SYSTEMS OF LINEAR EQUATIONS (1.1 AND 1.2)

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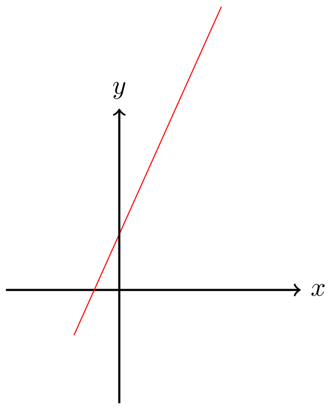
1. Number of solutions

Let us motivate the topic through some examples of "linear objects" in $\mathbb{R}^2$; the 2 D-space and also in $\mathbb{R}^3$; the 3 D-space.

Example 1.1 (lines in $\mathbb{R}^2$)**.** We know that any equation of the form

$$a\,x + b\,y = c$$

such that the $a \neq 0$ or $b \neq 0$ is represented by a *line* in the xy-plane.

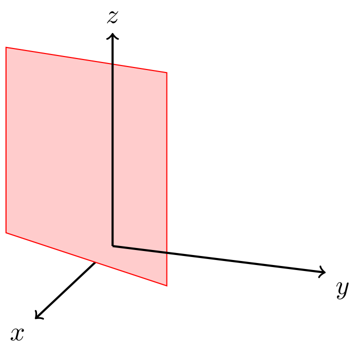


Algebraically, we call such an equation "a linear equation in two variables". Now imagine that you have a collection of these kind of equations in the xy-plane. Then, geometrically you have a set of 2 D lines, and algebraically you have what we call "a system of linear equations in two variables".

Example 1.2 (planes in $\mathbb{R}^3$)**.** Similarly, any equation of the form

$$a\,x + b\,y + c\,z = d$$

with a, b or c non-zero is a *plane* in the xyz-plane.



Algebraically, we call it "a linear equation in three variables". A collection of these equations is "a system of linear equations in three variables".

Moreover, we can extend the above construction to any number of variables.

A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation of the form:

$$a_1\, x_1 + a_2\, x_2 + \cdots + a_n\, x_n = b,$$

for some constants $a_1,\ a_2, \cdots, a_n$ known as *the coefficients* of the equation, and a constant b known as *the constant term* of the equation.

Example 1.3. The following are examples of linear equations.

(i) $3x + 2y = 5$ is linear in x, y.
(ii) $2x - y + \pi z = \sqrt{2}$ is linear in x, y, z.
(iii) $x_1 + \sin(\pi/7)x_2 - 12x_3 = e^2$ is linear in x_1, x_2, x_3.

The following are non-examples.

(i) $xy+z = 2$ is **not** linear in x, y, z, but it is linear in the two variables xy and z.
(ii) $e^x - 2y = 4$ is **not** linear in x, y, but it is linear in the two variables e^x and y.
(iii) $\sin(x_1) - 2y + \frac{1}{x_2} = -3$ is **not** linear in x_1, y, x_2 but it is linear in $\sin(x_1)$, y and $\frac{1}{x_2}$,
(iv) $\frac{1}{\sqrt{x}} - y^3 = 5$ **not** linear in x, y, but it is linear in the two variables $\frac{1}{\sqrt{x}}$ and y^3.

So even if an equation is not linear, we can apply a change of variables to view it as a linear equation in the new varibales.

A system of linear equations in the variables $x_1, x_2, \ldots, x_n$ is a collection of linear equation in $x_1, x_2, \ldots, x_n$.

In general, a system consisting of m linear equations in n variables $x_1, x_2, \ldots, x_n$ looks like

$$\begin{array}{rcl}
\text{Eqn. } 1 & : & a_{11}\, x_1 + a_{12}\, x_2 + \cdots + a_{1n}\, x_n = b_1, \\
\text{Eqn. } 2 & : & a_{21}\, x_1 + a_{22}\, x_2 + \cdots + a_{2n}\, x_n = b_2, \\
\text{Eqn. } 3 & : & a_{31}\, x_1 + a_{32}\, x_2 + \cdots + a_{3n}\, x_n = b_3, \\
\vdots & & \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
\text{Eqn. } m & : & a_{m1}\, x_1 + a_{m2}\, x_2 + \cdots + a_{mn}\, x_n = b_m,
\end{array}$$

Matrix Representation. Furthermore, we can use matrices to represent systems of linear equations. More precisely, any system of linear equations can be expressed in the form

$$A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

where A denotes *the coefficient matrix*, $\underline{\mathbf{x}}$ *the variables column-vector*, and $\underline{\mathbf{b}}$ *the constants column-vector.*

The matrix $[A : \underline{\mathbf{b}}]$ is called *the augmented matrix* of the system.

