Inverses

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Inverse of a transformation?

* $T: A \to B \quad T(\vec{a}_1) = \vec{b}_1$

  $T^{-1} = B \to A \quad T^{-1}(\vec{b}_1) = \vec{a}_1$

  $I_A : A \to A \quad I_A(\vec{a}_1) = \vec{a}_1$

  ↳ Identity transformation

  $I_B : B \to B \quad I_B(\vec{b}_1) = \vec{b}_1$

If T is inversible
(If T has an inverse)

$$\left.\begin{array}{l} \\ \end{array}\right\} \begin{array}{l} T^{-1}(T(\vec{a}_1)) = I_A(\vec{a}_1) \\ T^{-1}(T(\vec{b}_1)) = I_B(\vec{b}_1) \end{array}$$

3 conditions for inverse of a transformation

① If exists, T^{-1} is unique

$$T(\vec{a}) = [\quad] \cdot \vec{a}$$

$$T^{-1}(\vec{b}) = [\quad] \cdot \vec{b}$$

② T^{-1} only exists if every $\vec{a}_1$ maps to only one $\vec{b}_1$.

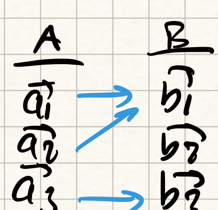
③ T^{-1} only exists if every $\vec{b}_1$ maps to only one $\vec{a}_1$

Surjective/injective?

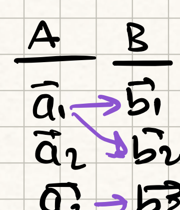
* Surjective (onto): Every $\vec{b}$ is being mapped to
* Injective (one-to-one): Only one $\vec{a}$ is mapping to a given $\vec{b}$.

* If T is invertible, T is both surjective and one-to-one. (and vice versa).

Ex:



NOT SURJECTIVE NOR INJECTIVE!



SURJECTIVE BUT NOT INJECTIVE!

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* Surjective (onto): Every vector $\vec{b}$ is being map to.
* Injective (one-to-one): Every $\vec{a}$ maps to a unique $\vec{b}$.
* If a transformation is both surjective and injective, then it is invertible.

Invertibility from the matrix-vector product

$T(\vec{x}) = A.\vec{x}$, and $T: \mathbb{R}^m \to \mathbb{R}^n$

* T can only be invertible when A is square and $\mathbb{R}^m = \mathbb{R}^n$

WHY? (1) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \Rightarrow \text{rref}(A) = \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1 \end{bmatrix}$.

$\Rightarrow N(A) = \text{span}\left(\begin{bmatrix} -1/3 \\ 1 \\ 1 \end{bmatrix}\right) \Rightarrow$

$A.\begin{bmatrix} -1/3 \\ 1 \\ 1 \end{bmatrix} = \vec{0}$, $A.\begin{bmatrix} -2/3 \\ 2 \\ 2 \end{bmatrix} = \vec{0}$, $A.\vec{0} = \vec{0}$, etc etc.

$\Rightarrow$ T not injective!

(2) $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 0 & -2 \end{bmatrix} \Rightarrow \text{rref}(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$C(B) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}\right) \Rightarrow T: \mathbb{R}^3 \to \mathbb{R}^2$

$\Rightarrow$ T not surjective!

For $A_{\text{row} \times \text{column}}$:

* Row < Column $\Rightarrow$ Not injective
* Row > Column $\Rightarrow$ Not surjective

* For non-square matrices: Does it look like letter I? If not, not injective. If yes, not surjective.

