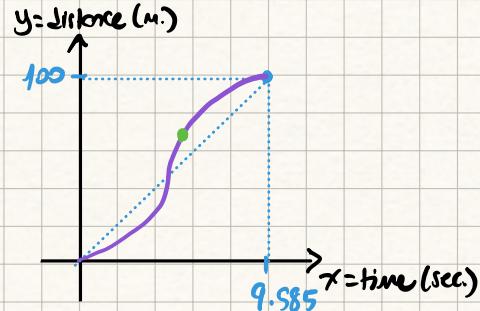


# Defining Average and Instantaneous Rates of Change at a Point

06.06.2025

## Newton, Leibniz, and Usain Bolt

- \* what is the instantaneous rate of change of something? → DIFFERENTIAL CALCULUS
- \* How fast is Usain Bolt running at this instant? (not on average!)



$$\text{avg. speed} = \frac{\Delta \text{distance}}{\Delta \text{time}} = \frac{\Delta y}{\Delta x} = \frac{\text{(rise)}}{\text{(run)}}$$

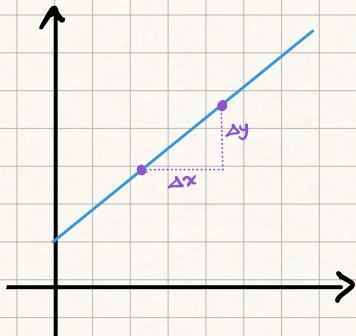
$$= \frac{100}{9.585} \approx 10.4 \text{ m/s}$$

$$\approx 37.58 \text{ km/h}$$

\* To find the speed of Bolt at the ●, we need to find the slope of the tangent line to that point, which is  $\frac{\Delta y}{\Delta x}$ . But we need the smallest  $\Delta x$  possible. Otherwise it's not instantaneous speed, it's still avg speed with a smaller interval. How do we find the smallest  $\Delta x$ ? LIMITS!

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \rightarrow \begin{array}{l} \text{infinitely small change in } y \\ \text{infinitely small change in } x \end{array}$$

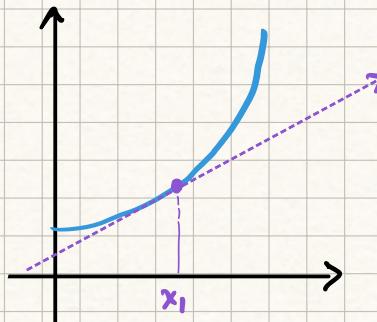
## Derivative as a concept



$$\text{Slope} = \frac{\Delta y}{\Delta x} = \text{instantaneous rate of change}$$

=

always same on the same line!



\* What is the slope of the tangent line to this point? Because that slope would tell us the instantaneous rate of change at that point.

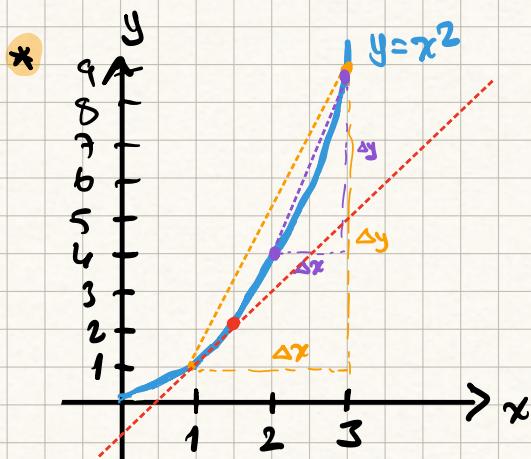
"DERIVATIVE"

- \* The derivative quantifies the sensitivity to change of a function's output with respect to its input.
- \* The derivative of a function of a single variable at a chosen value, when it exists, is the slope of the tangent line to the graph of the function at that point. Formula of which is  $\frac{\Delta y}{\Delta x}$ .

\* derivative =  $\frac{dy}{dx} = \frac{d}{dx} f(x)$  (Leibniz's notation) (very useful when dealing with integral calculus, differential equations, multivariable calc.)