For instance, the above system becomes

$$\begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

It is augmented matrix has the shape

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & \dots & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} & b_m \end{array}\right].$$

Example 1.4. The system of linear equations

$$\begin{aligned} 2x_1 - x_2 &= 4, \\ x_1 + 3x_2 &= 1, \\ x_1 + x_2 &= 0. \end{aligned}$$

can be written as

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}.$$

Equivalently, the augmented matrix of the system is $\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 3 & 1 \\ 1 & 1 & 0 \end{array}\right]$.

Example 1.5. The system of linear equations

$$\begin{aligned} 2x - y + z &= 4, \\ x + 3y \quad &= 1, \\ z &= 1. \end{aligned}$$

corresponds to

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

So its augmented matrix would be $\left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

Now let us talk about solutions to systems of linear equations.

A *solution* to a system of linear equation in $x_1, x_2, \dots, x_n$ is an n-tuple $(s_1, s_2, \dots, s_n)$ of real numbers that satisfies each equation in the system when we substitute s_1 for x_1, s_2 for x_2,..., s_n for x_n.

In matrix notation, a solution to $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ is a column-vector $\underline{\mathbf{s}}$ of size $n \times 1$ such that $A\underline{\mathbf{s}} = \underline{\mathbf{b}}$.

A system of linear equations $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ is said to be *consistent* if it has at least one solution. Otherwise, the system is said to be *inconsistent*.

Example 1.6. We can verify that $(2, 1, 1)$ is a solution to the system

$$\begin{aligned} 2x - y + z &= 4, \\ x + 3y &= 5, \\ z &= 1. \end{aligned}$$

Indeed,

$$\begin{aligned} 2(2) - 1 + 1 &= 4, \\ 2 + 3(1) &= 5, \\ 1 &= 1. \end{aligned}$$

Hence the above system is a consistent system.

On the other hand, $(2, 1, -1)$ is **not** a solution as it does not satisfy the 3rd equation in the system.

Now we aim to investigate the number of solutions that a given system of linear equations can have.

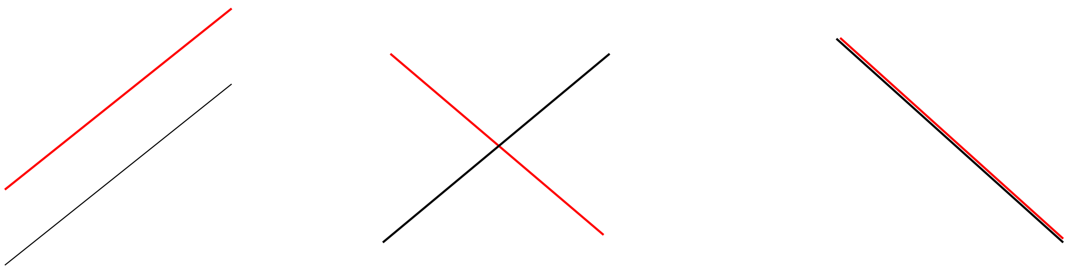
An observation on the solutions set.

Example 1.7. Consider a system of two linear equations in two variables. The system is given by two lines. Geometrically, a solution to the system is a point of intersection of the two lines. Therefore, the number of solutions to such a system is exactly the number of intersection points of the two lines.

Accordingly, we reduced the problem to understand:

Question

How two lines in the 2 D space can intersect?

