Derivatives

18.03.2025

What's the use case for derivatives in ML?

Derivatives in ML → Optimize functions by maximizing or minimizing them.

What is a derivative?

Derivative → Instantaneous rate of change of a function.

Ex Travelling car

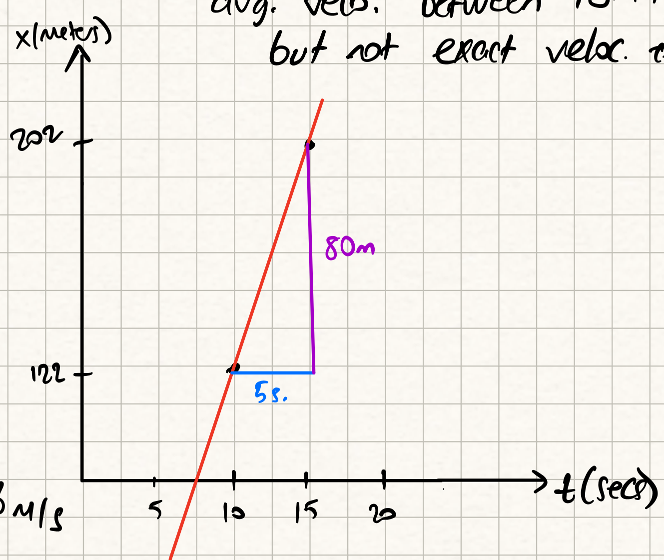
t (seconds)	x (meters)
0	0
5	36
10	122
15	202
20	265
25	351
30	441
35	551
40	591
45	716
50	816
55	900
60	1000

} 80 meters in 5 secs
} 63 meters in 5 secs
> Speed not constant

$$\text{Velocity} = \frac{\text{distance traveled}}{\text{time taken}} \Rightarrow$$

Q: Velocity at t = 12.5 secs?

A: Not enough info! Can find avg. velo. between 10-15 secs, but not exact veloc. at 12.5s.



Avg veloc between 10-15 secs. = slope

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t}$$

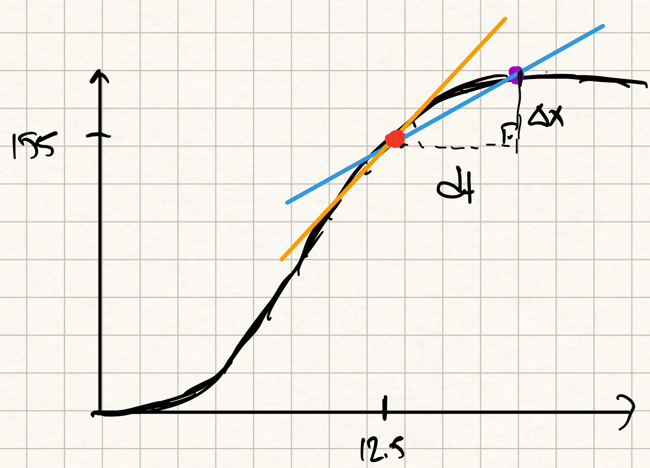
$$\frac{x(15) - x(10)}{t(15) - t(10)} = \frac{202-122}{15-10} = 16\ m/s$$

A **derivative** is the instantaneous rate of change of a function. We use derivatives to optimize our models.

What if more data?:

t(s)	10	11	12	13	14	15	16	17	18	19	20
X(m)	122	138	155	170	186	202	218	234	250	265	265

$$\text{slope} = \frac{x(13)-x(12)}{t(13)-t(12)} = \frac{170-155}{1} = 15\ m/s$$ → Better estimate



slope at a point

$$\frac{\Delta x}{\Delta t} \rightarrow \frac{dx}{dt}$$

dx → tiniest dist. possible

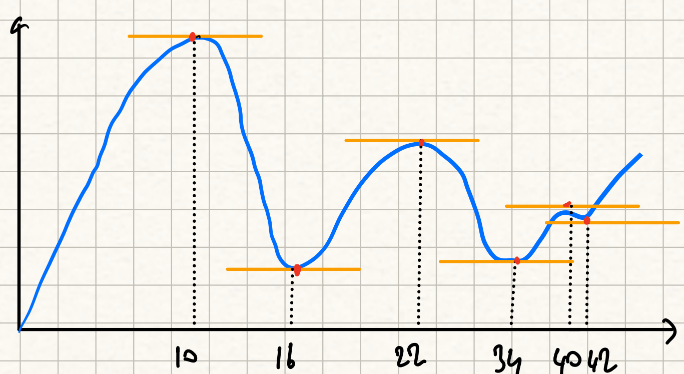
dt → tiniest dist. possible

slope of the tangent line = instantaneous rate of change of the func. at point 12.5

