Interpreting the Meaning of the Derivative in Context

15.06.2025

Examples

* Eddie drove from New York City to Philadelphia. The function D gives the total distance Eddie has driven (in kilometers) t hours after he left.

  What is the best interpretation for the following statement?

  $D'(2) = 100$

* $D(t)$ = Distance in kms after t hours
* $D'(t)$ = Instantenous rate of change of D with respect to t.

  = Speed at t!

  = "2 hours after leaving NY City, Eddie is driving at a speed of 100 km/h.

* A tank is being drained of water. The function V gives the volume of liquid in the tank, in liters, after t minutes.

  What is the best interpretation for the following statement?

  The slope of the line tangent to the graph of V at $t = 7$ is equal to -3. $\Rightarrow V'(t) = -3$

* $V'(7) = -3$ = After 7 minutes, the water is being drained at a speed of 3 liters/minute.

* $D'(x) = y$ means "After x [units for x], the D increases at a rate of y [units for y] per [unit for x].

* $D'(x) = y$ can be interpreted as "After x [units of x], the D increases at a rate of y [units of y] per [unit for x]."

Straight-line Motion: Connecting position, velocity, and acceleration

16.06.2025

Introduction to one-dimensional motion with calculus

* Position after t seconds

$$x(t) = t^3 - 3t^2 + 5, \ t \geq 0$$

t(s)	x
0	5
1	3
2	1
3	5

* Velocity after t seconds, which equals to the first derivative of x

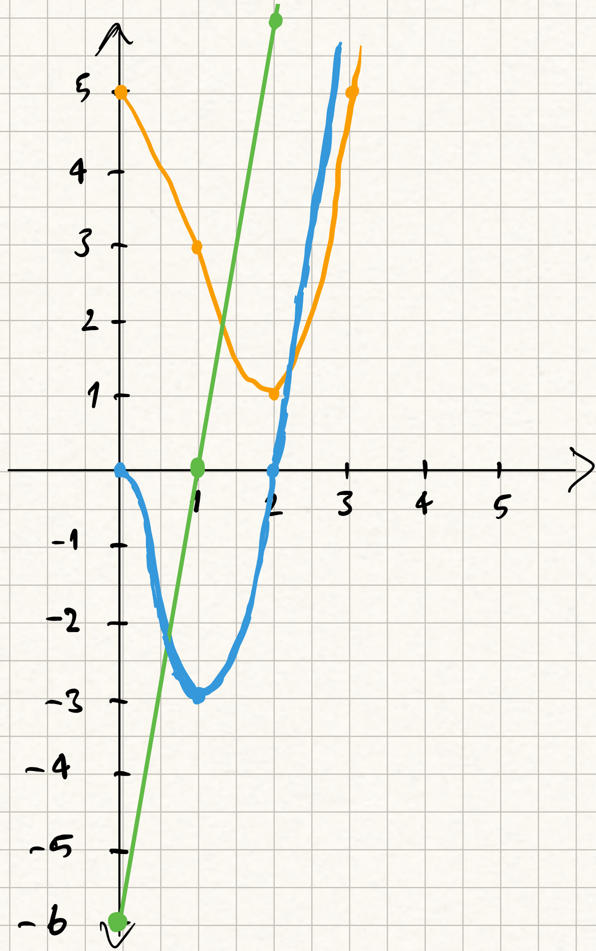
$$V(t) = x'(t) = 3t^2 - 6t, \ x \geq 0$$

t(s)	V
0	0
1	-3
2	0
3	9

* Acceleration at t seconds, which equals to the second derivative of x:

$$A(t) = 6t - 6, \ t \geq 0$$

t(s)	A
0	-6
1	0
2	6
3	12



* Velocity is the first derivative of position with respect to time. Acceleration is the first derivative of velocity with respect to time, which makes it also the second derivative of position with respect to time.

