

Orthogonality and Change of Basis

29.05.2024

Orthogonal Complements

$$\vec{v} \cdot \vec{x} = 0 \Rightarrow \vec{v} \perp \vec{x}$$

How about orthogonality of subspaces?

$$\begin{array}{l} \text{R}^n \\ \text{can't be different} \end{array} \quad V = \left\{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \right\}$$

$$\begin{array}{l} \text{R}^n \\ \text{can be different} \end{array} \quad V^\perp = \left\{ \vec{x}_1, \vec{x}_2, \dots, \vec{x}_m \right\}$$

* "V is orthogonal complement of V^\perp (and vice versa)

* Every vector in one subspace is orthogonal to every vector in the other.

$$V^\perp = \left\{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \right\}$$

for every vector $\vec{v} \in V$

* If V is a subspace, V^\perp is also a subspace.

$$\begin{array}{l} * \vec{x}_1 \cdot \vec{v} = 0 \\ * \vec{x}_2 \cdot \vec{v} = 0 \end{array} \quad \vec{x}_1 \cdot \vec{v} + \vec{x}_2 \cdot \vec{v} = 0$$

$$\begin{array}{l} x_2 \cdot v = 0 \\ \Rightarrow (\vec{x}_1 + \vec{x}_2) \cdot \vec{v} = 0 \end{array} \quad \Rightarrow \text{closed under addition}$$

$$\begin{array}{l} * c \cdot \vec{x}_1 \cdot \vec{v} = 0 \\ \Rightarrow c \cdot \underbrace{(\vec{x}_1 \cdot \vec{v})}_{0} = 0 \end{array} \quad \Rightarrow \text{closed under multiplication}$$

$$* V = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right)$$

$$V^\perp = \left\{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = 0 \right\}$$

$$\vec{x} = (x_1, x_2, x_3)$$

$$\begin{aligned} & \Rightarrow 1 \cdot x_1 + 0 \cdot x_2 + 2 \cdot x_3 = 0 \\ & -1 \cdot x_1 + 1 \cdot x_2 + 3 \cdot x_3 = 0 \end{aligned} \quad \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 1 & 3 & 0 \end{array} \right]$$

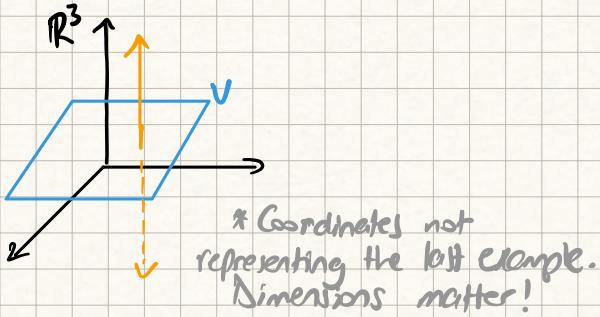
 * $V^\perp = \left\{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V \right\}$

* $(V^\perp)^\perp = V$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3 \\ x_2 + 5x_3 = 0 \Rightarrow x_2 = -5x_3 \end{array}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \Rightarrow V^T = \text{span} \left(\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \right)$$

* Conclusion: Our space is 3D (\mathbb{R}^3). Our subspace is a 2D plane. The orthogonal complement is a 1D line, which is orthogonal to the plane.



Qn12

① Find the orthogonal complement of $W = \text{span} \left(\begin{bmatrix} -1 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ -5 \end{bmatrix} \right)$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 0 \\ 2 & 0 & 3 & -5 & 0 \end{array} \right] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array} \right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array} \right]$$

$$\begin{aligned} \Rightarrow x_1 &= -2x_4 \\ x_3 &= 3x_4 \end{aligned} \Rightarrow W^T = \text{span} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right)$$

② Rewrite the orthogonal complement of $V = \begin{bmatrix} -2y+z \\ y \\ z \end{bmatrix}$.

$$V = \left\{ y \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mid y, z \in \mathbb{R}^3 \right\}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = -2x_3 \end{array}$$

$$V^\top = \text{span} \left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right)$$

③ $V = \text{span} \left(\begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -8 \\ 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -5 \\ -1 \\ -1 \end{bmatrix} \right) \Rightarrow V^\top = ?$

