Orthonormal Bases and Gram-Schmidt

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What's an orthonormal set?

* Orthonormal set:

  (1) All vectors are normal, and
  (2) All vectors are orthogonal

- orthogonal $\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = 0$
- normal $\Rightarrow \vec{v}_1 \cdot \vec{v}_1 = \|v_1\|^2 = 1$

Orthonormal Bases

Ex:

$$\left.\begin{aligned} \vec{v}_1 &= (0,0,-1) \\ \vec{v}_2 &= \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) \\ \vec{v}_3 &= \left(\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\right) \end{aligned}\right\} \quad \begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &= 0 \qquad & \vec{v}_1 \cdot \vec{v}_1 &= 1 \\ \vec{v}_1 \cdot \vec{v}_2 &= 0 & \vec{v}_2 \cdot \vec{v}_2 &= 1 \\ \vec{v}_2 \cdot \vec{v}_3 &= 0 & \vec{v}_3 \cdot \vec{v}_3 &= 1 \end{aligned}$$

* Orthogonal Matrix: Square, and columns form an orthonormal set.
* Orthonormal Matrix: Rectangular, and columns form an orthonormal set

Orthogonal Matrix

$$\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1 & 0 & 0 \end{bmatrix} \cdot [\vec{x}]_B = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} \Rightarrow [\vec{x}]_B = ?$$

No need for rref of augmented matrix!

$$[\vec{x}]_B = \left(\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} \right)$$

$$= \left(3, 7/\sqrt{2}, 3/\sqrt{2}\right) = \begin{bmatrix} 3 \\ 7/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}$$

* Orthonormal Basis: Every vector is normal (length = 1) and orthogonal to every other vector

* Orthogonal Matrix: Square matrix whose columns form an orthonormal set.
* Orthonormal Matrix: Rectangular matrix whose columns form an orthonormal set.

Quiz ① Which of the vector sets is orthonormal?

$$\vec{v}_1 = \left(\frac{-1}{2}, \frac{1}{2}, \frac{-1}{\sqrt{2}}\right), \quad \vec{v}_2 = \left(\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}\right), \quad \vec{v}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\|\vec{v}_1\|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1 \qquad \|\vec{v}_2\|^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1 \qquad \vec{v}_3 = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

$$\vec{v}_2 \cdot \vec{v}_3 = 0$$

② Convert $\vec{x} = \begin{bmatrix} -12 \\ 6 \end{bmatrix}$ from the standard basis to the alternate basis $B = \text{Span}\left(\begin{bmatrix} 5/6 \\ -\frac{\sqrt{11}}{6} \end{bmatrix}, \begin{bmatrix} \sqrt{11}/6 \\ 5/6 \end{bmatrix} \right)$

$$[\vec{x}]_B = \begin{bmatrix} -10 - \sqrt{11} \\ 5 - 2\sqrt{11} \end{bmatrix}$$

③ Convert $\vec{x} = (\sqrt{66}, \sqrt{6}, \sqrt{11})$ from the standard basis to the alternate basis $B = \text{Span}\left(\begin{bmatrix} 4/\sqrt{66} \\ -7/\sqrt{66} \\ 1/\sqrt{66} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{11} \\ -1/\sqrt{11} \\ -3/\sqrt{11} \end{bmatrix} \right)$

$$\Rightarrow [\vec{x}]_B = \begin{bmatrix} 4 + \frac{-7\sqrt{6}}{\sqrt{66}} + \frac{\sqrt{11}}{\sqrt{66}} \\ \frac{-2\sqrt{66}}{\sqrt{6}} - 1 + \frac{\sqrt{11}}{\sqrt{6}} \\ \frac{-\sqrt{66}}{\sqrt{11}} - \frac{\sqrt{6}}{\sqrt{11}} - 3 \end{bmatrix} = \begin{bmatrix} \frac{4\sqrt{66} - 7\sqrt{6} + \sqrt{11}}{\sqrt{66}} \\ \frac{-2\sqrt{66} - \sqrt{6} + \sqrt{11}}{\sqrt{6}} \\ \frac{-\sqrt{66} - \sqrt{6} - 3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$$

