Final (Fall 2019)

- **Problem 1**

$$\underline{u} = (\overset{u_1}{1}, \overset{u_2}{0}, \overset{u_3}{-1})$$

$$\underline{v} = (\overset{v_1}{2}, \overset{v_2}{-1}, \overset{v_3}{3})$$

(a) $\langle \underline{u}, \underline{v} \rangle = 2u_1v_1 + u_2v_2 + 3u_3v_3$

$$= 2(1)(2) + (0)(-1) + 3(-1)(3)$$
$$= 4 + 0 - 9$$
$$= \underline{-5}$$

$$\|\underline{u}\|^2 = \langle \underline{u}, \underline{u} \rangle = 2u_1^2 + u_2^2 + 3u_3^2$$
$$= 2(1)^2 + (0)^2 + 3(-1)^2$$
$$= 2 + 0 + 3$$
$$= 5$$

$\therefore$ $\|\underline{u}\| = \sqrt{5}$

$$d(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\| = \|(-1, 1, -4)\|$$

$$\|(-1,1,-4)\|^2 = \langle (-1,1,-4), (-1,1,-4) \rangle$$
$$= 2(-1)^2 + (1)^2 + 3(-4)^2$$
$$= 2 + 1 + 48 = 51$$

$\therefore$ $d(\underline{u}, \underline{v}) = \sqrt{51}$

$$\text{Proj}_{\underline{u}}\, \underline{v} = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\|^2} \cdot \underline{u} = \frac{-5}{5} \cdot (1, 0, -1)$$
$$= (-1, 0, 1)$$

(b) Suppose that $\underline{w} = (a, b, c)$ is a vector in $\mathbb{R}^3$ that is orthogonal to $\underline{u}$ and $\underline{v}$

$\therefore \langle \underline{u}, \underline{w} \rangle = 0$ & $\langle \underline{v}, \underline{w} \rangle = 0$

$\therefore$
$$2a + 0b - 3c = 0$$
$$4a - b + 9c = 0$$

$\therefore$ $\underline{a = \frac{3}{2}c}$ & $\underline{b = 15c}$, $\underline{c \text{ "free"}}$

$\rightarrow \underline{w} = c\left(\frac{3}{2}, 15, 1\right)$ such that c free

$\therefore$ there are infinitely many vectors orthogonal to $\underline{u} = (1, 0, -1)$ and $\underline{v} = (2, -1, 3)$ simultaneously. Each such a vector is a scaler multiple of the vector $\left(\frac{3}{2}, 15, 1\right)$.

(c) $B = \{\underset{\underline{v}_1}{(0,1,2)}, \underset{\underline{v}_2}{(2,0,0)}, \underset{\underline{v}_3}{(1,1,1)}\}$

B is linearly independent, so can be transformed to an orthonormal basis via Gram-Schmidt.

$\boxed{\underline{w}_1} = \underline{v}_1 = \boxed{(0,1,2)}$

$\boxed{\underline{w}_2} = \underline{v}_2 - \text{Proj}_{\underline{w}_1} \underline{v}_2 = \underline{v}_2 - \dfrac{\langle \underline{v}_2, \underline{w}_1 \rangle}{\|\underline{w}_1\|^2} \cdot \underline{w}_1$

$2(2)(0) + (1)(0) + 3(0)(2) = 0$

$2(0)^2 + (1)^2 + 3(2)^2 = 13$

$= (2,0,0) - \frac{0}{13} \cdot (0,1,2)$

$= \boxed{(2,0,0)}$

$$\boxed{\underline{w}_3} = \underline{v}_3 - \text{Proj}_{\underline{w}_1} \underline{v}_3 - \text{Proj}_{\underline{w}_2} \underline{v}_3$$

$$= \underline{v}_3 - \frac{\langle \underline{v}_3 , \underline{w}_1 \rangle}{\|\underline{w}_1\|^2} \cdot \underline{w}_1 - \frac{\langle \underline{v}_3 , \underline{w}_2 \rangle}{\|\underline{w}_2\|^2} \cdot \underline{w}_2$$

