Linear Algebra
Final Exam
December 14, 2022

Name: ______________________ **UID:** ____________________

- **The exam consists of SIX problems.**
- **Unsupported answers will receive little or no credit.**
- **Missing or blank pages will result in an automatic zero for the question.**
- **Time: 100 minutes.**

Problem	Score	Points
1		9
2		6
3		8
4		7
5		18
6		16
Total		64

Best wishes!

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Problem 1. (1.5 point each) Choose the **unique correct answer**.

(i) Are the vectors $\underline{\mathbf{v}}_1 = (2, 0, -1)$, $\underline{\mathbf{v}}_2 = (4, 0, 7)$, and $\underline{\mathbf{v}}_3 = (1, 1, 1)$ linearly independent in $\mathbb{R}^3$?

A. Yes.

B. No.

C. Data not complete.

D. None of these.

(ii) Which of the following is a subspace of $\mathbb{R}^3$?

A. All vectors of the form $(x, 0, 0)$.

B. All vectors of the form $(x, 1, 1)$.

C. All vectors of the form (x, y, z) such that $y = x + z + 1$.

D. None of these.

(iii) The polynomials $(x - 1)$, $(x - 1)^2$, $(x - 1)^3$, $x(x - 1)$.

A. span $\mathcal{P}_{\leq 3}$.

B. linearly dependent in $\mathcal{P}_{\leq 3}$.

C. linearly independent in $\mathcal{P}_{\leq 3}$.

D. None of these.

(iv) Which of the following is **false**?

A. Every subspace of a vector space is itself a vector space.

B. Every vector space is a subspace of itself.

C. The union of any two subspaces of a vector space V is a subspace of V.

D. None of these.

(v) Find the rank of a 5×6 matrix A for which $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$ has a two-dimensional solution space.

A. 5.

B. 6.

C. 8.

D. 4.

(vi) Which of the following holds in an inner product space V?

A. $\langle \underline{\mathbf{0}}\,, \underline{\mathbf{v}} \rangle = 0$ for any $\underline{\mathbf{v}} \in V$.

B. $\langle \underline{\mathbf{u}}\,, \underline{\mathbf{v}} \rangle = 0$ iff $\underline{\mathbf{u}} = \underline{\mathbf{v}} = \underline{\mathbf{0}}$.

C. Both **A** and **B** hold.

D. Neither **A** nor **B** holds.

Problem 2. (6 points) Let U be the subspace of $\mathbb{R}^3$ **spanned** by the three vectors:

$$\underline{\mathbf{v}}_1 = (1, 2, -1),\ \underline{\mathbf{v}}_2 = (1, 1, 3),\ \underline{\mathbf{v}}_3 = (1, 0, 7).$$

Show that $\{\underline{\mathbf{v}}_1,\ \underline{\mathbf{v}}_2\}$ form an orthogonal basis for U. Then, determine whether the vector $\underline{\mathbf{u}} = (1, -3, 4)$ belongs to U or not. **Argue your answer.**

$\underline{v}_1 \cdot \underline{v}_2 = (1,2,-1)\cdot(1,1,3) = 1+2-3 = 0 \rightsquigarrow \underline{v}_1 \perp \underline{v}_2$

$\therefore$ $\underline{v}_1, \underline{v}_2$ are linearly indep. $\rightsquigarrow$ ①

Since $\underline{v}_3 = -1\,\underline{v}_1 + 2\,\underline{v}_2$, then $\{\underline{v}_1, \underline{v}_2\}$ spans U $\rightsquigarrow$ ②

From ①, ②, we deduce that $\{\underline{v}_1, \underline{v}_2\}$ form an $\perp$ basis for U.

$\underline{u} \in U \rightsquigarrow \underline{u} = \alpha\,\underline{v}_1 + \beta\,\underline{v}_2$ for some α, β

$(1,-3,4) = \alpha\,(1,2,-1) + \beta\,(1,1,3)$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 3 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 4 & 5 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -15 \end{array}\right]$$

↳ a contradiction

$\therefore\ \underline{u} \notin U$

Problem 3. Consider the matrix

$$A = \begin{pmatrix} 3 & a & a \\ 4 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(i) (4 points) Find all value(s) of a such that A has exactly the three eigenvalues -1, 2, and 5.

Characteristic eqn. $|\lambda I - A| = \begin{vmatrix} \lambda-3 & -a & -a \\ -4 & \lambda-1 & -1 \\ 0 & 0 & \lambda-2 \end{vmatrix}$

$= (\lambda-2)\begin{vmatrix} \lambda-3 & -a \\ -4 & \lambda-1 \end{vmatrix}$

$= \boxed{(\lambda-2)}\left[(\lambda-3)(\lambda-1) - 4a\right]$

$\lambda = 2$ is always an eigenvalue for any a

$\therefore\ (\lambda+1)(\lambda-5) = (\lambda-3)(\lambda-1) - 4a$

$\rightsquigarrow\ \lambda^2 - 4\lambda - 5 = \lambda^2 - 4\lambda + 3 - 4a$

$\rightsquigarrow\ -5 = 3 - 4a$

$\rightsquigarrow\ 4a = 8 \rightsquigarrow \boxed{a = 2}$

(ii) (4 points) Select one of the value(s) of a you found above, and find a basis of the **eigenspace** of A relative to the eigenvalue $\lambda = -1$.

