Solving Systems of Linear Equations

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How do we solve a system of equations with two variables?

Method 1: Manipulating equations so that we can get rid of one of the variables.

$$3(5a + b = 17) \qquad 4a - 3b = 6 \Rightarrow \begin{aligned} 15a + 3b &= 51 \\ + \quad 4a - 3b &= 6 \\ \hline 19a + 0b &= 57 \\ \boxed{a = 3} \end{aligned}$$

$$5a + b = 17 \qquad 5.3 + b = 17 \qquad \boxed{b = 2}$$

Method 2: Re-organize one of the equations so that you can define one of the variables in terms of other.

$$5a + b = 17 \Rightarrow b = 17 - 5a$$
$$4a - 3b = 6 \longrightarrow 4a - 3(17 - 5a) = 6$$
$$4a - 51 + 15a = 6$$
$$19a = 57$$
$$\boxed{a = 3}$$

$$5a + b = 17 \qquad 5.3 + b = 17 \qquad \boxed{b = 2}$$

What if the system is redundant?

$$(a + b = 10) \cdot -2 \qquad 2a + 2b = 20$$

$$\begin{aligned} -2a - 2b &= -20 \\ + \quad 2a + 2b &= 20 \\ \hline \boxed{0 = 0} \end{aligned}$$

Solved System:
$$\boxed{a = x} \qquad \boxed{b = 10 - x}$$

"Solution has 1 degree of freedom, which is x. The solutions form a line."

$$(a + b = 10) \cdot -2 \qquad 2a + 2b = 24$$

$$-2a - 2b = -20 \qquad 2a + 2b = 24$$

$$\boxed{0 = -4}$$

CONTRADICTION!

Solved System

N/A

Original System

$$5a + b = 17$$
$$4a - 3b = 6$$

Intermediate System

$$a + 0.2b = 3.4$$
$$b = 2$$

Solved System

$$a = 3$$
$$b = 2$$

Original Matrix

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$$

Upper Diagonal Matrix

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

Row Echolon Form

Diagonal Matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reduced Row Echolon Form

Row Echolon Form Rules

* Main diagonal can be 0s or 1s.
* Below the main diagonal, everything must be 0.
* Any number can exist on the right-side of 1s.
* Only zeros are allowed on the right side of zeros.

$$\begin{matrix} 1 & n & n & n & n \\ 0 & 1 & n & n & n \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

The first non-zero element of a row is called the pivot. For a matrix to be in row echelon form:

1. All rows without pivots must be at the bottom
2. All pivots have to be at the right-side of the pivot of the row above.

* Some texts say that the pivots also have to be reduced to 1. Even if this is not a "must" it's very handy because we must do it for the reduced row echelon form anyways.

What are elementary row operations?

① Switching rows

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 & 3 \\ 5 & 1 \end{bmatrix}$$

② Multiplying a row by a non-zero scalar

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 50 & 10 \\ 4 & 3 \end{bmatrix}$$

③ Adding a row to another row

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\begin{array}{r} \begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \\ + \quad \\ \hline \begin{bmatrix} 9 & 4 \end{bmatrix} \end{array}$$

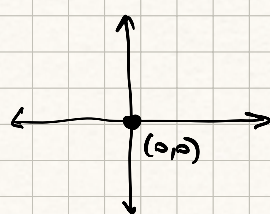
System 1	System 2	System 3
The dog is black.	The dog is black.	The dog
The cat is orange.	The dog is black.	The dog
Two Sentences	Two sentences.	Two sentences
Two pieces of information.	One piece of info.	Zero pieces of info*
Rank = 2	Rank = 1	Rank = 0

*About the color.

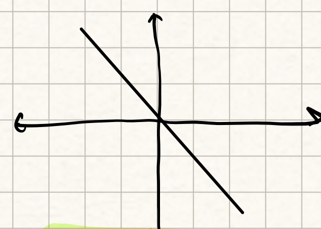
When we want to turn a matrix into a row echelon form and/or reduced row echelon form, we can perform the following elementary operations:

① Switch the order of rows
② Multiply the elements of a row by a non-zero scalar.
③ Add a row to another.

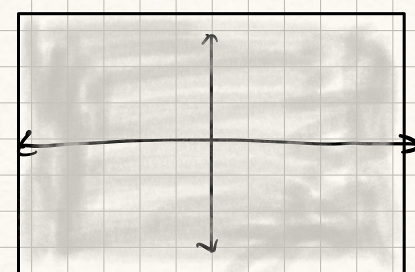
System 1	System 2	System 3
$a+b=0$	$a+b=0$	$0a+0b=0$
$a+2b=0$	$2a+2b=0$	$0a+0b=0$
Rank = 2	Rank = 1	Rank = 0
$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$



Dimension of solution space = 0 (it's a point)



Dimension of solution space = 1 (it's a line)



Dimension of solution space = 2 (It's a plane)

$$\text{Rank}^* = 2 - \left(\begin{matrix}\text{Dimension of} \\ \text{solution space}\end{matrix}\right)$$

*for 2x2

If the rank is equal to the number of rows, the matrix is non-singular. B/c this means it carries as many as information as the number of equations it has.

