Fall 2018

Problem 1

(a) det $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & t & 9 \\ 0 & 2 & 7 \end{bmatrix}$ Coeff. matrix

A singular (non-invertible) iff $|A| = 0$

But $|A| = \begin{vmatrix} 1 & 2 & -3 \\ -3 & t & 9 \\ 0 & 2 & 7 \end{vmatrix} \overset{C_1}{=} +1 \begin{vmatrix} t & 9 \\ 2 & 7 \end{vmatrix} - (-3) \begin{vmatrix} 2 & -3 \\ 2 & 7 \end{vmatrix}$

$= 7t - 18 + 3(14+6) = 7t + 42$

$\therefore \boxed{t = -6}$

(b) To have infinitely many solns., A should be singular, i.e $t = -6$. We need to see what really happens in this case.

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ -3 & t=-6 & 9 & -3 \\ 0 & 2 & 7 & -2q \end{array}\right]}_{\text{augmented matrix}} \xrightarrow{R_2 + 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 7 & -2q \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 2 & 7 & -2q \\ 0 & 0 & 0 & 0 \end{array}\right]$$

any value of q is allowed!

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -10 & 1+2q \\ 0 & 2 & 7 & -2q \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ infinitely many solns. independently from q

From R_3: $0 = 0$ ✓

From R_2: $2x_2 + 7x_3 = -2q \to x_2 = -q - \frac{7}{2}x_3$

From R_1: $x_1 - 10x_3 = 1 + 2q \to x_1 = 1 + 2q + 10x_3$, x_3 free

$\boxed{\text{Problem 2}}$

(a) $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 9 \\ 0 & 2 & 7 \end{bmatrix} \xrightarrow[\boxed{R_2 + 3R_1 \to R_2}]{m_{21} = 3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & 0 \\ 0 & 2 & 7 \end{bmatrix}$

$\xrightarrow[\boxed{R_3 - \frac{2}{7}R_2 \to R_3}]{m_{32} = \frac{-2}{7}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = U$

EF without normalizing Pivots

$L = \begin{bmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & -m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & \frac{2}{7} & 1 \end{bmatrix}$

$\therefore \; A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & \frac{2}{7} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

is an LU-factorization for A

(b) To solve $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$ we use Part (a)

$(LU) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

Let $U \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ & $L \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

$\Downarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & \frac{2}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

$y_1 = 1$

$-3y_1 + y_2 = -3 \to y_2 = 0$

$\frac{2}{7}y_2 + y_3 = -2 \to y_3 = -2$

$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

$7x_3 = -2 \to x_3 = \frac{-2}{7}$

$7x_2 = 0 \to x_2 = 0$

$x_1 + 2x_2 - 3x_3 = -2$

$x_1 + 0 + \frac{6}{7} = -2$

$x_1 = -2 - \frac{6}{7} = \frac{-20}{7}$

$\therefore \left(\frac{-20}{7}, 0, \frac{-2}{7}\right)$ is the unique soln. for the system.

(c) Yes, it is a unique LU-factorization
one way to see this is to notice the submatrices

$$A_1 = [1] \quad , \quad A_2 = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

Both are invertible, hence Existence & unique
are guaranteed.

$\boxed{\text{Problem } 3}$

(a) let $B = \begin{bmatrix} 1 & 0 \\ 5 & -2 \end{bmatrix}$

$|B| = -2 \neq 0$

$\rightarrow B$ invertible

$B^{-1} = \frac{1}{|B|} \begin{bmatrix} -2 & 0 \\ -5 & 1 \end{bmatrix}$

$= \frac{-1}{2} \begin{bmatrix} -2 & 0 \\ -5 & 1 \end{bmatrix}$

let $C = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix}$

$|C| = 2 \neq 0$

$\rightarrow C$ invertible

$C^{-1} = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix}$

Now, we've $\quad B\,A\,C = D = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$\longrightarrow \cancel{B^{-1}}(\cancel{B}A\cancel{C})\cancel{C^{-1}} = B^{-1}\,D\,C^{-1}$

$\longrightarrow \boxed{A} = B^{-1}\,D\,C^{-1}$

$= \frac{-1}{2} \begin{bmatrix} -2 & 0 \\ -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix}$

$= \frac{-1}{2} \begin{bmatrix} -2 & -2 \\ -4 & -5 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix}$

$= \frac{-1}{4} \begin{bmatrix} -2 & -2 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix}$

$= \frac{-1}{4} \begin{bmatrix} -4 & -6 \\ -10 & -16 \end{bmatrix}$

$= \boxed{\begin{bmatrix} 1 & 3/2 \\ 5/2 & 4 \end{bmatrix}}$

(b) $B^2 - 2B - 5I_n = 0$

$B^2 - 2B = 5I_n$

$B(B - 2I_n) = 5I_n \longrightarrow$ (1)

$\therefore |B(B - 2I_n)| = |5I_n| = 5^n \neq 0$

$= |B|\,|B - 2I_n|$

$\therefore |B| \neq 0 \longrightarrow \boxed{B^{-1} \text{ exists.}}$

Now, from (1) : $B(B-2I) = 5I$

$\longrightarrow \cancel{B^{-1}}(\cancel{B}(B-2I)) = 5B^{-1}$

$\longrightarrow B - 2I = 5B^{-1}$

$\longrightarrow \boxed{B^{-1} = \frac{1}{5}(B - 2I)}.$

(c) $|A| = 7$, A is 4×4

$\longrightarrow |7\underbrace{(-A^5)^{-1}A^t}_{4\times 4}| = 7^4 |(-A^5)^{-1}A^t|$

$= 7^4 |(-A^5)^{-1}| \cdot |A^t|$

$|A| = |A^t|$

$|B^{-1}| = \frac{1}{|B|}$

$= 7^4 \dfrac{+1}{|\underbrace{-A^5}_{4\times4}|} \cdot |A^t|$

$= 7^4 \cdot \dfrac{1}{(-1)^4 |A^5|} \cdot |A|$

$= 7^4 \dfrac{1}{1\cdot|A|^5} \cdot |A| = 7^4 \cdot \dfrac{1}{7^5} \cdot 7 = 1.$