
4



* $\cos(330°) = ?$ $\cos(30) = \frac{\sqrt{3}}{2}$

* $\sin(330°) = ?$ $-\sin(30°) = \frac{-1}{2}$

5



* $\cos(240°) = ?$ $-\cos 60° = \frac{-1}{2}$

* $\sin(240°) = ? = -\sin 60° = \frac{-\sqrt{3}}{2}$

Inverse Trigonometric Functions

09.05.2025

Intro to arcsine

* $\sin(\theta) = x \Rightarrow \arcsin(x) = \theta$
* D: $[-1,1]$ R: $[\frac{-\pi}{2}, \frac{\pi}{2}]$

* we have to restrict the domain so it's invertible. (one-to-one mapping is a must!)

* $\arcsin(\frac{-\sqrt{3}}{2}) = ?$ $\frac{\sqrt{3}}{2} \Leftrightarrow 60°\ (\frac{\pi}{3})$, but it's negative

$\Rightarrow \arcsin(\frac{-\sqrt{3}}{2}) = \frac{-\pi}{3}$ rad.

* It's also $\frac{4\pi}{3}$ but $\frac{4\pi}{3}$ is not in our restricted range!

Intro to arctangent

* $\tan(\theta) = x \Rightarrow \arctan(x) = \theta$
* D: $(-\infty, +\infty)$, R: $(-\frac{\pi}{2}, \frac{\pi}{2})$

* $\arctan(-1) = ?$ $\tan(\theta) = -1 \Rightarrow \theta = \frac{-\pi}{4}, \cancel{\frac{3\pi}{4}}$

out of our restricted range

$\Rightarrow \arctan(-1) = \frac{-\pi}{4}$

Intro to arccosine

* $\cos(\theta) = x \Rightarrow \arccos(x) = \theta$
* D: $[-1, 1]$ R: $[0, \pi]$

* $\arccos(\frac{-1}{2}) = ?$ $= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

because our x is negative

because $\cos(\frac{\pi}{3}) = \frac{1}{2}$

Composing trigonometric functions with their inverses

* $\cos(\arccos(x)) = x$, $\sin(\arcsin(x)) = x$
* When it comes to $\arccos(\cos(\theta))$ and $\arcsin(\sin(\theta))$, we need to find the θ equivalent of the result that is in our range.

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* $\sin(\theta) = x \Rightarrow \sin^{-1}(x) = \theta$; Domain: $[-1,1]$, Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$ * where $\cos(x)$ is zero or +
* $\cos(\theta) = x \Rightarrow \cos^{-1}(x) = \theta$; Domain: $[-1,1]$, Range: $[0, \pi]$ * where $\sin(x)$ is zero or +
* $\tan(\theta) = x \Rightarrow \tan^{-1}(x) = \theta$; Domain: $(-\infty, \infty)$, Range: $(\frac{-\pi}{2}, \frac{\pi}{2})$ * where $\cos(x)$ is + (zeros would have made the slope undefined)

Domain and Range of inverse tangent function

* Given $g(x) = \tan\left(x - \frac{3\pi}{2}\right) + 6$,

① Find $g^{-1}(x)$

② What is the domain of $g^{-1}(x) = ?$

③ What is the range of $g^{-1}(x) = ?$

①
$$x = \tan\left(g^{-1}(x) - \frac{3\pi}{2}\right) + 6$$

$$x - 6 = \tan\left(g^{-1}(x) - \frac{3\pi}{2}\right)$$

$$\tan^{-1}(x-6) = \cancel{\tan^{-1}}\left(\cancel{\tan}\left(g^{-1}(x) - \frac{3\pi}{2}\right)\right)$$

$$\tan^{-1}(x-6) = g^{-1}(x) - \frac{3\pi}{2}$$

$$g^{-1}(x) = \tan^{-1}(x-6) + \frac{3\pi}{2}$$

② Domain of $\tan^{-1}(x)$ = Range of $\tan(x)$

* Range for $\tan(x) = \{x \in \mathbb{R}\}$, b.c. the slope can be any $\mathbb{R}$.

$$\Rightarrow g^{-1}(x) = \tan^{-1}(\underbrace{x-6}_{\text{any real number}}) + \frac{3\pi}{2}$$

any real number $\rightarrow (-\infty, \infty)$

③ Range of $\tan^{-1}(x)$ = (Restricted) Domain of $\tan(x)$

* Restricted domain for $\tan(x) = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\times\ g^{-1}(x) = \tan^{-1}(x-6) + \frac{3\pi}{2} \Rightarrow$$

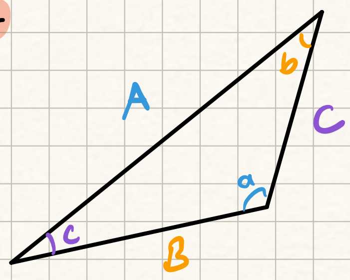
$$\frac{-\pi}{2} + \frac{3\pi}{2} \qquad \frac{\pi}{2} + \frac{3\pi}{2}$$

