Exam1New

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Linear Algebra
Exam 1
October 15, 2022

Name: ______________________ **UID:** ____________________

- **The exam consists of FOUR problems.**
- **Unsupported answers will receive little or no credit.**
- **Anyone caught writing after time has expired will be given a mark of zero.**
- **Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.**
- **Missing or blank pages will result in an automatic zero for the question.**
- **Time: 75 minutes.**

Problem	Score	Points
1		12
2		8
3		6
4		16
Total		42

Best wishes!

Dr. Isabel and Dr. Eslam

Problem 1, Part 1 (6 points) Find the inverse of A or show that it does not exist.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 6 & 2 & 3 \\ 2 & 5 & 2 & -2 \\ 3 & 8 & 4 & -3 \end{bmatrix}.$$

1) We use Gauß-Jordan:

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 3 & 6 & 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 5 & 2 & -2 & 0 & 0 & 1 & 0 \\ 3 & 8 & 4 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - 3R_1} \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & -2 & 0 & 1 & 0 \\ 0 & 2 & 1 & -9 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|cccc} 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 2 & 1 & -9 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 \to -R_3 \\ R_4 \to R_4 - 2R_2}]{R_1 \to R_1 - 2R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 14 & 5 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 3 & -2 & 0 & 1 \end{array}\right), \quad \underbrace{\boxed{\begin{matrix} 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{matrix}}}_{=:B}$$

As B has two identical rows, $\det(B) = 0$.

As A and B are row equivalent, also $\det(A) = 0$.

Hence, A does not have an inverse.

2) Different approach:

$$\det(A) = \begin{vmatrix} 6 & 2 & 3 \\ 5 & 2 & -2 \\ 8 & 4 & -3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & -2 \\ 3 & 4 & -3 \end{vmatrix} + \begin{vmatrix} 3 & 6 & 3 \\ 2 & 5 & -2 \\ 3 & 8 & -3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 2 \\ 3 & 8 & 4 \end{vmatrix}$$

$$= (-36 - 32 + 60 - 48 + 48 + 30) - 2(-18 - 12 + 24 - 18 + 24 + 12)$$
$$+ (-45 - 36 + 48 - 45 + 48 + 36) - 2(60 + 36 + 32 - 30 - 48 - 48)$$
$$= 22 - 2 \cdot (12) + 6 - 2(2)$$
$$= 0.$$

$\Rightarrow \det(A) = 0$, whence A has no inverse.

Problem 1, Part 2 (6 points) Let A and B be 2×2 matrices such that $\det(A) = 3$ and $\det(B) = 1/3$.

(i) Find the value of $\det\big(4\,A\,(A\,B)^T\,(BA)^{-1}\,B\big)$.

We use that for $A, B \in M_{n\times n}$, $c \in \mathbb{R}$:

1) The determinant is multiplicative
2) $|AB| = |BA|$
3) $(AB)^T = B^T A^T$ and $(AB)^{-1} = B^{-1}A^{-1}$
4) $|A^T| = |A|$
5) $|c \cdot A| = c^n |A|$.

Hence: $$|4 \cdot A\,(AB)^T\,(BA)^{-1}B| \overset{3)+5)}{=} 4^2\,|A B^T A^T A^{-1} B^{-1} B| \overset{2)+4)}{=} 16 \cdot |A \cdot A \cdot \cancel{A^{-1}} \cdot \cancel{B B^{-1}} B| \overset{1)}{=} 16 \cdot |A| \cdot |B| = 16 \cdot 3 \cdot \tfrac{1}{3} = \underline{\underline{16}}.$$

(ii) Does it follow that B is the inverse of A? **Explain your answer**.

This does not necessarily follow. Consider for example $A = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$ in $M_{2\times 2}$. Then $\det(A) = 3$ and $\det(B) = 1/3$, but $A \cdot B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix} \neq I_2$,

whence B is <u>not</u> the inverse of A.

Problem 2. (8 points)

(i) Find the value(s) of λ for which the system of linear equations in x and y

$$\begin{aligned} x + \lambda y &= 1 \\ \lambda x + y &= 1 \\ x - y &= \lambda \end{aligned}$$

has **(a)** no solutions, **(b)** a unique solution, **(c)** infinitely many solutions.

We create the augmented matrix and transfer it into REF.

1

$$\left(\begin{array}{cc|c} 1 & \lambda & 1 \\ \lambda & 1 & 1 \\ 1 & -1 & \lambda \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - \lambda R_1} \left(\begin{array}{cc|c} 1 & \lambda & 1 \\ 0 & 1-\lambda^2 & 1-\lambda \\ 0 & -1-\lambda & \lambda-1 \end{array}\right) =: (*)$$

1

Now, we want to divide R_2 by $1-\lambda^2$, which is possible only if $\lambda \neq \pm 1$. We distinguish cases.

Case 1: $\lambda = 1$. Then

1

$$(*) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[R_2 \to -\frac{1}{2}R_2]{R_1 \to R_1 + \frac{1}{2}R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The linear system has the unique solution $\binom{1}{0}$.