* If #rows > #columns $\Rightarrow$ Not surjective
* If #rows < #columns $\Rightarrow$ Not injective

* For non-square matrices: Does it look like letter I?
  No: Not injective
  Yes: Not surjective

Quiz ① Say whether or not T is surjective of injective.

$$T(\vec{x}) = \begin{bmatrix} 3 & -4 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \vec{x}$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 8 \\ 0 & 1 & -2 & 6 \end{bmatrix}$$

$\Rightarrow x_1 = 2x_3 - 8x_4$
$\quad x_2 = 2x_3 - 6x_4$

$$\Rightarrow N(A) = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

NOT INJECTIVE

$C(A) = \text{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right)$ can span $\mathbb{R}^2 \Rightarrow$ SURJECTIVE

Inverse transformations are linear

* If T is linear and invertible, then T^{-1} is linear. So:
  ① $T^{-1}(\vec{a}+\vec{b}) = T^{-1}(\vec{a}) + T^{-1}(\vec{b})$
  ② $T^{-1}(c.\vec{a}) = c.T^{-1}(\vec{a})$

* $T(\vec{x}) = A\vec{x}$
  $\Rightarrow T^{-1}(\vec{x}) = A^{-1}\vec{x}$

why? Let $T(\vec{x}) = A$, and $T^{-1}(\vec{x}) = B$.
$T^{-1}(T(x)) = I\vec{x}$
$\Rightarrow B.A = I \Rightarrow B = A^{-1}$

Ex: $T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \vec{x} \Rightarrow T^{-1}(\vec{x}) = ?$

$$\left[\begin{array}{cc|cc} \overset{A}{1} & 0 & \overset{I}{1} & 0 \\ -1 & 2 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} \overset{I}{1} & 0 & \overset{A^{-1}}{1} & 0 \\ 0 & 1 & 1/2 & 1/2 \end{array}\right]$$

* Inverse transformations are linear. Therefore:
  ① $T^{-1}(\vec{a}+\vec{b}) = T^{-1}(\vec{a}) + T^{-1}(\vec{b})$
  ② $T^{-1}(c.\vec{a}) = c.T^{-1}(\vec{a})$

Quiz

① $B = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \Rightarrow B^{-1} = ?$

$$\left[\begin{array}{cc|cc} -4 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 2 & -3 & 0 & 1 \\ -4 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -3/2 & 0 & 1/2 \\ 0 & -5 & 1 & 2 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & -3/2 & 0 & 1/2 \\ 0 & 1 & -1/5 & -2/5 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & -3/10 & -1/10 \\ 0 & 1 & -1/5 & -2/5 \end{array}\right]$$

② $M = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \\ 4 & -2 & 0 \end{bmatrix} \Rightarrow M^{-1} = ?$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 4 & -2 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 6 & 10 & 3 & 1 & 0 \\ 0 & -10 & -12 & -4 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & 1/2 & 1/6 & 0 \\ 0 & -10 & -12 & -4 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & -1/3 & 0 & -1/3 & 0 \\ 0 & 1 & 5/3 & 1/2 & 1/6 & 0 \\ 0 & 0 & 14/3 & 1 & 5/3 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/14 & -3/14 & 1/14 \\ 0 & 1 & 0 & 2/14 & -18/42 & -5/14 \\ 0 & 0 & 1 & 3/14 & 5/14 & 3/14 \end{array}\right]$$

Matrix inverses, and invertible and singular matrices

Matrix inverse is actually matrix division.

$$\frac{3}{3} = 1 \Rightarrow 3 \cdot \frac{1}{3} = 1 \Rightarrow x \cdot \frac{1}{x} = 1 \Rightarrow x \cdot x^{-1} = 1 \Rightarrow A \cdot A^{-1} = I$$

$$[A \mid I] \Rightarrow [I \mid A^{-1}]$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$|A| = ad - bc$

- Let $T(\vec{x}) = M \cdot \vec{x}$. If $rref(M) = I$, then T is invertible
- $[M \mid I] = [I \mid M^{-1}]$
- $M^{-1} = \frac{1}{|M|} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- $|M| = 0 \Rightarrow M$ is singular (not invertible)
- $|M| \neq 0 \Rightarrow M$ is non-singular (invertible)

Ex: $A = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{-2-0} \cdot \begin{bmatrix} 1 & 0 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 2 & 1 \end{bmatrix}$

* $|A| = 0 \Rightarrow A^{-1} = \underbrace{\dfrac{1}{|A|}}_{\text{undefined!}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ A not invertible!

