Systems of Equations

04/03/2025

What is the formula of a linear equation?

$y = w \cdot x + b$

w: weight, b: bias

weight: wind speed → power output

$y = \underbrace{w_1}\cdot x_1 + \underbrace{w_2}\cdot x_2 + b$

wind speed + temperature → power output

How do we formulate linear regression in ML? (as multiple terms)

$$w_1 \cdot x_1 + w_2 \cdot x_2 + \dots + w_n \cdot x_n + b = y$$

A dataset will have lots of rows, therefore ⇒

How do we formulate system of linear equations in ML?

$$\begin{aligned}
w_1 \cdot x_1^{(1)} + w_2 \cdot x_2^{(1)} + \dots + w_n x_n^{(1)} + b &= y^{(1)} \\
w_1 \cdot x_1^{(2)} + w_2 \cdot x_2^{(2)} + \dots + w_n x_n^{(2)} + b &= y^{(2)} \\
w_1 \cdot x_1^{(3)} + w_2 \cdot x_2^{(3)} + \dots + w_n x_n^{(3)} + b &= y^{(3)} \\
&\vdots \\
w_1 \cdot x_1^{(m)} + w_2 \cdot x_2^{(m)} + \dots + w_n x_n^{(m)} + b &= y^{(m)}
\end{aligned}$$

What are the values we aim to find so that we can get as close as possible to the best fitting line?

Now we have a lot of equations, a system of equations, and we aim to find values for weights and bias that gets us as close as possible to solve all these equations at once.

How do we formulate linear regression in ML (as vectors and matrices)?

$$W \cdot X + b = \hat{y}$$

Vector of weights: $[w_1 \; w_2 \; w_3 \dots w_n]$

Matrix of features:

$$\begin{bmatrix} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \dots & x_n^{(1)} \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & \dots & x_n^{(2)} \\ \vdots & & & & \\ x_1^{(m)} & x_2^{(m)} & x_3^{(m)} & \dots & x_n^{(m)} \end{bmatrix}$$

bias (a constant)

Vector of Target Variables: $[y^{(1)} \; y^{(2)} \; y^{(3)} \dots y^{(m)}]$

When we apply a linear regression model, we actually try to solve a system of equations in a way that our formula ($W \cdot X + b = \hat{y}$) would give us the best solution.

How do we convert a system of equations into vectors and matrices?

Example:

① Linear algebra score added to your calculus score minus your probability score was 6.

② Your algebra score minus your calculus score plus double your probability score was 4.

③ Four times your linear algebra score minus double your calculus score added to your probability score was 10

$$+1a \quad +1c \quad -1p = 6$$
$$+1a \quad -1c \quad +2p = 4$$
$$+4a \quad -2c \quad +1p = 10$$

$W = [a \ c \ p]$

$X = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 4 & -2 & 1 \end{bmatrix}$

$\hat{y} = [6 \ 4 \ 10]$

$b = 0$

What is a system of sentences?

System 1	System 2	System 3
. The dog is black	. The dog is black	. The dog is black
. The cat is orange	. The dog is black	. The dog is white
"Complete"	"Redundant"	"Contradictory"
Non-singular system	Singular system	Singular system

What type of systems are there based on completeness and singularity?

- A non-singular system carries as many pieces of information as sentences.

"System of equations is basically a system of sentences with numbers"

Sentences with numbers ⇒ equations

- . You bought an apple and a banana for \$10. Your wife bought an apple and two bananas for \$12. How much each fruit cost?

$$\Downarrow$$
$$a + b = 10$$
$$a + 2b = 12$$

Non-singular systems are complete. They carry as many pieces of information as equations, and they have one solution. Singular systems carry either redundant information (infinite number of solutions) or contradictory information (no solutions).

① You bought an apple and a banana for \$10.
② You bought two apples and two bananas for \$20. → Redundant → INFINITE NUMBER OF SOLUTIONS

· How much does each cost?

· Any two numbers that add to 10 are solutions.

① You bought an apple and a banana for \$10.
② You bought two apples and two bananas for \$24. → Contradictory → NO SOLUTIONS

How much each cost?

