Reducing Rational Expressions to Lowest Terms

16.05.2025

Intro to rational expressions

* A rational expression is a fraction whose numerator and denominator are polynomials.

  * $\frac{1}{x}$ * $\frac{x+5}{x^2-4x+4}$ * $\frac{x(x+1)(2x-3)}{x-6}$

* Domain of a rational expression includes all real numbers except for those that make its denominator zero.

Reducing rational expressions to lowest forms

* $\frac{9x+3}{12x+4} = \frac{3\cancel{(3x+1)}}{4\cancel{(3x+1)}} = \frac{3}{4}$

  * If this was a fraction, we should have stated the condition, $x \neq \frac{-1}{3}$

* $\frac{x^2-9}{5x+15} = \frac{\cancel{(x+3)}(x-3)}{5\cancel{(x+3)}} = \frac{x-3}{5}, \; x \neq -3$

* $\frac{x^2+6x+5}{x^2-x-2} = \frac{\cancel{(x+1)}(x+5)}{\cancel{(x+1)}(x-2)} = \frac{x+5}{x-2}, \; x \neq 2$

* A rational expression is reduced to its lowest terms if the numerator and denominator have no factors in common.

  1. Factor the numerator and denominator
  2. List restricted values
  3. Cancel common factors
  4. Reduce the lowest terms and note any restricted values not implied by the expression.

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* Domain of a rational expression includes all real numbers except for those that make its denominator zero.
* A rational expression is reduced to its lowest terms if numerator and denominator have no factors in common.

Quiz

① $\frac{x^2+12x+36}{x^2-36} = \frac{\cancel{(x+6)}(x+6)}{\cancel{(x+6)}(x-6)} = \frac{x+6}{x-6}, \quad x \neq \mp 6$

② $\frac{2x+14}{x^2+6x-7} = \frac{2(x+7)}{(x+7)(x-1)} = \frac{2}{x-1}, \quad x \neq 1, \; x \neq -7$

③ $\frac{x^2-14x+49}{6x^2-42x} = \frac{(x-7)\cancel{(x-7)}}{6x\cancel{(x-7)}} = \frac{x-7}{6x}, \quad x \neq 7, \; x \neq 0$

④ $\frac{x^2+3x-18}{7x-21} = \frac{\cancel{(x-3)}(x+6)}{7\cancel{(x-3)}} = \frac{x+6}{7}, \quad x \neq 3$

End Behavior of Rational Functions

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* $f(x) = \dfrac{7x^2 - 2x}{15x + 5}$, what does $f(x)$ approach as x approaches $-\infty$?

  * Let's divide the numerator and denominator by the highest degree term of x in the denominator. Which is x in this case:

$$f(x) = \frac{(7x^2 - 2x)\cdot\frac{1}{x}}{(15x+5)\cdot\frac{1}{x}} = \frac{7x - 2}{15 + \frac{5}{x}}$$

$7x$ → This is going to $-\infty$

$\frac{5}{x}$ → This is going to zero

going to $-\infty$

for "larger" x (meaning higher absolute values):

* $f(x) = \dfrac{7x^2}{15x} = \dfrac{7x}{15}$

  * Only the highest degrees of x matter, because that part will grow the fastest.

* $q(x) = \dfrac{6x^5 - 2}{3x^2 + x^9}$ ⇒ Find the <u>horizontal asymptote</u> of q.

  x^9 → highest in the denom.

  = "What is the $q(x)$ approaching as x approaches to $+\infty$ and $-\infty$"

$$q(x) = \frac{\frac{6}{x^4} - \frac{2}{x^9}}{\frac{3}{x^7} + 1}$$

$\frac{6}{x^4}$ → to zero

$\frac{2}{x^9}$ → to zero

$\frac{3}{x^7}$ → to zero

numerator: to zero

denominator: to zero

to zero

* Horizontal asymptote = $y=0$

* $f(x) = \dfrac{3x^4 - 7x^2 - 1}{x^4 - 2x^3 + 3}$, what does $f(x)$ approaches as $x \to -\infty$?

$$= \frac{3 - \cancel{\frac{7}{x^2}}^{\,0} - \cancel{\frac{1}{x^4}}^{\,0}}{1 - \cancel{\frac{2}{x}}^{\,0} + \cancel{\frac{3}{x^4}}^{\,0}} = \frac{3}{1} = 3$$

* If the degree of the denominator is larger than the degree of the numerator, there is a horizontal asymptote of $y=0$, which is the end behavior of the function.
* If the degrees of the denominator and numerator are equal, then there's a horizontal asymptote of $y = \frac{a}{b}$, where a is the leading coefficient of the numerator, b is the lead. coef. of denominator.
* If the degree of the numerator is greater than the degree of the denominator, long divide the polynomials to find the quotien (what you have w/o the remainder), then $y = q(x)$, where $q(x)$ is the quotient that provides the end behavior.