$\Rightarrow$ Range of $g^{-1}(x) = (\pi, 2\pi)$

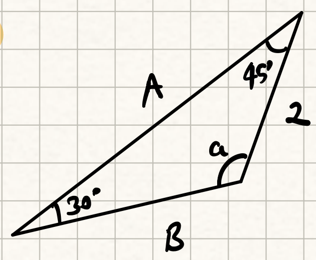
Law of Sines and Law of Cosines

03.05.2025

Solving for a side with the law of Sines



$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$



$\Rightarrow$ Find a, A, and B

$* \; a = 180 - (30+45) = 105°$

$$\frac{\sin 30°}{2} = \frac{\sin 45°}{B} = \frac{\sin 105}{c}$$

$$\Rightarrow \frac{1/2}{2} = \frac{1}{4} = \frac{\sin 45°}{B} = \frac{\sin 105°}{c} \Rightarrow$$

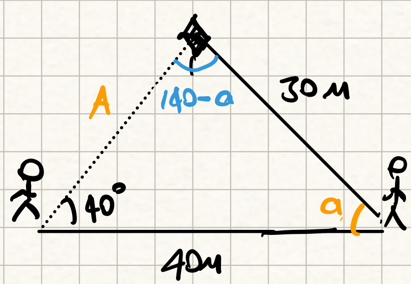
$B = (4)\sin 45°$

$= 4\frac{\sqrt{2}}{2} = 2\sqrt{2} \approx 2.83$

$C = (4)\sin 105°$

≈ 3.86

Solving for an angle with the law of Sines



$a = ?$

$$\frac{\sin(140-a)}{40} = \frac{\sin(40)}{30}$$

$\Rightarrow \sin(140-a) = \frac{4}{3}\sin(40°)$

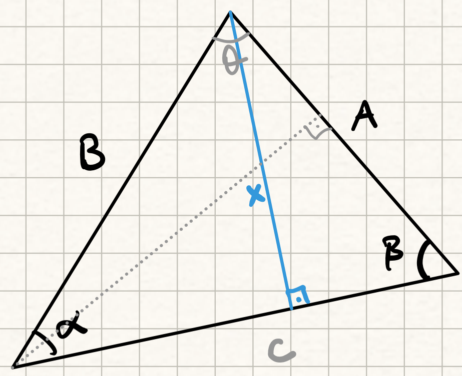
$\Rightarrow 140 - a = \sin^{-1}\left(\frac{4}{3}\sin(40°)\right)$

$\Rightarrow 140 - a \approx 58.99°$

$a \cong 81.01$

$*$ Law of Sines: $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

Proof of the Law of Sines



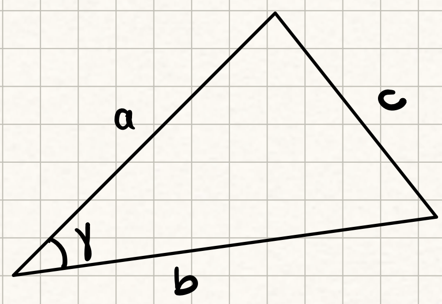
$$\sin\beta = \frac{x}{A} \Rightarrow x = A \cdot \sin\beta$$

$$\sin\alpha = \frac{x}{B} \Rightarrow x = B \cdot \sin\alpha$$

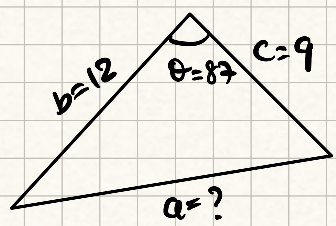
$$\Rightarrow A \cdot \sin\beta = B \cdot \sin\alpha$$

$$\Rightarrow \frac{\sin\alpha}{A} = \frac{\sin\beta}{B} \left(= \frac{\sin\theta}{C} \right)$$

Solving for a side with the law of cosines



$$\boxed{c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma)}$$



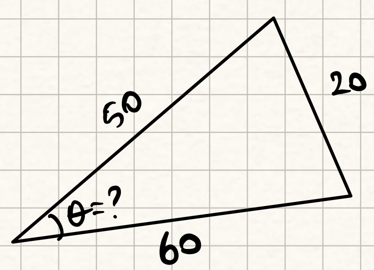
$$a^2 = 9^2 + 12^2 - 2(9)(12)(\cos(87))$$

$$a^2 = 81 + 144 - 216(\cos 87^\circ)$$

$$a = \sqrt{225 - 216(\cos(87^\circ))}$$

$$\approx 14.62$$

Solving for an angle with the law of cosines



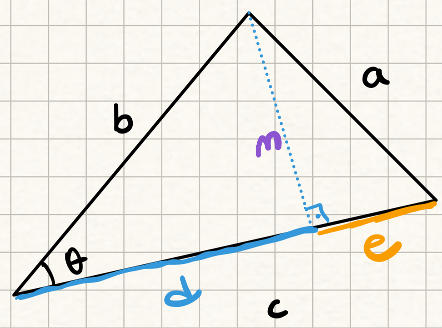
$$(20)^2 = (50)^2 + (60)^2 - 2(50)(60)(\cos\theta)$$

