Vectors Introduction

21.05.2025

Intro to vectors and scalars

- Vectors have magnitude (size) AND direction.
- Scalars only have magnitude (size)

- "I moved my bag 5 meters." → Scalar (A distance)
- "I moved my bag 5 meters to the right." → Vector (A displacement)

- "I moved my bag 5 meters in 2 seconds" → Scalar (2.5 m/s is speed)
- "I moved my bag 5 meters in 2 seconds to the right." → Vector (velocity)

Representing quantities with vectors

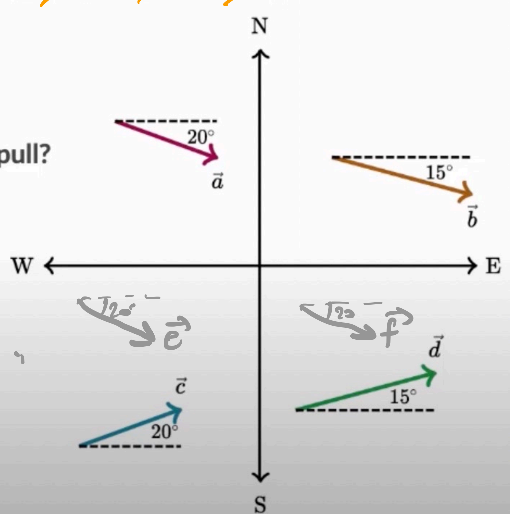
A powerful magnet is attracting a metal ball on a flat surface. The magnet is pulling the ball at a force of 15 Newtons, and the magnet is 20° to the south from the eastward direction relative to the ball.

Here are a few vectors, where $||\vec{a}|| = ||\cancel{\vec{c}}|| = 15$ N and $||\cancel{\vec{b}}|| = ||\cancel{\vec{d}}|| = \cancel{20}$ N:

Which vectors can represent the force of the team's pull?

$||\vec{a}||$

(also $\vec{e}$ and $\vec{f}$. Starting point not important! Only magnitude and the direction matters!)



Interpreting statements about vectors

- $||\vec{a}|| = ||\vec{b}||$ means that $\vec{a}$ and $\vec{b}$ have the same magnitude, not necessarily the same direction.

- A vector is a mathematical object that has both a magnitude (length) and a direction. They can be used to represent physical quantities such as velocity, force, and acceleration.

Vector Components

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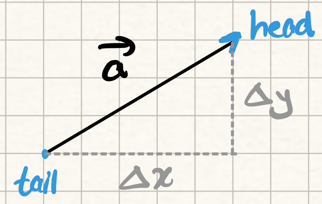
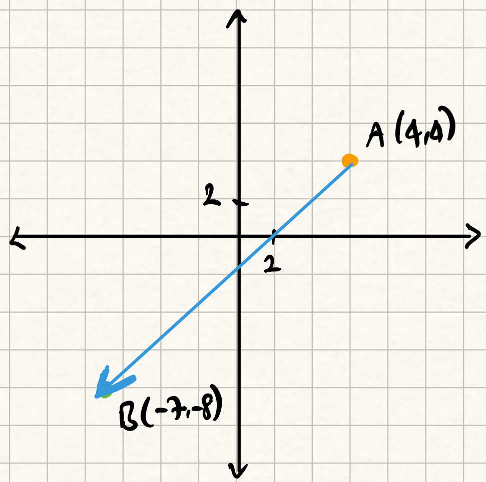
* 

vector components

$$\vec{a} = (\Delta x, \Delta y)$$
$$= (5,3)$$

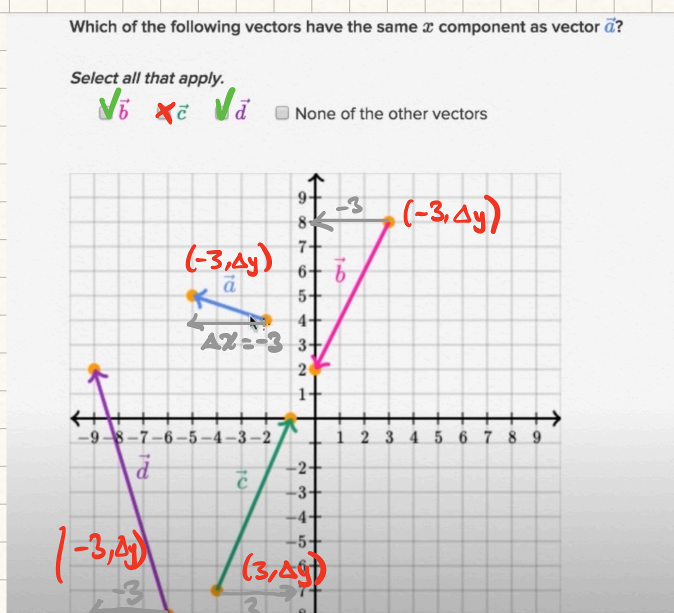
* Δx and Δy are not coordinates. However if the tail sits on the origin of a coordinate plane then Δx is the same as the x-coordinate and Δy is the same as the y-coordinate

