The Complex Plane

13.05.2025

Plotting numbers on the complex plane

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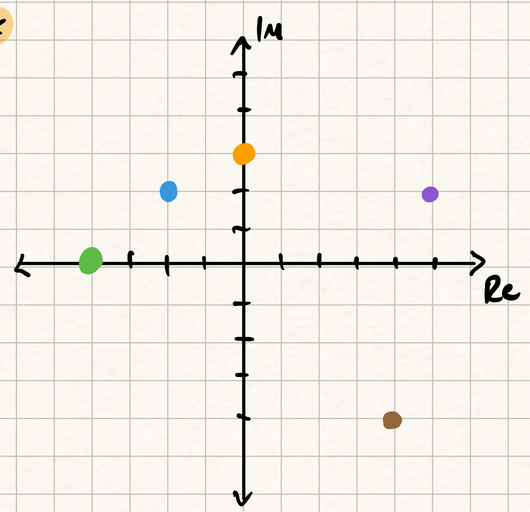

* $-2+2i$
* $2i+5$ $(=5+2i)$
* $3i$ $(=0+3i)$
* $4-4i$
* -4 $(=-4+0i)$

The complex plane

* The imaginary unit (i) is the number with the following equivalent properties:
  * $i^2 = -1$
  * $i = \sqrt{-1}$

* A complex number is any number that can be written as $a+bi$, where i is the imaginary unit and $a, b \in \mathbb{R}$. a is called the "real part" of the number, and b is called the "imaginary part" of the number (not bi!)

* The complex plane consists of real axis (like the x-axis on a Cartesian plate) and imaginary axis.

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* A complex number is any number that can be written as $a+bi$ (Rectangular Form), where i is the imaginary unit ($i^2=-1$, $i=\sqrt{-1}$), and $a, b \in \mathbb{R}$. "a" is called the real part, "b" is called the imaginary part of the complex number.

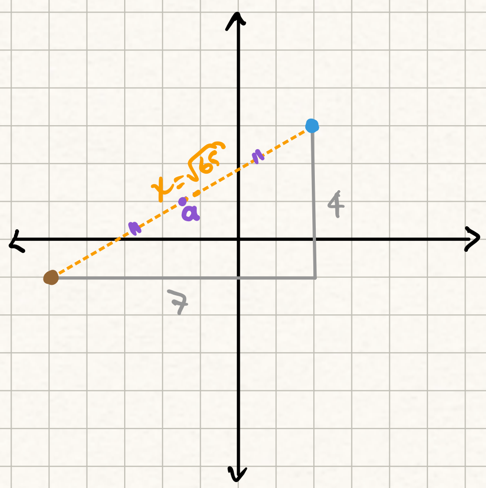
* The complex plane consists of real axis (horizontal) and imaginary axis (vertical) which intercept at zero.

Distance and Midpoint of Complex Numbers

13.05.2025

* $z = 2+3i$

$w = -5-i$



$x = \sqrt{4^2+7^2}$

$= \sqrt{65}$

$a = \left(\frac{2+(-5)}{2}\right) + \left(\frac{3-1}{2}\right)i$

$= \frac{-3}{2} + i$

Quiz: ① $z_1 = 7+i$, $z_2 = 2-7i \Rightarrow d = |z_1 - z_2|$

$= \sqrt{(7-2)^2 + (1-(-7))^2}$

$= \sqrt{25+64} = \sqrt{89}$

② $z_1 = (3-6i)$, $z_2 = (-2+9i) \Rightarrow |z_1 - z_2| = ?$

$= \sqrt{(3-(-2))^2 + (-6-9)^2} = \sqrt{25+225} = \sqrt{250} = 5\sqrt{10}$

③ $z_1 = 1+6i$, $z_2 = -4-3i$, the midpoint x.

