Linear Algebra
Final Exam - Makeup
December 15, 2022

Name: ______________________ **UID:** ________________

- **The exam consists of SIX problems.**
- **Unsupported answers will receive little or no credit.**
- **Missing or blank pages will result in an automatic zero for the question.**
- **Time: 100 minutes.**

Problem	Score	Points
1		9
2		6
3		8
4		7
5		18
6		16
Total		64

Best wishes!

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Problem 1. (1.5 point each) Choose the **unique correct answer**.

(i) Find the rank of a 6×5 matrix A for which $A\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has a two-dimensional solution space.

A. 5.

B. 4.

C. 7.

D. 3.

(ii) Which of the following form a subspace of $\mathbb{R}^3$?

A. The set of all vectors of the form $(x, 1, z)$.

B. The set of all vectors of the form (x, y, z) with $x+y+z+1 = 0$.

C. The set of all vectors of the form (x, y, z) with $y = x + 2z$.

D. None of these.

(iii) Are the polynomials $p_1(x) = 2 + 3x - x^2$, $p_2(x) = 1 + x^2$, and $p_3(x) = 5 + 6x - x^2$ linearly independent in $\mathcal{P}_{\leq 2}$?

A. Yes.

B. No.

C. Data not complete.

D. None of these.

(iv) The polynomials $(x-1)$, $(x-1)^2$, $x(x-1)$.

A. span $\mathcal{P}_{\leq 2}$.

B. linearly dependent in $\mathcal{P}_{\leq 2}$.

C. linearly independent in $\mathcal{P}_{\leq 2}$.

D. None of these.

(v) Which of the following is **false**?

A. Every vector space is a subspace of itself.

B. The intersection of two subspaces is again a subspace.

C. The nullspace of any matrix A is always contained in its row space.

D. None of these.

(vi) Which of the following holds in an inner product space V?

A. If $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in V$ are parallel, then $\langle \underline{\mathbf{u}}\,, \underline{\mathbf{v}} \rangle = \|\underline{\mathbf{u}}\|\|\underline{\mathbf{v}}\|$.

B. $\langle \underline{\mathbf{u}}\,, \underline{\mathbf{v}} \rangle = 0$ iff $\underline{\mathbf{u}} = \underline{\mathbf{v}} = \underline{\mathbf{0}}$.

C. Both **A** and **B** hold.

D. Neither **A** nor **B** holds.

Problem 2. (6 points) Let U be the subspace of $M_{2\times 2}$ **spanned** by the four vectors

$$A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Show that $\{A, B, D\}$ is an orthogonal basis for U. Then, determine whether the vector $E = \begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$ belongs to U or not. **Argue your answer.**

Problem 3. Consider the matrix

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -5 & 3 & a \\ 4 & -2 & -1 \end{pmatrix}.$$

(i) (4 points) Find all value(s) of a such that A has exactly the three eigenvalues -3, 0 and 3.

(ii) (4 points) Select one of the value(s) of a you found above, and find a basis of the **eigenspace** of A relative to the eigenvalue $\lambda = -3$.

Problem 4. Suppose that $A \in M_{3\times 3}$ such that $\lambda = -2$, 0 and 3 are its eigenvalues, i.e. with characteristic polynomial
$\chi_A(\lambda) = (\lambda+2)(\lambda)(\lambda-3)$.

(i) (3 points) Show that $A^4 = 7A^2 + 6A$.

(ii) (2 points) If $\underline{\mathbf{v}}$ is an eigenvector for A relative to the eigenvalue $\lambda = 0$, then $\underline{\mathbf{v}}$ is an eigenvector for $3A^2 + 5A - I$ relative to the eigenvalue ....

(iii) (2 points) If $\underline{\mathbf{v}}$ is an eigenvector for A relative to the eigenvalue $\lambda = 3$, then $\underline{\mathbf{v}}$ is an eigenvector for $3A^2 + 5A - I$ relative to the eigenvalue ....

Problem 5. Consider the transformation

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$T(x_1, x_2, x_3, x_4) = (-x_3, x_1 - x_2 - x_4, x_3).$$

(i) (4 points) Show that T is a linear transformation.

(ii) (2 points) Construct the representation matrix of T.

(iii) (4 points) Describe the **kernel** of T, by providing a **basis** for it.

(iv) (2 points) Is T injective (1-1)? **Justify.**

(v) (4 points) Describe the **range** of T, by providing a **basis** for it.

(vi) (2 points) Is T surjective (onto)? **Justify.**

Problem 6. (4 points each) True or False (Circle one and state your reason):

(i) The map $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x, y, z) = (x + y, y + z, x(y + z))$$

is a linear transformation.

Reason: True False

(ii) If A is an invertible matrix and λ an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

Reason: True False

(iii) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V such that $\langle \mathbf{u}\,,\,\mathbf{v}\rangle = -4$, $||\mathbf{u}|| = ||\mathbf{v}|| = 2$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Reason: True False

(iv) Assume $T_1(v) = Av$ and $T_2(v) = Bv$ are linear transformations with representation matrices A and B respectively. If A and B are row equivalent, then $ker(T_1) = ker(T_2)$.

Reason: True False

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