01 OPERATIONS ON ONE MATRIX

* Using substitution, elimination, and graphing to solve a linear system gets very expensive as the system gets larger. That's why we use matrices.
* To solve a linear system, we change the matrix into reduced row-echelon form by using row operations (swap, addition, scalar multiplication.

02 OPERATIONS ON TWO MATRICES

* Matrix addition:
    * Dimensions must be identical
    * Add corresponding entries
    * Is commutative and associative
* Matrix subtraction:
    * Dimensions must be identical
    * Subtract corresponding entries
    * Not commutative nor associative
* Scalar multiplication : Each entry is multiplied by the scalar.
* Opposite matrices: $A + B = 0 \Rightarrow B = -A$
* Matrix multiplication:

$$\underset{\substack{\uparrow \\ \#\text{ of rows}}}{A_{m,}}\underset{\substack{\uparrow \\ \#\text{ of columns}}}{{}_{n}} \cdot B_{n,p} = C_{m,p} = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$$

$R_1 \cdot C_1$ = Dot product of Row 1 (of A) and Column 1 (of B)

$$= R_{1,1} \cdot C_{1,1} + R_{1,2} \cdot C_{2,1}$$

* Matrix multiplication is associative and distributive, but not commutative.

* Zero Matrix : Matrix with only zero entries

$$O_{2x2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

* Identity Matrix :

  * $A \cdot I = A$, $I \cdot A = A$

  * Main diagonal : 1s , Other entries : 0s

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

03 MATRICES AS VECTORS

* Vector has 2 pieces of information contained within it:
  1. Direction
  2. Magnitude (Length) : $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$
* Row vector is a one-row matrix, Column vector is a one-column matrix
* Addition, subtraction, and scalar multiplication : Same as matrices
* Vector multiplication = Dot product
* Unit vector = A vector with length of 1.

$$\hat{u} = \frac{1}{\|\vec{v}\|} \cdot \vec{v}$$

* Standard Basis Vectors = For $\mathbb{R}^n$, n number of vectors where each vector has a single entry equal to 1, and the other entries equal to 0.
* Linear combination : The sum of scaled vectors.
* Span of a vector set : The collection of all vectors which can be represented by linear combinations of the set.
* Linear Independence : A set is linearly independent if none of the vectors in the set can be represented by a linear combination of the other vectors in the set. $(c_1, c_2, \dots, c_n) = (0, 0, \dots 0)$ is the only solution to : $c_1 \cdot v_1 + c_2 \cdot v_2 + \dots + c_n \cdot v_n = \vec{O}_n$
* A set of vectors (w) is a subspace of V if:
  1. $\vec{O} \in W$, and 2. $\vec{u}, \vec{v} \in W : \vec{u} + \vec{v} \in W$, and 3. $\vec{u} \in W, k \in \mathbb{R} : k \cdot \vec{u} \in W$

* The span of a vector set is all the linear combinations of that set. A span is always a subspace.
* A vector set is a basis for a space if:
  1. Spans the space, and 2. Is linearly independent

04 DOT PRODUCTS AND CROSS PRODUCTS

* Dot product $= \vec{a} \cdot \vec{b} = [a_1\ a_2 \dots a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n$
  * Is commutative, distributive, and associative.
  * Measures how similar the dotted vectors are.
* Cross product : $\vec{a} \times \vec{b} = i\,(a_2 \cdot b_3 - a_3 \cdot b_2) - j\,(a_1 b_3 - a_3 b_1) + k\,(a_1 b_2 - a_2 b_1)$
  * It's a vector that's orthogonal to the two vectors crossed.
  * Length of the cross product vector measures how different the crossed vectors are
* Angle between vectors :
  * $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\theta$
    * If $\vec{a} \perp \vec{b}$: dot product is 0.
    * If $\vec{a}$ and $\vec{b}$ point the same direction : dot product is maximized
    * If $\vec{a}$ and $\vec{b}$ point the opposite direction : dot product is minimized
  * $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin\theta$
    * If $\vec{a} \perp \vec{b}$: length of the cross product is maximized.
    * If $\vec{a}$ and $\vec{b}$ are collinear, length of the cross product is 0.

* Cauchy - Schwarz Inequality : $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \cdot \|\vec{b}\|$
* Vector Triangle Inequality $= \|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

  * For both equations, equality means $\vec{a}$ and $\vec{b}$ are linearly dependent.

* A plane is the set of all vectors that are perpendicular to one given normal vector, which is the vector that is perpendicular to the plane.

* Normal vector $= \vec{n} = (A, B, C)$
* Plane $= Ax + By + Cz = D$
* Plane $= a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, where (x_0, y_0, z_0) is a point in the plane.

