# PRECALCULUS

## 01 - COMPOSITE AND INVERSE FUNCTIONS

* Function composition is the action of combining two functions in such a way so that the outputs of one function becomes the input of the other.

$$f(g(x)) = (f \circ g)(x) = \text{"f composed with g"}$$

* We can evaluate a composite function either by "inside out evaluation" or by finding the composite function by substitution."
* We can use "The last output as a function of the first input" expression to express a composit function in terms of the model it generates.
* When we compose functions, we must make sure that it makes sense to plug the output of the inner function as an input for the outer function.
* Inverse functions reverse each other: $f(a) = b \Rightarrow f^{-1}(b) = a$
* A function is invertible only if there is a one-to-one relationship between its domain and range.
* If we can draw a horizontal line that intercepts the function on multiple points on the graph, then the function is invertible.
* If the function has both increasing and decreasing intervals, it is not invertible within a domain that includes a minima or maxima. In this case, we can restrict its domain to make it invertible.
* The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the line $y = x$.
* $f(g(x)) = g(f(x)) = x \Rightarrow f(x)$ and $g(x)$ are inverses of each other.
* We can use specific values to prove that two continuous functions are not inverses of each other, but we can not use specific values to prove that two continuous functions are inverses of each other.

# 02 TRIGONOMETRY

* The $(x,y)$ coordinate on the unit circle corresponding to an angle of $\theta$ degrees gives us $(\cos\theta, \sin\theta)$
* $\vec{S} = [0\ 1\ 2\ 3\ 4]$, $\vec{C} = [4\ 3\ 2\ 1\ 0]$, $\vec{\theta} = [0\ 30\ 45\ 60\ 90] \Rightarrow$
  * $\sin(\theta_i) = \frac{\sqrt{S_i}}{2}$
  * $\cos(\theta_i) = \frac{\sqrt{C_i}}{2}$
* $\sin(\theta) = x \Rightarrow \sin^{-1}(x) = \theta$; Domain: $[-1,1]$, Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$ * where $\cos(\theta) \geq 0$
* $\cos(\theta) = x \Rightarrow \cos^{-1}(x) = \theta$; Domain: $[-1,1]$, Range: $[0, \pi]$ * where $\sin(\theta) \geq 0$
* $\tan(\theta) = x \Rightarrow \tan^{-1}(x) = \theta$; Domain: $(-\infty, \infty)$, Range: $(\frac{-\pi}{2}, \frac{\pi}{2})$ * where $\cos(\theta) > 0$
* Law of Sines: $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$
* Law of Cosines: $c^2 = a^2 + b^2 - 2ab(\cos(\gamma))$
* A sinusoidal equation is a mathematical expression that describes a smooth periodic oscillation resembling the sine or cosine function. It's used to model repetitive phenomenia such as sound waves, light waves, tides, etc.
  * $y = A\sin(B(x-C)) + D$
  * $y = A\cos(B(x-C)) + D$
  * A: Amplitude (The peak deviation from the center)
  * B: Frequency (The period of wave)
  * C: Phase Shift ($C>0 \Rightarrow$ shifts right, $C<0 \Rightarrow$ shifts left)
  * D: Vertical Shift (Moves the entire graph up (+) or down (-))
* When solving sinusoidal equations, take into account that the trigonometric identities on the unit circle as well as the repetitive nature (periodicity) of the trigonometric functions.
* We can derive all trig. angle addition identities using:
  1. $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
  2. $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
* $\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$ $\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x) + \tan(y)}$

# 03 COMPLEX NUMBERS ($\mathbb{C}$)

* A complex number is any number that can be written as $a + bi$ (a.k.a. the rectangular form), where $i$ is the imaginary unit ($i^2 = -1$, $i = \sqrt{-1}$), and $a, b \in \mathbb{R}$. "a" is called the "real part", "b" is called the "imaginary part" of the complex number.
* The complex plane consists of real axis (horizontal) and imaginary axis (vertical) which intercept at zero.
* Distance between two complex numbers z and w is:

$$|z - w| = \sqrt{(Re(z) - Re(w))^2 + (Im(z) - Im(w))^2}$$

* Midpoint of z and w is $\dfrac{Re(z) + Re(w)}{2} + i\,\dfrac{Im(z) + Im(w)}{2}$
* Conjugate of the $z = a + bi$ $= \bar{z} = a - bi$
  * $z + \bar{z} = 2\,Re(z)$
  * $z \cdot \bar{z} = |z|^2 = Re(z)^2 + Im(z)^2$
* To divide a complex number by another, we multiply both numbers by the conjugate of the denominator.
* Modulus of a complex number ($r = |z|$) is its absolute value, its distance from the origin: $z = a + bi \Rightarrow r = |z| = \sqrt{a^2 + b^2}$
* Argument of a complex number is its angle: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
* If we're given r and $\theta$:
  * $z = r(\cos(\theta) + \sin(\theta))$ (a.k.a the polar form)
  * $z = r \cdot e^{i\theta}$ (a.k.a the exponential form)
* Complex multiplication: When we multiply z with w, we scale it by $|w|$ and rotate it by $\theta_w$. Therefore, it helps to express w in its polar form to easily visualize the rotation.
* Complex division: When we divide z by w, we scale it by $\frac{1}{|w|}$ and rotate it by $-\theta_w$. It's same as multiplying by $\frac{\bar{w}}{|w|^2}$.