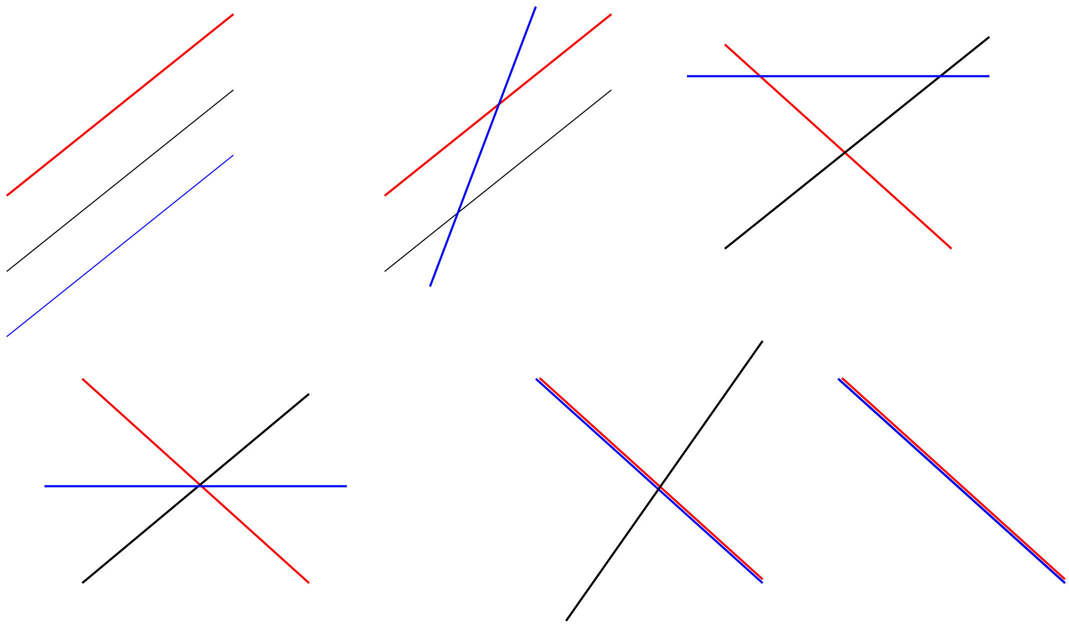


In the first situation, the two lines are non-identical parallel lines. So they do not intersect and the system has **no solutions**.
In the second situation, the two lines do intersect at a single point, and the system has exactly **one solution**.
In the third situation, the two lines are identical, so we have infinitely many intersection points. That is, the system has **∞-many solutions.**

Example 1.8. Consider a system of three linear equations in two variables. So the system is represented by three lines. Again a solution to the system corresponds to a point of intersection of the three lines. It remains to count the number of intersection points of the three lines.

Question

How three lines in the 2 D space can intersect?

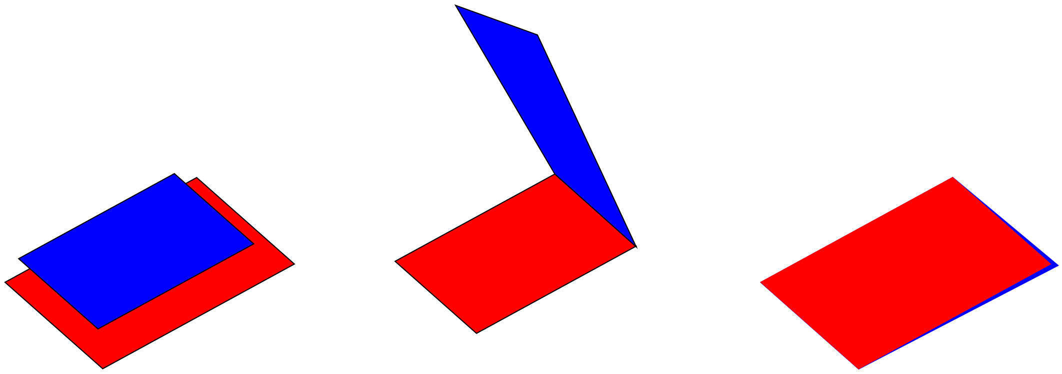


Obviously, each of the first three systems has **no solutions**. The 4th and the 5th systems have exactly **one solution**. And the last system has **∞-many solutions** as the three lines coincide.

Example 1.9. Consider a system of two linear equations in three variables. The system represents two planes in the 3 D-space. Similarly, an intersection point of the two planes corresponds to a solution to the system and vice versa.

Question

How two planes in the 3 D space can intersect?



For the 1st system, the two planes are non-identical and parallel, so the system has **no solutions**. For the 2nd system, the two planes intersect in a line, so the system has **∞-many solutions**. For the 3rd system, the two planes are identical, so the system also has **∞-many solutions**.

Enjoy yourself by discussing the intersection of three planes, in particular, how many solutions a system of three linear equations in three variables can have!

The above geometric examples encourage us to believe in the following fact regarding the number of solutions of a given system of linear equations.

Theorem 1.10. *Let $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ be a system of linear equations, and let N be the number of its solutions. Then, N is either* 0, 1 *or* ∞.

Proof. First, you need to see that the Theorem is equivalent to say

"if a system of linear equations has at least two solutions, then it should have infinitely many solutions".

Now, let us prove this property by assuming that there are two solutions namely, $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$ for the system. In particular,

$$A\,\underline{\mathbf{x}}_1 = \underline{\mathbf{b}} \text{ and } A\,\underline{\mathbf{x}}_2 = \underline{\mathbf{b}}.$$

Now we are going to use these two particular solutions to construct ∞-many other ones.

Let us try the linear combination of $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$ given by

$$\underline{\mathbf{x}}_3 = \underline{\mathbf{x}}_1 + 2\left(\underline{\mathbf{x}}_1 - \underline{\mathbf{x}}_2\right).$$

In this way we have that

$$\begin{aligned} A\,\underline{\mathbf{x}}_3 &= A\,\underline{\mathbf{x}}_1 + 2\,A\,\left(\underline{\mathbf{x}}_1 - \underline{\mathbf{x}}_2\right) \\ &= A\,\underline{\mathbf{x}}_1 + 2\,\left(A\,\underline{\mathbf{x}}_1 - A\,\underline{\mathbf{x}}_2\right) \\ &= \underline{\mathbf{b}} + 2\left(\underline{\mathbf{b}} - \underline{\mathbf{b}}\right) \\ &= \underline{\mathbf{b}} + \mathbf{0} \\ &= \underline{\mathbf{b}}. \end{aligned}$$

That is, $\underline{\mathbf{x}}_3$ is a third solution to the system, different from both $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$.

