Introduction, and Using Matrices to Represent Data

23.05.2025

Introduction to Matrices

* Matrix: A rectangular arrangement of numbers into rows and colums.

* $A = \begin{bmatrix} 1 & 0 & -7 \\ \pi & 5 & 11 \end{bmatrix}$ } A is a 2×3 matrix
  ↙ rows ↘ columns

$B = [8] \Rightarrow$ B is a 1×1 matrix

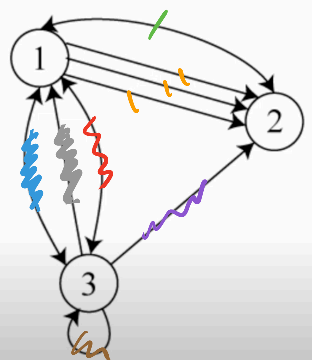
* $A_{i,j}$ represents the element on the i^{th} row and j^{th} column of matrix A.

Using Matrices to Represent Data

* Matrices provide a structured and efficient way to organize, manipulate, and analyze large datasets, enabling easy computation and transformation. They also align well with linear algebra operations, making them ideal for machine learning, statistics, and other data-driven applications.

Example: Networks

* This network diagram represents the different train routes between three cities. Each node is a city and each directed arrow represents a direct bus route from city to city.



Start \ End	City 1	City 2	City 3
City 1		XX	X
City 2	/		
City 3	X /	/	/

$$\begin{bmatrix} 0 & 4 & 2 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

Which city has the most incoming routes? → 2 5 routes

Which city has the most outgoing routes? 1 6 routes

Complete the matrix so it represents the number of direct routes between the cities, where rows are starting points and columns are end points.

Example: Payoffs

* Violet and Lennox play an elaborated version of rock paper scissors where each combination of shape choices earns a different number of points for the winner.

- When Violet wins, she gets 2 points.
- When Lennox wins with rock, she gets 3 points.
- When Lennox wins with paper, she gets 2 points.
- When Lennox wins with scissors, she gets 1 point.
- If both players choose the same shape, nobody gets any points.

Complete the matrix so it represents their scoring system. It shows the number of points Violet gets (a negative number means Lennox gets those points), where rows are Violet's chosen shape and columns are Lennox's chosen shape.

Violet \ Lennox	Rock	Paper	Scissors	
Rock	0	−2	+2	0
Paper	+2	0	−1	1
Scissors	−3	+2	0	−1

* Violet should choose Paper!

Assuming Lennox picks her shape entirely at random, what shape should Violet choose to maximize her chances of getting the most points?

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* A matrix is a rectangular arrangement of numbers into rows and columns.
* $A_{i,j}$ represents the element on the i^{th} row and j^{th} column of the matrix A.
* Matrices provide a structured and efficient way to organize, manipulate, and analyze large datasets, enabling easy computation and transformation. They also align well with linear algebra operations, making them ideal for machine learning, statistics, and other data-driven applications.

Matrix Operations: Scalar multiplication, matrix addition/subtraction

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Multiplying matrices by scalars

* Scalar multiplication is the product of a real number (scalar) and a matrix. In scalar multiplication, each matrix entry is multiplied by the scalar.

* $5 \cdot \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 15 & 0 \end{bmatrix}$

Adding and subtracting matrices

* To add/subtract matrices, we add/subtract the corresponding entries. Therefore, in order to add/subtract matrices, they must be of same dimensions.

* $\begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3+1 & 4+0 \\ 0+0 & -1+4 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 3 \end{bmatrix}$

* $\begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3-1 & 4-0 \\ 0-0 & -1-4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -5 \end{bmatrix}$

* $\begin{bmatrix} 3 & 4 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$ UNDEFINED

Zero Matrix and Opposite Matrix

* Zero matrix has zero for all its elements.

* $O_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

* When we subtract a matrix from the zero matrix, we find the opposite of that matrix.

