Venn Diagrams and the Addition Rule

26.05.2025

Probability with Venn diagrams

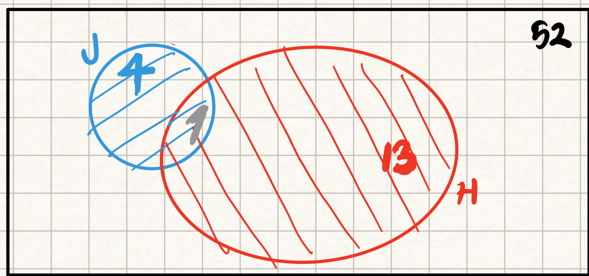
* A deck of cards: 4 suits, 13 ranks $\Rightarrow 4 \cdot 13 = 52$ cards.

* $P(\text{Jack}) = \dfrac{\#\text{ of Jacks}}{\#\text{ of cards}} = \dfrac{4}{52} = \dfrac{1}{13}$

* $P(\text{Heart}) = \dfrac{13}{52} = \dfrac{1}{4}$

* $P(\text{Jack of Hearts}) = \dfrac{1}{52}$

* $P(J \text{ or } H) = \dfrac{4+13-1}{52} = \dfrac{16}{52} = \dfrac{4}{13}$



Addition rule for probability

* I have a bag with 8 green cubes, 9 green spheres, 5 yellow cubes, 7 yellow spheres.

* $P(\text{Cube}) = \underbrace{\dfrac{8}{29}}_{\text{green cubes}} + \underbrace{\dfrac{5}{29}}_{\text{yellow cubes}} = \dfrac{13}{29}$

* $P(\text{yellow}) = \underbrace{\dfrac{5}{29}}_{\text{yellow cubes}} + \underbrace{\dfrac{7}{29}}_{\text{yellow spheres}} = \dfrac{12}{29}$

* $P(\text{yellow cube}) = \dfrac{5}{29}$

* $P(\text{yellow or a cube}) = \underbrace{\dfrac{12}{29}}_{\text{yellow}} + \underbrace{\dfrac{13}{29}}_{\text{cube}} - \underbrace{\dfrac{5}{29}}_{\text{yellow cube}}$

* $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

* If $P(A \text{ and } B) = 0$, A and B are mutually exclusive, therefore
$P(A \text{ or } B) = P(A) + P(B)$

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* $P(A) = \dfrac{\#\text{ of favorable outcomes}}{\#\text{ of all outcomes}} \in [0,1]$
* $P(\text{not } A) = P(A^c) = 1 - P(A)$
* $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
* $P(A \cup B) = P(A) + P(B)$ (if A and B are mutually exclusive)

Multiplication Rule for Probabilities

26.05.2025

Compound probability of independent events

* We have a coin:

* $P(H) = \frac{1}{2}$ * $P(T) = \frac{1}{2}$

* $P(HH) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(H_1) = P(H_2)$ because they are independent events.

* $P(\underbrace{THT}_{\text{in this order!}}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

* The propability of the ticket that won the lottery last week was the same as the probability of a ticket with sequential numbers such as 1234567...

Independent events example: test taking

* Multiple choice test. Problem 1 has 4 choices, problem 2 has 3 choices. Each problem has only one correct answer. If we randomly chose our answers, what's the prob. we got both right?

* Q_1 and Q_2 are independent events. $\Rightarrow P(Q_1 \text{ and } Q_2) = P(Q_1) \cdot P(Q_2)$

$$= \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

Dependent probability introduction

* We have a bag of 3 green marbles and 2 red marbles. We'll grab two marbles at once, if they are both green, we'll win 0.35 \$ otherwise we'll lose 1\$. Should we play?

$$P(GG) = \underbrace{\frac{3}{5}}_{\text{first time}} \cdot \underbrace{\frac{2}{4}}_{\text{second time}} = \frac{6}{20} = 0.30 < 0.35 \Rightarrow \text{NO!}$$

Dependent events!

* $P(A \text{ and } B) = P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$
* $P(A \cap B) = P(A) \cdot P(B)$ (if A and B are independent)

* What if we were allowed to put the first marble back?

$$P(GG) = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25} = 0.36 > 0.35 \Rightarrow \text{YES!}$$

Example 2:

* Maya and Doug are finalists in a crafting competition. For the final round, each of them will randomly select a card *without replacement* that will reveal what the star material must be in their craft. Here are the available cards:

↳ dependent events!

Leather	Wood	Plastic
Felt	Silk	Fleece

Maya and Doug both want to get silk as their star material. Maya will draw first, followed by Doug.

What is the probability that NEITHER contestant draws silk?

