Problem 1

1st system

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$
$x_1 \quad x_2 \quad x_3$

2nd system

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$x_1 \quad x_2 \quad x_3$

3rd system

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & -8 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$x_1 \quad x_2 \quad x_3 \quad x_4$

(a) 1st system: $A\underline{x} = \underline{b}$ $\quad x_1 = 3,\ x_2 = 0,\ x_3 = -1$

$\therefore\ \underline{x} = (3, 0, -1)$ unique soln.

2nd system: $B\underline{y} = \underline{c}$ $\quad 0 = 0$ ✓

$x_3 = -1$

$x_1 + 3x_2 = 5$

$\therefore\ x_1 = 5 - 3x_2,\ x_2$ free $\left(\underline{y} = (5 - 3x_2, x_2, -1) \right)$

$\longrightarrow$ infinitely many solns.

3rd system: $C\underline{z} = \underline{d}$ $\quad 0 = 0$ ✓

$0 = 1$!! $\quad \therefore$ inconsistent

(b) A is the only one row-equivalent to the identity matrix $\therefore$ invertible.

(c) Non-zero rows in REF form a basis for Rowspace

$\therefore$ Basis for $RS(A) = \{(1,0,0), (0,1,0), (0,0,1)\}$

Basis for $RS(B) = \{(1,3,0), (0,0,1)\}$

Basis for $RS(C) = \{(0,1,0,-8), (0,0,1,0)\}$

(d) Rank = dim(Rowspace)

$\therefore$ $\text{Rank}(A) = \underline{\underline{3}}$ $\quad \text{Rank}(B) = \underline{\underline{2}}$ $\quad \text{Rank}(C) = \underline{\underline{2}}$

e) Only A, because each column in A is a Pivot-column

f) Pivot-columns in the original matrix form a basis for the columnspace

$\therefore$ Basis for $CS(A) = \{ col_{1,A}, col_{2,A}, col_{3,A} \}$

Basis for $CS(B) = \{ col_{1,B}, col_{3,B} \}$

Basis for $CS(C) = \{ col_{2,C}, col_{3,C} \}$

Remark: ERO's don't change the Rowspace, but the columnspace may be different.

· Problem 2

(a) First, we find the eigenvalues of A :

the characteristic eqn.

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 2 & 0 & 0 \\ 2 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda-1 & 2 \\ 0 & 0 & 2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\begin{vmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 2 \\ 0 & 2 & \lambda-1 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 & 0 \\ 0 & \lambda-1 & 2 \\ 0 & 2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda-1)\left((\lambda-1)^2-4\right)\right] - 2\left[2\left((\lambda-1)^2-4\right)\right]$$

$$= \left[(\lambda-1)^2-4\right]\left[(\lambda-1)^2-4\right] = \left[(\lambda-1)^2-4\right]^2$$

$$= \left[\left((\lambda-1)-2\right)\left((\lambda-1)+2\right)\right]^2 = (\lambda-3)^2(\lambda+1)^2$$

$\therefore$ $\lambda = 3, 3, -1, -1$ are the eigenvalues for A

Second, we find the corresponding eigenvectors :

- For $\lambda = 3$: $(\lambda I - A)\underline{x} = \underline{0}$

$$\longrightarrow \underbrace{\begin{bmatrix} \overset{2}{3-1} & 2 & 0 & 0 \\ 2 & \overset{2}{3-1} & 0 & 0 \\ 0 & 0 & \overset{2}{3-1} & 2 \\ 0 & 0 & 2 & \underset{2}{3-1} \end{bmatrix}}_{4\times4} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{4\times1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times1}$$

From Row 4 : $2x_3 + 2x_4 = 0 \longrightarrow x_3 = -x_4$

From Row 3 : $2x_3 + 2x_4 = 0 \longrightarrow x_3 = -x_4$ ✓

From Row 2 or Row 1 : $2x_1 + 2x_2 = 0 \longrightarrow x_1 = -x_2$

x_2, x_4 "free"

(No need to reduce the system, it is really simple!)

$\therefore$ Any eigenvector for A relative to $\lambda = 3$ has the form

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_1} + x_4 \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\underline{v}_2}$$

for some $x_2, x_4 \in \mathbb{R}$ not both zeros.

$\therefore \left(\text{EigenSpace}\right)_{\lambda=3} = \text{Span}\{\underline{v}_1, \underline{v}_2\}$

- For $\lambda = -1$: $(\lambda I - A)\underline{x} = \underline{0}$

$$\begin{bmatrix} -1-1 & 2 & 0 & 0 \\ 2 & -1-1 & 0 & 0 \\ 0 & 0 & -1-1 & 2 \\ 0 & 0 & 2 & -1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From Row 4 or Row 3 : $-2x_3 + 2x_4 = 0$

$\therefore x_3 = x_4$ $\quad x_4$ free

From Row 1 or Row 2 : $-2x_1 + 2x_2 = 0$

$\therefore x_1 = x_2$ $\quad x_2$ free

$\therefore$ Any eigenvector for A relative to $\lambda = -1$ has the form :

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\underline{v}_3} + x_4 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{\underline{v}_4}$$

for some x_2, x_4 real numbers, not both zeros.

