Orthogonality and Change of Basis

29.05.2024

Orthogonal Complements

$\vec{v} \cdot \vec{x} = 0 \Rightarrow v \perp x$

How about orthogonality of subspaces?

$\mathbb{R}^n \quad V = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

$\mathbb{R}^n \quad V^\perp = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$

cannot be different

can be different

* "V is orthogonal complement of $V^\perp$ (and vice versa)
* Every vector in one subspace is orthogonal to every vector in the other.

$$V^\perp = \{\vec{x} \in \mathbb{R}^n \mid \underbrace{\vec{x} \cdot \vec{v} = 0}_{\text{for every vector } \vec{v} \in V}\}$$

* If V is a subspace, $V^\perp$ is also a subspace.
  * $\left.\begin{array}{l} \vec{x}_1 \cdot v = 0 \\ x_2 \cdot v = 0 \end{array}\right\}$ $\quad \vec{x}_1 \cdot v + \vec{x}_2 \cdot v = 0$
    $\Rightarrow (\vec{x}_1 + x_2) \cdot \vec{v} = 0 \Rightarrow$ closed under addition
  * $c \cdot \vec{x}_1 \cdot \vec{v} = 0 \quad \Rightarrow c \cdot \underbrace{(\vec{x}_1 \cdot \vec{v})}_{0} = 0 \Rightarrow$ closed under multiplication

* $V = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right)$

$$V^\perp = \left\{ \vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = 0 \right\}$$

$\vec{x} = (x_1, x_2, x_3)$

$$\Rightarrow \begin{array}{l} 1 \cdot x_1 + 0 x_2 + 2 x_3 = 0 \\ -1 x_1 + 1 x_2 + 3 x_3 = 0 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 1 & 3 & 0 \end{array}\right]$$

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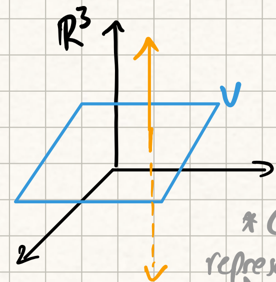
* $V^\perp = \{x \in \mathbb{R}^n \mid \vec{x} \cdot \vec{v} = 0 \text{ for every } \vec{v} \in V\}$
* $(V^\perp)^\perp = V$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 2x_3 = 0 \\ x_2 + 5x_3 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = -5x_3 \end{array}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \Rightarrow V^{\perp} = \text{span}\left(\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} \right)$$

* Conclusion: Our space is 3D ($\mathbb{R}^3$). Our subspace is a 2D plane. The orthogonal complement is a 1D line, which is orthogonal to the plane.



* Coordinates not representing the last example. Dimensions matter!

Quiz

① Find the orthogonal complement of $W = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ -5 \end{bmatrix} \right)$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 0 \\ 2 & 0 & 3 & -5 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 2 & -4 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_1 = -2x_4 \\ x_3 = 3x_4 \end{array} \Rightarrow W^{\perp} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right)$$

(2) Rewrite the orthogonal complement of $V = \begin{bmatrix} -2y+z \\ y \\ z \end{bmatrix}$.

$$V = \left\{ y \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; y,z \in \mathbb{R}^3 \right\}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = -2x_3 \end{array}$$

$$V^T = \text{span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right)$$

(3)

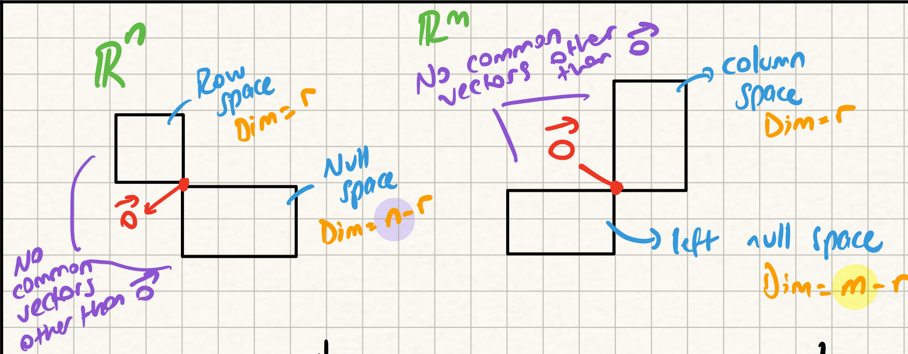
$$V = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -8 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -5 \\ -1 \end{bmatrix} \right) \Rightarrow V^T = ?$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 4 & -8 & 8 & 0 \\ 1 & 3 & -5 & -1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & -2 & 3 & 5 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 5 & -8 & -6 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & 9 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & -8 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -14 & 0 \\ 0 & 0 & 1 & -8 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_1 = -x_4 \\ x_2 = 14x_4 \\ x_3 = 8x_4 \end{array} \Rightarrow V^T = \text{span}\left(\begin{bmatrix} -1 \\ 14 \\ 8 \\ 1 \end{bmatrix} \right)$$