$2(1)(0) + (1)(1) + 3(1)(2)$ $\quad$ $2(1)(2) + (1)(0) + 3(1)(0)$

$2(0)^2 + (1)^2 + 3(2)^2$ $\quad$ $2(2)^2 + (0)^2 + 3(0)^2 = 8$

$$= (1,1,1) - \frac{7}{13}(0,1,2) - \frac{4}{8}(2,0,0)$$

$$= \boxed{\left(0, \frac{6}{13}, \frac{-1}{13}\right)}$$

$$\|\underline{w}_3\|^2 = 2(0)^2 + \left(\frac{6}{13}\right)^2 + 3\left(\frac{-1}{13}\right)^2 = \frac{39}{13^2} = \frac{3}{13}$$

$\therefore$ $\{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$ orthogonal basis

It remains to normalize each of them to get an orthonormal basis :

$$\boxed{\underline{u}_1} = \frac{1}{\|\underline{w}_1\|} \cdot \underline{w}_1 = \frac{1}{\sqrt{13}}(0,1,2) = \boxed{\left(0, \frac{1}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)}$$

$$\boxed{\underline{u}_2} = \frac{1}{\|\underline{w}_2\|} \cdot \underline{w}_2 = \frac{1}{\sqrt{8}}(2,0,0) = \boxed{\left(\frac{2}{\sqrt{8}}, 0, 0\right)}$$

$$\boxed{\underline{u}_3} = \frac{1}{\|\underline{w}_3\|} \cdot \underline{w}_3 = \frac{\sqrt{13}}{\sqrt{3}}\left(0, \frac{6}{13}, \frac{-1}{13}\right)$$

$$= \boxed{\left(0, \frac{2\sqrt{3}}{\sqrt{13}}, \frac{-1}{\sqrt{3}\sqrt{13}}\right)}$$

$\therefore$ $\{\underline{u}_1, \underline{u}_2, \underline{u}_2\}$ orthonormal basis.

• **Problem 2**

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Eigenvalues : $0 = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 0 & 0 \\ 0 & \lambda-1 & -5 & 10 \\ -1 & 0 & \lambda-2 & 0 \\ -1 & 0 & 0 & \lambda-3 \end{vmatrix}$

$$= (\lambda-1)\begin{vmatrix} \lambda-1 & -5 & 10 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-1)(\lambda-2)(\lambda-3)$$

Upper triangular

$\therefore$ 1, 1, 2, 3 are the eigenvalues

Eigenvectors : For $\lambda = 1$: $\qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\underbrace{(\overset{1}{\lambda} I - A)}_{4\times4} \underbrace{\underline{x}}_{4\times1} = \underbrace{\underline{0}}_{4\times1}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Row 1 : $0 = 0$ ✓

Row 2 : $-5x_3 + 10x_4 = 0 \longrightarrow x_3 = 2x_4$ $\qquad$ Note if $x_2 = 0$, then $x_4 \neq 0$

Row 3 : $-x_1 - x_3 = 0 \longrightarrow x_1 = -x_3 = -2x_4$ $\qquad$ if $x_4 = 0$, then $x_2 \neq 0$

Row 4 : $-x_1 - 2x_4 = 0 \longrightarrow x_1 = -2x_4$

$\therefore$ $x_1 = -2x_4$, x_2 free , $x_3 = 2x_4$, x_4 free

$$\therefore \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = x_4 \underbrace{\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{\underline{v}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_2}$$

$\therefore (\text{Eigenspace})_{\lambda=1} = \text{Span}\{\underline{v}_1, \underline{v}_2\}$

Scanned with CamScanner

For $\lambda = 2$: $(\underset{2}{\underset{\shortparallel}{\lambda}} I - A)\underline{x} = \underline{0}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Row 1 : $x_1 = 0$

Row 4 : $-\underset{0}{\underset{\shortparallel}{x_1}} - x_4 = 0 \quad \therefore x_4 = 0$