Let me tell you that the above calculations hold if we replace the constant 2 with any constant α. In other words, for any scalar α, the vector

$$\underline{\mathbf{x}} = \underline{\mathbf{x}}_1 + \alpha\left(\underline{\mathbf{x}}_1 - \underline{\mathbf{x}}_2\right)$$

is also solution to the system, which means that the system has infinitely many solutions, where each of these solutions is obtained from the two fixed solutions $\underline{\mathbf{x}}_1$ and $\underline{\mathbf{x}}_2$.

This shows the result. □

2. Using EROs to reduce a matrices to REF or RREF

In order to be able to solve a given system of linear equations, we should be familiar with the some terminologies related to matrices.

Question

More precisely, we aim to understand the following.

- What does it mean that a matrix A is in Row Echelon Form (REF)?
- What does it mean a matrix A is in Reduced Row Echelon Form (RREF)?
- How can we use Elementary Row Operations (EROs) to reduce a matrix A to REF or RREF?

By *a zero row* of a matrix we mean a row that consists entirely of zeros. A *non-zero row* is a row that has at least one non-zero entry.

Example 2.1. For the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

the 2nd row is a zero row as all entries of this row are 0s.

Definition 2.2 (REF)**.** A matrix A of size $m \times n$ is said to be in REF if it satisfies the following three axioms.

(1) All zero rows (if exist) are arranged below the non-zero rows (if exist).
(2) For any non-zero row (if exists), the 1st non-zero entry equals 1; *a leading one* or *a pivot*.
(3) For any two successive non-zero rows (if exist), say R_i and R_{i+1}, the pivot of R_i (the upper row) is to the left of the pivot of R_{i+1} (the lower row).

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Definition 2.3 (RREF)**.** For a matrix A of size $m \times n$ to be in RREF, it should be REF in the first place such that
(4) each pivot is the only non-zero entry in its column.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}.$$

Example 2.4. Let us check some examples.

The matrix
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$
is **not** REF (hence not RREF) as R_2 violates Axiom (2) in Definition 2.3. In other words, the 1st non-zero entry in R_2 (being a non-zero row) is not 1!

The matrix
$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$
is **not** REF (hence not RREF) as it violates Axiom (1) in Definition 2.3. In other word, R_2 is a zero row that occurs before the non-zero row R_3!

The matrix
$$\begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 1 & \pi \\ 0 & 1 & 0 & 0 & 4 \end{bmatrix}$$
is **not** REF (hence not RREF) as it violates Axiom (3) in Definition 2.3. In other words, the pivot in R_3 is not to the left of the piovot of R_4!

The matrix
$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
is REF for which each column is a pivot column. However, it is **not** RREF as it violates Axiom (4) in Definition 2.3. For example, the pivot appearing in the 2nd column is not the only non-zero entry in this column. Similarly, the 3rd pivot does not satisfy Axiom (4).

The matrix

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is RREF with two pivot columns namely, the 2nd and the 3rd columns.

The zero matrix $O_{m\times n}$ is both REF and RREF (why?) Also, the identity matrix I_n is both REF and RREF (why?)

Example 2.5. In this example we aim to classify 2×2 matrices A that are REF.

The best way to do this is based on the number of pivots in A.
- If A has **no pivots** then it must be the zero matrix $O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
- If A has exactly **one pivot** then we have two possibilities for A. Either

$$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For the first type, there exists infinitely many values for A as $*$ can be any real number. For the second type, A is not only REF but also RREF.
- If A has two pivots then it has the form

$$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}.$$

In particular, A of this type is RREF if and only if $* = 0$.

Question

Enjoy your self by classifying 3×3 matrices that are REF (respectively RREF) and have exactly two pivots :)

Elementary Row Operations (EROs)

Any matrix A can be reduced to REF or RREF via a sequence of Elementary Row Operations (EROs).

An ERO is one of the following row operations:

(I) Swap two rows R_i and R_j. Such an ERO is denoted by $R_i \leftrightarrow R_j$.
(II) Replace one row R_i by a **non-zero** scalar multiple $\alpha\, R_i$ of itself. Such an ERO is deonted by $R_i \to \alpha\, R_i$.
(III) Replace one row R_i by the result of addinig this row to a scalar multiple $\alpha\, R_j$ of another row R_j. Such an ERO is denoted by $R_i \to R_i + \alpha\, R_j$.

An algorithm

To reduce a given matrix to REF, you may like to follow the algorithm below.
- STEP 1: start with the leftmost non-zero column of the matrix and fix any non-zero entry $*$ of this column to be your pivot.
- STEP 2: If this non-zero entry $* \neq 1$, then apply an ERO of Type (II), which means dividing the corresponding row by $*$ in order to transform $*$ into 1. Otherwise, proceed to STEP 3.
- STEP 3: if needed, apply swapping to move the pivot from STEP 1 to the first row R_1 of the matrix A.
-STEP 4: apply EROs of the Type (III) in order to reduce the entries below the pivot in its column to zeros.
- STEP 5: Repeat the above process to the submatrix obtained from A by imagining that R_1 and all the columns from the beginning till the pivot column do not exist.

