Introduction to Logarithms

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NOTE: * "Logarithms are not a part of Precalculus, but Algebra 2. I realized I was not too comfortable with this subject, so I added as a bonus section."

Intro to logarithms

* $2^4 = 2\cdot2\cdot2\cdot2 = 16$

$\underbrace{2}_{\text{base}}^{\overset{\text{exponent}}{x}} = \underbrace{16}_{\text{argument}} \Rightarrow x = 4$

$\log_{\underset{\text{base}}{2}} \overset{\text{exponent}}{16} = \overset{\text{argument}}{x}$ } "The power I need to raise 2 to get to 16"

* $\log_3 81 = ? \Rightarrow 3^x = 81 \Rightarrow x = 4$

* $\log_6 216 = ? \Rightarrow 6^x = 216 \Rightarrow x = 3$

* $\log_{100} 1 = x \Rightarrow 100^x = 1 \Rightarrow x = 0$

Evaluating logarithms (advanced)

* $\log_8 2 = x \Rightarrow 8^x = 2 \Rightarrow x = \frac{1}{3}$ $\quad (\sqrt[3]{8} = 2)$

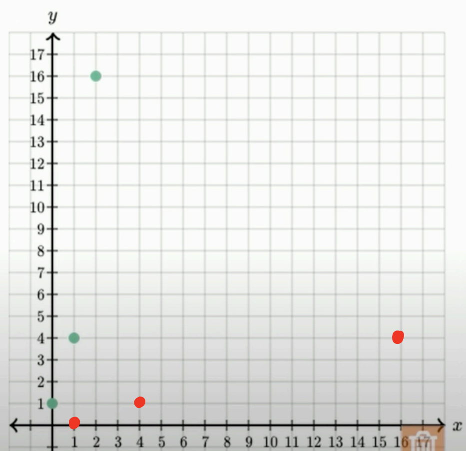
* $\log_2 \frac{1}{8} = x \Rightarrow 2^x = \frac{1}{8} \Rightarrow x = -3$

* $\log_8 \frac{1}{2} = x \Rightarrow 8^x = \frac{1}{2} \Rightarrow x = \frac{-1}{3}$

Relationship between exponentials and logs: graph

* The 3 points plotted below are on the graph of $y = b^x$.

Based only on these 3 points, plot the 3 corresponding points that must be on the graph of $y = \log_b x$ by clicking on the graph.



x	b^x
0	1
1	4
2	16

$y = \log_b x$

x	$\log_b x$
1	0
4	1
16	2

* Logarithms are the inverse operation of exponentiation. $2^x = 16 \Rightarrow \log_2 16 = x$

Relationship between expo.s and logs: tables

*

x	1.585	2.322	2.807	3.169
b^x	3.0	5.0	7.0	9.0

y	a	2	$2c$	$10d$
$\log_b y$	0	1.0	1.585	2.322

$a, b, c, d = ?$

$\log_b 2 = 1$

$\Rightarrow b^1 = 2 \Rightarrow b = 2$

$\log_b a = 0 \Rightarrow a = 1$

$\log_b 2c = 1.585 \Rightarrow b^{1.585} = 2c$

$\Rightarrow 3 = 2c \Rightarrow c = 3/2$

$\log_b 10d = 2.322 \Rightarrow b^{2.322} = 10d \Rightarrow 5 = 10d \Rightarrow d = \frac{1}{2}$

The Constant e and the Natural Logarithm

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e and compound interest

- 100 % interest rate for 1-year-term — 1\$ → 2\$
- 50 % " " " 6-months-" — 1\$ → 1.50\$ → 2.25 \$
- 25 % " " " 3-months-" — 1 → 1.25 → 1,5625 → 1,953125 → 2,44140625 \$

⋮

- 0,2739..% " " " 1-day -" — 1\$ → .. → .. → .. → → 2,7146\$
- Incredibly small interest rate for incredibly short payment terms → e = 2.718281828...... (Euler's number)

e as a limit

- $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \approx 2.71828182845...$ (never repeating!)

Evaluating natural logarithm with calculator

- $\log_e x = \ln x$
- Use a calculator to find $\log_e 67$ to the nearest thousandth.
- We type ln, then enter 67, and press the "=" sign.

