Operations on One Matrix

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Why learn linear algebra?

Goal → Solving systems of linear equation

↳ each term has a degree of one or zero

How? → Algebra: Substitution, elimination, or graphing

Linear Algebra: Matrices

How to use substitution to solve linear equations?

① Get a variable by itself in one of the equations
② Plug it into another equation
③ Solve the equation in step 2 for the remaining variable
④ Plug the result into the first equation

Ex:

$$y = x + 3$$
$$2x - 3y = 10$$

$$\Rightarrow \quad 2x - 3(x+3) = 10$$
$$2x - 3x - 9 = 10$$
$$x = -19$$
$$y = x + 3 = -16$$
$$(x, y) = (-19, -16)$$

How to use elimination to solve linear equations?

① Multiply one or both equations by a constant that will allow either the x-terms or the y-terms to cancel when the equations are added or subtracted.
② Add or subtract the equations.
③ Solve for the remaining variable
④ Plug it into one of the original equations and solve for the other variable.

Using substitution, elimination and/or graphing to solve a linear system gets very expensive as the system gets larger. That's why we use matrices.

Ex:

$$y = 3x - 4 \qquad -x + 2y = 12$$

$$\Rightarrow \quad 2/\ 3x - y = 4$$
$$+\quad -x + 2y = 12$$
$$\overline{\quad 5x = 20 \quad}$$
$$x = 4$$

$$\Rightarrow \quad 3 \cdot 4 - y = 4$$
$$12 - y = 4$$
$$y = 8$$
$$(x, y) = (4, 8)$$

How to use graphing to solve linear equations?

① Solve for y in each equation

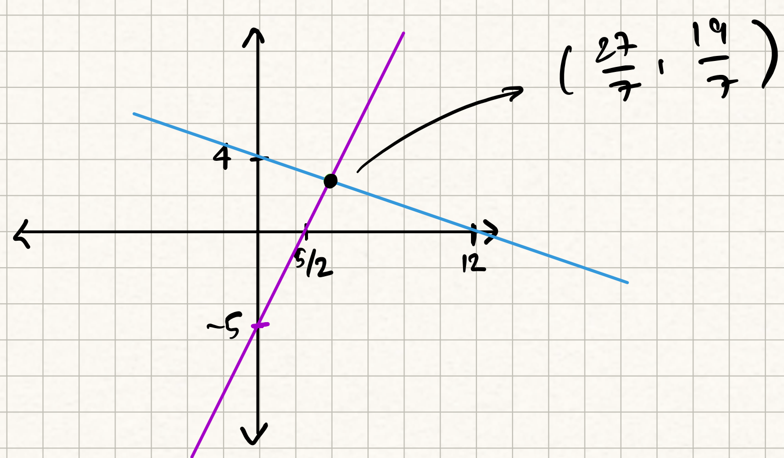
② Graph both equations on the same Cartesian coordinate sys.

③ Find the point of intersection of the lines.

Ex

$$x + 3y = 12 \qquad 2x - y = 5$$

$$\Rightarrow \quad y = \frac{12 - x}{3} = \frac{-1}{3}x + 4$$
$$y = 2x - 5$$



What is a matrix and why do we need matrices?

Matrix → A rectangular array of values, where each value is an entry in both a row and a column.

Why? Larger the system → Harder to solve with substit°, eliminat°, and graphing. Easier with matrices.

Matrix notation, dimensions and entries

$$A_{2 \times 3} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}$$

A: name of the matrix (in capital)

2: # of rows

3: # of columns

How to represent systems with matrices?

$$3x + 2y = 7$$
$$1x - 6y = 0$$

$$\Rightarrow \begin{array}{cc} x & y \\ \end{array}$$
$$\begin{bmatrix} 3 & 2 \\ 1 & -6 \end{bmatrix} \Rightarrow \text{Regular matrix}$$

$$\begin{array}{ccc} & & \text{Constant} \end{array}$$
$$\Rightarrow \left[\begin{array}{cc|c} 3 & 2 & 7 \\ 1 & -6 & 0 \end{array}\right] \Rightarrow \text{Augmented Matrix}$$

How to solve equations with matrices?

With Row Operations:

1. We can switch rows. Ex: $R_1 \longleftrightarrow R_2$
2. We can multiply a row with a constant. Ex: $\frac{1}{2} R_1 \rightarrow R_1$

   "Multiply Row 1 with 1/2 and replace the result wit Row 1"
3. We can add rows.

   Ex: $R_1 + R_2 \rightarrow R_2$

   : $R_1 + (-1) \cdot R_2 \rightarrow R_2 = R_1 - R_2 \rightarrow R_2$

What's the goal with row operations?

To change the matrix into row-echelon form (REF) or reduced row-echelon form (RREF)

1. All the pivot entries (1st non-zero entry of a row) are equal to 1.
2. Any rows with only zeros are at the bottom
3. Pivot entries in staircase pattern
4. In each pivot column, the pivot is the only non-zero entry

To solve a linear system, we change the matrix into reduced row-echelon form by using row operations such as swapping, adding, and scalar multiplication.

What is Gauss-Jorden Elimination?

An algorithm to get the matrix to RREF.

① Pull out any scalars from each row

② Make sure the first entry of the first row is 1. (Swap with another row if necessary)

③ Multiply through the first row by a scaler to make the leading entry equal to 1.

④ Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

⑤ Go back to step 2 and repeat until the matrix is in RREF.

Ex:

$$\begin{aligned} -x - 5y + z &= 17 \\ -5x - 5y + 5z &= 5 \\ 2x + 5y - 3z &= -10 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} -1 & -5 & 1 & 17 \\ -5 & -5 & 5 & 5 \\ 2 & 5 & -3 & -10 \end{array}\right]$$

$-R_1 \to R_1$
$\frac{1}{5}R_2 \to R_2$

$R_1 + R_2 \to R_2$
$2R_2 + R_3 \to R_3$

$R_2 \cdot (1/4) \to R_2$

$$\begin{bmatrix} 1 & 5 & -1 & -17 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & -3 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -1 & -17 \\ 0 & 4 & 0 & -16 \\ 0 & 3 & -1 & -8 \end{bmatrix} \begin{bmatrix} 1 & 5 & -1 & -17 \\ 0 & 1 & 0 & -4 \\ 0 & 3 & -1 & -8 \end{bmatrix}$$

$R_3 - 3R_2 \to R_3$

$$\begin{bmatrix} 1 & 5 & -1 & -17 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

$-R_3 \to R_3$

$$\begin{bmatrix} 1 & 5 & -1 & -17 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$R_1 + R_3 \to R_1$

$$\begin{bmatrix} 1 & 5 & 0 & -21 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$R_1 - 5R_2 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix} \Rightarrow (x, y, z) = (-1, -4, -4)$$

Number of solutions to the linear system

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \Rightarrow \text{Unique solution}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & c \end{array}\right] \Rightarrow$$

$c = 0 \Rightarrow$ Infinitely many solutions

$c \neq 0 \Rightarrow$ No solution