Orthogonal complements of the fundamental spaces



$A_{m\times n}$

* $N(A) = (C(A^T))^\perp$
* $C(A^T) = (N(A))^\perp$
* $C(A) = (N(A^T))^\perp$
* $N(A^T) = (C(A))^\perp$

* $\mathbb{R}^n$: V $\quad V^\perp$

$$\mathrm{Dim}(V) + \mathrm{Dim}(V^\perp) = n$$

Ex

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ -1 & 1 & 2 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \text{What are the dimensions of the fundamental spaces?}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{array}{l} r = 3 \text{ (rank)} \\ m = 3 \text{ (\# of rows)} \\ n = 4 \text{ (\# of columns)} \end{array}$$

Column space $\quad \mathrm{Dim}(C(A)) = r = 3 \quad \mathbb{R}^3 \quad = \mathrm{span}\left(\begin{bmatrix} \vdots \end{bmatrix}, \begin{bmatrix} \vdots \end{bmatrix}, \begin{bmatrix} \vdots \end{bmatrix}\right)$

Row space $\quad \mathrm{Dim}(C(A^T)) = r = 3 \quad \mathbb{R}^4 \quad = \leftrightarrow$

Null Space $\quad \mathrm{Dim}(N(A)) = n - r = 1 \quad \mathbb{R}^4 \quad = \mathrm{span}\left(\begin{bmatrix} \vdots \end{bmatrix}\right)$

Left Null Space $\quad \mathrm{Dim}(N(A^T)) = m - r = 0 \quad \mathbb{R}^3 \quad = \vec{0}_3 = (0,0,0)$

* "en güzel kürek, boş denizde çekilir."

$\mathbb{R}$"n" — "row space" — "null space"

* "En güzel kürek, boş denizde çekilir."

$A_{m\times n}$: n — row space — null space

$\Rightarrow C(A^T) = N(A)^\perp$, $\mathrm{Dim}(C(A^T)) = r$, $\mathrm{Dim}(N(A)) = n - r$, $C(A^T), N(A) \in \mathbb{R}^n$

$\Rightarrow C(A) = N(A^T)^\perp$, $\mathrm{Dim}(C(A)) = r$, $\mathrm{Dim}(N(A^T)) = m - r$, $C(A), N(A^T) \in \mathbb{R}^m$

Quiz

(1) Find the dimensions of 4 fundamental spaces for $A = \begin{bmatrix} -1 & 3 & 5 & -1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -14 & -6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \Rightarrow r = 3 \qquad m = 3, \quad n = 4$$

$Dim(C(A)) = 3$ in $\mathbb{R}^3$

$Dim(C(A^T)) = 3$ in $\mathbb{R}^4$

$Dim(N(A)) = 1$ in $\mathbb{R}^4$

$Dim(N(A^T)) = 0$ in $\mathbb{R}^3$

(2) Dimensions of 4 fundamental spaces of $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 1 & -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow r = 2 \qquad m = 3, \quad n = 3$$