05 MATRIX-VECTOR PRODUCTS

* For $A.\vec{x}$, $\vec{x}$ must be a column vector. For $\vec{x}.\vec{A}$, $\vec{x}$ must be a row vector.
* Null Space = All the vectors that satisfy $A.\vec{x} = \vec{0}$
* If $N(A)$ has only the zero vector, the columns of A are linearly dependent (and vice versa).
* Column Space = The linear combinations (span) of the column vectors of A. If the columns of A are linearly independent, then $C(A)$ is also the basis.
* The general solution of $A.\vec{x} = \vec{b}$ is $\vec{X_p} + \vec{X_n}$, where:
  * $\vec{A}.\vec{x}_n = \vec{0}$ (Called complementary solution)
  * $\vec{A}.\vec{x}_p = \vec{b}$ (For a particular $\vec{b}$. Called particular solution.)
* Dimension of a vector space = Number of basis vectors
* Nullity = Dimension of the null space = Number of free columns
* Rank = Dimension of the column space = Number of pivot columns.

06 TRANSFORMATIONS

* Transformations map vectors from one space (Domain) to specific vectors (Range) in another space (Codomain).
* Subset: The vector set that's being transformed
* Preimage: The vector set before being transformed.
* Image: The vector set after transformation is applied.
* Kernel of T: $Ker(T) = \{ \vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0}_n \}$

* T is linear if

  1. $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$, and
  2. $T(c.\vec{a}) = c.T(\vec{a})$

* Rotation Matrices

  * $Rot_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

  * $Rot_{\theta \text{ around } y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

  * $Rot_{\theta \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

  * $Rot_{\theta \text{ around } z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* $S(\vec{x}) = A.\vec{x}$, $T(\vec{x}) = B.\vec{x} \Rightarrow$
  * $S(\vec{x}) + T(\vec{x}) = (S+T)(\vec{x}) = (A+B)\vec{x}$
  * $c.T(\vec{x}) = c.(B.\vec{x}) = (c.B)\vec{x}$

* The projection of $\vec{v}$ onto L where L is given as scaled version of $\vec{x}$ is :

  1. $Proj_L(\vec{v}) = \left(\dfrac{\vec{v}.\vec{x}}{\vec{x}.\vec{x}}\right).\vec{x}$
  2. $Proj_L(\vec{v}) = (\vec{v}.\hat{u}).\hat{u}$
  3. $Proj_L(\vec{v}) = \begin{bmatrix} \hat{u}_1^2 & \hat{u}_1.\hat{u}_2 \\ \hat{u}_1.\hat{u}_2 & \hat{u}_2^2 \end{bmatrix}$

* Projections are linear transformations. Therefore :

  1. $Proj_L(\vec{a}+\vec{b}) = Proj_L(\vec{a}) + Proj_L(\vec{b})$, and
  2. $Proj_L(c.\vec{a}) = c.Proj_L(\vec{a})$

* $T \circ S(\vec{x}) = \vec{x}$ is transformed first by $S(\vec{x}) = A.\vec{x}$, then $T(\vec{x}) = B.\vec{x}$

  = a composition of linear transformations S and T

  $= B.A.\vec{x}$

07 INVERSES

* For a transformation to be invertible, it must be surjective and injective.
  * Surjective (onto): Every vector $\vec{b}$ is being map to.
  * Injective (one-to-one): Every $\vec{a}$ maps to a unique $\vec{b}$.

* If # rows > # columns ⇒ Not surjective
  If # rows < # columns ⇒ Not injective

* Inverse transformations are linear. Therefore:
  1. $T^{-1}(\vec{a}+\vec{b}) = T^{-1}(\vec{a}) + T^{-1}(\vec{b})$, and
  2. $T^{-1}(c.\vec{a}) = c.T^{-1}(\vec{a})$
* Let $T(\vec{x}) = M.\vec{x}$:
  * If $rref(M) = I$, then T is invertible.
  * If $|M| \neq 0$, then T is invertible (non-singular)

* Two methods to find M^{-1}:
  1. $[M \; I \mid I \; M^{-1}]$
  2. $M^{-1} = \frac{1}{|M|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* We can use inverse matrices to solve linear equations: $\vec{x} = A^{-1}.\vec{b}$
  This method allows us to calculate $\vec{x}$ for different $\vec{b}$'s easily.

