Problem 1, Part 1. (4 points) Given that the matrix below is the **augmented matrix** of a system of linear equations.

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 1 & k & 1 \end{bmatrix}$$

(i) Determine the number of equations and the number of variables, and **(ii)** find the value(s) of k such that the system is **inconsistent**.

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 1 & k & 1 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & k+1 & -1 \end{array}\right]$$

FROM R_2: $0 = 1$

a contradiction $\to$ ⨳

$$\boxed{\therefore \text{ The system is ALWAYS inconsistent } (k \in \mathbb{R})}$$

Problem 1, Part 2. (4 points) Given that the matrix below is the **coefficient matrix** of a **homogeneous** system of linear equations.

$$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -1 \\ 1 & k & 1 \end{bmatrix}$$

$\boxed{\text{3 eq., 3 variables}}$

(i) Determine the number of equations and the number of variables, and **(ii)** find the value(s) of k such that the system has **infinitely many solutions.**

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & k & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 + R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & k+1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & k+1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The system is always consistent being homogeneous

$\Rightarrow$ If $k = -1$, we have < 3 pivots

$\boxed{\Rightarrow \therefore \text{'}\infty\text{' many solutions if } k = -1}$

Problem 2, Part 1 (4 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & -2 & 2x \\ 20 & 5 & 5x^2 \end{bmatrix}.$$

Determine the values of x for which A is **invertible**.

Using R_1: Find $|A|$ and set $=0$ to find x s.t. A is singular

$$|A| = \begin{vmatrix} -2 & 2x \\ 5 & 5x^2 \end{vmatrix} - \begin{vmatrix} 4 & 2x \\ 20 & 5x^2 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ 20 & 5 \end{vmatrix}$$

$$= -10x^2 - 10x - 20x^2 + 40x + 20 + 40$$

$$= (-30x^2 + 30x + 60 = 0) \div -30$$

$$= x^2 - x - 2 = 0$$

$$= (x-2)(x+1) = 0$$

$$\rightarrow x = 2, x = -1$$

$$\boxed{\therefore A^{-1} \text{ exists if } x \neq -1, 2}$$

$\frac{1}{6} - \frac{1}{3}\left(-\frac{1}{6}\right) = \frac{1}{6} + \frac{1}{18}$

$\frac{3}{18} + \frac{1}{18} = \frac{4}{18}$

Problem 2, Part 2 (4 points) Select one of the values of x you found in Part (1) and obtain the **inverse** of A.

Let $x = 0$ $\quad [A : I] \longrightarrow [I : A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & -2 & 0 & 0 & 1 & 0 \\ 20 & 5 & 0^2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 20R_1 \to R_3]{R_2 - 4R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -6 & -4 & -4 & 1 & 0 \\ 0 & -15 & -20 & -20 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \cdot -\frac{1}{6} \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{6} & 0 \\ 0 & -15 & -20 & -20 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + 15R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1/6 & 0 \\ 0 & 0 & -10 & -10 & -\frac{15}{6} & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \cdot -\frac{1}{10} \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2/3 & 2/3 & -1/6 & 0 \\ 0 & 0 & 1 & 1 & 1/4 & -1/10 \end{array}\right]$$

$$\xrightarrow[R_1 - R_3 \to R_1]{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -1/12 & 1/10 \\ 0 & 1 & 2/3 & 2/3 & -1/6 & 0 \\ 0 & 0 & 1 & 1 & 1/4 & -1/10 \end{array}\right]$$

$$\xrightarrow{R_2 - \frac{2}{3}R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -1/12 & 1/10 \\ 0 & 1 & 0 & 0 & -1/3 & 1/15 \\ 0 & 0 & 0 & 1 & 1/4 & -1/10 \end{array}\right]$$

I

$$\boxed{\therefore A^{-1} = \begin{bmatrix} -2/3 & -1/12 & 1/10 \\ 0 & -1/3 & 1/15 \\ 1 & 1/4 & -1/10 \end{bmatrix}}$$

Problem 3. (8 points) Let A be a square matrix that satisfies

$$A^2 + 2A - 3I = O,$$

where O denotes the zero matrix and I denotes the identity matrix.

(i) Show that the matrix A must be invertible.