= DERIVATIVE OF THE FUNCTION

$$\text{slope} = \frac{x(20)-x(19)}{20-19} = \frac{265-265}{1} = \underline{\underline{0}}\ m/s$$ zero slope → horizontal line

What does it tell us if the derivative is zero?



In all the maximum or minimum points (local or not) the slope is zero!

The derivative of a function at a specific point is the slope of the tangent line. If the tangent line is horizontal, the derivative (slope) is zero. If the derivative is zero, that point is a local minimum (minima) or maximum (maxima).

How do we calculate the derivative?

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta y}{\Delta x} \qquad \text{slope at a point} = \frac{dy}{dx}$$

How do we notate derivatives?

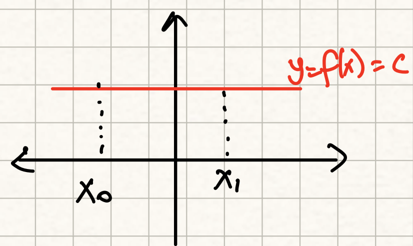
$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) = \frac{d}{dx} f(x)$$

$f'(x)$: Lagrange's Notation

$\frac{d}{dx} f(x)$: Leibniz's Notation

Derivatives of linear functions?

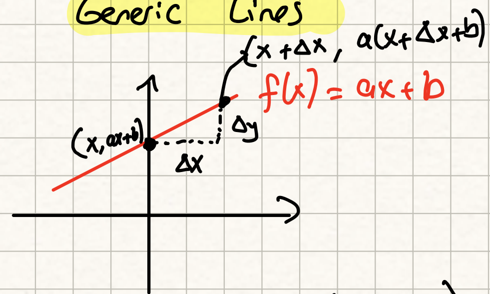
Horizontal Line (constant)



$$f'(x) = \frac{c-c}{x_1 - x_0} = 0$$

$$f(x) = c \Rightarrow f'(x) = 0$$

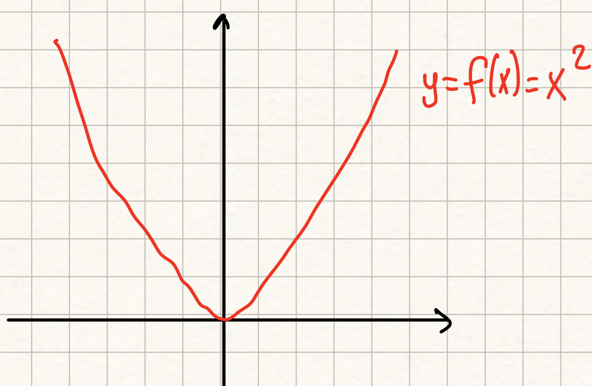
Generic Lines



$$f'(x) = \frac{\cancel{a}(\cancel{x}+\Delta x)+\cancel{b} - (\cancel{ax}+\cancel{b})}{\Delta x} = a$$

$$f(x) = ax+b \Rightarrow f'(x) = a$$

Derivatives of Quadratic functions?



$$\text{slope} = \frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

Δx	1	0.5	0.25	0.125	0.0625	0.001
Δf	3	1.25	0.562	0.265	0.128	0.002
slope	3	2.5	2.25	2.125	2.065	2.0001

$$f(x) = x^2 \Rightarrow f'(x) = 2x$$

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$y = f(x) = x^3$

slope $= \frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^3 - x^3}{\Delta x}$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$\Rightarrow \frac{\cancel{x^3} + 3x(\Delta x)^{\cancel{2}} + 3x^2 \cancel{\Delta x} + \Delta x^{\cancel{3}\,2} - \cancel{x^3}}{\cancel{\Delta x}}$