Discontinuities of Rational Functions

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* $f(x) = \dfrac{x^2 - 2x - 24}{x^2 + 10x + 24}$

At each of the following values of x, select whether f has a zero, a vertical asymptote or a removable discontinuity.

	Zero	Vertical Asymp.	Remov. Disc.
$x = -6$	×	✓	×
$x = -4$	×	×	✓
$x = 6$	✓	×	×

$$f(x) = \frac{(x-6)\cancel{(x+4)}}{(x+6)\cancel{(x+4)}}$$

* Num = 0 if $x = 6$ or $x = -4$
* Den = 0 if $x = -6$ or $x = -4$

$f(x) = 0$ → $x = 6$

$$= \frac{x-6}{x+6}, \quad \boxed{x \neq -4} \rightarrow \text{removable discontinuity}$$

vertical asymptote : $x = -6$

* $h(x) = \dfrac{x^2 + 4x - 32}{x^2 - 8x + 16} = \dfrac{(x+8)\cancel{(x-4)}}{(x-4)\cancel{(x-4)}} = \dfrac{x+8}{x-4}$

$x = 4$: Vertical asymptote

$x = -8$: zero

* Inputs where the function is defined and the numerator is equal to zero are zeros of the function.
* After cancelling out the common factors, any undefined input that no longer makes the denominator equal to zero is a removable discontinuity. The remaining undefined inputs are vertical asymptotes.

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Graphs of Rational Functions

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Graphing rational functions according to asymptotes

* $f(x) = \dfrac{3x^2 - 18x - 81}{6x^2 - 54}$

* Horizontal asymptote : $|x| \to \infty$

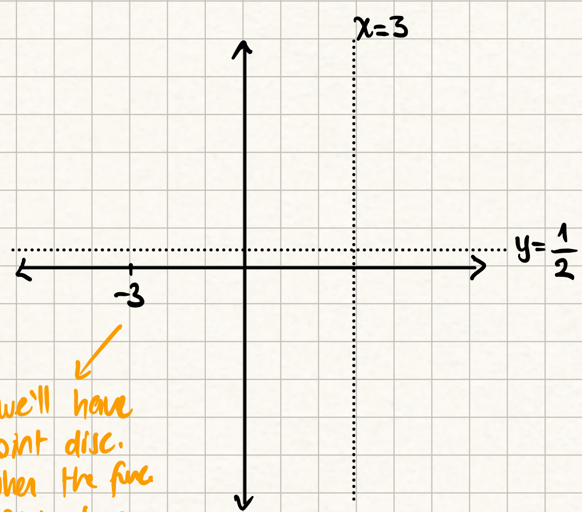
$\dfrac{3x^2 - 18x - 81}{6x^2 - 54}$ $\quad 2 = 2 \Rightarrow$ Horizontal asymptote $= y = \dfrac{3}{6} = \dfrac{1}{2}$

* Vertical asymptote(s) :

$= \dfrac{3(x^2 - 6x - 27)}{6(x^2 - 9)} = \dfrac{3(x-9)\cancel{(x+3)}}{6(x-3)\cancel{(x+3)}}$

$x \neq -3$ (Removable Discontinuity)

$x \neq 3$ (Vertical asymp.)



we'll have point disc. when the func passes here

* Having only these info. (hor. asy., vert. asy., rmv. disc.) is not enough to graph the function. We'll need to find y-intercepts, zeros, etc.

