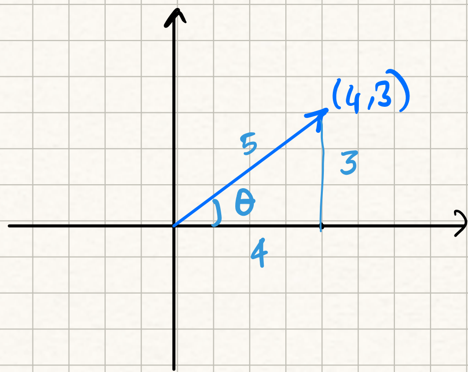
# Vector Algebra

11.03.2025

## Norm and direction




$$L_1 = |(a,b)|_1 = |a| + |b| = 7$$

$$L_2 = |(a,b)|_2 = \sqrt{a^2+b^2} = 5$$

$$\tan(\theta) = \frac{3}{4}$$

$$\theta = \arctan(3/4) = 0.64 = 36.87^\circ$$

## Notation

Row Vector

$$x = [x_1 \ x_2 \ \dots \ x_n]$$

Column Vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

## Vector Operations

$$u = (4,1) \qquad u+v = (5,4)$$

$$v = (1,3) \qquad u-v = (3,-2)$$

$$x = [x_1 \ x_2 \ \dots \ x_n] \qquad y = [y_1 \ y_2 \ \dots \ y_n]$$

$$x+y = [x_1+y_1 \ \ x_2+y_2 \ \dots \ x_n+y_n]$$

$$x-y = [x_1-y_1 \ \ x_2-y_2 \ \dots \ x_n-y_n]$$

$$\lambda . x = [\lambda x_1 \ \ \lambda x_2 \ \dots \ \lambda x_n]$$

* Every vector must have a magnitude and direction. Magnitude can be calculated in 2 different ways, L1 norm (aka taxicab) and L2 norm (Euclidean).
* We can add and subtract vectors from each other and multiply a vector with a scaler as if we're doing the math operations on regular numbers.

**The Dot Product**

$x = [x_1 \ x_2 \ \dots \ x_n]$ $\qquad$ $y = [y_1 \ y_2 \ \dots \ y_n]$

$$x \cdot y = (x_1 \cdot y_1) + (x_2 \cdot y_2) + \dots + (x_n \cdot y_n)$$

$$\langle x, y \rangle = x \cdot y$$

**Transposing a vector / matrix**

$$x = [1 \ 4 \ 8] \Rightarrow x^T = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

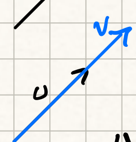
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

**Geometric Dot Product**

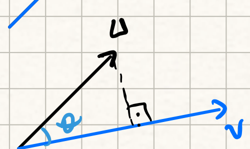
* Orthogonal vectors have dot product 0

$$\langle u, v \rangle = 0$$


$$\langle u, u \rangle = |u|^2 = |u| \cdot |u|$$




$$\langle u, v \rangle = |u| \cdot |v|$$




$$\langle u, v \rangle = |u| \cdot |v| \cdot \cos(\theta)$$

$$\langle u, v \rangle > 0 \Rightarrow 0 < \theta < 90$$

$$\langle u, v \rangle < 0 \Rightarrow 90 < \theta < 180$$

The dot product is a mathematical operation that takes two vectors and returns a scalar, representing the projection of one vector onto another, or, equivalently the sum of the products of their corresponding components.

Multiplying a matrix by a vector

## Equations as dot product

$a + b + c = 10$

$$[1\ 1\ 1] \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 10$$

$a + 2b + c = 15$

$$[1\ 2\ 1] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 15$$

$a + b + 2c = 12$

$$[1\ 1\ 2] \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 12$$

$$[1\ \ 1\ \ 1] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 10$$

$$[1\ \ 2\ \ 1] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 15$$

$$[1\ \ 1\ \ 2] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 12$$

$$\underset{3\times3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}} \cdot \underset{3\times1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix}}$$

$3\times\mathbf{3} \longleftrightarrow{=} \mathbf{3}\times1$

Number of colums of the matrix has to match the number of rows of the vector.

$$\underset{4\times3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}} \cdot \underset{3\times1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \underset{4\times1}{\begin{bmatrix} 10 \\ 15 \\ 12 \\ 13 \end{bmatrix}}$$