$\Rightarrow 3x\boxed{\Delta x}^{\to 0} + 3x^2 + \boxed{\Delta x}^{2\,\to 0}$

$\Rightarrow 3x^2$

$f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} = -x^{-2}$

$f(x) = x^2 \Rightarrow f'(x) = 2x = 2x^1 = 2x^{2-1}$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2 = 3x^{3-1}$

$f(x) = x^{-1} \Rightarrow f'(x) = -1 \cdot x^{-2} = -1 \cdot x^{-1-1}$

$$\boxed{f(x) = x^n \Rightarrow f'(x) = n \cdot x^{(n-1)}}$$

The inverse function and its derivative

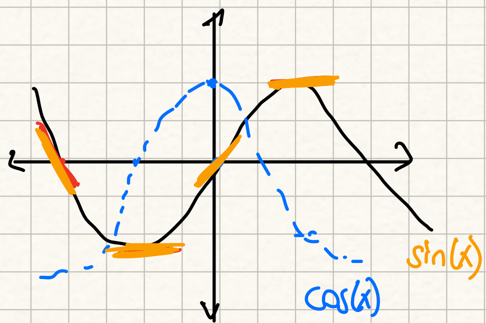
input → $\boxed{\text{function}}_{f}$ → output ⇒ output → $\boxed{\text{inverse function}}_{g}$ → input

$g(x) = f^{-1}(x)$ ⇒

$g(f(x)) = x$

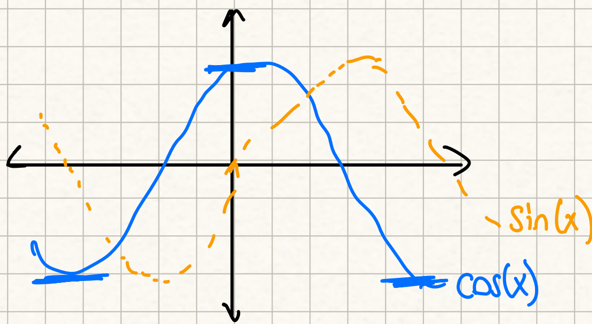
$g'(x) = \frac{1}{f'(x)}$

Derivative of trigonometric functions



x	$\pi/2$	$-\pi/2$	0	$-\pi$
$f'(x)$	0	0	1	-1
$\cos(x)$	0	0	1	-1

$$f(x)=\sin(x) \Rightarrow f'(x)=\cos(x)$$



x	$\pi/2$	$-\pi/2$	0	π
$f'(x)$	0	0	-1	-1
$\sin(x)$	0	0	1	1

$$f(x)=\cos(x) \Rightarrow f'(x)=-\sin(x)$$

The derivative of e^x? $\Rightarrow$

$$= e^x \Rightarrow f'(x)=e^x$$

The derivative of $\log(x)$? $\Rightarrow$

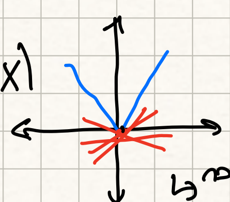
$$e^{\log(3)} = 3 \qquad e^{\log(x)} = x$$

$$f(x)=e^x \qquad f^{-1}(y)=\log(y)$$

Which functions are not differentiable?

- If a derivative does not exist for all points in the domain

Ex $f(x)=|x|$



↳ no derivative for $x=0$

Ex $f(x)=\begin{cases} 2, & \text{if } x<-1 \\ x+1, & \text{if } x \geq -1 \end{cases}$

↳ No derivative at $x=-1$

21.03.2025

1. Multiplication by scalers

$$f(x) = c \cdot g(x) \Rightarrow f'(x) = c \cdot g'(x)$$

2. The sum rule

$$f(x) = g(x) + h(x) \Rightarrow f'(x) = g'(x) + h'(x)$$

3. The product rule

$$f(x) = g(x) \cdot h(x) \Rightarrow f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x)$$

4. The chain rule

$$\frac{d}{dx} f(g(h(x))) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$