AP CALCULUS BC

01 LIMITS AND CONTINUITY

* A limit is the value that a function (or sequence) approaches as the argument (or index) approaches some value.
* When a limit doesn't approach the same value from both sides, then the limit doesn't exist: $\lim_{x\to c} f(x) = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$
* Just because a function is undefined for some x-value doesn't mean there's no limit. On the other hand, just because a function is defined for some x-value doesn't mean that limit exists.
* Let $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$,
  * $\lim_{x\to c}(f(x)+g(x)) = L+M$
  * $\lim_{x\to c}(f(x)-g(x)) = L-M$
  * $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
  * $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
  * $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$
  * $\lim_{x\to c}\left(f(x)^{\frac{r}{s}}\right) = L^{\frac{r}{s}}$
* When evaluating the limits of combined functions, we must verify whether the left-hand and right-hand limits are equal. If they are, the overall limit exist, even if the individual limit of one of the component functions does not.
* $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$ if and only if:

  ① $\lim_{x\to a} g(x) = L$ (exists)

  ② AND $f(x)$ is continuous at L.

* Selecting procedures for determining limits:

Calculating $\lim_{x \to a} f(x)$

A. Direct substitution

Try to evaluate the function directly.

$$f(a)$$

$$f(a) = \frac{b}{0}$$

where ***b*** is not zero

B. Asymptote (probably)

example:

$$\lim_{x \to 1} \frac{1}{x-1}$$

Inspect with a graph or table to learn more about the function at ***x=a***.

$$f(a) = b$$

where ***b*** is a real number

C. Limit found (probably)

example:

$$\lim_{x \to 3} x^2 = (3)^2 = 9$$

$$f(a) = \frac{0}{0}$$

D. Indeterminate form

example:

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

Try rewriting the limit in an equivalent form.

E. Factoring

example:

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

can be reduced to

$$\lim_{x \to -1} \frac{x - 2}{x - 3}$$

by factoring and cancelling.

F. Conjugates

example:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

can be rewritten as

$$\lim_{x \to 4} \frac{1}{\sqrt{x} + 2}$$

using conjugates and cancelling.

G. Trig identities

example:

$$\lim_{x \to 0} \frac{\sin(x)}{\sin(2x)}$$

can be rewritten as

$$\lim_{x \to 0} \frac{1}{2\cos(x)}$$

using a trig identity.

Try evaluating the limit in its new form.

H. Approximation

When all else fails, graphs and tables can help approximate limits.

* If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$

* f is discontinuous if we have to pick up our pencil while graphing it. There are 3 types of discontinuity:
  1. Removable (Point) Discontinuity: Two-sided limit exists, which means one-sided limits exist and are equal to each other. However, two-sided limit is not equal to the function's value at that point.
  2. Jump Discontinuity: Two-sided limit does not exist because one-sided limits are not equal to each other, even though they exist.
  3. Asymptotic Discontinuity: Two-sided limit does not exist because one-sided limits are unbounded, therefore they don't exist.
* f is continuous at $x = c$ $\iff$ $\lim_{x\to c} f(x) = f(c)$
* f is continuous over (a,b) $\iff$ f is continuous over every point in the interval
* f is continuous over $[a,b]$ $\iff$ f is continuous over (a,b) **AND** $\lim_{x\to a^+} f(x) = f(a)$ **AND** $\lim_{x\to b^-} f(x) = f(b)$
* f is continuous on all real numbers $\iff$ it has no types of discontinuity (removable, jump, or asymptotic).
* When the one-sided limits are unbounded to the same direction, we can say that limit is going to infinity in that direction.
* A function can not cross its vertical asymptote, but it can cross its horizontal asymptote (even multiple times).
* Functions with horizontal asymptotes have one-sided limits for x approaches ∞ and $-\infty$. These limits are finite real numbers.
* The intermediate value theorem states that for any function f that is continous over the interval $[a,b]$, the function will take any value between $f(a)$ and $f(b)$ over the interval. More formally, for any value L between $f(a)$ and $f(b)$, there's a value c in $[a,b]$ for which $f(c) = L$.

02 DIFFERENTIATION: DEFINITION AND BASIC DERIVATIVE RULES

* The derivative quantifies the sensitivity to change of a function's input with respect to its output.
* The derivative of a function of a single variable at a chosen value, when it exists, is the slope of the tangent line to the graph of the function at that point, formula of which is $\frac{\Delta y}{\Delta x}$.
* Lagrange's notation: $f'(x)$: Pronounced "f prime". Meaning "the slope of $f(x)$ for this specific x-value"
* Leibniz's notation: $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$: "The derivative of $f(x)$ (or y) with respect to x." Seems more complicated but very useful when dealing with integral calculus, differential equations, and multivariable calculus.
* The equation of tangent line of a curve $y=f(x)$ at a point (x_0, y_0) is $y-y_0=m(x-x_0)$, where m is the slope of $f(x)$ at $x=x_0$.