\*  $y = f(x) \Rightarrow$  derivative =  $f'(x)$  (Lagrange's notation) ("f prime")  
"The slope of  $f(x)$  for an  $x$ -value"

Secant lines & average rate of change



\* Avg. rate of change over  $[1, 3]$

$$= \frac{\Delta y}{\Delta x} = \frac{9-1}{3-1} = 4 \quad (\text{slope of the secant line!})$$

\* Avg. rate of change over  $[2, 3]$

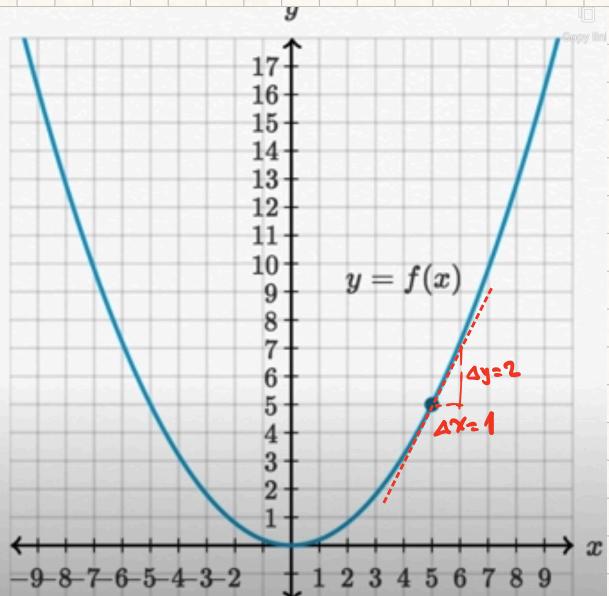
$$\frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5$$

\* Avg. rate of change at  $x=1.5$ ?  
Not avg. anymore! It's instantaneous rate of change at  $x=2$  and it's the slope of the tangent line.

Derivative as slope of curve

\* Estimate  $f'(5)$ . \*Slope of tangent line at  $x=5$

- 2
- 0.1
- 2
- 0.1
- 0



\* Lagrange's notation:  $f'(x)$  (Pronounced "f prime". Meaning "the slope of  $f(x)$  for an  $x$ -value". Simple notation.)

\* Leibniz's notation:  $\frac{dy}{dx}$  or  $\frac{d}{dx} f(x)$  ("the derivative of  $f(x)$  with respect to  $x$ ".)

Sound more complex, but very useful when dealing with integral calculus, differential equations and multivariable calculus.

X

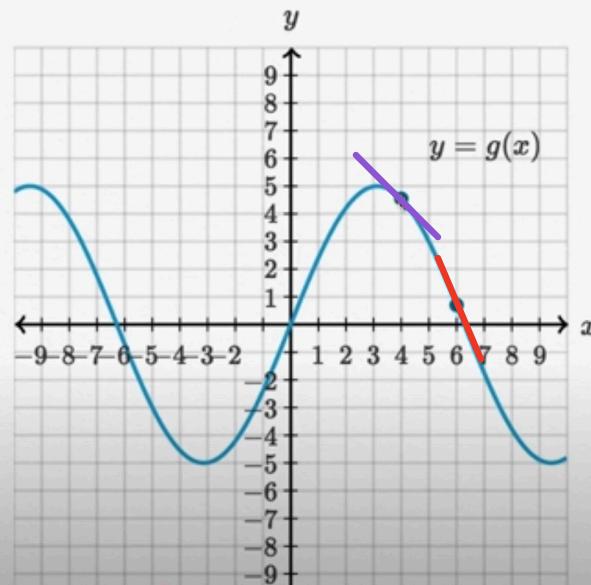
Compare.

$$g'(4) > g'(6)$$

<

>

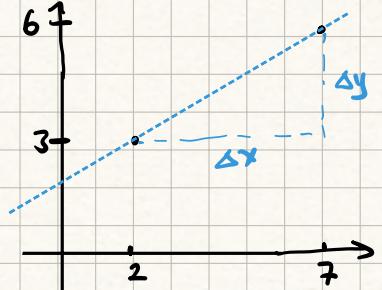
red line is steeper but  
in negative direction



