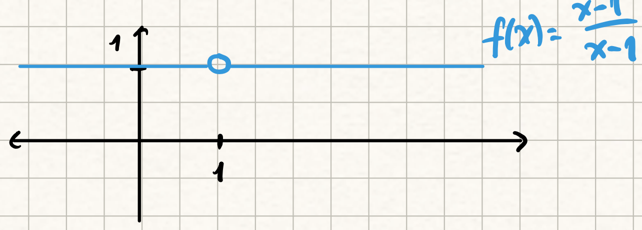
Defining Limits and Using Limit Notation

02.06.2025

Introduction

* $f(x) = \dfrac{x-1}{x-1}$ $f(1) = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ = Undefined!

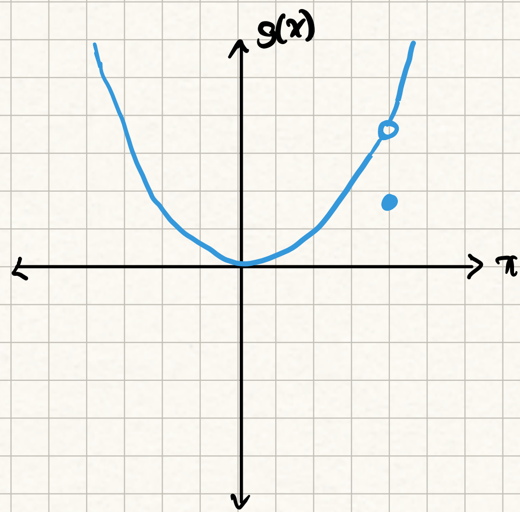
$\Rightarrow f(x) = 1,\ x \neq 1$



$\lim\limits_{x \to 1} f(x) = 1$

"limit of f(x) as x approaches 1."

* $g(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$



$g(2) = 1$

$\lim\limits_{x \to 2} g(x) = 4$

$g(1.9) = 3.61$

$g(1.99) = 3.9601$

$g(1.999) = 3.996001$

$g(\underbrace{1.999999999\ldots}_{\text{as } x \text{ approaches to } 2}) \approx \underbrace{4}_{\text{limit is } 4}$

* A limit is the value that a function (or sequence) approaches as the argument (or index) approaches some value. When a limit doesn't approach the same value from both sides, then the limit doesn't exist.

* $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

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* A limit is the value that a function (or sequence) approaches as the argument (or index) approaches some value.
* When a limit doesn't approach the same value from both sides, then the limit does not exist (DNE).
* $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x)$

Estimating Limit Values from Graphs

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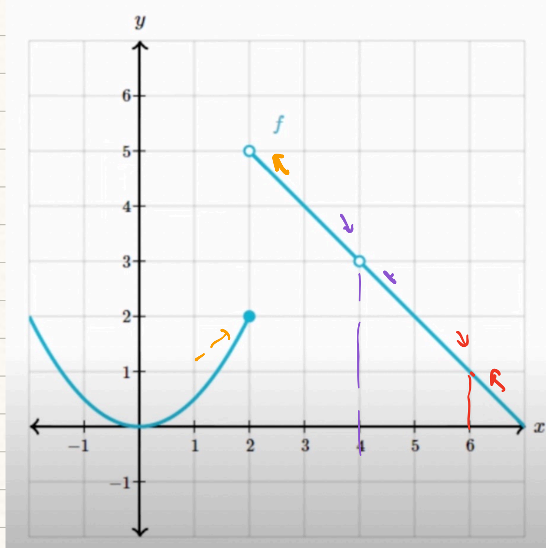


$\lim_{x\to 6} f(x) = -1$

$\lim_{x\to 4} f(x) = 3$

$\lim_{x\to 2} f(x) =$ DNE

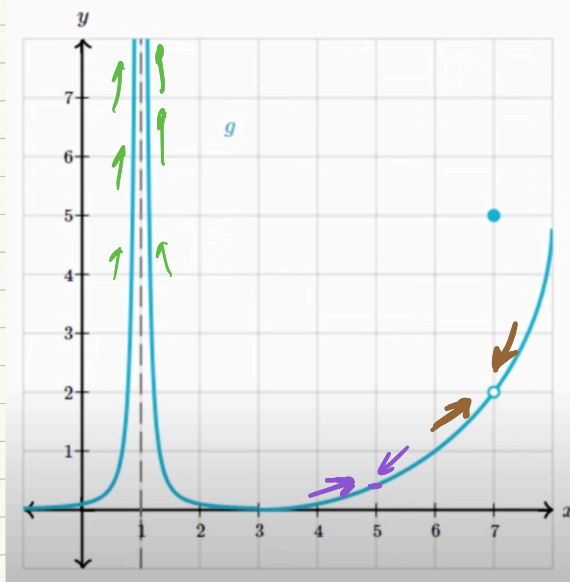
$\lim_{x\to 2^-} f(x) = 2 \neq \lim_{x\to 2^+} f(x) = 5$

as x approaches 2
* from left
* from below
* from negative direction

as x approaches 2
* from right
* from the positive direction
* from above

Unbounded Limits

*

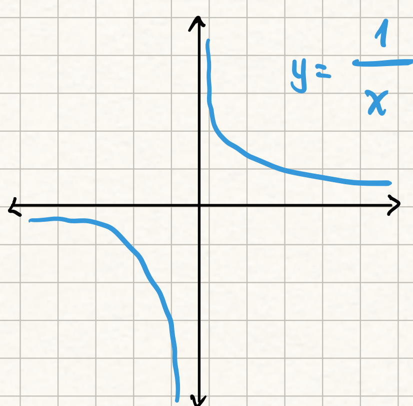


$\lim_{x\to 5} g(x) \approx 0.4$

$\lim_{x\to 7} g(x) = 2$

$\lim_{x\to 1} g(x) =$ Unbounded (b/c going to ∞)

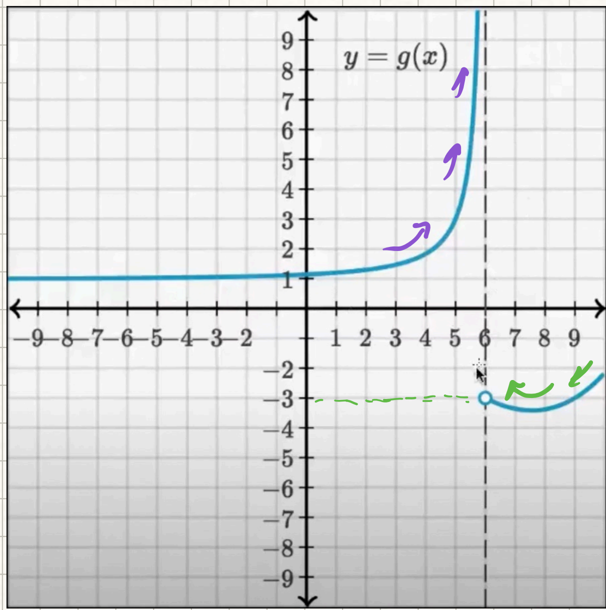
*



$\lim_{x\to 0} \frac{1}{x^2} = ?$ Does not exist!