* Formal form of the derivative: $\lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$
* When we're interested for one specific x only: $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$
* If f is not continuous at $x=c$, then f is not differentiable at $x=c$.
* If f is continuous at $x=c$, it doesn't necessarily mean that it's differentiable at $x=c$. The slopes of the tangent lines might approach to different values as $x \to c^-$ and $x \to c^+$.
* Power Rule: $f(x)=x^n$, $n=0 \Rightarrow f'(x)=n \cdot x^{n-1}$
* Constant Rule: $\frac{d}{dx}[k]=0$
* Multiplication by a Constant Rule: $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}[f(x)]$
* Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
* Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)]$

* Quotient Rule : $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)]\cdot g(x) - f(x)\frac{d}{dx}[g(x)]}{(g(x))^2}$

* $\frac{d}{dx}[\sin(x)] = \cos(x)$
* $\frac{d}{dx}[\cos(x)] = -\sin(x)$

* $\frac{d}{dx}[\tan(x)] = \frac{1}{\cos^2(x)} = \sec^2(x)$
* $\frac{d}{dx}[\cot(x)] = \frac{-1}{\sin^2(x)} = -\csc^2(x)$

* $\frac{d}{dx}[\sec(x)] = \sec(x)\cdot\tan(x)$
* $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

* $\frac{d}{dx}[e^x] = e^x$
* $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

03 DIFFERENTIATION OF COMPOSITE, IMPLICIT AND INVERSE FUNCTIONS

* Chain Rule: We take the derivative of the composite function with respect to inner function and multiply that with the derivative of the inner function with respect to x.

* $\frac{d}{dx}[f(g(x))] = \frac{d[f(g(x))]}{d\,g(x)}\cdot\frac{d[g(x)]}{dx}$

* $\frac{d}{dx}[a^x] = a^x \cdot \ln(a)$
* $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\cdot\ln(a)}$

* In implicit differentiation, we differentiate each side of an equation with two variables by treating one of the variables as a function of the other, not as a constant. This calls for using the chain rule.

* $f^{-1}(x) = g(x) \Rightarrow f'(x) = \frac{1}{g'(f(x))}$

* $\frac{d}{dx}[\sin^{-1}(x)] = \frac{1}{\sqrt{1-x^2}}$
* $\frac{d}{dx}[\cos^{-1}(x)] = \frac{-1}{\sqrt{1-x^2}}$
* $\frac{d}{dx}[\tan^{-1}(x)] = \frac{1}{1+x^2}$

* When selecting a procedure for calculating derivatives, ask "Do I see a product, quotient, or composition of functions?"
* Consider manipulating the function if it will make the calculation easier.
* When determining the order in which to apply differentiation rules to a complex expression, begin by identifying the outmost operation (based on the structure of the equation) and proceed inward, applying the appropriate rules step by step.
* The second derivative of a function is simply the derivative of the function's derivative. Notation: $f''(x)$ OR $\frac{d^2}{dx^2}[f(x)]$

04 CONTEXTUAL APPLICATIONS OF DIFFERENTIATION

* $D'(x) = y$ can be interpreted as "After x [units of x], the D increases at a rate of y [units of y] per [unit of x]."
* Velocity is the first derivative of position with respect to time. Acceleration is the first derivative of velocity with respect to time, which makes it the second derivative of position with respect to time.
* In one-dimensional motion, speed equals to the absolute value of velocity.
* Related rate problems are applied problems where we find the rate at which one quantity is changing, by relating it to the other quantities whose rates are known.
* When analyzing problems involving related rates, it's crucial to correctly distinguish variables and constants. It's also crucial to select the equation that correctly represents the problem.
* Local linearization approximates a non-linear function near a specific point "a" using its tangent line. It relies on the function's derivative to provide a linear approximation, therefore the function has to be differentiable at that specific point "a". The formula for local linearization ⇒ $L(x) = f(a) + f'(a)(x-a)$

* L'Hôpital's Rule helps us find limits in the form $\lim_{x \to c} \frac{u(x)}{v(x)}$ where direct substitution ends in the indeterminate forms $\left(\frac{0}{0}, \frac{\infty}{\infty}, \text{etc}\right)$. The rule says that if the limit $\lim_{x \to c} \frac{u'(x)}{v'(x)}$ exists, it's equal to $\lim_{x \to c} \frac{u(x)}{v(x)}$. We can apply l'Hôpital's Rule repeatedly.