Matrix - vector products

17.04.2025

Multiplying matrices by vectors

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix} \quad \vec{b} = (5, -1) \Rightarrow$$

$$A \cdot \vec{b} = \underbrace{\begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 0 & 1 \end{bmatrix}}_{3\times 2} \cdot \underbrace{\begin{bmatrix} 5 \\ -1 \end{bmatrix}}_{2\times 1} = \begin{bmatrix} 1\cdot(5) + 0(-1) \\ -2\cdot(5) + 4\cdot(-1) \\ 0\cdot(-1) + 1\cdot(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ -14 \\ -1 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -4 \\ 1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \underset{1\times 3}{\vec{w}} \cdot \underset{3\times 2}{B} = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

1×2

Quiz

① $A = \begin{bmatrix} -1 & 5 & 4 \\ 3 & 2 & 7 \\ -1 & 0 & 1 \end{bmatrix}$ $\Rightarrow \underset{3\times 3}{A} \cdot \underset{3\times 1}{\vec{v}} = \begin{bmatrix} -1 & 5 & 4 \\ 3 & 2 & 7 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 22 \\ 6 \end{bmatrix}$

$\vec{v} = (-2, 0, 4)$

② $M = \begin{bmatrix} -5 & -3 & 1 & 6 \\ 0 & 4 & -2 & 1 \end{bmatrix}$ $\Rightarrow \underset{2\times 4}{M} \cdot \underset{4\times 1}{\vec{v}} = \begin{bmatrix} -5 & -3 & 1 & 6 \\ 0 & 4 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ 5 \\ -4 \end{bmatrix}$

$\vec{v} = (1, -3, 5, -4)$

$$= \begin{bmatrix} -15 \\ -26 \end{bmatrix}$$

③ $M = \begin{bmatrix} -4 & -5 & 6 \\ 8 & 3 & -4 \end{bmatrix}$ $\Rightarrow \underset{1\times 2}{\vec{v}} \cdot \underset{2\times 3}{M} = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} -4 & -5 & 6 \\ 8 & 3 & -4 \end{bmatrix}$

$\vec{v} = (-2, 1)$

$$= \begin{bmatrix} 16 & 13 & -16 \end{bmatrix}$$

* For $A \cdot \vec{x}$, $\vec{x}$ must be a column vector. For $\vec{x} \cdot \vec{A}$, $\vec{x}$ must be a row vector

The null space and $A\cdot\vec{x}=\vec{0}$

* $A\cdot\vec{x}=\vec{0} \Rightarrow \vec{x}\in N(A)$

* null space is always a subspace

✓ 1. zero vector $\in N(A)$

✓ 2. closed under addition

$\vec{x}_1, \vec{x}_2 \in N(A) \Rightarrow$

$$\left.\begin{array}{l} A\cdot\vec{x}_1=\vec{0} \\ A\cdot\vec{x}_2=\vec{0} \end{array}\right\} + \quad A\vec{x}_1 + A\vec{x}_2=\vec{0}$$

$$A(\vec{x}_1+\vec{x}_2)=\vec{0} \Rightarrow \vec{x}_1+\vec{x}_2 \in N(A)$$

✓ 3. closed under scalar multiplication

$\vec{x}_1 \in N(A) \Rightarrow c\cdot\vec{x}_1 \in N(A)$?

$A\cdot(c\cdot\vec{x}_1)=\vec{0}$

$c\cdot \underbrace{A\cdot\vec{x}_1}_{\vec{0}}=\vec{0}$

$c\cdot\vec{0}=\vec{0}$

$\vec{0}=\vec{0}$

Ex

$$A=\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow[R_1\cdot(\frac{1}{4})\to R_1]{R_1-2R_2\to R_2} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$\Rightarrow x_1 - \frac{1}{2}x_2 = 0$

$\Rightarrow x_1 = \frac{1}{2}x_2 \qquad \Rightarrow \begin{bmatrix} x \\ 2x \end{bmatrix} \in N(A)$

Ex $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ ... etc.

* If the zero vector is the only vector in the null space of A, the column vectors of A is linearly independent (and vice versa).