* Not invertible = Singular

$ad - bc = 0 \Rightarrow ad = bc \Rightarrow \dfrac{a}{b} = \dfrac{c}{d}$

Quiz ① Are the matrices inverses of one another? $A \cdot B \overset{?}{=} I$

$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1/13 & 5/13 \\ 3/13 & -2/13 \end{bmatrix}$ YES!

$$A \cdot B = \begin{bmatrix} -2/13 + 15/13 & 10/13 - 10/13 \\ -3/13 + 3/13 & 15/13 - 2/13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

② Find the inverse of $M = \begin{bmatrix} 0 & -2 \\ -4 & 5 \end{bmatrix}$

$$M^{-1} = \frac{1}{0-8} \cdot \begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -5/8 & -1/4 \\ -1/2 & 0 \end{bmatrix}$$

③ $L = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$. Is L invertible or singular?

$|L| = (-3-0) = -3 \neq 0 \Rightarrow$ L is invertible

Solving systems with inverse matrices

$$7x + 5y = -4$$
$$-6x + 3y = -33 \Rightarrow \begin{bmatrix} 7 & 5 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -33 \end{bmatrix} \Rightarrow$$

$$A \cdot \vec{x} = \vec{b} \Rightarrow A^{-1} \cdot A \cdot \vec{x} = A^{-1} \cdot \vec{b} \Rightarrow I \cdot \vec{x} = A^{-1} \cdot \vec{b}$$

$$\Rightarrow \vec{x} = A^{-1} \cdot \vec{b}$$

$$A^{-1} \cdot \vec{b} = \frac{1}{21-(-30)} \begin{bmatrix} 3 & -5 \\ 6 & 7 \end{bmatrix} \cdot \vec{b} = \begin{bmatrix} 1/17 & -5/51 \\ 2/17 & 7/51 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -33 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

Why? This method allows us to change $\vec{b}$ and still easily calculate $A^{-1} \cdot \vec{b}$ because we already know A^{-1}.

Quiz 1. Use an inverse matrix to find the solution to:

$$3x + 12y = 51$$
$$-2x + 6y = -6 \qquad A = \begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 51 \\ -6 \end{bmatrix} \Rightarrow$$

$$A^{-1} \cdot \vec{b} = \frac{1}{18+24} \begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix} \cdot \vec{b} = \begin{bmatrix} 1/7 & -4/14 \\ 1/21 & 1/14 \end{bmatrix} \begin{bmatrix} 51 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{51}{7} + \frac{24}{14} \\ 51/21 - 6/14 \end{bmatrix} = \begin{bmatrix} \frac{126}{14} \\ \frac{102-18}{42} \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

* We can use inverse matrices to solve linear equations: $\vec{x} = A^{-1} \cdot \vec{b}$

This method allows us to calculate $\vec{x}$ for different $\vec{b}$'s very easily.

2. Use an inverse matrix to find the solution to:

$$y - 5x = -15$$
$$3x + 8y = 95 \rightarrow A = \begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

$$\Rightarrow A^{-1} \cdot \vec{b} = \frac{1}{-40-3} \begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix} \cdot \vec{b} = \begin{bmatrix} -8/43 & 1/43 \\ 3/43 & 5/43 \end{bmatrix} \cdot \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

$$= \begin{bmatrix} 120/43 + \frac{95}{43} \\ -45/43 + \frac{475}{43} \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

3. Use an inverse matrix to find the solution to:

$$4x + 8y = -20$$
$$-12x - 3y = -66 \rightarrow \underset{/3}{A} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}, \quad \underset{/3}{\vec{b}} = \begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

$$A^{-1} \cdot \vec{b} = \frac{1}{1-8} \begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix} \cdot \vec{b} = \begin{bmatrix} -1/7 & 2/7 \\ 4/7 & -1/7 \end{bmatrix} \begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

$$= \begin{bmatrix} 5/7 + \frac{44}{7} \\ -20/7 - 22/7 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$