Final Exam
May 22, 2021

Name: ______________________________ **UID:** ______________________

- The exam consists of FIVE problems.
- Unsupported answers will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.
- Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.
- Missing or blank pages will result in an automatic zero for the question.
- Time: 100 minutes.

Problem	Score	Points
1		15
2		10
3		10
4		8
5		20
Total		63

Problem 1. Consider the matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(i) (12 pts) Find the eigenvalues and the corresponding eigenspaces of A.

(ii) (3 pts) Is A diagonalizable? why?

Problem 2. Let $\mathcal{P}_3$ be the space of all polynomials in x of degree ≤ 3. Define a linear transformation $T : \mathcal{P}_3 \to \mathbb{R}^3$ by

$$T\left(p(x)\right) = \left(p(-1),\, p(0),\, p(1)\right).$$

(i) (5 pts) Determine a basis for the **kernel** of T.

(ii) (3 pts) Is T one-to-one? onto? why?

(iii) (2 pts) Describe the **range** of T.

Problem 3. (10 pts) Determine whether the sets S_1 and S_2 span the same subspace of $\mathcal{P}_2$ (the space of all polynomials in x of degree ≤ 2).

$$\begin{aligned} S_1 &= \left\{\mathbf{v}_1 = 1 + 3\,x - 2\,x^2,\, \mathbf{v}_2 = 2\,x + 2\,x^2,\, \mathbf{v}_3 = -2 + 10\,x^2\right\}, \\ S_2 &= \left\{\mathbf{v}_4 = 1 + 5\,x,\, \mathbf{v}_5 = -2 + x + 11\,x^2\right\}. \end{aligned}$$

Problem 4. (4 pts each) Let V, $\langle\, , \,\rangle$ be an inner product space.

(i) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors in V such that $\langle\, \mathbf{u}\, , \, \mathbf{w}\, \rangle = \langle\, \mathbf{v}\, , \, \mathbf{w}\, \rangle$ for all vectors $\mathbf{w} \in V$. Show that $\mathbf{u} = \mathbf{v}$.

(ii) Suppose that $B = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis of V. Find the length of the vector $\mathbf{v}$ if $\mathbf{v} = 2\,\mathbf{q}_1 - 3\,\mathbf{q}_2 + 4\,\mathbf{q}_3$.

Problem 5. (5 pts each) Prove or disprove the following statements.

(i) If A is a 2×2 matrix with eigenvalues $\lambda = -1, 3$, then $A^4 = 20\,A + 21\,I$.

Reason: True False

(ii) If two matrices A and B are row-equivalent, then they have the same eigenvalues.

Reason: True False

(iii) There are infinitely many unit vectors in $\mathbb{R}^3$ that are orthogonal to $(0, 2, -1)$.

Reason: True False

(iv) A linear transformation $T : \mathbb{R}^3 \to \mathcal{P}_2$ is one-to-one if and only if it is onto.

Reason: True False

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