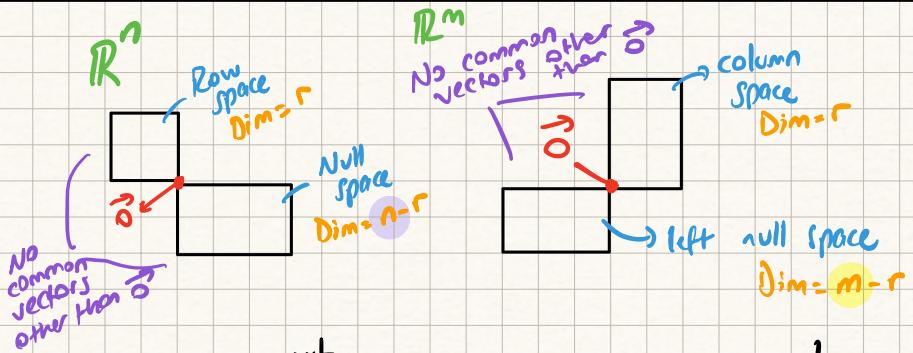
$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 4 & -8 & 8 & 0 \\ 1 & 3 & -5 & -1 & 0 \end{array} \right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 5 & -8 & -6 & 0 \end{array} \right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & 9 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & -8 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -14 & 0 \\ 0 & 0 & 1 & -8 & 0 \end{array} \right]$$

$$\Rightarrow \begin{array}{l} x_1 = -x_4 \\ x_2 = 14x_4 \\ x_3 = 8x_4 \end{array} \Rightarrow V^\top = \text{span} \left(\begin{bmatrix} -1 \\ 14 \\ 8 \\ 1 \end{bmatrix} \right)$$



Orthogonal complements of the fundamental spaces



$A_{m \times n}$

$$* N(A) = (C(A^\top))^\perp$$

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$$* \text{For } \mathbb{R}^n: V + V^\perp \\ \dim(V) + \dim(V^\perp) = n$$

\cong

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix}$$

→ what are the dimensions of the fundamental spaces?

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{array}{l} r = 3 \text{ (rank)} \\ m = 3 \text{ (# of rows)} \\ n = 4 \text{ (# of columns)} \end{array}$$

$$\text{Column space } \dim(C(A)) = r = 3 \quad \mathbb{R}^3 = \text{span}(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}, \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}, \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix})$$

$$\text{Row space } \dim(C(A^\top)) = r = 3 \quad \mathbb{R}^4 = \text{span}(\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \end{bmatrix})$$

$$\text{Null space } \dim(N(A)) = n - r = 1 \quad \mathbb{R}^4 = \text{span}(\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix})$$

$$\text{Left Null space } \dim(N(A^\top)) = m - r = 0 \quad \mathbb{R}^3 = \vec{0}_3 = (0,0,0)$$

* "en güzel kürk, boş denizde çekilir."

R

"row
space"

"null
space"

* "En güzel kürk, boş denizde çekilir."

$A_{m \times n}: \downarrow n \quad \text{row space} \quad \downarrow n \quad \text{null space}$

$$\Rightarrow C(A^\top) = N(A)^\perp, \dim(C(A^\top)) = r, \dim(N(A)) = n - r, C(A^\top), N(A) \subseteq \mathbb{R}^n$$

$$\Rightarrow C(A) = N(A^\top)^\perp, \dim(C(A)) = r, \dim(N(A^\top)) = m - r, C(A), N(A^\top) \subseteq \mathbb{R}^m$$

Quiz 2

① Find the dimensions of 4 fundamental spaces for $A = \begin{bmatrix} -1 & 3 & 5 & -1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -14 & -6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \Rightarrow r=3 \quad m=3 \quad n=4$$

$$\dim(C(A)) = 3 \text{ in } \mathbb{R}^3$$

$$\dim(C(A^\top)) = 3 \text{ in } \mathbb{R}^4$$

$$\dim(N(A)) = 1 \text{ in } \mathbb{R}^4$$

$$\dim(N(A^\top)) = 0 \text{ in } \mathbb{R}^3$$

② Dimensions of 4 fundamental spaces of $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow r=2 \quad m=3 \quad n=3$$

