Composing Functions (Bileşke Fonksiyonlar)

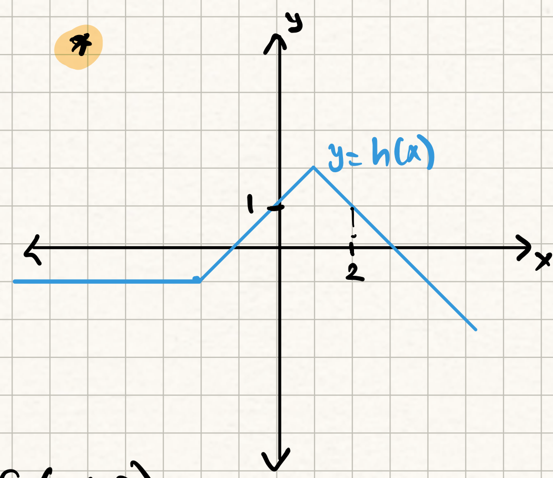
07.05.2025

Introduction

* $f(x) = x^2 - 1$

*

t	$g(t)$
1	3
2	-3
3	4
4	-1

* 

* $f(g(2)) = ?$

$2 \to \boxed{g} \to g(2) \to \boxed{f} \to f(g(2))$

$-3 \to f \to (-3)^2 - 1 = 8$

* $f(h(2)) = ? = f(1) = 1^2 - 1 = 0$

* $\underbrace{h(\underbrace{g(\underbrace{f(2)}_{3})}_{4})}_{-1} = ?$

* $f(g(x)) = (f \circ g)(x) =$ "f composed with g"

What's a real-world example for composite functions?

* Can is a farmer. Each year he plants seed that turn into corn. The function below gives the amount of corn, C, in kilograms (kg), that he expects to produce if he plants corn on a acres of land:

$$C(a) = 7500a - 1500$$

* Function composition is the action of combining two functions in such a way so that the outputs of one function becomes the input of the other.

$$f(g(x)) = (f \circ g)(x) = \text{"f composed with g"}$$

The function belows predicts how much money, M, in dollars, he will earn from selling c kilograms of corn:

$$M(c) = 0.9c - 50$$

How much money does he expect to make if he plants corn seed on a a acres of land?

$$M(C(a)) = (M \circ C)(a) = ?$$

$$M(C(A)) = 0.9\,C(a) - 50$$
$$= 0.9\,(7500a - 1500) - 50$$
$$= 6750a - 1350 - 50$$
$$= 6750a - 1400 = (M \circ C)(a)$$

* How much money can Cau expect to earn if he sells corn produced on 1.5 acres?

$$(M \circ C)(1.5) = 6750(1.5) - 1400$$
$$= \$8725$$

How to evaluate composite functions?

* If $f(x) = 3x - 1$ and $g(x) = x^3 + 2$, what is $f(g(3)) = ?$

1. "Inside out" evaluation:

$g(3) = 3^3 + 2 = 11 \qquad f(11) = 3(11) - 1 = 32$

2. Finding the composite function

$$(f \circ g)(x) = 3(x^2 + 2) - 1 \qquad \Rightarrow (f \circ g)(3) = 3(3)^2 + 5$$
$$= 3x^2 + 5 \qquad\qquad = 32$$

* We can evaluate a composite function either by "inside out evaluation" or by "finding the composite function by substitution."

Modeling with Composite Functions

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How can we express a composit function in terms of the model it creates?

* Carter has noticed a few quantitative relationships related to the success of his football team and has modeled them with the following functions:

Function	Input	Output
N	Winning percentage, w	Average number of fans per game, $N(w)$
W	Average daily practice time, x	Winning percentage, $W(x)$
P	Number of rainy days, r	Average daily practice time $P(r)$

What does the expression $N(W(x))$ represents?

input → output

① $W(x) \Rightarrow$ "average daily practice time" → "winning percentage"

first input

② $N(W(x)) \Rightarrow$ "winning percentage" → "average number of fans per game"

last output

$N(W(x))$ = "The average number of fans per game, as a function of the average practice time."

* "The last output as a function of the first input."

* We can use "The last output as a function of the first input" expression to express a composit function in terms of the model it generates

Not every model makes sense!

