Dot products and cross products

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How to calculate the dot product?

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$$

Length vs. dot product

* $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$: The square of the length of a vector is equal to the vector dotted to itself.

Properties of dot products

* Commutative : $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
* Distributive : $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
* Associative : $(c \cdot \vec{u}) \cdot \vec{v} = c \cdot (\vec{u} \cdot \vec{v})$

Cauchy-Schwarz Inequality

* $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\|$

$<$: Vectors are linearly independent

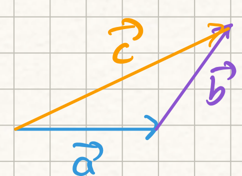
$=$: Vectors are linearly dependent

EX : $\vec{u} = (3, 4)$, $\vec{v} = (-6, -8)$ $\Rightarrow$ $|(3 \cdot (-6)) + (4 \cdot (-8))| \overset{?}{=} \sqrt{3^2 + 4^2} \cdot \sqrt{(-6)^2 + (-8)^2}$

$40 = 5 \cdot 8$

$\Rightarrow$ $\vec{u}$ and $\vec{v}$ are L.D.

Vector Triangle Inequality



$\|\vec{c}\| \leq \|\vec{a}\| + \|\vec{b}\| \Rightarrow$

* $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

$<$: Vectors are linearly independent

$=$: Vectors are linearly dependent

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* Dot product measures how similar two vectors are.

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 \cdot v_1 + u_2 \cdot v_2 + \dots + u_n \cdot v_n$$

* Dot product is commutative ($\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$), distributive ($\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$) and associative: $(c \cdot \vec{u}) \cdot \vec{v} = c \cdot (\vec{u} \cdot \vec{v})$

Ex:

$\vec{u} = (2,-1)$
$\vec{v} = (-1,4)$ $\Rightarrow$

$\|\vec{u}+\vec{v}\| \overset{?}{=} \|\vec{u}\| + \|\vec{v}\|$

$\|(1,3)\| \overset{?}{=} \sqrt{2^2+(-1)^2} + \sqrt{(-1)^2+4^2}$

$\sqrt{1+9} \overset{?}{=} \sqrt{5} + \sqrt{17}$

$\sqrt{10} \neq \sqrt{5} + \sqrt{17} \Rightarrow$ $\vec{u}$ and $\vec{v}$ are L.I.

Angle between vectors

* $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$

* If $\theta = 90^\circ \Rightarrow \cos 90^\circ = 0 \Rightarrow \vec{u} \perp \vec{v}$

Equation of a plane, and normal vectors

* A plane is the set of all vectors that are perpendicular (orthogonal) to one given normal vector, whic is the vector that is perpendicular (orthogonal) to the plane.

$Ax + By + Cz = D$: The standard equation of a plane where the normal vector is:

$\vec{n} = (A,B,C)$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$: Alternative equation which is the dot product of the normal vector and a vector in the plane

E1: Let $(2,5-1)$ be a point in the plane and $\vec{n} = (-1,-2,1)$ is the norm vector, the plane's equation is:

$a(x-2) + b(y-5) + c(z+1) = 0$

$-1(x-2) - 2(y-5) + 1(z+1) = 0$

$-x - 2y + z = -13$

Notice the coefs! $(-1,-2,1)$

* Cauchy-Schwarz Inequality : $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\|$ (= means lineerly dependent)
* Vector triangle inequality : $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (= means " ")
* $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos\theta$
* A plane is the set of all vectors that are perpendicular (orthogonal) to one given normal vector, whic is the vector that is perpendicular (orthogonal) to the plane.
* $Ax + By + Cz = D$ → Standard equation of a plane
* $\vec{n} = (A,B,C)$ → Normal vector
* $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

↳ (x_0, y_0, z_0) a point on a plane

Ex: $\vec{n} = (-3,2,5)$, passes through $(1,0,-2)$ $\Rightarrow$

$$-3x + 2y + 5z = D$$

$$-3 \cdot 1 + 2 \cdot 0 + 5 \cdot (-2) = -13 \Rightarrow$$

$$-3x + 2y + 5z = -13$$

Cross products

* Cross product is a vector that's orthogonal to the two vectors that are crossed.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 \cdot b_2 - a_2 \cdot b_1)$$

* $[\ ]$: Matrix $|\ |$: Determinent (more on determinants later)

Ex: $\vec{a} = (1,2,3)$, $\vec{b} = (-1,0,3)$ $\Rightarrow$ $\vec{a} \times \vec{b} = ?$

$$\vec{a} \times \vec{b} = i \begin{vmatrix} 2 & 3 \\ 0 & 3 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ -1 & 3 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= i(6-0) - j(3+3) + k(0+2)$$

$$= 6i - 6j + 2k$$

$$= (6,-6,2)$$

$$= [6 \ \ -6 \ \ 2]$$

$$= \begin{bmatrix} 6 \\ -6 \\ 2 \end{bmatrix}$$

Different notations for the same vector.

* Cross product is a vector that's orthogonal to the two vectors that are crossed.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 \cdot b_2 - a_2 \cdot b_1)$$

* $\underbrace{\|\vec{u} \times \vec{v}\|}_{\text{length of cross product}} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \sin\theta$

Length of the cross product vector vs. Dot product

* Length of the cross product vector is a scaler that tells us how different two crossed vectors are:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| . \|\vec{v}\| . \sin\theta$$

* If $\vec{u} \perp \vec{v}$, length of the cross product is maximized.
* If $\vec{u}$ and $\vec{v}$ are collinear, length of the cross product is zero.

* Dot Product tells us how similar two dotted vectors are:

$$\vec{u} . \vec{v} = \|\vec{u}\| . \|\vec{v}\| . \cos\theta$$

* If $\vec{u} \perp \vec{v}$, dot product is 0.
* If $\vec{u}$ and $\vec{v}$ point the same direction, dot product is max.
* If $\vec{u}$ and $\vec{v}$ point the opposite direction, dot product is min.

* This page is its own summary. Copy all of it.