$$\Rightarrow \dim(C(A)) = 2, \text{ in } \mathbb{R}^3$$

$$\dim(C(A^\top)) = 2, \text{ in } \mathbb{R}^3$$

$$\dim(N(A)) = 1, \text{ in } \mathbb{R}^3$$

$$\dim(N(A^\top)) = 1, \text{ in } \mathbb{R}^3$$

(3) Find the dimensions of 4 fundamental spaces of

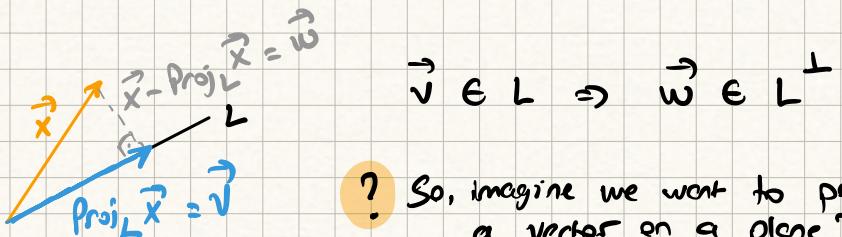
$$A = \begin{bmatrix} 2 & -3 & 6 & -5 & -6 \\ 4 & -5 & 12 & -11 & -14 \\ 2 & -2 & 6 & -6 & -8 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 3 & -5/2 & -3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -2 \end{bmatrix}$$

$$\Rightarrow \text{rank } A = 2 \quad \text{Dim}(C(A)) = 2 \text{ in } \mathbb{R}^3 \quad \text{Dim}(N(A)) = 3 \text{ in } \mathbb{R}^5$$

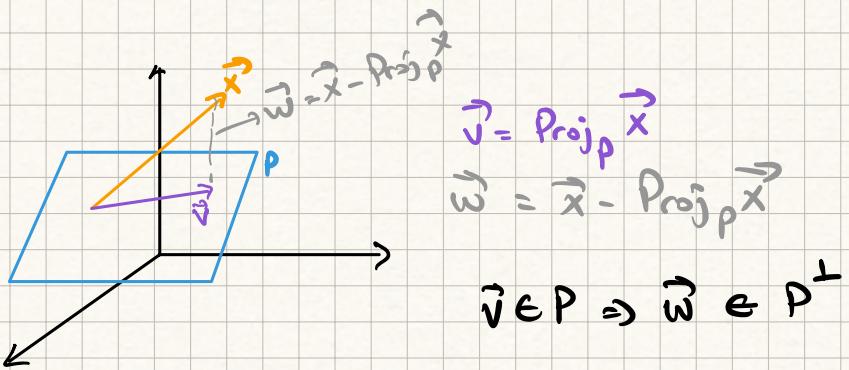
$$m = 3 \quad \text{Dim}(C(A^\top)) = 2 \text{ in } \mathbb{R}^5 \quad \text{Dim}(N(A^\top)) = 1 \text{ in } \mathbb{R}^3$$

$$n = 5$$

Projection onto the subspace



? So, imagine we want to project a vector on a plane?



* V in \mathbb{R}^n , A is the matrix of the column vectors that form the basis for V :

$$* \text{Proj}_V \vec{x} = A (A^\top A)^{-1} \cdot A^\top \vec{x}$$

! Note that we use this formula when A is not invertible. If it is, it means we are projecting \vec{x} to the all space \mathbb{R}^n , which gives us the \vec{x} itself.

* $\text{Proj}_V \vec{x} = A \cdot (A^\top A)^{-1} A^\top \vec{x}$

* If A is invertible, then $\text{Proj}_V \vec{x} = \vec{x}$

Ex:

$$V = \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \Rightarrow$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^T A)^{-1} = \left[\begin{array}{cc|cc} 6 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 0 & 2/11 & -1/11 \\ 0 & 1 & -1/11 & 6/11 \end{array} \right]$$

$$\Rightarrow A \cdot (A^T A)^{-1} \cdot A^T \cdot \vec{x} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2/11 & -1/11 \\ -1/11 & 6/11 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \cdot \vec{x}$$

$$= \begin{bmatrix} 3/11 & 4/11 \\ 2/11 & -1/11 \\ 3/11 & -7/11 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} 10/11 & 3/11 & -1/11 \\ 3/11 & 2/11 & 3/11 \\ -1/11 & 3/11 & 10/11 \end{bmatrix} \cdot \vec{x}$$

$$= \frac{1}{11} \begin{bmatrix} 10 & 3 & -1 \\ 3 & 2 & 3 \\ -1 & 3 & 10 \end{bmatrix} \vec{x}$$