$\Rightarrow Dim(C(A)) = 2$, in $\mathbb{R}^3$

$Dim(C(A^T)) = 2$, in $\mathbb{R}^3$

$Dim(N(A)) = 1$, in $\mathbb{R}^3$

$Dim(N(A^T)) = 1$, in $\mathbb{R}^3$

Projection onto the subspace

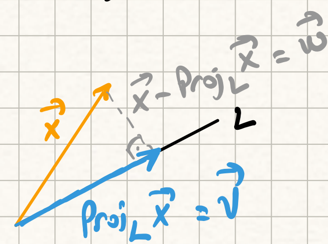
③ Find the dimensions of 4 fundamental spaces of

$$A=\begin{bmatrix} 2 & -3 & 6 & -5 & -6 \\ 4 & -5 & 12 & -11 & -14 \\ 2 & -2 & 6 & -6 & -8 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 & 3 & -5/2 & -3 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & \cancel{1}\ 0 & 0 & \cancel{-1}\ 0 & \cancel{-2}\ 0 \end{bmatrix}$$

$\Rightarrow$ rank = 2 $\quad Dim(C(A)) = 2$ in $\mathbb{R}^3 \quad Dim(N(A)) = 3$ in $\mathbb{R}^5$

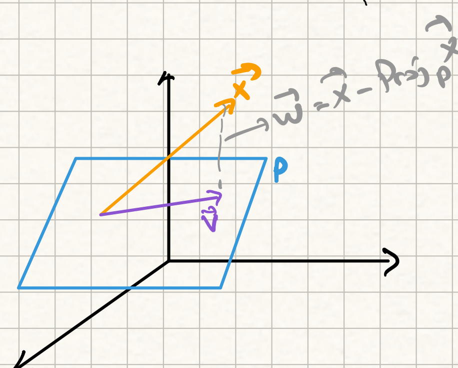
$m = 3 \quad Dim(C(A^T)) = 2$ in $\mathbb{R}^5 \quad Dim(N(A^T)) = 1$ in $\mathbb{R}^3$

$n = 5$



$\vec{v} \in L \Rightarrow \vec{w} \in L^{\perp}$

? So, imagine we want to project a vector on a plane?



$\vec{v} = Proj_P \vec{x}$

$\vec{w} = \vec{x} - Proj_P \vec{x}$

$\vec{v} \in P \Rightarrow \vec{w} \in P^{\perp}$

* V in $\mathbb{R}^n$, A is the matrix of the column vectors that form the basis for V:

$$* \ Proj_V \vec{x} = A\,(A^T A)^{-1} \cdot A^T \cdot \vec{x}$$

! Note that we use this formula when A is not invertible. If it is, it means we are projecting $\vec{x}$ to the all space $\mathbb{R}^n$, which gives us the $\vec{x}$ itself.

* $Proj_V \vec{x} = A \cdot (A^T A)^{-1} A^T \vec{x}$

* If A is invertible, then $Proj_V \vec{x} = \vec{x}$

Ex: $V = \text{span}\left(\begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right) \Rightarrow$

$$A = \begin{bmatrix}2 & 1\\1 & 0\\1 & -1\end{bmatrix} \quad A^T = \begin{bmatrix}2 & 1 & 1\\1 & 0 & -1\end{bmatrix}$$

$$A^TA = \begin{bmatrix}2 & 1 & 1\\1 & 0 & -1\end{bmatrix} \cdot \begin{bmatrix}2 & 1\\1 & 0\\1 & -1\end{bmatrix} = \begin{bmatrix}6 & 1\\1 & 2\end{bmatrix}$$

$$(A^TA)^{-1} = \left[\begin{array}{cc|cc}6 & 1 & 1 & 0\\1 & 2 & 0 & 1\end{array}\right] = \left[\begin{array}{cc|cc}1 & 0 & 2/11 & -1/11\\0 & 1 & -1/11 & 6/11\end{array}\right]$$

$$\Rightarrow A \cdot (A^T A)^{-1} \cdot A^T \cdot \vec{x} = \begin{bmatrix}2 & 1\\1 & 0\\1 & -1\end{bmatrix} \cdot \begin{bmatrix}2/11 & -1/11\\-1/11 & 6/11\end{bmatrix} \cdot \begin{bmatrix}2 & 1 & 1\\1 & 0 & -1\end{bmatrix} \cdot \vec{x}$$

$$= \begin{bmatrix}3/11 & 4/11\\2/11 & -1/11\\3/11 & -7/11\end{bmatrix} \cdot \begin{bmatrix}2 & 1 & 1\\1 & 0 & -1\end{bmatrix} \cdot \vec{x} = \begin{bmatrix}10/11 & 3/11 & -1/11\\3/11 & 2/11 & 3/11\\-1/11 & 3/11 & 10/11\end{bmatrix} \cdot \vec{x}$$

$$= \frac{1}{11}\begin{bmatrix}10 & 3 & -1\\3 & 2 & 3\\-1 & 3 & 10\end{bmatrix} \vec{x}$$