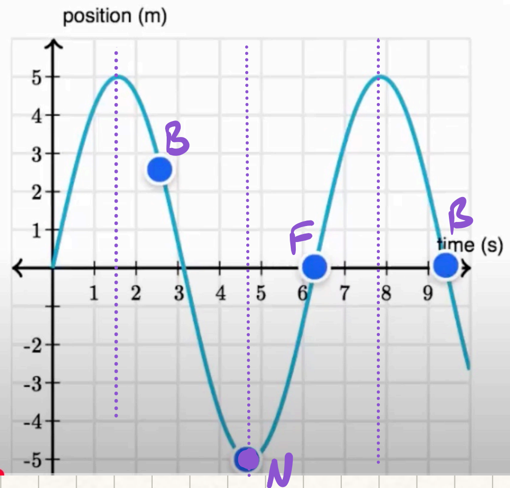
* Speed is the absolute value of velocity (only in one-dimensional motion).

Interpreting direction of motion from position-time graph

An object is moving along a line. The following graph gives the object's position, relative to its starting point, over time.

F B

For each point on the graph, is the object moving forward, backward, or neither?

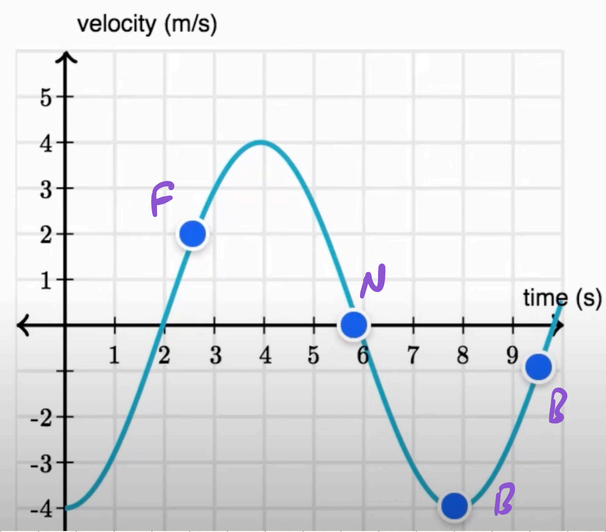


* Slope (P') $> 0 \Rightarrow$ F
  Slope (P') $< 0 \Rightarrow$ B
  Slope (P') $= 0 \Rightarrow$ N

Interpreting direction of motion from velocity-time graph

An object is moving along a line. The following graph gives the object's velocity over time.

For each point on the graph, is the object moving forward, backward, or neither?



* $V < 0 \Rightarrow$ B
  $V > 0 \Rightarrow$ F
  $V = 0 \Rightarrow$ N

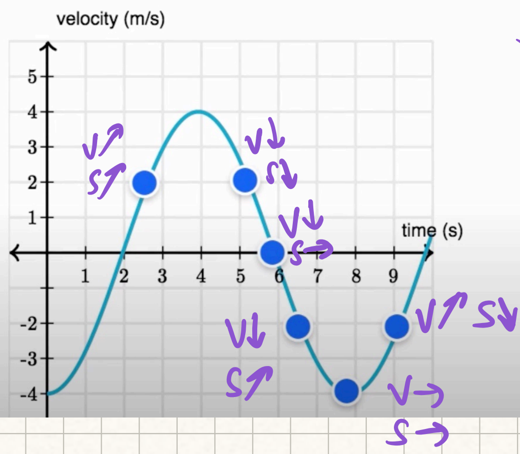
* The slope of V is acceleration, which doesn't tell us about the direction. we care about the velocity itself!
  (|velocity| = speed)

Interpreting change in speed from velocity-time graph

An object is moving along a line. The following graph gives the object's velocity over time.

For each point on the graph, is the object speeding up, slowing down, or neither?

Speed = |velocity|



Motion problems with derivatives

A particle moves along the x-axis. The function $x(t)$ gives the particle's position at any time $t \geq 0$:

$$x(t) = t^3 - 4t^2 + 3t - 2$$

What is the direction of the particle's motion at $t = 2$?

- (A) Left (B/c $v(2) < 0$)
- (B) Right
- (C) Neither

What is the particle's velocity $v(t)$ at $t = 2$? -1

What is the particle's acceleration $a(t)$ at $t = 3$? 10

At $t = 3$, is the particle's speed increasing, decreasing, or neither?

