MACT 2132, Summer 2022 Final Exam (30%)

[redacted]

TOTAL POINTS

64 / 64

QUESTION 1

1 Problem 1, Part 1 **5 / 5**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.
- **- 3 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 0.25 pts** Click here to replace this description.

QUESTION 2

2 Problem 1, Part 2 **7 / 7**

- ✓ **- 0 pts** Correct
- **- 1.5 pts** Click here to replace this description.
- **- 0.5 pts** Click here to replace this description.
- **- 7 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.

QUESTION 3

3 Problem 2, Part 1 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.
- **- 3 pts** Click here to replace this description.

QUESTION 4

4 Problem 2, Part 2 **6 / 6**

- ✓ **- 0 pts** Correct
- **- 1 pts** Click here to replace this description.
- **- 0.25 pts** Click here to replace this description.
- **- 0.5 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.

QUESTION 5

5 Problem 3 **20 / 20**

- ✓ **- 0 pts** Correct
- **- 6 pts** Click here to replace this description.
- **- 0.25 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.
- **- 2.5 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.

QUESTION 6

6 Problem 4 **16 / 16**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Click here to replace this description.
- **- 7 pts** Click here to replace this description.
- **- 3 pts** Click here to replace this description.
- **- 4 pts** Click here to replace this description.
- **- 6 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.
- **- 2 pts** Click here to replace this description.

Problem 1, Part 1. (1 point each) Complete **briefly** the following statements.

(i) A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is an **orthogonal basis** for an inner product space V if

every $v \in S$ is linearly independent, and the span of S is V
and $\langle v_i, v_j \rangle = 0$ as long as $i \neq j$

(ii) The **span** of $\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3\}$ has dimension **two** if

only one of the vectors is redundant and the rest are linerally independent

(iii) The **rank** of an $m \times n$ matrix A can be $0, 1, ..., k$ where $k =$

is the minimum value of m and n i.e. $\min\{m, n\}$

(iv) Any linear transformation $T : V \to W$ transforms a **subspace** of V into a

Subspace of W

(v) The **dimension of the eigenspace** of a square matrix A relative to an eigenvalue λ of multiplicity m is

less than or equal to m

1 Problem 1, Part 1 5 / 5

✓ - **0 pts** Correct

- **0.5 pts** Click here to replace this description.
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- **3 pts** Click here to replace this description.
- **2 pts** Click here to replace this description.
- **0.25 pts** Click here to replace this description.

Problem 1, Part 2. (7 points) Let U be the subspace **spanned** by the four vectors:

$$\mathbf{v}_1 = (1,2,1),\ \mathbf{v}_2 = (1,1,3),\ \mathbf{v}_3 = (1,0,5),\ \mathbf{v}_4 = (1,1,0).$$

Find an **orthonormal basis** for U.

$w_1 = v_1 \quad w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \quad w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$

$w_4 = v_4 - \frac{\langle v_4, w_3\rangle}{\langle w_3, w_3\rangle} w_3 - \frac{\langle v_4, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \frac{\langle v_4, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$w_1 = \boxed{(1,2,1)}$

$w_2 = (1,1,3) - \frac{(1,1,3)\cdot(1,2,1)}{(1,2,1)\cdot(1,2,1)}(1,2,1) = (1,1,3) - \frac{1+2+3}{1+4+1}(1,2,1) = \boxed{(0,-1,2)}$

$w_3 = (1,0,5) - \frac{(1,0,5)\cdot(1,2,1)}{(1,2,1)\cdot(1,2,1)}(1,2,1) - \frac{(1,0,5)\cdot(0,-1,2)}{(0,-1,2)\cdot(0,-1,2)}(0,-1,2)$

$= (1,0,5) - \frac{1+5}{1+4+1}(1,2,1) - \frac{1+10}{1+4}(0,-1,2) = \cancel{(0,0.2,-0.4)}\ (0,0,0)$

$w_4 = (1,1,0) - \cancel{\frac{(1,1,0)\cdot(0,0.2,-0.4)}{(0,0.2,-0.4)\cdot(0,0.2,-0.4)}(0,0.2,-0.4)} - \frac{(1,1,0)\cdot(0,-1,2)}{(0,-1,2)\cdot(0,-1,2)}(0,-1,2) - \frac{(1,1,0)\cdot(1,2,1)}{(1,2,1)\cdot(1,2,1)}(1,2,1)$