To multiply a matrix and a vector, # of columns of the matrix must be equal to the # of rows of the vector. The i-th element of the resulting vector is the dot product of the i-th row of the matrix and the input vector.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

# Linear Transformations

11.03.2025

Linear Transformation

1. Lines must remain lines
2. Origin must remain fixed in place.

= function

input vector $\rightarrow \boxed{f} \rightarrow$ output vector

Linear transformation as a matrix-vector multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \cdot \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$$

$$= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$\begin{bmatrix} a \\ c \end{bmatrix}$ is where $\hat{\imath}$ lands

$\begin{bmatrix} b \\ d \end{bmatrix}$ is where $\hat{\jmath}$ lands

$\begin{bmatrix} x \\ y \end{bmatrix}$ is the input vector

"We applied a linear transformation on the [x y] vector. The linear transformation we applied can be shown as a matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ tells us that:

1. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ goes to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ goes to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Linear transformations as matrix multiplication

Let's add a new transformation to the new basis

$$\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

* A linear transformation is a function between two vector spaces that preserves the operations of vector addition and scalar multiplication.
  * $T(u+v) = T(u) + T(v)$
  * $T(c \cdot u) = c \cdot T(u)$
* Linear transformation maintains the "straightness" of lines and keep the origin fixed.
* We can show linear transformations as matrices. Linear Transformations can be applied on vectors as well as matrices.

This means $\Rightarrow$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}}_{\text{SECOND! Transformation Matrix}} \cdot \underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{FIRST! Transformation Matrix}} = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}$$

$\Updownarrow$

How do we perform matrix multiplication?

$$\begin{bmatrix} \boxed{2 \quad -1} \\ \boxed{0 \quad 2} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix}3\\1\end{matrix}} & \boxed{\begin{matrix}1\\2\end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{2 \cdot 1}\ \boxed{\begin{matrix}3\\1\end{matrix}} & \boxed{2 \quad -1}\ \boxed{\begin{matrix}1\\2\end{matrix}} \\ \boxed{0 \quad 2}\ \boxed{\begin{matrix}3\\1\end{matrix}} & \boxed{0 \quad 2}\ \boxed{\begin{matrix}1\\2\end{matrix}} \end{bmatrix}$$

# of rows don't need to match.
# of columns of matrix 1 has to match # of rows of the matrix 2

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 5 & 2 & 0 \\ -2 & 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 21 & -6 \\ 1 & -3 & 8 & -4 \end{bmatrix}$$

$2 \times 3 \quad = \quad 3 \times 4 \qquad\qquad 2 \times 4$

What happens if we use an identity matrix as the transformation matrix?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$$

What is an inverse matrix?

$$[\text{Matrix}] \cdot \begin{bmatrix}\text{Inverse} \\ \text{Matrix}\end{bmatrix} = \begin{bmatrix}\text{Identity} \\ \text{Matrix}\end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$3a + 1c = 1$$
$$3b + 1d = 0$$
$$1a + 2c = 0$$
$$1b + 2d = 1$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 1 & 0 & 1 \\ 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{array}\right]$$

$a = 2/5$

$b = -1/5$

$c = -1/5$

$d = 3/5$

Matrices and linear transformations in Natural Language Processing

| Spam | Lottery | Win |
|---|---|---|
| Yes | 1 | 1 |
| Yes | 2 | 1 |
| No | 0 | 0 |
| Yes | 0 | 2 |
| No | 0 | 1 |
| No | 1 | 0 |
| Yes | 2 | 2 |
| Yes | 2 | 0 |
| Yes | 1 | 2 |

Scores:

Lottery = ...... points / occurance

Win = ...... points / occurance

Rule:

If the number of points of the sentence is bigger than ......, then the email is spam.

Goal:

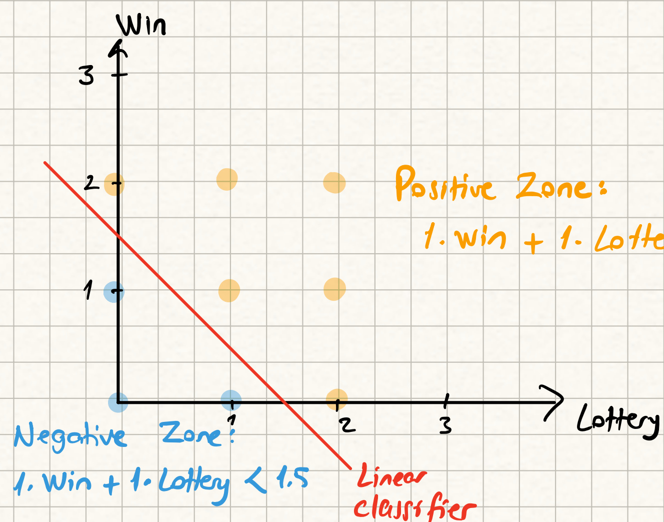
Find the best points and threshold.