Row 3 : $-x_1 = 0 \quad \therefore x_1 = 0$

Row 2 : $x_2 - 5x_3 + 10\underset{0}{\underset{\shortparallel}{x_4}} = 0 \quad \therefore x_2 = 5x_3$, x_3 free, $\neq 0$

$$\therefore \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_3}$$

$$\therefore (\text{Eigenspace})_{\lambda=2} = \text{span}\{\underline{v}_3\}$$

For $\lambda = 3$: $(\underset{3}{\underset{\shortparallel}{\lambda}} I - A)\underline{x} = \underline{0}$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Row 1 : $2x_1 = 0 \quad \therefore x_1 = 0$

Row 3 : $-\underset{0}{\underset{\shortparallel}{x_1}} + x_3 = 0 \quad \therefore x_3 = 0$

Row 2 : $2x_2 - 5\underset{0}{\underset{\shortparallel}{x_3}} + 10x_4 = 0 \quad \therefore x_2 = -5x_4$ $\quad x_4$ free, $\neq 0$

Row 4 : $-x_1 = 0 \longrightarrow x_1 = 0$

$$\therefore \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_4 \underbrace{\begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_4} \quad x_4 \text{ free}, \neq 0$$

$\therefore (\text{Eigenspace})_{\lambda=3} = \text{Span}\{\underline{v}_4\}$

(b) A size 4×4

The eigenvectors $\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ form a basis for $\mathbb{R}^4$

$\therefore$ A is diagonalizable

(c) Because $\lambda = 0$ is not an eigenvalue, then the matrix A is invertible. In particular, the rows / columns of A are linearly independent.

$\therefore \text{Rank}(A) = 4$ (no. of rows)

$\therefore \text{Nullity}(A) = \text{no. of columns} - \text{Rank}(A)$

$= 4 - 4 = 0$

• Problem 3

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a)

this is a homogeneous system, so it is always solvable

In particular, no restrictions on a, b can be found so that the system is inconsistent

(iii) no solution "No conditions"

Now, the system either has a unique soln. or infinitely many solns. This depends on the determinant

$$\begin{array}{ccc} a^2b & a^2b & a^2b \end{array}$$

$$\left|\begin{array}{ccc|cc} a & b & b & a & b \\ a & a & b & a & a \\ a & a & a & a & a \end{array}\right.$$

$$\begin{array}{ccc} a^3 & ab^2 & a^2b \end{array}$$

$$= a^3 + ab^2 + a^2b - 3a^2b$$

$$= a^3 - 2a^2b + ab^2$$

$$= a(a^2 - 2ab + b^2)$$

$$= a(a-b)^2$$

Exactly one soln. if $a(a-b)^2 \neq 0 \rightarrow$ (i)

inf. many solns. if $a(a-b) = 0 \longrightarrow$ (ii)

(b) if $a(a-b)^2 \neq 0$, the system has exactly one soln., the trivial soln. $x = y = z = 0$

if $a(a-b)^2 = 0$, then $a = 0$ or $a = b$

and we solve the system in both cases :

Scanned with CamScanner

Case 1: $a = 0$

$$\begin{bmatrix} 0 & b & b \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row 3: $\therefore\ 0 = 0$ ✓

Row 2: $by = 0$

$b = 0$

$\therefore\ x, y, z$ are free

$\therefore$ Soln. set $= \mathbb{R}^3$

$b \neq 0$

$y = 0$

Row 1

$\underset{= 0}{by} + bz = 0$

$\therefore\ bz = 0$

$\therefore\ z = 0$ because $b \neq 0$

$\therefore\ x$ free

Soln. set $= \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \text{ free} \right\}$

Case 2: $a = b$

$$\begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Row 1 = Row 2 = Row 3

$\longrightarrow\ a(x + y + z) = 0$

$a = 0$

$\therefore\ x, y, z$ free

$\therefore$ Soln. set $= \mathbb{R}^3$

$a \neq 0$

$\therefore\ x + y + z = 0$

$\therefore\ x = -y - z \qquad y, z$ free

$\therefore$ Soln. set $= \left\{ \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix},\ y, z \text{ free} \right\}$

