

01 OPERATIONS ON ONE MATRIX

- * Using substitution, elimination, and graphing to solve a linear system gets very expensive as the systems gets larger. That's why we use matrices.
- * To solve a linear system, we change the matrix into reduced row-echelon form by using row operations (swap, addition, scalar multiplication).

02 OPERATIONS ON TWO MATRICES

- * Matrix addition:

- * Dimensions must be identical
- * Add corresponding entries
- * Is commutative and associative

- * Matrix subtraction:

- * Dimensions must be identical
- * Subtract corresponding entries
- * Not commutative nor associative

- * Scalar multiplication : Each entry is multiplied by the scalar.

- * Opposite matrices: $A + B = 0 \Rightarrow B = -A$

- * Matrix multiplication :

$$A_{m,n} \cdot B_{n,p} = C_{m,p} = \begin{bmatrix} R_1 C_1 & R_1 C_2 \\ R_2 C_1 & R_2 C_2 \end{bmatrix}$$

Diagram illustrating matrix multiplication: $A_{m,n}$ (m rows, n columns) is multiplied by $B_{n,p}$ (n rows, p columns) to produce $C_{m,p}$. The result is a matrix with m rows and p columns. The entry in the first row and first column of C is the dot product of Row 1 of A and Column 1 of B .

$$\begin{aligned} R_1 \cdot C_1 &= \text{Dot product of Row 1 (of } A \text{) and Column 1 (of } B \text{)} \\ &= R_{1,1} \cdot C_{1,1} + R_{1,2} \cdot C_{2,1} \end{aligned}$$

- * Matrix multiplication is associative and distributive, but not commutative.

* Zero Matrix: Matrix with only zero entries

$$O_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

* Identity Matrix:

* $A \cdot I = A$, $I \cdot A = A$

* Main diagonal: 1s, Other entries: 0s

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

03 MATRICES AS VECTORS

* Vector has 2 pieces of information contained within it:

① Direction

② Magnitude (Length): $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

* Row vector is a one-row matrix, Column vector is a one-column matrix

* Addition, subtraction, and scalar multiplication: Same as matrices

* Vector multiplication = Dot product

* Unit vector = A vector with length of 1.

$$\hat{u} = \frac{1}{\|\vec{v}\|} \cdot \vec{v}$$

* Standard Basis vectors = For \mathbb{R}^n , n number of vectors where each vector has a single entry equal to 1, and the other entries equal to 0.

* Linear combination: The sum of scaled vectors.

* Span of a vector set: The collection of all vectors which can be represented by linear combinations of the set.

* Linear Independence: A set is linearly independent if none of the vectors in the set can be represented by a linear combination of the other vectors in the set. $(c_1, c_2, \dots, c_n) = (0, 0, \dots, 0)$ is the only solution to: $c_1 \cdot v_1 + c_2 \cdot v_2 + \dots + c_n \cdot v_n = \vec{0}_n$

* A set of vectors (w) is a subspace of V if:

① $\vec{0} \in w$, and ② $\vec{u}, \vec{v} \in w: \vec{u} + \vec{v} \in w$, and ③ $\vec{u} \in w, k \in \mathbb{R}: k \cdot \vec{u} \in w$

- * The span of a vector set is all the linear combinations of that set.
A span is always a subspace.
- * A vector set is a basis for a space if:
 - (1) Spans the space, and
 - (2) Is linearly independent

04 DOT PRODUCTS AND CROSS PRODUCTS

- * Dot product = $\vec{a} \cdot \vec{b} = [a_1, a_2, \dots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n$
 - * Is commutative, distributive, and associative.
 - * Measures how similar the dotted vectors are.
 - * Cross product : $\vec{a} \times \vec{b} = i(a_2 \cdot b_3 - a_3 \cdot b_2) - j(a_1 \cdot b_3 - a_3 \cdot b_1) + k(a_1 \cdot b_2 - a_2 \cdot b_1)$
 - * It's a vector that's orthogonal to the two vectors crossed.
 - * Length of the cross product vector measures how different the crossed vectors are
 - * Angle between vectors :
 - * $\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos \theta$
 - * If $\vec{a} \perp \vec{b}$: dot product is 0.
 - * If \vec{a} and \vec{b} point the same direction : dot product is maximized
 - * If \vec{a} and \vec{b} point the opposite direction : dot product is minimized
 - * $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin \theta$
 - * If $\vec{a} \perp \vec{b}$: length of the cross product is maximized.
 - * If \vec{a} and \vec{b} are collinear, length of the cross product is 0.
 - * Cauchy-Schwarz Inequality : $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \cdot \|\vec{b}\|$
 - * Vector Triangle Inequality : $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$
 - * A plane is the set of all vectors that are perpendicular to one given normal vector, which is the vector that is perpendicular to the plane.
- * For both equations, equality means \vec{a} and \vec{b} are linearly dependent.