$v(3) = 6$
$A(3) = 10$

Both in the same direction which means speed is increasing

$$V(x) = P'(x) = 3t^2 - 8t + 3$$
$$V(2) = 12 - 16 + 3 = -1$$

$$A(x) = V'(x) = 6t - 8$$
$$A(3) = 10$$

Rates of Change in Non-Motion Problems

16.06.2025

Applied rate of change: Forgetfulness

I studied for an English test today and learned 80 vocabulary words. In 10 days, I will have forgotten every word. The number of words that I remember t days after studying is modeled by

$$W(t) = 80(1-0.1t)^2 \quad \text{for } t \in [0, 10].$$

What is the rate of change of the the number of known words per day 2 days after studying for the test?

$\Rightarrow W'(2) = ?$

$$\frac{d}{dt}[W(t)] = 80 \cdot \frac{d}{dt}[1-0.1t)^2$$

$$= 80 \cdot 2 \cdot (1-0.1t) \cdot (-0.1)$$

$$= -16(1-0.1t)$$

$$W'(2) = -16(1-0.2) = -12.8 \text{ words/day}$$

Example

Carbon-14 is an element which loses half of its mass every 5730 years. The following function gives the mass, in grams, of a sample of carbon-14 after t years:

$$M(t) = 65 \cdot e^{-0.00012t}$$

$M'(1) = ?$

What is the instantaneous rate of change of sample's mass after 1 year?

$$M'(t) = 65 \cdot e^{-0.00012t} \cdot (-0.00012)$$

$$M'(1) = -0.0078 \text{ gr/year}$$

Introduction to Related Rates

16.06.2025

Related Rates Intro

* We throw a stone to a lake. The radius of the circle at this moment is 3 cm and it grows at a rate of 1cm/s. At what rate is the area of the circle growing?

$$r = 3\text{cm}, \quad \frac{d}{dt}[r] = 1\,\text{cm/s}, \quad A = \pi r^2 \Rightarrow \frac{d}{dt}[A] = ?$$

$$\frac{d}{dt}[A] = \frac{d}{dt}[\pi r^2] = \pi \frac{d}{dt}[r(t)^2]$$

* Just to emphasize that r is not a constant, but a function of time!

$$= \pi \cdot 2 \cdot r(t) \cdot \frac{d}{dt}[r(t)]$$

$$= \pi \cdot 2 \cdot 3 \cdot 1$$

$$\frac{d}{dt}[A] = 6\pi$$

* Related rate problems are applied problems where we find the rate at which one quantity is changing by relating it to the other quantities whose rates are known.

Analyzing related rates problems: Expressions

* The base $b(t)$ of a triangle is decreasing at a rate of $13\ \mathrm{m/h}$ and the height $h(t)$ of the triangle is increasing at a rate of $6\ \mathrm{m/h}$. At a certain instant t_0, the base is $5\ \mathrm{m}$ and the height is $1\ \mathrm{m}$. What is the rate of change of the area $A(t)$ of the triangle at that instant?

Match each expression with its units.

	m	m/h	m²	m²/h
$b'(t)$	○	●	○	○
$A(t_0)$	○	○	●	○
$h(t_0)$	●	○	○	○
$\frac{dA}{dt}$	○	○	○	●

	−13	6	5	1	Not given
$b(t_0)$	○	○	●	○	○
$\frac{db}{dt}$	●	○	○	○	○
$A'(t_0)$	○	○	○	○	●
$h'(t)$	○	●	○	○	○

$$A = \frac{b \cdot h}{2}$$

$$\frac{d}{dt}[A] = \frac{1}{2}\frac{d}{dt}[b \cdot h]$$

$$= \frac{1}{2}\left[\frac{d}{dt}[b] \cdot h + b \cdot \frac{d}{dt}[h]\right]$$

$$= \frac{1}{2}[-13 \cdot 1 + 5 \cdot 6]$$

$$= \frac{1}{2}(17)$$

$$= \frac{17}{2}\ \frac{m^2}{h}$$

* Related rate problems are applied problems where we find the rate at which one quantity is changing, by relating it to the other quantities whose rates are known.