Finding the components of a vector

* 

$$\vec{AB} = (\Delta x, \Delta y)$$
$$= (-7-4, -8-4)$$
$$= (-11, -12)$$

Comparing the components of vectors



* Vector components are the horizontal and vertical parts of a vector.
$$\vec{a} = (\Delta x, \Delta y)$$

Magnitude of Vectors

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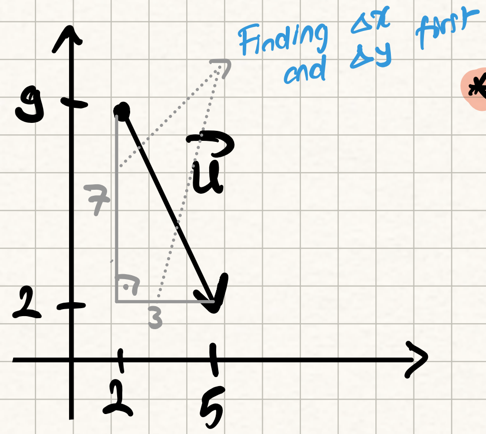
Vector magnitude from graph



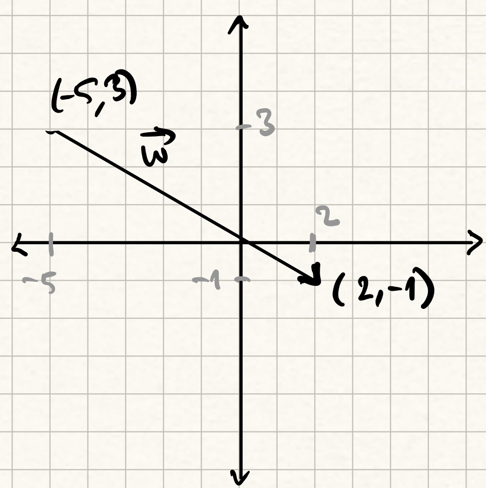
$$\|\vec{u}\| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$= \sqrt{49+9}$$
$$= \sqrt{58}$$

Vector magnitude from components

$\underbrace{\vec{a} = (5,-3)}_{\text{we're already given the } \Delta x \text{ and } \Delta y} \Rightarrow \|\vec{a}\| = \sqrt{25+9}$

$$= \sqrt{34}$$

Vector magnitude from initial and terminal points.



$$\|\vec{w}\| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$= \sqrt{(2-(-5))^2 + (-1-3)^2}$$
$$= \sqrt{7^2 + (-4)^2} = \sqrt{49+16}$$
$$= \sqrt{65}$$

$$\vec{w} = (\Delta x, \Delta y) = (7,-4)$$

* Magnitude of a vector can be found with $\|\vec{a}\| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Scalar Multiplication

21.05.2025

Scalar multiplication: Component form

* $\vec{w} = (1,2)\ ,\ c = 2 \Rightarrow c \cdot \vec{w} = (2 \cdot 1\ ,\ 2 \cdot 2) = (2,4)$
* Multiplying a vector by a positive scalar doesn't change its direction, just its magnitude. Multiplying a vector by a negative scalar flips its direction by 180° and scales it. The vector still lies on the same line though.

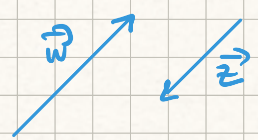
Example:

* $\vec{v} = (x,y)$ and $\|\vec{v}\| = 5$

$\vec{w} = (3x, 3y) \Rightarrow \|\vec{w}\| = ?$ $\quad c=3 \Rightarrow \|\vec{w}\| = |3 \cdot 5| = 15$

$\vec{z} = (-2x, -2y) \Rightarrow \|\vec{z}\| = ?$ $\quad c=-2 \Rightarrow \|\vec{w}\| = |-2 \cdot 5| = 10$

$\vec{v} = \nearrow \Rightarrow \vec{w} = ?\ ,\ \vec{z} = ?$



* Multiplying a vector by a positive scalar doesn't change its direction, only its magnitude. Multiplying by a negative scalar flips the vector by 180° and scales it. The resulting vector still lies on the same line though.