Remark 2.6. *To reduce a matrix to RREF, the same process works, however, in STEP 4, you need transform the elements below and above the pivot in its column to zeros.*

Now we are going to practice the above reduction process through some examples.

Example 2.7. Let us reduce the matrix below to REF then to RREF.

$$A = \begin{bmatrix} 1 & 1 & -8 & 6 \\ 2 & 3 & -21 & 14 \\ -1 & 0 & 4 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & -8 & 6 \\ 2 & 3 & -21 & 14 \\ -1 & 0 & 4 & -5 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 1 & -8 & 6 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & -8 & 6 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \text{REF}$$

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 + 3R_3]{R_2 \to R_2 + 5R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix} = \text{RREF}\,.$$

Example 2.8. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -8 & 6 \\ 2 & 3 & -21 & 14 \\ -1 & 0 & 4 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & -8 & 6 \\ 2 & 3 & -21 & 14 \\ -1 & 0 & 4 & -5 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_1]{R_3 \to -R_3} \begin{bmatrix} 1 & 0 & -4 & 5 \\ 2 & 3 & -21 & 14 \\ 0 & 1 & -8 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 0 & -4 & 5 \\ 0 & 3 & -13 & 4 \\ 0 & 1 & -8 & 6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & -4 & 5 \\ 0 & 1 & -8 & 6 \\ 0 & 3 & -13 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 0 & -4 & 5 \\ 0 & 1 & -8 & 6 \\ 0 & 0 & 11 & -14 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow \frac{R_3}{11}} \begin{bmatrix} 1 & 0 & -4 & 5 \\ 0 & 1 & -8 & 6 \\ 0 & 0 & 1 & -14/11 \end{bmatrix} = \text{REF}$$

$$\xrightarrow[R_1 \leftrightarrow R_1 + 4R_3]{R_2 \to R_2 + 8R_3} \begin{bmatrix} 1 & 0 & 0 & -1/11 \\ 0 & 1 & 0 & -46/11 \\ 0 & 0 & 1 & -14/11 \end{bmatrix} = \text{RREF}\,.$$

Example 2.9. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -8 & 6 \\ 0 & 3 & -21 & 14 \\ 0 & 0 & 4 & -5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & -8 & 6 \\ 0 & 3 & -21 & 14 \\ 0 & 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_2 - 3R_1} \begin{bmatrix} 0 & 1 & -8 & 6 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -3 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 0 & 1 & -8 & 6 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow \frac{R_2}{3}} \begin{bmatrix} 0 & 1 & -8 & 6 \\ 0 & 0 & 1 & -4/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{REF}$$

$$\xrightarrow{R_1 \to R_1 + 8R_2} \begin{bmatrix} 0 & 1 & 0 & -14/3 \\ 0 & 0 & 1 & -4/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{RREF}\,.$$

As a remark, a matrix A can have more than one REF, however, it has unique RREF.

Example 2.10. The matrix $\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$ can be reduced to $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \text{REF}$ using the EROs

(1) $R_1 \to -R_1$,
(2) $R_2 \to R_2 - R_1$,
(3) $R_2 \to \frac{R_2}{5}$.

Also the same matrix $\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$ can be reduced to a different REF namely, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ via the EROs

(1) $R_1 \leftrightarrow R_2$,
(2) $R_2 \to R_2 + R_1$,
(3) $R_2 \to \frac{R_2}{5}$.

However, reducing any of these REF to RREF leads to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We finish this subsection by the following definition.

Row- Equivalent matrices

Definition 2.11. Two matrices A and B of the same size are said to be *row- equivalent* if we can obtain B from A using a sequence of EROs.

Example 2.12. Any two matrices that appear in Example 2.9 are row-equivalent.

3. Gaussian Elimination Method

Now we have all what we need to digest one of the most efficient algorithms to solve systems of linear equations. I am talking about the very famous “Gaussian Elimination Method” and “Gauss-Jordan Elimination Method”.

To approach the idea behind this method, let us consider the following three systems of linear equations.

- 1st system:

$$\begin{array}{rcrcrcr} x & + & y & - & 8z & = & 6, \\ 2x & + & 3y & - & 21z & = & 14, \\ -x & & & + & 4z & = & -5. \end{array}$$

- 2nd system:

$$\begin{array}{rcrcrcr} x & + & y & - & 8z & = & 6, \\ & & y & - & 5z & = & 2, \\ & & & & z & = & -1. \end{array}$$

- 3rd system:

$$\begin{array}{rrrcr} x & & & = & 1, \\ & y & & = & -3, \\ & & z & = & -1. \end{array}$$

- You can see that the 1st system requires some algebraic manipulation “Elimination Theory” to find its solution set!