Analyzing Related Rates Problems: Equations (Pythagoras)

* Two cars are driving towards an intersection from perpendicular directions. The first car's velocity is 50 km/h and the second car's velocity is 90 km/h. At a certain instant t_0, the first car is a distance $x(t_0)$ of 0.5 km from the intersection and the second car is a distance $y(t_0)$ of 1.2 km from the intersection. What is the rate of change of the distance $d(t)$ between the cars at that instant?

Which equation should be used to solve the problem?

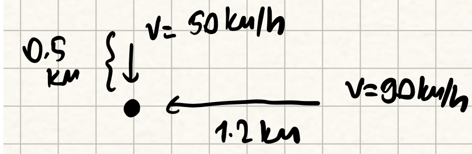
Choose 1 answer:

(A) $\tan[d(t)] = \frac{y(t)}{x(t)}$

(B) $d(t) = \frac{x(t) \cdot y(t)}{2}$

(C) $d(t) + x(t) + y(t) = 180$

(D) $[d(t)]^2 = [x(t)]^2 + [y(t)]^2$

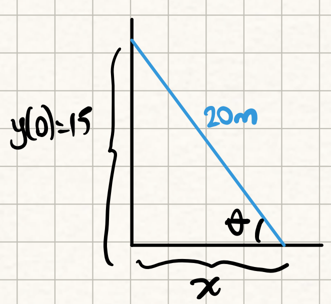


$$[D(t)]^2 = [x(t)]^2 + [y(t)]^2$$

* When analyzing problems involving related rates, it's important to correctly identify variables and constants. It's also crucial to select the equation that correctly represents the problem.

.... (Trig)

* A 20-meter ladder is leaning against a wall. The distance $x(t)$ between the bottom of the ladder and the wall is increasing at a rate of 3 meters per minute. At a certain instant t_0, the top of the ladder is a distance $y(t_0)$ of 15 meters from the ground. What is the rate of change of the angle $\theta(t)$ between the ground and the ladder at that instant?



$y(0) = 15 \qquad \frac{d}{dt}[x] = 3 \text{ m/min.} \qquad \theta'(t_0) = ?$

$x(t) = 20 \cdot \cos(\theta(t))$ (we're choosing $x(t)$ b/c we know its derivative already)

$\frac{d}{dt} x(t) = 20 \cdot (-\sin(\theta(t))) \cdot \theta'(t)$

at $t = t_0$ $\qquad 3 = -20 \cdot \sin(\theta(t)) \cdot \theta'(t)$

$\Rightarrow 3 = -20 \cdot \frac{15}{20} \cdot \theta'(t) \Rightarrow \theta'(t) = \frac{-1}{5}$ radians/minute

* When analyzing problems involving related rates, it's crucial to correctly identify variables and constants. It's also crucial to select the equation that correctly represents the problem.

Differentiating Related Functions : Intro

* The differentiable functions x and y are related by the following:
$y = \sqrt{x}$, $\frac{dx}{dt} = 12$. Find $\frac{dy}{dt}$ when $x = 9$.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{2\sqrt{x}} \cdot 12 = \frac{6}{\sqrt{x}} \qquad x = 9 \Rightarrow \frac{6}{\sqrt{9}} = 2$$

* The differentiable functions x and y are related by the following:
$\sin(x) + \cos(y) = \sqrt{2}$. Also, $\frac{dx}{dt} = 5$
Find $\frac{dy}{dt}$ when $y = \frac{\pi}{4}$, and $0 < x < \frac{\pi}{2}$.

