The American University in Cairo | **MACT 2132**
Mathematics and Actuarial Science | **Fall 2018**
Linear Algebra | **Final Exam**
December 15, 2018 | **Time Limit: 120 Minutes**

Name: ______________________ **UID:** __________

- This exam contains 12 pages (including this cover page).
- Answer **ALL** the questions (total of points is 80).
- Unsupported answers are considered miracles and will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.

Problem	Score	Points
1		16
2		16
3		16
4		16
5		16
Total		80

Problem 1. The augmented matrices in reduced row echelon form of three linear systems are given below.

- System 1: $A\mathbf{x} = \mathbf{b}$, $\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$
- System 2: $B\mathbf{y} = \mathbf{c}$, $\begin{pmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
- System 3: $C\mathbf{z} = \mathbf{d}$, $\begin{pmatrix} 0 & 1 & 0 & -8 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

a) (3 pts) How many solutions does each system have? Justify your answer.

b) (2 pts) Which of the matrices A, B, C is invertible? Justify your answer.

c) (3 pts) Find a basis for the row spaces of A, B, C.

d) (2 pts) Find the rank of each matrix A, B, C. Justify your answer.

e) (3 pts) Which of the matrices A, B, C have linearly independent columns? Explain why?

f) (3 pts) Which columns of A, B, C form a basis for the column space of each matrix, respectively?

Problem 2.

a) (8 points) Find an orthonormal basis for $\mathbb{R}^4$ consisting of eigenvectors of A, where

$$A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

b) (4 pts) Find (if possible) a 2×2 matrix A that satisfies the equation:

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} (A - 2I)^T \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

c) (4 pts) Suppose that A, B, and C are $n \times n$ matrices such that $ABC = I_n$. Prove that B is invertible with $B^{-1} = CA$.

Problem 3.

a) (8 pts) Let $T : M_{n\times n} \to M_{n\times n}$ be the transformation represented by

$$T(A) = A + A^t, \text{ for } A \in M_{n\times n}.$$

(i) Show that T is a linear transformation.

(ii) Find the kernel $\operatorname{Ker} T$.

(iii) Is T one-to-one? onto? Justify your answer.

b) (8 pts) For $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, consider the function

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 5u_2v_2.$$

(i) Show that $\langle \, , \, \rangle$ defines an inner product on $\mathbb{R}^2$.

(ii) Find the angle between the two vectors $\mathbf{u} = (-1, 1)$ and $\mathbf{v} = (1, 1)$.

Problem 4.

a) (8 pts) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that satisfies

$$T(1,1,0) = (1,0,-2),\ T(-1,1,0) = (1,2,-4),\ T(0,0,1) = (0,3,7).$$

(i) Compute $T(1,0,0)$ and $T(0,1,0)$.

(ii) Write down the standard matrix representation for T.

(iii) Find the form of $T(x,y,z)$.

b) (8 pts) Let $(V, \langle\ ,\ \rangle)$ be an inner product space, and $\mathbf{u}, \mathbf{v} \in V$. Prove the following facts.

(i) $||\mathbf{u} + \mathbf{v}||^2 - ||\mathbf{u} - \mathbf{v}||^2 = 4\langle \mathbf{u}, \mathbf{v} \rangle$.

(ii) $\mathbf{u} - \mathrm{Proj}_{\mathbf{v}}\, \mathbf{u}$ is orthogonal to $\mathbf{v}$ provided that $\mathbf{v} \neq \mathbf{0}$.

Problem 5. True/False (only **Four** items are required). Justify your answer.

a) (4 pts) If A is a diagonalizable matrix with nonnegative eigenvalues, then $\sqrt{A}$ exists.

b) (4 pts) If $\mathbf{b}$ is in the column space of a matrix A, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

c) (4 pts) Suppose that A, B are invertible 2×2 matrices with $|A| = 3, |B| = -5$. Then,

$$|6(AB)^t(BA)^{-1}| = 120.$$

d) (4 pts) If a matrix A is similar to B and B is similar to C, then A is similar to C.

e) (4 pts) The set $W = \{A \in M_{n,n} : A^t = -A\}$ of skew-symmetric matrices is a vector subspace of $M_{n,n}$.

f) (4 pts) A set S of five vectors in $\mathbb{R}^3$ always spans $\mathbb{R}^3$.

g) There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose image is the same as its kernel.

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