* Jaylen modeled the following relationships about their bus ride:

  - $P(b)$: Probability that Jaylen gets to work on time as a function of the time bus arrives (b).
  - $N(k)$: Number of people at the bus stop when the bus arrives as a function of the time the bus arrives (k)
  - $T(x)$: Time the bus arrives as a function of centimetors of precipitation (yagis) per hour.

? Which **two** of the following composed functions **make sense**?

Ⓐ $P(T(x))$ Ⓑ $P(N(k))$ Ⓒ $N(T(x))$ Ⓓ $T(P(b))$

Ⓐ Precipitation $\xrightarrow{T}$ arrival time $\xrightarrow{P}$ prob. on time ✓

Ⓑ arrival time $\xrightarrow{N}$ # of people at the stop $\xrightarrow{P}$ prob. on time ✗

Ⓒ precipitation $\xrightarrow{T}$ arrival time $\xrightarrow{N}$ # people at the stop ✓

Ⓓ arrival time $\xrightarrow{P}$ prob. on time $\xrightarrow{T}$ arrival time ✗

* When we compose functions, we must make sure that it makes sense to plug the output of the inner function to the outer function.

* When we compose functions, we must make sure that it makes sense to plug the output of the inner function as an input for the outer function.

Invertible Functions

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What are inverse functions?

- Inverse functions reverse each other:

$$f(a)=b \Rightarrow f^{-1}(b)=a$$

- For a function to be invertible, its inverse must be a function too.

x	1	2	3	4
$h(x)$	2	1	2	5

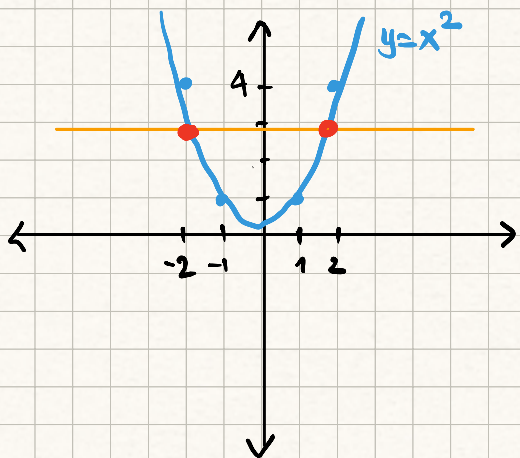
$\Rightarrow h^{-1}(2)=\{1,3\}$ ✗ NOT A FUNC!

h is non-invertible

- A function is invertible only if each input has a unique output (one-to-one mapping).

What is a horizontal line test?

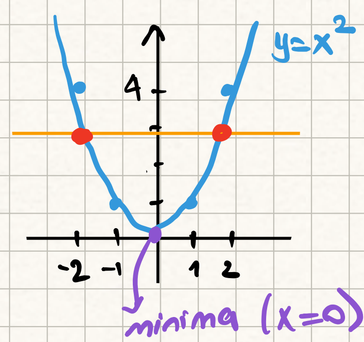
- If we can draw a horizontal line that intercepts the function on multiple points, the function is non-invertible.



$f(x)=x^2$ is non-invertible!

Restricting domains of functions to make them invertible

- If the function is not strictly monotonic (so it has both increasing and decreasing intervals) it is not invertible within a domain which includes a minima or maxima



- If we restrict our domaint to $-\infty < x \leq 0$ or $0 \leq x < \infty$, the function becomes invertible.

- Inverse functions reverse each other: $f(a)=b \Rightarrow f^{-1}(b)=a$
- A function is invertible only if there's a one-to-one relationship between its domain and its range
- If we can draw a horizontal line that intercepts the function on multiple points, the function is non-invertible.
- If the function has both increasing and decreasing intervals, it is not invertible within a domain that includes a minima or maxima.

Inverse Functions in Graphs and Tables

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Reading inverse values from a graph

The following graph shows $y = f(x)$.

What appears to be the value of $f^{-1}(2)$? → 4

Sketch the graph of $y = f^{-1}(x)$.