$$400 = 2500 + 3600 - 6000(\cos\theta)$$

$$\cos\theta = \frac{57\cancel{00}}{60\cancel{00}} \approx 18.19^\circ$$

Law of Cosines : $c^2 = a^2 + b^2 - 2ab(\cos(\gamma))$

Proof of the law of cosines



* $\cos\theta = \frac{d}{b} \Rightarrow d = b.\cos\theta$

* $e = c - d$

$\Rightarrow e = c - b.\cos\theta$

* $\frac{m}{b} = \sin\theta \Rightarrow m = b.\sin\theta$

* $a^2 = m^2 + e^2$

$\Rightarrow a^2 = (b.\sin\theta)^2 + (c - b.\cos\theta)^2$

$\Rightarrow a^2 = b^2.(\sin\theta)^2 + c^2 - 2bc\cos\theta + b^2(\cos\theta)^2$

$\Rightarrow a^2 = b^2.(\sin\theta)^2 + b^2(\cos\theta)^2 + c^2 - 2bc(\cos\theta)$

$\Rightarrow a^2 = b^2\left(\underbrace{(\sin\theta)^2 + (\cos\theta)^2}_{1}\right) + c^2 - 2bc(\cos\theta)$

$\Rightarrow a^2 = b^2 + c^2 - 2bc(\cos\theta)$

Solving General Triangles

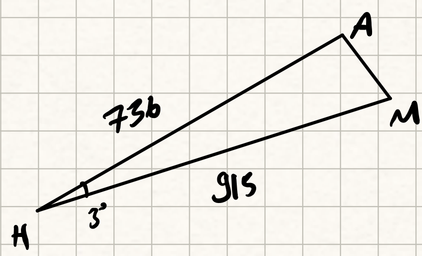
12.05.2025

Trig world problems:
Stars

Artemis seeks knowledge of the width of Orion's Belt, which is a pattern of stars in the Orion constellation. She has previously discovered the distances from her house to Alnitak (736 light years) and to Mintaka (915 light years), which are the endpoints of Orion's Belt. She also knows the angle between these stars in the sky is $3°$.

What is the width of Orion's Belt? That is, what is the distance between Alnitak and Mintaka? ______ light years

Round your answer to the nearest light year.

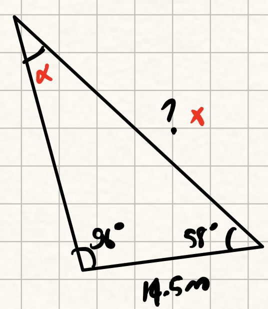


$|AM| = ?$

$x^2 = 736^2 + 915^2 - 2 \cdot (736)(915) \cdot \cos(3°)$

≈ 184

Quiz ①



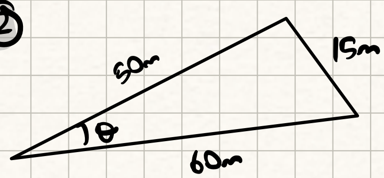
$\alpha = 180 - (96 + 58)$
$= 26°$

$\dfrac{\sin 26°}{14.5} = \dfrac{\sin 96°}{x}$

$x = 14.5 \left(\dfrac{\sin 96°}{\sin 26°} \right)$

≈ 32.9

②



$\theta = ?$

$15^2 = 50^2 + 60^2 - 2(50)(60)\cos\theta$

$225 = 2500 + 3600 - 6000(\cos\theta)$

$\cos\theta = \dfrac{5875}{6000}$

$\theta \cong 12°$

Sinusoidal Equations

12.05.2025

Sinusoidal equations of the form sin(x)=d

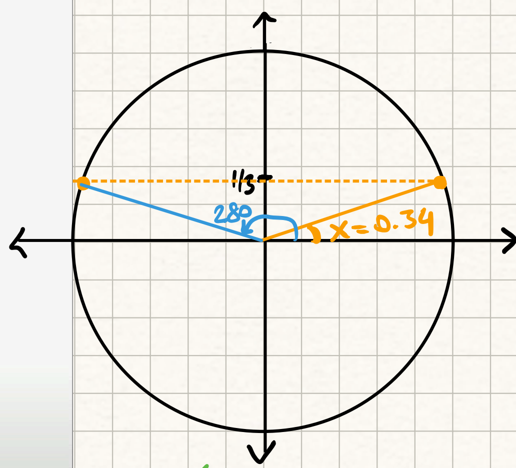
Which of these are contained in the solution set to $\sin x = \frac{1}{3}$?

Answers are rounded to the nearest hundredth.

