Linear Algebra
Final Exam
December 15, 2021

Name: The Model Answer UID: ____________

- The exam consists of FIVE problems.
- Unsupported answers will receive little or no credit.
- Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.
- Missing or blank pages will result in an automatic zero for the question.
- Time: 100 minutes.

Problem	Score	Points
1		10
2		8
3		10
4		20
5		15
Total		63

Best wishes!

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Problem 1. For $\underline{\mathbf{u}} = (a, b, c)$, $\underline{\mathbf{v}} = (x, y, z) \in \mathbb{R}^3$, let

$$\langle \underline{\mathbf{u}}\,,\, \underline{\mathbf{v}} \rangle = 2ax + by + cz.$$

1. (6 points) Apply Gram-Schmidt process to transform the basis

$$B = \{(0, 0, 1),\, (1, -1, 1),\, (1, 1, 1)\}$$

into an **orthonormal** basis B' for $\mathbb{R}^3$ relative to the above inner product function.

We have $v_1 = (0,0,-1)$, $v_2 = (1,-1,1)$, $v_3 = (1,1,1)$.
The Gram-Schmidt orthonormalization process produces

$w_1 = v_1 = (0,0,1)$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

Since $\langle w_1, w_1 \rangle = \langle (0,0,1), (0,0,1) \rangle = 0 + 0 + 1 = 1$

and $\langle v_2, w_1 \rangle = \langle (1,-1,1), (0,0,1) \rangle = 0 + 0 + 1 = 1$.

Hence $w_2 = (1,-1,1) - (\frac{1}{1})(0,0,1) = (1,-1,0)$.

Finally, $$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

now,
$\langle v_3, w_1 \rangle = \langle (1,1,1), (0,0,1) \rangle = 0 + 0 + 1 = 1$

$\langle v_3, w_2 \rangle = \langle (1,1,1), (1,-1,0) \rangle = 2 - 1 + 0 = 1$

$\langle w_2, w_2 \rangle = \langle (1,-1,0), (1,-1,0) \rangle = 2 + 1 + 0 = 3$

hence $w_3 = (1,1,1) - (\frac{1}{1})(0,0,1) - (\frac{1}{3})(1,-1,0) = (\frac{2}{3}, \frac{4}{3}, 0)$.

Therefore, by normalizing the set $\{w_1, w_2, w_3\}$, we get

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{w_1}{\sqrt{\langle w_1, w_1\rangle}} = \frac{(0,0,1)}{\sqrt{1}} = (0,0,1).$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{w_2}{\sqrt{\langle w_2, w_2\rangle}} = \frac{(1,-1,0)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, 0\right)$$

Since $\|w_3\|^2 = \langle w_3, w_3\rangle = \left\langle \left(\frac{2}{3}, \frac{4}{3}, 0\right), \left(\frac{2}{3}, \frac{4}{3}, 0\right)\right\rangle = \frac{8}{9} + \frac{16}{9} = \frac{24}{9}$

$$= \frac{8}{3}$$

Hence, $\|w_3\| = \frac{2\sqrt{2}}{\sqrt{3}}$.

Thus $u_3 = \frac{w_3}{\|w_3\|} = \frac{(2/3, 4/3, 0)}{2\sqrt{2}/\sqrt{3}} = \frac{\sqrt{3}}{2\sqrt{2}}\left(\frac{2}{3}, \frac{4}{3}, 0\right)$

$$= \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)$$

Consequently, the orthonormal set B' is

$$\boxed{B' = \left\{ (0,0,1), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)\right\}}.$$

2. (4 points) Express the vector $\underline{\mathbf{v}} = (1, -2, 4)$ as a linear combination of the new **orthonormal** basis B'.

Sol $\underline{V} = c_1 \underline{u}_1 + c_2 \underline{u}_2 + c_3 \underline{u}_3$

our aim is to find c_1, c_2 and c_3.

since B' is orthonormal set, hence

$c_1 = \langle v, u_1 \rangle$, $c_2 = \langle v, u_2 \rangle$, $c_3 = \langle v, u_3 \rangle$,

now.