Vector Addition and Subtraction

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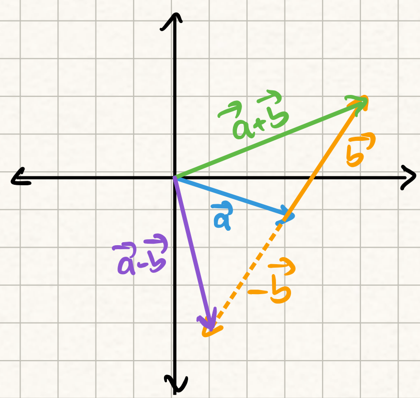
Adding and subtracting vectors

* $\vec{a} = (3,-1)$ $\vec{b} = (2,3)$

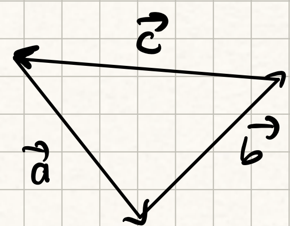
$\Rightarrow \vec{a} + \vec{b} = (3+2, -1+3) = (5,2)$

$\Rightarrow \vec{a} - \vec{b} = (3-2, -1-3) = (1,-4)$



* $\vec{a}+\vec{b}$ means we start $\vec{b}$ from the tip of $\vec{a}$ and connect the tail of $\vec{a}$ with the tip of $\vec{b}$.
* $\vec{a}-\vec{b} = \vec{a} + (-\vec{b})$, so we start from the tip of $\vec{a}$ and add $-\vec{b}$. Then connect the tail of $\vec{a}$ with the tip of $-\vec{b}$.

*

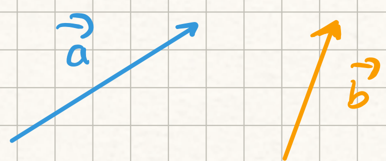


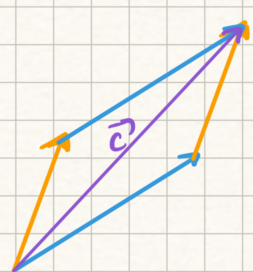
$\Rightarrow \vec{a}+\vec{b} = -\vec{c}$

$\vec{a}+\vec{b}+\vec{c} = 0$

Parallelogram rule for vector addition

*





* This parallelogram proves that

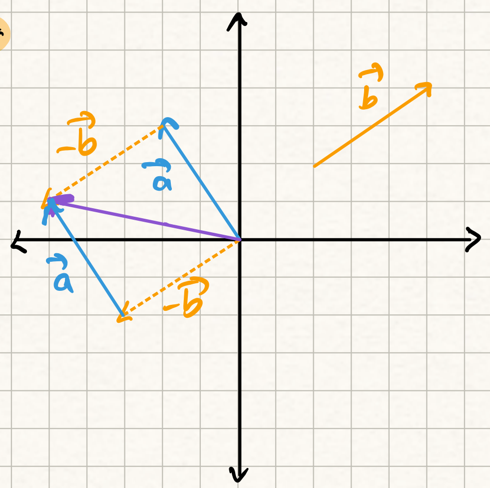
$\vec{a}+\vec{b} = \vec{b}+\vec{a}$

(b/c: $\vec{c} = \vec{c}$)

* To add to vectors, we simpl add their components.

$\vec{a} = (x_a, y_a)$ $\vec{b} = (x_b, y_b)$ $\Rightarrow$ $\vec{a}+\vec{b} = (x_a + x_b, y_a + y_b)$

Subtracting vectors with parallelogram rule

* 

* $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$
* $\vec{a}-\vec{b}=-\vec{b}+\vec{a}$

(b/c $\vec{c}=\vec{c}$)

Vector addition and magnitude

* $\vec{a}+\vec{b}=\vec{c}$

? $\|\vec{c}\| = \|\vec{a}\| + \|\vec{b}\|$

✓ "Only if $\vec{a}$ and $\vec{b}$ have the same direction."

$\vec{a} = \longrightarrow$

$\vec{b} = \longrightarrow$

$\Rightarrow$ $\vec{c} = \longrightarrow$

$\|\vec{a}\| + \|\vec{b}\| = \|\vec{c}\|$

? $\|\vec{c}\| > \|\vec{a}\| + \|\vec{b}\|$

✗ "In a triangle, no side can be longer than the sum of the other two."

