Transformations

18.04.2025

What is a transformation?

= Sending vectors or figures from their original locations into a new location. We can represent transformations as matrices.

Functions vs. transformations

$f(x) = x^2 \Rightarrow$

x	$f(x)$
0	0
1	1
2	4
3	9
$\vdots$	$\vdots$

$f: x \longmapsto x^2$

mapping

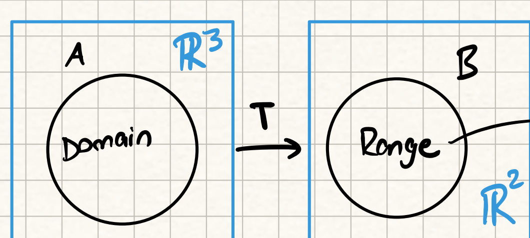
vector valued function

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

When working on matrices, we don't use the word function, but transformation

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$
$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} -5 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7 \\ -15 \end{bmatrix}$



Subset (subspace) we're mapping to

$\mathbb{R}^2 \rightarrow$ Codomain (space we're mapping to)

Sometimes Range = Codomain

$T: A \rightarrow B$

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

* Transformations map vectors from one space (Domain) to specific vectors (Range) in another space (Codomain)

Transformation matrices and the image of the subset

Quiz

1. T maps every vector in $\mathbb{R}^3$ to every vector in $\mathbb{R}^2$.

Domain = ? = $\mathbb{R}^3$

Codomain = ? = $\mathbb{R}^2$

Range = ? = $\mathbb{R}^2$

2. T maps every vector in $\mathbb{R}^4$ to the zero vector $\vec{0}$ in $\mathbb{R}^2$.

Domain = ? = $\mathbb{R}^4$

Codomain = ? = $\mathbb{R}^2$

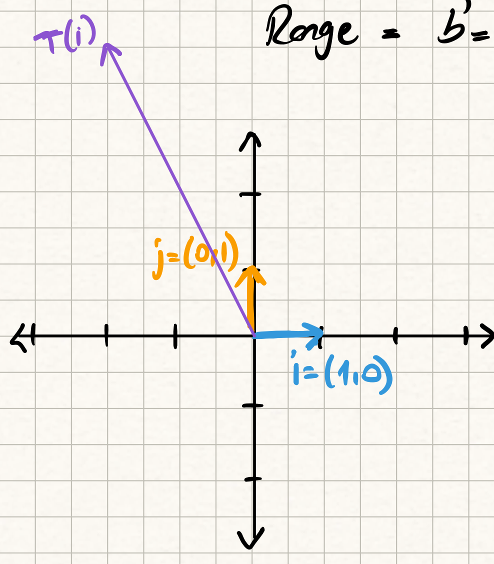
Range = $\vec{?}$ = $(0,0)$

3. T maps $\vec{a} = (1,2,-4)$ to $\vec{b} = (-3,0,4)$

Domain = $\mathbb{R}^3$

Codomain = $\mathbb{R}^3$

Range = $\vec{b} = (-3,0,4)$



$$T = \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}$$

i is going to (1,0) (it already is)

j is going to (-2,4)

$$T = \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \end{array}$$

i : (1,0,0) → (1,1,1)

j : (0,1,0) → (2,2,2)

k : (0,0,1) → (3,3,3)

Formula for the linear transformation using a transformation matrix

$\circledast$ $T(\vec{v}) = \vec{v} \cdot T$ (This is only valid if T is a linear transf. We'll learn about this soon.)

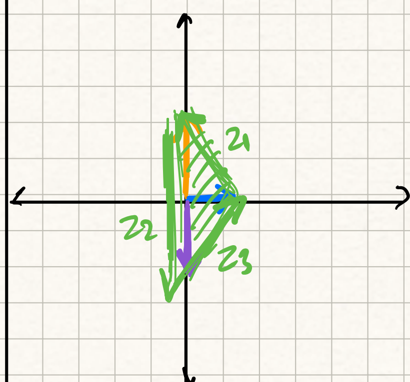
Ex: $T = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow T(\vec{v}) = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$

