Linear Algebra
Exam 2
July 16, 2022

Name: ______________________ UID: ________________

- The exam consists of FOUR problems.
- Unsupported answers will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.
- Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.
- Missing or blank pages will result in an automatic zero for the question.
- Time: 75 minutes.

Problem	Score	Points
1		10
2		8
3		8
4		16
Total		42

Best wishes!

Dr. Eslam Badr

Problem 1, Part 1. (5 points) Determine if each of the following sets is a **subspace** of $\mathbb{R}^3$. If so, find a **basis and the dimension** of the subspace. **Explain your answer.**

(i) $W = \{(x, y, z) : 2xz = y\}$.

Clearly, W is not a subspace of $\mathbb{R}^3$. To prove this, let us take $\vec{u} = (1,2,1)$ and $\vec{v} = (1,4,2)$ from W, then

$\vec{u} + \vec{v} = (1,2,1) + (1,4,2) = (2,6,3) \notin W$

because the condition $2xz = y$ does not hold, i.e.,

$2(2)(3) \neq 6$.

(ii) $W = \text{Span}\{\overset{v_1}{(1,1,0)}, \overset{v_2}{(1,-1,3)}, \overset{v_3}{(1,5,-6)}\}$.

According to the theorem 4.7, the span of a set of vectors in a vector space V is a subspace of V.

Therefore, W is a subspace of $\mathbb{R}^3$. To find its basis, we consider the matrix A having its rows as v_1, v_2 and v_3

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 3 \\ 1 & 5 & -6 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 4 & -6 \end{bmatrix} \xrightarrow{R_3 - 2R_2 \to R_3} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the nonzero row vectors $(1,1,0)$ and $(0,-2,3)$ form a basis of the row space of A. That is, they form a basis of the subspace spanned by $\{(1,1,0), (1,-1,3), (1,5,-6)\}$.

Consequently $\dim W = 2$.

Problem 1, Part 2. (5 points) Determine the value(s) of k for which the set of vectors

$$S = \{1 - x,\ 1 + x - kx^2 + x^3,\ x - x^3\}$$

is linearly independent in $\mathcal{P}_{\leq 3}$, the vector space of polynomials in x of degree ≤ 3.

Can we consider S as a basis for $\mathcal{P}_{\leq 3}$ for some values of k? **Justify.**

Consider the equation $c_1(1-x) + c_2(1+x-kx^2+x^3) + c_3(x-x^3) = 0$,

This eq. gives rise to a homogeneous system of equations having the augmented matrix,

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array}\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \end{array} \xrightarrow{R_2+R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_2+R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

Now, from the 2^{nd} row, we have $3c_2 = 0 \Rightarrow c_2 = 0$.

From the 3^{rd} row, we have $c_2 - c_3 = 0 \Rightarrow c_3 = 0$.

From the 1^{st} row, we have $c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 = 0$.

Consequently, the system has a unique solution $(0,0,0)$ regardless the value of k.

This shows that the vectors $1-x$, $1+x-kx^2+x^3$, $x-x^3$ are linearly independent for all $k \in \mathbb{R}$.

Since S contains 3 linearly independent vectors, but $\dim P_{\leq 3} = 4$, So, S can not be a basis for $P_{\leq 3}$ (whatever the value of k).

Problem 2. (8 points) Let $V = \mathbb{R}^2$, the set of all ordered pairs of real numbers. Define an **addition** and **scalar multiplication** by

$$\begin{aligned}(x, y) + (x', y') &= (x + x' + 1, y + y' - 2),\\ \alpha \cdot (x, y) &= (\alpha x, \alpha y),\end{aligned}$$

for all (x, y), $(x', y') \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$.

First, we remark that $V = \mathbb{R}^2$ is **NOT** a vector space under the operations $+$ and $\cdot$.

Verify if each of the following axioms holds; **(i)** Additive identity, **(ii)** Additive inverse, **(iii)** Associativity, and **(iv)** Distributivity. **Justify your answer.**

(i) Additive identity.

Let $\vec{0} = (x_0, y_0)$, the element $\vec{0} \in V$ is the additive identity if $\vec{u} + \vec{0} = \vec{u}$ for all $\vec{u} \in V$.

Now, $(x,y) + (x_0, y_0) = (x, y) \Rightarrow (x + x_0 + 1, y + y_0 - 2) = (x, y)$

which gives, $x + x_0 + 1 = x \Rightarrow x_0 = -1$

$y + y_0 - 2 = y \Rightarrow y_0 = 2$.

This establishes the existence of an additive identity, i.e., $\vec{0} = (-1, 2)$.

So, the axiom holds.

(ii) Additive Inverse Let $\vec{u} = (x, y) \in V$, to show that each $\vec{u} \in V$ has an additive inverse $-\vec{u} = (a, b)$, we must have:

$\vec{u} + (-\vec{u}) = \vec{0}$.

Now, $(x, y) + (a, b) = (-1, 2) \Rightarrow (x + a + 1, y + b - 2) = (-1, 2)$,

This gives, $x + a + 1 = -1 \Rightarrow a = -x - 2$

and $y + b - 2 = 2 \Rightarrow b = -y - 4$.