Quiz 2

① Find the formula for $\text{Proj}_V \vec{x}$ for $V = \text{span} \left(\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \right)$

$$\Rightarrow A \cdot (A^T A)^{-1} \cdot A^T \cdot \vec{x} = ?$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix} \quad A^T = \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix}$$



$$A^T \cdot A = \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 20 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \left[\begin{array}{cc|cc} 8 & -4 & 1 & 0 \\ -4 & 20 & 0 & 1 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & -1/2 & 1/18 & 0 \\ 0 & 1 & 1/36 & 1/18 \end{array} \right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 10/72 & 1/36 \\ 0 & 1 & 1/36 & 1/18 \end{array} \right]$$

$$\Rightarrow A \cdot (A \cdot A^T)^{-1} \cdot A^T \cdot \vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10/72 & 1/36 \\ 1/36 & 1/18 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} \cdot \vec{x}$$

$$= \begin{bmatrix} -20/72 + 0 & -2/36 + 0 \\ 0 + 4/36 & 0 + 4/18 \\ -20/72 + 2/36 & -2/36 + 2/18 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} \cdot \vec{x}$$

$$= \begin{bmatrix} -5/18 & -1/18 \\ 1/9 & 2/9 \\ -2/9 & 1/18 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 10/18 & -4/18 & 10/18 - 2/18 \\ -2/9 & 8/9 & -2/9 + 4/9 \\ 4/9 & 4/18 & 4/9 + 2/18 \end{bmatrix}$$

$$= \begin{bmatrix} 5/9 & -2/9 & 4/9 \\ -2/9 & 8/9 & 2/9 \\ 4/9 & 2/9 & 5/9 \end{bmatrix} \cdot \vec{x} = \frac{1}{9} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \cdot \vec{x}$$



② If \vec{x} is a vector in \mathbb{R}^4 , find an expression for the projection of any \vec{x} onto the subspace S , if S is spanned by \vec{x}_1 and \vec{x}_2 .

$$\vec{x}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \text{ and } \vec{x}_2 = \frac{1}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$* \text{Proj}_S \vec{x} = A \cdot (A^T A)^{-1} \cdot A^T \cdot \vec{x}$$

$$A^T = \begin{bmatrix} 1/3 & 0 & -1/3 & 2/3 \\ 0 & 1/3 & 1/3 & -1/3 \end{bmatrix}, \quad A = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \\ -1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1/9 + 1/9 + 4/9 & -1/9 - 2/9 \\ -1/9 - 2/9 & 1/9 + 1/9 + 1/9 \end{bmatrix} = \begin{bmatrix} 6/9 & -3/9 \\ -3/9 & 3/9 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \left[\begin{array}{cc|cc} 2 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 3 \\ -1 & 1 & 0 & 3 \end{array} \right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 6 \end{array} \right]$$

$$\Rightarrow \left[\begin{array}{cc|cc} 1/3 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 \\ -1/3 & 1/3 & 2/3 & -1/3 \end{array} \right] \left[\begin{array}{cc|cc} 3 & 3 & 3 & 3 \\ 3 & 6 & 6 & 6 \end{array} \right] \cdot \left[\begin{array}{cccc} 1/3 & 0 & -1/3 & 2/3 \\ 0 & 1/3 & 1/3 & -1/3 \end{array} \right]$$



$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 0 & -1/3 & 2/3 \\ 0 & 1/3 & 1/3 & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & -1/3 \\ 1/3 & 0 & -1/3 & 2/3 \end{bmatrix} \cdot \vec{x} = \frac{1}{3} \cdot \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix} \cdot \vec{x}$$

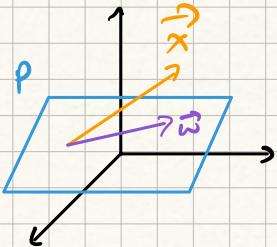
③ If \vec{x} is a vector in \mathbb{R}^3 , find an expression for the projection of any \vec{x} onto the subspace V .

$$V = \text{Span} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right)$$

* A is a square matrix, therefore :

$$\text{Proj}_V \vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \vec{x} = \vec{x}$$

Least squares solution



$\vec{w} = \text{Proj}_P \vec{x}$ is the vector that exists in P and also closest to \vec{x} .