  $\ln 67 \approx 4.205$

- Euler's number, e, is an irrational constant number which equals to 2.71828..... without repetition (just like π)

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

- $\log_e(x) = \ln(x)$ ("Natural log of x")

Properties of Logarithms

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Log of 1	* $\log_a 1 = 0$
Log of the same number as base	* $\log_a a = 1$
Product Rule	* $\log_a (m \cdot n) = \log_a m + \log_a n$
Quotient Rule	* $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
Power Rule	* $\log_a m^n = n \cdot \log_a m$
Change of base rule	* $\log_b a = \frac{\log_c a}{\log_c b}$
Equality Rule	* $\log_b a = \log_b c \Rightarrow a = c$
Number raised to log	* $a^{\log_a x} = x$
Other rules	* $\log_{b^n} a^m = \frac{m}{n} \log_b a$
	* $-\log_b a = \log_b \frac{1}{a} = \log_{\frac{1}{b}} a$
Example with multiple steps	* Simplify $\log_5 \frac{25^x}{y} = \log_5 25^x - \log_5 y$
	$= \log_5 5^{2x} - \log_5 y = (2x)(\overbrace{\log_5 5}^{=1}) - \log_5 y$
	$= 2x - \log_5 y$

* $\log_a(1) = 0$ * $\log_a(a) = 1$ * $\log_a(m \cdot n) = \log_a(m) + \log_a(n)$ * $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$

* $\log_a(m^n) = n \cdot \log_a(m)$ * $\log_b(a) = \frac{\log_c(a)}{\log_c(b)}$ * $a^{\log_a(x)} = x$

* $\log_{b^n}(a^m) = \frac{m}{n} \cdot \log_b(a)$ * $-\log_b a = \log_b \frac{1}{a} = \log_{\frac{1}{b}} a$

Example : Change of base

* $\log_2 50 = ?$ (we can't calculate this easily because 50 is not a rational power of 2.)

$$\Rightarrow \log_2 50 = \frac{\log(50)}{\log(2)}$$ = we can now use our calculator.

OR $$\Rightarrow \log_2 50 = \frac{\ln(50)}{\ln(2)}$$ = " " " " "

* $\log(a) \cdot \log_a(5) = ?$

$$= \log(a) \cdot \frac{\log(5)}{\log(a)} = \log(5)$$

* $$\frac{\log_c(6)}{\log(6)} = ? = \frac{\log(6)}{\log(c)} \cdot \frac{1}{\log(6)} = \frac{1}{\log(c)}$$

Solving Exponential Equations with Logs

* $10^{2t-3} = 7 \Rightarrow t = ? \Rightarrow \log_{10} 7 = 2t-3$

$$\Rightarrow \frac{\log(7)}{\log(10)} = 2t-3 \Rightarrow \log(7) = 2t-3 \Rightarrow t = \frac{\log(7)+3}{2}$$

* $5 \cdot 2^t = 1111 \Rightarrow t = ?$

$$\Rightarrow 2^t = \frac{1111}{5} \Rightarrow \log_2\left(\frac{1111}{5}\right) = t$$

$$\Rightarrow t = \frac{\log\left(\frac{1111}{5}\right)}{\log 2} \approx 7.796$$

Solving Exponential Models

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Real-world Problem: Medication Dissolve

* Carlos has taken an initial dose of a prescription medication.

The relationship between the elapsed time t, in hours, since he took the first dose, and the amount of medication, $M(t)$, in milligrams (mg), in his bloodstream is modeled by the following function.

$$M(t) = 20 \cdot e^{-0.8t}$$

In how many hours will Carlos have 1 mg of medication remaining in his bloodstream?
Round your answer, if necessary, to the nearest hundredth.

[ ] hours

$$\Rightarrow 1 = 20 \cdot e^{-0.8t} \Rightarrow e^{-0.8t} = \frac{1}{20} \Rightarrow \ln\left(\frac{1}{20}\right) = -0.8t$$

$$\Rightarrow t = \frac{\ln\left(\frac{1}{20}\right)}{-0.8} \approx 3.74 \text{ hours}$$

Real-World Problem: Bacteria Growth

* The bacteria in a Petri dish culture are self-duplicating at a rapid pace.

The relationship between the elapsed time t, in minutes, and the number of bacteria, $B(t)$, in the Petri dish is modeled by the following function.

$$B(t) = 10 \cdot 2^{\frac{t}{12}}$$

How many bacteria will make up the culture after 120 minutes?
Round your answer, if necessary, to the nearest hundredth.

[ ] bacteria

$$= 10 \cdot 2^{\frac{120}{12}}$$
$$= 10 \cdot 2^{10}$$
$$= 10240$$