(b/c going to $-\infty$ and ∞)

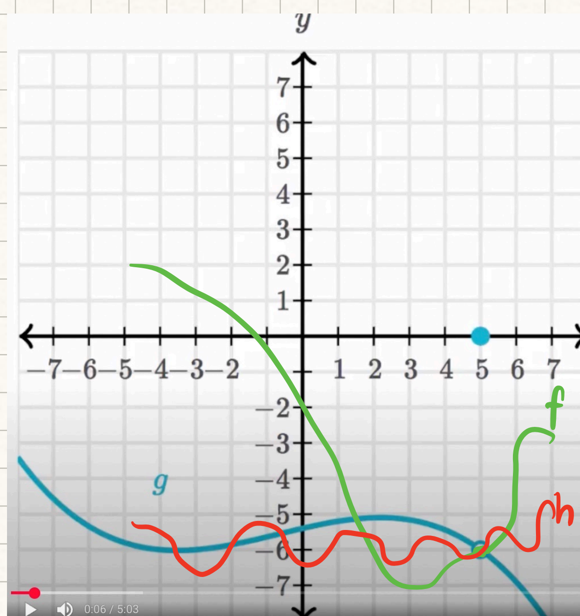
One sided limits from graphs: asymptote



$$\lim_{x \to 6^-} g(x) = \text{DNE}$$
(unbounded)

$$\lim_{x \to 6^+} g(x) = -3$$

Connecting limits and graphical behaviors



* $\lim_{x \to 5} g(x) = -6$
* $\lim_{x \to 5} f(x) = -6$
* $\lim_{x \to 5} h(x) = -6$

* limit doesn't tell us anything about the rest of the function. Only for the point x approaches!

* Just because a function is undefined for some x-value doesn't mean there's no limit. On the other hand, just because a function is defined for some x-value doesn't mean that the limit exists.

Estimating Limit Values from Tables

02.06.2025

Creating tables.

* $\lim_{x\to 3} \frac{x^3-3x^2}{5x-15}$

$f(3) = \frac{3^3 - 3\cdot 3^2}{5\cdot 3 - 15} = \frac{0}{0}$ = Undefined!

x	$\frac{x^3-3x^2}{5x-15}$
2.9	1.682
2.99	~ 1.788
2.999	~ 1.7988

x	$\frac{x^3-3x^2}{5x-15}$
3.1	1.922
3.01	~ 1.812
3.001	~ 1.801

$$\lim_{x\to 3} \frac{x^3-3x^2}{5x-15} = 1.8$$

Estimating from tables

* The function g is defined over the real numbers. This table gives select values of g.

x	$g(x)$
4	3.37
4.9	3.5
4.99	3.66
4.999	3.68
5	6.37
5.001	3.68
5.01	3.7
5.1	3.84
6	3.97

What is a reasonable estimate for $\lim_{x\to 5} g(x)$?

Choose 1 answer:

- (A) 3.68
- (B) 4
- (C) 5 — this is $g(5)$. Not $\lim_{x\to 5} g(x)$
- (D) 6.37
- (E) The limit doesn't exist

One-sided limit from tables

The function f is defined over the real numbers. This table gives select values of f.

(−) (+)

x	0.9	0.99	0.999	$\boxed{1}$	1.001	1.01	1.1
$f(x)$	2.5	2.1	2.02	$\boxed{5}$	−0.99	−0.92	−0.81

2 −1

What is a reasonable estimate for $\lim_{x\to 1^-} f(x)$?

$f(1) = 5$

$\lim_{x\to 1^-} f(x) = 2$

$\lim_{x\to 1^+} f(x) = -1$

$\lim_{x\to 1} f(x) = \text{DNE!}$

Determining Limits Using Algebraic Properties : Limit Properties

02.06.2025

Limit properties

* $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$ $\Rightarrow$

* $\lim_{x\to c} (f(x)+g(x)) = L+M$

* $\lim_{x\to c} (f(x)-g(x)) = L-M$

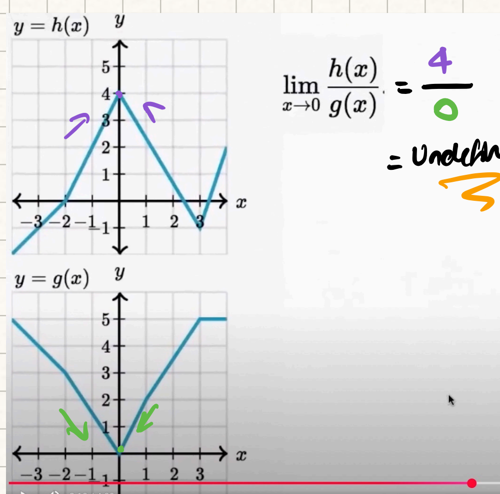
* $\lim_{x\to c} (f(x)\cdot g(x)) = L\cdot M$

* $\lim_{x\to c} (k\cdot f(x)) = k\cdot L$

* $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

* $\lim_{x\to c} (f(x))^{\frac{r}{s}} = L^{\frac{r}{s}}$

Limits of combined functions

* 

$\lim_{x\to 0} \frac{h(x)}{g(x)} = \frac{4}{0}$

$=$ Undefined

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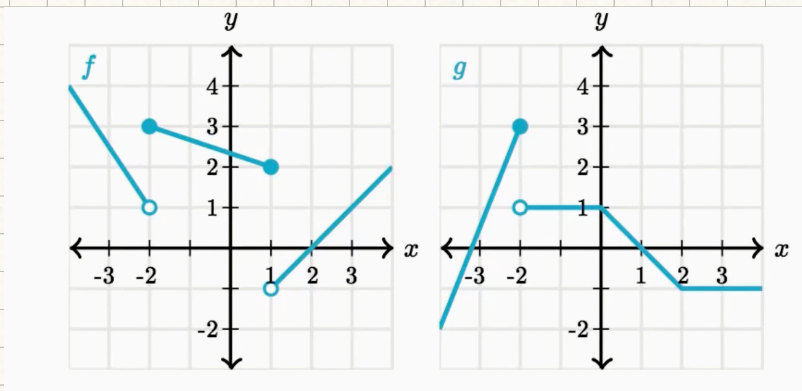
* Let $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$:

* $\lim_{x\to c} (f(x)+g(x)) = L+M$
* $\lim_{x\to c} (f(x)-g(x)) = L-M$
* $\lim_{x\to c} (f(x)\cdot g(x)) = L\cdot M$
* $\lim_{x\to c} (k\cdot f(x)) = k\cdot L$
* $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$
* $\lim_{x\to c} \left(f(x)^{\frac{r}{s}}\right) = L^{\frac{r}{s}}$

Limits of combined functions: piecewise functions



① $\lim_{x \to -2} (f(x) + g(x)) = ?$

$\lim_{x \to -2} f(x) = DNE$ & $\lim_{x \to -2} g(x) = DNE$ HOWEVER

$\lim_{x \to -2^-} (\underbrace{f(x)}_{1} + \underbrace{g(x)}_{3}) = \lim_{x \to -2^+} (\underbrace{f(x)}_{3} + \underbrace{g(x)}_{1}) = 4$

② $\lim_{x \to 1} (f(x) + g(x)) = ?$

$\lim_{x \to 1^-} (f(x) + g(x)) = 2 + 0 = 2$

$\lim_{x \to 1^+} (f(x) + g(x)) = -1 + 0 = -1$

$\neq$, so: DNE!

③ $\lim_{x \to 1} (f(x) \cdot g(x)) = ?$

$\lim_{x \to 1^-} (f(x) \cdot g(x)) = 2 \cdot 0 = 0$

$\lim_{x \to 1^+} (f(x) \cdot g(x)) = -1 \cdot 0 = 0$

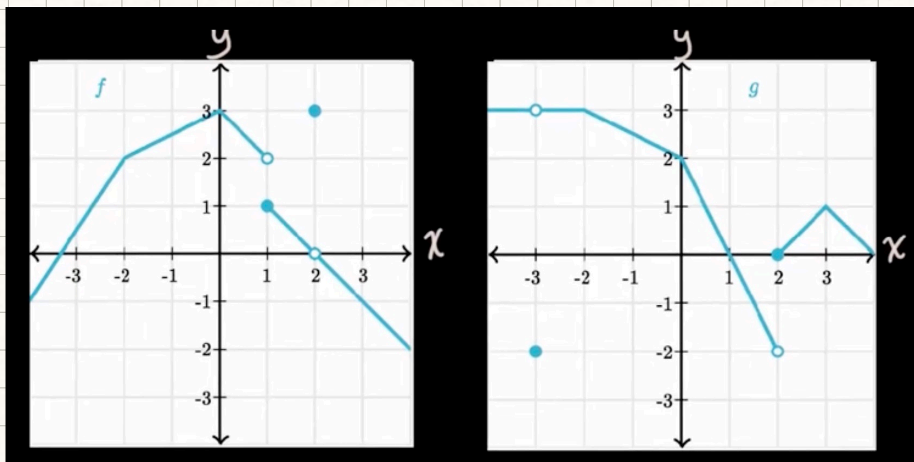
$\lim_{x \to 1} (f(x) \cdot g(x)) = 0$

* When evaluating the limits of combined functions, we must verify whether the left-hand and right-hand limits are equal. If they are, the overall limit exists, even if the individual limit of one of the component functions does not.

Theorem for limits of composite functions (when conditions are met)

* $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ if and only if:

1. $\lim_{x\to a} g(x) = L$ (exists)
2. AND $f(x)$ is continuous at L.



1. $\lim_{x\to -3} f(g(x)) = ?$

* $\lim_{x\to -3} (g(x)) = 3$ ✓
* $f(x)$ is continuous at $x=3$ ✓

$\lim_{x\to -3} f(g(x)) = f\left(\lim_{x\to -3} g(x)\right)$

$= f(3) = -1$

Limits of composite functions: Internal limit DNE

2. $\lim_{x\to 0} f(g(x)) = ?$

* $\lim_{x\to 0} g(x) = 2$ ✓
* $f(x)$ is NOT continuous at $x=2$ ✗

does not mean $\lim_{x\to 0} f(g(x))$ DNE

* $\lim_{x\to 0^-} (f(g(x)) = 0$

$\lim_{x\to 0^+} (f(g(x)) = 0$

$\lim_{x\to 0} f(g(x)) = 0$

* $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} (g(x)\right)$ if and only if:

1. $\lim_{x\to a} g(x) = L$ (exists)
2. AND $f(x)$ is continuous at L.

Limits of composite functions : External limit DNE

③ $\lim_{x \to 2} f(g(x)) = ?$

* $\lim_{x \to 2} g(x) =$ DNE ✗

* Second condition can not be evaluated here.

* $\lim_{x \to 2^-} f(g(x)) = 2$
* $\lim_{x \to 2^+} f(g(x)) = 3$

$\left.\begin{array}{l} \\ \\ \end{array}\right\} \lim_{x \to 2} f(g(x)) =$ DNE !

Determining Limits Using Algebraic Properties: Direct Substitution

07.06.2025

Limits by direct substitution

* $\lim_{x \to -1} (\underbrace{6x^2+5x-1}_{\text{we know that this is continuous. So:}}) = ?$

$= 6(-1)^2+5(-1)-1 = 0$

* f is continuous at $x=a$ iff (if and only if) $\lim_{x \to a} f(x) = f(a)$

Undefined limits by direct substitution

* $\lim_{x \to 1} \frac{x}{\ln(x)} = \frac{\lim_{x \to 1} x}{\lim_{x \to 1} \ln(x)} = \frac{1}{\ln(1)} = \frac{1}{0} = \text{DNE}$

Limits of trigonometric function

* $\lim_{x \to \pi} \sin(x) = \sin(\pi) = 0$
* $\lim_{x \to \pi} \cos(x) = \cos(\pi) = -1$

b/c we know that sin and cos are defined for all $\mathbb{R}$ and they are continuous.