* $A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \Rightarrow -A = \begin{bmatrix} -2 & -5 \\ -3 & 0 \end{bmatrix}$

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* Scalar multiplication is the product of a real number (scalar) and a matrix, where each entry is multiplied by the scalar.
* To add/subtract matrices, we add/subtract the corresponding entries. Therefore, the dimensions of the matrices should match.
* Zero Matrix has zero for all its elements : $O_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
* When we subtract a matrix from the zero matrix, we find the opposite of that matrix.

Properties of Matrix Addition	* Matrix addition is commutative : $A+B=B+A$ * Matrix addition is associative : $(A+B)+C = A+(B+C)$ * For any matrix A, there's a unique matrix O such that $A+O=A$. * For each A, there's a unique matrix $-A$ such that $A+(-A)=O$. * $A+B$ is a matrix of the same dimensions as A and B.
Properties of Matrix Scaler Multiplication	* Associative property: $(cd)A = c(dA)$ * Distributive properties: $c(A+B) = cA+cB$ $(c+d)A = cA+dA$ * Multiplicative identity property : $1A = A$ * Multiplicative properties of zero : $0.A = 0$ $c \cdot O = O$ * Closure property : cA is a matrix of same dimensions as A.

* $A+B = B+A$ // $A+(B+C) = (A+B)+C$ // $A+(-A) = O$ // $A+O = A$
* $c \cdot d \cdot A = c \cdot (d \cdot A)$ // $c(A+B) = c \cdot A + c \cdot B$ // $(c+d)A = c \cdot A + d \cdot A$ // $A \cdot O = O$

Using Matrices to Manipulate Data

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Example: Pet Store

A certain pet store chain has three types of dog food, and each comes in bags of two different sizes.

Matrix A represents the store's inventory at location A, where rows are food types and columns bag sizes.

Matrix A

Type \ Size	Size 1	Size 2
Type 1	5	7
Type 2	3	9
Type 3	10	15

Matrix B

Type \ Size	Size 1	Size 2
Type 1	8	6
Type 2	10	12
Type 3	5	9

Matrix C

Type \ Size	Size 1	Size 2
Type 1	-3	1
Type 2	-7	-3
Type 3	5	6

Matrix B represents the store's inventory at location B.

Matrix C represents how many more (or less) bags of each type and size there are in location A relative to location B. **Complete matrix C.** $C = A - B$

$D = A + B \Rightarrow$ what does D represent?

* Total inventory!

Example: Game Show

In the beginning of each episode of a certain gameshow, each contestant picks a certain door out of three doors. Then, the gameshow host randomly picks one of two prize bundles. After each round, each contestant receives a prize based on the door they picked and the bundle the host picked.

Matrix A represents the possible prizes for the first round.

Matrix A

Door \ Bundle	Bundle 1	Bundle 2
Door 1	\$100	\$400
Door 2	\$200	\$200
Door 3	\$300	\$0

Matrix B

Door \ Bundle	Bundle 1	Bundle 2
Door 1	\$600	\$200
Door 2	\$300	\$300
Door 3	\$0	\$400

Matrix C

Door \ Bundle	Bundle 1	Bundle 2
Door 1	1200	400
Door 2	600	600
Door 3	0	800

Matrix B represents the possible prizes for the second round.

The second round can also be a lightning round. In this case, the prizes are doubled.

Matrix C represents the possible prizes during a lightning round. **Complete matrix C.** $C = 2B$

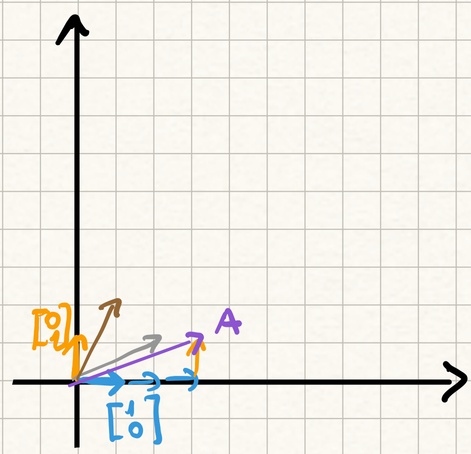
* $D = A + B \Rightarrow$ what does D represent?