$= \frac{1+(-4)}{2} + \frac{6+(-3)}{2}i = \frac{-3}{2} + \frac{3}{2}i$

④ Find the average of $z_1 = -10+2i$ and $z_2 = 2+3i$

$m = \frac{-10+2}{2} + \frac{2+3}{2}i = -4 + \frac{5}{2}i$

* Distance between two complex numbers z and w is $|z-w| = \sqrt{(Re(z)-Re(w))^2 + (Im(z)-Im(w))^2}$
* Midpoint of two complex numbers is $\frac{Re(z)+Re(w)}{2} + \frac{Im(z)+Im(w)}{2}i$

Complex Conjugates and Dividing Complex Numbers

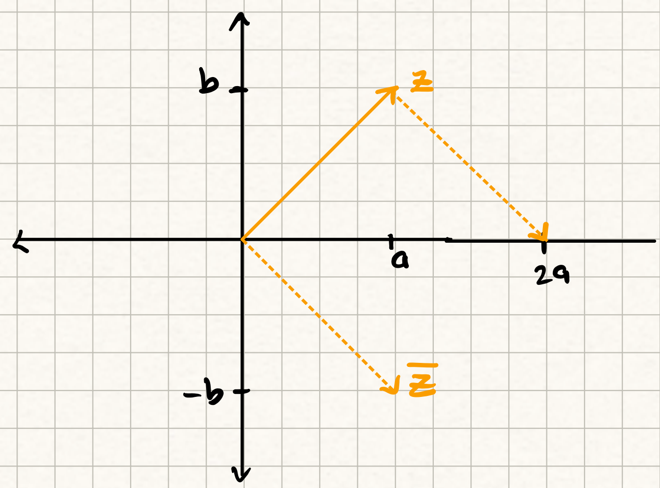
13.05.2025

Intro to complex number conjugates

* $z = a + bi \Rightarrow Re(z) = a \;,\; Im(z) = b$

$\Rightarrow$ conjugate of $z = \bar{z} = z^* = a - bi$

* $z + \bar{z} = 2(Re)z$



* $z \cdot \bar{z} = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2 = |z|^2$

* $\frac{1+2i}{4-5i} = ? = \frac{1+2i}{4-5i} \cdot \frac{(4+5i)}{(4+5i)} = \frac{4+5i+8i\overbrace{-10}^{=10i^2}}{16+20i-20i\underbrace{+25}_{=-25i^2}}$

$= \frac{-6+13i}{41} = \frac{-6}{41} + \frac{13}{41}i$

Complex number conjugates

* $z = 7-5i \Rightarrow z \cdot \bar{z} = ?$

$(7-5i) + (7+5i) \overset{*}{=} (7)^2 - (5i)^2$

$= 49 + 25$

$= 74$

* $(a-b)(a+b) = a^2 - b^2$

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* Conjugate of the complex number $z = a+bi$ is $\bar{z} = a - bi$.
  * $z + \bar{z} = 2\,Re(z)$
  * $z \cdot \bar{z} = |z|^2 = Re(z)^2 + Im(z)^2$

Dividing complex numbers

$$\star \quad \frac{6+3i}{7-5i} = ? = \frac{(6+3i)(7+5i)}{(7-5i)(7+5i)} = \frac{42+30i+21i-15}{49+25}$$

$$= \frac{27+51i}{74} = \frac{27}{74} + \frac{51}{74}i$$

Quiz

① $$\frac{-32+8i}{5+3i} = \frac{(-32+8i)(5-3i)}{25+9} = \frac{-160+96i+40i+24}{34}$$

$$= \frac{-136}{34} + \frac{136i}{34} = -4+4i$$

② $$\frac{-23+11i}{5+i} = \frac{(-23+11i)(5-i)}{25+1} = \frac{-115+23i+55i+11}{26}$$

$$= \frac{-104+78i}{26} = -4+3i$$

③ $$\frac{9-2i}{1-4i} = \frac{(9-2i)(1+4i)}{1+16} = \frac{9+36i-2i+8}{17} = \frac{17}{17} + \frac{34}{17}i$$

$$= 1+2i$$

④ $$\frac{-6+8i}{-4-3i} = \frac{(-6+8i)(-4+3i)}{16+9} = \frac{24-18i-32i-24}{25}$$

$$= \frac{-50i}{25} = -2i$$

* To divide a complex number by another, we multiply both numbers by the conjugate of the denominator.