System 1	System 2	System 3	System 4
$a+b+c=0$	$a+b+c=0$	$a+b+c=0$	$0a+0b+0c=0$
$a+2b+c=0$	$a+b+2c=0$	$2a+2b+2c=0$	$0a+0b+0c=0$
$a+b+2c=0$	$a+b+3c=0$	$3a+3b+3c=0$	$0a+0b+0c=0$
Rank 3	Rank 2	Rank 1	Rank 0

The number of pivots in a row echelon matrix is called its rank. If the rank equals to the number of rows, the matrix is non-singular.

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \xrightarrow{} \begin{bmatrix} 1 & 0.2 \\ 4 & -3 \end{bmatrix}$$

Let's turn 5 into 1 b/c main diag. can't have 5.

* 4 has to be converted into zero b/c bottom of the main diag. has to be zero. How?

$$\rightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & -0.75 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix}$$

"first, make it 1", then subtract the first row from the second

* Now we need 1 on the bottom right. So we divide the row by -0.95

$$\rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$ ⇒ Row Echolon Form   Rank = 2

(2) $$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$ Rank = 1

(3) $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ Rank = 0

* The rank of the matrix is the sum of the values in the main diagonal of the row echolon form!

*For 2x2

$$\begin{matrix} 2 & n & n & n & n \\ 0 & 1 & n & n & n \\ 0 & 0 & 3 & n & n \\ 0 & 0 & 0 & -5 & n \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Rank 5

① Rows with only zeros must go to the bottom.

② The left-most non-zero entry of a row is called a pivot. Every pivot must be to the right of the pivots on the rows above.

③ Rank of the matrix is the number of pivots. (a general rule, not just for 2x2)

$$\begin{matrix} 3 & n & n & n & n \\ 0 & 0 & -1 & n & n \\ 0 & 0 & 0 & -4 & n \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} \div 3 \\ \div -1 \\ \div -4 \\ \\ \\ \end{matrix}$$

Rank = 3

* We can make all pivots ones if we want.

$$\left.\begin{matrix} 1 & n & n & n & n \\ 0 & 0 & 1 & n & n \\ 0 & 0 & 0 & 1 & n \\ 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 \end{matrix}\right\}$$

Rank = 3

n's are now different of course, but the number of pivots is the same.

Reduced Row Echolon form

* The matrix must be in row echolon form
* Pivots must be converted to 1s.
* Any number above a pivot must be converted to 0.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

* we need this 2 to be 0. So we multiply the second row by 2 and subtract it from the first row

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we must get rid of this -5. Let's multiply 3^{rd} row by 5 and add it to the 1st row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now let's multiply 3rd row by 4 and subtract from the second.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is our reduced row echolon form.

The Gaussian Elimination Algorithm

$$2a - b + c = 1$$
$$2a + 2b + 4c = -2$$
$$4a + b + 0c = -1$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & -1 \end{array}\right]$$

Augmented Matrix

1. Turn R1 into a 1
by $R1 = R1/2$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & -1 \end{array}\right]$$

2. Turn R2 into a 0 by:
$R2 = R2 - (R1 \cdot 2)$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 3 & 3 & -3 \\ 4 & 1 & 0 & -1 \end{array}\right]$$

3. Turn R3 into a 0 by:
$R3 = R3 - (R1 \cdot 4)$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 3 & 3 & -3 \\ 0 & 3 & -2 & -3 \end{array}\right]$$

4. Turn R2 into 1 by
$R2 = R2/3$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \\ 0 & 3 & -2 & -3 \end{array}\right]$$

5. Turn R3 into 0 by
$R3 = R3 - (R2 \cdot 3)$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & 0 \end{array}\right]$$

⑥ Turn R3 into 1 by

$R3 = R3 / -5$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]}_{\text{Row Echelon Form}}$$

⇒ Back substitution

⑦ Turn R2 into 0 by

$R2 = R2 - R3$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

⑧ Turn R1 into 0 by

$R1 = R1 - (1/2\, R3)$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

⑨ Turn R1 into 0 by

$R1 = R1 + (1/2\, R_2)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} = a \\ = b \\ = c \end{array}$$

Identity matrix (only 1s in the diagonal)

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 0 & -7 & 9 \\ 0 & 0 & 0 & X \end{array}\right]$$

$X = 0$ ⇒ Infinite # of solutions

$X \neq 0$ ⇒ No solutions

When we have a reduced row echelon of an augmented matrix, the last column is the vector of constants.

If we have a column with all-zero elements:

① If the augmented column is also zero, the system has infinite solutions.

② If the augmented column is not zero, the system has no solutions.