Projection onto an orthonormal basis

* $\text{Proj}_V \vec{x} = A.(A^T A)^{-1}. A^T. \vec{x}$

* If A is orthonormal, then $(A^T . A)^{-1} = I$, therefore

* $\text{Proj}_V \vec{x} = A . A^T . \vec{x}$

Ex:

$\vec{V_1} = (0, 0, 1)$

$\vec{V_2} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$

$\{\vec{V_1}, \vec{V_2}\}$ is orthonormal.

$$A = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$\Rightarrow \text{Proj}_V \vec{x} = A . A^T . \vec{x} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \vec{x}$$

Quiz ① Find the projection of $\vec{x} = (-15, -75, 25)$ on to the subspace $V = \text{Span}\left(\begin{bmatrix} 5/\sqrt{50} \\ -3/\sqrt{50} \\ 4/\sqrt{50} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \right)$

```python
import sympy as sym


V = sym.Matrix(
    [
        [5 / sym.sqrt(50), -3 / sym.sqrt(50), 4 / sym.sqrt(50)],  # type: ignore
        [1 / sym.sqrt(6), -1 / sym.sqrt(6), -2 / sym.sqrt(6)],  # type: ignore
    ]
)
A = V.T

x = sym.Matrix([-15, -75, 25])

print(A @ A.T @ x)
```

$$= \begin{bmatrix} 80/3 \\ -50/3 \\ 50/3 \end{bmatrix}$$

* If A is orthonormal : $\text{Proj}_V \vec{x} = A . A^T . \vec{x}$

Gram-Schmidt process for change of basis

GOAL

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$
$$= \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$
$$= \text{Span}(\vec{u}_1, \vec{u}_2, \vec{v}_3)$$
$$= \text{Span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$$

How?

STEP 1: Normalize the first vector.

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

STEP 2-n: Change the next vector to another vector that is orthogonal to all vectors before it. And then normalize it.

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1) \cdot \vec{u}_1$$
$$\vec{w}_3 = \vec{v}_3 - \left[(\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2) \cdot \vec{u}_2\right]$$
$$\vdots$$
$$\vec{w}_n = \vec{v}_n - \left[(\vec{v}_n \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_n \cdot \vec{u}_2) \cdot \vec{u}_2 + \dots + (\vec{v}_n \cdot \vec{u}_{n-1}) \cdot \vec{u}_{n-1}\right]$$

EX:

$$V = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right)$$

$$\vec{u}_1 = \frac{(-1,1,0)}{\sqrt{2}} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right) \cdot \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + 2/\sqrt{2} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \vec{u}_2 = \frac{(1,1,-1)}{\sqrt{3}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \Rightarrow \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & -1/\sqrt{3} \end{bmatrix}$$

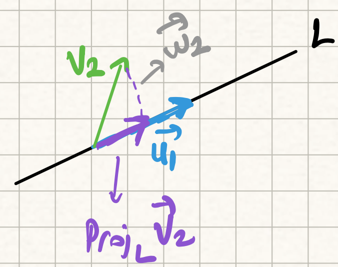
orthonormal matrix

* Gram-Schmidt Process is an iterative process that allows us to change a basis to an orthonormal basis. For the algorithm, check the first half of this page.

? How come: $\text{Span}\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-1\end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}\right)$

They don't point the same direction?

A The idea of projection!



* We know that $\vec{u}_1 \in V$ because it's just the normalized $\vec{v}_1$
* We know that $\vec{v}_2 \in V$, because it's defined like that ☺
* We know that $\vec{w}_2 \in V$ because it connects $\vec{u}_1$ and $\vec{v}_2$, which means it's on the same plane, even though it points a different direction than $\vec{v}_2$.
* We know that $\vec{u}_2 \in V$ because it's just normalized $\vec{w}_2$