Select all that apply.

- ✓ $0.34 + 2\pi n$, for n an integer
- ✗ $0.34 + \pi n$, for n an integer
- ✗ $-0.34 + 2\pi n$, for n an integer
- ✗ $-0.34 + \pi n$, for n an integer
- ✓ $2.80 + 2\pi n$, for n an integer
- ✗ $2.80 + \pi n$, for n an integer

$\sin^{-1}\left(\frac{1}{3}\right) \approx 0.34$



$$\left\{ 0.34 + 2\pi n,\ 2.80 + 2\pi n \right\}$$

Cosine equation algebraic solution set

* $-6\cos(8x) + 4 = 5 \Rightarrow x = ?$ (solution set)

$\Rightarrow \cos(8x + 2\pi n) = \frac{-1}{6}$ & $\cos(-8x + 2\pi n) = \frac{-1}{6}$

$$8x + 2\pi n = \cos^{-1}\left(\frac{-1}{6}\right) \qquad -8x + 2\pi n = \cos^{-1}\left(\frac{-1}{6}\right)$$

$$x = \frac{1}{8}\cos^{-1}\left(\frac{-1}{6}\right) - \frac{\pi}{4}n \qquad -8x = \cos^{-1}\left(\frac{-1}{6}\right) - 2\pi n$$

$$x = \frac{-1}{8}\cos^{-1}\left(\frac{-1}{6}\right) + \frac{\pi}{4}n$$

Cosine equation solution set in an interval

* We'll continue with the last question but we'll limit our solution set to the interval $\left[\frac{-\pi}{2}, 0\right] \approx [-1.57, 0]$

$$x = \frac{1}{8}\cos^{-1}\left(\frac{-1}{6}\right) - \frac{\pi}{4}n \qquad x = \frac{-1}{8}\cos^{-1}\left(\frac{-1}{6}\right) + \frac{\pi}{4}n$$

$$x \approx 0.22 - 0.785n \qquad x \approx -0.22 + 0.785n$$

n	x	
0	6.22	too high
1	-0.57	✓
2	-1.35	✓
3	-2.14	too low

n	x	
0	-0.22	✓
1	0.57	too high
-1	-1.01	✓
-2	-1.79	too low

* So, our solution set in the interval of $\left[\frac{-\pi}{2}, 0\right]$ is

$$\left\{ \begin{array}{ll} \overset{n=1}{x = \frac{1}{8}\cos^{-1}\left(\frac{-1}{6}\right) - \frac{\pi}{4}}, & \overset{n=0}{x = \frac{-1}{8}\cos^{-1}\left(\frac{-1}{6}\right)} \\ \overset{n=2}{x = \frac{1}{8}\cos^{-1}\left(\frac{-1}{6}\right) - \frac{\pi}{2}}, & \overset{n=-1}{x = \frac{-1}{8}\cos^{-1}\left(\frac{-1}{6}\right) - \frac{\pi}{4}} \end{array} \right\}$$

Sine equation, algebraic solution set

* $8\sin\left(\frac{x}{4}\right) + 11 = 14 \Rightarrow x = ?$

$$\sin\left(\frac{x}{4}\right) = \frac{3}{8} \qquad \sin\left(\pi - \frac{x}{4}\right) = \frac{3}{8}$$

$$\sin\left(\frac{x}{4} + 2\pi n\right) = \frac{3}{8} \qquad \sin\left(\pi - \frac{x}{4} + 2\pi n\right) = \frac{3}{8}$$

$$\frac{x}{4} + 2\pi n = \sin^{-1}\left(\frac{3}{8}\right) \qquad \pi - \frac{x}{4} + 2\pi n = \sin^{-1}\left(\frac{3}{8}\right)$$

$$\frac{x}{4} = \sin^{-1}\left(\frac{3}{8}\right) - 2\pi n \qquad \frac{-x}{4} = \sin^{-1}\left(\frac{3}{8}\right) - \pi - 2\pi n$$

$$x = 4\sin^{-1}\left(\frac{3}{8}\right) - 8\pi n \qquad x = -4\sin^{-1}\left(\frac{3}{8}\right) + 4\pi + 8\pi n$$

Solving $\cos(\theta) = 1$ and $\cos(\theta) = -1$

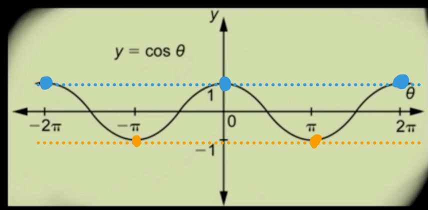
In the graph below, for what values of θ does $\cos\theta = 1$ and for what values of θ does $\cos\theta = -1$?