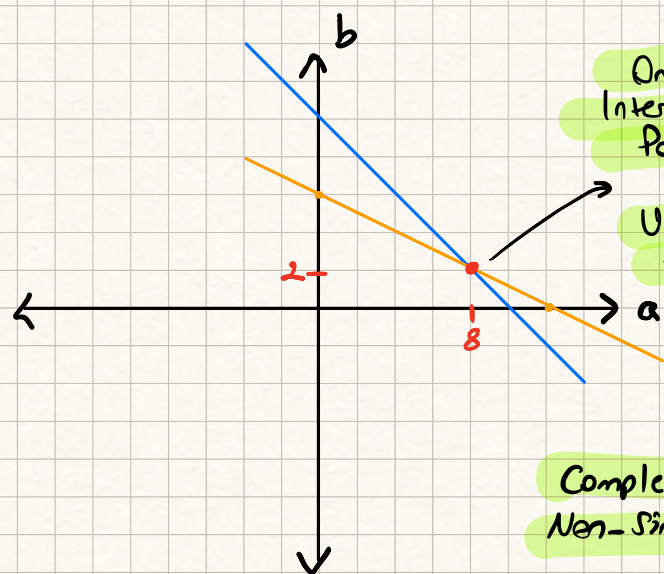
Graphical representation of system of equations

Two-variable linear equations can be visualized as lines in the coordinate plane. (Three-variables → planes in space, more variables → high-dimensional things we won't worry about now!)

Linear Equations ⟶ Lines

System of Linear Equations ⟶ Arrangements of lines

$a + b = 10$
$a + 2b = 12$

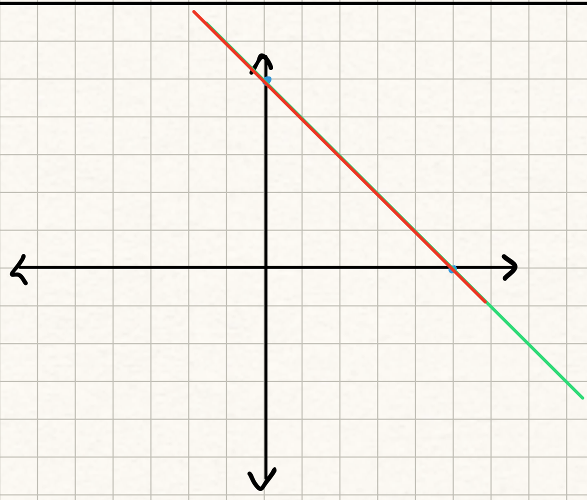


One Interception Point → Unique Solution

Complete
Non-Singular

$a + b = 10$

$2a + 2b = 20$



Overlapping Lines

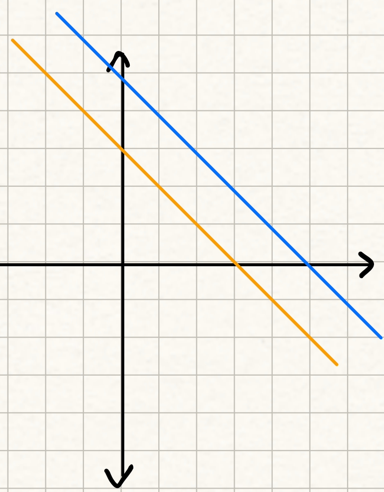
$\Downarrow$

Infinitely many Solutions

Redundant Singular

$a + b = 10$

$2a + 2b = 12$

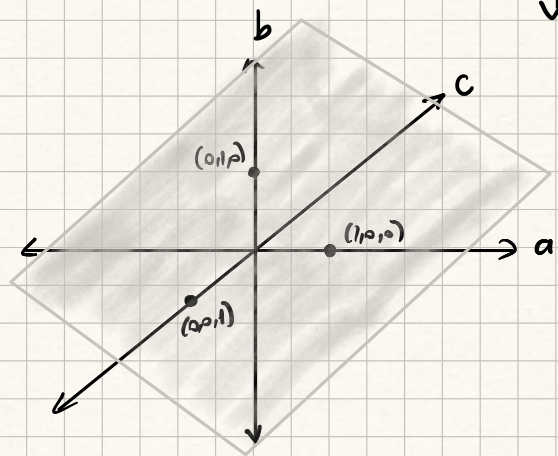


Parallel Lines

$\Downarrow$

No Solutions

Contradictory Singular



$a + b + c = 1 \Rightarrow$

$1 + 0 + 0 = 1$
$0 + 1 + 0 = 1$
$0 + 0 + 1 = 1$

Do constants matter for singularity?