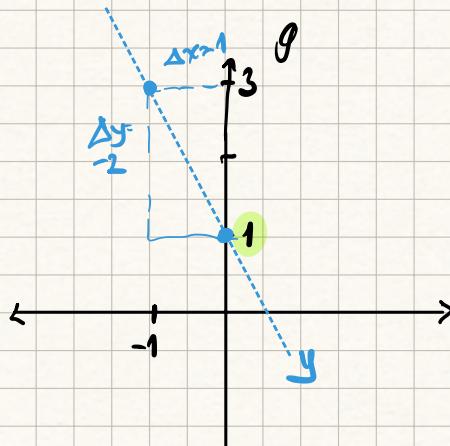
## The derivative and tangent line equations

- \* The tangent line to the graph of  $f$  at the point  $(2,3)$  passes through  $(7,6)$ . Find  $f'(2)$ .

$$= \frac{6-3}{7-2} = \frac{3}{5}$$



- \*  $g(-1)=3$   $g'(-1)=-2 \Rightarrow$  what's the equation of the tangent line to the graph of  $g$  at  $x=-1$ ?



\*  $y - y_1 = m(x - x_1)$

$$y - 3 = -2(x - (-1))$$

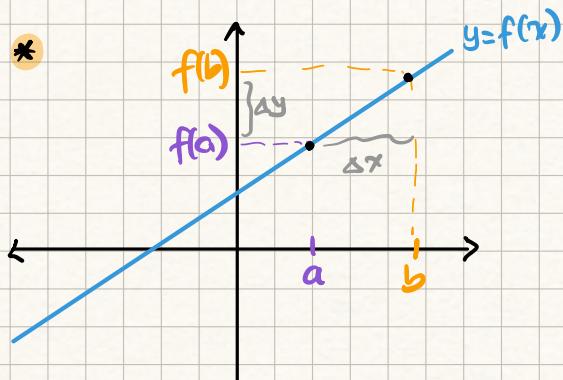
$$y - 3 = -2x + 2$$

$$y = -2x + 3$$

# Defining the Derivative of a Function and Using Derivative Notation

10.06.2025

Formal definition of the derivative as a limit

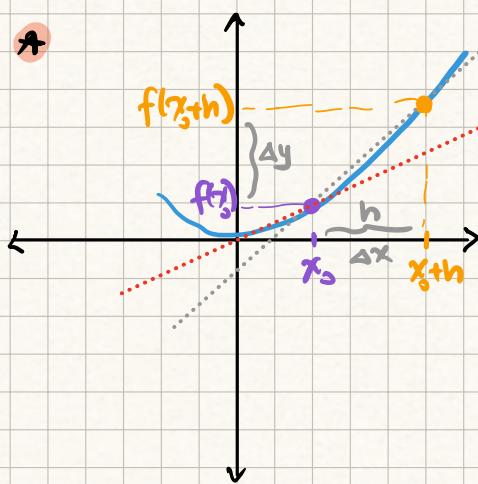


$$f(x) = mx + b$$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

\* The slope at every point on a straight line is constant. Therefore, the derivative of the linear function  $f(x) = mx + b$  with respect to  $x$  is equal to  $m$  for all  $x$  in its domain.

Formal form of derivative



$$\begin{aligned} \text{slope of secant} &= \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} \\ &= \frac{f(x_0+h) - f(x_0)}{h} \end{aligned}$$

$$\begin{aligned} \text{slope of the tangent line for } x = x_0 &= \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} \\ &= \underbrace{f'(x_0)}_{=f'(x)} \end{aligned}$$

Alternate form of derivative

\* If we don't need a generalized equation, we only want to find the derivative for one specific point:

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

Example 1:  
Derivative as a limit

$$* f(x) = \ln x \Rightarrow f'(e) = ?$$

$$\lim_{h \rightarrow 0} \frac{f(e+h) - f(e)}{h} = \lim_{h \rightarrow 0} \frac{\ln(e+h) - \ln(e)}{h}$$

\* The slope at every point on a straight line is constant. Therefore, the derivative of the linear function  $f(x) = mx + b$  with respect to  $x$  is equal to  $m$  for all  $x$  in its domain.

\* Formal form of the derivative:  $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

\* If we don't need a generalized equation, we only want to find the derivative for one specific  $x$ :

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

## Example 2: Derivative from limit expression



The alternate form of the derivative of the function  $f$  at a number  $a$ , denoted  $f'(a)$ , is given by

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$



provided the limit exists.

