The American University in Cairo
Mathematics and Actuarial Science
Linear Algebra
March 19, 2020

MACT 2132
Spring 2020
Exam 1, V2
Time Limit: 75 Minutes

Name: ______________________ UID: __________

- This exam contains 8 pages (including this cover page and the draft page).
- Answer all the problems.
- Unsupported answers are considered miracles and will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.

Problem	Score	Points
1		7
2		11
3		14
4		20
Total		52

Key

Problem 1. The augmented matrix of a system of linear equations is given by

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & k & k \\ -4 & 3 & 1 \end{bmatrix}$$

a) (1 pts) Determine the number of equations and the number of variables.

b) (6 pts) Find the value(s) of k that make the system consistent, and then solve the system.

(a) $2x_1 - x_2 = 3$
$4x_1 + kx_2 = k$
$-4x_1 + 3x_2 = 1$ $\Longrightarrow$ 2 variables
3 equations

(b) $\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 4 & k & k \\ -4 & 3 & 1 \end{array}\right]$

$\xrightarrow[R_2 - 2R_1 \to R_2]{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -1/2 & 3/2 \\ 0 & k+2 & k-6 \\ -4 & 3 & 1 \end{array}\right] \xrightarrow{R_3 + 4R_1 \to R_3} \left[\begin{array}{cc|c} 1 & -1/2 & 3/2 \\ 0 & k+2 & k-6 \\ 0 & 1 & 7 \end{array}\right]$

$\xrightarrow[k+2 \neq 0]{R_3 - \frac{1}{k+2}R_2 \to R_3} \left[\begin{array}{cc|c} 1 & -1/2 & 3/2 \\ 0 & k+2 & k-6 \\ 0 & 0 & 7 - \frac{k-6}{k+2} \end{array}\right] \Rightarrow$

$x_1 - 1/2 x_2 = 3/2$ ①
$(k+2)x_2 = (k-6)$ ②
$0 = 7 - \frac{k-16}{k+2}$ ③

from ③ : $7 - \frac{k-16}{k+2}$ must be equal to 0 for system to be consistent

$k+2 \neq 0 \to \underline{k \neq -2}$

$7 - \frac{k-6}{k+2} = 0 \to 7 = \frac{k-6}{k+2}$

$\to 7(k+2) = k-6$

$\to 7k + 14 = k - 6$

$\to 6k = -20$

$\to \boxed{k = -10/3}$

② : $x_2 = \frac{k-6}{k+2}$, $k+2 \neq 0$, no restriction here

$\therefore$ k must be $\frac{-10}{3}$ for system to be consistent

Problem 2. Let s, t and u be fixed real numbers. Consider the matrix

$$B = \begin{bmatrix} 1 & s & s^2 \\ 1 & t & t^2 \\ 1 & u & u^2 \end{bmatrix}.$$

a) (4 pts) Show that $|B| = (t-s)(u-s)(u-t)$. $= (tu - ts - su + s^2)(u-t)$
$= tu^2 - t^2u - tsu + t^2s - su^2 + tsu + s^2u - ts^2$

b) (5 pts) Find the inverse of B when $s = 0$, $t = 1$ and $u = 2$ or show that it does not exist.

c) (2 pts) Determine the condition(s) on s, t and u for which B is invertible.

(a) $\det(B) \overset{c_1}{=} 1\begin{vmatrix} t & t^2 \\ u & u^2 \end{vmatrix} - 1\begin{vmatrix} s & s^2 \\ u & u^2 \end{vmatrix} + 1\begin{vmatrix} s & s^2 \\ t & t^2 \end{vmatrix}$

$= tu^2 - ut^2 - su^2 + us^2 + st^2 - ts^2 + \underbrace{(tsu - tsu)}_{=0}$

$= (u-t)\left(\frac{tu(u-t)}{u-t} + \frac{s^2(u-t)}{u-t} - \frac{su(u-t)}{u-t} - \frac{ts(u-t)}{u-t}\right)$

$= (u-t)(tu + s^2 - su - ts)$

$= (u-t)(u-s)(t-s)$

(b) $\det(B) = (1)(2)(1) = 2$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. Compute $cof(B)$

$c_{11} = (-1)^2 \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 2$, $c_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -3$

$c_{13} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$, $c_{21} = \begin{vmatrix} 0 & 0 \\ 2 & 4 \end{vmatrix} = 0$, $c_{22} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = 4$

$c_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = -2$, $c_{31} = 0$, $c_{32} \overset{(-1)^5}{=} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1$,

$c_{33} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$ $\rightarrow cof(B) = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 4 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

$adj(B) = [cof(B)]^T = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

$$\boxed{B^{-1} = \frac{1}{2} adj(A) = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}}$$

(c) $\det(B) \neq 0 \rightarrow (t-s)(u-t)(u-s) \neq 0$

$\rightarrow (t-s) \neq 0$ and $(u-t) \neq 0$ and $(u-s) \neq 0$

$\rightarrow s \neq t$ and $u \neq t$ and $u \neq s$

Problem 3. Let A be the 3×3 matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}.$$

a) (8 pts) Determine the eigenvalues for A and the corresponding eigenvectors.