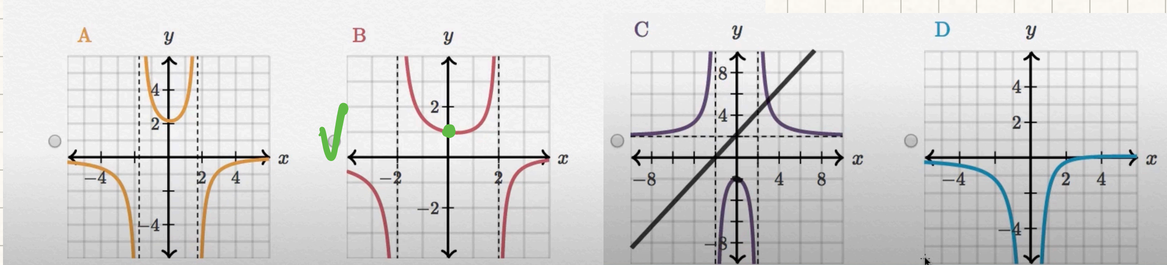
Graphs of rational functions : y-intercept

Let $f(x) = \dfrac{ax^n + bx + 12}{cx^m + dx + 12}$, where m and n are integers and a, b, c and d are unknown constants.

Which of the following is a possible graph of $y = f(x)$?
Dashed lines indicate asymptotes.

* y-intercept = $f(0)$

$f(0) = \dfrac{12}{12} = 1$



✓ B

Graphs of rational functions: horizontal asymptote

$\ast$ $f(x) = \dfrac{-x^{2} + ax + b}{1x^{2} + cx + d}$, where a, b, c, d are unknown constants.

$-1x$

what's the horizontal asymptote for $f(x) = ?$

$\ast$ $\underbrace{2 = 2}_{}$ $\Rightarrow$ $y = \dfrac{-1}{1} = -1$

highest degree terms are equal

Graphs of rational functions: vertical asymptote

$\ast$ $f(x) = \dfrac{g(x)}{x^2 - x - 6}$, where $g(x)$ is a polynomial.

Find the possible vertical asymptote for $f(x)$.

$f(x) = \dfrac{g(x)}{(x-3)(x+2)}$ $\Rightarrow$ $x \neq 3$, $x \neq -2$

$\ast$ If we have $(x-3)$ or $(x+2)$ as factors in the numerator, that one won't determine the vert. asymp., only a point of disc.

Graph of rational functions: zeros

$\ast$ $f(x) = \dfrac{2x^2 - 18}{g(x)}$ $\Rightarrow$ Find the x values for $f(x) = 0$

$= \dfrac{2(x^2 - 9)}{g(x)} = \dfrac{2(x+3)(x-3)}{g(x)}$ $\Rightarrow$ $f(-3) = 0$ or undefined

$f(3) = 0$ or undefined

Modeling with Rational Functions

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Analyzing structure word problem: cat store

* cats = c , dogs = d , bears = b

$c > d > b \Rightarrow$ which is greater : $\frac{b}{c+d+b}$ or $\frac{1}{3}$

* if $b=1,\ d=2,\ c=3 \Rightarrow \frac{1}{6} < \frac{1}{3}$

if $b=0,\ d=1,\ c=2 \Rightarrow \frac{0}{3} < 0$

* $\frac{1}{3} = \frac{1}{3b} = \frac{b}{b+b+b} \Rightarrow \frac{b}{\boxed{b+b}+b} \overset{?}{>} \frac{b}{\boxed{c+d}+b}$

$\Rightarrow b+b \overset{?}{<} c+d$ $\boxed{\text{YES!}}$

Then, 1/3 is greater

Combining mixtures example

* A partially-filled tank holds 30 L gasoline with a 18% concentration of ethanol. A fuel station is selling gasoline with a 25% concentration of ethanol.

What volume, in liters, of the fuel station gasoline would we need to add to tank to get gasoline with a 20% concentration of ethanol?

* $$\frac{3\cancel{0}\cdot\frac{18}{10\cancel{0}} + \frac{x}{4}}{30+x} = \frac{1}{5}$$

$$\frac{5.4 + 0.25x}{30+x} = \frac{1}{5}$$

$$27 + 1.25x = 30 + x$$

$$0.25x = 3$$

$$x = 12$$

Rational equations word problem: combined rates

(Example 1:)

* Ian can rake a lawn and bag the leaves in 5 hours. Xyandre can do the same in 3. Working together, how long?

