Problem 1, Part 1. (6 points) Determine if each of the following sets is a subspace of $\mathbb{R}^3$. If so, find a **basis and the dimension** of the subspace.

(i) $W = \{(a, b, c) : a + 2b - c = 3\}$.

$a = b = c = 0 \rightarrow 0 + 2(0) - 0 = 3$ XX

$\therefore 0 \notin W$

$\boxed{\therefore W \text{ is NOT a subspace of } \mathbb{R}^3}$

(ii) $W = \{(a - 2b, a + b, -2c) : a, b, c \in \mathbb{R}\}$.

$(a - 2b, a + b, -2c)$

$= (a, a, 0) + (-2b, b, 0) + (0, 0, -2c)$

$= a(1, 1, 0) + b(-2, 1, 0) + c(0, 0, -2)$

$= \text{Span}\{\underset{\vec{v}_1}{(1,1,0)}, \underset{\vec{v}_2}{(-2,1,0)}, \underset{\vec{v}_3}{(0,0,-2)}\} \leq \mathbb{R}^3$

since any span is a subspace

we show that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \rightarrow c_3 = 0, \quad \begin{array}{l} c_1 = 2c_2 \\ c_1 = -c_2 \end{array} \Rightarrow c_1 = c_2 = 0$$

$\boxed{\therefore \mathcal{B} = \{(1,1,0), (-2,1,0), (0,0,-2)\} \text{ basis for } W}$

$\boxed{\dim(W) = 3}$

Problem 1, Part 2. (4 points) Determine whether the set of vectors

$$S = \{\overset{\vec{v}_1}{1 - x},\ \overset{\vec{v}_2}{1 + x - 3x^2},\ \overset{\vec{v}_3}{2x + x^2}\}$$

is a basis for $\mathcal{P}_{\leq 2}$, the space of polynomials in x of degree ≤ 2.

$\dim(P_{\leq 2}) = 3$, $|S| = 3$ ✓

It remains to show that the vectors in S are linearly independent.

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 \quad\quad &= 0 \\ -c_1 + c_2 + 2c_3 &= 0 \\ -3c_2 + c_3 &= 0 \end{aligned}\right\} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right] \xrightarrow{R_3 + \frac{3}{2}R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

from $R_3 \to \boxed{c_3 = 0} \to R_2 \to \boxed{c_2 = 0} \to R_1 \to \boxed{c_1 = 0}$

trivial soln $(0, 0, 0)$ only solution for $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent

$$\boxed{\therefore\ S \text{ basis for } P_{\leq 2}}$$

Problem 2, Part 1 (4 points) Let A be a 5×8 matrix with **at least two pivot columns.**

What are the maximum and the minimum dimensions of RowSpace(A), ColumnSpace(A) and NullSpace(A)? **Explain your answer.**

$\text{Rank}(A) + \text{Nullity}(A) = 8$

at least two pivot columns $\rightarrow \text{Rank}(A) \geq 2$

$\rightarrow \text{Nullity}(A) \leq 6$

5 pivot columns at most $\rightarrow \text{Rank}(A) \leq 5$

$\rightarrow \text{Nullity}(A) \geq 3$

$\text{Rank}(A) = \dim(RS(A)) = \dim(CS(A))$

$$\boxed{\begin{array}{l} \therefore\ 2 \leq \dim(RS(A)) \leq 5 \\ 2 \leq \dim(CS(A)) \leq 5 \\ 3 \leq \text{Nullity}(A) \leq 6 \end{array}}$$

Problem 2, Part 2 (6 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 4 & -3 \\ 4 & 0 & 4 \end{bmatrix}.$$

Determine whether the vector $\mathbf{v} = (-2, 1, 2)$ is in NullSpace(A), RowSpace(A), or both.

$$\xrightarrow[R_3 - 4R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$RS(A) = \text{Span}\{(1,0,1), (0,2,-1)\}$$

$$\begin{aligned} c_1 \quad\quad &= -2 \\ c_2 &= 1 \\ c_1 - c_2 &= 2 \end{aligned} \longrightarrow \begin{aligned} c_1 &= -2 \\ c_2 &= 1 \\ -2 - 1 &= 2 \quad \times\times \end{aligned}$$

No solution

$$\boxed{\therefore \vec{v} \notin RS(A)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{aligned} x_1 &= -x_3 \\ 2x_2 &= x_3 \end{aligned} \xrightarrow[\neq 0]{x_3 = t \in \mathbb{R}} \begin{aligned} x_1 &= -t \\ x_2 &= \tfrac{1}{2}t \end{aligned}$$

$$NS(A) = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix} \right\}$$

$$2(-1, 1/2, 1) = (-2, 1, 2) = \vec{v}$$

$$\boxed{\therefore \vec{v} \in NS(A)}$$

Problem 3, Part 1. (8 points) Let W be the subspace of $\mathbb{R}^3$ spanned by the three vectors $(1,1,0)$, $(0,1,1)$, $(3,1,-2)$.

(i) Find an orthogonal basis to W relative to the inner product function given by: For $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1\nu_1 + u_2\nu_2 + 2u_3\nu_3.$$

(ii) Verify if $\mathbf{v} = (1,-1,2) \in W$.