08 DETERMINANTS

* Rule of Sarrus for $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow |A| = a.e.i + bfg + cdh - afh - bdi - ceg$

* Cramer's Rule: $a_1.x + b_1.y = d_1$, $a_2x + b_2.y = d_2 \Rightarrow$
  $x = \frac{Dx}{D}$, $y = \frac{Dy}{D}$, with $D \neq 0$, where
  $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $Dx \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}$, $Dy = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

* Determinant Modification Rules
  1. Multiplying a row of the matrix by a scalar requires that we multiply the determinant by the same scalar.
  2. when two rows of a matrix are swapped, the determinant must be multiplied by -1

* Upper Triangular Matrix: All entries below main diagonal are zeros.
* Lower Triangular Matrix: All entries above main diagonal are zeros.
* $\text{Det}(L) = \text{Det}(U) =$ product of the entries in the main diagonal.
* $\text{Area}_F = \text{Det}(A)$ (F is the figure created by the column vectors of A)
* $T: F \to G \Rightarrow \text{Area}_G = |\text{Area}_F \cdot \text{Det}(T)|$

09 TRANSPOSES

* $A_{m\times n} \Rightarrow A^T_{n\times m}$
* $|A| = |A^T|$
* $(A^T)^T = A$
* $(A \cdot B)^T = B^T \cdot A^T$
* $(A+B)^T = A^T + B^T$
* $(A^T)^{-1} = (A^{-1})^T$
* $A \cdot A^T$ is invertible if $\text{Det}(A) \neq 0$
* The row space of A is $C(A^T)$
* The left null space of A is $N(A^T)$
* A can be factorized as $A = L \cdot U$, where L is the lower triangular matrix, and U is upper triangular matrix.

10 ORTHOGONALITY AND CHANGE OF BASIS

* $V^\perp = \{ x \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V \}$
* $(V^\perp)^\perp = V$

* "En güzel kürek, boş denizde çekilir."

$A_{m\times n}$ — En → n, kürek → row space, boş → null space

* $C(A^T) = N(A)^{\perp}$, $\text{Dim}(C(A^T)) = r$, $\text{Dim}(N(A)) = n - r$, $C(A^T)$ and $N(A) \in \mathbb{R}^n$
* $C(A) = N(A^T)^{\perp}$, $\text{Dim}(C(A)) = r$, $\text{Dim}(N(A^T)) = m - r$, $C(A)$ and $N(A^T) \in \mathbb{R}^m$

* $\text{Proj}_V \vec{x} = A \cdot (A^T \cdot A)^{-1} \cdot A^T \cdot \vec{x}$
  * If A is invertible, then $\text{Proj}_V \vec{x} = \vec{x}$
* Least Square Solution: "There is no solution ($\vec{x}$) to this system, but this ($\vec{x}^*$) is as closest as we can get: * $A^T A \vec{x}^* = A^T \vec{b}$

* Domain / Codomain

$$\begin{tikzcd}
\text{Standard Basis} & \vec{x} \arrow[r, "A\cdot\vec{x}"] \arrow[d, "C^{-1}\cdot\vec{x}"'] & T(\vec{x}) \arrow[d, "C^{-1}\cdot T(x)"] \\
\text{Alternate Basis} & {[\vec{x}]_B} \arrow[r, "M\cdot[x]_B"'] & {[T(\vec{x})]_B}
\end{tikzcd}$$

11 ORTHONORMAL BASES AND GRAM-SCHMIDT

* Orthonormal Basis: Every vector is normal (length = 1) and orthogonal to every other vector.
* Orthogonal Matrix: Square matrix whose columns form an orthonormal set.
* Orthonormal Matrix: Rectangular matrix whose columns form an orthonormal set.
* If A is orthonormal: $\text{Proj}_V \vec{x} = A \cdot A^T \cdot \vec{x}$
* Gram-Schmidt Process: An iterative process to change a basis to an orthonormal basis: $V = \text{Span}(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n) \Rightarrow$

① $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

② Change the next vector so that it is orthogonal to all before it.

* $\vec{w}_n = \vec{v}_n - \left[(\vec{v}_n \cdot \vec{u}_1) u_1 + (\vec{v}_n \cdot u_2) u_2 + \ldots + (\vec{v}_{n-1} \cdot \vec{u}_{n-1}) \vec{u}_{n-1}\right]$

③ $\vec{u}_n = \dfrac{\vec{w}_n}{\|\vec{w}_n\|}$

④ Back to step 2 until $V = \text{Span}(\vec{u}_1, \vec{u}_2 \ldots \vec{u}_n)$

12 EIGENVALUES AND EIGENVECTORS

* Eigenspace is the set of eigenvectors. Eigenvectors are special set of input vectors for which the action of the matrix is described as a simple scaling by the eigenvalue of that specific eigenvector: $A \cdot \vec{e}_\lambda = \lambda \cdot \vec{e}_\lambda$
* Eigenvalues are all the values that satisfy $|\lambda I - A| = 0$. Therefore:

$$\begin{vmatrix} \lambda - A_{1,1} & (-1) A_{1,2} & \cdots & (-1) A_{1,n} \\ -(1) A_{2,1} & \lambda - A_{2,2} & \cdots & (-1) A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1) A_{m,1} & (-1) A_{m,2} & \cdots & \lambda A_{m,n} \end{vmatrix} = 0$$

* When we find the eigenvalues, we can verify them in two ways:

① $\text{Trace}^{*}(A) = \sum_{i=1}^{n} \lambda_i$ ② $\text{Det}(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$

* Sum of the values on the main diagonal