In ML, we care about if a system is non-singular or not. If not, we don't care much about the reason (redundant/contradictory) therefore we can neglect the constants in the equations.

What is "linear dependency" in the context of matrices?

SYSTEM 1

$$\underbrace{\begin{aligned} a + b &= 0 \\ a + 2b &= 0 \end{aligned}}_{\text{Non-Singular system}} \qquad \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{Non-Singular Matrix}}$$

* No equation is a multiple of the other one

* No row is a multiple of the other one

Rows are linearly independent

SYSTEM 2

$$\underbrace{\begin{aligned} a + b &= 0 \\ 2a + 2b &= 0 \end{aligned}}_{\text{Singular System}} \qquad \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}}_{\text{Singular Matrix}}$$

*Second equation is a multiple of the first one

* Second row is a multiple of the first one

Rows are linearly dependent

$$\begin{aligned} a &= 1 \\ b &= 2 \\ a + b &= 3 \end{aligned} \qquad \begin{aligned} 1a + 0b + 0c &= 1 \\ 0a + 1b + 0c &= 2 \\ 1a + 1b + 0c &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

+ =

Row 3 depends on rows 1 and 2.

$$\begin{aligned} a + b + c &= 0 \\ a + b + 2c &= 0 \\ a + b + 3c &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Row 2 is the average of the rows 1 and 3.

$\Rightarrow$ Row 2 depends on rows 1 and 3.

$$\begin{aligned} a + b + c &= 0 \\ a + 2b + c &= 0 \\ a + b + 2c &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

No linear relations between rows $\Rightarrow$ Non-Singular System

What is the determinant and how do we calculate it?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \underbrace{a \cdot d - b \cdot c}_{\text{the determinant of the matrix}} \quad \begin{array}{l} = 0 \Rightarrow \text{singular} \\ \neq 0 \Rightarrow \text{non-singular} \end{array}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{array}{l} (a \cdot e \cdot i) + (b \cdot f \cdot g) + (c \cdot d \cdot h) \\ -(c \cdot e \cdot g) - (f \cdot h \cdot a) - (i \cdot b \cdot d) \\ = 0 \Rightarrow \text{singular} \\ \neq 0 \Rightarrow \text{non-singular} \end{array} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

#1 ↓ #2 ↓ #3 + #2 ↓ #3 ↓ #1 + #3 ↓ #1 ↓ #2

− #3 ↓ #2 ↓ #1 − #2 ↓ #1 ↓ #3 − #1 ↓ #3 ↓ #2

How do we present and solve linear systems as matrices in NumPy?

$$\begin{array}{l} -x_1 + 3x_2 = 7 \\ 3x_1 + 2x_2 = 1 \end{array} \qquad A = \begin{bmatrix} -1 & 3 \\ 3 & 2 \end{bmatrix} \qquad b = [7 \ \ 1]$$

```
A = np.array([[-1,3],[3,2]], dtype = np.dtype(float))
b = np.array([7,1], dtype = np.dtype(float))

x = np.linalg.solve(A,b)
print(x)  # Output: [-1, 2]

d = np.linalg.det(A)
print(d)  # Output: -11.00

A_system = np.hstack((A, b.reshape((2,1))))
```

we need this because: b.shape was (2,) we need (2,1)

* Determinant tells us if the matrix is non-singular (np.linalg.det(A) ≠ 0) or singular (np.linalg.det(A) = 0)

* np.linalg.solve(A,b) returns an array of solutions for X_1 and X_2 (or throws an LinAlgError if A is singular.