With the Alternate Form of the Derivative as an aid, make sense of the following limit expression by identifying the function  $f$  and the number  $a$ .

$$f'(5) = \lim_{x \rightarrow 5} \frac{x^3 - 125}{x - 5} \Rightarrow$$

is the derivative of the function  $f(x) = x^3$

at the number  $a = 5$

Derivative of  $x^2$   
at  $x=3$

\*  $f(x) = x^2 \Rightarrow f'(3) = ?$

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{(3 + \Delta x)^2 - 3^2}{(3 + \Delta x) - 3} &= \lim_{\Delta x \rightarrow 0} \frac{9 + 6\Delta x + (\Delta x)^2 - 9}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left( \frac{6\Delta x}{\Delta x} + \frac{(\Delta x)^2}{\Delta x} \right) = \lim_{\Delta x \rightarrow 0} \underbrace{6}_{\text{"0"}/\cancel{\Delta x}} + \underbrace{\Delta x}_{\cancel{\Delta x}} = 6 \end{aligned}$$

Derivative of  $x^2$   
at any point

\*  $f(x) = x^2 \Rightarrow f'(x) = \frac{d}{dx} f(x) = ?$

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{x + \Delta x - x} &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left( \frac{2x\Delta x}{\Delta x} + \frac{(\Delta x)^2}{\Delta x} \right) = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) \\ &= 2x \end{aligned}$$



# Estimating Derivatives of a Function at a Point

10.06.2025

This table gives select values of the differentiable function  $f$ .

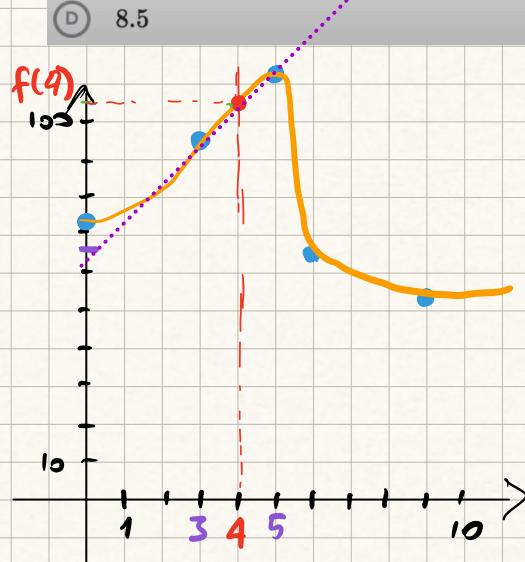
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$x$	0	3	5	6	9
$f(x)$	72	95	112	77	54

What is the best estimate for  $f'(4)$  we can make based on this table?

Choose 1 answer:

- A 17
- B -2
- C 103.5
- D 8.5



\* we don't know the function's formula, our orange line is an estimation.

The closest points to 4, of which we know the outputs, are 3 and 5.

$$f(3) = 95 \quad f(5) = 112$$

slope of the closest second line =  $\frac{f(5) - f(3)}{5 - 3} = 8.5$

# Determining When Derivatives Do and Don't Exist

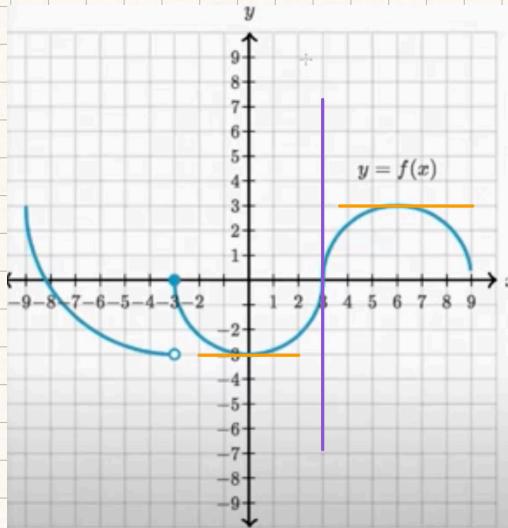
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## Differentiability and Continuity

- \* If  $f$  not continuous at  $x=c$ , then  $f$  is not differentiable at  $x=c$ .
- \* If there's jump or removable discontin. the slopes of the tangent lines will approach to different values as  $x \rightarrow c^-$  and  $x \rightarrow c^+$ .  
If  $f(c)$  is undefined, then  $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$  won't even work.
- \* If  $f$  is continuous at  $x=c$ , it doesn't necessarily mean that it's differentiable at  $x=c$ .
- \*  $f(x) = |x-c| \Rightarrow$  the slopes of the tangent lines will approach to different values as  $x \rightarrow c^-$  and  $x \rightarrow c^+$

## Example: Graphical

\*



The graph of function  $f$  is given below. It has a vertical tangent at the point  $(3, 0)$  and a horizontal tangent at the points  $(0, -3)$  and  $(6, 3)$ .