* $\lim_{x \to a} \sin(x) = \sin(x)$
* $\lim_{x \to a} \cos(x) = \cos(x)$
* $\lim_{x \to \pi} \tan(x) = \lim_{x \to \pi} \frac{\sin(x)}{\cos(x)} = \frac{\sin(\pi)}{\cos(\pi)} = \frac{0}{-1} = 0$
* $\lim_{x \to \frac{\pi}{2}} \tan(x) = \lim_{x \to \frac{\pi}{2}} \frac{\sin(x)}{\cos(x)} = \frac{\sin(\pi/2)}{\cos(\pi/2)} = \frac{1}{0} = \text{DNE}!$
* $\lim_{x \to \pi} \cot(x) = \lim_{x \to \pi} \frac{\cos(x)}{\sin(x)} = \frac{\cos(\pi)}{\sin(\pi)} = \frac{-1}{0} = \text{DNE}!$

Limits of piecewise functions

* $f(x) = \begin{cases} \frac{x+2}{x-1} & \text{for } 0 < x \leq 4 \\ \sqrt{x} & \text{for } x > 4 \end{cases}$

① $\lim_{x \to 4^+} f(x) = \sqrt{4} = 2$

② $\lim_{x \to 4^-} f(x) = \frac{4+2}{4-1} = 2$

③ $\lim_{x \to 4} f(x) = 2$

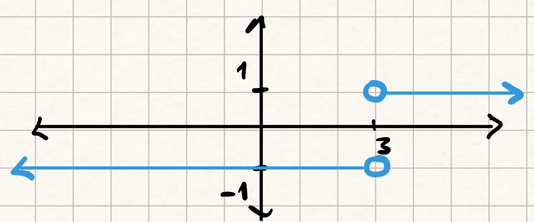
④ $\lim_{x \to 2} f(x) = \frac{2+2}{2-1} = 4$

* $g(x) = \begin{cases} \sin(x+1) & \text{for } x < -1 \\ 2^x & \text{for } -1 \leq x < 5 \end{cases}$

Limits of piecewise functions : absolute value

* $f(x) = \frac{|x-3|}{x-3} \Rightarrow \lim_{x \to 3} f(x) = ?$

$f(x) = \begin{cases} \frac{x-3}{x-3} & \text{for } x > 3 \\ \frac{-(x-3)}{x-3} & \text{for } x < 3 \end{cases} = \begin{cases} 1 & \text{for } x > 3 \\ -1 & \text{for } x < 3 \end{cases}$



$\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\Rightarrow \lim_{x \to 3} f(x) = \text{DNE!}$

Determining Limits Using Algebraic Manipulation

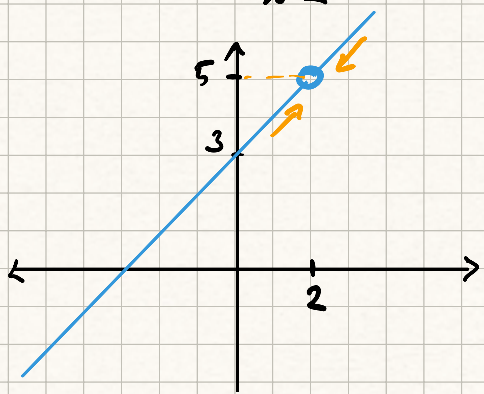
03.06.2025

Limits by factoring

* $f(x) = \dfrac{x^2+x-6}{x-2} \Rightarrow \lim_{x\to 2} f(x) = ?$

$f(2) = \dfrac{4+2-6}{2-2} = \dfrac{0}{0}$ = Undefined. $\frac{0}{0}$ DOES NOT MEAN lim DNE B/C .....

$\Rightarrow f(x) = \dfrac{(x+3)(x-2)}{x-2} = x+3 \,,\ x \neq 2 = \begin{cases} x+3 \,, & x \neq 2 \\ \text{undefined}\,, & x = 2 \end{cases}$



$\Rightarrow \lim_{x\to 2} f(x) = 5$

Limits by rationalizing

* $\lim_{x\to -1} \dfrac{x+1}{\sqrt{x+5}-2} = ?$

$= \dfrac{\lim_{x\to -1} x+1}{\lim_{x\to -1} \sqrt{x+5}-2} = \dfrac{-1+1}{\sqrt{4}-2} = \dfrac{0}{0}$ does not mean lim DNE

* $g(x) = \dfrac{x+1}{\sqrt{x+5}-2} \Rightarrow \lim_{x\to -1} g(x) = ?$

$g(x) = \dfrac{(x+1)(\sqrt{x+5}+2)}{(\sqrt{x+5}-2)(\sqrt{x+5}+2)} = \dfrac{(x+1)(\sqrt{x+5}+2)}{x+5-(2)^2} = \dfrac{\cancel{(x+1)}(\sqrt{x+5}+2)}{\cancel{x+1}}$

$= \sqrt{x+5}+2 \,,\ x \neq -1 \Rightarrow \lim_{x\to -1} g(x) = \sqrt{-1+5}+2 = 4$

Trig. limits using Pythagorean identity

* $\lim_{\theta\to 0} \frac{1-\cos\theta}{2\sin^2\theta} = ? = \frac{1-\cos(0)}{2\sin^2(0)} = \frac{0}{0}$ NOT HELPFUL!

* $\sin^2\theta+\cos^2\theta=1$

$$f(x) = \frac{1-\cos\theta}{2\sin^2\theta} = \frac{1-\cos\theta}{2\cdot(1-\cos^2\theta)^{*}}$$

$$= \frac{(\cancel{1-\cos\theta})}{2\cdot(\cancel{1-\cos\theta})(1+\cos\theta)} = \frac{1}{2(1+\cos\theta)}, \quad \theta \neq 0$$

$$\Rightarrow \lim_{\theta\to 0} \frac{1-\cos\theta}{2\sin^2\theta} = \lim \frac{1}{2(1+\cos\theta)}$$

$$= \frac{1}{2(1+\cos(0))} = \frac{1}{4}$$

Trig. limit using double angle identity

* $\lim_{\theta\to\frac{-\pi}{4}} \frac{1+\sqrt{2}\sin\theta}{\cos 2\theta} = ?$

* $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$f(x) = \frac{1+\sqrt{2}\sin\theta}{\cos 2\theta} = \frac{1+\sqrt{2}\sin\theta}{1-2\sin^2\theta^{*}}$$

$$= \frac{\cancel{1+\sqrt{2}\sin\theta}}{(\cancel{1+\sqrt{2}\sin\theta})(1-\sqrt{2}\sin\theta)} = \frac{1}{1-\sqrt{2}\sin\theta}, \quad \theta \neq \frac{-\pi}{4}$$

$$\lim_{\theta\to\frac{-\pi}{4}} \frac{1+\sqrt{2}\sin\theta}{\cos 2\theta} = \lim \frac{1}{1-\sqrt{2}\sin\theta}$$

$$= \frac{1}{2}$$

Selecting Procedures for Determining Limits

04.06.2025

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Calculating $\lim_{x \to a} f(x)$

A. Direct substitution

Try to evaluate the function directly.

$$f(a)$$

$$f(a) = \frac{b}{0}$$

where ***b*** is not zero

B. Asymptote (probably)

example:

$$\lim_{x \to 1} \frac{1}{x-1}$$

Inspect with a graph or table to learn more about the function at ***x=a***.

$$f(a) = b$$

where ***b*** is a real number

C. Limit found (probably)

example:

$$\lim_{x \to 3} x^2 = (3)^2 = 9$$

$$f(a) = \frac{0}{0}$$

D. Indeterminate form

example:

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

Try rewriting the limit in an equivalent form.