* $A\vec{x} = \vec{b} \Rightarrow$ If there is no real solution for \vec{x} , we use the least squares solution.



Ex1

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{array}{l} x_1 = 2 \\ x_2 = 1 \end{array} \quad A \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{b}$$

* we get \vec{b} , exactly. which means \vec{b} is in $c(A)$.

* If there is no \vec{x} that solves $A \cdot \vec{x} = \vec{b}$ for a specific \vec{b} (because \vec{b} is not in $c(A)$), we look for the least squares solution, which is $A \cdot \vec{x}^* = \text{Proj}_{c(A)} \vec{b}$. The formula is

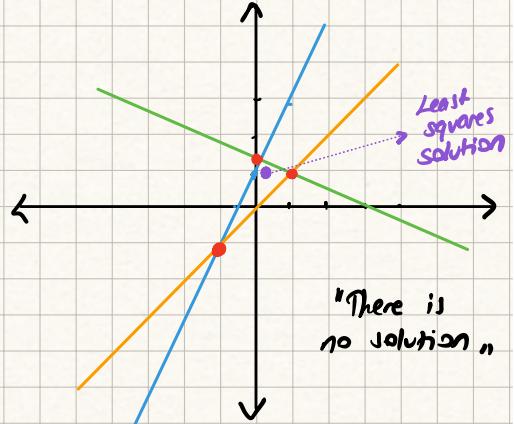
$$A^T A \vec{x}^* = A^T \vec{b}$$

Ex:

$$-2x + y = 1 \quad y = 2x + 1$$

$$-x + y = 0 \quad y = x$$

$$x + 2y = 3 \quad y = -\frac{1}{2}x + \frac{3}{2}$$



\Rightarrow Least squares solution:

$$\begin{array}{c} A \\ \xrightarrow{\quad} \\ \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad \begin{array}{c} \vec{B} \\ \xrightarrow{\quad} \end{array}$$

$$A^T A \cdot \vec{x}^* = A^T \cdot \vec{b}$$

$$\begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

* Least Square Solution: "There is no solution (\vec{x}) to this system, but this (\vec{x}^*) is as closest as we can get": $A^T A \vec{x}^* = A^T \vec{b}$

$$\Rightarrow \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 6 & -1 & 1 \\ -1 & 6 & 7 \end{array} \right]$$

$$= \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 35 & 43 \end{array} \right] = \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 1 & 43/35 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & 13/35 \\ 0 & 1 & 43/35 \end{array} \right]$$

$$\Rightarrow \vec{x}^* = \begin{bmatrix} 13/35 \\ 43/35 \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.23 \end{bmatrix}$$

The purple dot on the graph.
 The point where the sum of the distances to the other 3 points is the minimum.
 "There is no solution to this system, but this is the closest we can get."

Quiz ① Find the least squares solution to the system:

$$x - 2y = 5$$

$$3x + y = -6$$

$$-x - 2y = -2$$

$$\star A^T \cdot A \cdot \vec{x}^* = A^T \cdot \vec{b} \Rightarrow$$

$$\begin{bmatrix} 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & -2 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -6 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11 & 3 \\ 3 & 9 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} -11 \\ -12 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 11 & 3 & -11 \\ 1 & 3 & -4 \end{array} \right]$$

$$= \left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -11/10 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & -7/10 \\ 0 & 1 & -11/10 \end{array} \right] \Rightarrow \vec{x}^* = \begin{bmatrix} -7/10 \\ -11/10 \end{bmatrix}$$



2 Find the least squares solution to the system:

$$3x - 2y = -6$$

$$x - 5y = -5$$

$$x + y = 4$$

$$\begin{bmatrix} 3 & 1 & 1 \\ -2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & -5 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 11 & -10 \\ -10 & 30 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} -19 \\ 41 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & -41/10 \\ 11 & -10 & -19 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & -41/10 \\ 0 & 23 & 261/10 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -41/10 \\ 0 & 1 & 261/230 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 16/23 \\ 0 & 1 & 261/230 \end{bmatrix}$$