* Null space: All the vectors that satisfy $A\cdot\vec{x}=\vec{0}$
* If $N(A)$ has only the zero vector, the columns of A are linearly independent (and vice versa)

The null space of a matrix

Quiz

(1) $A = \begin{bmatrix} 1 & -4 & 3 \\ 2 & 4 & 2 \\ -1 & -5 & 0 \end{bmatrix}, \quad \vec{x} = (-5,1,3) \Rightarrow \vec{x} \in N(A)\ ?$

$A \cdot \vec{x} \overset{?}{=} 0 \qquad \begin{bmatrix} 1 & -4 & 3 \\ 2 & 4 & 2 \\ -1 & -5 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{x} \in N(A)$

(2) Which of the vectors is in the null space of $A = \begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & 9 & 0 \end{bmatrix} \xrightarrow{3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 4 & 12 & 0 \end{bmatrix} \xrightarrow{R_2/4 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x - 2z = 0 & x = 2z \\ y + 3z = 0 & y = -3z \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 2x \\ -3x \\ x \end{bmatrix} = N(A)$$

$$A = \begin{bmatrix} -2 & 0 & -1 & 2 \\ -4 & 0 & 2 & -4 \\ -6 & 0 & -3 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↳ rref(A)

* Pivot columns
* Free columns

$\Rightarrow$ Solve for pivot variables $\left\} \begin{matrix} x_1 + x_4 = 0 \\ x_1 = -x_4 \\ x_3 = 0 \end{matrix} \right.$ $\Rightarrow$ Write the linear combination in terms of free variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow N(A) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Quiz:

① $A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 4 \\ 6 & -6 & -12 \end{bmatrix} \Rightarrow N(A) = ?$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+R_2 \to R_1} \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 6x_3 \\ x_2 = 4x_3 \end{matrix} \Rightarrow N(A) = \text{span}\left(\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \right)$$

($R_3 - 6R_1 \to R_3$)

② $M = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -3 & -6 & 9 & -15 \\ 4 & 1 & -12 & 6 \end{bmatrix} \Rightarrow N(M) = ?$

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & -12 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -7 & 0 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 3x_3 - x_4 \\ x_2 = -2x_4 \end{matrix} \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \Rightarrow N(M) = \text{span}\left(\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

The column space and $A\vec{x} = \vec{b}$

(3) $B = \begin{bmatrix} 2 & 2 & -4 & 10 \\ -1 & -1 & 2 & -5 \\ 3 & 3 & -6 & 15 \end{bmatrix} \Rightarrow N(B) = ?$

$= \begin{bmatrix} 2 & 2 & -4 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = -x_2 + 2x_3 - 5x_4$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

* $C(A)$ = Linear combinations of columns of A
  = span of columns of A

$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \Rightarrow C(A) = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right)$

* $A \cdot \vec{x} = \vec{b}$: if $\vec{b} \in C(A)$, then $\vec{x}$ exists as a solution.

$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} \Rightarrow$

$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix} \Rightarrow (x_1, x_2) = (3, 0)$

* Column space is a span. A span is always a subspace. Therefore column space is always a subspace.

* If $N(A)$ tells us that the columns of A are linearly independent, then $C(A)$ is the basis for the column space.

* Column space of A is the linear combinations (span) of the column vectors of A. If $N(A)$ tells us that the columns of A are linearly independent, then $C(A)$ is also the basis.

Ex:

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad x_1 = -2x_2$$

$$N(A) = \text{span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) \Rightarrow N(A) \neq \vec{0} \Rightarrow \text{Columns of } A \text{ are linearly dependent}$$

$\Rightarrow$ columns of A do not form a basis for $C(A)$

How to find the basis?

1. $x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2. We know that x_2 is a free variable. let's make it zero.

3. $x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0 \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 = 0 \Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is linearly independent $\Rightarrow$ Basis for $C(A) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\Rightarrow C(A) = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$

Quiz

① $A = \begin{bmatrix} 1 & -5 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 1 & -1 & 2 & 4 \end{bmatrix} \Rightarrow N(A) = ?\ ,\ C(A) = ? = \text{span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$

$$\begin{bmatrix} 1 & -5 & 2 & 4 \\ 0 & 15 & -5 & -10 \\ 0 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = -2x_4 \end{matrix}$$

$$N(A) = x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \Rightarrow N(A) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$$

② $M = \begin{bmatrix} -1 & 2 & 6 & 5 \\ 0 & 3 & -7 & 9 \\ 3 & -6 & -18 & -15 \end{bmatrix} \Rightarrow$ Basis for $C(M) = ?$

$$\begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 1 & -7/3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -32/3 & 1 \\ 0 & 1 & -7/3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ x_3 and x_4 are free variables

$$\Rightarrow C(M) = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \right)$$

③ $A = \begin{bmatrix} 1 & -2 & 4 & 5 \\ 0 & 3 & 5 & 7 \\ -3 & 6 & 3 & 9 \\ 2 & -4 & -2 & -6 \end{bmatrix} \Rightarrow C(A)$ in terms of basis $= ?$

$3R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & -2 & 4 & 5 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 15 & 24 \\ 0 & 0 & -10 & -16 \end{bmatrix} = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \begin{bmatrix} 1 & -2 & 4 & 5 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 15 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \Rightarrow C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix} \right)$$

