Eigenvalues and Eigenvectors

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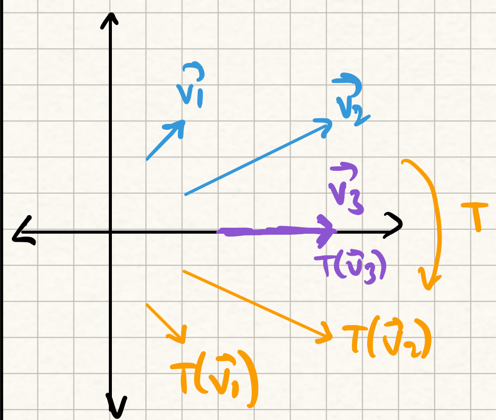
Eigenvalues, eigenvectors, eigenspaces

$T(\vec{v}) = A \cdot \vec{v}$: A is a matrix, the transformation matrix.

$T(\vec{v}) = \lambda \cdot \vec{v}$: λ is a constant, the eigenvalue.



* T flips vectors among the x-axis.
* $\vec{v}_3$ is an eigenvector because $T(\vec{v}_3)$ and $\vec{v}_3$ are on the same line. λ in $T(\vec{v}_3) = \lambda \cdot \vec{v}_3$ is the eigenvalue.

* $$\left.\begin{aligned} T(\vec{v}_3) &= A \cdot \vec{v}_3 \\ T(\vec{v}_3) &= \lambda \cdot \vec{v}_3 \end{aligned}\right\} \Rightarrow A \cdot \vec{v}_3 = \lambda \cdot \vec{v}_3$$

* $A \cdot \vec{v} = \lambda \cdot \vec{v}$ # Our general formula for λ

$\vec{0} = \lambda \cdot \vec{v} - A \cdot \vec{v}$ # Just rearranging.

$\vec{0} = \lambda \cdot I_n \vec{v} - A \cdot \vec{v}$ # Adding I_n

$\vec{0} = (\lambda \cdot I_n - A) \vec{v}$ # Factoring out $\vec{v}$.

$\vec{0} = B \cdot \vec{v}$ # $\lambda \cdot I_n - A$ is a matrix. Let's call it B

$B \cdot \vec{v} = \vec{0}$ # Now this looks like the null space equation.

we want to find non-zero vectors for $\vec{v}$, because zero vector don't give us new information.

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* The set of eigenvectors (Eigenspace) of a matrix is a special set of input vectors (eigenvector) for which the action of the matrix is described as a simple scaling by a scaler (eigenvalue)

* $A \cdot \vec{e}_\lambda = \lambda \vec{e}_\lambda \Rightarrow |\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda - A_{1,1} & (-1) A_{1,2} \\ (-1) A_{2,1} & \lambda - A_{2,1} \end{vmatrix} = 0$

$B \cdot \vec{v} = \vec{0}$

If we have non-zero $\vec{v}$, it means B will have linearly dependent columns (because $N(B) = \{\vec{0}\}$ won't be true), which means B is not invertible which means $|B| = 0$,

$|B| = 0$

* $|\lambda I_n - A| = 0$

Because $B = \lambda . I_n - A$. Now this is the formula to find eigenvalues.

Ex:

$T(\vec{v}) = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$

$$\left| \lambda \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 2 & 4 \\ 1 & \lambda + 1 \end{vmatrix} = 0 \quad \Rightarrow (\lambda - 2)(\lambda + 1) - 4 = 0$$

$\lambda^2 - \lambda - 6 = 0$ # characteristic equation

$(\lambda - 3)(\lambda + 2) = 0$

$\lambda_1 = 3 \quad \lambda_2 = -2$

Eigenvalues for the matrix A

* Trace : Sum of the values on the main diagonal.

* $\text{Trace} = \sum_{i=1}^{n} \lambda_i$ (Rule #1)

#1 $\text{Trace}(A) = 2 - 1 = 1$
$\lambda_1 + \lambda_2 = 3 - 2 = 1$ ✓

* $\text{Det}(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$ (Rule #2)

#2 $\text{Det}(A) = 2(-1) - (-4)(-1)$
$= -2 - 4 = -6$,
$\lambda_1 \cdot \lambda_2 = 3 \cdot (-2) = -6$ ✓

* When we find the eigenvalues we can verify them in two ways:

(1) $\text{Trace}^*(A) = \sum_{i=1}^{n} \lambda_i$ (2) $\text{Det}(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

$$E_\lambda = N(\lambda . I_n - A) = N\left(\begin{bmatrix}\lambda-2 & 4\\ 1 & \lambda+1\end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix}-4 & 4\\ 1 & -1\end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c}-4 & 4 & 0\\ 1 & -1 & 0\end{array}\right] = \left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$v_1 - v_2 = 0$