$b = f(a) \Rightarrow (a, b)$

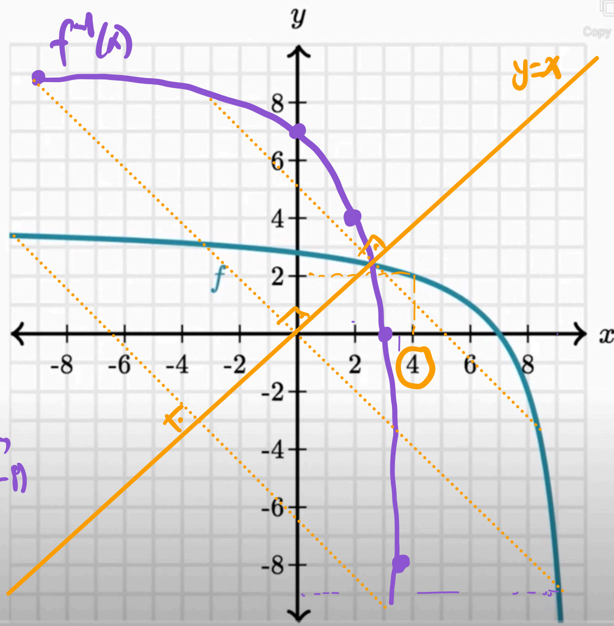
$a = f^{-1}(b) \Rightarrow (b, a)$

$(9,-9) \to (-9,9)$ $(-8,3.5) \to (3.5,-8)$

$(4,2) \to (2,4)$

$(7,0) \to (0,7)$

$(0,3) \to (3,0)$



* The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

Reading inverse values from a table

x	50	54	58	62	67	71
$g(x)$	70	65	59	54	49	44

$g^{-1}(54) = ?$

$g(x) = 54 \Rightarrow$

$x = 62$

* The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.

Verifying Inverse Functions by Composition

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Verifying inverse functions from tables

The following tables give all of the input-output pairs for the functions s and t.

x	-5	-1	2	7	8
$s(x)$	100	12	18	70	61

x	12	18	61	70	100
$t(x)$	-1	2	8	7	-5

Complete the table for the composite function $s(t(x))$.

x	12	18	61	70	100
$s(t(x))$	12	18	61	70	100

Are functions s and t inverses? YES

* $f \circ f^{-1}(x) = x$

$\Rightarrow s(x) = t^{-1}(x)$

$s^{-1}(x) = t(x)$

Using specific values to test for inverses

* You can not use specific values to prove that two continuous functions are inverses of each other (we have other methods for this), but you can use specific values to prove that two continuous functions are not inverses of each other.

* $f(x) = x^2 + 3$ $\quad g(x) = \sqrt{x-3}$ $\quad \rightarrow \quad f^{-1}(x) \overset{?}{=} g(x)$

$\rightarrow$ let x be 2, then

$f(x) = 7$ $\quad g(7) = \sqrt{4} = \mp 2 \quad \Rightarrow \quad f^{-1}(x) \neq g(x)$

Verifying inverse functions by composition

* $f(x) = (x+7)^3 - 1$ $\qquad g(x) = \sqrt[3]{x+1} - 7$

$(f \circ g)(x) = \left(\sqrt[3]{x+1} - 7 + 7\right)^3 - 1$

$= x + 1 - 1$

$= x$

$(g \circ f)(x) = \sqrt[3]{(x+7)^3 - 1 + 1} - 7$

$= x + 7 - 7$

$= x$

* $(f \circ g)(x) = (g \circ f)(x) = x \quad \Rightarrow \quad f(x)$ and $g(x)$ are inverses

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* $f(g(x)) = g(f(x)) = x \quad \Rightarrow \quad f^{-1}(x) = g(x)$ and $g^{-1}(x) = f(x)$

* We can use specific values to prove that two continuous functions are not inverses of each other, but we can not use specific values to prove that two continuous functions are inverses.

* $f(x) = 2x-3$, $g(x) = \frac{1}{2}x + 3$ $\Rightarrow$

$(f \circ g)(x) = 2\left(\frac{1}{2}x+3\right)-3$ $(g \circ f)(x) = \frac{1}{2}(2x-3) + 3$

$= x+3$ $= x + \frac{3}{2}$

$\Rightarrow (f \circ g)(x) \neq x$, $(g \circ f)(x) \neq x$, $(f \circ g)(x) \neq (g \circ f)(x)$

$\Rightarrow f(x) \neq g^{-1}(x)$, $g(x) \neq f^{-1}(x)$

Quiz

$f(x) = 4x-3$ and $g(x) = \frac{1}{4}x + 3$ $\Rightarrow$

① $(f \circ g)(x) = ? = 4\left(\frac{1}{4}x+3\right)-3 = x+9$

② Are $f(x)$ and $g(x)$ inverses? NO! $((f \circ g)(x) \neq x)$