$-2R_1 + R_4 \to R_4$

Solving $A \cdot \vec{x} = \vec{b}$

$A \cdot \vec{x} = \vec{0}$ ← complementary solution (stays the same)

$A \cdot \vec{x} = \vec{b}$ ← particular solution (can change)

$A \cdot \vec{x_n} = \vec{0}$ (n for null)

$+ \; A \cdot \vec{x_p} = \vec{b}$ (p for particular)

$A\vec{x_n} + A\vec{x_p} = \vec{0} + \vec{b}$

$$A(\underbrace{\vec{x_n} + \vec{x_p}}_{\text{General solution}}) = \vec{b}$$

* The general solution of $A \cdot \vec{x} = \vec{b}$ is the sum of:
  1. $\vec{A} \cdot \vec{x_n} = \vec{0}$ (Called complementary solution.)
  2. $\vec{A} \cdot \vec{x_p} = \vec{b}$ (For a particular $\vec{b}$. Called particular solution)

$G = P + C$ · General solution = Particuler Sol. + Complementery sol.

$\vec{X} = \vec{X}_p + \vec{X}_n$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 4 \\ 3 & 4 & 4 & 7 \end{bmatrix} \Rightarrow$ Find the general solution with $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Step 1 : Verify that $\vec{b}$ will produce a solution vector $\vec{x}$ and find $\vec{X_p}$

$-2R_1 + R_2 \to R_2$

$-3R_1 + R_3 \to R_3$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 3 & b_1 \\ 2 & 2 & 1 & 4 & b_2 \\ 3 & 4 & 4 & 7 & b_3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 3 & b_1 \\ 0 & -2 & -5 & -2 & b_2 - 2b_1 \\ 0 & -2 & -5 & -2 & b_3 - 3b_1 \end{array}\right]$$

$R_3 - R_2 \to R_3$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 3 & 3 & b_1 \\ 0 & -2 & -5 & -2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 3 & b_1 \\ 0 & 1 & 5/2 & 1 & \frac{-1}{b^2} + b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

$-R_2 + R_1 \to R_1$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & -b_1 + b_2 \\ 0 & 1 & 5/2 & 1 & \frac{-1}{2} b_2 + b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right] \Rightarrow$$

$b_3 - b_2 - b_1 = 0$

$\underbrace{b_1 + b_2 = b_3}$

✓ So our $\vec{B} = (1, 2, 3)$ will produce a solution vector

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 1 \\ 0 & 1 & 5/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} X_1 - 2X_3 + X_4 = 1 \\ X_2 + \frac{5}{2} X_3 + X_4 = 0 \end{array}$$

For $(X_3, X_4) = (0, 0)$

$X_1 = 1$

$X_2 = 0$

$\Rightarrow \vec{X_p} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

#Step 2 Find complementary solution. (we already have rref(A))

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 5/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 - 2x_3 + x_4 = 0 \\ x_1 = 2x_3 - x_4 \end{array} \qquad \begin{array}{l} x_2 + \frac{5}{2}x_3 + x_4 = 0 \\ x_2 = \frac{-5}{2}x_3 - x_4 \end{array}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x}_n = c_1 \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \left.\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{General solution for } A \cdot \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Quiz: Find the general solution to $\begin{bmatrix} 3 & -6 & 6 \\ -3 & 7 & -9 \\ -6 & 8 & 0 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & -6 & 6 & b_1 \\ -3 & 7 & -9 & b_2 \\ -6 & 8 & 0 & b_3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 2 & \frac{1}{3}b_1 \\ 0 & 1 & -3 & b_1 + b_2 \\ 0 & -4 & 12 & 2b_1 + b_3 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 2 & \frac{1}{3}b_1 \\ 0 & 1 & -3 & b_1+b_2 \\ 0 & 0 & 0 & 4b_1+4b_2+2b_1+b_3 \end{array}\right] \Rightarrow$$

$6b_1+4b_2+b_3=0$

$b=(1,-1,-2)$

$6-4-2=0$

$\checkmark\ 0=0$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -4 & \frac{1}{3}b_1+2b_1+2b_2 \\ 0 & 1 & -3 & b_1+b_2 \\ 0 & 0 & 0 & 6b_1+4b_2+b_3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -4 & 1/3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_1 = \frac{1}{3} \\ x_2 = 0 \end{array} \Rightarrow \vec{x}_p = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$x_1 - 4x_3 = 0$ $\quad$ $x_2 - 3x_3 = 0$

$x_1 = 4x_3$ $\quad$ $x_2 = 3x_3$

$$\Rightarrow \vec{x}_n = c_1 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \qquad \Rightarrow \vec{x} = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + c_1 \cdot \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