Sol 1:

$A^2 + 2A = 3I$

$\rightarrow \frac{1}{3}(A^2 + 2A) = I$

$\rightarrow \frac{1}{3}A(A + 2I) = I$

$\rightarrow A(\frac{1}{3}A + \frac{2}{3}I) = I$

$\rightarrow \boxed{\frac{1}{3}A + \frac{2}{3}I = A^{-1}}$

$A_{n \times n}$

Sol 2:

$A^2 + 2A = 3I$

$|A^2 + 2A| = 3$

$|A(A + 2I)| = 3$

$\rightarrow |A||A + 2I| = 3$

$\rightarrow |A| \neq 0$

$\boxed{\therefore A^{-1} \text{ exists}}$

(ii) Show that the matrix A satisfies $A^4 = 21I - 20A$.

$A^2 = 3I - 2A$

$(A^2)^2 = (3I - 2A)^2$

$= (3I - 2A)(3I - 2A)$

$= 9I - 6A - 6A + 4A^2$

$= 9I - 12A + 4(3I - 2A)$

$= 9I - 12A + 12I - 8A$

$\boxed{\therefore A^4 = 21A - 20A}$

Problem 4. (4 points each) True or False (circle one and state your reason):

(i) If A is an invertible matrix such that $A^{-1} = A$, then $|A| = 1$.

Reason: True $\boxed{\text{False}}$

$$A^{-1} = A$$
$$\rightarrow |A^{-1}| = |A|$$
$$\rightarrow \frac{1}{|A|} = |A| \rightarrow |A||A| = 1$$
$$\rightarrow (|A|)^2 = 1$$
$$\rightarrow \boxed{|A| = \pm 1}$$

ex/ consider $\underline{A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A^{-1}}$; $\underline{|A| = |A^{-1}| = -1}$

(ii) Let A and B be square matrices of the same size such that

$$|B| = \frac{1}{|A|}.$$

Then,, $B = A^{-1}$.

Reason: True $\boxed{\text{False}}$

Let $B = I$, $|B| = 1$

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

$|A| = 4 - 3 = 1$

$|B| = \frac{1}{|A|}$ but $B \neq A^{-1}$

(iii) For any invertible matrix A, we have

$$\operatorname{adj}\left(A^{-1}\right) = (\operatorname{adj}(A))^{-1}.$$

$\boxed{\text{True}}$ False

Reason:

$B \operatorname{adj}(B) = |B| I$

① $B = A^{-1}$;
$\rightarrow A^{-1} \operatorname{adj}(A^{-1}) = |A^{-1}| I$
$\rightarrow \boxed{\operatorname{adj}(A^{-1}) = |A^{-1}| A = \frac{1}{|A|} A \quad (*)}$

② $B = A$;
$\rightarrow A \operatorname{adj}(A) = |A| I$
$\rightarrow \left(\frac{1}{|A|} A\right) \operatorname{adj}(A) = I$
$\rightarrow \boxed{(\operatorname{adj}(A))^{-1} = \frac{1}{|A|} A \quad (**)}$

$\Rightarrow (*), (**)$,
$\boxed{\rightarrow \operatorname{adj}(A^{-1}) = (\operatorname{adj}(A))^{-1}}$

(iv) For a matrix A to have more than one LU-Factorization, it is necessary that A is singular.

$\boxed{\text{True}}$ False

Reason:

Assume A has more than one LU factorization and that A^{-1} exists
$\rightarrow \therefore A$ has a unique LU factorization since it is invertible and an LU factorization exists
but A has more than one LU factorization
$\rightarrow$ contradiction, so A cannot be invertible $\rightarrow \therefore A$ singular

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Bonus. (2 points) Discuss the following statement:

" For any square matrix A, there exist a **symmetric** matrix B and a **skew-symmetric** matrix C such that $A = B + C$ ".

we will show that the property is valid is true for any A

Let $B = \frac{1}{2}(A + A^T)$

$C = \frac{1}{2}(A - A^T)$

$\rightarrow B^T = [\frac{1}{2}(A + A^T)]^T = \frac{1}{2}(A^T + A) = \frac{1}{2}(A + A^T) = B$

$\therefore$ B symmetric

$\rightarrow C^T = [\frac{1}{2}(A - A^T)]^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C$

$\therefore$ C skew-symmetric

$$B + C = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$
$$= \frac{1}{2}A + \cancel{\frac{1}{2}A^T} + \frac{1}{2}A - \cancel{\frac{1}{2}A^T}$$
$$= A$$

$$\boxed{\therefore A = B + C}$$