3 Find the least squares solution to the system:

$$x + 2y = -4$$

$$x - y = 3$$

$$y = 2$$

$$A^T A \cdot \vec{x}^* = A^T \vec{b}$$

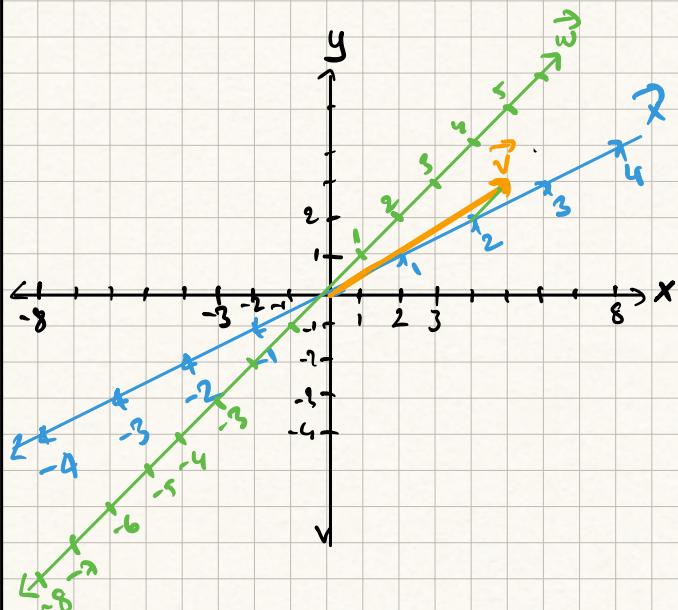
$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} -1 \\ -9 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 6 & -9 \\ 2 & 1 & -1 \end{array} \right]$$

$$= \left[\begin{array}{cc|c} 1 & 6 & -9 \\ 0 & -11 & 17 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 6 & -9 \\ 0 & 1 & -17/11 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & 3/11 \\ 0 & 1 & -17/11 \end{array} \right]$$

Coordinates in a new basis

* Why change basis? It might help solve problems quicker.



$$i = (1,0), j = (0,1)$$

$$B = \{i, j\}$$

$$\vec{v} = (5, 3) \Rightarrow \vec{v} = 5i + 3j$$

Let's change the basis to

$$\vec{x} = (2, 1), \vec{w} = (1, 1)$$

$$B = \{\vec{x}, \vec{w}\}$$

Now:

$$\vec{v} = 2\vec{x} + \vec{w}$$

$$[\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

* How to solve $[\vec{v}]_B$?

$\vec{x} \quad \vec{w}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot [\vec{v}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array} \right]$$

"change of basis matrix"

$$= \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right] \Rightarrow [\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$* A \cdot [\vec{v}]_B = \vec{v} .$$

$$\Rightarrow A^{-1} \cdot A \cdot [\vec{v}]_B = A^{-1} \cdot \vec{v}$$

$$\Rightarrow I \cdot [\vec{v}]_B = A^{-1} \cdot \vec{v}$$

$$\Rightarrow [\vec{v}]_B = A^{-1} \cdot \vec{v}$$

This means if we find the inverse of the change of basis matrix, we can easily calculate $[\vec{v}]_B$ for any \vec{v} .

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ? \quad \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow [\vec{v}]_B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \vec{v}$$

Quiz 2

① The vectors $\vec{v} = (2, 2, 3)$, $\vec{w} = (-6, 0, 2)$, and $\vec{z} = (2, 2, 5)$ form an alternate basis for \mathbb{R}^3 . Use them to transform

$\vec{x} = -2i + k$ into the alternate basis.

$$A = \begin{bmatrix} 2 & -6 & 2 \\ 2 & 0 & -2 \\ 3 & 2 & -5 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 3 & 2 & -5 & 1 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & 1 & -1 & 1/2 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 3 & -2 & 1 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{array} \right]$$



$$\textcircled{2} \quad A = \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & -10 & -10 \end{array} \right] = \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 1 & 1 \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right]$$

$$\textcircled{3} \quad A = \begin{bmatrix} 1 & -2 \\ -5 & 0 \end{bmatrix} \quad A^{-1} \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} \Rightarrow ?$$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & -10 & 5 & 1 \end{array} \right]$$

$$= \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -1/2 & -1/10 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1/5 \\ 0 & 1 & -1/2 & -1/10 \end{array} \right]$$

$$\Rightarrow \begin{bmatrix} 0 & -1/5 \\ -1/2 & -1/10 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Transformation
matrix for a basis