- The 2nd system is **simpler** as it only requires “back-substitution” to solve the system. That is, we have $z = -1$ from the last equation. If we back-substitution $z = -1$ into the second equation, then we obtain $y - 5(-1) = 2$, so $y = -3$. Finally, back-substitution $z = -1$ and $y = -3$ into the first equation yields $x + (-3) - 8(-1) = 6$, so $x = 1$. Hence, we conclude that the system has only the 3-tuple $(1, -3, -1)$ as its unique solution.

- The 3rd system is **the simplest** because it is so obvious that $(1, -3, -1)$ is the only solution to it. We do not even need back-substitution to see this!

- In Matrix Notations, the augmented matrices of these systems become

- 1st system: $\left[\begin{array}{ccc|c} 1 & 1 & -8 & 6 \\ 2 & 3 & -21 & 14 \\ -1 & 0 & 4 & -5 \end{array}\right]$ Not REF
- 2nd system: $\left[\begin{array}{ccc|c} 1 & 1 & -8 & 6 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$ REF
- 3rd system: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$ RREF

By Example 2.7, we deduce that any two of these matrices are row-equivalent in the sense of Definition 2.11. This allows us to say the any two of the three systems are "equivalent".

Formally speaking,

Definition 3.1. Two systems of linear equations $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ and $A'\underline{\mathbf{x}} = \underline{\mathbf{b}}'$ are said to be *equivalent* if they have the same solution set.
In Matrix Notations, this means that we can reduce the augmented matrix $[A \mid \underline{\mathbf{b}}]$ of the first system to $[A' \mid \underline{\mathbf{b}}']$ via a sequence of EROs.

For the mathematician among you, can you see why using EROs to simplify a given system of linear equations does not change the solution set of the system?

The above example suggests the following algorithm to solve a given system of linear equations $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$.

Gaussian Elimination Method.

- Construct the augmented matrix $[A \mid \underline{\mathbf{b}}]$ of the system,
- Apply EROs to reduce the coefficient matrix A to REF, and in parallel use the same EROs to change the constants $\underline{\mathbf{b}}$. That is, simplify $[A \mid \underline{\mathbf{b}}]$ to $[\text{REF} \mid \underline{\mathbf{b}}']$.
- Use back-substitution to solve the new system $[\text{REF} \mid \underline{\mathbf{b}}']$, which is equivalent to the old system.

Remark 3.2. *If we slightly modify the above method by reducing the system to RREF instead of REF, then we are applying* ***Gauss-Jordan Elimination Method.***

Example 3.3. We aim to solve the system:

$$\begin{array}{rcrcrcr} & & x_2 & - & x_3 & = & 0, \\ x_1 & & & - & 3x_3 & = & -1, \\ -x & + & 3x_2 & & & = & 2. \end{array}$$

According to the algorithm, we first construct the augmented matrix

$$[A \mid \underline{\mathbf{b}}] = \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -3 & -1 \\ -1 & 3 & 0 & 2 \end{array}\right]$$

Second, we reduce it to REF or RREF using a sequence of EROs

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -3 & -1 \\ -1 & 3 & 0 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 3 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \text{New equivalent system}$$

Since the new equivalent system is in REF (actually, RREF), we can solve it using back-substitution. One observes that the last equation in the new system has the shape

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right],$$

which can be read as $0\,x_1 + 0\,x_2 + 0\,x_3 = 1$, "a contradiction". From this we conclude that the above system has **no solutions**.

Geometrically, the system represents three planes that do not intersect!

Fact 3.4. *A system of linear equations is inconsistent if and only if it can be reduced via EROs to a system that has a contradiction equation, that is, an equation of the shape*

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & b \neq 0 \end{array}\right],$$

All the coefficients are zeros, but the corresponding constant term is non-zero!

Example 3.5. We aim to solve the system

$$\begin{array}{rcrcrcr} & & x_2 & - & x_3 & = & 0, \\ x_1 & & & - & 3x_3 & = & -1, \\ -x_1 & + & x_2 & & & = & 1. \end{array}$$

The augmented matrix is

$$[A \mid \underline{\mathbf{b}}] = \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -3 & -1 \\ -1 & 1 & 0 & 1 \end{array}\right]$$

A sequence of EROs to reduce it to REF is described below

$$\begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 1 & 0 & -3 & | & -1 \\ -1 & 1 & 0 & | & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & -3 & | & -1 \\ 0 & 1 & -1 & | & 0 \\ -1 & 1 & 0 & | & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 0 & -3 & | & -1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 1 & -3 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & -3 & | & -1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to \frac{R_3}{-2}} \begin{bmatrix} 1 & 0 & -3 & | & -1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} = \text{REF}\,.$$

We note that the reduced system does not have any contradiction, hence it is consistent. Moreover, each column in the coefficient matrix is a "pivot column", which indicates that the system has **unique solution**.