$\therefore \left(\text{Eigenspace}\right)_{\lambda=-1} = \text{Span}\{\underline{v}_3, \underline{v}_4\}$

Third, the eigenvectors $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$\underline{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

form a basis for $\mathbb{R}^4$. Also, this basis is orthogonal relative to the standard inner product (Dot Product)

Indeed,

$$\langle \underline{v}_1, \underline{v}_2 \rangle = (-1)(0) + (1)(0) + (0)(-1) + (0)(1) = 0$$

$$\langle \underline{v}_1, \underline{v}_3 \rangle = (-1)(1) + (1)(1) + (0)(0) + (0)(0) = 0$$

$$\langle \underline{v}_1, \underline{v}_4 \rangle = (-1)(0) + (1)(0) + (0)(1) + (0)(1) = 0$$

$$\langle \underline{v}_2, \underline{v}_3 \rangle = (0)(1) + (0)(1) + (-1)(0) + (1)(0) = 0$$

$$\langle \underline{v}_2, \underline{v}_4 \rangle = (0)(0) + (0)(0) + (-1)(1) + (1)(1) = 0$$

$$\langle \underline{v}_3, \underline{v}_4 \rangle = (1)(0) + (1)(0) + (0)(1) + (0)(1) = 0$$

To get an orthonormal basis, we divide each vector by its length: $\|\underline{v}_i\| = \sqrt{2}$ for $i = 1, 2, 3, 4$

$\therefore$ $\underline{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

$\underline{u}_2 = \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ form an orthonormal basis for $\mathbb{R}^4$ from the eigenvectors of A.

$\underline{u}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

$\underline{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

• Problem 3 :

a.

[i] T is linear transformation if

$$T(A+B) = T(A) + T(B)$$

$$T(\lambda \cdot A) = \lambda \cdot T(A)$$

for all $A, B \in M_{n\times n}$, c scaler

① $T(A+B) = (A+B) + (A+B)^t$

Property of t $= (A+B) + (A^t + B^t)$

Associativity $= A + (B + A^t) + B^t$

Commutativity $= A + (A^t + B) + B^t$

Associativity $= (A + A^t) + (B + B^t)$

$= T(A) + T(B)$ ✓

② $T(\lambda \cdot A) = (\lambda \cdot A) + (\lambda \cdot A)^t$

Property of t $= (\lambda \cdot A) + (\lambda \cdot A^t)$

distributivity $= \lambda \cdot (A + A^t)$

$= \lambda \cdot T(A)$ ✓

[ii] To find Ker(T), we solve the eqn. $T(A) = O_{n\times n}$

$\therefore\ T(A) = O_{n\times n}$

$A + A^t = O_{n\times n}$

$A = O_{n\times n} - A^t$

$A = -A^t$

$\therefore\ Ker(T) = \{ A \in M_{n\times n} : \underline{A = -A^t} \}$

skew-symmetric matrices

$\boxed{iii}$ $\underline{n=2}$: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in Ker(T)$

$\underline{n=3}$: $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in Ker(T)$

$\underline{n=4}$: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in Ker(T)$

In General, $[a_{ij}]$ such that $a_{12} = 1$, $a_{21} = -1$,
$a_{ij} = 0$ otherwise

belongs to $Ker(T)$.

So $Ker(T) \neq \{O_{n\times n}\}$ $\therefore$ T not 1-1

On the otherhand, a linear transformation

$$T : V \longrightarrow W$$

(such that $\dim(V) = \dim(W) = m < +\infty$)

is 1-1 iff it is onto.

Because $T : M_{n\times n} \longrightarrow M_{n\times n}$ is such a linear transformation, then it is not Onto being not 1-1.

