Matrices as Vectors

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What is a vector?

* An array of numbers
* Has two pieces of information:
  1. Direction
  2. Magnitude (length)

$\vec{a} = (2,3)$ → coordinate point

$\vec{a} = \begin{bmatrix} 2 & 3 \end{bmatrix}$ → row vector

$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ → column vector

Relationship between matrices and vectors

* Matrices are composed of row vectors and column vectors.
* # of components in a vector = number of dimensions of the space they're in
* # of linearly independent vectors in a matrix = number of dimensions of the plane they form

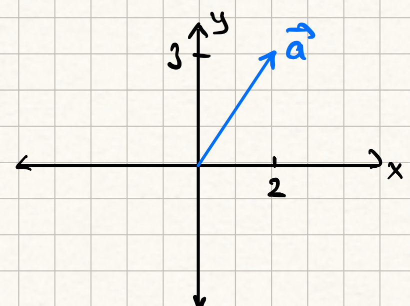
Ex:

$$A = \begin{bmatrix} 4 & -6 & 1 & -8 & 5 \\ 1 & 1 & -2 & 9 & 0 \end{bmatrix} \Rightarrow$$

* A has two row vectors, $\vec{a_1}$ and $\vec{a_2}$, so they form a two-dimensional plane.
* a_1 and a_2 have 5 elements each. So they are vectors in a five-dimensional space.

How to sketch vectors?

$\vec{a} = \begin{bmatrix} 2 & 3 \end{bmatrix}$



* Vector has 2 pieces of information contained within it: ① Direction ② Magnitude
* Row vector: One-row matrix
* Column vector: One-column matrix
* Addition, subtraction and scalar multiplication: same as the vectors.
* Vector multiplication = Dot product
* Magnitude = Length = $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots a_n^2}$

Vector Operations	* Addition : $(a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2)$ ↳ we add $\vec{b}$ to the tip of $\vec{a}$ * Scalar Multiplication : $k.(a_1, a_2) = (k.a_1, k.a_2)$ * Dot product : $\vec{a}.\vec{b} = (a_1.b_1 + a_2.b_2)$
Length of a vector	* $\|\|\vec{a}\|\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$
What is a unit vector?	$\|\|\vec{a}\|\| = 1 \Rightarrow \vec{a}$ = unit vector $(\hat{u})$ "hat" means the length is 1. $\|\|\vec{a}\|\| \neq 1 \Rightarrow \hat{u} = \frac{1}{\|\|\vec{a}\|\|} . \vec{a}$
	Ex: Find the unit vector pointing in the same direction as $\vec{v} = (1, 4, -2)$. $\vec{u} = \frac{1}{\sqrt{1+16+4}} . (1, 4, -2) = \left(\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)$
What are basis vectors?	* A set of linearly independent vectors that can span the whole space. Ex: $i = [2, 3]$, $j = [1, 4]$ → We can span $\mathbb{R}^2$ with just these two vectors (by addition and scalar multiplication), so they are the basis vectors for $\mathbb{R}^2$.
What are standard basis vectors?	* For $\mathbb{R}^2$: $\hat{i} = [1\ 0]$, $\hat{j} = [0\ 1]$ * For $\mathbb{R}^3$: $\hat{i} = [1\ 0\ 0]$, $\hat{j} = [0\ 1\ 0]$, $\hat{k} = [0\ 0\ 1]$

* Unit vectors : A vector with length 1.
$\hat{u} = \frac{1}{\|\|\vec{v}\|\|} . \vec{v}$
* Standard basis vectors : For $\mathbb{R}^n$, n number of vectors where each vector has a single entry equal to 1, and the other entries equal to 0.
* Linear combination : The sum of scaled vectors.
* Span of a vector set : The collection of all vectors which can be represented by linear combinations of the set

Linear combinations of the basis vectors

$\vec{a} = (6,4) = 6\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix} + \begin{bmatrix}0\\4\end{bmatrix} = \begin{bmatrix}6+0\\0+4\end{bmatrix} = \begin{bmatrix}6\\4\end{bmatrix}$

$\Rightarrow \underline{\vec{a} = 6\hat{i} + 4\hat{j}}$

↳ 4^{th} type of vector representation. 1. Coordinates 2. Row 3. Column 4. Combination of basis vectors.