* $w_1 \cdot w_2 = |w_1| \cdot |w_2| \left[\cos(\theta_{w_1} + \theta_{w_2}) + i \sin(\theta_{w_1} + \theta_{w_2})\right]$
* $\frac{w_1}{w_2} = \frac{|w_1|}{|w_2|} \cdot \left[\cos(\theta_{w_1} - \theta_{w_2}) + i \sin(\theta_{w_1} - \theta_{w_2})\right]$
* $w^n = |w|^n \cdot \left[\cos(\theta_w \cdot n) + i \sin(\theta_w \cdot n)\right]$
* $z^n = w \Rightarrow z = \sqrt[n]{|w|} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right], k = 0, 1, 2, \ldots, n-1$
* $f(x) = ax^2 + bx + c \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## 04 RATIONAL FUNCTIONS

* A rational expression is a fraction whose numerator and denominator are polynomials.
* Domain of a rational expression includes all real numbers except for those that make its denominator zero
* A rational expression is reduced to its lowest terms if numerator and denominator have no factors in common.
* End behaviour of rational functions:
    * If the degree of the denominator is larger : $y = 0$
    * If the degrees are equal $y = \frac{a}{b} = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$
    * If the degree of the numerator is larger, long divide the polynomials.
* Inputs where the function is defined and the numerator is equal to zero are the "zeros" of the function. After cancelling out the common factors, any undefined input that no longer makes the denominator zero is a removable discontinuity. The remaining undefined values are vertical asymptotes.
* When multiplying rational expressions, we must identify and exclude any values that would make the denominator of any expression equal to zero, as there are restrictions on the domain.

* When dividing rational expressions, we must additionally exclude any values that would make the numerator of the divisor zero, since division by zero is undefined.
* The Least Common Multiple (LCM) of the denominators in two or more fractions is called the least common denominator. Using the least common denominator is "nearly" a must with rational expressions in order to avoid performing arithmetics with higher degree polynomials.

## 05 CONIC SECTIONS

* A conic section is a curve obtained from a cone's surface intersecting a plane.
* The main types of conic sections are ellipses, parabolas, and hyperbolas. Circles are a special kind of ellipses, so they're conic sections as well.
* Ellipse:
  * has two radii of different size: The major radius is longer than the minor radius.
  * The vertices and co-vertices intersect at the center.
  * $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ is the equation of the ellipse centered at $(h,k)$, whose horizontal radius is "a" and vertical radius is "b."
  * The foci lie on the major radius. The sum of the distances from the two foci to each point on the ellipse is always the same. The equation of the foci is: $f^2=|a^2-b^2|$
* Hyperbola:
  * is made up by two distinct branches or curves that extend away from each other.
  * Center is the point equidistant from the two branches, around which the hyperbola is symmetrical.

* Transverse axis is the line segment connecting the two closest points on the two branches.
* Conjugate axis is the line segment perpendicular to the transverse axis, passing through the center.
* $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2}$ and $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2}$ are the equations for two types of hyperbolas, where the origin is $(h,k)$.
* Asymptotes are the two lines that the branches of the hyperbola will approach as they extend further and further away from the center. These asymptotes intersect at the center and their formula is $y = \mp \frac{b}{a}$.
* Foci are the two fixed points located inside each curve of the hyperbola such that, for any point on the hyperbola, the difference of distances to the foci is a constant. The formula for the foci is: $f^2 = a^2 + b^2$

## 06 VECTORS

* A vector is a mathematical object that has both a magnitude (length) and a direction. Vectors can be used to represent physical quantities such as velocity, force, and acceleration.
* Vector components are the horizontal and vertical parts of a vector: $\vec{a} = (\Delta x, \Delta y)$
* Magnitude of a vector: $\|\vec{a}\| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$
* Multiplying a vector by a positive scalar doesn't change its direction, only its magnitude. Multiplying by a negative scalar flips the vector by $180°$ and scales it. The resulting vector still lies on the same line.
  $\vec{a} = (x, y),\ c \in \mathbb{R} \Rightarrow c \cdot \vec{a} = (c \cdot x, c \cdot y)$
* To add vectors, we simply add their components:
  $\vec{a} = (x_a, y_a)\quad \vec{b} = (x_b, y_b) \Rightarrow \vec{a} + \vec{b} = (x_a + x_b, y_a + y_b)$

* Direction of the vector $\vec{v}$ is $\theta_v = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right)$. When we use a calculator, we need to remember that the tangent value is the same for the diagonally opposite quadrants and we might need to add $180^\circ$ to the result.
* $\vec{a} = (\|\vec{a}\| \cdot \cos(\theta_a), \|\vec{a}\| \cdot \sin(\theta_a))$