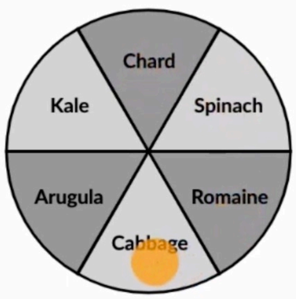
* $P(\text{No Silk}) = P(\text{Maya No Silk}) \cdot P(\text{Doug no Silk} \mid \text{Maya No Silk})$

↳ "given that"

$$= \frac{5}{6} \cdot \frac{4}{5} = \frac{20}{30} = \frac{2}{3}$$

Interpreting general multiplication rule

* Two contestants are finalists in a cooking competition. For the final round, each of them spin a wheel to determine what star ingredient must be in their dish:



Event	Meaning
K_1	The first contestant lands on kale.
K_2	The second contestant lands on kale.
K_1^C	The first contestant does *not* land on kale.
K_2^C	The second contestant does *not* land on kale.

1. Using the general multiplication rule, express symbolically the probability that neither contestant lands on kale. $P(K_1^C) \cdot P(K_2^C)$

2. Interpret what each part of this probability statement represents:

$$P(K_1^C \text{ and } K_2) = P(K_1^C) \cdot P(K_2 \mid K_1^C)$$

= Probability of first contestant not landing on kale AND probability of second contestant landing on kale GIVEN THAT first contestant not landing on kale.

Permutations

26.05.2025

Factorial and counting seat arrangements

* Permutations : # of ways we can arrange things.

* We have 3 chairs and 3 people. How many ways we can arrange?

  * ABC / ACB / BCA / BAC / CAB / CBA = 6 ways (permutations)

* 100 seats?

  $\underline{1}$ $\underline{2}$ $\underline{3}$ . . . . $\underline{100}$ = 100!

  100 possible choices, 99 poss. ch., 98 poss. ch., ..., 1 possible choice

Permutation formula

* 5 people, 3 chairs?

$$\underset{\text{Chair 1}}{5} \cdot \underset{\text{Chair 2}}{4} \cdot \underset{\text{Chair}}{3} = 60 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{\overset{\text{people}}{5!}}{(\underset{\text{people}}{5}-\underset{\text{chairs}}{3})!}$$

* $\underbrace{P(n,r)}_{\text{"\# of ways to permute n things into r spaces."}} = {}_nP_r = \frac{n!}{(n-r)!}$

Example: Possible 3 letter words

* How many possible 3-letter words are their in English?

  $\underset{\#1}{26} \cdot \underset{\#2}{26} \cdot \underset{\#3}{26} = 17.576$

  we can have multiple letters

* What if only different letters?

  $\underset{\#1}{26} \cdot \underset{\#2}{25} \cdot \underset{\#3}{24} = 15.600$

Zero factorial

* ${}_nP_k = \frac{n!}{(n-k)!}$ $\quad n = k \Rightarrow \frac{n!}{0!}$ → This should not be zero therefore mathematicians decided that:

  * $0! = 1$

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* Permutation is the number of ways we can arrange things. We use it when the order matters.
* Number of ways to permute n things into k spaces $= {}_nP_k = \frac{n!}{(n-k)!}$

Combinations

26.05.2025

Intro to combinations

* We have 6 people and 3 chairs but we don't care about the arrangement. How many different ways to choose 3 people out of 6 people?

$$= \frac{\#\text{ of permutations}}{\#\text{ of ways to arrange 3 people}} = \frac{\frac{6!}{3!}}{3!} = \frac{120}{6} = 20 \text{ combinations}$$

Combination formula

* $${}_nC_k = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

"# of k-element things within n things"

→ another notation

Example: Handshaking combinations

* How many different ways 4 people can shake hands?

$$= {}_4C_2 = \frac{4!}{2!\,(4-2)!} = \frac{\cancel{4}\cdot 3\cdot 2\cdot 1}{\cancel{2}\cdot 1\cdot \cancel{2}\cdot 1} = 6 \text{ combinations}$$

Example: 9-card hands

* Deck: 4 suits, 9 ranks = 36 cards. How many 9-card hands are possible?

$$= {}_{36}C_9 = \frac{36!}{9!\cdot 27!} = 94{,}143{,}280$$

* Combination is the selection of objects without considering order
* Number of k-object groups within n number of objects $= \binom{n}{k} = {}_nC_k = \frac{n!}{k!\,(n-k)!}$

Probability Using Combinatorics

26.05.2025

Probability using combinations

* Probability of getting 3 Heads out of 8 coin toss

$$= P\left(\frac{3}{8}H\right) = \frac{\text{\# favorable outcomes}}{\text{\# total outcomes}} = \frac{{}_8C_3}{2^8} = \frac{56}{2^8} = 0.2188$$

$$= 21.88\%$$

Example: Lottery

* Player chooses 4 numbers out of 1 to 60. Order doesn't matter. What's the probability of choosing 3-15-46-49?