* $\sin(x) + \cos\left(\frac{\pi}{4}\right) = \sqrt{2} \Rightarrow \sin(x) = \frac{\sqrt{2}}{2} \Rightarrow x = \frac{\pi}{4}$

* $\frac{d}{dt}\left[\sin(x(t))\right] + \frac{d}{dt}\left[\cos(y(t))\right] = 0$

$$\Rightarrow \frac{dx}{dt}\cos(x) - \frac{dy}{dt}\sin(y) = 0$$

$$\Rightarrow 5\cos\left(\frac{\pi}{4}\right) - \frac{dy}{dt}\sin\left(\frac{\pi}{4}\right) = 0$$

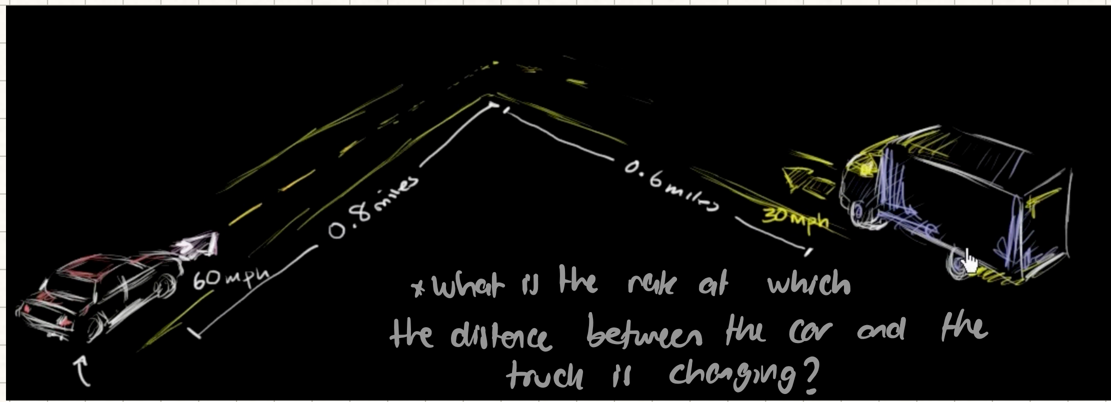
$$\Rightarrow 5\frac{\sqrt{2}}{2} - \frac{dy}{dt}\frac{\sqrt{2}}{2} = 0$$

$$\Rightarrow \frac{dy}{dt} = 5$$

Solving Related Rates Problems

16.06.2025

Approaching cars :

* 

* $c(t_0) = 0.8$ miles $t(t_0) = 0.6$ miles $D'(t) = ?$

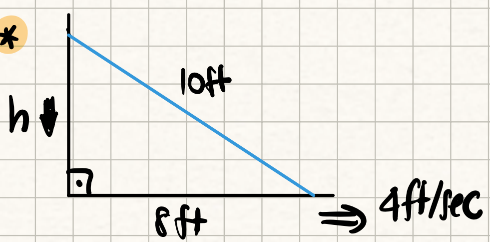
$c'(t) = -60$ m/h $t'(t) = -30$ m/h

$\Rightarrow D(t) = \sqrt{c(t_0)^2 + t(t_0)^2} = \left[c(t_0)^2 + t(t_0)^2\right]^{1/2}$

$\Rightarrow D'(t) = \dfrac{1}{2\sqrt{c(t_0)^2 + t(t_0)^2}} \cdot \left[2c(t_0) \cdot c'(t_0) + 2 \cdot t(t_0) t'(t_0)\right]$

$= \dfrac{\left[2 \cdot (0.8) \cdot (-60) + 2 \cdot (0.6)(-30)\right]}{2\sqrt{(0.8)^2 + (0.6)^2}} = -66$ m/h