Identities with Complex Numbers

13.05.2025

Complex numbers and Sum of squares factorization

* $x^2+y^2 = x^2-(-1)y^2 = x^2-i^2y^2 = x^2-(yi)^2$
$= (x+yi)(x-yi)$

* $36a^8+2b^6 = (6a^4)^2+(\sqrt{2}b^3)^2 = (6a^4)^2-(i\sqrt{2}b^3)^2$
$= (6a^4+i\sqrt{2}b^3)(6a^4-i\sqrt{2}b^3)$

Factoring polynomials using complex numbers

* $x^4+10x^2+9 = ?$
$= (x^2+9)(x^2+1)$
$= (x^2+3^2)(x^2+1^2)$
$= (x+3i)(x-3i)(x+i)(x-i)$

Modulus (absolute value) and Argument (angle) of Complex Numbers

13.05.2025

Absolute value of complex numbers

* Absolute value means "distance from zero" whether its a real number or imaginary number.

* $z = 3-4i \Rightarrow |z| = \sqrt{3^2+4^2} = 5$

Complex numbers with the same modulus

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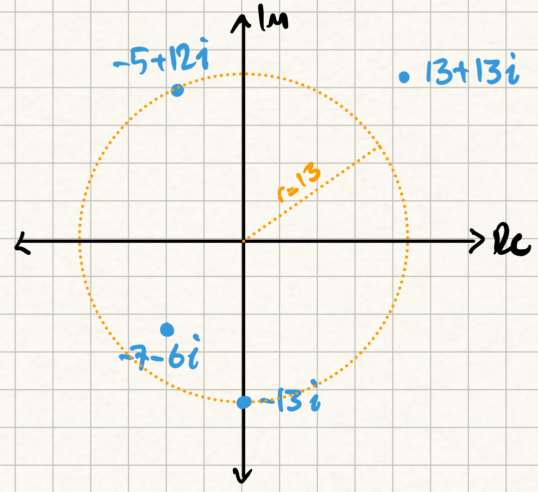
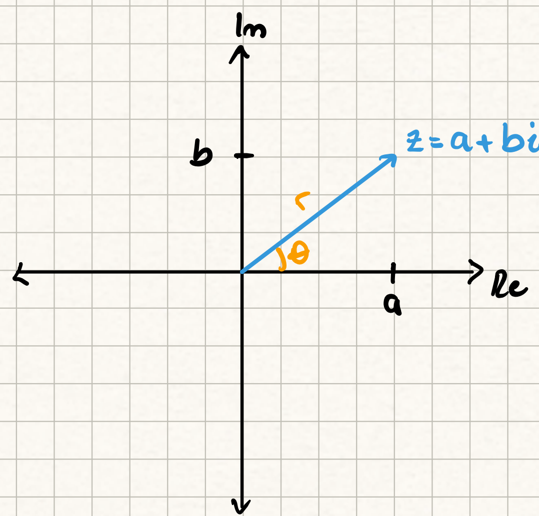


* Which of these complex numbers have a modulus of 13^2

$-5+12i$ and $-13i$

* All the complex numbers that are on the circle have a modulus of 13.

Absolute value and angle of complex numbers



* θ is called the "argument" of the complex numbers z.
* r is called the modulus of the complex number z.

* $r = \sqrt{a^2+b^2}$
* $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

* If we're given θ and r:

$a = r.\cos\theta$

$b = r.\sin\theta$

$\Rightarrow z = r.\cos\theta + r.\sin\theta\, i$

* $z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i \Rightarrow r = ?$ $\theta = ?$

$r = |z_1| = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° = \frac{\pi}{6}$

* Modulus of a complex number (r) is its absolute value, its distance from the origin $z = a+bi \Rightarrow r = \sqrt{a^2+b^2}$
* Argument of a complex number (θ) is its angle: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
* If we're given r and θ $\Rightarrow$ * $z = r(\cos\theta + i\sin\theta)$ = Polar Form
  * $z = r.e^{i\theta}$ = Exponential form