Preimage and Image

preimage : $i, j, \vec{v}$

image : $T(i), T(j), T(\vec{v})$

$= Ti, Tj, T\vec{v}$



$\vec{a} = (1,0)$
$\vec{b} = (0,2)$
$\vec{c} = (0,-2)$

They form the triangle z

$\vec{z}_1 = -\vec{a} + \vec{b}$
$= \vec{b} - \vec{a}$

$\vec{z}_1 = \{ \vec{a} + t(\vec{b} - \vec{a}) \mid 0 \leq t \leq 1 \}$

$\vec{z}_2 = \{ \vec{b} + t(\vec{c} - \vec{b}) \mid 0 \leq t \leq 1 \}$

$\vec{z}_3 = \{ \vec{c} + t(\vec{a} - \vec{c}) \mid 0 \leq t \leq 1 \}$

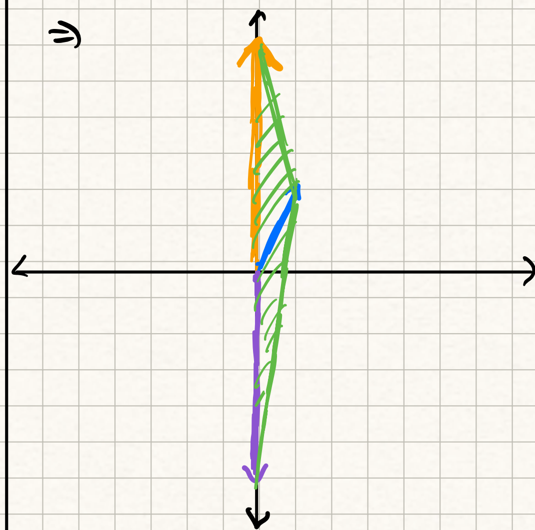
This makes sure that the origin of $\vec{z}_1$ is at the tip of $\vec{a}$.

Preimage

$T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \Rightarrow T\vec{a} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $T\vec{b} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

$T\vec{c} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$

$\Rightarrow$



$$T(\vec{a}+t(\vec{b}-\vec{a})) =$$
$$= T\vec{a} + T(t(\vec{b}-\vec{a}))$$
$$= T\vec{a} + t \cdot T(\vec{b}-\vec{a})$$
$$= T\vec{a} + t(T\vec{b} - T\vec{a})$$

$$\Rightarrow T\vec{b} + t(T\vec{c} - T\vec{b})$$
$$T\vec{c} + t(T\vec{a} - T\vec{c})$$

Image

Quiz

① If $\vec{a} = (-4,2)$ becomes $\vec{b}$ after $\Theta = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix}$, $b = ?$

$$b = T\vec{a} = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -42 \\ -12 \end{bmatrix}$$

② What are the vertices of the transf° of the polygon $(-2,1), (1,3), (2,-2), (-3,-1)$ after $P = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$

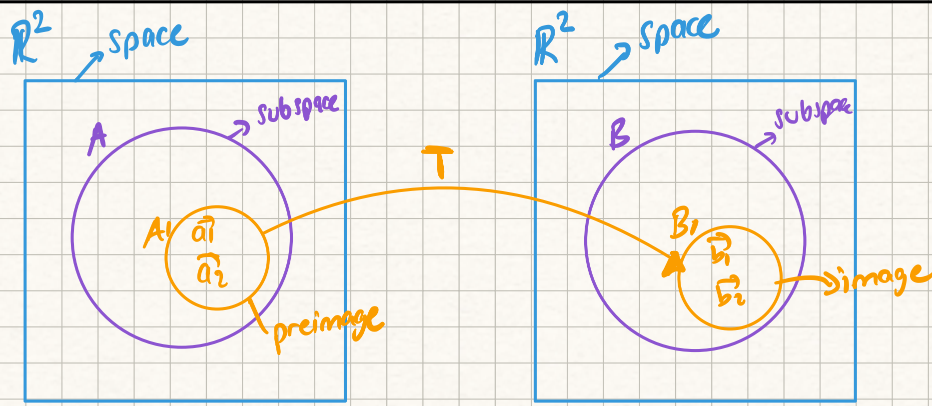
$$P(\vec{a},\vec{b},\vec{c},\vec{d}) = \left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ -4 \end{bmatrix}, \begin{bmatrix} 6 \\ -6 \end{bmatrix} \right)$$

③ What are the vertices of the triangle $(-3,0), (1,2), (1,-2)$ after transformed by $S = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$?

$$T(\vec{a},\vec{b},\vec{c}) = \left(\begin{bmatrix} 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)$$

- **Subset** : The vector set that's being transformed.
- **Preimage** : The vector set before being transformed.
- **Image** : The vector set after transformation is applied.