$\cos\theta = 1$ Ⓢ $\theta = 0, 2\pi, 4\pi, 6\pi \ldots$

$\cos\theta = 1$ Ⓢ $\theta = 0, -2\pi, -4\pi \ldots$

$\cos\theta = -1$ Ⓢ $\theta = \pi, 3\pi, 5\pi \ldots$

$\cos\theta = -1$ Ⓢ $\theta = -\pi, -3\pi, -5\pi \ldots$



Quiz ① Select one or more expressions that together represent all solutions to the equation. Your answer should be in degrees. (Assume $n \in \mathbb{Z}$): $4\cos(10x) + 2 = 2$

$\cos(10x \pm 2\pi n) = 0$

$10x \pm 360^\circ n = 90^\circ$

$x = 9^\circ \pm 36^\circ n$

$\cos(-10x \pm 2\pi n) = 0$

$-10x \pm 360^\circ n = 90^\circ$

$x = -9^\circ \pm 36^\circ n$

② $14\sin(20x) - 3 = 2$

$\sin(20x \mp 2\pi n) = \frac{5}{14}$

$20x = \sin^{-1}\left(\frac{5}{14}\right) \mp 2\pi n$

$x = \frac{1}{20}(0.3652) \mp \frac{\pi n}{10}$

$x = 0.018 \mp \frac{\pi n}{10}$

$\sin(\pi - 20x \pm 2\pi n) = \frac{5}{14}$

$\pi - 20x \pm 2\pi n = 0.3652$

$-20x = 0.3652 - \pi \pm 2\pi n$

$-20x = -2.777 \pm 2\pi n$

$x = 0.139 \pm \frac{\pi n}{10}$

(use this summary after the next one)

* When solving sinusoidal equations, take into account the trigonometric identities in the unit circle as well as the repetitive nature (periodicity) of the trigonometric functions.

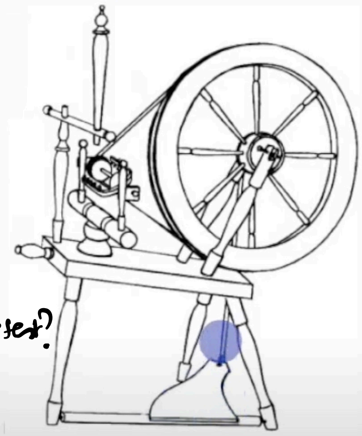
Sinusoidal Models

12.05.2025

Interpreting solutions of trigonometric equations

As Alvaro presses the treadle of a spinning wheel with his foot, it moves a bar up and down, making the wheel spin. The function $B(t)$ models the height, in centimeters, of the top of the bar when Alvaro has pressed the treadle for t seconds.

$$B(t) = 90 - 12\sin(5t)$$



(1) **What does the solution set to $y = 90 - 12\sin(5 \cdot 6)$ represent?**

* The height (in cm) of the top of the bar when Alvaro has pressed the treadle for 6 seconds.

(2) What does the solution set $95 = 90 - 12\sin(5t)$ represent?

* Times (in seconds) needed to press the treadle for top of the bar to be 95 cm in height.

(This won't happen only once. It's periodical!)

(3) What does the solution set to $y = 90 - 12\sin\left(\frac{\pi}{2}\right)$ represent?

$\sin\left(\frac{\pi}{2}\right) = 1$, which is the max. value $\sin\left(\frac{\pi}{2}\right)$ can take. Therefore $90 - 12\sin\left(\frac{\pi}{2}\right)$ is the min (bc substitution!) value y can take. So: "The lowest height for the top of the bar.

Quiz: (1)

The function $M(x)$ models the height, in meters above street level, of the tip of the minute hand of the Abraj Al Bait clock at x minutes after midnight.

The expression $H(x)$ models the height, in meters above street level, of the tip of the hour hand at x minutes after midnight.

$$M(x) = 430 + 23\cos\left(\frac{\pi}{30}x\right)$$

$$H(x) = 430 + 17\cos\left(\frac{\pi}{360}x\right)$$

Consider the following equation:

$$y = \left|\left(430 + 23\cos\left(\frac{\pi}{30} \cdot 45\right)\right) - \left(430 + 17\cos\left(\frac{\pi}{360} \cdot 45\right)\right)\right|$$

What does the solution set for the last equation represent?