Quiz

① Find the formula for $\text{Proj}_V \vec{x}$ for $V = \text{span}\left(\begin{bmatrix}-2\\0\\-2\end{bmatrix} \begin{bmatrix}0\\4\\2\end{bmatrix}\right)$

$$\Rightarrow A \cdot (A^TA)^{-1} \cdot A^T \cdot \vec{x} = ?$$

$$A = \begin{bmatrix}-2 & 0\\0 & 4\\-2 & 2\end{bmatrix} \quad A^T = \begin{bmatrix}-2 & 0 & -2\\0 & 4 & 2\end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 20 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \left[\begin{array}{cc|cc} 8 & -4 & 1 & 0 \\ -4 & 20 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -1/2 & 1/8 & 0 \\ 0 & 1 & 1/36 & 1/18 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 10/72 & 1/36 \\ 0 & 1 & 1/36 & 1/18 \end{array}\right]$$

$$\Rightarrow A \cdot (A \cdot A^T)^{-1} \cdot A^T \cdot \vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10/72 & 1/36 \\ 1/36 & 1/18 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -20/72 + 0 & -2/36 + 0 \\ 0 + 4/36 & 0 + 4/18 \\ -20/72 + 2/36 & -2/36 + 2/18 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} \cdot \vec{x}$$

$$= \begin{bmatrix} -5/18 & -1/18 \\ 1/9 & 2/9 \\ -2/9 & 1/18 \end{bmatrix} \cdot \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 10/18 & -4/18 & 10/18 - 2/18 \\ -2/9 & 8/9 & -2/9 + 4/9 \\ 4/9 & 4/18 & 4/9 + 2/18 \end{bmatrix}$$

$$= \begin{bmatrix} 5/9 & -2/9 & 4/9 \\ -2/9 & 8/9 & 2/9 \\ 4/9 & 2/9 & 5/9 \end{bmatrix} \cdot \vec{x} = \frac{1}{9} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \cdot \vec{x}$$

(2) If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \text{ and } \vec{x}_2 = \frac{1}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

* $\text{Proj}_S \vec{x} = A \cdot (A^T A)^{-1} \cdot A^T \cdot \vec{x}$

$$A^T = \begin{bmatrix} 1/3 & 0 & -1/3 & 2/3 \\ 0 & 1/3 & 1/3 & -1/3 \end{bmatrix}, \quad A = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \\ -1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 1/9 + 1/9 + 4/9 & -1/9 - 2/9 \\ -1/9 - 2/9 & 1/9 + 1/9 + 1/9 \end{bmatrix} = \begin{bmatrix} 6/9 & -3/9 \\ -3/9 & 3/9 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

$$(A^T \cdot A)^{-1} = \left[\begin{array}{cc|cc} 2 & -1 & 3 & 0 \\ -1 & 1 & 0 & 3 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 3 & 3 \\ -1 & 1 & 0 & 3 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 6 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \\ -1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 0 & -1/3 & 2/3 \\ 0 & 1/3 & 1/3 & -1/3 \end{bmatrix}$$

$$= \begin{bmatrix}1&1\\1&2\\0&1\\1&0\end{bmatrix}\cdot\begin{bmatrix}1/3&0&-1/3&2/3\\0&1/3&1/3&-1/3\end{bmatrix}$$

$$= \begin{bmatrix}1/3&1/3&0&1/3\\1/3&2/3&1/3&0\\0&1/3&1/3&-1/3\\1/3&0&-1/3&2/3\end{bmatrix}\cdot\vec{x} = \frac{1}{3}\cdot\begin{bmatrix}1&1&0&1\\1&2&1&0\\0&1&1&-1\\1&0&-1&2\end{bmatrix}\cdot\vec{x}$$