$c_1 = \langle (1,-2,4), (0,0,1) \rangle = 0+0+4 = 4$

$c_2 = \langle (1,-2,4), (\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, 0) \rangle = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} + 0 = \frac{4}{\sqrt{3}}$

$c_3 = \langle (1,-2,4), (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0) \rangle = \frac{2}{\sqrt{6}} - \frac{4}{\sqrt{6}} = \frac{-2}{\sqrt{6}}$

Consequently

$$\boxed{V = 4\,(0,0,1) + \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, 0\right) - \frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right)}.$$ #

Problem 2. Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & -2 \\ 2 & 2 & -4 \end{pmatrix}.$$

1. (4 points) Find the **eigenvalues** of A.

Consider the matrix $\lambda I - A = \begin{bmatrix} \lambda-3 & 0 & 0 \\ -2 & \lambda-1 & 2 \\ -2 & -2 & \lambda+4 \end{bmatrix}$.

To find the eigenvalues of A, we solve the characteristic eq.

$$\det(\lambda I - A) = 0.$$

That is $\begin{vmatrix} \lambda-3 & 0 & 0 \\ -2 & \lambda-1 & 2 \\ -2 & -2 & \lambda+4 \end{vmatrix} = 0$

(expand the determinant using the 1st row)

$$(\lambda-3)\begin{vmatrix} \lambda-1 & 2 \\ -2 & \lambda+4 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-3)\left[(\lambda-1)(\lambda+4)+4\right] = 0$$

$$\Rightarrow (\lambda-3)\left[\lambda^2+3\lambda\right] = 0$$

$$\Rightarrow (\lambda-3)(\lambda+3)\lambda = 0.$$

Hence $\boxed{\lambda = -3,\ 0,\ 3}$, which are the eigenvalues of the matrix A.

2. (4 points) Select one of the eigenvalues you found above, and find its corresponding **eigenspace**.

For $\lambda_1 = -3$.

To find the eigenspace corresponding to $\lambda_1 = -3$, we solve the homogeneous system represented by $(-3I - A)\underline{v} = \underline{0}$, where $\underline{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, that is

$$\left[\begin{array}{ccc|c} -6 & 0 & 0 & 0 \\ -2 & -4 & 2 & 0 \\ -2 & -2 & 1 & 0 \end{array}\right] \xrightarrow[-\frac{1}{6}R_1 \to R_1]{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & -4 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right],$$

Thus from the 1st row, we have $x_1 = 0$

from the 2nd (or 3rd) row, $2x_2 - x_3 = 0 \Rightarrow x_3 = 2x_2$.

Therefore, $\underline{v} = \begin{bmatrix} 0 \\ x_2 \\ 2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

Consequently, $\boxed{\text{the (eigenspace)}_{\lambda_1 = -3} = \left\{ t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \middle| t \in \mathbb{R} \right\}}$

For $\lambda_2 = 0$ Consider the homogeneous system represented by $(0I - A)\underline{v} = \underline{0}$, where $\underline{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, that is

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ -2 & -2 & 4 & 0 \end{array}\right] \xrightarrow[R_3 - R_2 \to R_3]{-\frac{1}{3}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

From the 1st row, $x_1 = 0$

From the 2nd (or 3rd row), $-x_2 + 2x_3 = 0 \Rightarrow x_2 = 2x_3$.

hence $\underline{V} = \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

Consequently, the $\boxed{(\text{eigenspace})_{\lambda_2=0} = \left\{ t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}}$.

For $\lambda_3 = 3$ Consider the homogeneous system represented by $(3I - A)\underline{V} = \underline{0}$,

where $\underline{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, that is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & 0 \\ -2 & -2 & 7 & 0 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & 0 \\ 0 & -4 & 5 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & -4 & 5 & 0 \end{array}\right]$$

clearly, $-4x_2 + 5x_3 = 0 \Rightarrow x_3 = \frac{4}{5}x_2$

and $x_1 - x_2 - x_3 = 0 \Rightarrow x_1 = x_2 + x_3$

$$= x_2 + \frac{4}{5}x_2 = \frac{9}{5}x_2.$$

Hence $\underline{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{9}{5}x_2 \\ x_2 \\ \frac{4}{5}x_2 \end{bmatrix} = \frac{1}{5}x_2 \begin{bmatrix} 9 \\ 5 \\ 4 \end{bmatrix}$,

Consequently, the eigenspace Corresponding to $\lambda_3 = 3$ is

$$\boxed{(\text{eigenspace})_{\lambda_3=3} = \left\{ t \begin{bmatrix} 9 \\ 5 \\ 4 \end{bmatrix} \mid t \in \mathbb{R} \right\}}. \quad \#$$

Problem 3. (1 point each) Complete **briefly** the following statements.