$w_4 = (1,1,0) - \cancel{(1)(0,0.2,-0.4)} - (-0.2)(0,-1,2) - (0.5)(1,2,1)$

$= \cancel{(0.5, -0.4,)}\ \cancel{(0.5, 2.3, -0.1)}\ (0.5, -0.2, -0.1)$

$u_1 = (1,2,1)\frac{1}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}(1,2,1) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$u_2 = (0,-1,2)\frac{1}{\sqrt{1+4}} = \left(0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$u_3 = (0,0,0) -$

~~$u_4 = (0.5, 2.3, -0.1)\frac{1}{\sqrt{5.55}} = \left(\frac{0.5}{\sqrt{5.55}}, \frac{2.3}{\sqrt{5.55}}, \frac{-0.1}{\sqrt{5.55}}\right)$~~

$u_4 = (0.5, -0.2, -0.1)\frac{1}{\sqrt{0.3}} = \left(\frac{0.5}{\sqrt{0.3}}, \frac{-0.2}{\sqrt{0.3}}, \frac{-0.1}{\sqrt{0.3}}\right)$

(0,0,0) is redundent So not part of basis

orthonormal basis is $\left\{\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{0.5}{\sqrt{0.3}}, \frac{-0.2}{\sqrt{0.3}}, \frac{-0.1}{\sqrt{0.3}}\right)\right\}$

2 Problem 1, Part 2 **7 / 7**

- ✓ **- 0 pts** Correct
- **- 1.5 pts** Click here to replace this description.
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- **- 7 pts** Click here to replace this description.
- **- 1 pts** Click here to replace this description.

Problem 2, Part 1. Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & 0 \\ a & 1 & -2 \\ a & a & -4 \end{pmatrix}.$$

(i) (4 points) Find all value(s) of a which will guarantee that A has eigenvalues 0, 3, and -3 simultaneously.

$|\lambda I - A| = 0$

$\left| \lambda I - \begin{bmatrix} 3 & 0 & 0 \\ a & 1 & -2 \\ a & a & -4 \end{bmatrix} \right| = 0$

$\begin{vmatrix} \lambda-1 & 2 \\ -a & \lambda+4 \end{vmatrix}$

$\begin{vmatrix} \lambda-3 & 0 & 0 \\ -a & \lambda-1 & 2 \\ -a & -a & \lambda+4 \end{vmatrix} = 0 = (\lambda-3)((\lambda-1)(\lambda+4) + 2a)$

$= (\lambda-3)(\lambda^2+3\lambda-4+2a)$

We need to have 0, 3, −3 so $\lambda=3$; $\lambda=0$, $\lambda=-3$

$\lambda^2+3\lambda-4+2a = \lambda(\lambda+3) = \lambda^2+3\lambda$

$-4+2a=0$

$2a=4$

$a=2$

When $a=2$ A has eigenvalus 0, 3, −3

(ii) (6 points) Select one of the value(s) of a you found above, and find the **eigenspaces** of A relative to the eigenvalues $\lambda = 0$ and $\lambda = 3$ respectively.

$a = 2$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix}$$

For $\lambda = 0$ $\quad (0I - A)x = 0 \quad \begin{bmatrix} -3 & 0 & 0 \\ -2 & -1 & 2 \\ -2 & -2 & 4 \end{bmatrix} x = 0$

$$\left[\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ -2 & -2 & 4 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = 0 \\ x_2 = 2x_3 \\ x_3 \text{ free} \end{array} \quad \begin{array}{l} -x_2 + 2x_3 = 0 \\ -2x_2 + 4x_3 = 0 \\ 2x_2 = 4x_3 \\ x_2 = 2x_3 \end{array} \quad \begin{bmatrix} 0 \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{Span} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{eigenspace}_{\lambda=0} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

For $\lambda = 3$

$$(3I - A)x = 0 \qquad \begin{bmatrix} 0 & 0 & 0 \\ -2 & 2 & 2 \\ -2 & -2 & 7 \end{bmatrix} x = 0$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & 0 \\ -2 & -2 & 7 & 0 \end{array}\right] \xrightarrow{\times -\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & -2 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{+2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free

$4x_2 = 5x_3$

$x_2 = \frac{5}{4}x_3$

$x_1 = x_2 + x_3 = \frac{9}{4}x_3$

$$\begin{bmatrix} \frac{9}{4}x_3 \\ \frac{5}{4}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 9/4 \\ 5/4 \\ 1 \end{bmatrix}$$

$$\text{eigenspace}_{\lambda=3} = \text{Span}\left\{ \begin{bmatrix} 9/4 \\ 5/4 \\ 1 \end{bmatrix} \right\}$$

3 Problem 2, Part 1 **10 / 10**

✓ **- 0 pts** Correct

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- 1 pts Click here to replace this description.

