Linear Algebra
Exam 1
June 25, 2022

Name: Model Answer UID: ____________

- The exam consists of FOUR problems.
- Unsupported answers will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.
- Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.
- Missing or blank pages will result in an automatic zero for the question.
- Time: 75 minutes.

Problem	Score	Points
1		8
2		10
3		8
4		16
Total		42

Best wishes!

Dr. Eslam Badr

Problem 1. (8 points) Find the value(s) of k for which the system of linear equations

$$\begin{aligned} x + ky &= 1 \\ kx + y &= k - 1 \end{aligned}$$

has

(i) no solutions ......

(ii) unique solution ......

(iii) infinitely many solutions.....

When there is exactly one solution, it is $x = \ldots$ and $y = \ldots$.

Consider the augmented matrix

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & k-1 \end{array}\right] \xrightarrow{R_2 - kR_1 \to R_2} \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & -1 \end{array}\right].$$

clearly, the system has no solution if $1-k^2=0 \Rightarrow k=\mp 1$.

• If $k \neq \mp 1$, in this case we have two pivots in row-equivalent matrix, so, we have a unique solution.

Moreover, from the 2nd row we have

$$(1-k^2)y = -1 \Rightarrow \boxed{y = \frac{-1}{1-k^2}}, \quad k \neq \mp 1,$$

from the 1st row, $x + ky = 1 \Rightarrow x = 1 - ky$

$$x = 1 + \frac{k}{1-k^2} = \frac{1+k-k^2}{1-k^2}, \quad k \neq \mp 1$$

Finally, the system Cannot have infinitely many Solutions as we Can not have a zero row in the row-equivalent augmented matrix.

Problem 2, Part 1 (5 points) Consider the matrix

$$A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \end{bmatrix}.$$

Determine the values of x for which A is singular.

It is easy to note that, if $x=-1$, then $R_1 = R_2$ on the matrix A, So $\det A = 0$.

Again, if $x=2$, then $R_1 = R_3$ on A

So $\det A = 0$

Moreover, if $x=-2$, then $R_1 = R_3$ on A

So $\det A = 0$

That is, if $x = -1, 2$ or -2 then $\det A = 0$. (i.e., A is singular)

Finally, we need to show that there are no other values at which $\det A = 0$.

If we find $\det A$ by expansion, so we get a third degree polynomial which has at most three roots.

As we have seen, we get three values of x, at which $\det A = 0$.

This leads to, there are no other values of x

Consequently A is singular iff $x = -1, 2$ or -2.

Note: iff: If and only if.

Another Solution

In this method, we find the value of the determinant of the matrix A.

$$\det(A) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \longrightarrow R_2 \\ R_3 - R_1 \longrightarrow R_3 \\ R_4 - R_1 \longrightarrow R_4 \end{matrix}$$

$$= \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & -1-x & 1-x^2 & -1-x^3 \\ 0 & 2-x & 4-x^2 & 8-x^3 \\ 0 & -2-x & 4-x^2 & -8-x^3 \end{vmatrix}$$

factor out $(1+x)$, $(2-x)$, $(2+x)$ from the 2^{nd}, 3^{rd} and 4^{th} rows respectively, we get

$$\det(A) = (1+x)(2-x)(2+x) \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & -1 & 1-x & -1+x-x^2 \\ 0 & 1 & 2+x & 4+2x+x^2 \\ 0 & -1 & 2-x & -4+2x-x^2 \end{vmatrix}$$

$$R_3 + R_2 \longrightarrow R_3$$
$$R_4 - R_2 \longrightarrow R_4$$

$$= (1+x)(2-x)(2+x) \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & -1 & 1-x & -1+x-x^2 \\ 0 & 0 & 3 & 3+3x \\ 0 & 0 & 1 & -3+x \end{vmatrix}$$

$$R_4 - \tfrac{1}{3} R_3 \longrightarrow R_4$$

$$= (1+x)(2-x)(2+x) \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & -1 & 1-x & -1+x-x^2 \\ 0 & 0 & 3 & 3+3x \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$$\det A = (1)(-1)(3)(-4)(1+x)(2-x)(2+x) \Rightarrow \boxed{\det A = 0 \text{ iff } x = -1, 2, -2.}$$