To solve the system, we use back-substitution. From the last equation, we get $x_3 = 0$. From the second equation, we get $x_2 - 1(0) = 0$, so $x_2 = 0$. From the first equation, we get $x_1 + 0(0) - 3(0) = -1$, so $x_1 = -1$. Therefore, $(-1, 0, 0)$ is the only solution to the system.

Geometrically, the system represents three planes that intersect at a single point!

Example 3.6. We aim to solve the system

$$\begin{array}{rcrcrcr} & & x_2 & - & x_3 & = & 0, \\ x_1 & & & - & 3x_3 & = & -1, \\ -x_1 & + & 3x_2 & & & = & 1. \end{array}$$

The augmented matrix is

$$[A \mid \underline{\mathbf{b}}] = \begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 1 & 0 & -3 & | & -1 \\ -1 & 3 & 0 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 & | & 0 \\ 1 & 0 & -3 & | & -1 \\ -1 & 3 & 0 & | & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & -3 & | & -1 \\ 0 & 1 & -1 & | & 0 \\ -1 & 3 & 0 & | & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 0 & -3 & | & -1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 0 & -3 & | & -1 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} = \text{RREF}$$

Since contradictions do not appear in the REF/RREF, then the system is consistent. Moreover, the 3rd column is a "non-pivot column", which indicates that the corresponding variable x_3 can be taken as a free variable and that the system has **∞-many solutions**.

To solve the system, we use back-substitution. The third equation is redundant because it says $0 = 0$!. From the second equation, we have $x_2 = x_3$, $x_1 = -1 + 3x_3$

from the first equation, and x_3 is free (assumes any value). Calling $x_3 = t$ (a parameter) leads to the fact that any solution $\underline{\mathbf{x}}$ to the system has the form

$$\underline{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+3t \\ t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

for some $t \in \mathbb{R}$. The above form is a **parametric vector representation** of the solution set using the parameter t.

Have you seen such a parametric vector equation before? For those who took Calculus II, these equations describe a 3 D line; the line passing through the point $(-1, 0, 0)$ and parallel to the vector $\langle 3, 1, 1 \rangle$. Thus the system represents three planes intersecting in a line.

Now it is time to speak about *homogeneous systems* of linear equations.

Definition 3.7. A *homogeneous* system of linear equations is a system whose constants terms are all zeros (that is, a system of the form $A\,\underline{\mathbf{x}} = \mathbf{0}$).

Such a system is **always consistent** because "the trivial solution" (known also as the "the zero solution") $\underline{\mathbf{x}} = \mathbf{0}$ is an obvious solution to the system.

In particular, we have:

Fact 3.8. *Any homogeneous system of linear equations has either unique solution; the trivial one, or ∞-many solutions.*

Example 3.9. We aim to solve the homogeneous system

$$\begin{array}{rcrcrcl} & & x_2 & - & x_3 & = & 0, \\ x_1 & & & - & 3x_3 & = & 0, \\ -x_1 & + & 3x_2 & & & = & 0. \end{array}$$

$$\begin{aligned}
[A \mid \underline{\mathbf{b}}] \quad &= \quad \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & -3 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right] \\
&\xrightarrow{R_3 \to R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&= \quad \text{RREF}
\end{aligned}$$

Since the 3rd column is a "non-pivot column", then the we consider x_3 as a free variable. Hence the system has **∞-many solutions**.

If we let $x_3 = t$ (a parameter), then we can express any solution $\underline{\mathbf{x}}$ to the system in the form

$$\underline{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

for some $t \in \mathbb{R}$.

Geometrically, this means that the three plane intersect in the line that passes through the origin and parallel to the vector $\langle -3, 1, 1 \rangle$.

Example 3.10. For each fixed λ, consider the system

$$\begin{array}{ccccccc} x & + & y & + & \lambda z & = & \lambda^2, \\ x & + & \lambda y & + & z & = & \lambda, \\ \lambda x & + & y & + & z & = & 1. \end{array}$$

Question

For which value(s) of λ, are there (i) no solutions, (ii) unique solution, (iii) ∞-many solutions?

To approach this example, we construct the augmented matrix and reduce it to REF.

$$[A \mid \underline{\mathbf{b}}] = \begin{bmatrix} 1 & 1 & \lambda & | & \lambda^2 \\ 1 & \lambda & 1 & | & \lambda \\ \lambda & 1 & 1 & | & 1 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - \lambda R_1} \begin{bmatrix} 1 & 1 & \lambda & | & \lambda^2 \\ 0 & \lambda - 1 & 1 - \lambda & | & \lambda - \lambda^2 \\ 0 & 1 - \lambda & 1 - \lambda^2 & | & 1 - \lambda^3 \end{bmatrix}$$

To proceed, we should distinguish between whether $\lambda = 1$ or $\lambda \neq 1$.

