Write all of your work on the answer sheet. **Exam time: 90 minutes.**

QUESTION 1. [3 marks] Define the following ***bold*** terms.

a) A ***solution*** of a system of linear equations in the variables $x_1, x_2, \dots, x_n$.
b) Two systems of linear equations are ***equivalent***.
c) An ***elementary matrix*** of size $n \times n$.

QUESTION 2. [10 marks] Consider the following system of linear equation.

$$\begin{aligned} x - 3z &= -2 \\ 3x + y - 2z &= 5 \\ 2x + 2y + z &= 4 \end{aligned}$$

a) Write the augmented matrix of the system above. Then use either Gauss Elimination or Gauss-Jordan Elimination method to solve the system.

b) Write the system in the form of
$$AX = B$$
where A is the coefficient matrix, X is the variables column matrix, and B is the column matrix of the constant terms of the system.

c) Calculate the determinant of A to show that the matrix A is invertible.

d) Given that $A^{-1} = \frac{1}{7}\begin{bmatrix} -5 & 6 & -3 \\ 7 & -7 & 7 \\ -4 & 2 & -1 \end{bmatrix}$.

Use A^{-1} to find the unique solution of the system above.

QUESTION 3. [2 marks] Is the following system of linear equations consistent? If yes, solve the system.

$$\begin{aligned} x + 2y + 3z &= 0 \\ 3x + 2y + z &= 0 \end{aligned}$$

QUESTION 4. [4 marks] There are four types of 2×2 matrices in *row-echelon form*. One of them is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ where k is any real number. What are the other three types?

QUESTION 5. [6 marks] Consider the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

a) Let $C = [c_{ij}] = AB$. What are the entries $c_{11},\ c_{12},\ c_{13}$?

b) Let $D = [d_{ij}] = BA$. What is the entry d_{21}?

c) Let $R = [r_{ij}] = AB^2$. What is the entry r_{13}? [***Hint***: use part (a).]

d) What is the minor M_{23} of the element a_{23} in matrix A.

QUESTION 6. [6 marks] Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$. Write both matrices A and A^{-1} as the product of elementary matrices.

QUESTION 7. [4 marks] Suppose that A is an $n \times n$ matrix whose i^{th} and j^{th} rows are equal. Prove that $\det(A) = 0$.

QUESTION 8. [5 marks] An $n \times n$ matrix A is called ***orthogonal*** if it is invertible and $A^{-1} = A^T$.

a) Is $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ orthogonal?

b) Is $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ orthogonal?

c) Prove that if A is an orthogonal matrix then either $\det(A) = 1$ or $\det(A) = -1$.

All the best ☺

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