Quiz ① $z = 1-2i$, $-180^\circ < \theta < 180$ $\Rightarrow$ $\theta = ?$

$\theta = \tan^{-1}(-2) \cong -63.4^\circ$

② $z = 8-9i$, $-\pi < \theta < \pi$ $\Rightarrow$ $\theta = ?$ (in radians)

$\theta = \tan^{-1}\left(\frac{-9}{8}\right) \approx -0.844$

③ $r = 3$, $\theta = 20^\circ$ $\Rightarrow$ $z = ?$ (round to the nearest thousandth)

$z = r\cos(20^\circ) + r\sin(20^\circ)\,i$

$= 2.82 + 1.026i$

④ $|z_1| = 12$, $\theta_1 = 45^\circ$ $\Rightarrow$ $z_1 = ?$

$\cos(45^\circ) = \frac{\sqrt{2}}{2}$

$\sin(45^\circ) = \frac{\sqrt{2}}{2}$

$\Rightarrow$ $z_1 = 12 \cdot \frac{\sqrt{2}}{2} + 12 \cdot \frac{\sqrt{2}}{2}\,i$

$= 6\sqrt{2} + 6\sqrt{2}\,i$

⑤ $|z_1| = 11$, $\theta_1 = 180^\circ$ $\Rightarrow$ $z_1 = ?$

$\cos(180^\circ) = -1$

$\sin(180^\circ) = 0$

$\Rightarrow$ $z_1 = 11(-1) + 11(0)$

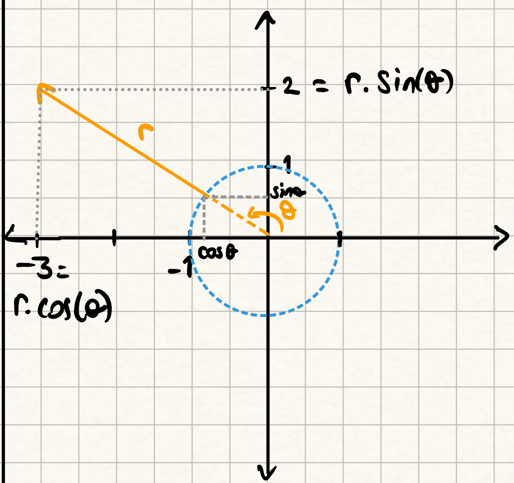
$= -11$

Polar Form of Complex Numbers

14.05.2025

Polar and rectangular form of complex numbers

* $\underbrace{z = -3 + 2i}_{\text{Rectangular Form}}$



* $z = r.\cos(\theta) + r\sin(\theta)\,i$
* $\theta = \tan^{-1}\left(\frac{b}{a}\right) \approx 2.55$
* $r = \sqrt{a^2+b^2} = \sqrt{4+9} = \sqrt{13}$

$\Rightarrow z \cong \sqrt{13}\cos(2.55) + \sqrt{13}\sin(2.55)\,i$

$\underbrace{\cong \sqrt{13}\,(\cos(2.55) + \sin(2.55)\,i)}_{\text{Polar Form}}$

"Go to the direction of 2.55 radians counter clock-wise for $\sqrt{13}$ units."

Converting a complex number from polar to rectangular form

* $z = \sqrt{17}\,(\cos(346°) + i\sin(346°))$ $\Rightarrow$ Write z in rectangular form (Round to the nearest Integer.)

$r = \sqrt{17} \Rightarrow a^2 + b^2 = 17$

1. $\theta = 346° \Rightarrow \tan(\theta) = -0.2493 = \frac{b}{a} \cong \frac{-1}{4}$

$\Rightarrow a \cong -4b \qquad (4b)^2 + b^2 \cong 17 \Rightarrow (a,b) = (4,-1), (\cancel{-4},1)$

B.C $\theta = 346°$

$\Rightarrow z \cong 4 - i$

2. $\theta = 346° \Rightarrow \cos\theta \cong 4$
$\sin\theta \cong -1$
$\Rightarrow z = 4 - i$

Quiz: ① $z=\sqrt{29}\left(\cos(248^\circ)+i(\sin(248^\circ)\right) \Rightarrow z=?$ (rect.)