Another Approach $\dim(Range(T)) = \dim(M_{n\times n}) - Nullity(T)$

$= n^2 - \underbrace{Nullity(T)}_{+ve \text{ because } Ker(T) \neq \{0\}}$

$\leq n^2 - 1 < n^2$

So $Range(T) \neq M_{n\times n}$

So T not Onto.

b. [i] $\langle \underline{u}, \underline{v} \rangle = 2u_1 v_1 + 5 u_2 v_2$

① $\langle \underline{u}, \underline{v} \rangle = 2u_1 v_1 + 5 u_2 v_2$

$= 2 v_1 u_1 + 5 v_2 u_2 = \langle \underline{v}, \underline{u} \rangle$. ✓

② $\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle (u_1 + v_1, u_2 + v_2), (w_1, w_2) \rangle$

$= 2(u_1 + v_1) w_1 + 5(u_2 + v_2) w_2$

$= (2u_1 w_1 + 5 u_2 w_2) + (2 v_1 w_1 + 5 v_2 w_2)$

$= \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$ ✓

③ $\langle \lambda \circ \underline{u}, \underline{v} \rangle = \langle (\lambda u_1, \lambda u_2), (v_1, v_2) \rangle$

$= 2(\lambda u_1)(v_1) + 5(\lambda u_2) v_2$

$= \lambda (2u_1 v_1 + 5 u_2 v_2) = \lambda \langle \underline{u}, \underline{v} \rangle$ ✓

④ $\langle \underline{u}, \underline{u} \rangle = 2 u_1^2 + 5 u_2^2 \geq 0$

Moreover, $\langle \underline{u}, \underline{u} \rangle = 0$ iff $2u_1^2 + 5 u_2^2 = 0$

iff $u_1^2 = u_2^2 = 0$

iff $u_1 = u_2 = 0$

iff $\underline{u} = \underline{0}$ ✓

$\therefore \langle , \rangle$ is an inner Product on $\mathbb{R}^2$.

[ii] $\underline{u} = (-1, 1)$ $\underline{v} = (1, 1)$

$$\cos(\theta) = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \|\underline{v}\|} = \frac{2(-1)(1) + 5(1)(1)}{\sqrt{2(-1)^2 + 5(1)^2} \sqrt{2(1)^2 + 5(1)^2}} = \frac{3}{7}$$

$\therefore \theta = \cos^{-1}\left(\frac{3}{7}\right)$

Problem 4 a.

[i] $(1,0,0) = \frac{1}{2}\cdot(1,1,0) - \frac{1}{2}\cdot(-1,1,0)$

$$\therefore T(1,0,0) \overset{T \text{ linear}}{=} \frac{1}{2}T(1,1,0) - \frac{1}{2}T(-1,1,0)$$
$$= \frac{1}{2}(1,0,-2) - \frac{1}{2}(1,2,-4)$$
$$= (0,-1,1)$$

$(0,1,0) = \frac{1}{2}\cdot(1,1,0) + \frac{1}{2}\cdot(-1,1,0)$

$$\therefore T(0,1,0) \overset{T \text{ linear}}{=} \frac{1}{2}T(1,1,0) + \frac{1}{2}T(-1,1,0)$$
$$= \frac{1}{2}(1,0,-2) + \frac{1}{2}(1,2,-4)$$
$$= (1,1,-3)$$

[ii] The standard matrix representation for T:

$$A_T = \left[\underset{\text{Col}_1}{T(1,0,0)} \;\vdots\; \underset{\text{Col}_2}{T(0,1,0)} \;\vdots\; \underset{\text{Col}_3}{T(0,0,1)}\right]$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 3 \\ 1 & -3 & 7 \end{bmatrix}$$

[iii] $(x,y,z) = x\cdot(1,0,0) + y\cdot(0,1,0) + z\cdot(0,0,1)$

$$\therefore T(x,y,z) = x\cdot T(1,0,0) + y\cdot T(0,1,0) + z\cdot T(0,0,1)$$
$$= x\cdot(0,-1,1) + y\cdot(1,1,-3) + z\cdot(0,3,7)$$
$$= (y, -x+y+3z, x-3y+7z)$$

b. $V, \langle \, , \, \rangle$ IPS

(i) $\|\underline{u}+\underline{v}\|^2 = \langle \underline{u}+\underline{v}, \underline{u}+\underline{v} \rangle$

Distributivity $\rightarrow = \langle \underline{u}, \underline{u} \rangle + \langle \underline{u}, \underline{v} \rangle + \langle \underline{v}, \underline{u} \rangle + \langle \underline{v}, \underline{v} \rangle \longrightarrow (1)$

$\|\underline{u}-\underline{v}\|^2 = \langle \underline{u}-\underline{v}, \underline{u}-\underline{v} \rangle$

$= \langle \underline{u}, \underline{u} \rangle - \langle \underline{u}, \underline{v} \rangle - \langle \underline{v}, \underline{u} \rangle + \langle \underline{v}, \underline{v} \rangle \longrightarrow (2)$

$\therefore \|\underline{u}+\underline{v}\|^2 - \|\underline{u}-\underline{v}\|^2 = (1) - (2)$

$= 2\langle \underline{u}, \underline{v} \rangle + 2\langle \underline{v}, \underline{u} \rangle$

Commutativity $\rightarrow = 2\langle \underline{u}, \underline{v} \rangle + 2\langle \underline{u}, \underline{v} \rangle$