Inverse matrix as a linear transformation transforms the plane to its original state. Only non-singular matrices have inverses. They are also called invertible.

a) Lottery: 1 pt , Win: 2 pts , Threshold: 3 pts.

b) Lottery: 1 pt , Win: 1 pt , Threshold: 1.5 pts.

c) Lottery: 2 pts , Win: 3 pts , Threshold: 1.5 pts.




One-Layer Neural Network Model

Input Data

$$\begin{bmatrix} \#\text{word Lottery} & \#\text{word Win} \end{bmatrix}$$

Model

$$\begin{bmatrix} \text{Pts. for Lottery} \\ \text{Pts. for win} \end{bmatrix} > \text{Threshold ?}$$

$$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 > 1.5 \quad \text{(spam)}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \not> 1.5 \quad \text{(not spam)}$$

Data · Model = Product > 1.5 ? Target

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 0 \\ 0 & 2 \\ 0 & 1 \\ 1 & 0 \\ 2 & 2 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \\ 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} > 1.5 \ ? \quad \begin{bmatrix} \text{YES} \\ \text{YES} \\ \text{NO} \\ \text{YES} \\ \text{NO} \\ \text{NO} \\ \text{YES} \\ \text{YES} \\ \text{YES} \end{bmatrix}$$

$1 \cdot Win + 1 \cdot Lottery > 1.5$

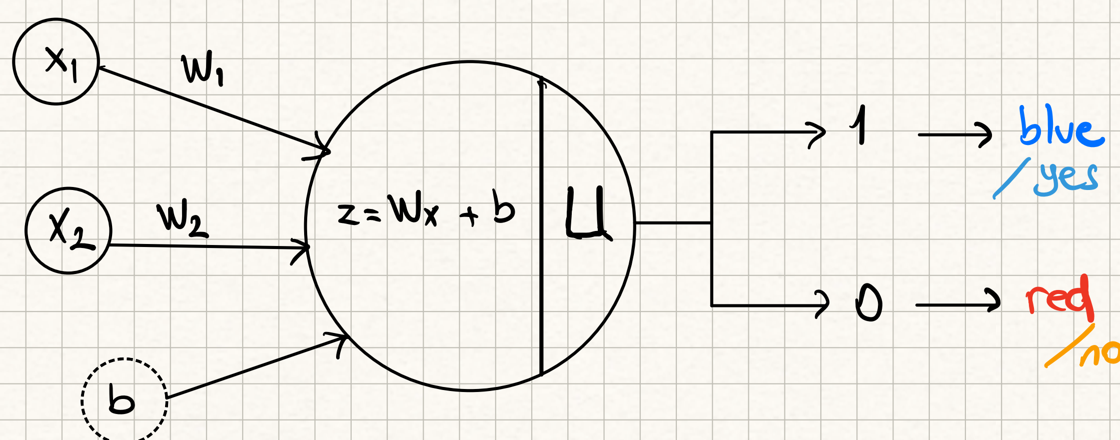
$1 \cdot Win + 1 \cdot Lottery - \underbrace{1.5}_{BIAS} > 0$

"AND" logic with linear tranformation.

| AND | x | y |
|---|---|---|
| No | 0 | 0 |
| No | 1 | 0 |
| No | 0 | 1 |
| yes | 1 | 1 |

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{Model}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}}_{\text{Dot Product}} > 1.5\,?$$

How does a single perceptron neural network work?




A perceptron is the simplest type of artificial neural network, designed to model a single neuron. It takes multiple inputs, each with an associated weight; it calculates a weighted sum of these inputs; it applies an activation function (typically a step function) to the weighted sum; and outputs a binary value indicating the category the input belongs to. Perceptrons use the concept of a linear transformation, which can be expressed as a matrix multiplication, to calculate its output and make decisions.