Linear Algebra
Final Exam
July 24, 2022

Name: ______________________ **UID:** ____________________

- **The exam consists of FOUR problems.**
- **Unsupported answers will receive little or no credit.**
- **Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.**
- **Missing or blank pages will result in an automatic zero for the question.**
- **Time: 100 minutes.**

Problem	Score	Points
1		12
2		16
3		20
4		16
Total		64

Best wishes!

Dr. Eslam Badr

Problem 1, Part 1. (1 point each) Complete **briefly** the following statements.

(i) A set of vectors $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ ...,\mathbf{v}_k\}$ is an **orthogonal basis** for an inner product space V if

....

(ii) The **span** of $\{\underline{\mathbf{v}}_1,\ \underline{\mathbf{v}}_2,\ \underline{\mathbf{v}}_3\}$ has dimension **two** if

....

(iii) The **rank** of an $m \times n$ matrix A can be $0, 1, ..., k$ where $k =$

....

(iv) Any linear transformation $T : V \to W$ transforms a **subspace** of V into a

....

(v) The **dimension of the eigenspace** of a square matrix A relative to an eigenvalue λ of multiplicity m is

....

Problem 1, Part 2. (7 points) Let U be the subspace **spanned** by the four vectors:

$$\mathbf{v}_1 = (1, 2, 1),\ \mathbf{v}_2 = (1, 1, 3),\ \mathbf{v}_3 = (1, 0, 5),\ \mathbf{v}_4 = (1, 1, 0).$$

Find an **orthonormal basis** for U.

Problem 2, Part 1. Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ a & 1 & -2 \\ a & a & -4 \end{pmatrix}.$$

(i) (4 points) Find all value(s) of a which will guarantee that A has eigenvalues 0, 3, and -3 simultaneously.

(ii) (6 points) Select one of the value(s) of a you found above, and find the **eigenspaces** of A relative to the eigenvalues $\lambda = 0$ and $\lambda = 3$ respectively.

Problem 2, Part 2. (6 points) Suppose that A is a 3×3 matrix such that $\lambda = 1,\ 2,\ -2$ are its eigenvalues.

By the aid of **Cayley-Hamilton Theorem**, find A^5 and A^{-1} in the form $\alpha\, A^2 + \beta\, A + \gamma\, I$ for some constants α, β, γ.

Problem 3. Let $\mathcal{P}_{\leq 3}$ be the space of polynomials of degree ≤ 3 in the variable t. Consider the transformation $T : \mathcal{P}_{\leq 3} \longrightarrow \mathbb{R}^2$ given by

$$T\left(a_0 + a_1\,t + a_2\,t^2 + a_3\,t^3\right) = (a_0 + a_2\,,\, a_1 - a_2)\,.$$

(i) (2 points) Find the images of the two vectors $\mathbf{u} = 2 - x - x^2$ and $\mathbf{v} = x - x^3$.

(ii) (4 points) Show that T is a linear transformation.

(iii) (4 points) Describe the **kernel** of T, and a **basis** for it.

(iv) (4 points) Describe the **range** of T, and a **basis** for it.

(v) (3 points) Determine the **rank** and the **nullity** of T. **Justify**.

(vi) (3 points) Is T one-to-one? onto? **Justify**.

Problem 4. (4 points each) True or False (Circle one and state your reason):

(i) The set $\mathcal{P}_{=3}$ of polynomials $p(x)$ of degree exactly 3 is a vector space relative to the standard addition and scalar multiplication.

Reason: True False

(ii) There is a linear transformation $T : \mathbb{R}^2 \to \mathcal{P}_{\leq 2}$ that is onto.

Reason: True False

(iii) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are **unit** vectors in an inner product space V such that $\langle \mathbf{u}\,,\ \mathbf{v}\rangle = 1$. Then $\mathbf{u}$ and $\mathbf{v}$ are **not** linearly independent.

Reason: True False

(iv) If A and B are square matrices such that $PBP^{-1} = A$ for some invertible matrix P, then A and B have the same charactertistic equation.

Reason: True False

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