$= 4\langle \underline{u}, \underline{v} \rangle$

(ii) $\operatorname{Proj}_{\underline{v}} \underline{u} = \dfrac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{v}\|^2} \cdot \underline{v} = \lambda \cdot \underline{v}$

$\therefore \underline{u} - \operatorname{Proj}_{\underline{v}} \underline{u} = \underline{u} - \lambda \cdot \underline{v}$

$\therefore \langle \underline{u} - \operatorname{Proj}_{\underline{v}} \underline{u}, \underline{v} \rangle = \langle \underline{u} - \lambda \cdot \underline{v}, \underline{v} \rangle$

$= \langle \underline{u}, \underline{v} \rangle - \lambda \langle \underline{v}, \underline{v} \rangle$

$= \langle \underline{u}, \underline{v} \rangle - \dfrac{\langle \underline{u}, \underline{v} \rangle}{\cancel{\|\underline{v}\|^2}} \cancel{\langle \underline{v}, \underline{v} \rangle}$

$= \langle \underline{u}, \underline{v} \rangle - \langle \underline{u}, \underline{v} \rangle = 0$ ✓

$\therefore \underline{u} - \operatorname{Proj}_{\underline{v}} \underline{u} \perp \underline{v}$.

- Problem 5

[a] True if A diagonalizable, then there is an invertible matrix P (formed from the eigenvectors) such that $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$

where $\lambda_1, \lambda_2, —, \lambda_n$ are the eigenvalues of A (not necessarily distinct)

Now, if $\lambda_1, \lambda_2, —, \lambda_n$ are all non-negative then the matrix $\sqrt{D} = \begin{bmatrix} \sqrt{\lambda_1} & & 0 \\ & \sqrt{\lambda_2} & \\ 0 & & \ddots \ \sqrt{\lambda_n} \end{bmatrix}$ is a well-defined matrix

$$\therefore A = PDP^{-1} = (P\sqrt{D}P^{-1})(P\sqrt{D}P^{-1})$$
$$= (P\sqrt{D}P^{-1})^2$$

$\therefore P\sqrt{D}P^{-1} = \sqrt{A}$ ($\therefore$ it exists)

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[b] False if $\underline{b} \in CS(A)$, then the system $A\underline{x} = \underline{b}$ has a solution, but not necessarily a unique soln.

For example, $\underline{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ belongs to the Columnspace of $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Thus, $A\underline{x} = \underline{b}$ is consistent but it has infinitely many solns. namely

$$\underline{x} = \begin{bmatrix} 1 - x_2 \\ x_2 \end{bmatrix}, \quad x_2 \text{ free}$$

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[c] False $|6(AB)^t(BA)^{-1}| = 6^2 |(AB)^t(BA)^{-1}|$.

$$= 6^2 |(AB)^t| \, |(BA)^{-1}|$$

$$= 6^2 |AB| \frac{1}{|BA|}$$

$$= 6^2 |A||B| \frac{1}{|B||A|} = 6^2 \neq 120$$

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[d] True

$A \sim B \therefore B = P^{-1}AP$ for some P

$B \sim C \therefore C = Q^{-1}BQ$ for some Q

$$\therefore C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q$$

$$= (Q^{-1}P^{-1}) A (PQ)$$

$$= (PQ)^{-1} A (PQ)$$

$$= R^{-1}AR \quad \text{with } R = PQ$$

$\therefore A \sim C$.

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[e] True We've seen in Problem 3 that $W = \operatorname{Ker}(T)$ where $T : M_{n\times n} \to M_{n\times n}$ is the linear transformation $T(A) = A + A^t$

$\therefore W$ subspace in $M_{n\times n}$

(the kernel of any L.T. does)

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f) False $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{v}_2 = 2 \cdot \underline{v}_1$

$\underline{v}_3 = 3 \cdot \underline{v}_1$

$\underline{v}_4 = 4 \cdot \underline{v}_1$

$\underline{v}_5 = 5 \cdot \underline{v}_1$

$\therefore$ $\text{Span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_5\} = \text{Span}\{\underline{v}_1\} \neq \mathbb{R}^3$

because $\mathbb{R}^3$ needs at least $\underline{3}$ vectors to be generated.

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g) True let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T(x, y) = (y, 0)$$

$\therefore$ $\text{Range}(T) = \{(y, 0) : y \text{ any number}\}$

$= \text{Span}\{(1, 0)\}$

$\therefore$ $\text{Ker}(T) = \{(x, y) : T(x, y) = \underline{0}\}$

$= \{(x, y) : (y, 0) = \underline{0}\}$

$= \{(x, y) : y = 0\}$

$= \{(x, 0) : x \text{ any number}\}$

$= \text{Span}\{(1, 0)\}$

$\therefore$ $\text{Range}(T) = \text{Ker}(T)$.