2 $$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 \\ -1 & 0 & 0 & 2 & 1 \\ 0 & 1 & -3 & -1 & -4 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = ?$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 3 & -1 & 3 & b_1 \\ 0 & 1 & -3 & -1 & -4 & -b_1-b_2 \\ 0 & 1 & -3 & -1 & -4 & b_3 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & -1 & 3 & -1 & 3 & b_1 \\ 0 & 1 & -3 & -1 & -4 & -b_1-b_2 \\ 0 & 0 & 0 & 0 & 0 & b_3+b_2+b_1 \end{array}\right]$$

$$= \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & -b_2 \\ 0 & 1 & -3 & -1 & -4 & -b_1-b_2 \\ 0 & 0 & 0 & 0 & 0 & b_3+b_2+b_1 \end{array}\right]$$

$$\vec{b} = (1,-2,1) \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & -3 & -1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 2 \\ x_2 = 1 \end{array}$$

$$\Rightarrow \vec{x}_p = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & -3 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$x_1 - 2x_4 - x_5 = 0$$
$$x_1 = 2x_4 + x_5$$
$$x_2 - 3x_3 - x_4 - 4x_5 = 0$$
$$x_2 = 3x_3 + x_4 + 4x_5$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

③ $\begin{bmatrix} 2 & 3 & 4 & -4 \\ 2 & 3 & 8 & -10 \\ 6 & 9 & 16 & -18 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \vec{x} = ?$

$$\left[\begin{array}{cccc|c} 1 & 3/2 & 2 & -2 & \frac{1}{2}\cdot b_1 \\ 0 & 0 & 4 & -6 & b_2 - b_1 \\ 0 & 0 & 4 & -6 & b_3 - 3b_1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 3/2 & 2 & -2 & \frac{1}{2} b_1 \\ 0 & 0 & 4 & -6 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & 2b_1 + b_2 - b_3 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 3/2 & 2 & -2 & \frac{1}{2} b_1 \\ 0 & 0 & 1 & -3/2 & \frac{1}{4}(b_2 - b_1) \\ 0 & 0 & 0 & 0 & 2b_1 + b_2 - b_3 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 3/2 & 0 & 1 & b_1 - \frac{1}{2} b_2 \\ 0 & 0 & 1 & -3/2 & \frac{1}{4}(b_2 - b_1) \\ 0 & 0 & 0 & 0 & 2b_1 + b_2 - b_3 \end{array}\right]$$

$\vec{b} = (1, -1, 1) \Rightarrow \left[\begin{array}{cccc|c} 1 & 3/2 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{x}_p \begin{bmatrix} 3/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cccc|c} 1 & 3/2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 3/2\, x_2 + x_4 = 0 \\ x_1 = -3/2\, x_2 - x_4 \end{array} \qquad x_3 = 3/2\, x_4$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3/2 \\ 1 \end{bmatrix} \Rightarrow \vec{x}_n = c_1 \cdot \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -1 \\ 0 \\ 3/2 \\ 1 \end{bmatrix}$

Dimensionality, nullity, and rank

* Dimension of a space = # of basis vectors

Ex

$$A = \begin{bmatrix} 2 & 0 & 4 & -2 \\ 1 & 3 & -1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -5/3 \end{bmatrix} \end{array}$$

* Dimension of a vector space = Number of basis vectors.
* Nullity = Dimension of the null space = Number of free columns in ref(A)
* Rank = Dimension of the column space = Number of pivot columns in ref(A)

* Nullity of a space = Dimension of the null space

$= \text{Dim}(N(A)) = \text{nullity}(A) = \#$ of free columns in ref(A)

* nullity (A) = 1 $\quad N(A) = \text{span}\left(\begin{bmatrix} -7/3 \\ 4/3 \\ 5/3 \\ 1 \end{bmatrix}\right)$

* Rank of a space = Dimension of the column space.

$= \text{Dim}(C(A)) = \text{rank}(A) = \#$ of pivot columns in ref(A)

* rank (A) = 3 $\quad C(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}\right)$

* Nullity + Rank = # of columns

Quiz

1. $$M = \begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ -3 & 6 & -9 & 3 & -6 \\ -5 & 9 & -7 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -22 & 9 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ 0 & 1 & 22 & -9 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nullity (M) = 3

rank (M) = 2

2. $$X = \begin{bmatrix} -2 & 10 & 4 \\ 1 & 3 & -6 \\ 1 & -5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -5 & -2 \\ 0 & -8 & 15 \\ 0 & 0 & 22 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & -15/8 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ nullity (X) = 0 $\quad$ rank(X) = 3