Fall 2020

Problem 1 $\underline{v}_1 = (4,3,0,0)$, $\underline{v}_2 = (0,5,0,4)$

$\underline{v}_3 = (3,1,0,-1)$, $\underline{v}_4 = (0,0,6,0)$

$\underline{w} = (8,1,3,-4)$

Let $A = [\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3 \mid \underline{v}_4 \mid \underline{w}] = \left[\begin{array}{cccc|c} 4 & 0 & 3 & 0 & 8 \\ 3 & 5 & 1 & 0 & 1 \\ 0 & 0 & 0 & 6 & 3 \\ 0 & 4 & -1 & 0 & -4 \end{array}\right]$ (1)

$\Downarrow$ ERO's (2)

$$\left[\begin{array}{cccc|c} 1 & 0 & 3/4 & 0 & 2 \\ 0 & 1 & -1/4 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ (1)

$\uparrow$ 1st, $\uparrow$ 2nd, $\uparrow$ 4th

$\therefore$ $\underline{v}_1, \underline{v}_2, \underline{v}_4$ form a basis for W

$\boxed{\#\text{Pivots} = 3 \ \therefore \dim(W) = \underline{\underline{3}} \quad \therefore W \neq \mathbb{R}^4 \text{ because } \mathbb{R}^4 \text{ has } \dim = \underline{\underline{4}}}$ (2) (ii)

$\boxed{\underline{w} = 2\,\underline{v}_1 - 1\,\underline{v}_2 + 1/2\,\underline{v}_4}$ (2) (iii)

(i) Clearly, $\underline{v}_4 = (0,0,6,0)$ is orthogonal to $\underline{v}_1 = (4,3,0,0)$ & $\underline{v}_2 = (0,5,0,4)$

$\Downarrow$ We only need to replace $\underline{v}_2$ with $\underline{w}_2 = \underline{v}_2 - \text{Proj}_{\underline{v}_1} \underline{v}_2$ (1)

$$\underline{w}_2 = (0,5,0,4) - \frac{3}{5}(4,3,0,0) = \left(\frac{-12}{5}, \frac{16}{5}, 0, 4\right)$$

$\therefore$ $\underline{v}_1, \underline{w}_2, \underline{v}_4$ form an orthogonal basis for W (1)

Problem 2 (i) $Nul(A) = Nul(B)$

$$B = \begin{array}{c} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array} \quad RREF$$

x_1, x_3, x_4 are basic variables

x_2, x_5 free 1

$x_4 = -2x_5$

$x_3 = x_5$ 1

$x_1 = -x_2 - 8x_5$

$$\therefore \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - 8x_5 \\ x_2 \\ x_5 \\ -2x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -8 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \quad 1$$

$\therefore \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -8 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$ form a basis for $Nul(A)$ 1

(ii) Non-zero row vectors in B form a basis for $Row(A)$:

$\{ (1,1,0,0,8), (0,0,1,0,-1), (0,0,0,1,2) \}$ 2

(iii) The corresponding pivot columns in A form a basis for $Col(A)$: $Col_{1,A} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $Col_{3,A} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 1 \end{bmatrix}$, $Col_{4,A} = \begin{bmatrix} 3 \\ 8 \\ 10 \\ 2 \end{bmatrix}$ 2

(iv) 1 False $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis for $Col(B)$

$\Downarrow$ any vector in $Col(B)$ has a zero in the last compo.

This is not the case for vectors in $Col(A)$

$\therefore Col(A) \neq Col(B)$ 2

Problem 3 $\quad T(a_0+a_1x+a_2x^2) = 3a_0 - a_1 + a_2$

(i) let $\underline{u} = a_0 + a_1 x + a_2 x^2$

$\underline{v} = b_0 + b_1 x + b_2 x^2$, c scaler

- $T(\underline{u}+\underline{v}) = T\left[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2\right)$

$= 3(a_0+b_0) - (a_1+b_1) + (a_2+b_2)$

$= (3a_0 - a_1 + a_2) + (3b_0 - b_1 + b_2)$ (2)

$= T(\underline{u}) + T(\underline{v})$ Axiom 1

- $T(c\underline{u}) = T\left((ca_0) + (ca_1)x + (ca_2)x^2\right)$

$= 3(ca_0) - (ca_1) + (ca_2)$

$= c(3a_0 - a_1 + a_2)$

$= c\,T(\underline{u})$ Axiom 2 (1)

T satisfies Axiom 1, Axiom 2 $\therefore$ T is a L.T.