Select all the  $x$ -values for which  $f$  is not differentiable.

Select all that apply:

-3

0

3

6

$$f'(0) = 0$$
$$f'(6) = 0$$

not continuous

vertical tangent

(Bonus: sharp turns/edges)

- \* If  $f$  is not continuous at  $x=c$ , then  $f$  is not differentiable at  $x=c$ .
- \* If  $f$  is continuous at  $x=c$ , it doesn't necessarily mean that it's differentiable at  $x=c$ .  
The slopes of the tangent lines might approach to different values as  $x \rightarrow c^-$  and  $x \rightarrow c^+$ .

## Example : Algebraic

$$* f(x) = \begin{cases} x^2, & x < 3 \\ 6x-9, & x \geq 3 \end{cases}$$

Is  $f(x)$  continuous and differentiable at  $x=3$ ?  
 YES YES

$$* \lim_{x \rightarrow c^-} f(3) = \lim_{x \rightarrow c^+} f(3) = f(3)$$

$$g = g = g$$

$$* \lim_{x \rightarrow 3} \frac{f(x)-f(3)}{x-3}$$

$$\lim_{x \rightarrow 3^-} \frac{f(x)-g}{x-3} = \frac{x^2-g}{x-3} = \frac{(x+3)(x-3)}{x-3} = x+3 = 6$$

$$\lim_{x \rightarrow 3^+} \frac{f(x)-g}{x-3} = \frac{6x-9-g}{x-3} = \frac{6(x-3)}{x-3} = 6$$

## Example 2: Algebraic

$$* \text{Is } f(x) = \begin{cases} x-1, & x < 1 \\ (x-1)^2, & x \geq 1 \end{cases}$$

continuous / differentiable at  $x=1$ ?  
 YES NO

$$* f(1) = (1-1)^2 = 0$$

$$\lim_{x \rightarrow 1^-} f(x) = (1-1) = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = (1-1) = 0$$

$$* \lim_{x \rightarrow 1} \frac{f(x)-f(1)}{x-1} = \lim_{x \rightarrow 1} \frac{f(x)-0}{x-1}$$

$$\lim_{x \rightarrow 1^-} \frac{(x-1)-0}{(x-1)-1} = \lim_{x \rightarrow 1^-} \frac{x-1}{x-1} = 1$$

$$\lim_{x \rightarrow 1^+} \frac{(x-1)^2-0}{(x-1)^2-1} = 0$$

# Applying the Power Rule

10.06.2025

## Power Rule

\*  $f(x) = x^n, n \neq 0$   
 $f'(x) = n \cdot x^{n-1}$

\*  $f(x) = x^2 \Rightarrow f'(x) = 2x$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(x) = x^{-100} \Rightarrow f'(x) = -100x^{-101}$

$f(x) = x^{2.571} \Rightarrow f'(x) = 2.571 \cdot x^{1.571}$

... with rewriting  
the expression

\*  $\frac{d}{dx} \left( \frac{1}{x} \right) = \frac{d}{dx} (x^{-1}) = -1 \cdot x^{-2} = \frac{-1}{x^2}$

\*  $f(x) = \sqrt[3]{x} \Rightarrow f'(x) = f'(x^{1/3}) = \frac{1}{3} \cdot x^{-2/3}$

\*  $\frac{d}{dx} \left( \sqrt[3]{x^2} \right)$  at  $x=8$  ?

$$\begin{aligned} \frac{d}{dx} \left( x^{2/3} \right) &= \frac{2}{3} x^{-1/3} \Rightarrow x=8 \Rightarrow \frac{2}{3} \cdot 8^{-1/3} \\ &= \frac{2}{3} \cdot \frac{1}{\sqrt[3]{8}} = \frac{1}{3} \end{aligned}$$

