The American University in Cairo
Mathematics and Actuarial Science
Linear Algebra
November 26, 2019

MACT 2132
Fall 2019
Exam 2
Time Limit: 75 Minutes

Name: ______________________ UID: __________

- This exam contains 6 pages (including this cover page).
- Answer **ALL** the questions (total of points is 45).
- Unsupported answers are considered miracles and will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.

Problem	Score	Points
1		12
2		12
3		12
4		9
Total		45

Problem 1. Let $\mathcal{D}$ be the set of all 3×3 matrices of the shape

$$\begin{bmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & r \end{bmatrix},$$

where t, s and r are real numbers.

a) (6 pts) Show that $\mathcal{D}$ is a vector subspace of $\mathcal{M}_{3\times 3}$.

b) (4 pts) Find a basis for $\mathcal{D}$.

c) (2 pts) What is the dimension of $\mathcal{D}$? Justify your answer.

(a) Any vector $\underline{v}$ in D has the shape

$$\underline{v} = \begin{bmatrix} t & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & r \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{v}_1} + s \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\underline{v}_2} + r \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{v}_3}$$

$$= t \cdot \underline{v}_1 + s \cdot \underline{v}_2 + r \cdot \underline{v}_3$$

so $D = \text{span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$. As the span$(\cdot)$ yields a vector space then D is a subspace of $M_{3\times 3}$.

(b) it is clear from the 1st part that $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ spanning set of $\underline{D}$

But also, $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly indep. as the vector eqn.

$$\underline{0} = c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + c_3 \cdot \underline{v}_3 = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

has a unique soln. $c_1 = c_2 = c_3 = 0$

so $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ basis for D

(c) $\dim D$ = the number of vectors in any basis of D

$= \underline{\underline{3}}$ from (b)

Problem 2. (12 pts) Let $\mathcal{P}_3$ be the vector space of polynomials of degree at most three in the variable x. Consider the following inner product on $\mathcal{P}_3$:

$$< p(x), q(x) > := \int_{-1}^{1} p(x)q(x)\,dx.$$

Determine an orthonormal basis for $\mathcal{P}_3$, relative to the above inner product function.

First, we consider the well-known basis for P_3, namely

$$\{ \underset{\underline{v}_1}{1}, \underset{\underline{v}_2}{x}, \underset{\underline{v}_3}{x^2}, \underset{\underline{v}_4}{x^3} \}$$

Second, we transform this basis into an orthonormal basis by the aid of Gram-Schmidt algorithm:

① $\underline{w}_1 := \underline{v}_1 = 1$

$\therefore \|\underline{w}_1\|^2 = < \underline{w}_1, \underline{w}_1 > = \int_{-1}^{1} 1 \cdot 1\, dx = 2$

② $\underline{w}_2 := \underline{v}_2 - \text{Proj}_{\underline{w}_1} \underline{v}_2 = \underline{v}_2 - \dfrac{< \underline{v}_2, \underline{w}_1 >}{\|\underline{w}_1\|^2} \cdot \underline{w}_1$

$= x - \dfrac{\int_{-1}^{1} x\, dx}{2} \cdot 1 = x$ Rmk: $\int_{-1}^{1} x\, dx = 0$

$\|\underline{w}_2\|^2 = \int_{-1}^{1} x \cdot x\, dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

③ $\underline{w}_3 := \underline{v}_3 - \text{Proj}_{\underline{w}_1} \underline{v}_3 - \text{Proj}_{\underline{w}_2} \underline{v}_3 = \underline{v}_3 - \dfrac{< \underline{v}_3, \underline{w}_1 >}{\|\underline{w}_1\|^2} \cdot \underline{w}_1 - \dfrac{< \underline{v}_3, \underline{w}_2 >}{\|\underline{w}_2\|^2} \cdot$

$= x^2 - \dfrac{\cancel{\int_{-1}^{1} x^2 dx} \rightarrow \frac{2}{3}}{2} \cdot 1 - \dfrac{\cancel{\int_{-1}^{1} x^3 dx} \rightarrow 0}{2/3} \cdot x$

$= x^2 - \frac{1}{3}$

$\|\underline{w}_3\|^2 = \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = 2\int_{0}^{1} (x^2 - \frac{1}{3})^2 dx = 2\int_{0}^{1} (x^4 - \frac{2}{3}x^2 + \frac{1}{9})\, dx$

$= 2\left(\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right) = 2\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{8}{45}$

④ $\underline{w}_4 := \underline{v}_4 - \text{Proj}_{\underline{w}_1} \underline{v}_4 - \text{Proj}_{\underline{w}_2} \underline{v}_4 - \text{Proj}_{\underline{w}_3} \underline{v}_4$