* Total possible prizes assuming we don't have the lightning round.

Matrices as Transformations of the Plane

23.05.2025

Matrices as linear transformations of points on a plane



* $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

The first column tells what to do with the $\begin{bmatrix}1\\0\end{bmatrix}$ vector. "Take $\begin{bmatrix}1\\0\end{bmatrix}$ vector and transform it to $\begin{bmatrix}1\\0\end{bmatrix}$"

The second column tells what to do with the $\begin{bmatrix}0\\1\end{bmatrix}$ vector. "Take $\begin{bmatrix}0\\1\end{bmatrix}$ vector and transform it to $\begin{bmatrix}0\\1\end{bmatrix}$"

means "don't change"

therefore $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ is called the identity matrix.

* $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$ means that the $\begin{bmatrix}1\\0\end{bmatrix}$ vector will be mapped (transformed) to $\begin{bmatrix}2\\1\end{bmatrix}$ vector, and $\begin{bmatrix}0\\1\end{bmatrix}$ vector will be transformed to $\begin{bmatrix}1\\2\end{bmatrix}$ vector. Therefore if the point A was the weighted sum of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ before $\left(A = 3\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix}\right)$ will be transformed to the weighted sum of $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ now: $A' = 3\begin{bmatrix}2\\1\end{bmatrix} + 1\begin{bmatrix}1\\2\end{bmatrix}$.

$$\Rightarrow A' = \begin{bmatrix}6\\3\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$$

Intro to determinant notation and computation

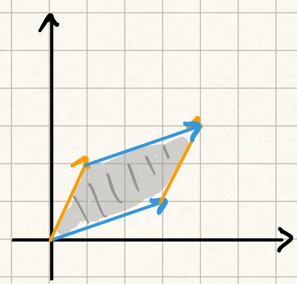
* $A = \begin{bmatrix}a & b\\c & d\end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix}a & b\\c & d\end{vmatrix} = a\cdot d - b\cdot c$

* $A = \begin{bmatrix}1 & -2\\3 & 5\end{bmatrix} \Rightarrow \det(A) = 1\cdot 5 - 3(-2) = 5+6 = \underline{\underline{11}}$

* Matrices represent linear transformations that map vectors from one space to another, such as scaling, rotating, shearing, or reflecting them. By multiplying a vector by a transformation matrix, we apply that transformation to the vector.
* The determinant is a scalar-valued function of the entries of a square matrix. For the 2x2 matrix $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, $\det(A) = a\cdot d - b\cdot c$

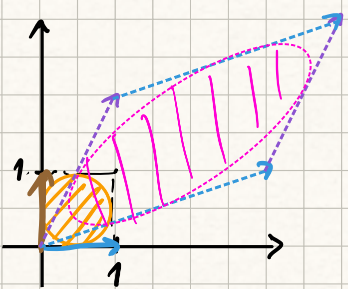
Interpreting determinants in terms of area

* The absolute value of det(A) is the area of the parallelogram that its column vectors create.



$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \qquad \det(A) = 5$$

* When we use A as a transformation matrix, the area of the figure it transforms scales by the absolute value of det(A).



$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \qquad \det(A) = 5$$

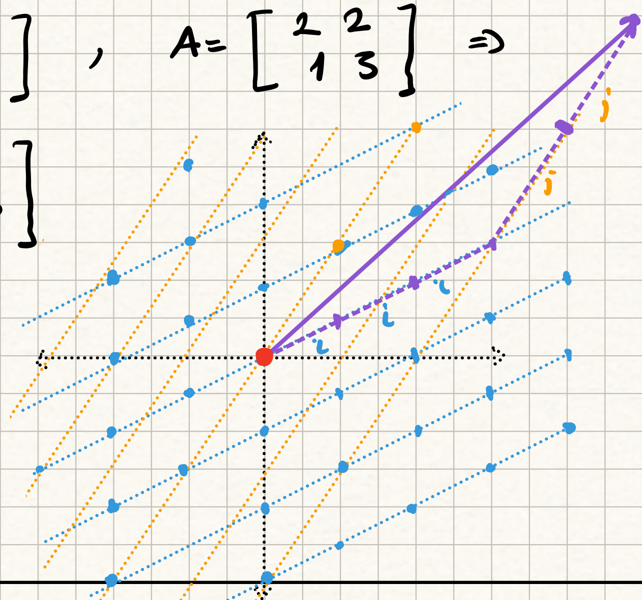
* Area of the is 5 times as the area of the