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Problem 2, Part 2 (5 points) Find an **LU-Factorization** for the matrix A or show that it does not exist.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 4 & 8 \\ 1 & -2 & 4 & -8 \end{bmatrix} \quad \begin{array}{l} R_2 - R_1 \longrightarrow R_2 \\ R_3 - R_1 \longrightarrow R_3 \\ R_4 - R_1 \longrightarrow R_4 \end{array} \quad : \quad \boxed{\begin{array}{l} \alpha_{21} = -1 \\ \alpha_{31} = -1 \\ \alpha_{41} = -1 \end{array}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -3 & 3 & -9 \end{bmatrix} \quad \begin{array}{l} R_3 + \frac{1}{2} R_2 \longrightarrow R_3 \\ R_4 - \frac{3}{2} R_2 \longrightarrow R_4 \end{array} \quad : \quad \boxed{\begin{array}{l} \alpha_{32} = \frac{1}{2} \\ \alpha_{42} = -\frac{3}{2} \end{array}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 3 & -6 \end{bmatrix} \quad R_4 - R_3 \longrightarrow R_4 \quad : \quad \boxed{\alpha_{43} = -1}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & -12 \end{bmatrix} = U$$

Therefore $\boxed{A = L\,U}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\alpha_{21} & 1 & 0 & 0 \\ -\alpha_{31} & -\alpha_{32} & 1 & 0 \\ -\alpha_{41} & -\alpha_{42} & -\alpha_{43} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1/2 & 1 & 0 \\ 1 & 3/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

Problem 3. (2 points each) Let A and B be $n \times n$ matrices such that

$$AB = O,$$

where O denotes the zero matrix.

Give a proof or counterexample for each of the following.

(i) Either A or B (or both) is O.

This statement is not true. let us take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

clearly $A \neq O$, $B \neq O$, but

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

(ii) $BA = O$.

This statement is not necessarily true. As we have seen in the part i.

$AB = O$, but $BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq O$

(iii) $A\,B^T A^T B = O$. This statement is true.

Since $A B^T A^T B = A\,(AB)^T B$

but $AB = O \Rightarrow (AB)^T = O$

Hence $A(AB)^T B = O$.

(iv) The system $(BA)\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has infintely many solutions.

The homogeneous system $(BA)\,\underline{x} = \underline{0}$ has infinitely many solution if $\det(BA) = 0$.

Now, Consider

$$\begin{aligned} \det(BA) &= (\det B)(\det A) \\ &= (\det A)(\det B) = \det(AB) \\ &= 0. \qquad (\text{as } AB = O) \end{aligned}$$

So, the statement is true.

Problem 4. (4 points each) True or False (circle one and state your reason):

(i) If A and B are symmetric $n \times n$ matrices, then $AB + BA$ is also symmetric.

Reason: True False

Since $A^T = A$ and $B^T = B$. Now Consider,

$$(AB+BA)^T = (AB)^T + (BA)^T$$
$$= B^T A^T + A^T B^T$$
$$= BA + AB = AB + BA.$$

Hence $AB + BA$ is symmetric.

(ii) If A is a square matrix such that $|A| \neq 0$, then A is invertible.

Reason: True False

We know that
$$A(\operatorname{adj} A) = |A|\, I, \text{ and } |A| \neq 0$$

then $A\left(\frac{1}{|A|} \operatorname{adj} A\right) = I$

this shows that A and $\left(\frac{1}{|A|} \operatorname{adj} A\right)$ are inverse two each other, so A is invertible.

(iii) There exists a 2×2 invertible matrix A satisfying

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} A = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}.$$

Reason: True **False**

We compute the determinants of both sides

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} A \right) = \det \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}$$

$$\Rightarrow \det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \det A = 0$$

$$\Rightarrow \quad -2 \det A = 0 \quad \Rightarrow \quad \det A = 0$$

Hence A is not invertible.

(iv) For any two square matrices A and B of the same size, we have

$$(AB)^T = B^T A^T.$$

Reason: **True** False

Since (i,j)-entry of $(AB)^T$

$= (j,i)$-entry of AB

$= \text{Row}_{j,A} \cdot \text{Col}_{i,B}$

$= \text{Col}_{j,A^T} \cdot \text{Row}_{i,B^T}$

$= \text{Row}_{i,B^T} \cdot \text{Col}_{j,A^T} = (i,j)$-entry of $B^T A^T$.

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