Preimage, image, and the kernel



* If we know $\vec{a_1}$, we calculate $T.\vec{a}$ to find $\vec{b_1}$.
* If we know $\vec{b_1}$, we turn T into an augmented matrix to find $\vec{a_1}$

Ex: $T = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}, \vec{b_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{b_2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \Rightarrow \vec{a_1}, \vec{a_2} = ?$

$$\left[\begin{array}{cc|c} 4 & 0 & 0 \\ -2 & 3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 6 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \Rightarrow \vec{a_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 0 & 4 \\ -2 & 3 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 6 & 6 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \vec{a_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow T: \begin{array}{l} \vec{a_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \vec{b_1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \vec{a_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \vec{b_2} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{array}$$

Ker(T) = all of the vectors that map to $\vec{0}$

↓

Kernel of the transformation

* Kernel of the transformation = $Ker(T) = \{ \vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0}_n \}$

Quiz ① Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$B_1 = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -2/3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & 5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 14/3 \\ 0 & 1 & 5/3 \end{array}\right]$$

$$A_1 = \left\{ \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 14/3 \\ 5/3 \end{bmatrix} \right\}$$

② Find the preimage A_1 of the subset B_1, under the transf° $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$B_1 = \left\{ \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 0 & -3 \\ 1 & 4 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 8 & -3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & -3/8 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 12/8 \\ 0 & 1 & -3/8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ -2 & 0 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 8 & 6 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 1 & 3/4 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3/4 \end{array}\right]$$

$$A_1 = \left\{ \begin{bmatrix} 3/2 \\ -3/8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3/4 \end{bmatrix} \right\}$$

Linear transformations as matrix-vector products

$T: \mathbb{R}^n \to \mathbb{R}^m$ ✱ T is a linear transformation if:

(1) $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$, and

(2) $T(c \cdot \vec{a}) = c \cdot T(\vec{a})$

Ex:

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} -a_1 + 2a_2 \\ a_2 - 3a_1 \\ a_1 - a_2 \end{bmatrix} \Rightarrow T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

How to find the transf' matrix

$i = (1,0) \to T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

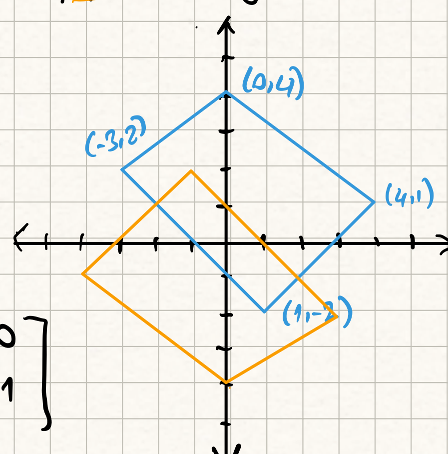
$j = (0,1) \to T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$T = \begin{bmatrix} -1 & 2 \\ -3 & 1 \\ 1 & -1 \end{bmatrix}$$

2 standard basis because T is from $\mathbb{R}^2$

Ex: Imagine we want our figure to flip across y-axis, and then flip across x-axis.

$x \to -x$
$y \to -y$ $\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} -x \\ -y \end{bmatrix}$



What's the $T(\vec{x}) = ?$

$$\left.\begin{aligned} i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} &\to \begin{bmatrix} -1 \\ -0 \end{bmatrix} \\ j = \begin{bmatrix} 0 \\ 1 \end{bmatrix} &\to \begin{bmatrix} -0 \\ -1 \end{bmatrix} \end{aligned}\right\} T(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \left(\begin{bmatrix} -4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

✱ A transformation is linear if:

(1) $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$, and

(2) $T(c \cdot \vec{a}) = c \cdot T(\vec{a})$

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Quiz ① Reflect the square with vertices $(-3,2),(4,2),(4,-5),(-3,-5)$ over the x-axis.