$$=-2-5i$$

② $z_1 = 6+2\sqrt{3}$ in polar form

$$r=\sqrt{36+12} = 4\sqrt{3} \qquad \Rightarrow z_1 = 4\sqrt{3}\left(\cos(30^\circ)+i\sin(30^\circ)\right)$$

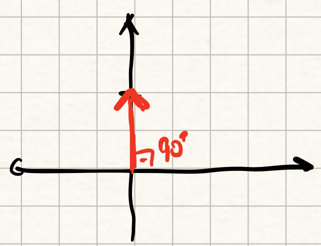
$$\tan^{-1}\left(\frac{2\sqrt{3}}{6}\right) = 30^\circ$$

③ Express $z_1 = 3\left[\cos(60^\circ)+i\sin(60^\circ)\right]$ in rectangular form:

$$z_1 = \frac{3}{2}+\frac{3\sqrt{3}}{2}i$$

④ $z_1 = 0+10i$ in polar form

$r=10$

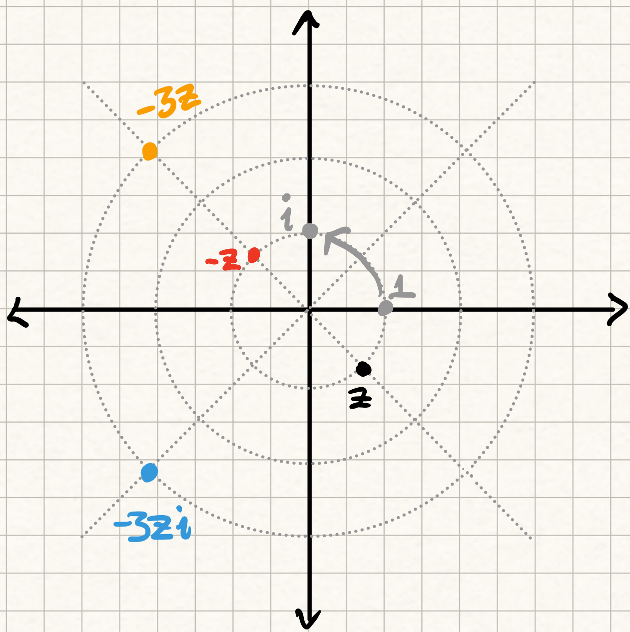


$$z_1 = 10\left(\cos(90^\circ)+i\sin(90^\circ)\right)$$

Graphically Multiplying Complex Numbers

14.05.2025

Multiplying complex numbers graphically: "$-3i$"



* If we multiply 1 by i, we flip it 90° counter clock wise, so:

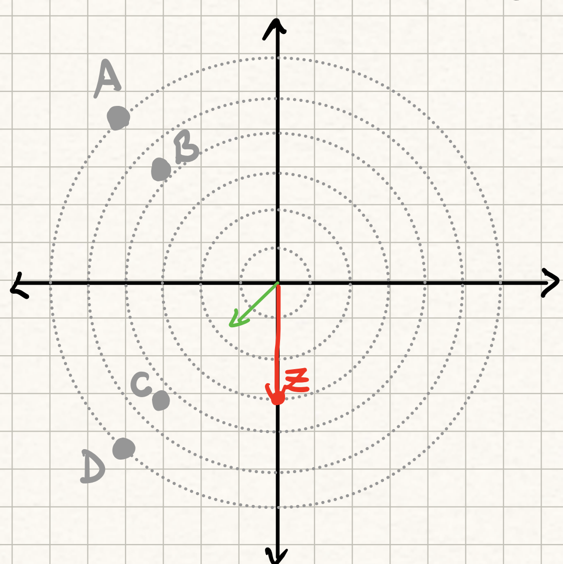
$-3iz =$

* Change the direction 180°
* Scale by 3
* Rotate 90° counter clock-wise

What is complex multiplication?

* When you multiply z by a complex number, you are going to rotate by the argument of that complex number and you are going to scale the modulus of z by the modulus of the complex number. Therefore it's helpful to express the comp. number in its polar form to easily visualize the multiplication.

"$-1-i$"



* Which point represents the product of z and $-1-i$?