$$P = \frac{1}{\binom{60}{4}} = \frac{1}{\frac{60!}{56!\,4!}} = \frac{1}{487{,}635}$$

Example: Officer selection

* A club of 9 people wants to choose the 3-member board in random. What's the P of Marsha being President, Sabita being VP, and Robert being Secretary.

$$P = \frac{\text{\# of fav. outcomes}}{\text{\# of all outcomes}} = \frac{1}{{}_9P_3} = \frac{1}{9\cdot 8\cdot 7} = \frac{1}{504}$$

Example: Taste testing

> Samara is setting up an olive oil tasting competition for a festival. From 15 distinct varieties, Samara will choose 3 different olive oils and blend them together. A contestant will taste the blend and try to identify which 3 of the 15 varieties were used to make it.
>
> Assume that a contestant can't taste any difference and is randomly guessing.
>
> **What is the probability that a contestant correctly guesses which 3 varieties were used?**

$$= \frac{1}{{}_{15}C_3} = \frac{1}{\frac{15!}{12!\,3!}} = \frac{1}{455}$$

Example: Choosing groups

* Kyra works on a team of 13 people. Her manager is randomly selecting a 3-member team. What's the P of Kyra being on the team

$$P = \frac{\text{\# of 3-member teams Kyra is in}}{\text{\# of 3-member teams}} = \frac{\binom{12}{2}}{\binom{13}{3}} = \frac{\frac{12\cdot 11}{2}}{\frac{13\cdot 12\cdot 11}{3}} = \frac{3}{13}$$

Example: Choosing cards

* Standard deck, no replacement. What is the P of 2 aces and 2 kings (in any order)? $\Rightarrow P = \frac{({}_2C_4)({}_2C_4)}{({}_{52}C_4)}$

Probability Distributions Introduction

27.05.2025

Constructing a prob. distr. for random var.

* X = number of "heads" after 3 flips of a fair coin.

Possible Scenarios

HHH THH
HHT THT
HTH TTH
HTT TTT

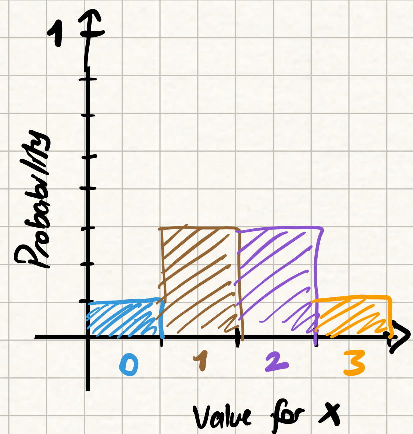
$P(X=0) = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$

Discrete Prob. Dist. for X



Valid discrete prob. dist. examples

* Anthony DeNoon is analyzing his basketball statistics. The following table shows a probability model for the results from his next two free throws.

Outcome	Probability
Miss both free throws	0.2 ≥ 0 ✓
Make exactly one free throw	0.5 ≥ 0 ✓
+ Make both free throws	+ 0.1 ≥ 0 ✓
All possible scenarios = 1 $\neq$	0.8 ✗

Is this a valid probability model?

* You are a space alien. You visit planet Earth and abduct 97 chickens, 47 cows, and 77 humans. Then, you randomly select one Earth creature from your sample to experiment on. Each creature has an equal probability of getting selected.

Create a probability model to show how likely you are to select each type of Earth creature.
Input your answers as fractions or as decimals rounded to the nearest hundredth.

97
47
77
221

Type of Earth creature	Estimated probability
Chicken	$97/221 \approx 0.44$
Cow	$47/221 \approx 0.21$
Human	$77/221 \approx 0.35$

* Hugo plans to buy packs of baseball cards until he gets the card of his favorite player, but he only has enough money to buy at most 4 packs. Suppose that each pack has probability 0.2 of containing the card Hugo is hoping for.

Let the random variable X be the number of packs of cards Hugo buys. Here is the probability distribution for X:

$X = \#$ of packs	1	2	3	4
$P(X)$	0.2	0.16	0.128	?

Find the indicated probability.

$P(X \geq 2) = 1 - P(X<2) = 1 - 0.2 = 0.8$

* A probability distribution describes the likelihood of all possible outcomes for a given event.

Theoretical and Empirical Probability Distributions

27.05.2025

Example: Tables

A board game has players roll two 3-sided dice (these exist!) and subtract the numbers showing on the faces. The game only looks at non-negative differences. For example, if a player rolls a 1 and a 3, the difference is 2.

Let D represent the difference in a given roll.

Construct the theoretical probability distribution of D.