$(x,y)\to(x,-y) \Rightarrow$

$$\left.\begin{array}{l} i=\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix} \\ j=\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}0\\-1\end{bmatrix}\end{array}\right\} T(\vec{x})=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$\Rightarrow T\left(\begin{bmatrix}-3\\2\end{bmatrix},\begin{bmatrix}4\\2\end{bmatrix},\begin{bmatrix}4\\-5\end{bmatrix},\begin{bmatrix}-3\\-5\end{bmatrix}\right)=\left(\begin{bmatrix}-3\\-2\end{bmatrix},\begin{bmatrix}4\\-2\end{bmatrix},\begin{bmatrix}4\\5\end{bmatrix},\begin{bmatrix}-3\\5\end{bmatrix}\right)$$

② Double the width of the rectangle $(3,-6),(3,1),(-1,1),(-1,-6)$

$$T=\begin{bmatrix}2&0\\0&1\end{bmatrix} \Rightarrow T(\vec{x})=[(6,-6),(6,1),(-2,1),(-2,-6)]$$

③ Reflect $(1,1),(0,-4),(-4,-4),(-3,1)$ over the y-axis and then compress it vertically by a factor of 3.

$$\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}-x\\y\end{bmatrix}\to\begin{bmatrix}-x\\y/3\end{bmatrix} \Rightarrow T=\begin{bmatrix}-1&0\\0&1/3\end{bmatrix}$$

$$\Rightarrow T(\vec{x})=\left[\left(-1,\tfrac{1}{3}\right),\left(0,-\tfrac{4}{3}\right),(4,-4/3),(3,1/3)\right]$$

Linear transformations as rotations

$\mathbb{R}^2$
$$Rot_\theta=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$$

$\mathbb{R}^3$
$$Rot_{\theta \text{ around } y}=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}$$

$\mathbb{R}^3$
$$Rot_{\theta \text{ around } x}=\begin{bmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{bmatrix}$$

$\mathbb{R}^3$
$$Rot_{\theta \text{ around } z}=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$$

* Rotation matrices:

$$Rot_\theta=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix} \quad Rot_{\theta \text{ around } x}=\begin{bmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{bmatrix} \quad Rot_{\theta \text{ around } y}=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}$$

$$Rot_{\theta \text{ around } z}=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$$

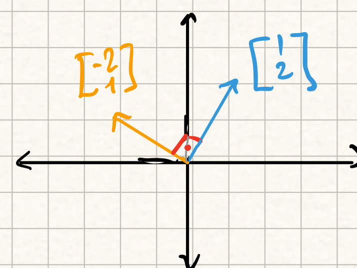
$$\left.\begin{array}{l} * \; Rot_\theta(\vec{a}+\vec{b}) = Rot_\theta(\vec{a}) + Rot_\theta(\vec{b}) \\ * \; Rot_\theta(c\cdot\vec{a}) = c\cdot Rot_\theta(\vec{a}) \end{array}\right\}$$ Because it's a linear transf°

Ex

$$Rot_{90°} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ 90°

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow$$

$$Rot_{90°}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Quiz

1. Rotate $(-1,4)$ by $270°$ $\Rightarrow$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

2. Rotate $(2,0,-3)$ by $60°$ around the x-axis $\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60 & -\sin 60 \\ 0 & \sin 60 & \cos 60 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3\sqrt{3}}{2} \\ -3/2 \end{bmatrix}$$

3. Rotate $(-2,3,-1)$ by $225°$ around z-axis.

$$R = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ -1 \end{bmatrix}$$

Adding and scaling linear transformations

* To add or scale linear transformations, the dimensions of the spaces should match

$$S : \mathbb{R}^n \to \mathbb{R}^m \qquad A_{m\times n} \cdot \vec{x}$$
$$T : \mathbb{R}^n \to \mathbb{R}^m \qquad B_{m\times n} \cdot \vec{x}$$

Adding →

* $(S+T)(\vec{x}) = S(\vec{x}) + T(\vec{x}) = A\vec{x} + B\vec{x} = (\vec{A}+\vec{B}) \cdot \vec{x}$

Ex:

$$S(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

↳ Rotates 90°

↳ Stretches horizontally by 2, and vertically by 3

Two transf° at once $= S+T = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$x = (1,1) \Rightarrow (S+T)(\vec{x}) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

Scaling →

$$c \cdot T(\vec{x}) = c(B\vec{x}) = (c \cdot B)\vec{x}$$

Ex: $T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $c = -2$, $\vec{x} = (1,1) \Rightarrow$

$$c \cdot T(\vec{x}) = (c \cdot T) \cdot \vec{x} = \left(-2 \cdot \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

* We can add transformations: $S(\vec{x}) + T(\vec{x}) = (S+T)(\vec{x}) = (A+B)\vec{x}$
* We can scale transformations: $c \cdot T(x) = c \cdot (B(\vec{x})) = (c \cdot B)\vec{x}$