$|w| = \sqrt{2}$ $\quad \theta_w = \tan^{-1}\left(\frac{-1}{-1}\right) = 135°$ (Quadrant 3)

$\Rightarrow w = \sqrt{2}\left(\cos(135°) + i\sin(135°)\right)$

$\Rightarrow$ "Rotate z 135° counter clock-wise and scale it by $\sqrt{2}$"

$\Rightarrow$ "Point B"

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* Complex multiplication: When we multiply z with w, we scale it by r_w and rotate it by w_θ. Therefore, it's helpful to express w in its polar form to easily visualize the rotation.
* Complex division: When we divide z by w, we scale it by $\frac{1}{r_w}$ and rotate it by $(-1)w_\theta$.
* Dividing by w is the same as multiplying by $\frac{\overline{w}}{|w|^2}$

Multiplying and Dividing Complex Numbers in Polar Form

14.05.2025

Multiplying complex numbers in polar form

$w_1 = 3(\cos(330°) + i\sin(330°))$

$w_2 = 2(\cos(120°) + i\sin(120°))$ $\Rightarrow w_1 \cdot w_2 = ?$

$\Rightarrow w_1 \cdot w_2 = \underbrace{6}_{3\cdot 2}(\cos(\underbrace{90}_{330+120}) + i\sin(90)) = 6i$

$w_1 \cdot w_2 = |w_1 \cdot w_2|\left(\cos(\theta_{w_1} + \theta_{w_2}) + i\sin(\theta_{w_1} + \theta_{w_2})\right)$

Dividing complex numbers in polar form

$w_1 = 8\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$

$w_2 = 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$ $\Rightarrow w_1 \cdot w_2 = ?$

$\frac{w_1}{w_2} = \underbrace{4}_{8/2}\left(\cos\left(\underbrace{\frac{\pi}{6}}_{\frac{4\pi}{3} - \frac{7\pi}{6}}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2\sqrt{3} + 2i$

$\frac{w_1}{w_2} = \frac{|w_1|}{|w_2|}\left(\cos(\theta_{w_1} - \theta_{w_2}) \cdot i\sin(\theta_{w_1} - \theta_{w_2})\right)$

Taking and visualizing powers of a complex number

$z = -1 + i\sqrt{3} \Rightarrow z^4$ in polar and rectangular form?

$|z| = \sqrt{1+3} = 2$

$\theta_z = \tan^{-1}(-\sqrt{3}) = 120°$

$\Rightarrow z = 2(\cos(120°) + i\sin(120°))$

$z^2 = 4(\cos(240°) + i\sin(240°))$

$z^3 = 8(\cos(360) + i\sin(360))$

$z^4 = 16(\cos(120°) + i\sin(120))$

$= -8 + 8\sqrt{3}\, i$

$w^n = |w|^n\left(\cos(\theta_w \cdot n) + i\sin(\theta_w \cdot n)\right)$

$w_1 \cdot w_2 = |w_1| \cdot |w_2| \cdot \left(\cos(\theta_{w_1} + \theta_{w_2}) + i\sin(\theta_{w_1} + \theta_{w_2})\right)$

$\frac{w_1}{w_2} = \frac{|w_1|}{|w_2|}\left(\cos(\theta_{w_1} - \theta_{w_2}) + i\sin(\theta_{w_1} - \theta_{w_2})\right)$

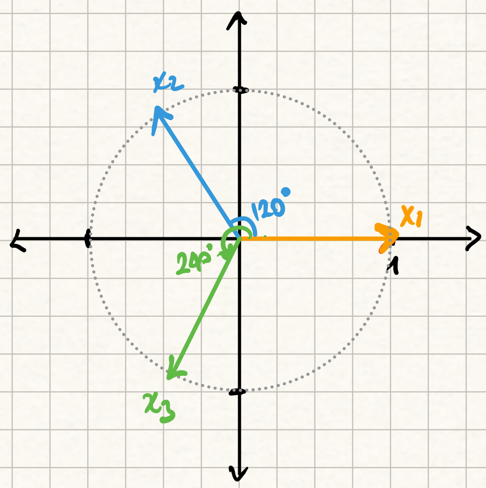
$w^n = |w|^n\left(\cos(\theta_w \cdot n) + i\sin(\theta_w \cdot n)\right)$