Thus, the additive inverse $-\vec{u} = (-x - 2, -y - 4)$.

So the axiom holds.

(iii) Associativity

Let $\vec{u}=(x_1,y_1)$, $\vec{v}=(x_2,y_2)$ and $\vec{w}=(x_3,y_3)\in V$

we need to check that

$$(\vec{u}+\vec{v})+w = \vec{u}+(\vec{v}+\vec{w}).$$

L.H.S , $(\vec{u}+\vec{v})+\vec{w} = \big((x_1,y_1)+(x_2,y_2)\big)+(x_3,y_3)$

$$= (x_1+x_2+1 , y_1+y_2-2)+(x_3,y_3)$$

$$= (x_1+x_2+1+x_3+1 , y_1+y_2-2+y_3-2)$$

$$= (x_1+x_2+x_3+2 , y_1+y_2+y_3-4)$$

R.H.S , $\vec{u}+(\vec{v}+\vec{w}) = (x_1,y_1)+\big((x_2,y_2)+(x_3,y_3)\big)$

$$=(x_1,y_1)+(x_2+x_3+1, y_2+y_3-2)$$

$$=(x_1+x_2+x_3+1+1, y_1+y_2+y_3-2-2)$$

$$=(x_1+x_2+x_3+2, y_1+y_2+y_3-4)$$

Thus the axiom holds.

(iv) Distributivity

let $\vec{u}=(x_1,y_1)$, $\vec{v}=(x_2,y_2)\in V$ and $\alpha\in\mathbb{R}$, then

$$\alpha(\vec{u}+\vec{v}) = \alpha\big((x_1,y_1)+(x_2,y_2)\big)$$

$$=\alpha(x_1+x_2+1, y_1+y_2-2) = (\alpha x_1+\alpha x_2-\alpha, \alpha y_1+\alpha y_2-2\alpha)$$

but

$$\alpha\vec{u}+\alpha\vec{v} = \alpha(x_1,y_1)+\alpha(x_2,y_2)$$

$$=(\alpha x_1,\alpha y_1)+(\alpha x_2,\alpha y_2)$$

$$=(\alpha x_1+\alpha x_2+1, \alpha y_1+\alpha y_2-2)$$

So, the axiom $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ <u>does NOT hold.</u>

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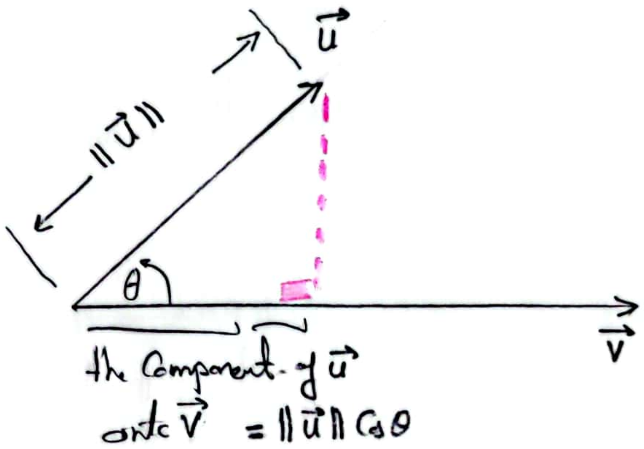
Problem 3, Part 1. (4 points) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two non-zero vectors in an inner product space V. Prove that the projection of $\mathbf{u}$ onto $\mathbf{v}$, denoted by $\text{Proj}_{\mathbf{v}}\, \mathbf{u}$, is given by

$$\text{Proj}_{\mathbf{v}}\, \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{v}||^2} \mathbf{v}.$$

As show in the figure,

The orthogonal projection of the vector $\vec{u}$ onto $\vec{v}$ is

$$\text{Proj}_{\vec{v}} \vec{u} = ||\vec{u}|| \cos\theta \cdot \frac{\vec{v}}{||\vec{v}||}$$



but $\cos\theta = \dfrac{\langle \vec{u}, \vec{v} \rangle}{||\vec{u}||\, ||\vec{v}||}$, then

$$\text{Proj}_{\vec{v}} \vec{u} = ||\vec{u}|| \frac{\langle \vec{u}, \vec{v} \rangle}{||\vec{u}||\, ||\vec{v}||} \quad \frac{\vec{v}}{||\vec{v}||}$$

$$= \frac{\langle \vec{u}, \vec{v} \rangle}{||\vec{v}||^2} \vec{v}.$$

Problem 3, Part 2. (4 points) Let W be the set that consists of all the vectors in $\mathbb{R}^3$ orthogonal to $(1, 4, -1)$.

Show that W is a **subspace** of $\mathbb{R}^3$, find a **basis** B for it, and determine the **dimension** of W.