Quiz ① The subspace V is a plane in $\mathbb{R}^3$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-3\\1\end{bmatrix}\right)$$

$$\vec{u}_1 = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix} \quad \vec{w}_2 = \vec{v}_2 - \left(\begin{bmatrix}-1\\-3\\1\end{bmatrix} \cdot \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}\right) \cdot \vec{u}_1$$

$$= \vec{v}_2 - \underbrace{\left(\frac{2}{3} - 2 + \frac{1}{3}\right)}_{(-1)} \cdot \vec{u}_1 = \begin{bmatrix}-1\\-3\\1\end{bmatrix} + \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix} = \begin{bmatrix}-5/3\\-7/3\\4/3\end{bmatrix}$$

$$\|w_2\| = \sqrt{\frac{25}{9} + \frac{49}{9} + \frac{16}{9}} = \sqrt{\frac{90}{9}} = \sqrt{10} \Rightarrow \vec{u}_2 = \left(\frac{-5}{3}\sqrt{10}, \frac{-7}{3}\sqrt{10}, \frac{4}{3}\sqrt{10}\right)$$

(2) $V = \text{Span}\left(\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}-1\\-2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right)$ to orthonormal basis?

$$\vec{u_1} = \left(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right) \qquad \vec{w_2} = \vec{v_2} - \left(\begin{bmatrix}-1\\-2\\2\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{5}\\0\\-2/\sqrt{5}\end{bmatrix}\right) \cdot \vec{u_1}$$

$$= \vec{v_2} - \left(\frac{-1}{\sqrt{5}} + 0 - \frac{4}{\sqrt{5}}\right) \cdot \vec{u_1} = \vec{v_2} + \frac{5}{\sqrt{5}}\begin{bmatrix}1/\sqrt{5}\\0\\-2/\sqrt{5}\end{bmatrix}$$

$$= \begin{bmatrix}-1\\-2\\2\end{bmatrix} + \begin{bmatrix}1\\0\\-2\end{bmatrix} = \begin{bmatrix}0\\-2\\0\end{bmatrix} \Rightarrow \vec{u_2} = (0, -1, 0)$$

$$\vec{w_3} = \vec{v_3} - \left[(\vec{v_3} \cdot \vec{u_1}) \cdot \vec{u_1} + (\vec{v_3} \cdot \vec{u_2}) \cdot \vec{u_2}\right]$$

$$= \vec{v_3} - \left[\left(\begin{bmatrix}0\\-1\\3\end{bmatrix} \cdot \begin{bmatrix}1/\sqrt{5}\\0\\-2/\sqrt{5}\end{bmatrix}\right) \cdot \vec{u_1} + \left(\begin{bmatrix}0\\-1\\3\end{bmatrix} \cdot \begin{bmatrix}0\\-1\\0\end{bmatrix}\right) \cdot \vec{u_2}\right]$$

$$= \vec{v_3} - \left(\frac{-6}{\sqrt{5}} \cdot \begin{bmatrix}1/\sqrt{5}\\0\\-2/\sqrt{5}\end{bmatrix} + \begin{bmatrix}0\\-1\\0\end{bmatrix}\right)$$

$$= \begin{bmatrix}0\\-1\\3\end{bmatrix} - \left(\begin{bmatrix}-6/5\\0\\12/5\end{bmatrix} + \begin{bmatrix}0\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\\3\end{bmatrix} - \begin{bmatrix}-6/5\\-1\\12/5\end{bmatrix}$$

$$= \begin{bmatrix}6/5\\0\\3/5\end{bmatrix} \qquad \|w_3\| = \sqrt{\frac{36}{25} + \frac{9}{25}} = \frac{\sqrt{45}}{5} = \frac{3\sqrt{5}}{5}$$

$$\Rightarrow \vec{u_3} = \left(\frac{6}{5} \cdot \frac{5}{3\sqrt{5}}, 0, \frac{3}{5} \cdot \frac{5}{3\sqrt{5}}\right) = \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$$

③ $V = \text{Span}\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\1\\3\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\0\end{bmatrix}\right)$ to orthonormal basis?

```python
from sympy import Matrix

v1 = Matrix([1, 1, -1, 1])
v2 = Matrix([0, 2, 1, 3])
v3 = Matrix([2, 0, -2, 0])

u1 = v1 / v1.norm()

w2 = v2 - (v2.T.dot(u1)) * u1
u2 = w2 / w2.norm()

w3 = v3 - ((v3.T.dot(u1)) * u1 + (v3.T.dot(u2)) * u2)
u3 = w3 / w3.norm()
```