* Normal vector = $\vec{n} = (A, B, C)$

* Plane = $Ax + By + Cz = D$

* Plane = $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, where (x_0, y_0, z_0) is a point in the plane.

05 MATRIX-VECTOR PRODUCTS

* For $A \cdot \vec{x}$, \vec{x} must be a column vector. For $\vec{x} \cdot A$, \vec{x} must be a row vector.

* Null Space = All the vectors that satisfy $A \cdot \vec{x} = \vec{0}$

* If $N(A)$ has only the zero vector, the columns of A are linearly dependent (and vice versa).

* Column Space = The linear combinations (span) of the column vectors of A . If the columns of A are linearly independent, then $C(A)$ is also the basis.

* The general solution of $A \cdot \vec{x} = \vec{b}$ is $\vec{x}_p + \vec{x}_n$, where:

* $\vec{A} \cdot \vec{x}_n = \vec{0}$ (Called complementary solution)

* $\vec{A} \cdot \vec{x}_p = \vec{b}$ (For a particular \vec{b} . Called particular solution)

* Dimension of a vector space = Number of basis vectors

* Nullity = Dimension of the null space = Number of free columns

* Rank = Dimension of the column space = Number of pivot columns.

06 TRANSFORMATIONS

* Transformations map vectors from one space (Domain) to specific vectors (Range) in another space (Codomain).

* Subset: The vector set that's being transformed

* Preimage: The vector set before being transformed.

* Image: The vector set after transformation is applied.

* Kernel of T : $\text{Ker}(T) = \left\{ \vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0}_n \right\}$

* T is linear if

$$\textcircled{1} \quad T(\vec{a} + \vec{b}) = T(\vec{a}) + T(\vec{b}), \text{ and}$$

$$\textcircled{2} \quad T(c \cdot \vec{a}) = c \cdot T(\vec{a})$$

* Rotation Matrices

$$* \text{Rot}_{\theta} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$* \text{Rot}_{\text{around } y} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$* \text{Rot}_{\text{around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$* \text{Rot}_{\text{around } z} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* $S(\vec{x}) = A \cdot \vec{x}, T(\vec{x}) = B \cdot \vec{x} \Rightarrow$

$$* S(\vec{x}) + T(\vec{x}) = (S + T)(\vec{x}) = (A + B)\vec{x}$$

$$* c \cdot T(\vec{x}) = c \cdot (B \cdot \vec{x}) = (c \cdot B)\vec{x}$$

* The projection of \vec{v} onto L where L is given as scaled version of \vec{x} is :

$$\textcircled{1} \quad \text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \cdot \vec{x} \quad \textcircled{2} \quad \text{Proj}_L(\vec{v}) = (\vec{v} \cdot \hat{u}) \cdot \hat{u} \quad \textcircled{3} \quad \text{Proj}_L(\vec{v}) = \begin{bmatrix} \hat{u}_1^2 & \hat{u}_1 \cdot \hat{u}_2 \\ \hat{u}_1 \cdot \hat{u}_2 & \hat{u}_2^2 \end{bmatrix}$$

* Projections are linear transformations. Therefore :

$$\textcircled{1} \quad \text{Proj}_L(\vec{a} + \vec{b}) = \text{Proj}_L(\vec{a}) + \text{Proj}_L(\vec{b}), \text{ and}$$

$$\textcircled{2} \quad \text{Proj}_L(c \cdot \vec{a}) = c \cdot \text{Proj}_L(\vec{a})$$

* $T \circ S(\vec{x}) = \vec{x}$ is transformed first by $S(\vec{x}) = A \cdot \vec{x}$, then $T(\vec{x}) = B \cdot \vec{x}$
= a composition of linear transformations S and T
= $B \cdot A \cdot \vec{x}$

