MACT 2132 Exam 2 Solutions

① For the three vectors $\{(1,1,2), (1,0,-1), (1,-1,k)\}$ to be a basis they must be linearly independent.

$R_1 \quad c_1 + c_2 + c_3 = 0$

$R_2 \quad c_1 \qquad - c_3 = 0 \Rightarrow c_1 = c_3$, plug in to R_3

$R_3 \quad 2c_1 - c_2 + kc_3 = 0$

$2c_3 - c_2 + kc_3 = 0$

$2c_3 + kc_3 = c_2 \Rightarrow c_2 = (2+k)c_3$

Plug in to R_1

$c_3 + (2+k)c_3 + c_3 = 0$

$c_3 + 2c_3 + kc_3 + c_3 = 0$

$4c_3 + kc_3 = 0$

$(4+k)c_3 = 0$ since $c_3 = 0$ in order to get a basis

$\Rightarrow \boxed{k \neq 4}$

(2)

(i) Since B is rref of A with three nonzero rows $\Rightarrow \text{rank}A = 3$

$\text{rank}A + \text{nullity}A = 5$

$3 + 2 = 5$ So $\text{nullity}\,A = 2$

(ii) Look at $B\vec{x} = 0$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_5 = 0$, $x_2 + 3x_3 = 0 \Rightarrow x_2 = -3x_3$ let $x_3 = s \Rightarrow x_2 = -3s$

$x_1 = 2s - t$ where $x_4 = t$

$\vec{x} = s\begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ The basis for $\text{Nul}\,A = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(iii) Since B has

$$\begin{pmatrix}-2\\3\\0\\0\end{pmatrix} = -2\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+3\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \left(\underset{\substack{\uparrow\\ \text{third}\\ \text{column}\\ \text{of } B}}{\mathrm{Col}_3(B)} = -2\,\underset{\substack{\uparrow\\ \text{first column}\\ \text{of } B}}{\mathrm{Col}_1(B)} + 3\,\underset{\substack{\uparrow\\ \text{fourth column}\\ \text{of } B}}{\mathrm{Col}_4(B)} \right)$$

then the third column of A

$$\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\\a_{43}\end{pmatrix} = -2\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}+3\begin{pmatrix}0\\1\\2\\1\end{pmatrix}$$

$$\Rightarrow a_{13} = -2$$
$$a_{23} = 2+3 = 5$$
$$a_{33} = 6$$
$$a_{43} = -2+3 = 1$$

$$\boxed{\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\\a_{43}\end{pmatrix} = \begin{pmatrix}-2\\5\\6\\1\end{pmatrix}}$$

(iv)

There are several reasons why $Col(A) \neq Col(B)$ & any valid reason is sufficient. One reason for example is any vector in $Col(B)$ has a zero in the last component which is not the case for vectors in $Col(A)$.

$\therefore Col(A) \neq Col(B)$

Problem 3. (8 points) Let $C[-1,1]$ be the space of all continuous functions on the closed interval $[-1,1]$, and consider the inner product function

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Apply Gram-Schmidt process to transform the set

$$\{1,\ 1+3x,\ \cos(\pi x)\}$$

into an orthonormal set.

$\underline{v}_1 = 1, \quad \underline{v}_2 = 1+3x, \quad \underline{v}_3 = \cos(\pi x)$

$\underline{w}_1 = \underline{v}_1 = \boxed{1} \qquad \|\underline{w}_1\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} 1\cdot 1\,dx = x\Big|_{-1}^{1} = 1-(-1) = 2$

$\underline{w}_2 = \underline{v}_2 - \mathrm{Proj}_{\underline{w}_1} \underline{v}_2 = (1+3x) - \dfrac{\langle \underline{v}_2, \underline{w}_1\rangle}{\|\underline{w}_1\|^2}\cdot \underline{w}_1 = (1+3x) - \dfrac{2}{2}\cdot 1 = \boxed{3x}$

$\left[\ \langle \underline{v}_2, \underline{w}_1\rangle = \langle 1+3x, 1\rangle = \int_{-1}^{1}(1+3x)\cdot 1\,dx = \int_{-1}^{1} 1\,dx + \underbrace{\int_{-1}^{1} 3x\,dx}_{\text{odd} \Rightarrow = 0} = 2 \right.$

$\left. \|\underline{w}_2\|^2 = \langle 3x, 3x\rangle = \int_{-1}^{1} 9x^2\,dx = 3x^3\Big|_{-1}^{1} = 3[1+1] = 6\ \right]$

$\underline{w}_3 = \underline{v}_3 - \mathrm{Proj}_{\underline{w}_1}\underline{v}_3 - \mathrm{Proj}_{\underline{w}_2}\underline{v}_3 = \cos(\pi x) - \dfrac{\langle \underline{v}_3, \underline{w}_1\rangle}{\|\underline{w}_1\|^2}\underline{w}_1 - \dfrac{\langle \underline{v}_3, \underline{w}_2\rangle}{\|\underline{w}_2\|^2}\underline{w}_2$