E. Factoring

example:

$$\lim_{x \to -1} \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

can be reduced to

$$\lim_{x \to -1} \frac{x-2}{x-3}$$

by factoring and cancelling.

F. Conjugates

example:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

can be rewritten as

$$\lim_{x \to 4} \frac{1}{\sqrt{x} + 2}$$

using conjugates and cancelling.

G. Trig identities

example:

$$\lim_{x \to 0} \frac{\sin(x)}{\sin(2x)}$$

can be rewritten as

$$\lim_{x \to 0} \frac{1}{2\cos(x)}$$

using a trig identity.

Try evaluating the limit in its new form.

H. Approximation

When all else fails, graphs and tables can help approximate limits.

! Add the table to the summary!

Determining Limits Using the Squeeze Theorem

04.06.2025

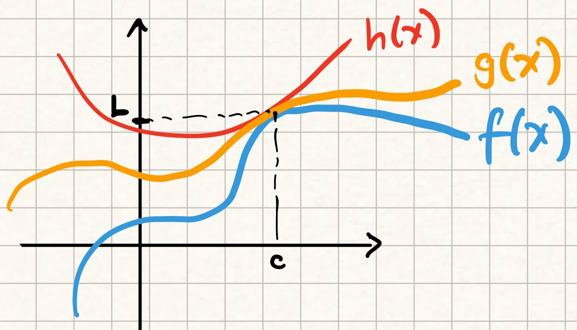
Squeeze Theorem Intro

* - Diya eats at least as much as Imran.
  - Sal eats at least as much as Diya.
  - Tuesday, Imran and Sal ate 1,500 calories each.

How much must have Diya eaten that day?

$$\left.\begin{array}{l} \text{Imran} \leq \text{Diya} \leq \text{Sal} \\ 1500 \leq \text{Diya} \leq \text{Sal} \end{array}\right\} \text{Diya} = 1{,}500$$

* $f(x) \leq g(x) \leq h(x)$

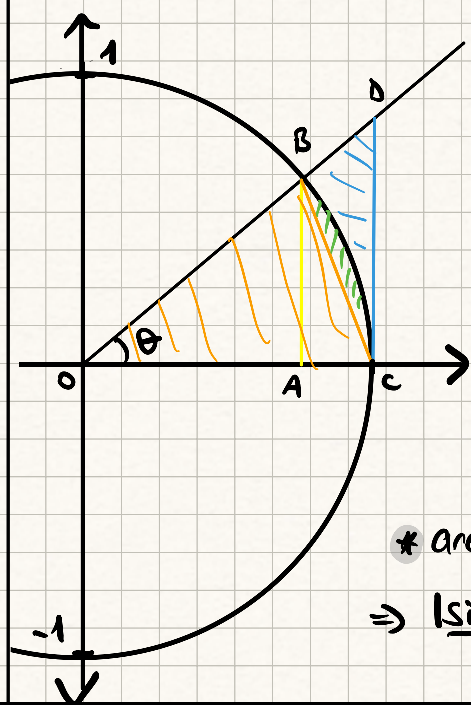


$$\left.\begin{array}{l} \lim\limits_{x\to c} f(x) = L \\ \lim\limits_{x\to c} h(x) = L \end{array}\right\} \Rightarrow \lim_{x\to c} g(x) = L$$

$\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$

* Let's prove $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

(* Abs. values are there so the equations work for both 1^{st} and 4^{th} quadrants)



* $|AB| = |\sin\theta|$ * $|CD| = \dfrac{\tan\theta}{|OC|} = |\tan\theta|$

* $\text{area}(\Delta OBC) = \dfrac{|AB| \cdot |OC|}{2} = \dfrac{|\sin\theta|}{2}$

* $\text{area}(\triangle OBC) = \dfrac{|\theta|}{2\pi} \cdot \pi (1)^2 = \dfrac{|\theta|}{2}$

* $\text{area}(\Delta OCD) = \dfrac{|CD| \cdot |OC|}{2} = \dfrac{|\tan\theta|}{2}$

* $\text{area}(\Delta OBC) \leq \text{area}(\triangle OBC) \leq \text{area}(\Delta OCD)$

$$\Rightarrow \frac{|\sin\theta|}{2} \leq \frac{|\theta|}{2} \leq \frac{|\tan\theta|}{2}$$

* We can use the squeeze theorem to determine limits when appropriate:

If $g(x) \leq f(x) \leq h(x)$, and $\lim\limits_{x\to a} g(x) \leq \lim\limits_{x\to a} h(x)$, then $\lim\limits_{x\to a} f(x) = L$

$\Rightarrow |\sin\theta| \leq |\theta| \leq |\tan\theta|$

$\Rightarrow \dfrac{|\sin\theta|}{|\sin\theta|} \leq \dfrac{|\theta|}{|\sin\theta|} \leq \dfrac{|\sin(\theta)|}{|\cos\theta|} \cdot \dfrac{1}{|\sin\theta|}$

$\Rightarrow 1 \leq \dfrac{|\theta|}{|\sin\theta|} \leq \dfrac{1}{|\cos\theta|}$

$\Rightarrow 1 \geq \dfrac{\sin\theta}{\theta} \geq \cos\theta$

* absolute value signs are no longer necessary b/c
- $\cos(\theta) = \cos(-\theta)$
- $\dfrac{|\theta|}{|\sin\theta|} = \dfrac{\theta}{\sin\theta}$

$\Rightarrow \lim\limits_{\theta\to 0} 1 \geq \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \geq \lim\limits_{\theta\to 0} \cos\theta$

$\Rightarrow 1 \geq \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} \geq 1 \quad \Rightarrow \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 1$

* let's prove $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$

$= \lim\limits_{x\to 0} \dfrac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x(1+\cos x)}$

$= \lim\limits_{x\to 0} \dfrac{\sin^2 x}{x(1+\cos x)} = \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{\sin x}{1+\cos x}$