Complex number equations: $x^3 = 1$

* $x^3 = 1$ $\qquad x^3 - 1 = 0$

$z = 1 \Rightarrow z = 1 + 0i$ $\qquad |z| = 1 \qquad \theta_z = 0^\circ = 2\pi = 4\pi \ldots$

$\Rightarrow 1 = 1e^{0i} = 1e^{2\pi i} = 1e^{4\pi i}$



$\Rightarrow x^3 = e^{2\pi i} = e^{4\pi i}$

$\Rightarrow (x^3)^{1/3} = (e^{2\pi i})^{1/3} = (e^{4\pi i})^{1/3}$

$\Rightarrow x_1 = 1 \qquad x_2 = e^{\frac{2\pi}{3} i} \qquad x_3 = e^{\frac{4\pi}{3} i}$

$|x_1| = 1 \qquad |x_2| = 1 \qquad |x_3| = 1$

$\theta_{x_1} = 0^\circ \qquad \theta_{x_2} = \frac{2\pi}{3} \qquad \theta_{x_3} = \frac{4\pi}{3}$

$\Rightarrow x_1 = 1 + 0i = 1$

$x_2 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

$x_3 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$

* $z^n = w \Rightarrow z = \sqrt[n]{r} \cdot \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \; k = 0, 1, 2 \ldots n-1$

Quiz

1. $z = -\frac{7\sqrt{2}}{2} - \frac{7\sqrt{2}}{2} i \Rightarrow z^3 = ?$ $\qquad (r_z = 7, \; \theta_z = 225^\circ)$

$\Rightarrow z = 7(\cos(225^\circ) + i \sin(225))$

$z^2 = 49(\cos(450)) + i \sin(450))$

$z^3 = 343(\cos(675^\circ)) + i \sin(675^\circ))$

$= 343(\cos(315^\circ)) + i \sin(315^\circ)) = 242.5 - 242.5 i$

* $z^n = w \Rightarrow z = \sqrt[n]{r_w} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \; k = 0, 1, 2 \ldots, n-1$

② $z = -3\sqrt{3} - 3i \Rightarrow z^3 = ?$ $(r_z = 6,\ \theta_z = 210^\circ)$

$\Rightarrow z = 6(\cos(210^\circ) + i\sin(210^\circ))$

$\Rightarrow z^3 = 216(\cos(270^\circ) + i\sin(270^\circ))$

$= -216i$

③ $z^3 = 27 \Rightarrow z_1, z_2, z_3 = ?$

$z_1 = 3$

$z_2 = \sqrt[3]{27}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 3(\cos(120^\circ) + i\sin(120^\circ))$

$z_3 = \sqrt[3]{27}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = 3(\cos(240^\circ) + i\sin(240^\circ))$