* I : 1/5 lawn/hour   X : 1/3 lawn/hour

$$I+X = \frac{1}{5}+\frac{1}{3} = \frac{8}{15} \text{ lawn/hour} \Rightarrow \frac{15}{8} = 1.875 \text{ hr}$$
$$= 112.5 \text{ minutes}$$

(Example 2:)

* Together Anya and Bill stained a large porch deck in 8 hours. Anya works twice as fast. What are their ind. durations?

$$\frac{1}{a}+\frac{1}{2a} = \frac{1}{8}$$

$$\frac{3}{2a} = \frac{1}{8} \qquad a = 12$$

Anya : 12 hours

Bill = 24 hours

Eliminating solutions :

* Two hoses fill a pond together in 12 minutes. If used alone one of them fills the pond 10 minutes faster. How long each hose to fill the pond by itself?

faster hose : $\frac{f \text{ minutes}}{1 \text{ pond}} \Leftrightarrow \frac{1}{f}$ pond/minutes

slower hose : $\frac{f+10 \text{ minutes}}{1 \text{ pond}} \Leftrightarrow \frac{1}{f+10}$ pond/minutes

$$\frac{1}{f}+\frac{1}{f+10} = \frac{1 \text{ (pond)}}{12 \text{ (minutes)}} \Rightarrow \frac{f+10+f}{f(f+10)} = \frac{1}{12}$$

$$f^2+10f = 24f+120$$
$$f^2-14f-120=0$$
$$(f-20)(f+6)=0$$

f = 20 b/c −6 doesn't make sense

Reasoning about unknown variables

* $a, b \in \mathbb{Z}$, $a>0$, $b<0$, $\frac{a}{b} > ab$ ⇒ tell me interesting things about a, b, $\frac{a}{b}$, and $a \cdot b$.

* $\frac{a}{b} < 0$ — $a=2$, $b=-3$ ⇒ $\frac{-2}{3} > -6$ ✓

* $ab < 0$ — $a=3$, $b=-2$ ⇒ $-\frac{3}{2} > -6$ ✓

* $\frac{a}{b} > a \cdot b$, $(a,b) \neq (1,-1)$ — $a=1$, $b=-2$ ⇒ $\frac{-1}{2} > -2$ ✓

* $b \cdot \frac{a}{b} < a \cdot b \cdot b$ — $a=1$, $b=-1$ ⇒ $\frac{-1}{1} > -1 \cdot 1$ ✗

$\Rightarrow a < a \cdot b^2$

$1 < b^2$

$b^2 > 1 \Rightarrow b < -1$ or $\cancel{b > 1}$ b/c $b<0$

Reasoning about unknown variables: divisibility

* $a, b, c \in \mathbb{Z}$, $a, b, c > 0$, $\frac{a+b}{c} \in \mathbb{Z}$, $\frac{a}{c} \in \mathbb{Z}$

$\Rightarrow \frac{b}{c} \in \mathbb{Z}$?

* $\underbrace{\frac{a+b}{c}}_{\in \mathbb{Z}} = \underbrace{\frac{a}{c}}_{\in \mathbb{Z}} + \underbrace{\frac{b}{c}}_{\text{must be } \in \mathbb{Z}}$

Structure in rational expression

* $P(t) = \frac{(t-a)(a-t)(t-a)(t-b)}{\sqrt{a^2+b^2}}$, $c>b>a>0$ ⇒ which one is larger? (A) $P(c)$ (B) # times $p(t)=0$?

(B) $t-a=0 \Rightarrow t=a$ or $t-b=0 \Rightarrow t=b$ $\Big\}$ 2 times — (B) = 2

(A) $P(c) = \frac{\overset{+}{(c-a)}\overset{-}{(a-c)}\overset{+}{(c-a)}\overset{+}{(c-b)}}{\sqrt{a^2+b^2}}$ — (A) < 0 ⇒ B > A

Multiplying and Dividing Rational Expressions

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Monomials:

* $\frac{\cancel{6}^{2}x^{\cancel{3}^{2}}}{5} \cdot \frac{2}{\cancel{3x}} = \frac{4x^2}{5}, \; x \neq 0$

* $\frac{2x^4}{7} \div \frac{5x^4}{4} = \frac{2\cancel{x^4}}{7} \cdot \frac{4}{5\cancel{x^4}} = \frac{8}{35}, \; x \neq 0$

Multiplying Rational Expressions

* $\frac{x^2-9}{x^2-10x+25} \cdot \frac{4x-20}{x^2+5x+6} = \frac{\cancel{(x+3)}(x-3)(4)\cancel{(x-5)}}{\cancel{(x-5)}(x-5)\cancel{(x+3)}(x+2)}$