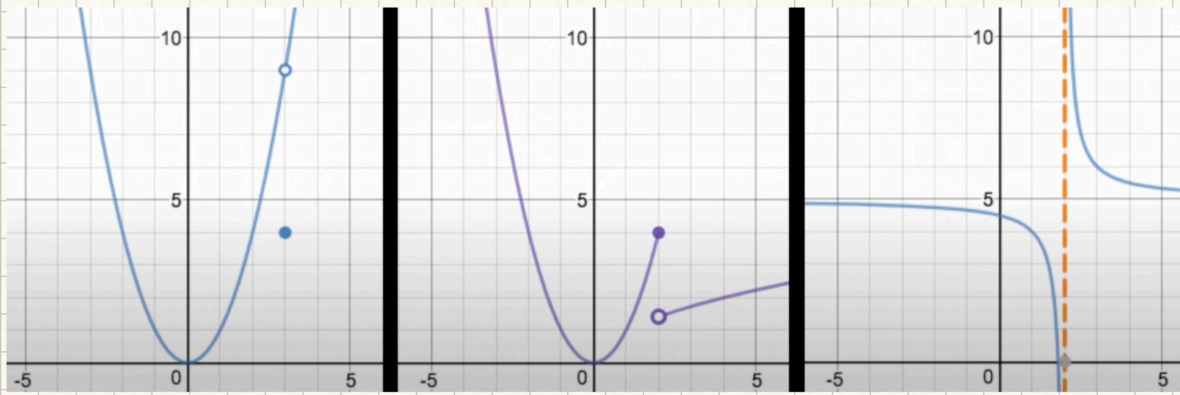
$= \underbrace{\lim\limits_{x\to 0} \dfrac{\sin x}{x}}_{=1} \cdot \lim\limits_{x\to 0} \dfrac{\sin x}{1+\cos x} = \lim\limits_{x\to 0} \dfrac{\sin x}{1+\cos x} = \dfrac{0}{1+1} = 0$

we've already proved this

Exploring Types of Discontinuities

04.06.2025

* f is discontinuous if I have to pick up my pencil while graphing.



* Point (Removable) Discontinuity

* Two-sided limit exists, which means one-sided limits exist and are equal to each other. However two-sided limit is not equal to function's value at that point.

* Jump Discontinuity

* Two-sided limit does not exist because one-sided limits are not equal to each other, even though they do exist.

* Asymptotic Discontinuity

* Two-sided limit does not exist because one-sided limits are unbounded, therefore they do not exist.

* f is discontinuous if we have to pick up our pencil while graphing it. There are 3 types of discontinuity:

1 Removable (Point) Discontinuity: Two-sided limit exists, which means one-sided limits exist and are equal to each other. However, two-sided limit is not equal to function's value at that point
2 Jump Discontinuity: Two-sided limit DNE because one-sided limits are not equal to each other, even though they exist. 3 Asymptotic Discontinuity: Two-sided limit DNE b/c one-sided limits are unbounded, therefore they DNE.

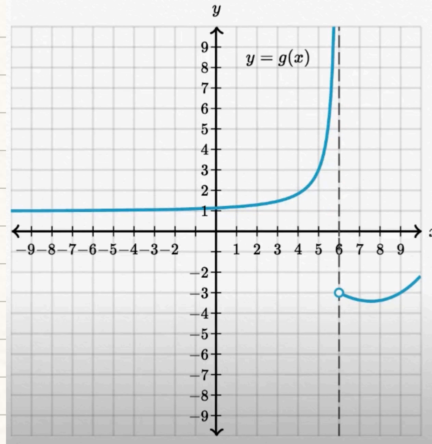
Defining Continuity at a Point

04.06.2025

Continuity at a point

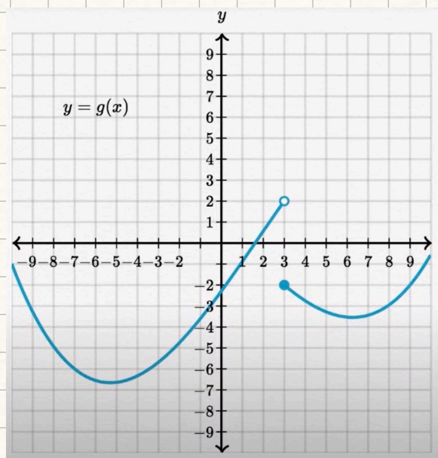
* f is continuous at $x=c \iff \lim_{x\to c} f(x) = f(c)$

*



- ✗ Both $\lim_{x\to 6^+} g(x)$ and $\lim_{x\to 6^-} g(x)$ exist — * unbounded * unbounded
- ✗ $\lim_{x\to 6} g(x)$ exists * b/c one-sided limits DNE
- ✗ g is defined at $x = 6$ * $g(6)$ is undefined
- ✗ g is continuous at $x = 6$ * $g(6)$ is undefined * $\lim_{x\to 6}$ DNE
- ✓ None of the above

*



- ✓ Both $\lim_{x\to 3^+} g(x)$ and $\lim_{x\to 3^-} g(x)$ exist
- ✗ $\lim_{x\to 3} g(x)$ exists b/c one-sided limits are not equal
- ✓ g is defined at $x = 3$ $g(3) = -2$
- ✗ g is continuous at $x = 3$ $\lim_{x\to 3} g(x)$ DNE
- ✗ None of the above

* $g(x) = \begin{cases} \log(3x) & \text{for } 0 < x < 3 \\ (4-x)\log(9) & \text{for } x \geq 3 \end{cases}$, $\lim_{x\to 3} g(x) = ?$ $\log(9)$

$\lim_{x\to 3^-} g(x) = \log(9) = \lim_{x\to 3^+} g(x) = 1\cdot\log(9) = g(3) = \log(9)$

* $f(x) = \begin{cases} \ln(x) & \text{for } 0 < x \leq 2 \\ x^2\ln(x) & \text{for } x > 2 \end{cases}$, $\lim_{x\to 2} f(x) = ?$ DNE

$f(2) = \ln(2) = \lim_{x\to 2^-} f(x) = \ln(2) \neq \lim_{x\to 2^+} 4\ln(2)$

* f is continuous at $x=c \iff \lim_{x\to c} f(x) = f(c)$

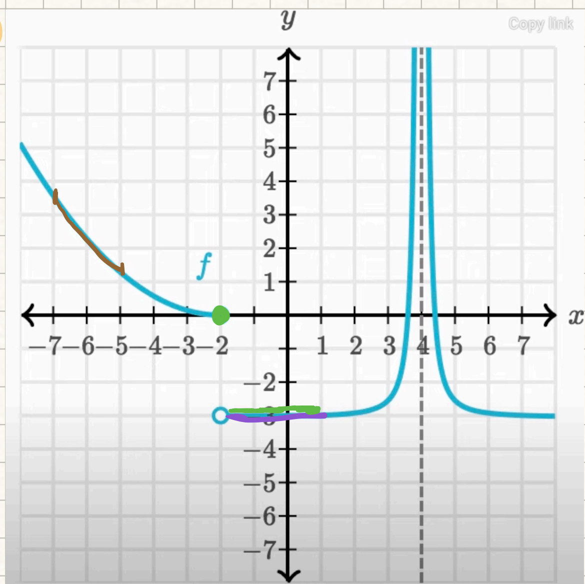
Confirming Continuity Over an Interval

04.06.2025

Continuity over an interval

* f is continuous over (a,b) $\iff$ f is continuous over every point in interval

* f is continuous over $[a,b]$ $\iff$ f is continuous over (a,b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

* 

* $(-7,-5) = \checkmark$
* $(-2,1) = \checkmark$ b/c 2 is not in the interval
* $[-2,1) = X$ b/c $\lim_{x \to 2^+} f(x) \neq f(2)$

Functions continuous on all real numbers

* A function is continuous on all real numbers if and only if it has no types of discontinuity (removable, jump, or asymptotic).