- 2 pts Click here to replace this description.

- 3 pts Click here to replace this description.

Problem 2, Part 2. (6 points) Suppose that A is a 3×3 matrix such that $\lambda = 1, 2, -2$ are its eigenvalues.

By the aid of **Cayley-Hamilton Theorem**, find A^5 and A^{-1} in the form $\alpha A^2 + \beta A + \gamma I$ for some constants α, β, γ.

$(\lambda-1)(\lambda+2)(\lambda-2)=0$

$(\lambda-1)(\lambda^2-4) = \lambda^3-\lambda^2-4\lambda+4=0$

$A^3-A^2-4A+4I=0$

$A^3 = A^2+4A-4I$

$A^5 = A^3A^2 = A^4+4A^3-4A^2 = A^3A+4(A^2+4A-4I)-4A^2$

$= (A^2+4A-4I)A + \cancel{4A^2}+16A-16I-\cancel{4A^2}$

$= A^3+4A^2-4A+16A-16I = A^2+4A-4I+4A^2-4A+16A-16I$

$= 5A^2+16A-20I$

0

A^{-1}

$A^3-A^2-4A=-4I$

$A(A^2-A-4I)=-4I$

$A\left(\frac{1}{-4}(A^2-A-4I)\right)=I$

A^{-1} Since $AB=I$ means $B=A^{-1}$

$A^{-1} = \frac{-1}{4}A^2+\frac{1}{4}A+I$

2

4 Problem 2, Part 2 **6 / 6**

- ✓ **- 0 pts** Correct
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- **- 0.5 pts** Click here to replace this description.
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Problem 3. Let $\mathcal{P}_{\leq 3}$ be the space of polynomials of degree ≤ 3 in the variable t. Consider the transformation $T : \underset{4}{\mathcal{P}_{\leq 3}} \longrightarrow \underset{2}{\mathbb{R}^2}$ given by

$$T\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3\right) = (a_0 + a_2\,;\, a_1 - a_2)\,.$$

(i) (2 points) Find the images of the two vectors $\mathbf{u} = 2 - x - x^2$ and $\mathbf{v} = x - x^3$.

$T(2-x-x^2) = (2-1, -1+1) = (1,0)$

$T(x-x^3) = (0+0, 1-0) = (0,1)$

(ii) (4 points) Show that T is a linear transformation.

$T(\vec{u}+\vec{v}) = (u_0+v_0+u_2+v_2,\ u_1+v_1-u_2-v_2) = (u_0+u_2, u_1-u_2) + (v_0+v_2, v_1-v_2)$

$=$

$T(\vec{u}) + T(\vec{v}) = (u_0+u_2, u_1-u_2) + (v_0+v_2, v_1-v_2) = T(\vec{u}+\vec{v})$

1 So ation one true

$T(c\vec{u}) = (cu_0 + cu_2, cu_1 - cu_2) = c(u_0+u_2, u_1-u_2) = cT(\vec{u})$

2 So axion 2 true

axion one and two true So T is a linar transforman

(iii) (4 points) Describe the **kernel** of T, and a **basis** for it.

$\ker(T) = \{ v \in V : T(v) = 0 \}$

$T(\vec{v}) = \underline{0}_w$

$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = (a_0 + a_2, a_1 - a_2) = (0, 0)$

$a_0 + a_2 = 0 \qquad a_1 - a_2 = 0$

$a_0 = -a_2 \qquad a_1 = a_2 \qquad a_2$ free $\quad a_3$ free

$\vec{v} = (-a_2 + a_2 t + a_2 t^2 + a_3 t^3) = a_2(-1 + t + t^2) + a_3(t^3)$

basis of $\ker(T) = \{ (-1 + t + t^2), (t^3) \}$

(iv) (4 points) Describe the **range** of T, and a **basis** for it.

Range(T) is image of T $\qquad$ $\text{Range}(T) = \{ T(v) \mid v \in V \}$

$\begin{array}{l|l} a_0 + + a_2 & x \\ 0 + a_1 - a_2 & y \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 1 & & x \\ 0 & 1 & -1 & & y \end{array}\right] \qquad a_0 +$

rank + nullity = $\dim(P_{\le 3}) = 4$

rank + 2 $= 4 \qquad$ rank = 2

$\text{rank} = \dim(\text{Range}(T)) = \dim(\mathbb{R}^2)$

$\therefore$ So T is onto therefor W = Codomain

$\text{range}(T) = \mathbb{R}^2$

basis of range $= \{ (1,0), (0,1) \}$

(v) (3 points) Determine the **rank** and the **nullity** of T. **Justify**.

nullity = $\dim(\ker(T)) = 2$

linear

rank + nullity = $\dim(P_{\leq 3}) = 4$

rank + 2 = 4 rank = 2

(vi) (3 points) Is T one-to-one? onto? **Justify**.