 ✗

Direction of Vectors

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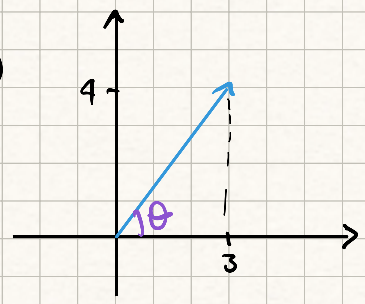
1st and 2nd quadrants

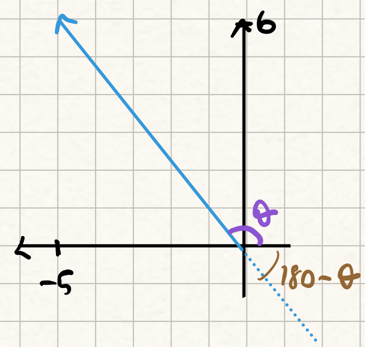
* $\vec{u} = (3,4)$



$\Rightarrow \tan(\theta) = \frac{\Delta y}{\Delta x} = \frac{4}{3}$

$\theta \approx 53.13^\circ$

* $\vec{w} = (-5,6)$



$\Rightarrow \tan(\theta) = \frac{\Delta y}{\Delta x} = \frac{-6}{5}$

$\theta \approx \underbrace{-50.2^\circ}_{180-\theta}$ ✗

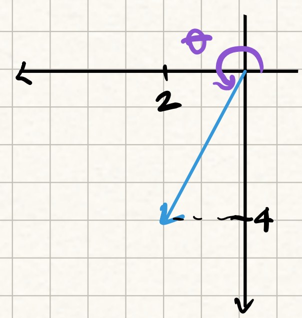
$\theta \cong -50.2^\circ + 180^\circ$

$\approx 129.80^\circ$

* When we're calculating the direction of vectors, we need to remember that the tangent value is the same for the $1^{st}/3^{rd}$ and $2^{nd}/4^{th}$ quadrants because they are the same lines! We should add 180° if necessary.

3rd and 4th quadrants

* $\vec{a} = (-2,-4)$



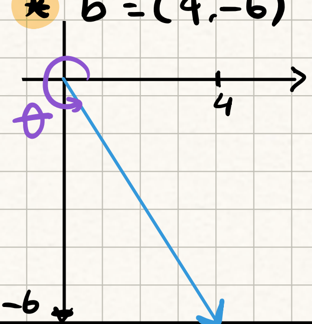
$\tan(\theta) = \frac{\Delta y}{\Delta x} = 2$

$\tan^{-1}(2) \approx 63.43^\circ$ ✗ ← 1st Quad.

$\approx 180 + 63.43$

$\approx 243.43^\circ$

* $\vec{b} = (4,-6)$



$\tan(\theta) = \frac{\Delta y}{\Delta x} = \frac{-6}{4}$

$\Rightarrow \theta \approx -56.31^\circ$ ✓

(we can do $360 - 56.31^\circ$ to make it positive, but it's optional.)

* Direction of the vector $\vec{v}$ is $\theta_v = \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right)$. When we use a calculator we need to remember that the tangent value is the same for the diagonally opposite quadrant and we might need to add 180° to the result.

Vector Components From Magnitude and Direction

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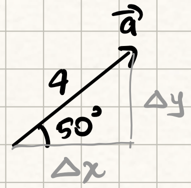


* $\|\vec{a}\| = 4$, $\theta = 50^\circ$

$\Rightarrow \frac{\Delta x}{4} = \cos(50^\circ)$ $\quad \frac{\Delta y}{4} = \sin(50^\circ)$

$\Rightarrow \Delta x \approx 2.57$ $\quad \Delta y \approx 3.06$

$\Rightarrow \vec{a} \approxeq (2.57, 3.06)$

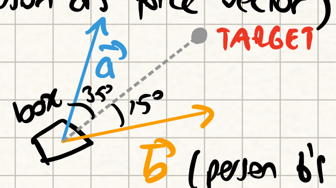
* $\vec{a} = (\|\vec{a}\| \cdot \cos(\theta), \|\vec{a}\| \cdot \sin(\theta))$

* $\|\vec{b}\| = 10$, $\theta_b = 135^\circ \Rightarrow \vec{b} = ?$

$\vec{b} = (10 \cdot \cos(135^\circ), 10 \cdot \sin(135^\circ)) = (-7.07, 7.07)$

Word problem:

*



$\|\vec{a}\| = 330N$

$\|\vec{b}\| = 300N$

$\theta_a = 35^\circ$

$\theta_b = 15^\circ$

? "Who's helping to the goal more?" :)

$a_x = 330 \cdot \cos(35^\circ) \approx 270.32$ N ✗

$b_x = 300 \cdot \underbrace{\cos(15^\circ)}_{= \cos(345^\circ)} \approx 318.76$ N ✓