④ $z = \sqrt{3} + i \Rightarrow z^5 \approx ?$ $(r_z = 2,\ \theta_z = 30^\circ)$

$\Rightarrow z = 2(\cos(30^\circ) + i\sin(30^\circ))$

$\Rightarrow z^5 = 2^5(\cos(30^\circ \cdot 5) + i\sin(30^\circ \cdot 5))$

$= 32(\cos(150^\circ) + i\sin(150^\circ))$

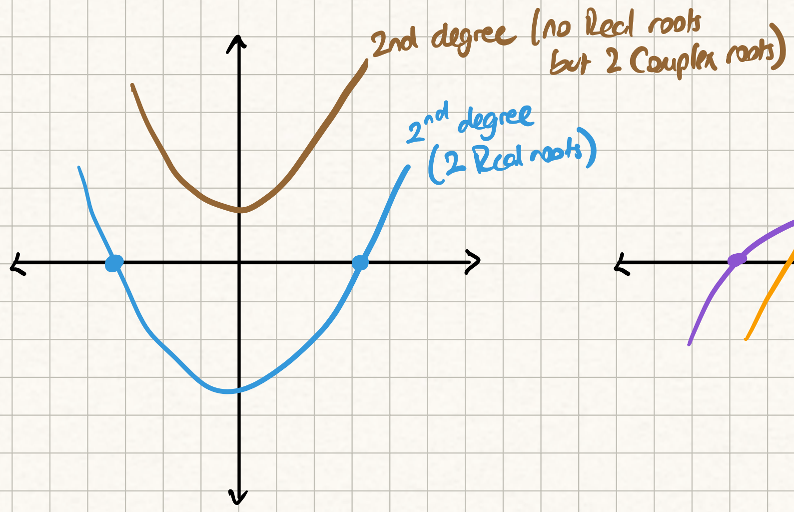
$= -27.7 + 16i$

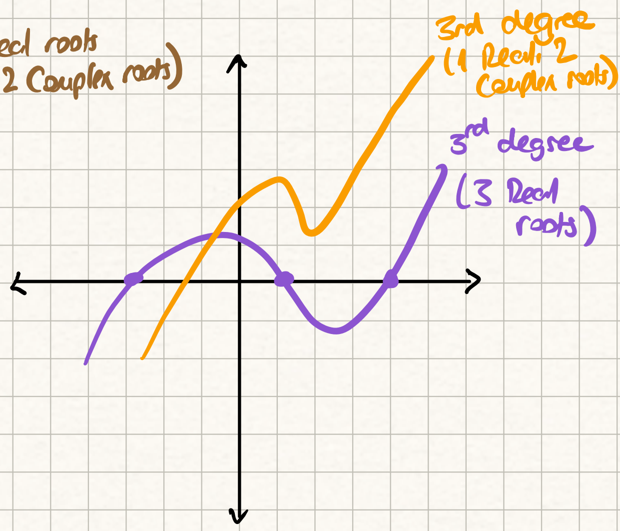
The Fundamental Theorem of Algebra

15.05.2025

$$P(x) = \underbrace{ax^n + bx^{n-1} + \dots + k}$$

n-roots (not only Real roots, but complex roots)





* The fundamental theorem of algebra: "Every polynomial of degree n, where n is a positive whole number, has exactly n-root, or solutions.
* Couplex roots are always come in pairs which are conjugates. So a 3^{rd} degree polynomial can have 3 Real roots or 1 Real and 2 Couplex roots, but it can't have 3 Couplex roots.

Quadratics and the Fundamental Theorem of Algebra

* $f(x) = 5x^2 + 6x + 5$ ⇒ we'll have 2 roots.

$5x^2 + 6x + 5 = 0$

$$\Rightarrow x = \frac{-6 \pm \sqrt{6^2 - 4.5.5}}{2.5} * = \frac{-6 \pm \sqrt{-64}}{10}$$

$$= \frac{-6 \pm 8i}{10} = \frac{-3}{5} \pm \frac{4}{5} i$$

$$\Rightarrow x_1 = \frac{-3}{5} + \frac{4}{5} i \qquad x_2 = \frac{-3}{5} - \frac{4}{5} i$$

* Quadratic formula:

$f(x) = ax^2 + bx + c$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Number of possible real roots of a polynomial

* $Ax^7 + Bx^6 + Cx^5 + Dx^4 + Ex^3 + Fx^2 + Gx + H$

→ we'll have 7 roots in total, possibly:

Real	Non-real, complex
7	0
5	2
3	4
1	6

must be even since complex roots come in pairs

Summary of identities with complex numbers

* $i^2 = -1$
* $i^3 = -i$
* $i^4 = 1$
* $(a+bi)^2 = a^2 + 2abi - b^2$
* $(a+bi)^3 = a^3 + 3a^2bi - 3ab^2 - b^3i$
* $(a+bi)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$
* $\overline{a+bi} = a - bi$ (complex conjugate)
* $|a+bi| = \sqrt{a^2+b^2}$ (modulus, absolute value)
* $(a+bi)(a-bi) = a^2 + b^2$
* $\overline{(a+bi)(c+di)} = \overline{(a+bi)} \cdot \overline{(c+di)}$
* $|a+bi| \cdot |c+di| = |(a+bi) \cdot (c+di)|$