Falling ladder

* 

$x(t_0) = 8$ ft $\dfrac{dx}{dt} = 4$ ft/sec

$\dfrac{dh}{dt} = ?$ $h(t_0)^2 + x(t_0)^2 = 100$

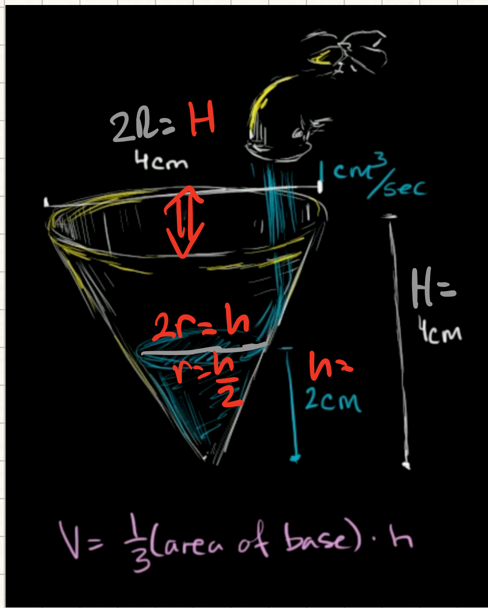
$h(t_0) = 6$ ft.

$h(t_0)^2 + x(t_0)^2 = 100$

$\Rightarrow 2 \cdot h(t_0)\dfrac{dh}{dt} + 2x(t_0)\dfrac{dx}{dt} = 0 \Rightarrow 2(6)\dfrac{dh}{dt} + 2(8)(4) = 0$

$\Rightarrow 12\dfrac{dh}{dt} + 64 = 0 \Rightarrow \dfrac{dh}{dt} = \dfrac{-16}{3}$ ft/sec.

Water pouring into a cone



$$V'(t) = 1 \text{ cm}^3/\text{sec}$$

$$2r = 4\text{cm} \quad H = 4\text{cm}$$

$$h(t_0) = 2\text{cm} \qquad \frac{d}{dt}h = ?$$

$$V_{water} = \frac{1}{3} \cdot A_{\substack{water\\surface}} \cdot h$$

$$= \frac{1}{3} 2\pi \left(\frac{h}{2}\right)^2 \cdot h$$

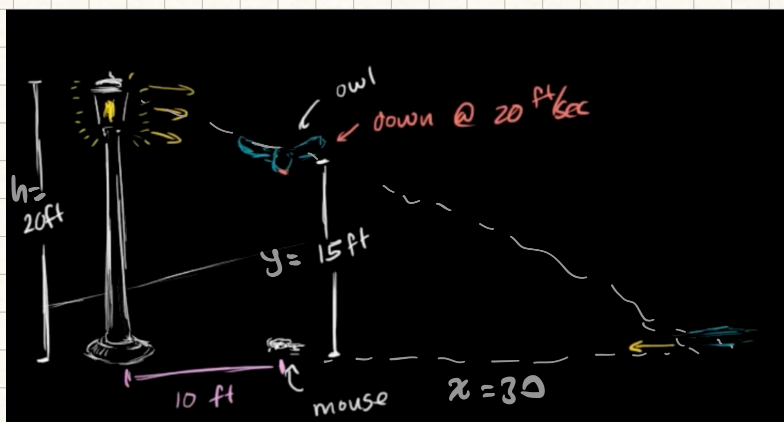
$$V_{water} = \frac{\pi h^3}{12}$$

$$V'_{water} = \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$1 = \frac{\pi}{\cancel{12}} \cdot \cancel{3 \cdot (2)^2} \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{\pi} \text{ cm/sec}$$

Shadow Problem



$$y = 15\text{ft}$$

$$\frac{x}{x+10} = \frac{\cancel{15}\,3}{\cancel{20}\,4} \Rightarrow 4x = 3x + 30$$

$$\Rightarrow x = 30\text{ft}$$

$$h = 20\text{ft}$$

$$\frac{dy}{dt} = -20 \text{ ft/sec}$$

$$\Rightarrow \frac{dx}{dt} = ?$$

$$\frac{x}{y} = \frac{x+10}{20} \Rightarrow 20x = xy + 10y$$

$$\Rightarrow 20\frac{dx}{dt} = \frac{dx}{dt} \cdot y + x\frac{dy}{dt} + 10\frac{dy}{dt}$$

