Problem 1. (3 pts each) Consider the following system of equations

$$x + 4y - 2z = 1$$

$$x + 7y - 6z = 6$$

$$3y + qz = t$$

a) Which number q makes the corresponding coefficient matrix noninvertible?

$$\begin{vmatrix} 1 & 4 & -2 \\ -1 & 7 & -6 \\ 0 & 3 & q \end{vmatrix} = (7q+18)-(4q+6) = 0$$

$$3q + 12 = 0$$

$$q = -4$$

b) For which value t will the system have infinitely many solutions?

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 1 & 7 & -6 & 6 \\ 0 & 3 & q & t \end{array}\right) \xrightarrow{R2-R1} \left(\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 3 & q & t \end{array}\right) \xrightarrow{R3-R2} \left(\begin{array}{ccc|c} 1 & 4 & -2 & 1 \\ 0 & 3 & -4 & 5 \\ 0 & 0 & q+4 & t-5 \end{array}\right)$$

$$q = -4 \qquad t = 5$$

c) Find the solution that has $z = 1$.

$$3y + q = t \quad \Rightarrow y = \tfrac{1}{3}(t-q)$$

$$x + \tfrac{7}{3}(t-q) - 6 = 6 \Rightarrow x = 12 - \tfrac{7}{3}(t-q)$$

$$12 - \tfrac{7}{3}(t-q) + \tfrac{4}{3}(t-q) - 2 = 1$$

$$10 - t + q = 1$$

$$t - q = 9 \qquad y = 9/3 = 3$$

$$x = 12 - \tfrac{7}{3}(9) = -9$$

Problem 2. (5 pts each)

a) Solve for A.

$$(A^{-1} - 2I)^T = -2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -2 & 0 \end{pmatrix}$$

$$A^{-1} - 2I = \begin{pmatrix} -2 & -2 \\ -2 & 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -2 & -2 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix}$$

$$A = \frac{-1}{4}\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 \\ -1/2 & 0 \end{pmatrix}$$

b) Prove that if A, B, and C are $n \times n$ matrices and $ABC = I$, then B is invertible and $B^{-1} = CA$

$$ABCA = A$$

$$\therefore\ BCA = I$$

$$\therefore\ B^{-1} = CA$$

$$CABC = C$$

$$(CAB)C = C$$

$$\therefore\ CAB = I$$

$$\therefore\ CA = B^{-1}$$

Problem 3. (4 pts each)

a) If the Gaussian elimination leads to $x + y = 1$ and $2y = 3$. Find two possible original problems whose solution set is equivalent to the latter.

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 3 \end{array}\right) \xrightarrow{R2+R1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 4 \end{array}\right) \qquad \begin{aligned} x+y&=1 \\ x+3y&=4 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 3 \end{array}\right) \xrightarrow[2R2]{2R1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 4 & 6 \end{array}\right) \qquad \begin{aligned} 2x+2y&=2 \\ 4y&=6 \end{aligned}$$

b) For which three numbers a will elimination fail to give three pivots?

$$A = \begin{pmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{pmatrix}.$$

$$\begin{pmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{pmatrix} \xrightarrow[R3-R1]{R2-R1} \begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{pmatrix} \xrightarrow{R3-R2} \begin{pmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & 0 & a-4 \end{pmatrix}$$

$a = 0, 2, 4$

Problem 4. Given $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ with $det(A) = -3$. Find

a) (2 pts) $det(4A^{-1}(A^T)^2)$,

$$|4A^{-1}(A^T)^2| = 4^3|A^{-1}||A^T|^2 = 4^3 \frac{1}{|A|}|A|^2 = 4^3|A|$$

$$= 4^3 \cdot (-3) = -192$$

b) (2 pts) $det((-A^4)^{-1} \times det(A))$,

$$|-3\,(-A^4)^{-1}| = (-3)^3\,|(-A^4)^{-1}| = (-3)^3 \frac{1}{|(-A)^4|}$$

$$= (-3)^3 \frac{1}{|-A|^4} = \frac{(-3)^3}{((-1)^3|A|)^4} = \frac{(-3)^3}{((-1)^3 \cdot (-3))^4} = -1/3$$

c) (3 pts) $det \begin{pmatrix} 5d & -a & 4g-7a \\ 5e & -b & 4h-7b \\ 5f & -c & 4i-7c \end{pmatrix}$.

$$\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \xrightarrow{5R1} \begin{pmatrix} 5d & 5e & 5f \\ a & b & c \\ g & h & i \end{pmatrix} \xrightarrow{-R2} \begin{pmatrix} 5d & 5e & 5f \\ -a & -b & -c \\ g & h & i \end{pmatrix}$$

det $\quad -(-3) = 3 \qquad 5(3) = 15 \qquad -1(15) = -15$

$$\xrightarrow{4R3} \begin{pmatrix} 5d & 5e & 5f \\ -a & -b & -c \\ 4g & 4h & 4i \end{pmatrix} \xrightarrow{R3+7R2} \begin{pmatrix} 5d & 5e & 5f \\ -a & -b & -c \\ 4g-7a & 4h-7b & 4i-7c \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 5d & -a & 4g-7a \\ 5e & -b & 4h-7b \\ 5f & -c & 4i-7c \end{pmatrix}$$

det $\quad 4(-15) = -60 \qquad -60 \qquad -60$

$\therefore$ det (___) $= -60$

Problem 5.

a) (4 pts) Represent $A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}$ as a product of elementary matrices.

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow[E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}]{R3-3R1} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & 2 \end{pmatrix} \xrightarrow[R1-3R2]{R3+5R2} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 12 \end{pmatrix} \xrightarrow{R3/12} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/12 \end{pmatrix}$$

$$\xrightarrow[R2-2R3]{R1+5R3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_5 = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 12 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

b) (2 pts)Is A an invertible matrix? Explain your answer.

Yes, because A can be expressed as the product of elementary matrices.

c) (2 pts) Is the LU factorization of A unique? Explain your answer.

Yes, because we do not need to swap any rows. (Given that we require the diagonal of L to be all 1's).