AP CALCULUS BC

01 LIMITS AND CONTINUITY

* A limit is the value that a function (or sequence) approaches as the argument (or index) approaches some value.
* When a limit doesn't approach the same value from both sides, then the limit doesn't exist: $\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$
* Just because a function is undefined for some x-value doesn't mean there's no limit. On the other hand, just because a function is defined for some x-value doesn't mean that limit exists.
* Let $\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$,
  * $\lim_{x \to c} (f(x) + g(x)) = L + M$
  * $\lim_{x \to c} (f(x) - g(x)) = L - M$
  * $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
  * $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$
  * $\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$
  * $\lim_{x \to c} \left(f(x)^{\frac{r}{s}}\right) = L^{\frac{r}{s}}$
* When evaluating the limits of combined functions, we must verify whether the left-hand and right-hand limits are equal. If they are, the overall limit exist, even if the individual limit of one of the component functions does not.
* $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$ if and only if:

  ① $\lim_{x \to a} g(x) = L$ (exists)

  ② AND $f(x)$ is continuous at L.

* Selecting procedures for determining limits:

Calculating $\lim_{x\to a} f(x)$

A. Direct substitution

Try to evaluate the function directly.

$$f(a)$$

$$f(a) = \frac{b}{0}$$

where ***b*** is not zero

B. Asymptote (probably)

example:

$$\lim_{x\to 1} \frac{1}{x-1}$$

Inspect with a graph or table to learn more about the function at ***x=a***.

$$f(a) = b$$

where ***b*** is a real number

C. Limit found (probably)

example:

$$\lim_{x\to 3} x^2 = (3)^2 = 9$$

$$f(a) = \frac{0}{0}$$

D. Indeterminate form

example:

$$\lim_{x\to -1} \frac{x^2-x-2}{x^2-2x-3}$$

Try rewriting the limit in an equivalent form.

E. Factoring

example:

$$\lim_{x\to -1} \frac{x^2-x-2}{x^2-2x-3}$$

can be reduced to

$$\lim_{x\to -1} \frac{x-2}{x-3}$$

by factoring and cancelling.

F. Conjugates

example:

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$$

can be rewritten as

$$\lim_{x\to 4} \frac{1}{\sqrt{x}+2}$$

using conjugates and cancelling.

G. Trig identities

example:

$$\lim_{x\to 0} \frac{\sin(x)}{\sin(2x)}$$

can be rewritten as

$$\lim_{x\to 0} \frac{1}{2\cos(x)}$$

using a trig identity.

Try evaluating the limit in its new form.

H. Approximation

When all else fails, graphs and tables can help approximate limits.

* If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$

* f is discontinuous if we have to pick up our pencil while graphing it. There are 3 types of discontinuity:
   1. Removable (Point) Discontinuity: Two-sided limit exists, which means one-sided limits exist and are equal to each other. However, two-sided limit is not equal to the function's value at that point.
   2. Jump Discontinuity: Two-sided limit does not exist because one-sided limits are not equal to each other, even though they exist.
   3. Asymptotic Discontinuity: Two-sided limit does not exist because one-sided limits are unbounded, therefore they don't exist.
* f is continuous at $x=c \iff \lim_{x\to c} f(x) = f(c)$

* f is continuous over $(a,b) \iff$ f is continuous over every point in the interval
* f is continuous over $[a,b] \iff$ f is continuous over (a,b) **AND** $\lim_{x\to a^+} f(x) = f(a)$ **AND** $\lim_{x\to b^-} f(x) = f(b)$
* f is continuous on all real numbers $\iff$ it has no types of discontinuity (removable, jump, or asymptotic).
* When the one-sided limits are unbounded to the same direction, we can say that limit is going to infinity in that direction.
* A function can not cross its vertical asymptote, but it can cross its horizontal asymptote (even multiple times).
* Functions with horizontal asymptotes have one-sided limits for x approaches ∞ and $-\infty$. These limits are finite real numbers.
* The intermediate value theorem states that for any function f that is continuous over the interval $[a,b]$, the function will take any value between $f(a)$ and $f(b)$ over the interval. More formally, for any value L between $f(a)$ and $f(b)$, there's a value c in $[a,b]$ for which $f(c) = L$.

02 DIFFERENTIATION: DEFINITION AND BASIC DERIVATIVE RULES

* The derivative quantifies the sensitivity to change of a function's input with respect to its output.
* The derivative of a function of a single variable at a chosen value, when it exists, is the slope of the tangent line to the graph of the function at that point, formula of which is $\frac{\Delta y}{\Delta x}$.
* Lagrange's notation: $f'(x)$: Pronounced "f prime". Meaning "the slope of $f(x)$ for this specific x-value"
* Leibniz's notation: $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$: "The derivative of $f(x)$ (or y) with respect to x." Seems more complicated but very useful when dealing with integral calculus, differential equations, and multivariable calculus.
* The slope at every point on a straight line is constant. Therefore the derivative of the linear function $f(x) = mx+b$ with respect to x is equal to m for all x in its domain.
* Formal form of the derivative: $\lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$
* When we're interested for one specific x only: $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$
* If f is not continuous at $x=c$, then f is not differentiable at $x=c$.
* If f is continuous at $x=c$, it doesn't necessarily mean that it's differentiable at $x=c$. The slopes of the tangent lines might approach to different values as $x \to c^-$ and $x \to c^+$.
* Power Rule: $f(x) = x^n, n=0 \Rightarrow f'(x) = n \cdot x^{n-1}$
* Constant Rule: $\frac{d}{dx}[k] = 0$
* Multiplication by a Constant Rule: $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}[f(x)]$
* Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
* Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)]$

* Quotient Rule : $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - f(x)\frac{d}{dx}[g(x)]}{(g(x))^2}$

* $\frac{d}{dx}[\sin(x)] = \cos(x)$ * $\frac{d}{dx}[\cos(x)] = -\sin(x)$

* $\frac{d}{dx}[\tan(x)] = \frac{1}{\cos^2(x)} = \sec^2(x)$ * $\frac{d}{dx}[\cot(x)] = \frac{-1}{\sin^2(x)} = -\csc^2(x)$

* $\frac{d}{dx}[\sec(x)] = \sec(x)\cdot\tan(x)$ * $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

* $\frac{d}{dx}[e^x] = e^x$ * $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

03 DIFFERENTIATION OF COMPOSITE, IMPLICIT AND INVERSE FUNCTIONS

* Chain Rule: We take the derivative of the composite function with respect to inner function and multiply that with the derivative of the inner function with respect to x.

* $\frac{d}{dx}[f(g(x))] = \frac{d[f(g(x))]}{d\,g(x)} \cdot \frac{d[g(x)]}{dx}$

* $\frac{d}{dx}[a^x] = a^x\cdot\ln(a)$ * $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\cdot\ln(a)}$

* In implicit differentiation, we differentiate each side of an equation with two variables by treating one of the variables as a function of the other, not as a constant. This calls for using the chain rule.

* $f^{-1}(x) = g(x) \Rightarrow f'(x) = \frac{1}{g'(f(x))}$

* $\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$ * $\frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$ * $\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$

* When selecting a procedure for calculating derivatives, ask "Do I see a product, quotient, or composition of functions?"
* Consider manipulating the function if it will make the calculation easier.
* When determining the order in which to apply differentiation rules to a complex expression, begin by identifying the outmost operation (based on the structure of the equation) and proceed inward, applying the appropriate rules step by step.
* The second derivative of a function is simply the derivative of the function's derivative. Notation: $f''(x)$ OR $\frac{d^2}{dx^2}[f(x)]$