---

* f is continuous over (a,b) $\iff$ f is continuous over every point in the interval.
* f is continuous over $[a,b]$ $\iff$ f is continuous over (a,b) AND $\lim_{x \to a^+} f(x) = f(a)$ AND $\lim_{x \to b^-} f(x) = f(b)$.

* f is continuous on all real numbers $\iff$ it has no types of discontinuity (removable, jump, or asymptotic).

Removing Discontinuities

04.06.2025

Removing discontinuities with factorization

* $f(x) = \dfrac{6x^2+18x+12}{x^2-4}$ is not defined at $x = \pm 2$. What value should be assigned to $f(-2)$ to make $f(x)$ continuous at that point?

* $f(x) = \dfrac{6(x^2+3x+2)}{(x+2)(x-2)} = \dfrac{6\cancel{(x+2)}(x+1)}{\cancel{(x+2)}(x-2)}$

$= \dfrac{6(x+1)}{x-2}, \ x \neq -2$

$\Rightarrow \dfrac{6(-2+1)}{-2-2} = \dfrac{-6}{-4} = \dfrac{3}{2}$

$$f(x) = \begin{cases} \dfrac{6x^2+18x+12}{x^2-4} & \text{for } x \neq \pm 2 \\ \dfrac{3}{2} & \text{for } x = -2 \end{cases}$$

Removing discontinuities with rationalization

* $f(x) = \begin{cases} \dfrac{\sqrt{x+4}-3}{x-5} & , \text{if } x \neq 5 \\ c & , \text{if } x = 5 \end{cases}$. If f is continuous at $x=5$, what is the value of c?

$f(5) = \lim\limits_{x \to 5} f(x) = \dfrac{\sqrt{x+4}-3}{x-5} \cdot \dfrac{(\sqrt{x+4}+3)}{(\sqrt{x+4}+3)} = \dfrac{\cancel{x+4-9}}{\cancel{(x-5)}(\sqrt{x+4}+3)}$

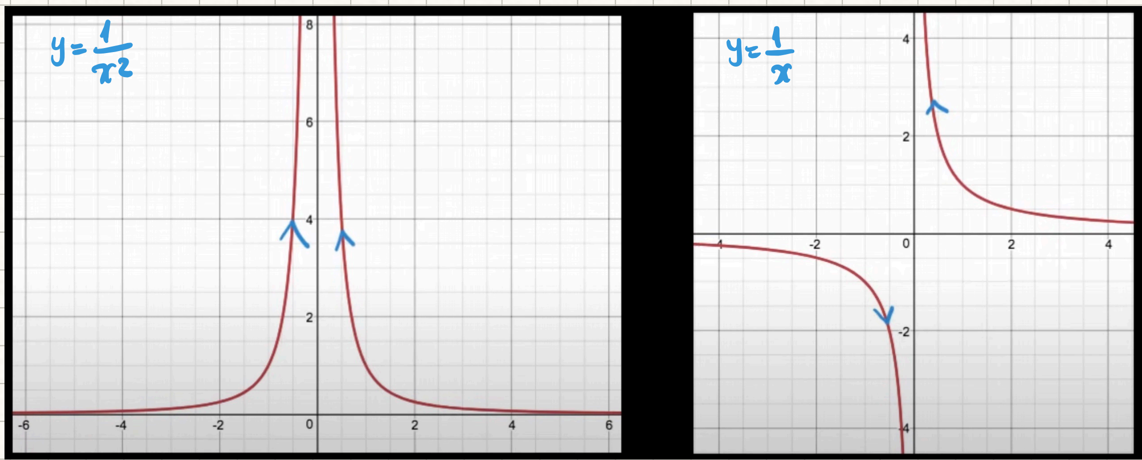
$= \dfrac{1}{\sqrt{x+4}+3}, \text{ for } x \neq 5$

Connecting Infinite Limits and Vertical Asymptotes

04.06.2025

Introduction to infinite limits

* When the one-sided limits are unbounded to the same direction, we can say that the limit is going to infinity.



* $\lim_{x\to 0} \frac{1}{x^2} = \infty$

* $\lim_{x\to 0} \frac{1}{x} = \text{DNE}$

Infinite limits and asymptotes

* A function can not cross its vertical asymptote, but it can cross its horizontal asymptote.

* The limit at a vertical asymptote is never a finite real number. The limit at a horizontal asympote is always a finite real number.

Analyzing unbounded limits: Mixed function

* $f(x) = \frac{x}{1-\cos(x-2)} \Rightarrow \lim_{x\to 2^-} f(x) = ?\ +\infty \quad \lim_{x\to 2^+} f(x) = ?\ +\infty$

$f(2) = \frac{2}{1-\cos(0)} = \frac{2}{0}$ = undefined. How about limits?

$f(2.1) \cong 1{,}378{,}779$
$f(2.01) \cong 131{,}969{,}879$ $\Big\}$ $+\infty$

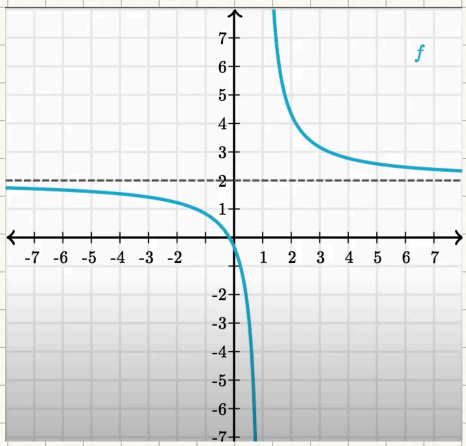
$f(1.9) \cong 1{,}247{,}467$
$f(1.99) \approx 130{,}656{,}745$ $\Big\}$ $+\infty$

* When the one-sided limits are unbounded to the same direction, we can say that the limit is going to infinity.
* A function can not cross its vertical asymptote but it can cross its horizontal asymptote (even multiple times.)