1

Case 2: $\lambda = -1$. Then

1

$$(*) = \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{array}\right)$$

contradiction

For $\lambda = -1$, the linear system has no solution.

Case 3: $\lambda \neq 1$ and $\lambda \neq -1$. Then

1

$$(*) \xrightarrow{R_2 \to \frac{1}{1-\lambda^2} R_2} \left(\begin{array}{cc|c} 1 & \lambda & 1 \\ 0 & 1 & 1/(1+\lambda) \\ 0 & -1-\lambda & \lambda-1 \end{array}\right) \xrightarrow{R_3 \to R_3 + (1+\lambda) R_2} \left(\begin{array}{cc|c} 1 & \lambda & 1 \\ 0 & 1 & 1/(1+\lambda) \\ 0 & 0 & \lambda \end{array}\right)$$

contradiction for $\lambda \neq 0$

If $\lambda \notin \{-1, 0, 1\}$, the linear system has no solution.

Case 3b: $\lambda = 0$

1

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

If $\lambda = 0$, the linear system has the unique solution $\binom{1}{1}$. 1

(ii) For all λ such that there is a unique solution, state the solution!

Answer: For $\lambda = 0$, the system has the unique solution $x=1, y=1$. For $\lambda=1$, the linear system has the unique solution $x=1, y=0$. For any other λ, it has no solutions.

Problem 3. (6 points) In this question, you are asked to prove that

"For any square matrix $\boldsymbol{A}$, $\boldsymbol{A}$ is invertible iff $\det(\boldsymbol{A}) \neq 0$"

(i) First, assume that A is invertible and deduce that $\det(A) \neq 0$.

If A is invertible, then ex. A^{-1} s.t. $A \cdot A^{-1} = I_n$.

By multiplicativity of det, we get that

$1 = \det(I_n) = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$, whence

$\det(A)$ cannot be zero.

(ii) Second, assume that $\det(A) \neq 0$ and deduce that A has inverse.

If $|A| \neq 0$, then from $A \cdot \mathrm{adj}(A) = |A| I_n$ we deduce

that $B := \frac{1}{|A|}\mathrm{adj}(A)$ is the inverse of A, as:

$$A \cdot B = A \cdot \frac{1}{|A|}\mathrm{adj}(A) = \frac{1}{|A|} A \cdot \mathrm{adj}(A) = \frac{1}{|A|} \cdot |A| \cdot I_n = I_n.$$

Hence, A is invertible, as desired.

Problem 4. (4 points each) True or False (circle one and state your reason):

(i) If A and B are symmetric $n \times n$ matrices, then AB is also symmetric.

Reason: True False

We provide a counter example.

e.g. Consider $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for $a, b, c \in \mathbb{R}$. Then

A and B are symmetric and $A \cdot B = \begin{pmatrix} b & c \\ a & b \end{pmatrix}$, which is not symmetric for $a \neq c$.

In particular, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ provide a counter example for the claim.

(ii) Let A, B and C be square matrices of the same size such that A is invertible and

$$B = A^{-1}CA.$$

Then, B is invertible if and only if C is invertible.

Reason: Direct Proof. True False

Note that $|B| = |A^{-1}CA| \overset{2) \text{ Prob } 1}{=} |A^{-1}AC| = |I_n C| = |C|$. $(*)$

Hence, B is invertible iff $|B| \neq 0$

iff $|C| \neq 0$ (by $(*)$)

iff C is invertible. ∎

(iii) If a square matrix B is obtained from A by swapping two rows and $\det(A) > \det(B)$, then A must be invertible.

Reason: True False

1

Proof by Contradiction.

Assume the statement was false, i.e. there are matrices A, B s.t.

1) B arises from A by swapping two rows
2) $\det(A) > \det(B)$ and
3) A is not invertible.

From 3) we know that $|A| = 0$. 1 From 1) we deduce that $|B| = -|A| = -0 = 0$. 1 Now 2) yields that $0 = \det(A) > \det(B) = 0$, which is a contradiction. Hence, if any matrices A, B satisfy 1) and 2), then they cannot satisfy 3), so A must be invertible. 1 ☒

(iv) For any invertible matrix A, we have

$$\operatorname{adj}\left(A^T\right) = (\operatorname{adj}(A))^T .$$

Reason: True False

1

Recall that if A is invertible, then

$\operatorname{adj}(A) = |A| \cdot A^{-1}$. Hence, $\operatorname{adj}(A)^T = (|A| A^{-1})^T$

1

For all B, c:

$= |A| \; (A^{-1})^T$ 1 as $(c \cdot B)^T = c \cdot B^T$

$= |A| \; (A^T)^{-1}$ as $(B^{-1})^T = (B^T)^{-1}$

$= |A^T| \; (A^T)^{-1}$ 1 as $|B| = |B^T|$

$= \operatorname{adj}(A^T)$. ☒

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