$\left[\ \langle \underline{v}_3, \underline{w}_1\rangle = \int_{-1}^{1}\cos(\pi x)\,dx = \frac{+1}{\pi}\sin(\pi x)\Big|_{-1}^{1} = \frac{1}{\pi}[0-0] = 0 \right.$

$\left. \langle \underline{v}_3, \underline{w}_2\rangle = \int_{-1}^{1} 3x\cos(\pi x)\,dx = 0 \text{ (odd fn.)}\ \right]$

$\therefore \underline{w}_3 = \cos(\pi x) - 0 - 0 = \cos(\pi x)$

$\|\underline{w}_3\|^2 = \langle \cos(\pi x), \cos(\pi x)\rangle = \int_{-1}^{1}\cos^2(\pi x)\,dx = \frac{1}{2}\int_{-1}^{1}[1+\cos(2\pi x)]\,dx$

$= \frac{1}{2}\left[x + \frac{1}{2\pi}\sin(2\pi x)\right]_{-1}^{1} = \frac{1}{2}\left[(1+0)-(-1+0)\right] = 1$

Let $\underline{u}_1 = \dfrac{1}{\|\underline{w}_1\|}\underline{w}_1 = \dfrac{1}{\sqrt{2}}\cdot 1, \quad \underline{u}_2 = \dfrac{1}{\|\underline{w}_2\|}\underline{w}_2 = \dfrac{1}{\sqrt{6}}(3x), \quad \underline{u}_3 = \dfrac{1}{\|\underline{w}_3\|}\underline{w}_3 = \cos(\pi x)$

$\therefore \underline{u}_1, \underline{u}_2, \underline{u}_3$ orthonormal

Problem 4. (4 points each) True or False (Circle one and state your reason):

i) The set $W = \{(x, y, z) : x + y - z \leq 1\}$ is a subspace of $\mathbb{R}^3$.

Reason: True $\boxed{\text{False}}$

Not closed under Add

$(0,0,1) \in W$ because $0+0-1 \leq 1$

$-2 \cdot (0,0,1) = (0,0,-2) \notin W$ because $0+0-(-2) > 1$

ii) If two unit vectors $\underline{u}$ and $\underline{v}$ are orthogonal, then so are $\underline{u} + \underline{v}$ and $\underline{u} - \underline{v}$.

Reason: $\boxed{\text{True}}$ False

$\langle \underline{u}, \underline{u} \rangle = 1 = \langle \underline{v}, \underline{v} \rangle$

orthogonal $\longrightarrow$ $\langle \underline{u}, \underline{v} \rangle = 0$

$\therefore \langle \underline{u}+\underline{v}, \underline{u}-\underline{v} \rangle = \langle \underline{u}, \underline{u} \rangle - \langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{u} \rangle - \langle \underline{v}, \underline{v} \rangle$

$= 1 - 0 + 0 - 1 = 0$

$\therefore \underline{u}+\underline{v} \perp \underline{u}-\underline{v}$

iii) The set $\{1, 3x, 1 - x^2, 1 - 7x\}$ is a basis for $\mathcal{P}_2$ (the space of all polynomials of degree ≤ 2).

Reason: True $\boxed{\text{False}}$

Vectors $= 4 > 3 = \dim(P_2)$

$\therefore$ linearly dependent

$\therefore$ Not basis

iv) There exists a 3×6 matrix A such that Nullity$(A) = 2$.

Reason: True $\boxed{\text{False}}$

Rank + Nullity = # Columns $= 6$

However, Rank $= \dim(\text{Rowspace}) \leq 3$

$\therefore$ Nullity ≥ 3

v) For $\underline{u} = (u_1, u_2, u_3)$, $\underline{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, the product

$$\langle \underline{u}, \underline{v} \rangle = u_1 v_1 + u_3 v_3$$

defines an inner product on $\mathbb{R}^3$.

Reason: True $\boxed{\text{False}}$

it violates Axiom $\underline{\underline{4}}$

For example, $\underline{u} = (0, 1, 0) \neq \underline{0}$

$\|\underline{u}\| = \sqrt{(0)(0) + (0)(0)} = 0$ ↯

vi) The set $\{1, 1 - \sin^2(x), 1 - \cos^2(x)\}$ is linearly independent in $C[-1, 1]$ (the space of all continuous functions on the closed interval $[-1, 1]$).

Reason: True $\boxed{\text{False}}$

$\underline{v}_1 = 1$, $\underline{v}_2 = 1 - \sin^2(x)$, $\underline{v}_3 = 1 - \cos^2(x)$

$\underline{v}_2 + \underline{v}_3 = 2 - (\sin^2(x) + \cos^2(x)) = 2 - 1 = 1 = \underline{v}_1$

$\therefore$ Not linearly indep.