Quiz ① $c = -3, \quad T = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \Rightarrow c \cdot T(\vec{x}) = ?$

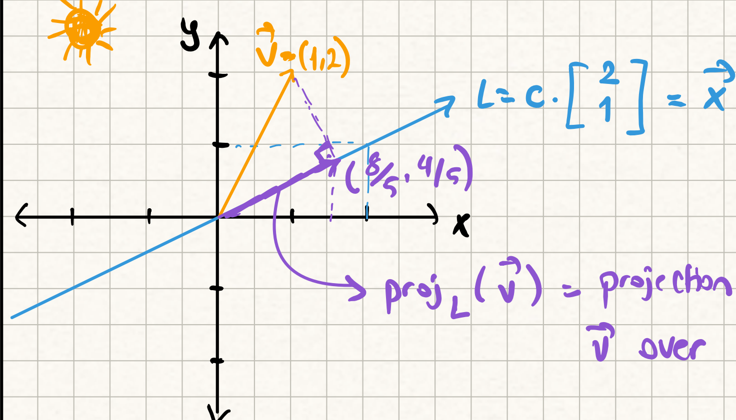
$$= \begin{bmatrix} -6 & 3 \\ 0 & -15 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

② $S = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} -1 & 4 \\ 2 & 4 \end{bmatrix} \Rightarrow (S+T)\vec{x} = ?$

$$= \begin{bmatrix} 2 & 10 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Projections as linear transformations

3 ways to calculate $Proj_L(\vec{v})$



$proj_L(\vec{v})$ = projection of $\vec{v}$ over L.

$$c = \frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}}$$

① $Proj_L(\vec{v}) = c \cdot \vec{x}$

$$= \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \cdot \vec{x}$$

$c = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow Proj_L(\vec{v}) = \left(\frac{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \right) \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$= \frac{4}{5} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 4/5 \end{bmatrix}$$

$\|\vec{x}\| = \sqrt{5} \Rightarrow \quad c = \frac{\vec{v} \cdot \vec{x}}{\|\vec{x}\|^2} = \frac{\vec{v} \cdot \vec{u}}{\|\hat{u}\|^2} = v \cdot \vec{u} \Rightarrow$

$\hat{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1\sqrt{5} \end{bmatrix}$ ② $\boxed{Proj_L(\vec{x}) = (\vec{v} \cdot \vec{u})\vec{u}}$

* The projection of $\vec{v}$ onto L, where L is given as scaled version of $\vec{x}$ is

① $Proj_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \cdot \vec{x}$ ② $Proj_L(\vec{v}) = (\vec{v} \cdot \hat{u}) \cdot \hat{u}$ ③ $Proj_L(\vec{v}) = \begin{bmatrix} u_1^2 & u_1 \cdot u_2 \\ u_1 \cdot u_2 & u_2^2 \end{bmatrix} \cdot \vec{v}$

$$\Rightarrow \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right) \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \left(\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}} \right) \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$= \frac{4}{\sqrt{5}} \cdot \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 8/5 \\ 4/5 \end{bmatrix}$$

3 $Proj_L(\vec{a} + \vec{b}) = Proj_L(\vec{a}) + Proj_L(\vec{b})$

$Proj_L(c \cdot \vec{a}) = c \cdot Proj_L(\vec{a})$

$Proj_L(\vec{v}) = A \cdot \vec{v}$ $\qquad \hat{u} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

$$A = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}$$

Quiz 1 $L = \left\{ c \begin{bmatrix} 2 \\ 0 \end{bmatrix} \mid c \in \mathbb{R} \right\}, \vec{v} = (1,5) \Rightarrow Proj_L(\vec{v}) = ?$

$\|\vec{x}\| = \sqrt{2+0} = 2$

$\hat{u} = (1,0)$

$$\Rightarrow A \cdot \vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2 $L = \left\{ c \begin{bmatrix} 3 \\ -4 \end{bmatrix} \mid c \in \mathbb{R} \right\}, \vec{v} = (-2,-1) \Rightarrow Proj_L(\vec{v}) = ?$

$\|\vec{x}\| = \sqrt{9+16} = 5$

$\vec{u} = \left(\frac{3}{5}, \frac{-4}{5} \right)$

$$\Rightarrow A \cdot \vec{v} = \begin{bmatrix} 9/25 & -12/25 \\ -12/25 & 16/25 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -6/25 \\ 8/25 \end{bmatrix}$$