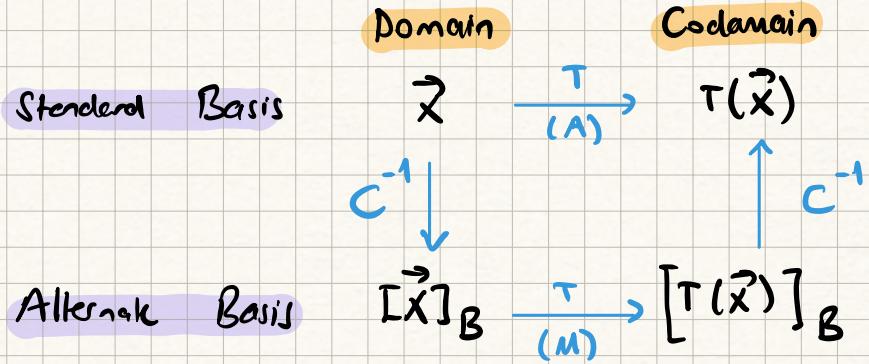
* $T(\vec{x}) = A \cdot \vec{x}$: we assume that T is in standard basis.

* $[T(\vec{x})]_B = M[\vec{x}]_B$: If we switch to alternate basis, the formula is the same.

* $C[\vec{x}]_B = \vec{x}$

*Change of
basis matrix*

* $M = C^{-1} \cdot A \cdot C$



$\underline{\underline{\text{E}}}: \quad T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \cdot \vec{x}$ $B = \text{Span} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right)$

$$[\vec{x}]_B = (1, 3) \quad \Rightarrow \quad M([\vec{x}]_B) = ?$$

$$C = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \quad C^{-1} = \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 3 & 0 & 1 \end{array} \right]$$

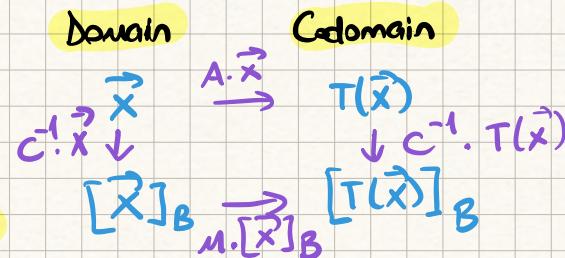
$$= \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 1/3 \end{array} \right] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & -2/3 \\ 0 & 1 & 0 & -1/3 \end{array} \right]$$

$$M = C^{-1} \cdot A \cdot C$$

$$= \left[\begin{array}{cc|cc} -1 & -2/3 & 1 & 0 \\ 0 & -1/3 & 2 & -2 \end{array} \right] \cdot \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 2 & -2 & 0 & 3 \end{array} \right]$$

$$= \left[\begin{array}{cc|cc} -1 & -2/3 & -1 & 2 \\ 0 & -1/3 & 2 & 10 \end{array} \right] = \left[\begin{array}{cc|cc} 7/3 & -26/3 \\ 2/3 & -10/3 \end{array} \right]$$

$$= \frac{1}{3} \cdot \begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix}$$



* $A = T(\vec{x})$

* $C = C(\vec{x})$

* $M = C^{-1} \cdot A \cdot C$

$$\Rightarrow [\vec{T}(\vec{x})]_B = M \cdot \vec{x} = \frac{1}{3} \cdot \begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -71/3 \\ -28/3 \end{bmatrix}$$

Quiz

Use the transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to transform $[\vec{x}]_B = (5, 4, -2)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \text{Span} \left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right)$$

```
[1] import numpy as np

[5] x = np.array([5, 4, -2])
    A = np.array([[−2, −2, 1], [1, 0, −2], [0, 1, 0]])
    C = np.array([[1, 0, 2], [−1, 1, 1], [1, −1, −2]])

[7] C_inv = np.linalg.inv(C)

[11] C_inv @ A @ C @ x
→ array([-15., -36., 12.])
```