\* Power Rule:  $f(x) = x^n, n \neq 0 \Rightarrow f'(x) = n \cdot x^{n-1}$

# Derivative Rules: Introduction

10.06.2025

## Basic derivative rules

$$* \frac{d}{dx} [k] = 0$$

$$* \frac{d}{dx} [k \cdot f(x)] = k \cdot \frac{d}{dx} f(x) = k \cdot f'(x)$$

$$* \frac{d}{dx} [2 \cdot x^5] = 2 \cdot \frac{d}{dx} [x^5] = (2)(5)x^4 = 10x^4$$

$$* \frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$$

$$* \frac{d}{dx} [x^3 + x^{-4}] = 3x^2 + (-4)x^{-5} = 3x^2 - 4x^{-5}$$

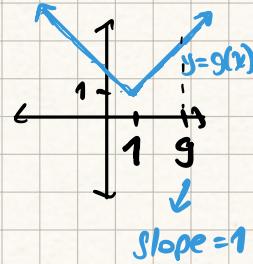
$$* \frac{d}{dx} [\underbrace{2x^2}_{4x} + \underbrace{3x}_3 + \underbrace{4}_0] = 4x + 3$$

## Example: Table

$x$	0	1	4	9	16	$g(x) =  x-1  + 1$
$f(x)$	2	1	-2	1	6	$h(x) = 3f(x) + 2g(x)$
$f'(x)$	-3	-1	1	3	4	$\frac{d}{dx} h(x) \text{ at } x=9 = ?$

$$* = 3 \cdot \frac{d}{dx} (f(x)) + 2 \cdot \frac{d}{dx} (g(x))$$

$$= 3(3) + 2 \cdot (1) = 11$$



- \* Constant Rule:  $\frac{d}{dx} [k] = 0$
- \* Multiplication by a constant:  $\frac{d}{dx} [k \cdot f(x)] = k \cdot \frac{d}{dx} [f(x)]$
- \* Sum Rule:  $\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} [f(x)] + \frac{d}{dx} [g(x)]$