D	0	1	2
$P(D)$	3/9	4/9	2/9

D2 \ D1	1	2	3
1	0	1	2
2	1	0	1
3	2	1	0

Example: Multiplication

Kai goes to a restaurant that advertises a promotion saying "1 in 5 customers get a free dessert!"

Suppose Kai goes to the restaurant twice in a given week, and each time, he has a $\frac{1}{5}$ probability of getting a free dessert.

Let X represent the number free desserts Kai gets in his two trips.

Construct the theoretical probability distribution of X.

X	0	1	2
$P(X)$	16/25	8/25	1/25

$$P(X=2) = \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{25}$$

$$P(X=0) = \frac{4}{5} \cdot \frac{4}{5} = \frac{16}{25}$$

$$P(X=1) = 1 - \frac{1}{25} - \frac{16}{25} = 8/25$$

Probability distributions from empirical data

"experience-based"

Jayda owns a restaurant where customers can make their orders using an app. She decides to offer a discount on appetizers to attract more customers, and she's curious about the probability that a customer orders a large number of appetizers.

Jayda tracked how many appetizers were in each of the past 500 orders:

# of appetizers	0	1	2	3	4	5	6
Frequency	40 /500	90 /500	160	120	50	30	10 /500

Let X represent the number of appetizers in a random order.

Based on these results, construct an approximate probability distribution of X.

X	0	1	2	3	4	5	6
$P(X)$	0.08	0.18	0.32	0.24	0.10	0.06	0.02

* Theoretical probability distributions are based on mathematical calculations, while empirical probability distributions are based on observed data.

Decisions with Probability and Expected Value

27.05.2025

Decisions with probability

Roberto and Jocelyn decide to roll a pair of fair 6-sided dice to determine who has to dust their apartment.

- If the sum is 7, then Roberto will dust. 6 samples
- If the sum is 10 or 11, then Jocelyn will dust. 5 samples
- If the sum is anything else, they'll roll again.

Is this a fair way to decide who dusts? Why or why not?

Choose 1 answer:

- No, there is a higher probability that Roberto dusts.
- ~~No, there is a higher probability that Jocelyn dusts.~~
- ~~Yes, they both have an equal probability of vacuuming.~~

D2 \ D1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Mean (Expected Value) of a discrete random variable

X = # of workouts I do in a week

X	$P(X)$
0	0.1
1	0.15
2	0.4
3	0.25
4	0.1

$$E(X) = \mu_X = 0(0.1) + 1(0.15) + 2(0.4) + 3(0.25) + 4(0.1)$$

$$= 0.15 + 0.8 + 0.75 + 0.4$$

$$= 2.10$$

"I'm expected to do 2.1 workouts per week (e.g. 21 w/o in 10 weeks)"

Interpreting EV

A certain lottery ticket costs \$2, and the back of the ticket says, "The overall odds of winning a prize with this ticket are 1 : 50, and the expected return for this ticket is \$0.95."

Which interpretations of the expected value are correct?

Choose all answers that apply:

- The probability that one of these tickets wins a prize is 0.95, on average.
- Someone who buys this ticket is most likely to win \$0.95.
- If we looked at many of these tickets, the average return would be about \$0.95 per ticket.
- If 1,000 people each bought one of these tickets, they'd expect a net ~~gain~~ return of about \$950 in total.

* Expected value (EV) is the mean of the possible values a random variable can take, weighted by the probability of those outcomes.

Example: Lottery ticket

A "Pick 4" lottery game involves drawing 4 numbered balls from separate bins each containing balls labeled from 0 to 9. So there are $10{,}000$ possible selections in total: $0000, 0001, 0002, \ldots, 9998, 9999$.

Players can choose to play a "straight" bet, where the player wins if they match all 4 digits in the correct order. The lottery pays $\$4{,}500$ on a successful $\$1$ straight bet.

Let X represent a player's net gain on a $\$1$ straight bet.

Calculate the expected net gain $E(X)$.

Hint: The expected net gain can be negative.

$$E(X) = \frac{1}{10000} \cdot 4499 + \frac{9999}{10000}(-1)$$

$$= -0.55\,\$$$

Example: Protection Plan

An electronics store gives customers the option of purchasing a protection plan when customers buy a new television. The customer pays $\$80$ for the plan, and if their television is damaged or stops working, the store will replace it for no additional charge. The store knows that 2% of customers who buy this plan end up needing a replacement that costs the store $\$1{,}200$ each.

Here is a table that summarizes the possible outcomes from the store's perspective:

Replacement?		Cost	Net gain (X)
Yes	0.02	$\$1{,}200$	$-\$1{,}120$
No	0.98	$\$0$	$\$80$

$$E(X) = 0.02(-1{,}120) + 0.98(80)$$

$$= 56\,\$$$

Let X represent the store's net gain from one of these plans.

Calculate the expected net gain $E(X)$.