Using matrices to transform the plane

* $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}' = 3 \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{i} + 2 \underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{j}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}' = 3i + 2j$$



* The absolute value of det(A) is the area of the parallelogram that its column vectors create.

* When we use A as a transformation matrix, the area of the figure it transforms scales by the absolute value of det(A).

Using matrices to transform the plane: Composing matrices

* $A = \begin{bmatrix} 0 & 2 \\ 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix}$ $\Rightarrow$

* $B \circ A$ is a matrix composition, just like composite functions, it says "Transform with A first, then transform the result with the matrix B.

$$B \circ A = \left[\underbrace{0\begin{bmatrix} -3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 4 \end{bmatrix}} \quad \underbrace{2\begin{bmatrix} -3 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 4 \end{bmatrix}} \right]$$

$$B \circ A = \begin{bmatrix} 0 \\ 20 \end{bmatrix} \qquad \begin{bmatrix} -6 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$B \circ A = \begin{bmatrix} 0 \\ 20 \end{bmatrix}\begin{bmatrix} -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 20 & -2 \end{bmatrix}$$

Using matrices to transform a 4D vector

* $\begin{bmatrix} -1 \\ -3 \\ -5 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (-5)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

* $\begin{bmatrix} 1 & 0 & -3 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 2 & 0 & 2 \\ 3 & -1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -3 \\ -5 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix} - 5\begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} -1 \\ -2 \\ -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -6 \\ 3 \end{bmatrix} + \begin{bmatrix} 15 \\ 15 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \\ -7 \\ 3 \end{bmatrix}$$

Composing 3x3 matrices

! The 2x2 example was helpful to understand the composition of transformations but doing the same arithmetics with 3x3 matrices is unnecessary. We'll learn the efficient way on the next lesson anyways!

* $B \circ A$ is a matrix composition, just like composite functions, it means "Transform with A first, then transform the result with matrix B.

Multiplying Matrices by Matrices

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* $\begin{bmatrix} 2 & -2 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 7 & -6 \end{bmatrix} = \begin{bmatrix} (2)(-1)+(-2)(7) & (2)(4)+(-2)(-6) \\ (5)(-1)+(3)(7) & (5)(4)+(3)(-6) \end{bmatrix}$

$= \begin{bmatrix} -16 & 20 \\ 16 & 2 \end{bmatrix}$

* Matrix multiplication combines two matrices by computing the dot product of rows from the first matrix with columns from the second, producing a new matrix.

* $E = \begin{bmatrix} 0 & 3 & 5 \\ 5 & 5 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 4 \\ 3 & -2 \\ 4 & -2 \end{bmatrix}$ $\Rightarrow$ $E \cdot D = ?$

$E \cdot D = \begin{bmatrix} 29 & -16 \\ 38 & 6 \end{bmatrix}$

Properties of Matrix Operations

26.05.2025

Defined Matrix operations

* For $A \cdot B$ to be defined, # of columns of A and # of rows of B must be equal: $A_{m\times n} \cdot B_{n \cdot p} = A_{m\times p}$

* $\begin{bmatrix} 2 & 4 \\ 1 & 0 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 6 & 0 \\ 7 & -1 & -3 & 4 \end{bmatrix} = A_{3\times 4}$

  3x2 -defined- 2x4

* For A+B or A-B to be defined, A and B must have same # of rows and columns

Intro to identity matrices

* $I \cdot A = A \cdot I = A \Rightarrow I$ = identity matrix

* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

  3x3 3x3 3x3

* Identity matrices (are square matrices that...) have ones on the main diagonal, and zeros at the other positions.