Connecting Limits at Infinity and Horizontal Asymptotes

04.06.2025

Introduction to limits at infinity



* $\lim_{x\to+\infty} f(x) = 2$
* $\lim_{x\to-\infty} f(x) = 2$

Functions with same limit at infinity

* $\lim_{x\to\infty} 3 = 3$
* $\lim_{x\to\infty} \frac{1}{x}(\sin x) + 3 = 3$
* $\lim_{x\to\infty} \frac{3x^2}{x^2+5} = 3$
* $\lim_{x\to\infty} \frac{3\ln(x)}{\ln(x+5)} = 3$

Limits at infinity of quotients

* $f(x) = \frac{4x^5 - 3x^2 + 3}{6x^5 - 100x^2 - 10}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^5}{6x^5} = \frac{4}{6}$$

* $\lim_{x\to-\infty} \frac{3x^3 - 2x^2 + 7}{6x^4 - x^3 + 2x - 100} = \lim_{x\to-\infty} \frac{3x^3}{6x^4} = \lim_{x\to-\infty} \frac{3}{6x} = 0$

* $\lim_{x\to\infty} \frac{4x^4 - 3x^3 + 7x^2 - 10}{250x^3 + 5x^2 - x + 1000} = \lim_{x\to\infty} \frac{4x^4}{250x^3} = \lim_{x\to\infty} \frac{4x}{250} = \infty$

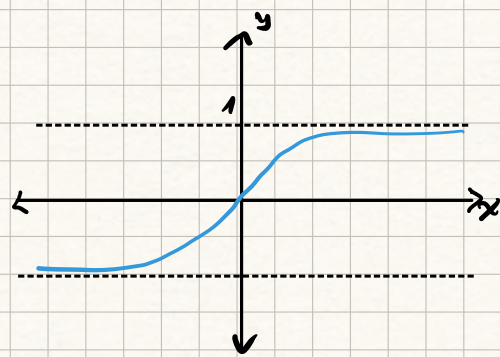
* Functions with horizontal asymptotes have one-sided limits for x approaches ∞ and $-\infty$. These limits are finite real numbers.

Limits at infinity of quotients with square roots (odd power)

$$f(x) = \frac{x}{\sqrt{x^2+1}} \Rightarrow f(x) \approx \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} \text{ for } \begin{matrix} x \to \infty \\ x \to -\infty \end{matrix}$$

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim \frac{x}{|x|} = 1$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim \frac{x}{|x|} = -1$$



... Even power

$$\lim_{x\to-\infty} \frac{\sqrt{4x^4-x}}{2x^2+3} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{1}{\sqrt{x^4}} \cdot \sqrt{4x^4-x}}{2+\frac{3}{x^2}} = \frac{\sqrt{\frac{4x^4-x}{x^4}}}{2+\frac{3}{x^2}}$$

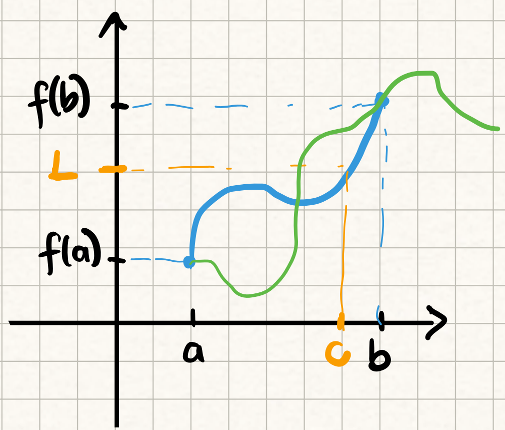
$$= \frac{\sqrt{4-\cancel{\frac{1}{x^3}}} \to \infty}{2+\cancel{\frac{3}{x^2}} \to \infty} = \frac{\sqrt{4}}{2} = 1$$

Working with the Intermediate Value Theorem

04.06.2025

Intermediate value theorem

* Suppose f is a function continuous at every point of the interval $[a,b]$.
  * f will take on every value between $f(a)$ and $f(b)$ over the interval
  * For any L between the values $f(a)$ and $f(b)$, there exists a number "c" in $[a,b]$ for which $f(c) = L$.



* Very verbose but extremely intuitive theorem

* Green line is OK too. It can take values outside of $f(a)$ and $f(b)$ as long as it takes on every value between them

* Let f be a continuous function on the closed interval $[-2, 1]$, where $f(-2) = 3$ and $f(1) = 6$.

  Which of the following is guaranteed by the Intermediate Value Theorem?

  - $f(c) = 4$ for at least one c between 3 ~~and~~ 6
  - $f(c) = \cancel{0}$ for at least one c between -2 and 1
  - $f(c) = \cancel{0}$ for at least one c between 3 ~~and~~ 6
  - ✓ $f(c) = 4$ for at least one c between -2 and 1

* $f(-2) = 3$

  $f(1) = 6$

  for any L between 3 and 6, there's at least one "c" in $[-2,1]$ such that $f(c) = L$.

* The intermediate value theorem: For any function f that is continuous over the interval $[a,b]$, the function will take any value between $f(a)$ and $f(b)$ over the interval. More formally, for any value L between $f(a)$ and $f(b)$, there's a value c in $[a,b]$ for which $f(c) = L$

Justification with the intermediate value theorem: Table

The table gives selected values of the continuous function f.

x	0	2	4	6
$f(x)$	0	-2	3	7

Can we use the intermediate value theorem to say that the equation $f(x) = 0$ has a solution where $4 \leq x \leq 6$? If so, write a justification?

$f(4) = 3$ $\quad 3 \leq \cancel{0} \leq 7$

$f(6) = 7$

Can we use the intermediate value theorem to say that there is a value c such that $f(c) = 0$ and $2 \leq c \leq 4$? If so, write a justification?

$f(2) = -2$ $\quad -2 \leq 0 \leq 3$ ✓

$f(4) = 3$

Justification with the intermediate value theorem: Equation

Let $g(x) = \frac{1}{x}$.

Can we use the intermediate value theorem to say that there is a value c such that $g(c) = 0$ and $-1 \leq c \leq 1$? If so, write a justification?

No. b/c $g(x)$ is not defined at $x=0$, therefore $g(x)$ not continuous over $[-1,1]$

Can we use the intermediate value theorem to say that the equation $g(x) = \frac{3}{4}$ has a solution where $1 \leq x \leq 2$? If so, write a justification?

$f(1) = 1$ $\quad f(2) = \frac{1}{2}$ $\quad \frac{1}{2} \leq \frac{3}{4} \leq 1$ ✓