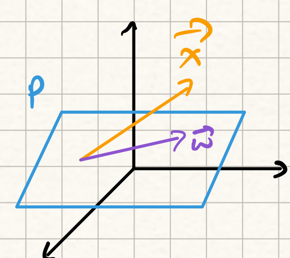
3) If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right)$$

* A is a square matrix, therefore:

$$\text{Proj}_V \vec{x} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot\vec{x} = \vec{x}$$

Least squares solution



$\vec{w} = \text{Proj}_P \vec{x}$ is the vector that exists in P and also closest to $\vec{x}$.

* $A\vec{x} = \vec{b}$ ⇒ If there is no real solution for $\vec{x}$, we use the least squares solution.

Ex 1

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$x_1 = 2$
$x_2 = 1$

$$A \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{b}$$

* we get $\vec{b}$, exactly. which means $\vec{b}$ is in $c(A)$.
* If there is no $\vec{x}$ that solves $A \cdot \vec{x} = \vec{b}$ for a specific $\vec{b}$ (because $\vec{b}$ is not in $c(A)$), we look for the least squares solution, which is $A \cdot \vec{x}^* = \mathrm{Proj}_{c(A)} \vec{b}$. The formula is

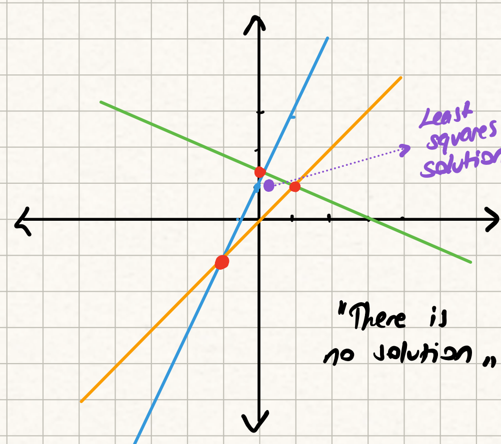
$$\boxed{A^T A \vec{x}^* = A^T \cdot \vec{b}}$$

Ex:

$-2x + y = 1$ $\quad y = 2x + 1$

$-x + y = 0$ $\quad y = x$

$x + 2y = 3$ $\quad y = -\frac{1}{2}x + \frac{3}{2}$



"There is no solution"

$\Rightarrow$ **Least squares solution:**

$$\begin{array}{ccccc} A & & \vec{x} & & \vec{b} \\ \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} & \cdot & \begin{bmatrix} x \\ y \end{bmatrix} & = & \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \end{array}$$

$$A^T \cdot A \cdot \vec{x}^* = A^T \cdot b$$

$$\begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

* Least Square Solution: "There is no solution ($\vec{x}$) to this system, but this ($\vec{x}^*$) is as closest as we can get": $A^T A \vec{x}^* = A^T \vec{b}$

$$\Rightarrow \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 6 & -1 & 1 \\ -1 & 6 & 7 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 35 & 43 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -6 & -7 \\ 0 & 1 & 43/35 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 13/35 \\ 0 & 1 & 43/35 \end{array}\right]$$

$$\Rightarrow \vec{x}^* = \begin{bmatrix} 13/35 \\ 43/35 \end{bmatrix} \cong \begin{bmatrix} 0.37 \\ 1.23 \end{bmatrix}$$

* The purple dot on the graph.
* The point where the sum of the distances to the other 3 points is the minimum.
* "There is no solution to this system, but this is the closest we can get."