* Projections are linear transformations. Therefore:

1. $Proj_L(\vec{a} + \vec{b}) = Proj_L(\vec{a}) + Proj_L(b)$, and,
2. $Proj_L(c \cdot \vec{a}) = c \cdot Proj_L(\vec{a})$

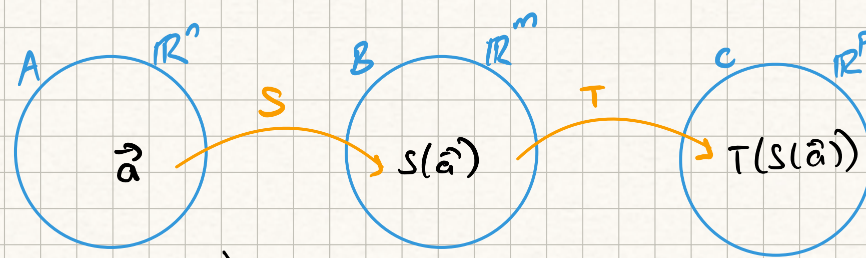
③ $L = \left\{ c\begin{bmatrix}-6\\2\end{bmatrix} \mid c\in\mathbb{R} \right\}, \vec{v} = (2,4) \Rightarrow \text{Proj}_L(\vec{v}) = ?$

$\|\vec{x}\| = \sqrt{36+4} = 2\sqrt{10}$

$\hat{u} = \left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

$A\cdot\vec{v} = \begin{bmatrix} 9/10 & -3/10 \\ -3/10 & 1/10 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}$

Compositions of linear transformations



$S: A \to B$
$T: B \to C$

$\cancel{S(T(\vec{a}))}$
$\cancel{S\circ T(\vec{a})}$

$T(S(\vec{a})) = T\circ S(\vec{a})$

composition sign

$S(\vec{a}) = A_{m\times n}\cdot\vec{a}$

$T(\vec{b}) = B_{p\times m}\cdot\vec{b}$

$\Rightarrow T(S(\vec{a})) = T(A\cdot\vec{a}) = B\cdot A\cdot\vec{a} = C\,\vec{a}$

$B\cdot A = C$

Ex:

$S(\vec{a}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix}\cdot\vec{a}, \quad T(\vec{a}) = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix}\cdot\vec{a}, \quad \vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow$

$C = \begin{bmatrix} -1 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 6 & 0 \end{bmatrix}$

$C\vec{a} = \begin{bmatrix} 0 & -8 \\ 6 & 0 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 \\ 18 \end{bmatrix}$

* $T\circ S(\vec{x})$ = $\vec{x}$ is transformed first by $S(\vec{a}) = A$, then $T(\vec{a}) = B$
  = a composition of linear transformations S and T
  $= B\cdot A\cdot\vec{x}$

Quiz (1) If $S: X \to Y$ and $T: Y \to Z$, then what is $T(S(\vec{x}))$ = ?

$$S(\vec{x}) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix} \qquad T(\vec{x}) = \begin{bmatrix} 2x_1 - x_2 \\ -2x_2 \end{bmatrix} \qquad \vec{x} = (1, -1)$$

$$S(\vec{x}) = A = \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} \qquad T(\vec{x}) = B = \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix}$$

$$T \circ S(\vec{x}) = B \cdot A \cdot \vec{x}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ -6 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$$

(2) $$S(\vec{x}) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ -4x_3 \\ x_2 - x_1 \end{bmatrix}, \quad T(\vec{x}) = \begin{bmatrix} -3x_1 \\ -2x_2 + x_3 \\ 4x_3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$$

$$\Rightarrow T \circ S(\vec{x}) = ?$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 & 3 & -3 \\ -1 & 1 & 8 \\ -4 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -24 \\ -6 \\ -24 \end{bmatrix}$$

(3) $$S(\vec{x}) = \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_1 + x_2 \end{bmatrix}, \quad T(\vec{x}) = \begin{bmatrix} 3x_2 \\ -2x_1 \\ 4x_3 \end{bmatrix}, \quad \vec{x} \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \Rightarrow$$

$$T \circ S(\vec{x}) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -5 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 10 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 \\ -20 \\ 20 \end{bmatrix}$$