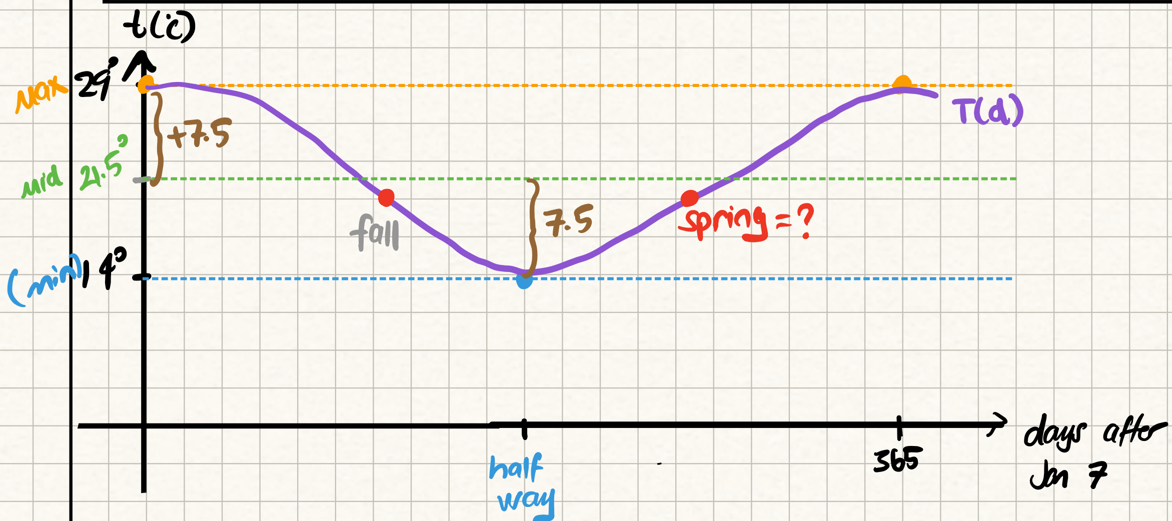
The difference between the tips of hour and minute hands of Abraj Al Bait clock at 00:45

Modeling annual temperature

The hottest day of the year in Santiago, Chile, on average, is January 7, when the average high temperature is $29^{\circ}\mathrm{C}$. The coolest day of the year has an average high temperature of $14^{\circ}\mathrm{C}$.

Use a trigonometric function to model the temperature in Santiago, Chile, using 365 days as the length of a year. Remember that January 7 is in the *summer* in Santiago.

How many days after January 7 is the first spring day when the temperature reaches $20^{\circ}C$?



* We chose a trigonometric function because our model needs to be periodical.

* After drawing the T(d) function, we ask ourselves which trig. func should I choose? Which one starts at maximum point? cosine, because $\cos(0^{\circ}) = 1$

$$T(d) = \text{Amplitude} \cos(\text{Arg.}\, d) + \text{Shift}$$

$$T(d) = 7.5 \cos\Big(\underbrace{\frac{2\pi}{365}}_{}d\Big) + 21.5^{\circ}$$

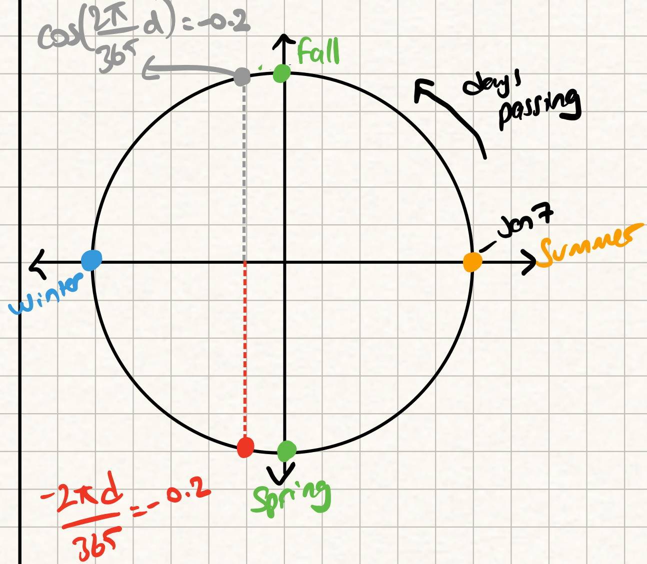
Because we want d=365 to give us the maximum solution.

* A sinusoidal equation is a mathematical expression that describes a smooth, periodic oscillation resembling the sine or cosine function. It is used to model repetitive phenomena such as sound waves, light waves, tides, and mechanical vibrations. The formula is:

$$\left.\begin{array}{l} y = A\sin(B(x-C)) + D \\ y = A\cos(B(x-C)) + D \end{array}\right\} \begin{array}{l} A = \text{Amplitude (The peak deviation from the centerline)} \\ B = \text{Frequency (The period of wave)} \\ C = \text{Phase shift } (C>0 \Rightarrow \text{shifts right}, C<0 \Rightarrow \text{shifts left}) \\ D = \text{Vertical shift (Moves the entire graph up (+) or down (-))} \end{array}$$

$$20 = 7.5\cos\left(\frac{2\pi}{365}d\right) + 21.5$$

$$\cos\left(\frac{2\pi}{365}d\right) = -0.2$$



$$\frac{-2\pi d}{365} = 1.77$$

$$d = \frac{-365(1.77)}{2\pi}$$

$$d = -102.8221$$

$$= 262.1779$$

$$\approx 262 \text{ days}$$

Quiz An ice cream truck that plays loud music is circling Bulan's neighbourhood. $C(t)$ models the volume of the music (in dB) that Bulan hears, t minutes after the truck arrives in the neighbourhood. Here t is in radians.