Characteristic equation: $\det(\lambda I - A) = 0$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} \lambda-1 & -2 & -1 \\ 0 & \lambda-1 & 0 \\ -4 & 0 & \lambda-1 \end{bmatrix} \right| \begin{matrix} \lambda-1 & -2 \\ 0 & \lambda-1 \\ -4 & 0 \end{matrix}$$

$$= (\lambda-1)^3 - 4(\lambda-1) = (\lambda-1)[(\lambda-1)^2 - 4]$$

$$= (\lambda-1)[\lambda^2 - 2\lambda + 1 - 4] = (\lambda-1)(\lambda^2 - 2\lambda - 3)$$

$$= (\lambda-1)(\lambda-3)(\lambda+1) = 0$$

$$\rightarrow \lambda = \{-1, 1, 3\}$$

for $\lambda = -1$:

$$\left[\begin{array}{ccc|c} -2 & -2 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & 0 & -2 & 0 \end{array}\right] \begin{pmatrix} -R_1 \to R_1 \\ -R_2 \to R_2 \\ -R_3 \to R_3 \end{pmatrix}$$

$$= \left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 4 & 0 & 2 & 0 \end{array}\right] \rightarrow x_2 = 0 \text{ from row 2}$$

from ① or ③, $x_1 = -\frac{1}{2}x_3$

let $x_3 = t \in \mathbb{R} - \{0\}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda = 1$:

$$\left[\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 0, \quad -2x_2 = x_3$$

let $x_3 = t \in \mathbb{R} - \{0\}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

for $\lambda = 3$:

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right] \xrightarrow[-R_2 \to R_2]{-\frac{1}{2}R_1 \to R_1}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right]$$

Remark
(you can see that
$x_2 = 0$ and from
① or ③
you can s
$x_1 = 1/2$

$$\xrightarrow{R_3 - 4R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 4R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = 0$ and $x_1 = \frac{1}{2}x_3$

let $x_3 = t \in \mathbb{R} - \{0\}$

$$\to \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}}$$

b) (6 pts) Use Cayley-Hamilton Theorem to express A^5 in terms of A^2, A, and I.

Any square matrix satisfies $|\lambda I - A| = 0$

from (a)

$(\lambda-1)(\lambda^2 - 2\lambda - 3)$

$= \lambda^3 - 2\lambda^3 - 3\lambda - \lambda^2 + 2\lambda + 4$

$= \lambda^3 - 3\lambda^2 - \lambda + 4 = 0$

CHT $\Rightarrow$ $A^3 - 3A^2 - A + 4I = 0$

$\boxed{A^3 = 3A^2 + A + 4I}$

$A^4 = A^3 A = (3A^2 + A + 4I)A$

$= 3A^3 + A^2 + 4A$

$= 3(3A^2 + A + 4I) + A^2 + 4A$

$= 9A^2 + 3A + 12I + A^2 + 4A$

$\Rightarrow \boxed{A^4 = 10A^2 + 7A + 12I}$

$A^5 = A^4 A = (10A^2 + 7A + 12I)A$

$= 10A^3 + 7A^2 + 12A$

$= 10(3A^2 + A + 4I) + 7A^2 + 12A$

$= 30A^2 + 10A + 40I + 7A^2 + 12A$

$A^5 = 37A^2 + 22A + 40I$

Problem 4. (5 pts each) Prove or disprove <u>four</u> of the following.

a) A homogeneous system of three linear equations in two variables has infinitely many solutions.

Reason:

True **False**

$x_1 + x_2 = 0 \rightarrow x_1 = -x_2$

$x_1 - x_2 = 0 \rightarrow x_1 = x_2$

$x_3 = 0$

$\Big\}$ $x_2 = 0$

$x_1 = 0$

$\therefore$ only valid for $(0,0,0)$

b) If A and B are $n \times n$ matrices and A is invertible, then $|(A\,B\,A^{-1})^2| = |B^2|$.

Reason:

True False

$\det(ABA^{-1})^2$

$= [\det(A)\det(B)\det(A^{-1})]^2$

$= [\det(A)\det(B)\frac{1}{\det(A)}]^2$ ($\because A^{-1}$ exists)

$= [\det(B)]^2$

$= \det(B)\det(B)$

$= \det(B^2)$

c) A square matrix can have two different row-echelon forms.

Reason:

True False

The REF of a matrix is a matrix in EF but with certain conditions, so any matrix has an REF and an EF.

ex/ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is in EF

and applying $R_1 + R_2 \rightarrow R_1$

$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

also in EF

d) If $\underline{x}$ is an eigenvector for A relative to an eigenvalue λ, then so is $c\underline{x}$ for any scaler $c \neq 0$.

Reason:

True False

$c \cdot (\lambda I - A)\vec{x} = \vec{0} \cdot c$

$\rightarrow (\lambda I - A) c\vec{x} = \vec{0}$

e) If A and B are 2×2 matrices such that $A + B$ is invertible, then A or B is invertible.

Reason:

True False

Let $A+B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}}_{B}$

$\det(A) = 0$ $\det(B) = 0$

but $\det(A+B) = -2 \neq 0$

$\therefore$ $A+B$ invertible,

but neither A or B invertible

f) If A is a symmetric invertible matrix, then $5A^{-1}$ is symmetric.

Reason:

True False

$A^T = A$

$(A^{-1})^T = (A^T)^{-1} = A^{-1}$

$(5A^{-1})^T = 5(A^T)^{-1} = 5(A)^{-1} = 5A^{-1}$ ✓

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