T is not one to one as nullity(T) = $2 \neq 0$

So $\ker(T) \neq \{\underline{0}\}$ and cannot be one 1-to one

T is Onto as rank(T) = dim(range(T)) = 2 = dim($\mathbb{R}^2$) = 2

So W = range So T is onto

5 Problem 3 **20 / 20**

✓ **- 0 pts** Correct

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- 2.5 pts Click here to replace this description.

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Problem 4. (4 points each) True or False (Circle one and state your reason):

(i) The set $\mathcal{P}_{=3}$ of polynomials $p(x)$ of degree exactly 3 is a vector space relative to the standard addition and scalar multiplication.

Reason: True False

not closed under addition

$p(x) = 1 + x + x^3$ $\quad q(x) = 1 + x + x^2 - x^3$

$p(x), q(x) \in P_{=3}$

but $p(x) + q(x) = 2 + 2x + x^2 \notin P_{=3}$

So not vector space as not closed under addition

(ii) There is a linear transformation $T : \mathbb{R}^2 \to \mathcal{P}_{\leq 2}$ that is onto.

Reason: True False

rank + nullity = dim $(\mathbb{R}^2)$

rank + nullity = 2

rank ≤ 2 so rank $\neq$ dim$(P_{\leq 2}) = 3$

rank ≤ 2 so never $= 3$ so never onto

(iii) Suppose that **u** and **v** are unit vectors in an inner product space V such that $\langle \mathbf{u}, \mathbf{v} \rangle = 1$. Then **u** and **v** are **not** linearly independent.

Reason:

True False

$\langle u, v \rangle = 1$

$\cos\theta = \dfrac{\langle u, v \rangle}{\|u\|\|v\|} = 1$

$\theta = 0$ so u and v are on top of each other so v can be expresed as αu

~~if not independent~~ ~~$v = \alpha u$~~

$\langle u, \alpha u \rangle = \langle \alpha u, u \rangle = \alpha \langle u, u \rangle = \alpha \cdot 1 = \alpha$

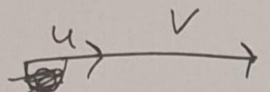
$u_1 v_1 + u_2 v_2 = 1$

$u_1 v_1 + u_2 v_2 = (u_1^2 + u_2^2 + v_1^2 + v_2^2)^{1/2}$

$u_1^2 + u_2^2 = 1$

$v_1^2 + v_2^2 = 1$

$u_1^2 - v^2 + u_2^2$



(iv) If A and B are square matrices such that $PBP^{-1} = A$ for some invertible matrix P, then A and B have the same charactertistic equation.

Reason:

True False

$PBP^{-1} = A$

$PB = AP$

$B = P^{-1}AP$

$P(P^{-1}AP)P^{-1} = A$

$(\lambda I - A)x = 0$

$|\lambda I - A| = 0$

$Ax = \lambda x$

$(\lambda I - A)x = 0$

$\lambda x = Ax$

$|\lambda I - A| = |\lambda I - B|$

$|\lambda I - PBP^{-1}| = 0$

$|\lambda PP^{-1} - PBP^{-1}| = 0$

$|(\lambda P - PB)P^{-1}| = 0$

$|P(\lambda I - B)P^{-1}| = 0$

$|P||\lambda I - B||P^{-1}| = 0$

$|P|$ and $|P^{-1}| \neq 0$ as invertible so

$|\lambda I - B| = 0$

$|\lambda I - A| = |\lambda I - B|$

So same characteristic equation

6 Problem 4 16 / 16

✓ - **0 pts** Correct

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MACT 2132 Summer 2022

Linear Algebra
Final Exam
July 24, 2022

Name: Omar Elfouly UID: 900211195

- The exam consists of FOUR problems.
- Unsupported answers will receive little or no credit.
- Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.
- Missing or blank pages will result in an automatic zero for the question.
- Time: 100 minutes.

Problem	Score	Points
1		12
2		16
3		20
4		16
Total		64

Best wishes!

Dr. Eslam Badr