$$C(t) = -15\cos\left(\frac{2\pi}{15}t\right) + 65$$

How many minutes after the truck arrives does the volume first each 75 dB?

$$75 = -15\cos\left(\frac{2\pi}{15}t\right) + 65$$

$$\frac{-2}{3} = \cos\left(\frac{2\pi}{15}t\right)$$

$$\frac{2\pi}{15}t = 2.3005$$

$$= 5.5$$

Angle Addition Identities

12.05.2025

Trig angle addition identities

$\ast$ $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$

$\ast$ $\sin(a-b) = \sin(a)\underbrace{\cos(-b)}_{=\cos(b)} + \underbrace{\sin(-b)}_{=-\sin(b)}\cos(a)$

$= \sin(a)\cos(b) - \sin(b)\cos(a)$

$\ast$ $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$\ast$ $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$

$\ast$ $\cos(2a) = \cos(a+a) = \cos(a)\cos(a) - \sin(a)\sin(a)$

$= \cos^2(a) - \sin^2(a)$ $\quad \ast \cos^2(a)+\sin^2(a)=1$

$= \cos^2(a) - 1 + \cos^2(a)$

$\cos(2a) = 2\cos^2(a) - 1$

$\frac{\cos(2a)+1}{2} = \cos^2(a)$

$\cos^2(a) = \frac{1}{2}(\cos(2a)+1)$

$\cos(2a) = (1-\sin^2(a)) - \sin^2(a)$

$\cos(2a) = 1 - 2\sin^2(a)$

$\Rightarrow \sin^2(a) = \frac{1}{2}(1-\cos(2a))$

$\ast$ $\sin(2a) = \sin(a) + \sin(a) = \sin(a)\cos(a) + \sin(a)\cos(a)$

$= 2\sin(a)\cos(a)$

$\ast$ We can derive all trig. angle addition identities using these 2 equations:

- $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

Using the cosine angle addition identity

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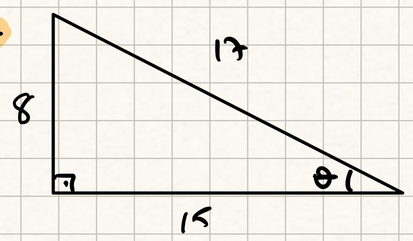

$\cos(\angle ABC + 60°) = ?$

$\cos(\theta + 60°) = ?$

$$\cos(\theta + 60) = \cos(\theta)\cdot\cos(60) - \sin(\theta)\cdot\sin(60)$$

$$= \frac{15}{17}\cdot\frac{1}{2} - \frac{8}{17}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{15 - 8\sqrt{3}}{34}$$

Using the cosine double-angle identity

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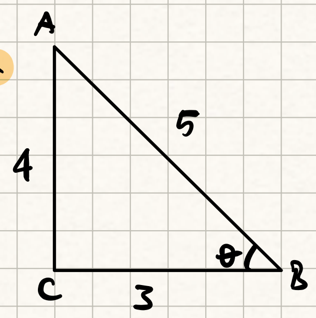


$\cos(2\cdot\theta) = ?$

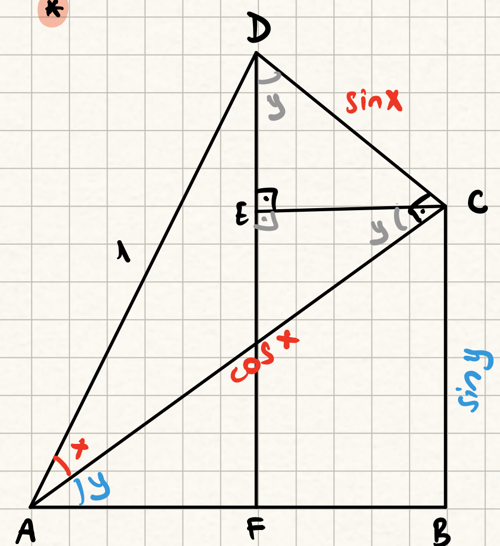
$$= \cos(\theta+\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)$$

$$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$

$$= \frac{9}{25} - \frac{16}{25} = \frac{-7}{25}$$

Proof of the sine angle addition identity

*



$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x+y) = \overline{DF} = \overline{DE} + \overline{EF} = \overline{DE} + \overline{CB}$

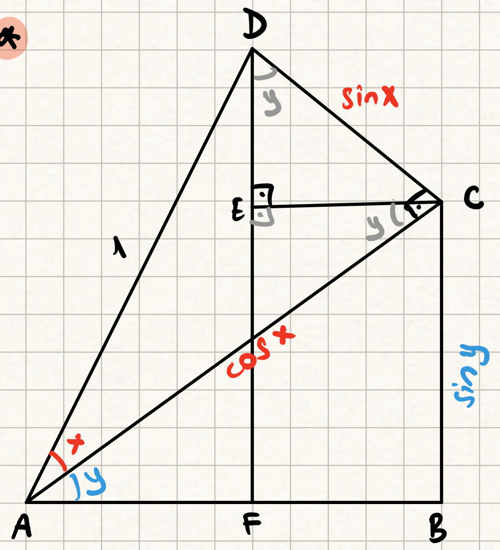
$\sin y = \frac{\overline{CB}}{\cos x} \Rightarrow \overline{CB} = \cos x \sin y$