## 07 MATRICES

* A matrix is a rectangular arrangement of numbers into rows and columns.
* Matrices provide a structured and efficient way to organize, manipulate, and analyze datasets, enabling easy computation and transformation. They also align well with linear algebra operations, making them ideal for machine learning, statistics, and other data-driven applications.
* $A_{i,j}$ represents the element on the $i^{th}$ row and $j^{th}$ column of matrix A.
* Scalar multiplication is the product of a real number (scalar) and a matrix, where each entry is multiplied by the scalar.
* To add/subtract matrices, we add/subtract the corresponding entries, therefore, the dimensions of the matrices should match.
* Zero matrix has zeros for all its elements : $O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
* When we subtract a matrix from the zero matrix, we find the opposite of that matrix.
* $A+B = B+A$ // $A+(B+C) = (A+B)+C$ // $A+(-A) = O$
* $c \cdot d \cdot A = c \cdot (d \cdot A)$ // $c(A+B) = c \cdot A + c \cdot B$ // $(c+d)+A = c \cdot A + d \cdot A$
* $A \cdot O = O$ // $A + O = A$
* Matrices can represent linear transformations that map vectors from one space to another, such as scaling, rotating, shearing, or reflecting them. By multiplying a vector by a transformation matrix, we apply that transformation to the vector.

* The determinent is a scalar-valued function of the entries of a square matrix. For the 2x2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = a \cdot d - b \cdot c$
* The absolute value of $\det(A)$ gives us the area of the parallelogram that its column vectors create.
* When we use A as a transformation matrix, the area of the figure it transforms scales by the absolute value of $\det(A)$.
* $B \circ A$ is a matrix composition. Just like composite functions, it means "Transform with A first, then transform the result with B."
* For $A \cdot B$ to be defined, number of columns of A and number of rows of B must be equal: $A_{m \times n} \cdot B_{n \times k} = C_{m \times k}$
* $I \cdot A = A \cdot I = A \Rightarrow I$ is an identity matrix. Identity matrices are square matrices that have ones on the main diagonal and zeros at the other positions.
* $A \cdot B \neq B \cdot A$
* $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
* $A \cdot (B+C) = A \cdot B + A \cdot C$ and $(A+B) \cdot C = A \cdot C + B \cdot C$
* $\left.\begin{matrix} ax+by=p \\ cx+dy=q \end{matrix}\right\}$ can be represented as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$, the generalized form of which is $A_{n \times n} \cdot \vec{x}_{n \times 1} = \vec{b}_{n \times 1}$
* Representing a system of equations as matrices helps us to solve it more efficiently, especially if the matrix is a transformation matrix and/or the system is large.
* $A \cdot A^{-1} = A^{-1} \cdot A = I \Rightarrow A^{-1}$ is the inverse of A.
* Not all matrices are invertible. A matrix is invertible only if:
  1. is a square matrix (so $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are both defined)

  AND 2. has a non-zero determinant (so that the area of the pre-image is not scaled by zero and turn into a line)

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ adjugate of $A$ = $adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* $A^{-1}_{2x2} = \frac{1}{det(A)} \cdot adj(A)$

* $A \cdot \vec{x} = b \Rightarrow A^{-1} \cdot A \cdot \vec{x} = A^{-1} \cdot \vec{b} \Rightarrow \vec{x} = A^{-1} \cdot \vec{b}$, which is one way of solving linear equations with matrices.

# 08 PROBABILITY AND COMBINATORICS

* Probability of event A occuring $= P(A) = \frac{\text{\# of favorable outcomes}}{\text{\# of all outcomes}} \in [0,1]$
* $P(\text{not } A) = P(A^c) = 1 - P(A)$
* $P(A \backslash B)$ = Probability of B occurring GIVEN that A has occurred.
* $P(A \text{ and } B) = P(A \cap B) = P(A \backslash B) \cdot P(B) = P(B \backslash A) \cdot P(A)$
* $P(A \cap B) = P(A) \cdot P(B)$ (if A and B are independent)
* $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
* $P(A \cup B) = P(A) + P(B)$ (if A and B are mutually exclusive)
* Permutation is the number of ways we can arrange things in a specific order : ${}_nK_k = \frac{n!}{(n-k)!}$
* Combination is the number of ways we can select things without considering order : ${}_nC_k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
* A probability distribution describes the likelihood of all possible outcomes for a given event based on mathematical calculations (theoratical) or observed data (empirical).
* Expected value (EV) is the mean of the possible values a random variable can take, weighted by the probability of those outcomes.

# 09 SERIES

* A series is an addition of infinitely many terms, one after another.
* In geometric series, the ratio of consecutive terms is constant.
$S_n = \frac{a(1-r^n)}{1-r}$, where "n" is the number of terms, "a" is the first term, and "r" is the ratio. It can also be represented as:
$\sum_{k=0}^{n} a \cdot r^k$
* The binomial theorem: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
* In arithmetic series, the difference of consecutive terms (the common difference) is constant: $S_n = n \cdot \left(\frac{a_1 + a_n}{2}\right)$. The number of terms, n, can be calculated with: $n = \frac{a_n - a_1}{d} + 1$, where "d" is the common difference.