Dimension of identity matrix

* $I_{m\times n} \cdot C_{a\times b} = C_{a\times b} \Rightarrow (m,n) = ?$

  $n=a$, $m=a$

  $\rightarrow (m,n) = (a,a) \Rightarrow I_{a\times a}$

* $C_{a\times b} \cdot I_{m\times n} = C_{a\times b} \Rightarrow (m,n) = ?$

  $m=b$, $n=b$

  $\Rightarrow (m,n) = (b,b) \Rightarrow I_{b\times b}$

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* For A·B to be defined, # of columns of A and # of rows of B must be equal.
* $A_{m\times n} \cdot B_{n\times k} = C_{m\times k}$
* $I \cdot A = A \cdot I = A \Rightarrow I$ is an identity matrix. Identity matrices are square matrices that have ones on the main diagonal and zeros at the other positions.

Is matrix multiplication commutative?

* Let $A_{5\times2}$ and $B_{2\times3}$ be matrices. $A \cdot B$ will be a 5x2 matrix. $B \cdot A$ will be undefined. Therefore matrix multiplication is not commutative.

* What if we had two matrice with same dimensions?

$$\begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -6 \\ 6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 9 & 12 \end{bmatrix}$$

NOT COMMUTATIVE

Associative property of matrix multiplication

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \quad C = \begin{bmatrix} i & j \\ k & l \end{bmatrix} \Rightarrow$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)\begin{bmatrix} i & j \\ k & l \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} i & j \\ k & l \end{bmatrix}\right)$$

$$\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\begin{bmatrix} i & j \\ k & l \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{bmatrix}$$

$$\begin{bmatrix} aei+bgi+afk+bhk & aej+bgj+afl+bhl \\ cei+dgi+cfk+dhk & cej+dgj+cfl+dhl \end{bmatrix} \stackrel{\checkmark}{=} \begin{bmatrix} aei+afk+bgi+bhk & aej+afl+bgj+bhl \\ cei+cfk+dgi+dhk & cej+cfl+dgj+dhl \end{bmatrix}$$

Zero matrix and matrix multiplication

* O = zero matrix $\Rightarrow O \cdot A = A \cdot O = O$

* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Using properties of matrix operations

* Which of the following equals to $A \cdot B \cdot C$ (which all are square matrices)
  * $B \cdot A \cdot C$ ✗
  * $A(C \cdot B) = ACB$ ✗
  * $A \cdot (B \cdot C) = A \cdot B \cdot C$ ✓
  * $A(B \cdot C + A) - A^2 = ABC + A \cdot A - A \cdot A = ABC$ ✓
  * $A(B+C) = AB + AC$ ✗

* $A \cdot B \neq B \cdot A$
* $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
* $A \cdot (B+C) = A \cdot B + A \cdot C$ and $(A+B) \cdot C = A \cdot C + B \cdot C$

Representing Systems of Equations with Matrices

26.05.2025

* $$\left.\begin{aligned} 3x - 2y - z &= -1 \\ 2x + 5y + z &= 0 \\ -4x - y &= 8 \end{aligned}\right\}$$ can be represented as: $$\underbrace{\begin{bmatrix} 3 & -2 & -1 \\ 2 & 5 & 1 \\ -4 & -1 & 0 \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 8 \end{bmatrix}$$

* $$\begin{aligned} 2x + y &= 9 \\ 3x - y &= 5 \end{aligned} \Rightarrow \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

* $\underset{n\times n}{A} \cdot \underset{n\times 1}{\vec{x}} = \underset{n\times 1}{\vec{b}}$ is the generalized form. Representing a system of equation this way will help us to solve it more efficiently, especially as the system gets larger.