$\triangle ABC$

$\cos y = \frac{\overline{DE}}{\sin x} \Rightarrow \overline{DE} = \sin x \cos y$

$\triangle DEC$

Proof of the cosine addition identity



$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x+y) = \overline{AF} = \overline{AB} - |\overline{FB}|$$

$$= \overline{AB} - \overline{EC}$$

$$\underbrace{\cos y = \frac{\overline{AB}}{\cos x}}_{\triangle ABC} \Rightarrow \overline{AB} = \cos x \cos y$$

$$\underbrace{\sin y = \frac{\overline{EC}}{\sin x}}_{\triangle EDC} \Rightarrow \overline{EC} = \sin x \sin y$$

Proof of the tangent angle sum and difference identities

* $$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$$

$$= \frac{\dfrac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y)}}{\dfrac{\cos(x)\cos(y) - \sin(x)\sin(y)}{\cos(x)\cos(y)}} = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

* $$\tan(x-y) = \tan(x + (-y)) = \frac{\tan(x) + \tan(-y)}{1 - \tan(x)\tan(y)}$$ $\tan(-y) = -\tan(y)^{*}$

$$= \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

* $$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

* $$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

* $$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

Using Trigonometric Identities

13.05.2025

Finding trig values using angle addition identities

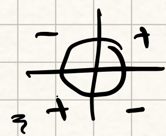
★ Find the value of $\sin\left(\frac{7\pi}{12}\right)$ without using a calculator.

$$\sin\left(\frac{7\pi}{12}\right) = \sin(105^\circ) = \sin(60+45)$$

$$= \sin(60)\cos(45) + \cos(60)\sin(45)$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$



Using the tangent angle addition identity

★ $\tan\left(\frac{13\pi}{12}\right) = \tan(195) = \tan(225 - \tan 30) = \tan(45-30)$

$$= \frac{\tan(45) - \tan(30)}{1 + \tan(45)(\tan 30)} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

Quiz

① $\cos\left(\frac{17}{12}\pi\right) = ? = \cos(255) = \cos(225+30)$

$$= \cos(225)\cos(30) - \sin(225)\sin(30)$$

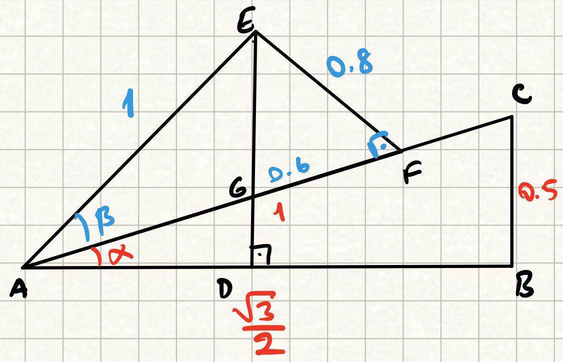
$$= -\frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\,\frac{1}{2}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

② $\tan\left(\frac{\pi}{12}\right) = \tan(15) = \tan(60-45)$

$$= \frac{\tan(60) - \tan(45)}{1 + \tan(60)\tan(45)} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

Using trig angle addition identities: Finding side lengths



$\overline{AB} = \frac{\sqrt{3}}{2}$ $\overline{AF} = 0.6$

$\overline{AC} = 1$ $\overline{AE} = 1$

$\overline{CB} = 0.5$ $\overline{EF} = 0.8$

$\Rightarrow \overline{ED} = ? = \sin(\alpha+\beta) = ?$

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$= (0.5)(0.6) + \left(\frac{\sqrt{3}}{2}\right)(0.8)$$
$$= 0.9928$$

Using trig angle addition identities: manipulating expressions

* $\cos 2\theta = C$, and $0 < \theta < \pi$. Write a formula for $\sin\theta$ in terms of C.

$$\cos(\theta+\theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin\theta$$
$$C = \cos^2(\theta) - \sin^2(\theta)$$

* $\cos^2(\theta) + \sin^2(\theta) = 1$

$\cos^2(\theta) = 1 - \sin^2(\theta)$

$$C = 1 - \sin^2(\theta) - \sin^2(\theta)$$
$$C = 1 - 2\sin^2(\theta)$$
$$\sin^2(\theta) = \frac{1-C}{2}$$
$$\sin(\theta) = \pm\sqrt{\frac{1-C}{2}}$$

$0 < \theta < \pi \Rightarrow \sin\theta > 0 \Rightarrow$

* $\sin(\theta) = \sqrt{\frac{1-C}{2}}$