Ex: $\vec{a} = (-3,2,-1) = -3\hat{i} + 2\hat{j} - \hat{k}$

Linear independence in two dimensions

* A set is linearly independent if none of the vectors in the set can be represented by a linear combination of the other vectors in the set.

* If a set has more than n number of vectors in $\mathbb{R}^n$, it can never be linearly independent.

Ex: $V_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $V_2 = \begin{bmatrix}4\\4\end{bmatrix} \Rightarrow V_2 = 4 \cdot V_1 \Rightarrow$ The set is L.D. (linearly dependent)

* V_1 and V_2 are linearly dependent
* V_1 and V_2 are collinear (they lie on the same line)

Spanning

* n number of linearly independent vectors always span the $\mathbb{R}^n$ space.

Testing for linear independence

* If $c_1 \cdot V_1 + c_2 \cdot V_2 + \dots + c_n \cdot V_n = [0\ 0\ \dots\ 0]$ is true only for $c_1 = c_2 = \dots = c_n = 0$, then the set $\{V_1, V_2 \dots V_n\}$ is linearly independent.

Linear Independence: A set is linearly independent if none of the vectors in the set can be represented by a linear combination of the other vectors in the set. * $(c_1, c_2, \dots, c_n) = (0, 0 \dots 0)$ is the only solution to: $c_1 \cdot V_1 + c_2 \cdot V_2 + \dots + c_n \cdot V_n = \vec{0}_n$

Ex

$\vec{a} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\Rightarrow$ $c_1 \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 4 & 2 & 0 \end{array}\right] \xrightarrow{R_4 - 4R_1 \to R_4} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -10 & 0 \end{bmatrix} \xrightarrow{R_2 \cdot \frac{-1}{10} \to R_2}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 - 3R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow (c_1, c_2) = 0$

$\Rightarrow \vec{a}$ and $\vec{b}$ are L.I.

Ex:

$\vec{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$ $\Rightarrow \begin{bmatrix} -2 & 6 & 0 \\ 1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ 1 & -3 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2 = R_2} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow 0 = 0$

infinitely many solutions

$\Rightarrow \vec{a}$ and $\vec{b}$ are L.D.

Linear Subspace

A set of vectors (W) is a subspace of V if:

1. $\vec{0} \in W$, and...
2. $\vec{u}, \vec{v} \in W$: $\vec{u} + \vec{v} \in W$, and...
3. $\vec{u} \in W$, $k \in \mathbb{R}$: $k \cdot \vec{u} \in W$

Ex:

$$M = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid y \leq 0 \right\}$$

1. $x = 0$, $y = 0$ $\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in M$ ✓
2. $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ If $b \leq 0$ and $d \leq 0$, $b+d \leq 0$ ✓
3. $k = -1 \Rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot (-1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin M$ ✗

Linear Subspaces: A set of vectors (W) is a subspace of V if:

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Difference between the span and the basis

* The zero vector is always a subspace of $\mathbb{R}^n$
* $\mathbb{R}^n$ is always a subspace of itself
* Any line through the origin in $\mathbb{R}^2$, any plane through the origin in $\mathbb{R}^3$, etc, is always a subspace.
* If n number of vectors span a space, they are always its subspace

In order for a set of vectors to form a basis for $\mathbb{R}^n$,

1. The vectors need to span $\mathbb{R}^2$, and
2. They need to be linearly independent.

Ex: $\vec{a} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$ $\Rightarrow$

$\{\vec{a}, \vec{b}, \vec{c}\}$: Spans $\mathbb{R}^2$, not a basis ($\vec{c} = 2 \cdot \vec{b}$)

$\{\vec{a}, \vec{b}\}$: Spans $\mathbb{R}$ and a basis for it.

Span : The span of a vector set is all the linear combinations of that set. A span is always a subspace.

Basis : A vector set is a basis for a space if:
1. spans the space, and
2. is linearly independent