Quiz (1) Find the least squares solution to the system:

$$\begin{aligned} x - 2y &= 5 \\ 3x + y &= -6 \\ -x - 2y &= -2 \end{aligned}$$

$$* \quad A^T \cdot A \cdot \vec{x}^* = A^T \cdot \vec{b} \Rightarrow$$

$$\begin{bmatrix} 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & -2 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -6 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11 & 3 \\ 3 & 9 \end{bmatrix} \cdot \vec{x} = \begin{bmatrix} -11 \\ -12 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 11 & 3 & -11 \\ 1 & 3 & -4 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 3 & -4 \\ 0 & 1 & -11/10 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -7/10 \\ 0 & 1 & -11/10 \end{array}\right] \Rightarrow x^* = \begin{bmatrix} -7/10 \\ -11/10 \end{bmatrix}$$

(2) Find the least squares solution to the system:

$$3x - 2y = -6$$
$$x - 5y = -5$$
$$x + y = 4$$

$$\begin{bmatrix} 3 & 1 & 1 \\ -2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & -5 \\ 1 & 1 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 11 & -10 \\ -10 & 30 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} -19 \\ 41 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & -41/10 \\ 11 & -10 & -19 \end{array}\right]$$

$$= \begin{bmatrix} 1 & -3 & -41/10 \\ 0 & 23 & 261/10 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -41/10 \\ 0 & 1 & 261/230 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 16/23 \\ 0 & 1 & 261/230 \end{array}\right]$$

(3) Find the least squares solution to the system:

$$x + 2y = -4$$
$$x - y = 3$$
$$y = 2$$

$$A^T \cdot A \cdot \vec{x}^* = A^T \cdot \vec{b}$$

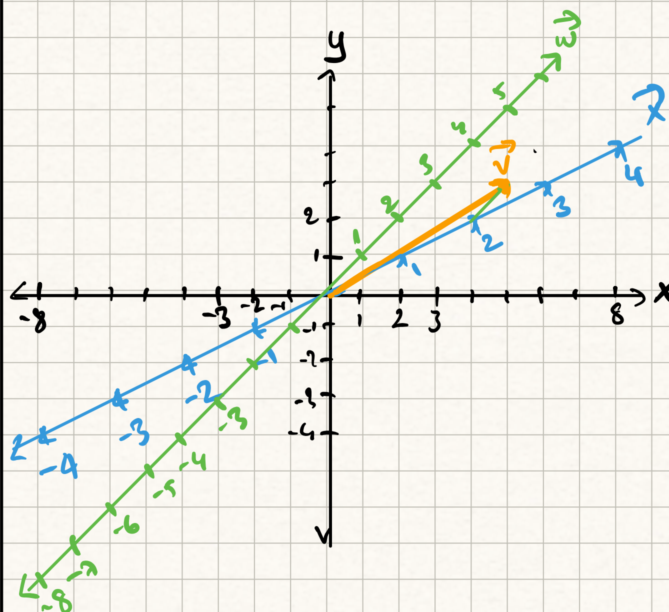
$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} \cdot \vec{x}^* = \begin{bmatrix} -1 \\ -9 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 6 & -9 \\ 2 & 1 & -1 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 6 & -9 \\ 0 & -11 & 17 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 6 & -9 \\ 0 & 1 & -17/11 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 3/11 \\ 0 & 1 & -17/11 \end{array}\right]$$

Coordinates in a new basis

* Why change basis? It might help solve problems quicker.



$i = (1,0)$, $j = (0,1)$

$B = \{i, j\}$

$\vec{v} = (5,3) \Rightarrow \vec{v} = 5i + 3j$

Let's change the basis to

$\vec{x} = (2,1)$, $\vec{w} = (1,1)$

$B = \{\vec{x}, \vec{w}\}$

Now:

$\vec{v} = 2\vec{x} + \vec{w}$

$$[\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

* How to solve $[\vec{v}]_B$?

$$\underbrace{\begin{array}{cc} \vec{x} & \vec{w} \\ \hline \multicolumn{2}{c}{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \end{array}}_{\text{"change of basis matrix"}} \cdot [\vec{v}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow [\vec{v}]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

* $A \cdot [\vec{v}]_B = \vec{v}$.

$\Rightarrow A^{-1} \cdot A \cdot [\vec{v}]_B = A^{-1} \cdot \vec{v}$

$\Rightarrow I \cdot [\vec{v}]_B = A^{-1} \cdot \vec{v}$

$\Rightarrow \boxed{[\vec{v}]_B = A^{-1} \cdot \vec{v}}$ → This means if we find the inverse of the change of basis matrix, we can easily calculate $[\vec{v}]_B$ for any $\vec{v}$.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = ?$ $\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right]$

$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow [\vec{v}]_B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \cdot \vec{v}$

Quiz (1) The vectors $\vec{v} = (2,2,3)$, $\vec{s} = (-6,0,2)$, and $\vec{w} = (2,-2,-5)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\vec{x} = -2i + k$ into the alternate basis.

