Exam 2
November 21, 2020

Name: ______________________________ **UID:** ______________________

- **The exam consists of FIVE problems.**
- **Unsupported answers will receive little or no credit.**
- **Anyone caught writing after time has expired will be given a mark of zero.**
- **Time: 90 minutes.**

Problem	Score	Points
1		10
2		11
3		8
4		8
5		16
Total		53

Problem 1. Let W be the subspace of $\mathbb{R}^4$ spanned by

$$S = \{(4,3,0,0),(0,5,0,4),(3,1,0,-1),(0,0,6,0)\}$$

(i) (5 points) Find an orthogonal basis for W.

(ii) (2 points) Determine, with justification, whether $W = \mathbb{R}^4$ or not.

(iii) (3 points) Express $(8,1,3,-4)$ as a linear combination of a basis of W.

Problem 2. Consider the row equivalent matrices A and B given by

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 12 \\ 2 & 2 & 5 & 8 & 127 \\ 3 & 3 & 6 & 10 & 38 \\ 1 & 1 & 1 & 2 & 11 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) (4 points) Find a basis for $\mathrm{Nul}(A)$, the null space of A.

(ii) (2 points) Find a basis for $\mathrm{Row}(A)$, the row space of A.

(iii) (2 points) Find a basis for $\mathrm{Col}(A)$, the column space of A.

(iv) (3 points) Is $\mathrm{Col}(A) = \mathrm{Col}(B)$? Justify.

Problem 3. Let $T : \mathcal{P}_2 \to \mathbb{R}$ be the transformation

$$T\left(a_0 + a_1\,x + a_2\,x^2\right) \;=\; 3\,a_0 - a_1 + a_2.$$

(i) (3 points) Show that T is a linear transformation.

(ii) (3 points) Find a basis for $\mathrm{Ker}(T)$, the kernel of T.

(iii) (2 points) Is T onto? Justify.

Problem 4. Let $\mathcal{P}_2$ be the set of all polynomials of degree ≤ 2. The set $\mathcal{P}_2$ with the operations of vectors addition and scaler multiplication defined below is a vector space.

For $\underline{\mathbf{u}} = a_0 + a_1 x + a_2 x^2$ and $\underline{\mathbf{v}} = b_0 + b_1 x + b_2 x^2$ in $\mathcal{P}_2$:

$$\underline{\mathbf{u}} \oplus \underline{\mathbf{v}} = (a_0 + b_0) + (a_1 + b_1 - 3)\,x + (a_2 + b_2 - 7)\,x^2,$$

$$\lambda \odot \underline{\mathbf{u}} = (\lambda\, a_0) + (\lambda\, a_1 - 3\lambda + 3)\,x + (\lambda\, a_2 - 7\lambda + 7)\,x^2.$$

(i) (2 points) What is the additive identity $\underline{\mathbf{0}}$ for this vector space? Explain your answer.

(ii) (2 pts) What is the additive inverse $-\underline{\mathbf{u}}$? Explain your answer.

(iii) (4 pts) Verify the axiom: For any $\underline{\mathbf{u}}$, $\underline{\mathbf{v}} \in \mathcal{P}_2$ and $\lambda \in \mathbb{R}$

$$\lambda \odot (\underline{\mathbf{u}} \oplus \underline{\mathbf{v}}) = (\lambda \odot \underline{\mathbf{u}}) \oplus (\lambda \odot \underline{\mathbf{v}}).$$

Problem 5. (4 points each) Determine whether the following statements are true or false. Proved justification if the statement is true and a counter-example if it is false.

(i) The set $W = \left\{(0, a^2, a^3) : a \in \mathbb{R}\right\}$ is a subspace of $\mathbb{R}^3$.

Reason: True False

(ii) A linear transformation $T : M_{2,4} \longrightarrow M_{3,3}$ is never onto.

Reason: True False

(iii) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation. If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent, then so is $\{T(\mathbf{v_1}), T(\mathbf{v_2}), T(\mathbf{v_3})\}$.

Reason: True False

(iv) It is possible to write a linearly independent set containing three polynomials all of degree 2 in $\mathcal{P}_3$. (Provide an example if the statement is true).

Reason: True False

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