07 INVERSES

- * For a transformation to be invertible, it must be surjective and injective.
 - * Surjective (onto) : Every vector \vec{b} is being mapped to.
 - * Injective (one-to-one) : Every \vec{a} maps to a unique \vec{b} .
- * If # rows > # columns \Rightarrow Not surjective
If # rows < # columns \Rightarrow Not injective
- * Inverse transformations are linear. Therefore :
 - ① $T^{-1}(\vec{a} + \vec{b}) = T^{-1}(\vec{a}) + T^{-1}(\vec{b})$, and
 - ② $T^{-1}(c \cdot \vec{a}) = c \cdot T^{-1}(\vec{a})$
- * Let $T(\vec{x}) = M \cdot \vec{x}$:
 - * If $\text{rref}(M) = I$, then T is invertible.
 - * If $|M| \neq 0$, then T is invertible (non-singular)
- * Two methods to find M^{-1} :
 - ① $[M \mid I \mid I \mid M^{-1}]$
 - ② $M^{-1} = \frac{1}{|M|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- * We can use inverse matrices to solve linear equations: $\vec{x} = A^{-1} \cdot \vec{b}$
This method allows us to calculate \vec{x} for different \vec{b} 's easily.

08 DETERMINANTS

- * Rule of Sarrus for $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow |A| = a \cdot e \cdot i + b \cdot f \cdot g + c \cdot d \cdot h - a \cdot f \cdot h - b \cdot d \cdot i - c \cdot e \cdot g$
- * Cramer's Rule : $a_1 \cdot x + b_1 \cdot y = d_1, a_2 \cdot x + b_2 \cdot y = d_2 \Rightarrow$ $x = \frac{Dx}{D}, y = \frac{Dy}{D}$, with $D \neq 0$, where
 $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, Dx = \begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}, Dy = \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}$

* Determinant Modification Rules

- (1) Multiplying a row of the matrix by a scalar requires that we multiply the determinant by the same scalar.
- (2) When two rows of a matrix are swapped, the determinant must be multiplied by -1

- * Upper Triangular Matrix: All entries below main diagonal are zeros.
- * Lower Triangular Matrix: All entries above main diagonal are zeros.
- * $\text{Det}(L) = \text{Det}(U) =$ product of the entries in the main diagonal.
- * $\text{Area}_F = \text{Det}(A)$ (F is the figure created by the column vectors of A)
- * $T: F \rightarrow G \Rightarrow \text{Area}_G = |\text{Area}_F \cdot \text{Det}(T)|$

09 TRANSPOSES

- * $A_{m \times n} \Rightarrow A_{n \times m}^T$
- * $|A| = |A^T|$
- * $(A^T)^T = A$
- * $(A \cdot B)^T = B^T \cdot A^T$
- * $(A+B)^T = A^T + B^T$
- * $(A^T)^{-1} = (A^{-1})^T$
- * $A \cdot A^T$ is invertible if $\text{Det}(A) \neq 0$
- * The row space of A is $C(A^T)$
- * The left null space of A is $N(A^T)$
- * A can be factorized as $A = L \cdot U$, where L is the lower triangular matrix, and U is upper triangular matrix.

10 ORTHOGONALITY AND CHANGE OF BASIS

- * $V^\perp = \{x \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V\}$
- * $(V^\perp)^\perp = V$

* "En güzel kirek, bors denizde celulir."
 $A_{m \times n} \Downarrow$ row space \Downarrow null space

- * $C(A^T) = N(A)^\perp$, $\dim(C(A^T)) = r$, $\dim(N(A)) = n-r$, $C(A^T)$ and $N(A) \in \mathbb{R}^n$
- * $C(A) = N(A^T)^\perp$, $\dim(C(A)) = r$, $\dim(N(A^T)) = m-r$, $C(A)$ and $N(A^T) \in \mathbb{R}^m$

* $\text{Proj}_V \vec{x} = A \cdot (A^T \cdot A)^{-1} \cdot A^T \cdot \vec{x}$

* If A is invertible, then $\text{Proj}_V \vec{x} = \vec{x}$

* Least Square Solution: "There is no solution (\vec{x}) to this system, but this (\vec{x}^*) is as closest as we can get: * $A^T A \vec{x}^* = A^T \vec{b}$

