The augmented matrix is

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & 0 & -11 & -16 & -1 \end{array}\right)$$

Exchange R_1 & R_3:

$$\sim \begin{array}{c} R_1^{*} = R_3 \\ R_2 \\ R_3^{*} = R_1 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & 2 & 1 & -4 & 1 \end{array}\right)$$

$$\sim \begin{array}{c} R_1^{*} \\ R_2^{*} = R_2 - R_1^{*} \\ R_3^{**} = R_3^{*} - R_1^{*} \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 3 & 18 & 18 & 3 \\ 0 & 2 & 12 & 12 & 2 \end{array}\right)$$

$$\sim \begin{array}{c} R_1^{*} \\ R_2^{**} = R_2^{*}/3 \\ R_3^{***} = R_3^{**}/2 \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & 1 & 6 & 6 & 1 \end{array}\right)$$

$$\sim \begin{array}{c} R_1^{*} \\ R_2^{**} \\ \overline{R_3} = R_3^{***} - R_2^{**} \end{array} \left(\begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_3 = s$, $x_4 = t \Rightarrow x_2 = 1 - 6x_3 - 6x_4 = 1 - 6s - 6t$

$x_1 = -1 + 11x_3 + 16x_4 = -1 + 11s + 16t$

In vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 + 11s + 16t \\ 1 - 6s - 6t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 11 \\ -6 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 16 \\ -6 \\ 0 \\ 1 \end{pmatrix}$$

Problem 2. (5 + 7 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & c \\ 0 & a & -b \\ -1/a & x & x^2 \end{bmatrix},$$

where a, b, c are given constants with $a \neq 0$.

1) Find all values of x in terms of a, b, c such that the matrix A is invertible.

$$|A| = \begin{vmatrix} 1 & 0 & c \\ 0 & a & -b \\ -\frac{1}{a} & x & x^2 \end{vmatrix} \overset{\text{Row 1}}{=} +1\begin{vmatrix} a & -b \\ x & x^2 \end{vmatrix} - 0\begin{vmatrix} 0 & -b \\ -\frac{1}{a} & x^2 \end{vmatrix} + c\begin{vmatrix} 0 & a \\ -\frac{1}{a} & x \end{vmatrix}$$

$$= ax^2 + bx + c$$

A invertible $\underline{\text{iff}}$ $|A| \neq 0$ $\underline{\text{iff}}$ $ax^2 + bx + c \neq 0$

$\underline{\text{iff}}$ $x \neq \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2) Find all possible LU-Factorizations of A when $a = b = c = x = 1$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow[\therefore \alpha_{31} = 1]{R_3 \to R_3 + R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\therefore \alpha_{32} = -1]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha_{21} & 1 & 0 \\ -\alpha_{31} & -\alpha_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Second, $|A| = |U| = 3 \neq 0 \rightsquigarrow$ A invertible

$\therefore$ A has a unique LU-factorization

(Existence + invertible $\Rightarrow$ uniqueness)

Problem 3. (5 points each) Consider the matrix

$$B = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

1) Find an invertible 3×3 matrix A satisfying $A^2 + AB = 3A$.

$$A^2 + AB = 3A$$

$$A(A+B) = 3A$$

$$\cdot A^{-1} \quad A^{-1}A(A+B) = A^{-1}(3A)$$

$$A + B = 3I$$

$$A = 3I - B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

2) Find the inverse of A.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 - R_3} \left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{array}}_{A^{-1}}\right]$$

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Problem 4. (5 points each) True or False (Circle one and state your reason):

1) If $x_1 = 0, x_2 = 0, x_3 = 1$ is a solution to a homogeneous system of linear equation in x_1, x_2, x_3, then the system has infinitely many solutions.

Reason: $\boxed{\text{True}}$ False

Homogeneous system always consistant

unique soln. $(x_1 = x_2 = x_3 = 0)$

∞-many solns.

Since $x_1 = x_2 = 0, x_3 = 1$ is a non-zero soln., then ∞-many solns.

2) Let A, B, C be $n \times n$ invertible matrices. Then,

$$\det\left(C^{-1}(AB^{-1})^{-1}(CA^{-1})^{-1}C^2\right) = \det(BC).$$

Reason: True $\boxed{\text{False}}$

$$|C^{-1}(AB^{-1})^{-1}(CA^{-1})^{-1}C^2| = |C^{-1}BA^{-1}AC^{-1}C^2|$$
$$= |C^{-1}BC| = |C^{-1}||B||C|$$
$$= \frac{1}{\cancel{|C|}}|B|\cancel{|C|} = |B|$$
$$\neq |BC|$$

3) Let A, B, and $A+B$ be invertible matrices. Then,

$$\left(A^{-1}+B^{-1}\right)^{-1} = A\,(A+B)^{-1}\,B.$$

$\boxed{\text{True}}$ False

Reason:

$$\left(A\,(A+B)^{-1}B\right)^{-1} = B^{-1}(A+B)A^{-1}$$
$$= B^{-1}AA^{-1} + B^{-1}BA^{-1}$$
$$= B^{-1} + A^{-1} = A^{-1} + B^{-1}$$

$$\therefore\ A(A+B)^{-1}B = \left(A^{-1}+B^{-1}\right)^{-1}$$

4) Let A be a 3×3 matrix such that

$$A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Then, there exists a 3×3 non-zero matrix B such that $AB = O$.

$\boxed{\text{True}}$ False

Reason:

for example, take $B = \begin{bmatrix}0 & 0 & 0\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} \neq 0$