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* $\left.\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}\right\}$ can be represented as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$
* $A_{n\times n} \cdot \vec{x}_{n\times 1} = \vec{b}_{n\times 1}$ is the generalized form.
* Representing a system of equation this way helps us to solve it more efficiently, especially if the matrix is a transformation matrix and/or the system is larger.

Introduction to Matrix Inverses

26.05.2025

* $A \cdot A^{-1} = I \Rightarrow A^{-1}$ is the inverse of A.
* If A is a transformation matrix, A^{-1} is also a transformation matrix. If A rotates a vector $90°$ clockwise ($\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$), A^{-1} should rotate a vector $90°$ counter clockwise ($\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$).

Let's verify:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

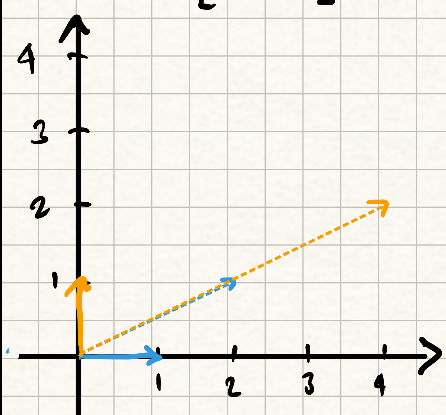
Invertible matrices and determinants

* It might be useful to think transformation as functions. Just like functions, some matrices are invertible, some are not.
* Invertible matrices are:
  1. Square matrices... (so $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are both defined)
  2. ...with non-zero determinants (so that the figure doesn't "disappear" (turn into a line))

Invertible matrices and transformations

* $B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow$ If we use B as a transformation matrix, all vectors will be transformed on to the same line, therefore any figures (pre-images) will turn into a line.

$Det(B) = 0 \iff$ Area of a line $= 0$



Inverse matrices and matrix equations

* $\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \Rightarrow A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$

$\Rightarrow A^{-1} \cdot A \cdot \begin{bmatrix} p \\ q \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} p \\ q \end{bmatrix} \Rightarrow$ 'A must be invertible, otherwise we won't have unique solution'

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* $A \cdot A^{-1} = A^{-1} \cdot A = I \Rightarrow A^{-1}$ is the inverse of A.
* Not all matrices are invertible. A matrix is invertible only if
  1. is a square matrix (so $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are both defined)

  AND

  2. has a non-zero determinant (so that the area of the pre-image is not scaled by zero)

Finding inverses of 2x2 matrices

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \underset{\text{adjugate (adjoint)}}{\operatorname{adj}(A)} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* $A^{-1}_{2x2} = \dfrac{1}{\det(A)} \cdot \operatorname{adj}(A)$

* $A = \begin{bmatrix} 3 & 5 \\ -7 & 2 \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{(3)(2)-(5)(-7)} \cdot \begin{bmatrix} 2 & -5 \\ 7 & 3 \end{bmatrix}$

$= \dfrac{1}{6+35} \cdot \begin{bmatrix} 2 & -5 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} 2/41 & -5/41 \\ 7/41 & 3/41 \end{bmatrix}$

Solving linear systems with matrix inverses

* $\left.\begin{aligned} 2x - 5y &= 7 \\ -2x + 4y &= -6 \end{aligned}\right\} \Rightarrow A^{-1} \cdot \underbrace{\begin{bmatrix} 2 & -5 \\ -2 & 4 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 7 \\ -6 \end{bmatrix}$

$A^{-1} = \dfrac{1}{8-10} \cdot \begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix}$

$= \dfrac{-1}{2} \cdot \begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2 & -5/2 \\ -1 & -1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -2 & -5/2 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow (x, y) = (1, -1)$

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ Adjugate of $A = \operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
* $A^{-1}_{2x2} = \dfrac{1}{\det(A)} \cdot \operatorname{adj}(A)$
* $A \cdot \vec{x} = \vec{b} \Rightarrow A^{-1} \cdot A \cdot \vec{x} = A^{-1} \cdot \vec{b} \Rightarrow \vec{x} = A^{-1} \cdot \vec{b}$ which is one way of solving linear equations with matrices