$\boxed{a=2}$ $\rightsquigarrow$ $-I-A = \begin{array}{c} \left[\begin{array}{ccc} -4 & -2 & -2 \\ -4 & -2 & -1 \\ 0 & 0 & -3 \end{array}\right] \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$

3^{rd} eqn. $\rightsquigarrow$ $x_3 = 0$

2^{nd} eqn. $\rightsquigarrow$ $-4x_1 - 2x_2 = 0 \rightsquigarrow x_2 = -2x_1$

x_1 free

$\therefore$ Any eigenvector for A relative to $\lambda = -1$ has the form

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \qquad x_1 \neq 0$$

$$\therefore \underbrace{(\text{Eigenspace})}_{\lambda=-1} = \text{Null space}(-I-A)$$

$$= \text{Span}\underbrace{\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\}}_{\text{basis}}$$

Problem 4. Suppose that A is a 2×2 matrix such that $\lambda = 1, \ -2$ are its eigenvalues, i.e. with characteristic polynomial $\chi_A(\lambda) = (\lambda-1)(\lambda+2)$.

(i) (3 points) Show that $A^{-1} = \frac{1}{2}(A+I)$.

Cayley-Hamilton $\rightsquigarrow$ $(A - I)(A+2I) = 0$

$\rightsquigarrow A^2 + A - 2I = 0$

$\rightsquigarrow A^2 + A = 2I$

$\rightsquigarrow A(A+I) = 2I$

$\rightsquigarrow A\left(\frac{A+I}{2}\right) = I$

$\rightsquigarrow A^{-1} = \frac{1}{2}(A+I)$ ✓

(ii) (2 points) If $\underline{\mathbf{v}}$ is an eigenvector for A relative to the eigenvalue $\lambda = 1$, then $\underline{\mathbf{v}}$ is an eigenvector for $7A^3 - 3A^2 + 5A - I$ relative to the eigenvalue ......$7(1)^3 - 3(1)^2 + 5(1) - 1 = 8$

Remark: $(7A^3 - 3A^2 + 5A - I)\underline{v} = 7(1)^3\underline{v} - 3(1)^2\underline{v} + 5(1)\underline{v} - \underline{v}$

$= 8\underline{v}$

(iii) (2 points) If $\underline{\mathbf{v}}$ is an eigenvector for A relative to the eigenvalue $\lambda = -2$, then $\underline{\mathbf{v}}$ is an eigenvector for $7A^3 - 3A^2 + 5A - I$ relative to the eigenvalue ......$7(-2)^3 - 3(-2)^2 + 5(-2) - 1 = -79$

Problem 5. Consider the transformation

$$T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$T(x_1, x_2, x_3, x_4) = (-x_3, x_1 - x_2 - x_4, x_3, x_3).$$

let $\underline{u} = (x_1, x_2, x_3, x_4)$, $\underline{v} = (a, b, c, d)$

let α scalar

(i) (4 points) Show that T is a linear transformation.

$$T(\underline{u}+\underline{v}) = T(x_1+a, x_2+b, x_3+c, x_4+d)$$
$$= (-x_3-c, x_1+a-x_2-b-x_4-d, x_3+c, x_3+c)$$
$$= (-x_3, x_1-x_2-x_4, x_3, x_3) + (-c, a-b-d, c, c)$$
$$= T(\underline{u}) + T(\underline{v}) \longrightarrow 1$$

$$T(\alpha \underline{u}) = T(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4)$$
$$= (-\alpha x_3, \alpha x_1 - \alpha x_2 - \alpha x_4, \alpha x_3, \alpha x_3)$$
$$= \alpha(-x_3, x_1 - x_2 - x_4, x_3, x_3)$$
$$= \alpha T(\underline{u})$$

(ii) (2 points) Construct the representation matrix of T.

$$A = \left[T\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \vdots T\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \vdots T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \vdots T\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right]$$

4×4

$$= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(iii) (4 points) Describe the **kernel** of T, by providing a **basis** for it.