1. The dimension of a vector space V is the number of vectors in any basis of V.

2. A vector $\underline{\mathbf{v}} \in \text{Span}(\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3)$ means that $\underline{v}$ can be written as a linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3$, i.e, $\underline{v} = c_1\underline{v}_1 + c_2\underline{v}_2 + c_3\underline{v}_3$.

3. The span of $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3\}$ has dimension 3 if and only if v_1, v_2 and v_3 are linearly independent vectors.

4. An orthogonal set of non-zero vectors is always linearly independent.

5. In any vector space V, if $c \cdot \underline{\mathbf{v}} = \underline{0}$, then $c = 0$ or $\underline{v} = \underline{0}$.

6. For any linear transformation $T : V \to W$, $T(\underline{0}_V) = \underline{0}_W$

7. The kernel of a linear transformation $T : V \to W$ is the set of all vectors $v \in V$ that map to $\underline{0}_W$, i.e., $Ker(T) = \{\underline{v} \in V \mid T(\underline{v}) = \underline{0}_W\}$.

8. Two vector spaces V and W are isomorphic if there is a one-to-one and onto linear transformation from V to W.
i.e., there is an isomorphism from V to W.

9. An eigenvalue of a square matrix A is a real number λ that satisfies the equation $A\underline{v} = \lambda \underline{v}$, where $\underline{v}$ is a non-zero vector called eigenvector corresponding to λ. Or simply, the eigenvalue of A is a root of the characteristic equation $|\lambda I - A| = 0$.

10. If λ is an eigenvalue for A, then its corresponding eigenspace is the union of the set of all of eigenvectors corresponding to λ and the zero vector, i.e.,

$$(\text{eigenspace})_\lambda = \{\underline{v} \in \mathbb{R}^n \mid A\underline{v} = \lambda \underline{v}\}.$$

Problem 4. (5 points each) True or False (Circle one and state your reason):

1. Let $V = C[0,1]$ be the space of continuous functions on the interval $[0,1]$. Then, the dimension of V is infinite.

   Reason: True False

   Consider the set $\{1, x, x^2, \dots, x^n\}$ of linearly independent vectors of cardinality n for any $n \in \mathbb{Z}^+$

   clearly this set is a subset of $V = C[0,1]$.

   Thus, we can form a linearly independent set of vectors in $C[0,1]$ of any size.

   Hence $\dim V = \infty$.

2. There is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ that is one-to-one.

   Reason: True False

   Let us consider the linear transformation

   $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

   defined by $T(x,y) = (x, y, 0)$

   clearly $Ker(T) = \{(0,0)\}$

   Hence T is 1-1. #

3. Suppose that $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$ are unit vectors in an inner product space V such that $\langle \underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle = 1$. Then, Distance $(\underline{\mathbf{u}}, \underline{\mathbf{v}}) = 0$.

Reason: True False

we have $\underline{u}$ and $\underline{v}$ are unit-vectors

$$\Rightarrow \langle \underline{u}, \underline{u} \rangle = \langle \underline{v}, \underline{v} \rangle = 1.$$

Now, the distance (u,v) is

$$\left(D(u,v)\right)^2 = \langle \underline{u}-\underline{v}, \underline{u}-\underline{v} \rangle$$
$$= \langle \underline{u}, \underline{u} \rangle - \langle \underline{u}, \underline{v} \rangle - \langle \underline{v}, \underline{u} \rangle + \langle \underline{v}, \underline{v} \rangle$$
$$= 1 - 1 - 1 + 1 = 0.$$

4. If a vector $\underline{\mathbf{v}}$ is an eigenvector for a square matrix A relative to eigenvalues λ_1 and λ_2, then $\lambda_1 = \lambda_2$.