# Derivative Rules: Connecting with the Power Rule

10.06.2025

## Differentiating polynomials

$$* f(x) = x^5 + 2x^3 - x^2$$

$$\frac{d}{dx} f(x) = 5x^4 + 6x^2 - 2x$$

$$* \text{ at } x=2 = ? \quad = 5(2)^4 + 6(2)^2 - 2(2) = 100 \quad \text{≈}$$

## Differentiating integer powers

$$* g(x) = \frac{2}{x^3} - \frac{1}{x^2} \Rightarrow g'(x) = ? \quad g'(2) = ?$$

$$g(x) = 2x^{-3} - x^{-2}$$

$$g'(x) = -6x^{-4} + 2x^{-3}$$

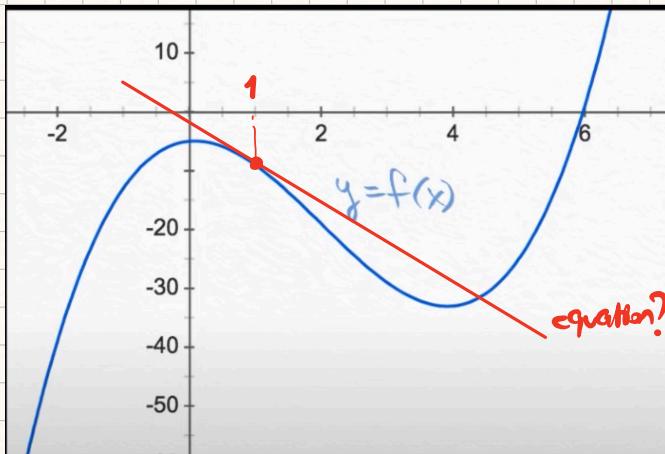
$$g'(2) = -6(2)^{-4} + 2(2)^{-3}$$

$$= \frac{-6}{16} + \frac{2}{8}$$

$$= \frac{-6+4}{16} = \frac{-2}{16} = \frac{-1}{8} \quad \approx$$

## Tangents of polynomials

$$* f(x) = x^3 - 6x^2 + x - 5$$



$$* f(1) = (1)^3 - 6(1)^2 + 1 - 5 \\ = -9$$

$$f'(x) = 3x^2 - 12x + 1$$

$$f'(1) = 3(1) - 12(1) + 1 \\ = -8$$

$$y = m \cdot x + b$$

$$-8 = -3(1) + b$$

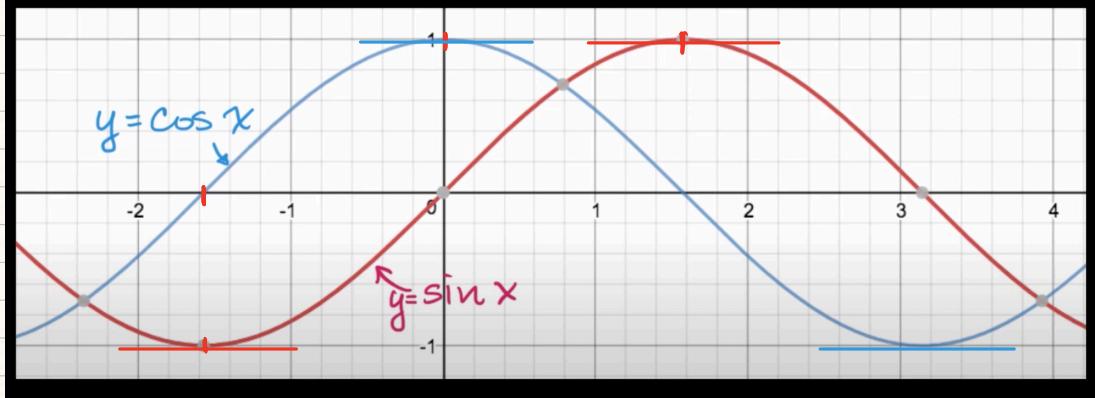
$$-1 = b \Rightarrow y = -8x - 1$$

Derivatives of  $\sin(x)$  and  $\cos(x)$ 

\*

$$\frac{d}{dx} [\sin x] = \cos(x)$$

$$\frac{d}{dx} [\cos x] = -\sin(x)$$



Example:

$$g(x) = 7 \sin(x) - 3 \cos(x) - \left(\frac{\pi}{3\sqrt{x}}\right)^2 \Rightarrow g'(x) = ?$$

$$\begin{aligned} g(x) &= 7 \sin(x) - 3 \cos(x) - \left(\pi \cdot x^{-1/3}\right)^2 \\ &= 7 \sin(x) - 3 \cos(x) - \pi^2 \cdot x^{-2/3} \end{aligned}$$

$$g'(x) = 7 \cos(x) + 3 \sin(x) + \frac{2\pi^2}{3} x^{-5/3}$$

Derivative of  $e^x$ 

\*

$$\frac{d}{dx} [e^x] = e^x$$

Derivative of  $\ln(x)$ 

\*

$$\frac{d}{dx} [\ln(x)] = \frac{1}{x}$$

\*  $\frac{d}{dx} \sin(x) = \cos(x)$  \*  $\frac{d}{dx} \cos(x) = -\sin(x)$

\*  $\frac{d}{dx} [e^x] = e^x$  \*  $\frac{d}{dx} [\ln(x)] = \frac{1}{x}$

# The Product Rule

10.06.2025

## Product Rule

$$* \frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$* \frac{d}{dx} [x^2 \sin(x)] = \underbrace{2x}_{\frac{d}{dx}(x^2)} \sin(x) + x^2 \underbrace{\cos(x)}_{\frac{d}{dx}[\sin(x)]}$$

## Differentiating products

$$* \frac{d}{dx} (e^x \cos(x)) = -e^x \sin(x) + e^x \cos(x) \\ = e^x (\cos(x) - \sin(x))$$

## Product rule with table

\* The following table lists the values of functions  $f$  and  $h$ , and of their derivatives,  $f'$  and  $h'$ , for  $x = 3$ .