* $\vec{a} = (\|\vec{a}\| \cdot \cos(\theta_a), \|\vec{a}\| \cdot \sin(\theta_a))$

Adding Vectors in Magnitude and Direction Form

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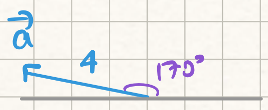
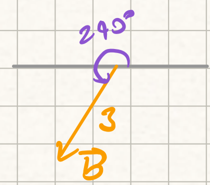




* Find the magnitude and direction of $\vec{a}+\vec{b}$.

$$\vec{a} = (4\cdot\cos(170°),\ 4\sin(170°))$$
$$\vec{b} = (3\cdot\cos(240°),\ 3\sin(240°))$$
$$\Rightarrow \vec{a}+\vec{b} \approx (-5.44,\ -1.90)$$

* $\tan(\theta_{a+b}) = \left(\frac{-1.9}{-5.44}\right)$

$$\theta_{a+b} \approx 19.25° + 180°$$

b/c 3rd Quad.

$$\approx 199.25°$$

* $\|\vec{a}+\vec{b}\| = \sqrt{(-5.44)^2+(-1.9)^2}$

$$\approx 5.76$$

Vector word problem: Resultant velocity

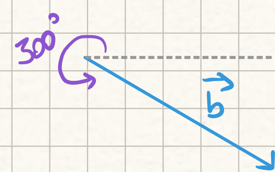
A boat is travelling at a speed of 26 $\frac{\text{km}}{\text{h}}$ in a direction that is a 300° rotation from east.

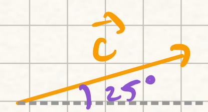
At a certain point it encounters a current at a speed of 15 $\frac{\text{km}}{\text{h}}$ in a direction that is a 25° rotation from east.

Answer two questions about the boat's velocity after it meets the current.

1) What is the boat's speed after it meets the current?
Round your answer to the nearest tenth. You can round intermediate values to the nearest hundredth.

2) What is the direction of the boat's velocity after it meets the current?
Round your answer to the nearest integer. You can round intermediate values to the nearest hundredth.





$$\vec{b}+\vec{c} = (26.59,\ -16.18)$$

① $\|\vec{b}+\vec{c}\| = \sqrt{(26.59)^2+(-16.18)^2} \cong 31.1$ km/h

② $\tan(\theta) = \dfrac{-16.18}{26.59} \approx 329°$

Vector word problem: hiking

Keita left camp three days ago on a journey into the jungle. The three days of his journey can be described by displacement (distance and direction) vectors $\vec{d_1}$, $\vec{d_2}$, and $\vec{d_3}$.

$\vec{d_1}+\vec{d_2}+\vec{d_3} = (15, 19)$

$\vec{d_1} = (7, 8)$

$\vec{d_2} = (6, 2)$

$\vec{d_3} = (2, 9)$

(Distances are given in kilometers, km.)

How far is Keita from camp at the end of day three?

km $\quad \|\vec{d_1}+\vec{d_2}+d_3\| \approx 24.2$ km

(Round to the nearest tenth.)

What direction is Keita from camp at the end of day three?

° $\quad \approx 52°$

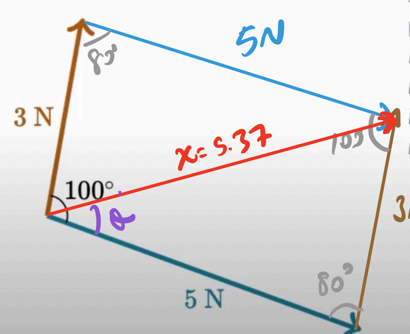
(Round to the nearest degree. Your answer should be between 0 and $180°$.)

Vector word problem: Resultant force

A metal ball lies on a flat horizontal surface. It is attracted by two magnets placed around it.

The first magnet's force on the ball is 5 N.

The second magnet's force on the ball is 3 N in a direction that is a 100° rotation from the first magnet's force.



1) What is the combined strength of the magnets' pulls?

Round your answer to the nearest tenth. You can round intermediate values to the nearest hundredth.

2) What is the direction of the magnets' combined pulls, relative to the direction of the first magnet's pull?

Round your answer to the nearest integer. You can round intermediate values to the nearest hundredth.

* $x^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos(80)$

$x \simeq 5.37$ N

* $\dfrac{\sin(\theta)}{3} = \dfrac{\sin(80°)}{5.37} \Rightarrow \sin(\theta) \simeq 0.55 \Rightarrow \theta \simeq 33.4°$