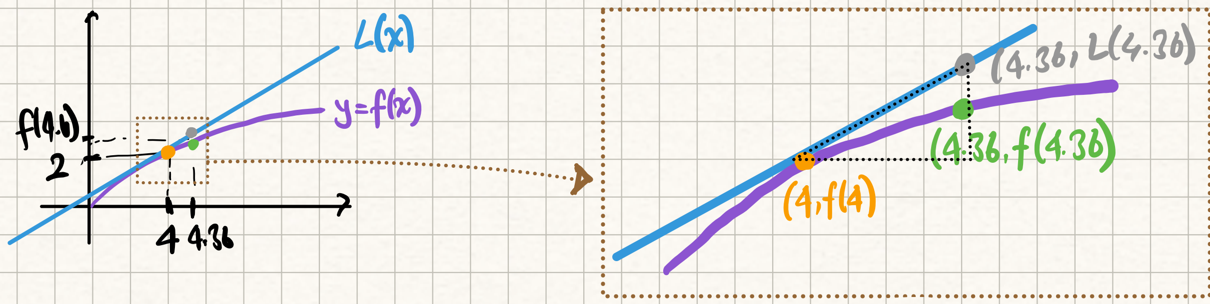
$$\Rightarrow 20\frac{dx}{dt} = \frac{dx}{dt}(15) + (30)(-20) + 10(-20)$$

$$\Rightarrow 5\frac{dx}{dt} = -600 - 200 \Rightarrow \frac{dx}{dt} = -160 \text{ ft/sec.}$$

Approximating Values of a Function Using Local Linearity and Linearization — 17.06.2025

Local linearity

* $f(x) = \sqrt{x} \Rightarrow f(4.36) \approx ?$ (without a calculator?)



$$L(x) = f(4) + \underbrace{f'(4)}_{\text{slope at } x=4} \underbrace{(x-4)}_{\text{difference from } x=4}$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(4) = \frac{1}{4}$$

$$\Rightarrow L(4.36) \approx 2 + \frac{1}{4} \cdot (0.36)$$

$$\approx 2.09$$

* $L(x) = f(a) + f'(a)(x-a)$

* Local linearization approximates a nonlinear function near a specific point using its tangent line (or tangent plane in higher dimensions), relying on the function's derivative to provide a linear approximation that is accurate close to the point of interest.

Local linearity and differentiability

* The idea behind local linearity is that if we zoom in sufficiently on a point, even a non-linear function that is differentiable at that point will look linear. Which means that if the function is not differentiable at that point, we can not use local linearity.

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* Local linearization approximates a non-linear function near a specific point using its tangent line. It relies on the function's derivative to provide a linear approximation, therefore the function has to be differentiable at that specific point.

* $L(x) = f(a) + f'(a)(x-a)$

Worked example: Approximation with local linearity	* f is twice differentiable with $f(2)=1$, $f'(2)=4$, and $f''(2)=3$. Find $\approx f(1.9)$ using the line tangent to the graph of f at $x=2$. $$f(1.9)=f(2)+f'(2)(1.9-2)$$ $$=1+4(-0.1)$$ $$=1-0.4=0.6$$
Quiz	* The local linear approximation to the function f at $x=-2$ is $y=3x-5$. What is $f(-2)+f'(-2)=?$ $$f(-2)=3(-2)-5=-11$$ $$f'(-2)=\frac{d}{dx}[3x-5]=3$$ $$f(-2)+f'(-2)=-11+3=-8$$
Linear approximation of a rational function	* $f(x)=\frac{1}{x-1}$ $\Rightarrow$ Linear approx. of f around $x=-1$. $$y=mx+b$$ $$b=f(0)=-1$$ $$m=f'(x)=\frac{d}{dx}[(x-1)^{-1}]=-(x-1)^{-2}(1)=\frac{-1}{(x-1)^2}$$ $$\Rightarrow f'(-1)=\frac{-1}{(-1-1)^2}=\frac{-1}{4}$$ $$f(-1)=\frac{-1}{2}$$ $$\Rightarrow (x_0,y_0)=(-1,\tfrac{-1}{2})$$ $$\Rightarrow y-(\tfrac{-1}{2})=\frac{-1}{4}(x-(-1))$$ $$y+\frac{1}{2}=\frac{-x-1}{4}$$ $$y=\frac{-1}{4}x-\frac{3}{4}$$

Using L'Hôpital's Rule for Finding Limits of Indeterminate Forms

17.06.2025

L'Hôpital's Rule Introduction

* We'll use derivatives to find limits such as $\frac{0}{0}, \frac{\infty}{\infty}$ ...

