Transposes

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Transposes and their determinants

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 0 & 6 \\ -1 & 2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 4 & 0 \\ -1 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} \quad B^T = \begin{bmatrix} 3 & 0 & -1 \\ -4 & 0 & 2 \\ 1 & 6 & 2 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

"A-transpose equals to ..."

* $A_{m\times n} \Rightarrow A^T_{n\times m}$
* $|A| = |A^T|$

Quiz (1) $\text{Det}\left(\begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix}\right) = ?$

$$= 7\cdot6\cdot3 + 3\cdot1\cdot2 + 4\cdot1\cdot2 - 4\cdot6\cdot2 - 3\cdot1\cdot3 - 7\cdot1\cdot2$$

$$= 126 + 6 + 8 - 48 - 9 - 14 = 69$$

Transposes of products, sums, and inverses

* Product $\rightarrow (A\cdot B)^T = B^T\cdot A^T$ // $(A.B.C)^T = C^T.B^T.A^T$
* Sums $\rightarrow (A+B)^T = A^T + B^T$ // $(A+B+C)^T = A^T + B^T + C^T$
* Inverses $\rightarrow (A^T)^{-1} = (A^{-1})^T$

Quiz (1) Find $(AB)^T$ for $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -6 & 1 \\ 0 & -8 & 5 \\ 1 & 1 & -2 \end{bmatrix}$

$$(AB)^T = B^T.A^T = \begin{bmatrix} 4 & 0 & 1 \\ -6 & -8 & 1 \\ 1 & 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -3 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 16 \\ -16 & -35 & 10 \\ 10 & 15 & -20 \end{bmatrix}$$

* $|A| = |A^T|$
* $(A^T)^T = A$
* $(A\cdot B)^T = B^T\cdot A^T$
* $(A+B)^T = A^T + B^T$
* $(A^T)^{-1} = (A^{-1})^T$
* $A^T\cdot A$ is invertible if $\text{Det}(A) \neq 0$

② Find $(A+B)^T$ for $A=\begin{bmatrix}1&1&-2\\2&3&1\\3&-3&4\end{bmatrix}$, $B=\begin{bmatrix}4&-6&1\\0&-8&5\\1&1&-2\end{bmatrix}$

$$(A+B)^T=A^T+B^T=\begin{bmatrix}4&0&1\\-6&-8&1\\1&5&-2\end{bmatrix}+\begin{bmatrix}1&2&3\\1&3&-3\\-2&1&4\end{bmatrix}=\begin{bmatrix}5&2&4\\-5&-5&-2\\-1&6&2\end{bmatrix}$$

③ Find $(A^{-1})^T$ for $A=\begin{bmatrix}1&1&-2\\2&3&1\\3&-3&4\end{bmatrix}$

$$(A^{-1})^T=(A^T)^{-1}=\left(\begin{bmatrix}1&2&3\\1&3&-3\\-2&1&4\end{bmatrix}\right)^{-1}=\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\1&3&-3&0&1&0\\-2&1&4&0&0&1\end{array}\right]$$

$$=\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-6&-1&1&0\\0&6&4&1&1&1\end{array}\right]=\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-6&-1&1&0\\0&0&1&7/40&-1/8&1/40\end{array}\right]$$

$$=\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&0&1/20&1/4&3/20\\0&0&1&7/40&-1/8&1/40\end{array}\right]=\left[\begin{array}{ccc|ccc}1&0&0&3/8&-1/8&-3/8\\0&1&0&1/20&1/4&3/20\\0&0&1&7/40&-1/8&1/40\end{array}\right]$$

Null and column spaces of the transpose

$$A_{m\times n}=\begin{bmatrix}2&1&1&0\\-1&0&1&-2\end{bmatrix}=\begin{bmatrix}1&0&-1&2\\2&1&1&0\end{bmatrix}=\begin{bmatrix}1&0&-1&2\\0&1&3&-4\end{bmatrix}$$

$x_1=x_3-2x_4$

$x_2=-3x_3+4x_4$

vector space is 2D

There are 2 vectors

* Column space	$C(A)$	$\mathbb{R}^m=\mathbb{R}^2$	$Dim=2$	$span\left(\begin{bmatrix}2\\-1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right)$
* Null space	$N(A)$	$\mathbb{R}^n=\mathbb{R}^4$	$Dim=n-r=4-2=2$	$span\left(\begin{bmatrix}1\\-3\\1\\0\end{bmatrix},\begin{bmatrix}-2\\4\\0\\1\end{bmatrix}\right)$

rank

* The row space of A is $C(A^T)$
* The left null space of A is $N(A^T)$

$$A^T = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

* Row Space | $C(A^T)$ | $\mathbb{R}^n = \mathbb{R}^4$ | Dim = m | $\text{span}\left(\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix}\right)$

Because columns of A = rows of A^T

Always same with Dim(C(A))

* Left Null Space | $N(A^T)$ | $\mathbb{R}^m = \mathbb{R}^2$ | Dim = m-r = 2-2 = 0 | $\text{span}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$

means it's a point

$$(A^T\vec{x})^T = (\vec{0})^T$$

$$\vec{x}^T \cdot (A^T)^T = (\vec{0})^T$$

$$\vec{x}^T \cdot A = \vec{0}^T$$

Quiz ① Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 4 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 12 \end{bmatrix}$$