(1) If $\lambda = 1$, then the system reduces to

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

In particular, y and z are free variables and **the system has ∞-many solutions.** Let $y = t$ (a parameter) and $z = s$ (a parameter) to get $x = 1 - t - s$ for some $s, t \in \mathbb{R}$. In particular, a solution to the system has the parametric vector form

$$\underline{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Equivalently, the solution set to the system forms a plane as it is described by two free parameters!

(2) If $\lambda \neq 1$, then we can apply $R_2 \to \frac{R_2}{\lambda - 1}$ and $R_3 \to \frac{R_3}{1 - \lambda}$ to obtain

$$\begin{bmatrix} 1 & 1 & \lambda & | & \lambda^2 \\ 0 & 1 & -1 & | & -\lambda \\ 0 & 1 & 1 + \lambda & | & 1 + \lambda + \lambda^2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & \lambda & | & \lambda^2 \\ 0 & 1 & -1 & | & -\lambda \\ 0 & 0 & 2 + \lambda & | & 1 + 2\lambda + \lambda^2 \end{bmatrix}$$

This gives us two subcases to deal with, depending on whether $2 + \lambda = 0$ or not.

- If $2 + \lambda = 0$, then the last equation of the system becomes

$$\begin{bmatrix} 0 & 0 & 0 & | & 1 \end{bmatrix},$$

which is a contradiction. Hence the system is inconsistent if $\lambda = -2$.

- If $2+\lambda \neq 0$ then we can use $R_3 \to \dfrac{R_3}{2+\lambda}$ to get

$$\left[\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & 1 & -1 & -\lambda \\ 0 & 0 & 1 & \dfrac{1+2\lambda+\lambda^2}{2+\lambda} \end{array}\right]$$

In particular, no contradiction appears and the system is consistent. Since we have three pivots, the system has a unique solution given by

$$\begin{aligned} z &= \frac{1+2\lambda+\lambda^2}{2+\lambda}, \\ y &= -\lambda + \frac{1+2\lambda+\lambda^2}{2+\lambda} = \frac{1}{2+\lambda}, \\ x &= \lambda^2 - \lambda\left(\frac{1+2\lambda+\lambda^2}{2+\lambda}\right) - \frac{1}{2+\lambda} = \frac{-\lambda}{2+\lambda} - \frac{1}{2+\lambda} = -\frac{1+\lambda}{2+\lambda}. \end{aligned}$$

Finally we deduce that the system has (i) no solutions if $\lambda = -2$, (ii) unique solution if $\lambda \neq 1, -2$, and (iii) ∞-many solutions if $\lambda = 1$.

Example 3.11. Decide whether the following statement is True or False.

"A system of linear equations that has more equations than the variables is always inconsistent."

FALSE. For example, take the system:

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 1, \\ x_1 & - & x_2 & = & 3, \\ x_1 & + & 2x_2 & = & 0, \end{array}$$

It is easy to verify that $(2, -1)$ solves the system. In particular, we have a system of THREE linear equations in TWO variables that is consistent!

Example 3.12. Decide whether the following statement is True or False.

"If a system of linear equations has fewer equations than the variables, then it must have ∞-many solutions."

FALSE. For example, take the system:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1, \\ x_1 & + & x_2 & + & x_3 & = & -1. \end{array}$$

It is impossible to find values for x_1, x_2 and x_3 such that $x_1+x_2+x_3$ equals 1 and -1 simultaneously. Thus we have a system of TWO linear equations in THREE variables that does not have solutions!

Example 3.13. Decide whether the following statement is True or False.

"If a homogeneous system of linear equations has fewer equations than the variables, then it must have ∞-many solutions."

TRUE. Being homogeneous implies that the system must be consistent. Now assume that the number of equations m is strictly less than the number of variables n. In the best case scenario, we will have a pivot in each row of the augmented matrix. So the number of pivots (in the best case scenario) is strictly smaller than the number of variables. Therefore, free variables do exist and it must be the case that the system has ∞-many solutions!

Summing up, we have

- To solve a given system of linear equations $A\,\underline{\mathbf{x}} = \mathbf{0}$, we first reduce it to REF or RREF through a sequence of EROs.
  - If the new system in REF/RREF has a contradiction
  $$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & b \neq 0 \end{array}\right],$$
  then the system is **inconsistent**.
  - Otherwise, the system is **consistent**. Moreover, it has unique solution if the number of pivots = the number of variables, and it has ∞-many solutions if the number of pivots < the number of variables.
- It is a convention that pivot columns correspond to "basic variables" and non-pivot columns correspond to "free variables".

  In the case that free variables exist, we call them t, s, $r, \dots$ as a way of parameterizing the solution set. For instance, having exactly one free variable means that the solution set is "a line", having exactly two variables means that the solution set is "a plane", and so on.