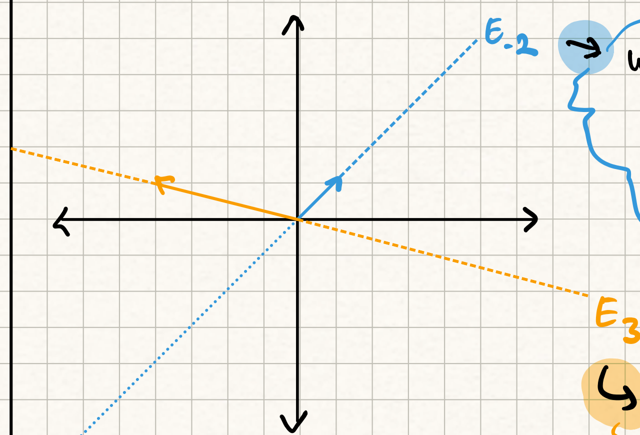
$v_1 = v_2$ $\Rightarrow E_{-2} = Span\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

$$E_3 = N\left(\begin{bmatrix}1 & 4\\ 1 & 4\end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c}1 & 4 & 0\\ 1 & 4 & 0\end{array}\right] = \left[\begin{array}{cc|c}1 & 4 & 0\\ 0 & 0 & 0\end{array}\right]$$

$v_1 = -4v_2$

$\Rightarrow E_3 = Span\left(\begin{bmatrix}-4\\1\end{bmatrix}\right)$



When we apply T, every vector on this line will be scaled by 2 and point the opposite direction.

when we apply T, every vector on this line will be scaled by 3, and point the same direction.

SO...

* -2 and 3 are eigenvalues
* (1,1) and (-4,1) are eigenvectors
* E_{-2} and E_3 are eigenspaces.

Quiz

(1) Find the eigenvalues for $A = \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix}$

$$\left| \begin{bmatrix} \lambda+3 & 0 \\ -1 & \lambda-4 \end{bmatrix} \right| = 0$$

$$(\lambda+3)(\lambda-4) - 0 = 0 \Rightarrow \begin{matrix} \lambda_1 = -3 \\ \lambda_2 = 4 \end{matrix}$$

(2) Find the eigenvectors for the question 1.

$$\lambda = -3 \Rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -7 & 0 \end{array}\right]$$

$$V_1 = -7V_2$$

$$E_{-3} = \text{Span}\left(\begin{bmatrix} -7 \\ 1 \end{bmatrix} \right)$$

$$\lambda = 4 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad V_1 = 0$$

$$E_4 = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

(3) Find the eigenvectors of $A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$

$$\begin{vmatrix} \lambda-2 & 3 \\ 0 & \lambda-5 \end{vmatrix} = 0 \Rightarrow (\lambda-2)(\lambda-5) = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 5$$

$$E_2 = \left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow V_2 = 0 \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$E_5 = \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad V_1 = -V_2 \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Eigen in three dimensions

$$T(\vec{v}) = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$|\lambda . I - A| = \begin{vmatrix} \lambda-4 & 0 & -1 \\ 1 & \lambda+6 & 2 \\ -5 & 0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 0 \begin{vmatrix} \quad \end{vmatrix} + (\lambda+6) \begin{vmatrix} \lambda-4 & -1 \\ -5 & \lambda \end{vmatrix} - 0 \begin{vmatrix} \quad \end{vmatrix}$$

$$\Rightarrow (\lambda+6)\left[(\lambda-4)\lambda - (-1)(-5)\right]$$

$$= (\lambda+6)\left[\lambda^2 - 4\lambda - 5\right] = (\lambda+6)(\lambda-5)(\lambda+1)$$

$$\Rightarrow \lambda_1 = -6 \qquad \lambda_2 = 5 \qquad \lambda = -1$$

$$E_{-6} = \left[\begin{array}{ccc|c} -10 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ -5 & 0 & -6 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathrm{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$E_{-1} = \left[\begin{array}{ccc|c} -5 & 0 & -1 & 0 \\ 1 & 5 & 2 & 0 \\ -5 & 0 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1/5 & 0 \\ 0 & 1 & 9/25 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathrm{Span}\left(\begin{bmatrix} -1/9 \\ -9/25 \\ 1 \end{bmatrix}\right)$$

$$E_5 = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 11 & 2 & 0 \\ -5 & 0 & 5 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3/11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathrm{Span}\left(\begin{bmatrix} 1 \\ -3/11 \\ 1 \end{bmatrix}\right)$$

Quiz ① Find the eigenvectors of $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix}$

$$\begin{vmatrix} \lambda-1 & 4 & -2 \\ 0 & \lambda-4 & 3 \\ 0 & 0 & \lambda+2 \end{vmatrix} = 0 \Rightarrow +(\lambda-1)\begin{vmatrix} \lambda-4 & 3 \\ 0 & \lambda+2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)\left[(\lambda-4)(\lambda+2) - 0\right] = 0$$

$$\Rightarrow \lambda_1 = 1 \quad \lambda_2 = 4 \quad \lambda_3 = -2$$

$$E_1 = \left[\begin{array}{ccc|c} 0 & 4 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$E_4 = \left[\begin{array}{ccc|c} 3 & 4 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{Span}\left(\begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$E_{-2} = \left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] =$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$$

② Find the eigenvectors for $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

$$\begin{vmatrix} \lambda+2 & 0 & 0 \\ -1 & \lambda-3 & 0 \\ 0 & 3 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda+2)(\lambda-3) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -2 \quad \lambda_3 = 3$$

$$E_1 = \left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathrm{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$E_{-2} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -5 & 0 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 5 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \mathrm{Span}\left(\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$E_3 = \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathrm{Span}\left(\begin{bmatrix} 0 \\ -2/3 \\ 1 \end{bmatrix}\right)$$

③ Find the eigenvectors for $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda-1)(\lambda+2) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -2$$

$$E_1 = \left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$$E_{-2} = \left[\begin{array}{ccc|c} -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \operatorname{Span}\left(\begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}\right)$$