The American University in Cairo
Mathematics and Actuarial Science
Linear Algebra
May 14, 2020

MACT 2132
Spring 2019
Final Exam
Time Limit: 120 Minutes

Name: ______________________________ **UID:** __________

- This exam contains 6 pages (including this cover page).
- Answer **all** the questions.
- Unsupported answers are considered miracles and will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.

Problem	Score	Points
1		20
2		20
3		14
4		18
5		20
Total		92

Problem 1. The two matrices A and B are row-equivalent.

$$A = \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 1 & -2 & -1 & 1 & 9 & 12 \\ -1 & 2 & 1 & 3 & -5 & -16 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) (3 pts) Find a basis for the RowSpace(A). Explain your answer.

b) (3 pts) Are the first three columns of A linearly independent? why?

c) (3 pts) Determine whether the columns of A spans $\mathbb{R}^4$ or not. Explain your answer.

d) (3 pts) Is the last column of A in the span of the 1st, 3rd, and 4th columns? why?

e) (4 pts) Describe the NullSpace(A).

f) (4 pts) Find Rank(A) and Nullity(A). Justify your answer.

Problem 2. Consider the following 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}$$

a) (10 pts) Find the eigenvalues of A and the corresponding eigenspaces.

b) (4 pts) Construct a basis for $\mathbb{R}^3$ consisting of eigenvectors of A. Justify your answer.

c) (6 pts) Use Gram-Schmidt to transform the basis in part b) into an orthonormal basis.

Problem 3. Let $\mathcal{C}[-1,2]$ be the space of all continuous functions on the interval $[-1,2]$. Suppose that a transformation $T : \mathcal{C}[-1,2] \to \mathbb{R}$ is given by

$$T(f) = \int_{-1}^{2} f(x)\, dx.$$

a) (4 pts) Show that T is a linear transformation.

b) (4 pts) Find the images of $f(x) = \cos(\pi x)$ and $g(x) = x^3$, respectively.

c) (6 pts) Is T one-to-one? onto? Justify your answer.

Problem 4. The set $\mathcal{M}_{2\times 2}$ of all 2×2 matrices with the operations of vectors addition and scaler multiplication defined below is a vector space.

$$\begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 + 8 \\ z_1 + z_2 - 3 & w_1 + w_2 \end{bmatrix},$$

$$\lambda \odot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \lambda\, x & \lambda\, y + 8\lambda - 8 \\ \lambda\, z - 3\lambda + 3 & \lambda\, w \end{bmatrix}.$$

a) (4 pts) What is the additive identity "$\underline{\mathbf{0}}$" for this vector space? Explain your answer.

b) (4 pts) For $\underline{\mathbf{v}} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, what is the additive inverse "$-\underline{\mathbf{v}}$"? Explain your answer.

c) (10 pts) Verify the axioms:

(i) "For any $\underline{\mathbf{u}}$, $\underline{\mathbf{v}} \in \mathcal{M}_{2\times 2}$, and $\lambda \in \mathbb{R}$, $\lambda \odot (\underline{\mathbf{u}} \oplus \underline{\mathbf{v}}) = (\lambda \odot \underline{\mathbf{u}}) \oplus (\lambda \odot \underline{\mathbf{v}})$".

(ii) "For any $\underline{\mathbf{u}} \in \mathcal{M}_{2\times 2}$, and λ, $\mu \in \mathbb{R}$, $(\lambda\,\mu) \odot \underline{\mathbf{u}} = \lambda \odot (\mu \odot \underline{\mathbf{u}})$".

Problem 5. (5 pts each) Prove or disprove <u>four</u> of the following.

a) If $\dim(V) = n < +\infty$, then any set of $n-1$ vectors in V must be linearly independent.

Reason: True False

b) If U and W are both subspaces of a vector space V, then their intersection $U \cap W$ is also a subspace.

Reason: True False

c) The set $W = \{(x, xy, y) \in \mathbb{R}^3 \, : \, x, y \in \mathbb{R}\}$ forms a subspace of $\mathbb{R}^3$.

Reason: True False

d) Let A be an $n \times n$ matrix such that $A^2 = O$. If B is similar to A, then $B^2 = O$.

e) Reason: True False

f) Let V, $\langle \, , \, \rangle$ be an inner product space. If $|| \, \underline{\mathbf{u}} \, || \leq 1$ and $|| \, \underline{\mathbf{v}} \, || \leq 1$, then $| \langle \, \underline{\mathbf{u}}, \, \underline{\mathbf{v}} \, \rangle | \leq 1$.

Reason: True False

g) The kernel of a linear transformation $T : V \to W$ is a subspace of V.

Reason: True False