Reason: True False

Since $\underline{v}$ is an eigenvector of A relative to λ_1 and λ_2, then we have

$$A\underline{v} = \lambda_1 \underline{v} \quad \text{and} \quad A\underline{v} = \lambda_2 \underline{v},$$

hence, $\lambda_1 \underline{v} = \lambda_2 v \Rightarrow \lambda_1 \underline{v} - \lambda_2 \underline{v} = \underline{0}$

$$\Rightarrow (\lambda_1 - \lambda_2)\underline{v} = \underline{0}$$

since $\underline{v}$ is an eigenvector, so $\underline{v} \neq 0$

which gives $\lambda_1 - \lambda_2 = 0 \Rightarrow \underline{\lambda_1 = \lambda_2}$.

Problem 5. Let $\underline{b} = (1,1,0)$ in $\mathbb{R}^3$. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(\underline{v}) = (\underline{v} \cdot \underline{b})\ \underline{b},$$

for every $\underline{v} \in \mathbb{R}^3$.

1. (2 points) Find the images of the two vectors $(1,0,1)$ and $(1,1,1)$.

$$T((1,0,1)) = [(1,0,1)\cdot(1,1,0)]\ (1,1,0)$$
$$= [1+0+0](1,1,0) = (1,1,0)$$
$$T((1,1,1)) = [(1,1,1)\cdot(1,1,0)]\ (1,1,0)$$
$$= [1+1](1,1,0) = 2(1,1,0)$$
$$= (2,2,0)$$

2. (4 points) Show that T is a linear transformation.

Let $\underline{u}$ and $\underline{v}$ be two vectors in $\mathbb{R}^3$, then

$$T(\underline{u}+\underline{v}) = [(\underline{u}+\underline{v})\cdot \underline{b}]\ \underline{b}$$
$$= [(\underline{u}\cdot\underline{b}) + (\underline{v}\cdot b)]\ \underline{b}$$
$$= (\underline{u}\cdot\underline{b})\underline{b} + (\underline{v}\cdot\underline{b})\underline{b}$$
$$= T(\underline{u}) + T(\underline{v}).$$

and for any $\alpha \in \mathbb{R}$, $\underline{v} \in \mathbb{R}^3$, we have

$$T(\alpha\underline{v}) = (\alpha\underline{v}\cdot\underline{b})\underline{b}$$
$$= \alpha(\underline{v}\cdot\underline{b})\ \underline{b}$$
$$= \alpha T(\underline{v})$$

which shows that T is a linear transformation.

3. (4 points) Determine the **kernel** of T, and a **basis** for it.

Since, Kernel of T is $\{ \underline{v} \in \mathbb{R}^3 \mid T(v) = \underline{0} \}$

, thus sol $T(\underline{v}) = 0$, this gives

$$(\underline{v} \cdot \underline{b})\, \underline{b} = \underline{0}$$

Since $\underline{b}$ is non-zero vector, thus we must have

$$\underline{v} \cdot \underline{b} = 0$$

Let $\underline{v} = (x, y, z)$, then

$$0 = \underline{v} \cdot \underline{b} = (x, y, z) \cdot (1, 1, 0)$$
$$= x + y$$
$$\Rightarrow x = -y .$$

Hence $\underline{v} = (x, -x, z)$

Consequently $\ker(T) = \{ (x, -x, z) \mid x, z \in \mathbb{R} \}$

$$= \{ x(1,-1,0) + z(0,0,1) \; ; \; x, z \in \mathbb{R} \},$$

clearly the vectors $(1,-1,0)$ and $(0,0,1)$ are linearly independent.

Hence the basis of $\ker(T)$ is

$$B_{\ker(T)} = \{ (1,-1,0), (0,0,1) \}.$$

4. (3 points) Determine the **range** of T, and a basis for it.

We have $T(\underline{v}) = (\underline{v} \cdot \underline{b})\, \underline{b}$

Since $(\underline{v} \cdot \underline{b})$ is a scaler, thus

$$\text{Range}(T) = \text{span}\{\underline{b}\}.$$

So, a basis of Range(T) is $\{\underline{b}\}$.

5. (2 points) Is T one-to-one? Onto?

Since $\text{Ker}(T) \neq \{\underline{0}\}$, hence T is not one-to-one.

Moreover, $\dim \text{Range}(T) = 1$

and $\dim(\text{Codomain})$ of $T = 3$

Thus T is not onto.

__Note__: T is onto if $\dim \text{Range}(T) = \dim(\text{Codomain})$ of T.

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