$A = \begin{bmatrix} 2 & -6 & 2 \\ 2 & 0 & -2 \\ 3 & 2 & -5 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 3 & 2 & -5 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & 1 & -1 & 1/2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -3 & 1 & -1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 3 & -2 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1/2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$

② $A = \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & -10 & -10 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -3 & -1 \\ 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

③ $A = \begin{bmatrix} 1 & -2 \\ -5 & 0 \end{bmatrix}$ $A^{-1} \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} = ?$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & -10 & 5 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -1/2 & -1/10 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1/5 \\ 0 & 1 & -1/2 & -1/10 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 0 & -1/5 \\ -1/2 & -1/10 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Transformation matrix for a basis

* $T(\vec{x}) = A \cdot \vec{x}$: we assume that T is in standard basis.

* $[T(\vec{x})]_B = M [\vec{x}]_B$: If we switch to alternate basis, the formula is the same.

* $C [\vec{x}]_B = \vec{x}$

change of basis matrix

* $M = C^{-1} \cdot A \cdot C$

$$\begin{tikzcd}
 & \text{Domain} & \text{Codomain} \\
\text{Standard Basis} & \vec{x} \arrow[r, "T"', "(A)"] \arrow[d, "C^{-1}"'] & T(\vec{x}) \\
\text{Alternate Basis} & {[\vec{x}]_B} \arrow[r, "T"', "(M)"] & {[T(\vec{x})]_B} \arrow[u, "C^{-1}"']
\end{tikzcd}$$

Ex:

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \cdot \vec{x} \qquad B = \mathrm{Span}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right)$$

$$[\vec{x}]_B = (1,3) \quad \Rightarrow \quad M([\vec{x}]_B) = ?$$

$$C = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \qquad C^{-1} = \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 3 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 1/3 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & -1 & -2/3 \\ 0 & 1 & 0 & -1/3 \end{array}\right]$$

$$M = C^{-1} \cdot A \cdot C$$

$$= \begin{bmatrix} -1 & -2/3 \\ 0 & -1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2/3 \\ 0 & -1/3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ -2 & 10 \end{bmatrix} = \begin{bmatrix} 7/3 & -26/3 \\ 2/3 & -10/3 \end{bmatrix}$$

$$= \frac{1}{3} \cdot \begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix}$$

---

*

$$\begin{tikzcd}
 & \text{Domain} & \text{Codomain} \\
\text{Standard basis} & \vec{x} \arrow[r, "A \cdot \vec{x}"] \arrow[d, "C^{-1} \cdot \vec{x}"'] & T(\vec{x}) \arrow[d, "C^{-1} \cdot T(\vec{x})"] \\
\text{Alternate basis} & {[\vec{x}]_B} \arrow[r, "M \cdot [\vec{x}]_B"'] & {[T(\vec{x})]_B}
\end{tikzcd}$$

* $A = T(\vec{x})$
* $B =$ Basis
* $C =$ Change of basis matrix $= B^T$
* $M = C^{-1} \cdot A \cdot C$

$$\Rightarrow [T(\vec{x})]_B = M \cdot \vec{x} = \frac{1}{3} \cdot \begin{bmatrix} 7 & -26 \\ 2 & -10 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -71/3 \\ -28/3 \end{bmatrix}$$

Quiz Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $[\vec{x}]_B = (5, 4, -2)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right)$$

```
[1]  import numpy as np
```

```
[5]  x = np.array([5, 4, -2])
     A = np.array([[-2, -2, 1], [1, 0, -2], [0, 1, 0]])
     C = np.array([[1, 0, 2], [-1, 1, 1], [1, -1, -2]])
```

```
[7]  C_inv = np.linalg.inv(C)
```

```
[11] C_inv @ A @ C @ x
```

```
array([-15., -36.,  12.])
```