$x_1 = -4x_3$

$x_2 = -12x_3$

$$N(A^T) = \text{span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}\right)$$

in $\mathbb{R}^3$, Dim = 1

$$C(A^T) = \left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

= in $\mathbb{R}^2$, Dim = 2

(2) Find the row space and left null space of

$$B = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 1 & 3 & -3 & -2 \\ 0 & 0 & 4 & -4 \end{bmatrix}$$, and their dimensions.

$$B^T = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 3 & 0 \\ 1 & -3 & 4 \\ 0 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 7 & -8 \\ 0 & -3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & -22 \\ 0 & 0 & -14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow N(A^T) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) \qquad C(A^T) = \text{span}\left(\begin{bmatrix} 2 & 1 & 0 \\ -2 & 3 & 0 \\ 1 & -3 & 4 \\ 0 & -2 & -4 \end{bmatrix}\right)$$

(3) Find the row space and left null space of $C = \begin{bmatrix} -1 & 5 & 0 \\ 1 & -2 & 3 \\ 0 & 0 & -4 \end{bmatrix}$ and their dimensions.

$$C^T = \begin{bmatrix} -1 & 1 & 0 \\ 5 & -2 & 0 \\ 0 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N(C^T) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) \qquad C(C^T) = \text{span}\left(\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}\right)$$

$\mathbb{R}^3$, Dimension: 0 $\qquad$ $\mathbb{R}^3$, Dim = 3

The product of a matrix and its transpose

* If the columns of A are linearly independent, $A^T \cdot A$ is always invertible.

Ex: $A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 1 & 0 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -3 & 0 \end{bmatrix}$ $\quad A^T \cdot A = \begin{bmatrix} 5 & 2 \\ 2 & 10 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2/5 \\ 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 \\ 0 & 46/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow A^T \cdot A = \text{invertible}$$

Quiz ① $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow$ Is $A^T \cdot A$ invertible?

$$\text{Det}(A) = 4 + 0 + 0 - 4 - 0 - 3 = -3 \neq 0 \Rightarrow A^T A = \text{invertible}$$

A=LU factorization

* $A = L \cdot U$

- A → Matrix
- L → Lower Triangular Matrix
- U → Upper Triangular Matrix

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$$

$$A \quad = \quad L \quad \cdot \quad U$$

First Method:

$$\underset{E_{2,1}}{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}} \cdot \underset{A}{\begin{bmatrix} 2 & 1 \\ 8 & 1 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}}$$

$R_2 - 4R_1 \rightarrow R_2$

$E_{2,1}$ means "elimination matrix that effects row 2, column 1

$$\Rightarrow A = E_{2,1}^T \cdot U$$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$$

we changed the sign

A L U

* With this method, we cannot swap rows unless its the very first thing we do!

$$\begin{array}{ccccc} E_{3,2} & E_{3,1} & E_{2,1} & A & U \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot & \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot & \begin{bmatrix} 2 & 3 & 3 \\ 8 & 14 & 12 \\ 4 & 4 & 5 \end{bmatrix} = & \begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ R_3+R_2 \rightarrow R_3 & R_3-2R_1 \rightarrow R_3 & R_2-4R_1 \rightarrow R_2 & & \end{array}$$

$$\Rightarrow A = E_{2,1}^T \cdot E_{3,1}^T \cdot E_{3,2}^T \cdot U$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 3 \\ 8 & 14 & 12 \\ 4 & 4 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{L} \cdot \begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 3 \\ 8 & 14 & 12 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A = L \cdot U$$

$$\Rightarrow \quad A = L \cdot D \cdot U$$

$$\begin{bmatrix} 2 & 3 & 3 \\ 8 & 14 & 12 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3/2 & 3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Second Method

$$\underset{A}{\begin{bmatrix} 2 & 3 & 3 \\ 8 & 14 & 12 \\ 4 & 4 & 5 \end{bmatrix}} = \underset{I \to L}{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}} \cdot \underset{A \to U}{\begin{bmatrix} 2 & 3 & 3 \\ 0 & 2 & 0 \\ 0 & \cancel{-2}\ 0 & \cancel{-1}\ -1 \end{bmatrix}}$$

$R_2 - 4R_1 \to R_2$

Notice the sign change

$R_3 - 2R_1 \to R_3$

$R_3 + R_2 \to R_3$

Quiz

(1) Rewrite $A = \begin{bmatrix} 3 & 6 \\ -6 & 10 \end{bmatrix}$ in LU form:

$$\begin{bmatrix} 3 & 6 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 6 \\ 0 & 22 \end{bmatrix}$$

$R_2 + 2R_1 \to R_2$

(2) Find L if $A = \begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix}$ is decomposed as LU:

$$\begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & -6 & 5 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2$

$R_3 + 3R_2 \to R_3$

(3) Rewrite $M = \begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix}$ in $L \cdot D \cdot U$ form, where

D is the diagonal matrix that factors pivots out of U.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$R_2 - 4R_1 \to R_2$

$R_3 + 2R_1 \to R_3$

$R_4 - 2R_1 \to R_4$

$R_3 - R_2 \to R_3$

$R_4 - 5R_3 \to R_4$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$