(i) NOTE :

$$3(1,1,0) - 2(0,1,1)$$
$$= (3,3,0) - (0,2,2)$$
$$= (3,1,-2)$$

$\therefore S = \{ \underset{\vec{v}_1}{(1,1,0)}, \underset{\vec{v}_2}{(0,1,1)} \}$ basis for W

we Apply Gram Schmidt:

$$\boxed{\vec{\omega}_1 = (1,1,0)}$$

$$\vec{\omega}_2 = (0,1,1) - \frac{\langle (0,1,1),(1,1,0) \rangle}{\langle (1,1,0),(1,1,0) \rangle}(1,1,0)$$

$$= (0,1,1) - \frac{1(1)}{2(1)(1) + (1)(1)}(1,1,0) = (0,1,1) - \frac{1}{3}(1,1,0)$$

$$= (0,1,1) - (\tfrac{1}{3}, \tfrac{1}{3}, 0) = \boxed{(-\tfrac{1}{3}, \tfrac{2}{3}, 1) = \vec{\omega}_2}$$

$$\boxed{B_\perp = \{ (1,1,0), (-\tfrac{1}{3}, \tfrac{2}{3}, 1) \}}$$

(ii) $\vec{v} \in W$ if $c_1(1,1,0) + c_2(-\frac{1}{3}, \frac{2}{3}, 1) = (1,-1,2)$ for some $c_1, c_2 \in \mathbb{R}$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 1 \\ 1 & \frac{2}{3} & -1 \\ 0 & 1 & 2 \end{array}\right] \rightarrow \frac{R_3}{c_2 = 2}$$

$$\longrightarrow \frac{R_2}{c_1 + \frac{2}{3}(2) = -1 \rightarrow c_1 = -\frac{7}{3}}$$

$$\xrightarrow{R_1} -7/3 - \tfrac{1}{3}(2) = -9/3 = -3 \underset{\neq 1}{} \times\times$$

$$\boxed{\therefore \vec{v} \notin W}$$

Problem 3, Part 2. (6 points) Let $(V, +, \cdot, <\,,\,>)$ be an inner product space. Prove **Cauchy-Schwarz inequality.** That is, for any two vectors $\mathbf{u}$, $\mathbf{v} \in V$, we have

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}||\,||\mathbf{v}||.$$

Hint: Show that, for any scalar α,

$$||\mathbf{u} - \alpha\mathbf{v}||^2 = ||\mathbf{u}||^2 + \alpha^2||\mathbf{v}||^2 - 2\alpha\langle \mathbf{u}, \mathbf{v} \rangle.$$

Next, substitute $\alpha = \dfrac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2}$ into the inequality $||\mathbf{u} - \alpha\mathbf{v}||^2 \geq 0$.

$$||\vec{u} - \alpha\vec{v}||^2 = \langle \vec{u} - \alpha\vec{v}, \vec{u} - \alpha\vec{v} \rangle$$

$$= \langle \vec{u}, \vec{u} \rangle - \alpha\langle \vec{u}, \vec{v} \rangle - \alpha\langle \vec{v}, \vec{u} \rangle + \alpha^2\langle \vec{v}, \vec{v} \rangle$$

$$= ||\vec{u}||^2 - 2\alpha\langle \vec{u}, \vec{v} \rangle + \alpha^2||\vec{v}||^2$$

$$\Rightarrow ||\vec{u}||^2 + \alpha^2||\vec{v}||^2 - 2\alpha\langle \vec{u}, \vec{v} \rangle \geq 0$$

$$||\vec{u}||^2 + \frac{(\cancel{\alpha}\langle \vec{u}, \vec{v} \rangle)^2}{||\vec{v}||^4}||\vec{v}||^2 - \frac{2\langle \vec{u}, \vec{v} \rangle}{||\vec{v}||^2}\langle \vec{u}, \vec{v} \rangle \geq 0$$

$$\Rightarrow \left(||\vec{u}||^2 + \frac{(\langle \vec{u}, \vec{v} \rangle)^2}{||\vec{v}||^2} \geq \frac{2(\langle \vec{u}, \vec{v} \rangle)^2}{||\vec{v}||^2}\right)||\vec{v}||^2$$

$$\Rightarrow ||\vec{u}||^2\,||\vec{v}||^2 + (\langle \vec{u}, \vec{v} \rangle)^2 \geq 2(\langle \vec{u}, \vec{v} \rangle)^2$$

$$\Rightarrow (||\vec{u}||\,||\vec{v}||)^2 \geq (\langle \vec{u}, \vec{v} \rangle)^2$$

$$\Rightarrow \boxed{|\langle \vec{u}, \vec{v} \rangle| \leq ||\vec{u}||\,||\vec{v}||}$$

Problem 4. (4 points each) True or False (circle one and state your reason):

(i) If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are linearly independent.

Reason: $\boxed{\text{True}}$ False

Assume $\vec{u}, \vec{v}$ L.I

$\Rightarrow c_1 \vec{u} + c_2 \vec{v} = 0 \rightarrow c_1 = c_2 = 0$

$\rightarrow \alpha_1(\vec{u}+\vec{v}) + \alpha_2(\vec{u}-\vec{v}) = \alpha_1\vec{u} + \alpha_1\vec{v} + \alpha_2\vec{u} - \alpha_2\vec{v} = 0$

$\rightarrow (\alpha_1+\alpha_2)\vec{u} + (\alpha_1-\alpha_2)\vec{v} = 0$

$\rightarrow \alpha_1 = -\alpha_2$ and $\alpha_1 = \alpha_2 \Rightarrow \boxed{\begin{matrix}\alpha_1 = 0 \\ \alpha_2 = 0\end{matrix}}$

(ii) Let $C(-1,1)$ be the space of continuous functions on $(-1,1)$. The map $\langle\, , \rangle$ defined by: For $f, g \in C(-1,1)$,

$$\langle f\, , g\rangle = f(0)g(0)$$

is an inner product function.

Reason: True $\boxed{\text{False}}$

$\langle f, f \rangle = 0$ if $f = 0$

let $f = x^2$

$\rightarrow \langle x^2, x^2 \rangle = 0(0) = 0$

but $f \neq 0$