(ii) $T(a_0 + a_1 x + a_2 x^2) = 0$ iff $3a_0 - a_1 + a_2 = 0$ (1)

iff $a_1 = 3a_0 + a_2$

$\therefore Ker(T) = \{ a_0 + (3a_0+a_2)x + a_2x^2 : a_0, a_2 \text{ free} \}$ (1)

$\Downarrow \; a_0(1+3x) + a_2(x+x^2)$ (1)

$\therefore$ the two vectors $1+3x$ & $x+x^2$ form a basis for Ker(T)

(iii) Ker(T) has dimension 2

$\therefore$ Nullity $= 2$ (1)

$\therefore$ Rank $= \dim(P_2) - 2 = 3 - 2 = 1 = \dim(\text{Codomain})$

$\therefore$ T onto. (1)

Problem 4

(i) $\underline{0} = 0 \odot \underline{u}$

$= 0a_0 + (0a_1 - 3\cdot 0 + 3)x + (0a_2 - 7\cdot 0 + 7)x^2$

$= \boxed{3x + 7x^2}$ (2)

(ii) $-\underline{u} = -1 \cdot \underline{u}$

$= -1a_0 + (-a_1 + 3 + 3)x + (-a_2 + 7 + 7)x^2$

$= \boxed{-a_0 + (-a_1 + 6)x + (-a_2 + 14)x^2}$ (2)

(iii) $\underline{u} = a_0 + a_1 x + a_2 x^2$ & $\underline{v} = b_0 + b_1 x + b_2 x^2$

$LHS = \lambda \odot (\underline{u} \oplus \underline{v})$

$= \lambda \odot \left((a_0 + b_0) + (a_1 + b_1 - 3)x + (a_2 + b_2 - 7)x^2 \right)$

$= \lambda(a_0 + b_0) + [\lambda(a_1 + b_1 - 3) - 3\lambda + 3]x + [\lambda(a_2 + b_2 - 7) - 7\lambda + 7]x^2$

$= \lambda(a_0 + b_0) + [\lambda a_1 + \lambda b_1 - 6\lambda + 3]x + [\lambda a_2 + \lambda b_2 - 14\lambda + 7]x^2$ (2)

$\longrightarrow (*)$

$RHS = (\lambda \odot \underline{u}) + (\lambda \odot \underline{v})$

$= \lambda a_0 + (\lambda a_1 - 3\lambda + 3)x + (\lambda a_2 - 7\lambda + 7)x^2$

$\oplus$

$\lambda b_0 + (\lambda b_1 - 3\lambda + 3)x + (\lambda b_2 - 7\lambda + 7)x^2$

$\therefore (*) = (**)$

(2)

$= (\lambda a_0 + \lambda b_0) + [(\lambda a_1 - 3\lambda + 3) + (\lambda b_1 - 3\lambda + 3) - 3]x$

$+ [(\lambda a_2 - 7\lambda + 7) + (\lambda b_2 - 7\lambda + 7) - 7]x^2$

$= \lambda(a_0 + b_0) + (\lambda a_1 + \lambda b_1 - 6\lambda + 3)x + (\lambda a_2 + \lambda b_2 - 14\lambda + 7)x^2$

$\longrightarrow (**)$

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Problem 5 (i) False 1

$(0,1,1)$ & $(0,1,-1) \in W$

But $(0,1,1)+(0,1,-1) = (0,2,0) \notin W$

because $(0,2,0) \neq (0,a^2,a^3)$ for any $a \in \mathbb{R}$

$\therefore$ W Not closed under Add

$\therefore$ W Not a subspace

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(ii) True 1 Rank + Nullity $= \dim(M_{2\times4}) = 8$

$\therefore$ max. Rank $= 8$

3

$\therefore$ Rank $\neq 9 = \dim(M_{3\times3})$

$\therefore$ T never Onto.

(iii) False 1 Counter example:

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(\underline{v}) = \underline{0}$ is a L.T. 3

$\Downarrow$ $T(\underline{v}_1), T(\underline{v}_2), T(\underline{v}_3)$ all are $\underline{0}$

$\therefore$ Not linearly indep.

(iv) True 1 $\{x^2, x+x^2, 1+x+x^2\}$

if $c_0 x^2 + c_1(x+x^2) + c_2(1+x+x^2) = 0$

$\therefore$ $c_0 + c_1 + c_2 = 0$ 3

$c_1 + c_2 = 0$

$c_2 = 0$

$\Downarrow$ $c_1 = c_2 = c_3 = 0$ $\therefore$ linearly indep.