$= \underline{v}_4 - \dfrac{< \underline{v}_4, \underline{w}_1 >}{\|\underline{w}_1\|^2} \cdot \underline{w}_1 - \dfrac{< \underline{v}_4, \underline{w}_2 >}{\|\underline{w}_2\|^2} \cdot \underline{w}_2 - \dfrac{< \underline{v}_4, \underline{w}_3 >}{\|\underline{w}_3\|^2} \cdot \underline{w}_3$

$$\underline{w}_4 = x^3 - \frac{\langle x^3, 1\rangle}{2}\cdot 1 - \frac{\langle x^3, x\rangle}{(2/3)}\cdot x - \frac{\langle x^3, x^2-\frac{1}{3}\rangle}{(8/45)}\cdot\left(x^2-\tfrac{1}{3}\right)$$

$$= x^3 - \frac{\cancelto{0}{\int_{-1}^{1} x^3\,dx}}{2} - \frac{\int_{-1}^{1} x^4\,dx}{(2/3)}\cdot x - \frac{\cancelto{0}{\int_{-1}^{1} x^3\left(x^2-\frac{1}{3}\right)dx}}{(8/45)}\cdot\left(x^2-\frac{1}{3}\right)$$

$$= x^3 - \frac{\int_{-1}^{1} x^4\,dx}{(2/3)}\cdot x$$

$$= x^3 - \frac{(2/5)}{(2/3)}\cdot x = x^3 - \frac{3}{5}x$$

$$\|\underline{w}_4\|^2 = \int_{-1}^{1}\underbrace{\left(x^3-\frac{3}{5}x\right)^2}_{\text{Even fn.}}dx = 2\int_{0}^{1}\left(x^3-\frac{3}{5}x\right)^2dx$$

$$= 2\int_0^1\left(x^6-\frac{6}{5}x^4+\frac{9}{25}x^2\right)dx = 2\left(\frac{1}{7}-\frac{6}{25}+\frac{9}{75}\right)$$

Now, $\{\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4\}$ orthogonal basis for P_3

Set $\underline{u}_i = \dfrac{1}{\|\underline{w}_i\|}\cdot\underline{w}_i$ for $i = 1, 2, 3, 4$

$\therefore$ the vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4$ form an orthonormal basis for P_3.

Problem 3. (3 pts each) Consider the matrix

$$A = \begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix}.$$

a) Find a basis for the row space of A.

b) Find a basis for the column space of A.

c) Find a basis for the null space of A.

d) What is the rank of A? What is the nullity of A? Justify your answer.

$$\begin{bmatrix} 3 & -6 & 21 \\ -2 & 4 & -14 \\ 1 & -2 & 7 \end{bmatrix} \xrightarrow[\frac{1}{2}R_2 \to R_2]{\frac{1}{3}R_1 \to R_1} \begin{bmatrix} 1 & -2 & 7 \\ -1 & 2 & -7 \\ 1 & -2 & 7 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EF

- the non-zero rows in EF form a basis for the RS(A)
- We've only one pivot that occurs in the first column of EF

$\therefore$ (a) Basis for RS(A) is $\{(1,-2,7)\}$

(b) Basis for CS(A) is $\left\{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}\right\}$

(d) Rank(A) = dim RS(A) = 1

Rank(A) + Nullity(A) = no. of columns of A

$\therefore$ 1 + Nullity(A) = 3

$\therefore$ Nullity(A) = 2

(c) To find the NS(A) : $\begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $x = 2y - 7z$, y, z free

$y \to t$, $z \to s$

$\therefore$ any vector in NS(A) has the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t - 7s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ NS(A) has $\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 1 \end{bmatrix}\right\}$ as a basis.

Problem 4. (3 pts each) Prove or disprove **Three** of the following.

a) The subset of $\mathbb{R}^2$ consisting of all points (x, y) on the ellipse $2x^2 + 3y^2 = 1$ is a subspace.

False Clearly, $\underline{0} = (0,0)$ doesn't lie on the ellipse then, it is not a subspace of $\mathbb{R}^2$, as any subspace should contain the zero vector $\underline{0}$

b) Let $\underline{u} = (u_1, u_2)$, $\underline{v} = (v_1, v_2) \in \mathbb{R}^2$. The function $< \underline{u}, \underline{v} >= 2u_1v_2 - u_2v_2$ is an inner product on $\mathbb{R}^2$.

False For example, take $\underline{u} = (0,1)$

$\therefore \|\underline{u}\|^2 = \langle \underline{u}, \underline{u} \rangle = 2(0)(1) - (1)(1) = -1 < 0$!!

c) If A is an $m \times n$ matrix, then Nullity(A) = Nullity(A^T)

$\underset{m\times n}{A}$ $\underset{n\times m}{A^t}$

False Rank(A) = Rank(A^t)

Nullity$(A) = n -$ Rank(A)

Nullity$(A^t) = m -$ Rank(A^t)

Since m, n can be different, then Nullity$(A) \neq$ Nullity(A^t)

d) Let V be an an inner product space. Then, for any $\mathbf{u}, \mathbf{v} \in V$,

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||.$$

True

$$\|\underline{u}+\underline{v}\|^2 = \langle \underline{u}+\underline{v}, \underline{u}+\underline{v} \rangle$$
$$= \langle \underline{u}, \underline{u} \rangle + 2\langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{v} \rangle$$
$$= \|\underline{u}\|^2 + 2\langle \underline{u}, \underline{v} \rangle + \|\underline{v}\|^2$$
$$\leq \|\underline{u}\|^2 + 2|\langle \underline{u}, \underline{v} \rangle| + \|\underline{v}\|^2$$

Cauchy-Shwarz

$$\leq \|\underline{u}\|^2 + 2\|\underline{u}\| \, \|\underline{v}\| + \|\underline{v}\|^2$$
$$= (\|\underline{u}\| + \|\underline{v}\|)^2$$

$\therefore \|\underline{u}+\underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ Triangle Inequality

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