* $\lim_{x\to c} f(x) = 0$ AND $\lim_{x\to c} g(x) = 0$ AND $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \Rightarrow$

$$\Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = L$$

* $\lim_{x\to c} f(x) = \pm\infty$ AND $\lim_{x\to c} g(x) = \pm\infty$ AND $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L \Rightarrow$

$$\Rightarrow \lim_{x\to c} \frac{f(x)}{g(x)} = L$$

* $\lim_{x\to 0} \frac{\sin(x)}{x} = ? = \frac{0}{0}$! Let's use l'Hôpital's Rule

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1 \quad \text{so} \quad \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Example:

* $\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x} = \frac{0}{0}$

1st Iter.

$$\Rightarrow \lim_{x\to 0} \frac{2\cdot\cos(x) - 2\cos(2x)}{1 - \cos(x)} = \frac{2\cdot\cos(0) - 2\cos(0)}{1-\cos(0)} = \frac{2-2}{1-1} = \frac{0}{0}$$

2nd Iter.

$$\Rightarrow \lim_{x\to 0} \frac{-2\sin(x) + 4\sin(2x)}{\sin(x)} = \frac{-2(\sin(0)) + 4(\sin(0))}{\sin(0)} = \frac{0}{0}$$

3rd Iter.

$$\Rightarrow \lim_{x\to 0} \frac{-2\cos(x) + 8\cos(2x)}{\cos(x)} = \frac{-2(\cos(0)) + 8\cdot\cos(0)}{1} = 6$$

* If $\lim_{x\to c} \frac{f(x)}{g(x)}$ is still indeterminate, the rule can be applied repeatedly.

* L'Hôpital's Rule helps us find limits in the form $\lim_{x\to c} \frac{u(x)}{v(x)}$ where direct substitution ends in the indeterminate forms $(\frac{0}{0}, \frac{\infty}{\infty} ..)$. The rule says that if the limit $\lim_{x\to c} \frac{u'(x)}{v'(x)}$ exists, then it's equal to $\lim_{x\to c} \frac{u(x)}{v(x)}$. We can apply l'Hôpital's rule repeatedly.

L'Hôpital's Rule:

Limit at infinity example

* $\lim_{x\to\infty} \frac{4x^2-5x}{1-3x^2} = \frac{\infty}{-\infty} \Rightarrow \lim_{x\to\infty} \frac{8x-5}{1-6x} = \frac{\infty}{-\infty}$

$\Rightarrow \lim_{x\to\infty} \frac{8}{-6} = \frac{-4}{3}$

* $\lim_{x\to\infty} \frac{3x^2}{e^x} = ? = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = \frac{6}{\infty} = 0$

* $\lim_{x\to\infty} \frac{4x^2-7x}{(\ln(x))^2} = \lim_{x\to\infty} \frac{8x-7}{2\ln(x)\cdot\frac{1}{x}} = \lim_{x\to\infty} \frac{8x^2-7x}{2\ln(x)}$

$= \lim_{x\to\infty} \frac{16x-7}{\frac{2}{x}} = \lim_{x\to\infty} \frac{16x^2-7x}{2} = \lim_{x\to\infty} \frac{32x-7}{2} = \infty$

* $\lim_{x\to\infty} \frac{e^{5x-3}}{\ln(x-2)} = \lim_{x\to\infty} \frac{e^{5x-3}\cdot 5}{\frac{1}{x-2}} = \lim_{x\to\infty} 5\cdot e^{5x-3}\cdot(x-2)$

$=\infty$

* $\lim_{x\to\infty} \frac{8-3x}{9x^2+11} = \lim_{x\to 0} \frac{-3}{18x} = 0$

* $\lim_{x\to\infty} \frac{6x}{15x-8} = \lim_{x\to\infty} \frac{6}{15} = 2.5$