MACT 2123

Fall 2020

Exam 1
October 3, 2020

Name: KEY

UID: FALL 2020

Problem 1. (5 points each) Consider the system of equations given by

$$\begin{aligned} x \quad\quad\;\; + 2z &= 4 \\ y + z &= -1 \\ 2x - 3y + z &= 11 \end{aligned}$$

1) Express the system in *augmented* matrix form, and perform row operations on it to get it in reduced row echelon form (RREF).

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & -1 \\ 2 & -3 & 1 & 11 \end{array}\right] \xrightarrow{R_3 - 2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & -3 & -3 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2 \to R_3]{-1/3 R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

in REF ✓ every pivot only non-zero entry in its column.

$$\therefore \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2) Find **all** solutions to the system of equations, and describe the geometric nature of the solution set in $\mathbb{R}^3$ (i.e: the solution set forms a point, line or plane?) Justify your answer.

① $x + 2z = 4$

② $y + z = -1 \Rightarrow z = -1 - y$

let $y = t \in \mathbb{R}$

in ①: $x + 2(-1-t) = 4 \rightarrow x - 2 - 2t = 4$

$\Rightarrow x = 6 + 2t$

$y = t$

$z = -1 - t$

The solution set is a line given by the parametric equations above.

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Problem 2. (6 points + 4 points + 4 points) Consider the matrix

$$A = \begin{bmatrix} a & b & c \\ 3 & 3 & 5 \\ 1 & 0 & 0 \end{bmatrix}.$$

1) Compute all the cofactors of A that are independent of the values of a, b, c.

$C_{11} = 0$, $C_{12} = (-1)^3(-5) = 5$, $C_{13} = (-1)^4(-3) = -3$

$C_{21} = 0$

2) Find four different triples (a, b, c) for which $|A| = 0$. Justify your answer.

row ① or Row ③ $\det(A) = 5b - 3c = 0 \rightarrow 5b = 3c \rightarrow \boxed{b = \frac{3}{5}c,}$ indep. of a

$c = 0 \rightarrow b = 0$

$c = 1 \rightarrow b = 3/5$

$c = 2 \rightarrow b = 6/5$

$c = 3 \rightarrow b = 9/5$

$\Rightarrow$ ① $(a, 0, 0)$ ② $(a, 3/5, 1)$ ③ $(a, 6/5, 2)$ ④ $(a, 9/5, 3)$ for ANY value of a

3) If A is invertible, use Cramer's rule to give a formula for x_2 in terms of a, b, c if

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 4 \\ 7 \end{bmatrix}$$

$$x_2 = \frac{\begin{vmatrix} a & a & c \\ 3 & 4 & 5 \\ 1 & 7 & 0 \end{vmatrix}}{|A|} \overset{\text{Row ③}}{=} \frac{(5a - 4c) - 7(5a - 3c)}{5b - 3c}$$

provided that $5b - 3c \neq 0$ (i.e., A^{-1} exists)

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Problem 3. (4 points + 8 points)

1) Let A and B be two 3×3 matrices such that $\det(A) = -3$ and $\det(B) = 4$. Evaluate $\det(A^T B^{-1} A^{-2} B^2)$.

$= \det(A) \det(B^{-1}) \det((A^{-1})^2) \det(B) \det(B)$

$= \det(A) \dfrac{1}{\det(B)} \dfrac{1}{[\det(A)]^2} [\det(B)]^2$

$= -3 (1/4)(1/(-3)^2)(4)^2 = (-3/4)(1/9)(16)$

$= -16/12$

$\therefore \det(A^T B^{-1} A^{-2} B^2) = \dfrac{-16}{12}$

2) Find the inverse of the following matrix or show that it is not invertible.

$$A = \begin{bmatrix} 1 & -6 & 0 \\ 0 & -3 & 3 \\ 2 & 5 & -1 \end{bmatrix}$$

$\det(A) \overset{\text{Row } ①}{=} 1(-12) + 6(-6)$

$\Rightarrow \det(A) = -48 \neq 0$

$\therefore A^{-1}$ exists

$c_{11} = -12$, $c_{12} = -(-6) = 6$, $c_{13} = 6$, $c_{21} = -6$

$c_{22} = -1$, $c_{23} = -17$, $c_{31} = -18$

$c_{32} = -3$ $c_{33} = -3$

$A^{-1} = \dfrac{1}{\det(A)} [\operatorname{cof}(A)]^T$

$= \dfrac{1}{\det(A)} [\operatorname{adj}(A)]$

$$\operatorname{cof}(A) = \begin{bmatrix} -12 & 6 & 6 \\ -6 & -1 & -17 \\ -18 & -3 & -3 \end{bmatrix}$$

$$\operatorname{adj}(A) = [\operatorname{cof}(A)]^T = \begin{bmatrix} -12 & -6 & -18 \\ 6 & -1 & -3 \\ 6 & -17 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{-1}{48} \begin{bmatrix} -12 & -6 & -18 \\ 6 & -1 & -3 \\ 6 & -17 & -3 \end{bmatrix}$$

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Problem 4. (6 points each)

1) Prove that if $A^3 - \frac{3}{2}A^2 + \frac{3}{4}A = O$, (where O denotes the zero matrix) then $(I - 2A)$ is invertible and $(I - 2A)^{-1} = (I - 2A)^2$.

$\Rightarrow \times 8$: $8A^3 - 12A^2 - 6A = O$

$\rightarrow -I$: $8A^3 - 12A^2 + 6A - I = -I$

$\circledast \rightarrow$ $-(I-2A)(I-2A)^2 = -I$

$\rightarrow$ $(I-2A)(I-2A)^2 = I$

$\det(I-2A)\det((I-2A)^2) = \det(I) \neq 0$

$\therefore$ $I-2A$ invertible

$(I-2A)(I-2A)^2 = I$

$\Rightarrow (I-2A)^{-1} = (I-2A)^2$

$\circledast$ $(I-2A)(I-2A)(I-2A)$

$= (I - 4A + 4A^2)(I-2A)$

$= I - 6A + 12A^2 - 8A^3$

$\circledast$

2) List all possible 3×3 matrices in RREF with exactly one zero row.

① $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$

③ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

② $\begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

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Problem 5. (5 points each) True or False (Circle one and state your reason):

1) There is a 2×2 invertible matrix A such that:

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} A = A \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}.$$

True False

Reason:

$\det\left(\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} A\right) = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} |A| = |A| \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix}$

$\Rightarrow 0 = 5|A|$

iff $|A| = 0$

iff A singular.

$\therefore$ No invertible 2×2 matrix will satisfy the eq.

2) If A is invertible and $AB = BA$, then B must be invertible as well.

True False

Reason: let $A = I$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$AB = B = BA$ but B is not invertible.

3) If A is an invertible matrix then $\text{adj}(A^2) = |A|^{n-1}\, \text{adj}(A)^2$.

True False

Reason: If A^{-1} exists

$\text{adj}(A^2) = \text{adj}(AA)$, we claim $\text{adj}(AA) = \text{adj}(A)\text{adj}(A)$

$\text{adj}(A)\text{adj}(A) = |A| A^{-1} |A| A^{-1}$

$= |A||A| A^{-1} A^{-1}$

$= |AA| (AA)^{-1}$

$= \text{adj}(AA)$

$\therefore \text{adj}(A^2) = \text{adj}(A) \cdot \text{adj}(A) = [\text{adj}(A)]^2$