Problem 4

$T : P_2 \longrightarrow \mathbb{R}$

$$T(P) = \int_{-1}^{1} P(x)\,dx$$

(a) Axiom 1: let $p, q \in P_2$ be any two vectors

$$\underline{T(P+q)} = \int_{-1}^{1} (P+q)(x)\,dx = \int_{-1}^{1} [p(x) + q(x)]\,dx$$

we add Polynomials Pointwise

both integrals are convergent

$$= \int_{-1}^{1} P(x)\,dx + \int_{-1}^{1} q(x)\,dx$$

$$= T(P) + T(q)$$

Axiom 2: let $P \in P_2$ any vector, and let c any scaler

$$\underline{T(c \cdot P)} = \int_{-1}^{1} (c \cdot P)(x)\,dx = \int_{-1}^{1} cP(x)\,dx$$

$\int_{-1}^{1} cP(x)\,dx$, $\int_{-1}^{1} P(x)\,dx$ are convergent

$$= c\int_{-1}^{1} P(x)\,dx = c \cdot T(P)$$

$\therefore$ T Preserves Add / Scaler mult.

$\therefore$ T is a L.T.

(b) $T(x) = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big]_{-1}^{1} = 0$

(c) We've from (b) that $P(x) = x$ is a non-zero vector in P_2 whose image $= \underline{0}$ in $\mathbb{R}$

$\therefore Ker(T) \neq \{\underline{0}\}$ $\therefore$ T not 1-1.

(d) Clearly, T is not the Zero transformation

For instance, $T(x^2) = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \neq 0$

$\therefore$ Range(T) (as a subspace in Codomain$(T) = \mathbb{R}$) is a non-trivial subspace. In particular, Range(T) has dimension ≥ 1. But also it has dimension $\leq \dim(\mathbb{R}) = 1$

$\therefore \dim(\text{Range}(T)) = 1$

$\therefore$ Range$(T) = \mathbb{R}$ $\quad \therefore T$ onto.

Problem 5

ⓐ **False** the zero function $f(x) = 0$ is not contained in W as $f(0) = 0 \neq 2$

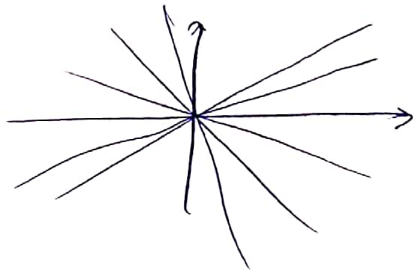
$\therefore$ W not a subspace of $C(-\infty, \infty)$

ⓑ **True** Similar matrices have same determinant

$\therefore$ $\det(A) = 0$ iff $\det(B) = 0$

$\therefore$ A singular iff B singular.

ⓒ **True** any line through the origin is a subspace of dimension 1, and we've in $\mathbb{R}^2$ infinitely many such lines.



ⓓ **True** By Rank–Nullity theorem

$\text{Rank}(T) + \text{Nullity}(T) = 2$

$\therefore$ $\text{Rank}(T) \leq 2$

$\therefore$ $\text{Rank}(T)$ is never $\dim(\mathbb{R}^3)$

$\therefore$ T never onto.

e) <u>False</u>

$$|(A^2B)^T B^2 A^{-1}| = |(A^2B)^T| \; |B|^2 \, |A^{-1}|$$

$$= |A^2B| \; |B|^2 \, |A^{-1}|$$

$$= |A^2| \, |B| \, |B|^2 \, |A^{-1}|$$

$$= |A|^2 \, |B|^3 \, |A|^{-1}$$

$$= |A| \, |B|^3 = (3)(-2)^3 = -24 \neq 24$$

f) <u>False</u>

For example, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We've $AB = O_{2\times 2} = AC$

However, $B \neq C$

g) <u>True</u>

if $\lambda = 0$ is an eigenvalue, then

$$|0 \cdot I - A| = 0$$

$$\therefore |-A_{n\times n}| = 0 \quad \therefore (-1)^n |A| = 0$$

$$\therefore |A| = 0$$

$\therefore A$ Singular

h) <u>True</u>

$$T(\underline{0}_V) = T(0 \cdot \underline{0}_V)$$

$$= 0 \cdot T(\underline{0}_V)$$

$$= \underline{0}_W$$