Symbolically, we can write W as

$$W = \{ w = (x, y, z) \in \mathbb{R}^3 \mid (1,4,-1) \cdot (x, y, z) = 0 \}.$$

Now, the condition of orthogonality gives

$$(1,4,-1) \cdot (x, y, z) = 0$$

$$\Rightarrow \quad x + 4y - z = 0 \qquad \text{or} \qquad z = x + 4y.$$

Consequently, we can rewrite W as

$$W = \{ (x, y, x+4y) \mid x, y \in \mathbb{R} \}$$

$$= \{ x(1,0,1) + y(0,1,4) \mid x, y \in \mathbb{R} \}$$

Clearly $W = \text{Span}\{ (1,0,1), (0,1,4) \}$

hence W is a subspace of $\mathbb{R}^3$.

Moreover, the vectors $(1,0,1)$ and $(0,1,4)$ are linearly independent because $(1,0,1) \neq \lambda(0,1,4)$, $\forall \lambda \in \mathbb{R}$,

hence $\dim W = 2$.

Problem 4. (4 points each) True or False (circle one and state your reason):

(i) If the three vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are **not** linearly independent in V, then so are the two vectors $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2\}$.

Reason: True **False**

let us take $\vec{V_1} = (1,0)$ $\vec{V_2} = (0,0)$, $\vec{V_3} = (0,1)$

which are linearly dependent vectors as $\vec{0} = (0,0)$ is one of them.

Now, $\vec{V_1} - \vec{V_2} = \vec{V_1} = (1,0)$

and $\vec{V_3} - \vec{V_2} = \vec{V_3} = (0,1)$

which are linearly independent vectors as they represent the standard basis of $\mathbb{R}^2$.

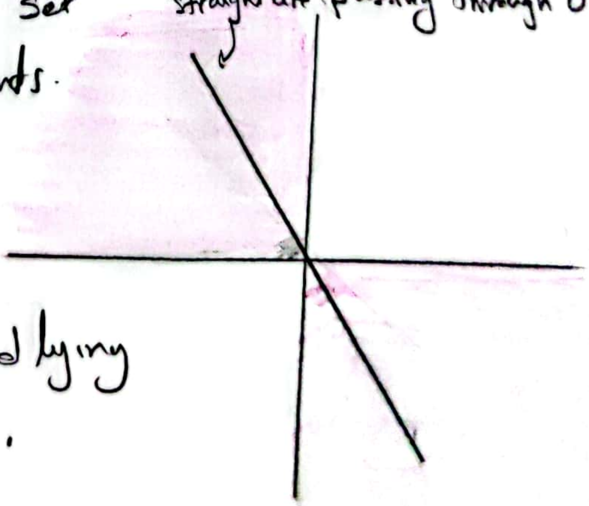
(ii) There exists a subset U of $\mathbb{R}^2$, which is not a subspace of $\mathbb{R}^2$ but it contains many subspaces of $\mathbb{R}^2$.

Reason: **True** False

Let us take $U \subset \mathbb{R}^2$ as the set represents the 2nd and 4th quadrants.

Clearly U is not a subspace of $\mathbb{R}^2$, but any straight line passing through the origin and lying in the set U is a subspace of U.

straight line passing through $\vec{0}$.



(iii) The map $\langle \, , \, \rangle$ defined by:

$$\langle p(x)\,,\, q(x)\rangle = p(0)q(0) + p(1)q(1)$$

for $p(x), q(x) \in \mathcal{P}_{\leq 3}$ defines an inner product function on $\mathcal{P}_{\leq 3}$.

Reason: True **False**

let us take $p(x) = x(x-1) = x^2 - x$,

clearly, $p(x)$ is not a zero vector, but

$\langle p(x), p(x)\rangle = p(0)\,p(0) + p(1)\,p(1)$

$= 0$

which contradicts the axiom $\langle p(x), p(x)\rangle > 0$

if $p(x) \neq 0$.

(iv) Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthonormal basis for an inner product space V. Then any $\mathbf{v} \in V$ can be written as $\mathbf{v} = \alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2$ with $\alpha = \langle \mathbf{v}, \mathbf{v}_1\rangle$ and $\beta = \langle \mathbf{v}, \mathbf{v}_2\rangle$.

Reason: **True** False

Since V_1 and V_2 represent a basis of V,

So any vector $v \in V$ can be written as a linear Combination of the basis. That is

$\vec{V} = \alpha \vec{v_1} + \beta \vec{v_2}$

Moreover, $\vec{v_1}$ and $\vec{v_2}$ are orthonormal, that is $\langle \vec{v_1}, \vec{v_2}\rangle = 0$

and $\langle v_1, v_1\rangle = 1$, $\langle v_2, v_2\rangle = 1$.

Now, $\langle \vec{V}, \vec{v_1}\rangle = \langle \alpha \vec{V_1} + \beta \vec{V_2}, \vec{v_1}\rangle = \alpha \langle \vec{v_1}, \vec{v_1}\rangle + \beta \langle \vec{v_2}, \vec{v_1}\rangle$

$= \alpha$

Similarly,

$\langle \vec{V}, \vec{V_2}\rangle = \langle \alpha \vec{V_1} + \beta \vec{V_2}, \vec{V_2}\rangle = \alpha \langle \vec{V_1}, \vec{V_2}\rangle + \beta \langle \vec{V_2}, \vec{V_2}\rangle$

$= \beta$.

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