$x$	$f(x)$	$h(x)$	$f'(x)$	$h'(x)$
3	6	0	6	4

Evaluate  $\frac{d}{dx}[f(x) \cdot h(x)]$  at  $x = 3$ .  $= f'(3) \cdot h(3) + f(3) \cdot h'(3)$   
 $= 6 \cdot 0 + 6 \cdot 4 = 24$

## Product rule with implicit and explicit

$$* f(-1) = 3, f'(-1) = 5, g(x) = \frac{1}{x}, F(x) = f(x) \cdot g(x)$$

$$\Rightarrow F'(-1) = ?$$

$$= f'(-1) \cdot g(-1) + f(-1) \cdot g'(-1)$$

$$= (5)(-1) + (3)(-1)$$

$$= -5 - 3 = -8$$

$$g'(x) = \frac{d}{dx} (x^{-1}) \\ = -x^{-2} \\ = \frac{-1}{x^2}$$

\* Product Rule:  $\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

# The Quotient Rule

10.06.2025

## Quotient Rule

$$* \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

$$* f(x) = \frac{x^2}{\cos(x)} \Rightarrow f'(x) = ?$$

$$= \frac{2x \cdot \cos(x) + x^2 \sin(x)}{\cos^2(x)}$$

## ... with table



- Let  $f$  be a function such that  $f(-1) = 3$  and  $f'(-1) = 5$ .

- Let  $g$  be the function  $g(x) = 2x^3$ .  $g'(x) = 6x^2$

Let  $F$  be a function defined as  $F(x) = \frac{f(x)}{g(x)}$ .

$$\begin{aligned} g(-1) &= -2 \\ g'(-1) &= 6 \end{aligned}$$

$$F'(-1) = \boxed{\phantom{00}}$$

$$F(-1) = \frac{f'(-1) \cdot g(-1) - f(-1) \cdot g'(-1)}{(g(-1))^2} = \frac{(5)(-2) - (3)(6)}{(-2)^2}$$

$$= \frac{-10 - 18}{4} = \frac{-28}{4} = \cancel{-7}$$

$$* \frac{d}{dx} \left[ \frac{5-3x}{x^2+3x} \right] = \frac{(-3)(x^2+3x) - (5-3x)(2x+3)}{(x^2+3x)^2}$$

$$= \frac{-3x^2 - 9x - 10x - 15 + 6x + 9x}{x^2(x+3)^2} = \frac{3x^2 - 4x - 15}{(x^2+3x)^2}$$

## Differentiating rational functions

\* Quotient Rule:  $\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

# Finding the Derivatives of $\tan(x)$ , $\cot(x)$ , $\sec(x)$ , and $\csc(x)$

10.06.2025

Derivatives of  $\tan(x)$  and  $\cot(x)$

$$\begin{aligned} * \frac{d}{dx} (\tan(x)) &= \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right) = \frac{\cos(x) \cdot \cos(x) + \sin(x) \cdot (-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \underline{\sec^2(x)} \end{aligned}$$
  

$$\begin{aligned} * \frac{d}{dx} (\cot(x)) &= \frac{d}{dx} \left( \frac{\cos(x)}{\sin(x)} \right) = \frac{-\sin(x) \cdot \sin(x) - \cos(x) \cdot \cos(x)}{\sin^2(x)} \\ &= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = \underline{-\csc^2(x)} \end{aligned}$$

Derivatives of  $\sec(x)$  and  $\csc(x)$

$$\begin{aligned} * \frac{d}{dx} (\sec(x)) &= \frac{d}{dx} \left[ \frac{1}{\cos(x)} \right] = \frac{0 \cdot \cos(x) + \sin(x) \cdot 1}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \end{aligned}$$
  

$$\begin{aligned} * \frac{d}{dx} (\csc(x)) &= \frac{d}{dx} \left[ \frac{1}{\sin(x)} \right] = \frac{0 \cdot \cos(x) + \cos(x)}{\sin^2(x)} \\ &= \frac{\cos(x)}{\sin^2(x)} \end{aligned}$$

 \*  $\frac{d}{dx} [\tan(x)] = \frac{1}{\cos^2(x)} = \sec^2(x)$    \*  $\frac{d}{dx} [\cot(x)] = \frac{-1}{\sin^2(x)} = -\csc^2(x)$

\*  $\frac{d}{dx} [\sec(x)] = \frac{\sin(x)}{\cos^2(x)}$    \*  $\frac{d}{dx} [\csc(x)] = \frac{\cos(x)}{\sin^2(x)}$