$x \neq 5$
$x \neq -3$
$x \neq -2$

$= \frac{4x-12}{x^2-3x-10}, \; x \neq \{-3,-2,5\}$

Dividing rational expressions

* $\frac{\frac{x^2-3x-4}{-3x-15}}{\frac{x^2-16}{x^2-x-30}} = \frac{x^2-3x-4}{-3x-15} \cdot \frac{x^2-x-30}{x^2-16} = \frac{\cancel{(x-4)}(x+1)(x-6)\cancel{(x+5)}}{-3\cancel{(x+5)}(x+4)\cancel{(x-4)}}$

$\neq 0$
$x \neq 6$
$x \neq -5$

$x \neq -5$
$x \neq -4$
$x \neq 4$

$= \frac{(x+1)(x-6)}{-3(x+4)}, \; x \neq \{-5,-4,4,6\}$

* When multiplying rational expressions, we must identify and exclude any values that would make the denominator of any expression equal to zero, as these are restrictions on the domain.

* When dividing rational expressions, we must additionally exclude any values that would make the numerator of the divisor zero, since division by zero is undefined.

Adding and Subtracting Rational Expressions

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Like denominators

* $$\frac{\boxed{6}}{2x^2-7} + \frac{\boxed{-3x-8}}{2x^2-7} = \frac{\boxed{6} + \boxed{-3x-8}}{2x^2-7} = \frac{-2-3x}{2x^2-7}$$

Unlike denominators

* $$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

* $$\frac{5x}{\underset{(3x+1)}{2x-3}} + \frac{-4x^2}{\underset{(2x-3)}{3x+1}} = \frac{5x(3x+1) + (-4x^2)(2x-3)}{(2x-3)(3x+1)}$$

$$= \frac{15x^2+5x-8x^3+12x^2}{(2x-3)(3x+1)} = \frac{-8x^3+27x^2+5x}{(2x-3)(3x+1)}$$

* $$\frac{-5x}{8x+7} - \frac{6x^3}{3x+1} = \frac{(-5x)(3x+1) - (6x^3)(8x+7)}{(8x+7)(3x+1)}$$

$$= \frac{-15x^2-5x-48x^4-42x^3}{(8x+7)(3x+1)} = \frac{-48x^4-42x^3-15x^2-5x}{(8x+7)(3x+1)}$$

Least common denominators

* The least common multiple of the denominators in two or more fractions is called the least common denominator. Using the least common denominator is "almost" a must with rational expressions in order to avoid performing arithmetics with higher degree polynomials

* The least common multiple (LCM) of the denominators in two or more fractions is called the least common denominator. Using the least common denominator is "almost" a must with rational expressions in order to avoid performing arithmetics with higher degree polynomials.

* Find the least common multiple (LCM) of:

$3z^3 - 6z^2 - 9z$ and $7z^4 + 21z^3 + 14z^2$

$(3z)(z^2 - 2z - 3)$ $(7z^2)(z^2 + 3z + 2)$

$(3)(z)(z-3)(z+1)$ $(7)\cancel{(z)}(z)\cancel{(z+1)}(z+2)$

b/c we have it on the lef

$$LCM = (3)(z)(z-3)(z+1)(7)(z)(z+2)$$
$$= 21z^2(z-3)(z+1)(z+2)$$

Subtracting rational expressions: factored denominators

* $$\frac{-x^3}{(x+8)(9x-5)} - \frac{3}{(x+6)(9x-5)} =$$

$(x+6)$ $(x+8)$

$$= \frac{(-x^3)(x+6) - (3)(x+8)}{(x+8)(9x-5)(x+6)} = \frac{-x^4 - 6x^3 - 3x - 24}{(x+8)(9x-5)(x+6)}$$

Subtracting rational expressions

* $$\frac{a-2}{a+2} - \frac{a-3}{a^2+4a+4}$$: Find the difference. Express as a simplified rational expression. and state the domain.

$$\frac{(a-2)}{(a+2)} - \frac{(a-3)}{(a+2)(a+2)} = \frac{(a+2)(a-2) - (a-3)}{(a+2)(a+2)}$$

$(a+2)$

$$= \frac{a^2 - 4 - a + 3}{(a+2)(a+2)} = \frac{a^2 - a - 1}{(a+2)^2}, \quad a \neq -2$$