$Ker(T) = \{ \underline{v} \in V : T(\underline{v}) = \underline{0}_V \}$

$T(x_1, x_2, x_3, x_4) = (0,0,0,0)$

$(-x_3, x_1 - x_2 - x_4, x_3, x_3) = (0,0,0,0)$

$\rightsquigarrow x_3 = 0$ & $x_1 - x_2 - x_4 = 0$

$\rightsquigarrow x_1 = x_2 + x_4$ x_2, x_4 free

$\therefore Ker(T) = \{ (x_2 + x_4, x_2, 0, x_4) : x_2, x_4 \text{ free} \}$

$= \{ x_2(1,1,0,0) + x_4(1,0,0,1) : x_2, x_4 \text{ free} \}$

$= \text{Span}\{ \underbrace{(1,1,0,0), (1,0,0,1)}_{\text{linearly indep.}} \}$

$\therefore (1,1,0,0), (1,0,0,1)$ form a basis for $Ker(T)$.

$\therefore \text{Nullity}(T) = 2$

(iv) (2 points) Is T injective (1-1)? **Justify.**

No as $Ker(T) \neq \{ \underline{0}_V \}$

For example, $(1,1,0,0) \in Ker(T)$.

(v) (4 points) Describe the **range** of T, by providing a **basis** for it. Any image has the form:

$$(x_3, x_1 - x_2 - x_4, x_3, x_3) = x_3(1,0,1,1) + x_1(0,1,0,0) + x_2(0,-1,0,0) + x_4(0,-1,0,0)$$

$$\therefore \text{Range}(T) = \text{Span}\{\underbrace{\overset{\underline{v}_1}{(1,0,1,1)}, \overset{\underline{v}_2}{(0,1,0,0)}}_{\text{lin. indep.}}, \underbrace{\overset{\underline{v}_3}{(0,-1,0,0)}}_{\text{redundant}}\}$$

$\underline{v}_2 \neq \alpha \underline{v}_1$ for any α $\qquad$ $\underline{v}_3 = -\underline{v}_2$

$\therefore$ $(1,0,1,1), (0,1,0,0)$ form a basis for Range(T)

$\therefore \text{Rank}(T) = 2$

(vi) (2 points) Is T surjective (onto)? **Justify.**

No as $\text{Rank}(T) = 2 \neq \dim(\text{Codomain}) = 4$

Problem 6. (4 points each) True or False (Circle one and state your reason):

(i) The map $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x, y, z) = (x + y + 1, y + z - 1)$$

is a linear transformation.

Reason: True **False**

Any linear transformation $T: V \to W$ must map $\underline{0}_V$ to $\underline{0}_W$

$T(\underline{0}_V) = T(0,0,0) = (0+0+1, 0+0-1)$
$= (1,-1) \neq \underline{0}_W$

$\therefore$ T not linear.

(ii) If A and B are square matrices such that $PAP^{-1} = B$ for some invertible matrix P, then every eigenvalue of A is also an eigenvalue of B.

Reason: **True** False

$|\lambda I - B| = |\lambda I - PAP^{-1}|$
$= |\lambda P I P^{-1} - PAP^{-1}| = |P(\lambda I - A)P^{-1}|$
$= \cancel{|P|}\, |\lambda I - A|\, \cancel{|P^{-1}|}$
$= |\lambda I - A|$

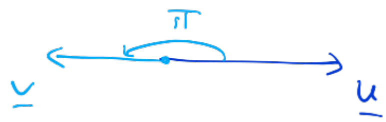
$\rightsquigarrow$ A, B have the same characteristic polyn.
$\rightsquigarrow$ A, B have the same eigenvalues (the roots of the charact. polyn.)

(iii) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space V such that $\langle \mathbf{u}, \mathbf{v} \rangle = -4$, $||\mathbf{u}|| = ||\mathbf{v}|| = 2$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Reason: True False

$$\cos(\theta) = \frac{\langle \underline{u}, \underline{v} \rangle}{||\underline{u}||\,||\underline{v}||} = \frac{-4}{2 \cdot 2} = -1 \rightsquigarrow \theta = \pi$$

$\therefore$ $\underline{u}, \underline{v}$ are // opposite direction



(iv) If A and B are row-equivalent matrices, then they have the same nullspace.

Reason: True False

A, B row-equivalent means that $E_k \cdots E_1 A = B$ for some elementary matrices $E_1, \dots, E_k$

If $\underline{x} \in NS(A)$ then $A\underline{x} = \underline{0}$

$\rightsquigarrow (E_k \cdots E_1 A)\underline{x} = E_k \cdots E_1 \underline{0} = \underline{0}$

$\rightsquigarrow B\underline{x} = \underline{0} \rightsquigarrow \underline{x} \in NS(B)$

Thus $\boxed{NS(A) \subseteq NS(B)}$

Conversely, if $\underline{x} \in NS(B)$ then $B\underline{x} = \underline{0}$

$\rightsquigarrow (E_k \cdots E_1 A)\underline{x} = \underline{0}$

$\rightsquigarrow A\underline{x} = E_1^{-1} \cdots E_k^{-1} \underline{0} = \underline{0}$

$\rightsquigarrow \underline{x} \in NS(A)$ that is $\boxed{NS(B) \subseteq NS(A)}$

Draft: