Problem 1, Part 1. (6 points) Let $V = \mathbb{R}^2$, the set of all ordered pairs (x, y) of real numbers. Define an operation of **addition** and **scalar multiplication** by

$$\begin{aligned}(x, y) \oplus (x', y') &= (x + x' + 1, y + y' - 2),\\ \alpha \odot (x, y) &= (\alpha x, \alpha y),\end{aligned}$$

for all (x, y), $(x', y') \in V$ and $\alpha \in \mathbb{R}$.

The set V is **NOT** a vector space under the operations $\oplus$ and $\odot$.

Determine **only two** of the vector space axioms which fail to hold. **Justify your answer.**

① Distributive Property: Let $c \in \mathbb{R}$, $\vec{u} = (x, y)$, $\vec{v} = (x', y')$

$c(\vec{u} \oplus \vec{v}) = c(x + x' + 1, y + y' - 2)$

$= (cx + cx' + c, y + y' - 2c)$

$c\vec{u} \oplus c\vec{v} = (cx, cy) \oplus (cx', cy')$

$= (cx + cx' + 1, cy + cy' - 2)$

$\neq c(\vec{u} \oplus \vec{v})$

② Distributive property: Let $c, d \in \mathbb{R}$

$(c+d)\vec{u} = ([c+d]x, [c+d]y) = (cx + dx, cy + dy)$

$c\vec{u} \oplus d\vec{u} = (cx, cy) \oplus (dx, dy)$

$= (cx + dx + 1, cy + dy - 2)$

$\neq (c+d)\vec{u}$

Problem 1, Part 2. (6 points) Give an example of the following or state that it does not exist. **Do not justify your answer.**

(i) A set of vectors S that spans $\mathbb{R}^3$ but not linearly independent.

$S = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$

(ii) 2-dimensional subspaces U and W of $\mathbb{R}^3$ that intersect in a line.

$U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ (xy plane)

$W = \{(0, y, z) \mid y, z \in \mathbb{R}\}$ (yz plane)

(iii) A matrix A such that RowSpace(A) = ColumnSpace(A).

Any A s.t $A = A^T$

ex/ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iv) An infinite set of linearly independent vectors in $C(-1, 1)$, which is not a basis for $C(-1, 1)$.

$P_\infty \subseteq C(-1,1) \rightarrow S = \{1, x, x^2, \ldots\}$

(v) A linear transformation $T : \mathbb{R}^2 \to M_{2\times2}$ which is $1-1$ but not onto.

Rank + 0 = 2 $\rightarrow$ Rank = 2 < 4

$T(x, y) = \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$

(vi) A 2×2 matrix A that has every vector in $\mathbb{R}^2$ as an eigenvector.

$A = 0$, $\vec{x} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Problem 2. (8 points) Find the eigenvalues and their algebraic multiplicities, a basis and the dimension of each eigenspace for the 3×3 matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 2 & 1 & -1 \end{pmatrix}.$$

$|\lambda I - A| = 0 \rightarrow \begin{vmatrix} \lambda-1 & -1 & -1 \\ -2 & \lambda & 2 \\ -2 & -1 & \lambda+1 \end{vmatrix} = 0$

$\rightarrow (\lambda-1)\begin{vmatrix} \lambda & 2 \\ -1 & \lambda+1 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ -2 & \lambda+1 \end{vmatrix} - \begin{vmatrix} -2 & \lambda \\ -2 & -1 \end{vmatrix} = 0$

$\rightarrow (\lambda-1)[\lambda(\lambda+1)+2] + [-2(\lambda+1)+4] - [2+2\lambda] = 0$

$\rightarrow (\lambda-1)[\lambda^2+\lambda+2] + [-2\lambda+2] - [2+2\lambda] = 0$

$\rightarrow (\lambda-1)[\lambda^2+\lambda+2] - 2\lambda + \cancel{2} - \cancel{2} - 2\lambda = 0$

$\rightarrow (\lambda-1)[\lambda^2+\lambda+2] - 4\lambda = 0$

$\rightarrow \lambda^3 + \cancel{\lambda^2} + 2\lambda - \cancel{\lambda^2} - \lambda - 2 - 4\lambda = 0$

$\rightarrow \lambda^3 - 3\lambda - 2 = 0$

$\rightarrow (\lambda+1)^2(\lambda-2) = 0$

$\rightarrow \boxed{\lambda_1 = -1 \ (\text{mult } 2)}$

$\rightarrow \boxed{\lambda_2 = 2 \ (\text{mult } 1)}$

$\vec{v}_{\lambda_1}: \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ -2 & -1 & 2 & 0 \\ -2 & -1 & 0 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \rightarrow R_2]{R_2 - R_1 \rightarrow R_2} \left[\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

from R_2, R_3: $\vec{v}_3 = 0$

R_1: $-2\vec{v}_1 - \vec{v}_2 = 0$

$\vec{v}_2 = -2v_1$ (let $v_1 = t$)

$\rightarrow v_2 = -2t$

$\therefore \vec{v}_{\lambda_1} = t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

$\rightarrow \boxed{(\text{Eigenspace } \lambda_1) = \text{span}\left\{ \begin{matrix} 1 \\ -2 \\ 0 \end{matrix} \right\}} \rightarrow \boxed{B_{ES_{\lambda_1}} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right\}}$

$\vec{v}_{\lambda_2}: \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & 2 & 2 & 0 \\ -2 & -1 & 3 & 0 \end{array}\right] \xrightarrow[R_3 + 2R_1 \rightarrow R_3]{R_2 + 2R_1 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right]$

From R_3: $-3\vec{v}_2 + \vec{v}_3 = 0 \rightarrow v_3 = 3v_2$

let $v_2 = t \in \mathbb{R}$

$\rightarrow v_3 = 3t$

From R_1: $v_1 - t - 3t = 0 \rightarrow v_1 = 4t$

$\vec{v}_{\lambda_2} = t \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

$\rightarrow \boxed{(\text{Eigenspace}_{\lambda_2}) = \text{span}\left\{ \begin{pmatrix} 4 \\ +1 \\ 3 \end{pmatrix} \right\} \rightarrow B_{ES_{\lambda_2}} = \left\{ \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \right\}}$

Problem 3, Part 1. (4 points) Find the value(s) of a which will guarantee that A has eigenvalues $0, 3$, and -3.

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -5 & 3 & a \\ 4 & -2 & -1 \end{pmatrix}.$$

$|\lambda I - A| = 0$

$\rightarrow \begin{vmatrix} \lambda+2 & 0 & -1 \\ 5 & \lambda-3 & -a \\ -4 & 2 & \lambda+1 \end{vmatrix} = 0$

$\rightarrow (\lambda+2)\begin{vmatrix} \lambda-3 & -a \\ 2 & \lambda+1 \end{vmatrix} - \begin{vmatrix} 5 & \lambda-3 \\ -4 & 2 \end{vmatrix} = 0$

$\rightarrow (\lambda+2)\left[(\lambda-3)(\lambda+1) + 2a\right]$

$- \left[10 + 4(\lambda-3)\right] = 0$

$\rightarrow (\lambda+2)(\lambda-3)(\lambda+1) + 2(\lambda+2)a - 10 - 4(\lambda-3) = 0$

$\lambda = 0 \rightarrow -6 + 4a - 10 + 2 = 0$

$\rightarrow \boxed{a = 1}$

$\lambda = 3 \rightarrow 10a - 10 = 0$

$\rightarrow \boxed{a = 1}$

$\lambda = -3 \rightarrow -12 - 2a - 10 + 24 = 0$

$\rightarrow \boxed{a = 1}$

$\therefore a = 1$ is the unique value that gaurantees $\lambda = 0, -3, 3$ a eigenvalues for A simultaneously

Problem 3, Part 2. (6 points) Let A be a 2×2 matrix that has eigenvalues 4 and -2. **Cayley-Hamilton Theorem** guarantees that there are constants b_k and c_k for every $k \geq 2$ such that

$$A^k = b_k A + c_k I.$$

Find b_k and c_k for $k = 2$ and 3, and then find a recursive relationship to find b_k and c_k for every $k \geq 2$.

Hint: Show that $b_{n+1} = 2b_n + c_n$ and $c_{n+1} = 8b_n$.

$(\lambda - 4)(\lambda + 2) = 0$

$\rightarrow \lambda^2 - 2\lambda - 8 = 0$

$\underline{CHT} \rightarrow A^2 - 2A - 8I = 0$

$\rightarrow \boxed{A^2 = 2A + 8I} \rightarrow \boxed{\begin{array}{l} b_2 = 2 \\ c_2 = 8 \end{array}}$

$\rightarrow A^3 = AA^2 = A(2A + 8I)$

$= 2A^2 + 8A$

$= 2(2A + 8I) + 8A$

$= 4A + 16I + 8A$

$\rightarrow \boxed{A^3 = 12A + 16I} \rightarrow \boxed{\begin{array}{l} b_3 = 12 \\ c_3 = 16 \end{array}}$

$A^{k+1} = b_{k+1} A + c_k I = AA^k$

$b_{k+1} A + c_{k+1} I = A(b_k A + c_k I)$

$= b_k A^2 + c_k A$

$= b_k (2A + 8I) + c_k A$

$= 2b_k A + 8b_k I + c_k A$

$= (2b_k + c_k)A + 8b_k I$

$\rightarrow b_{k+1} A + c_{k+1} I = (2b_k + c_k)A + 8b_k I$

$\rightarrow \boxed{\begin{array}{l} b_{k+1} = 2b_k + c_k \\ c_{k+1} = 8b_k \end{array}}$

Problem 4. Let $V = \mathcal{M}_{2\times 2}$ be the space of 2×2 matrices, and let $W = \mathcal{P}_{\leq 2}$ be the space of polynomials of degree 2 or less in x.

Let $T : V \longrightarrow W$ be the transformation that transforms any $\underline{\mathbf{v}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V$ to $T(\underline{\mathbf{v}}) = 2a + (b-d)x - (b+c)x^2 \in W$.

(i) (2 points) Find the images of $\underline{\mathbf{u}} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ and $\underline{\mathbf{v}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

$T(\vec{u}) = T\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = 2 - 2x^2$

$T(\vec{v}) = T\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2 - 2x^2$

(ii) (4 points) Show that T is a linear transformation. $\vec{u} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \vec{v} = \begin{pmatrix} c_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$T(\vec{u}+\vec{v}) = 2(a_1+a_2) + (b_1+b_2-d_1-d_2)x - (b_1+b_2+c_1+c_2)x^2$

$= 2a_1 + (b_1 - d_1)x - (b_1+c_1)x^2$
$+ 2a_2 + (b_2-d_2)x - (b_2+c_2)x^2$

$= T(\vec{u}) + T(\vec{v})$

$T(\alpha\vec{u}) = 2\alpha a + (\alpha b - \alpha d)x - (\alpha b + \alpha c)x^2$

$= 2\alpha a + \alpha(b-d)x - \alpha(b+c)x^2$

$= \alpha[2a + (b-d)x - (b+c)x^2]$

$= \alpha T(\vec{u})$

(iii) (4 points) Determine the **kernel** of T, and a **basis** for it.

$$\ker(T) = \{ \vec{v} \in M_{2\times2} \mid T(\vec{v}) = 0 \}$$

$$T(\vec{v}) = 2a + (b-d)x - (b+c)x^2 = 0$$

$$\rightarrow \left.\begin{array}{l} a = 0 \\ b = d \\ b = -c \end{array}\right\} \begin{array}{l} \text{let } b = t \in \mathbb{R} \\ \rightarrow a = 0, \rightarrow c = -t \\ \qquad\qquad \rightarrow d = t \end{array}$$

$$\rightarrow \boxed{\ker(T) = \left\{ \begin{bmatrix} 0 & t \\ -t & t \end{bmatrix} \mid t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}}$$

$$\boxed{B_{\ker(t)} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}}$$

$*$NOTE$*$ $\operatorname{Dim}(\ker(T)) = \underline{1}$

(iv) (3 points) Determine the **range** of T, and a basis for it.

$\text{Rank}(T) + \text{Nullity}(T) = \text{Dim}(M_{2\times 2})$

$\rightarrow \text{Rank}(T) + 1 = 4$

$\rightarrow \text{Rank}(T) = 3 = \text{Dim}(P_{\leq 2})$

$\therefore T$ onto

$\boxed{\therefore \text{Range}(T) = P_{\leq 2}}$

$\boxed{B_{\text{Range}(T)} = \{1, x, x^2\}}$

(v) (3 points) Is T one-to-one? onto? **Justify your answer.**

T is NOT 1–1 as $\text{Dim}(\ker(T)) = 1 \neq 0$

and $\ker(T)$ contains vectors other than $\vec{0}_V$

T is onto since $\text{Rank}(T) = \text{Dim}(P_{\leq 2})$

Problem 5. (4 points each) True or False (Circle one and state your reason):

(i) In any vector space V, there is unique additive inverse for each vector $\underline{\mathbf{v}} \in V$.

Reason: $\boxed{\text{True}}$ False

Assume that $\vec{u}, \vec{w}$ are inverses for $\vec{v}$

$\rightarrow \vec{u} = \vec{u} + 0 = \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} = 0 + \vec{w} = w$

$\boxed{\therefore \vec{u} = \vec{w}}$

(ii) Let V be a 3-dimensional vector space. Then, V has infinitely many subspaces of dimension 2.

Reason: $\boxed{\text{True}}$ False

V is isomorphic to $\mathbb{R}^3$ and $\mathbb{R}^3$ contains infinity many subspaces of dimension two (each of them is a plane through the origin)

(iii) There is a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, not necessarily linear, such that $\text{Ker}(T) = \{\underline{0}\}$ but T is not $1-1$.

Reason:

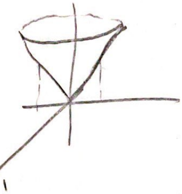
$\boxed{\text{True}}$ False

$T(x,y) = (x^2, y^2)$

Let $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$

$T(\vec{u}+\vec{v}) = ((x_1+x_2)^2, (y_1+y_2)^2)$ clearly not equal to $T(\vec{u})+T(\vec{v})$

Similarly for scaler multiplication.



$T(x,y) = 0$ iff $(x^2, y^2) = 0$

iff $x^2 = 0, y^2 = 0$

iff $x = 0, y = 0$

$\rightarrow \ker(T) = \{0\}$

$T(-1,1) = (1,1)$, $T(1,1) = (1,1)$ $\rightarrow$ T NOT 1-1

(iv) Let A and B be $n \times n$ matrices that commute ($AB = BA$). Then, A and B must share the same eigenvectors.

Reason:

True $\boxed{\text{False}}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$AB = BA$

but $x_A = t(1,1)$ eigenvector for A as $Ax = 1x$ but it is not an eigenvector of B as $x_B = t(0,1)$ is the form for eignvetors of B.