Fall 22 Original

Tuesday, November 29, 2022 10:23 PM

Linear Algebra
Exam 2
November 23, 2022

Name: ______________________ **UID:** ____________________

- **The exam consists of FIVE problems.**
- **Unsupported answers will receive little or no credit.**
- **Anyone caught writing after time has expired will be given a mark of zero.**
- **Time: 75 minutes.**

Problem	Score	Points
1		6
2		6
3		8
4		10
5		12
Total		42

Best wishes!

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Problem 1. (6 points) Determine if each of the following sets is a subspace.

(i) Is $W = \{(x, y, z) \in \mathbb{R}^3 : 2\,x\,z = y\}$ a subspace of $(\mathbb{R}^3, +, \cdot)$ with the standard addition and multiplication?

- W is not a subspace, as it is not closed under scalar multiplication.

To see that, consider $w = (1,2,1)$. and $\alpha = 2$.

Then $w \in W$, as $2 \cdot 1 \cdot 1 = 2$.

Nevertheless, $2 \cdot w = (2,4,2) \notin W$, as $2 \cdot 2 \cdot 2 = 8 \neq 4$.

(ii) Is $W = \{f(x) \in \mathcal{F}(-\infty, \infty) \,:\, f(-x) = -f(x)\}$ a subspace of $(\mathcal{F}(-\infty, \infty), +, \cdot)$ with the standard addition and multiplication?

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It is a subspace, as

1) W contains $id : \mathbb{R} \to \mathbb{R}$ defined via $id(x) = x$.

2) W is closed under scalar multiplication and addition, as for $f, g \in W$ and $\alpha \in \mathbb{R}$ we get

$$(\alpha \cdot f + g)(-x) = \alpha \cdot f(-x) + g(-x)$$

$$= -\alpha f(x) - g(x) \quad , \text{as } f, g \in W$$

$$= -(\alpha f + g)(x).$$

Hence $\alpha f + g \in W$.

Problem 2. Consider the set of vectors

$$S = \{1 - x, \, x - x^3, \, 1 + x - kx^2 + x^3\}$$

in $\mathcal{P}_{\leq 3}$, the space of polynomials in x of degree ≤ 3.

(i) (4 points) Determine the value(s) of k for which S is linearly independent. ↳ For any $k \in \mathbb{R}$, S is lin. independent.

Proof: S is lin. ind. iff the linear equation

(*) $c_1(1-x) + c_2(x - x^3) + c_3(1 + x - kx^2 + x^3) = 0$ implies $c_1 = c_2 = c_3$.

Note that (*) holds

iff $(c_1 + c_3) + (-c_1 + c_2 + c_3)x + (-kc_3)x^2 + (-c_2 + c_3)x^3 = 0$

iff I $c_1 + c_3 = 0$ and II $-c_1 + c_2 + c_3 = 0$ and III $(-kc_3) = 0$ and

IV $(-c_2 + c_3) = 0$ iff

iff I $c_1+c_3=0$ and II $-c_1+c_2+c_3=0$ and III $(-kc_3)=0$ and

IV $(-c_2+c_3)=0$ iff $\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & -k \\ 0 & -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$A \cdot \underline{c} = \underline{0}$

Now $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & -k \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -k \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -k \end{pmatrix} \xrightarrow[R_4 \to R_4 + \frac{k}{3} R_3]{R_3 \to \frac{1}{3} R_3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ in REF.

Hence, independently of k, there are as many pivots as columns, whence the system $A\underline{c}=\underline{0}$ only has the trivial solution $c_1=c_2=c_3=0$.

(ii) (2 points) Can S be a basis for $\mathcal{P}_{\leq 3}$? Justify your answer.

S cannot be a basis for $\mathcal{P}_{\leq 3}$, as

$\dim(\mathcal{P}_{\leq 3}) = 4$ and $|S| = 3$.

By definition, the dimension of a vector space is the size of any of its bases.

Problem 3. (8 points) Consider the set

$$\mathbb{R}^2 = \{(x,y) \,:\, x, y \text{ real numbers}\},$$

equipped with the following addition and scalar multiplication:

$$\begin{aligned} (x,y) + (x',y') &= (x\,x', y\,y') \\ \alpha \cdot (x,y) &= (\alpha\,x + 1, \alpha\,y + 1) \end{aligned}$$

The triple $(\mathbb{R}^2, +, \cdot)$ is **not** a vector space because ...

Choose all answers that apply and justify why the property fails.

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(i) it does not satisfy the additive commutativity axiom,

(ii) it does not satisfy the additive identity axiom,

(iii) it does not satisfy the scalar multiplicative identity axiom,

(iv) it does not satisfy the compatibility axiom for scalar multiplication, $2(2(1,0)) = 2\cdot(3,1) = (7,3) \neq 4(1,0) = (5,1)$

(v) it does not satisfy the additive inverse axiom.

Properties (iii), (iv) and (v) fail, while (i) and (ii) hold.
Note (for v) that the additive identity is $(1,1)$.
Note further that we prove their failure through counter examples.

iii) Consider $(x,y) = (0,0)$. Then $1\cdot(0,0) = (1\cdot 0+1, 1\cdot 0+1) = (1,1) \neq (0,0)$.
Hence $1(x,y) = (x,y)$ is not true for all $(x,y) \in \mathbb{R}^2$.

iv) Consider $(x,y) = (1,0)$ and $\alpha = \beta = 1$. Then
$(\alpha\cdot\beta)(x,y) = 1\cdot(1,0) = (2,1)$ and
$(\alpha(\beta(x,y)) = 1\cdot(1(1,0)) = 1\cdot(2,1) = (3,2)$. As $(2,1) \neq (3,2)$, we see
that $(\alpha\cdot\beta)(x,y) \neq \alpha(\beta(x,y))$ for all $(x,y) \in \mathbb{R}^2$, $\alpha,\beta \in \mathbb{R}$.

v) Consider $(x,y) = (0,0)$. Then for all $(x',y') \in \mathbb{R}^2$, we get $(x,y)+(x',y') =$
$= (0,0) + (x,y) = (0\cdot x, 0\cdot y) = (0,0) \neq \mathbf{0} = (1,1)$. Hence $(0,0)$ does not have an inverse.

Problem 4. Suppose that the following two matrices are row-equivalent

$$A = \begin{bmatrix} 1 & 2 & a \\ 1 & 1 & b \\ 1 & 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(i) (6 points) Describe NullSpace(A) and RowSpace(A) as vector

(i) (6 points) Describe NullSpace(A) and RowSpace(A) as vector spaces by finding a basis and the dimension of each of them.

Note that both the null space and the row space of any matrix stay invariant under ERO's, whence NullSpace(A) = NullSpace(B) and RowSpace(A) = RowSpace(B).

Hence: 1) A basis for RowSpace(A) is given by $\{(1,0,-3),(0,1,1)\}$. and dim(RowSpace(A)) = 2.

2) For NullSpace, we solve $B \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0}$. Set $x_3 := s$. Then $x_2 = -s$ and $x_1 = 3s$. Hence,

NullSpace(A) $= \left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} s \mid s \in \mathbb{R} \right\} = \text{span}\left(\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\} \right)$, so a basis is given by $\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$. Further, dim(NullSpace(A)) = 1.

(ii) (2 points) Find the column vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

We use that $A \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \underline{0}$, i.e.

$$\begin{pmatrix} 1 & 2 & a \\ 1 & 1 & b \\ 1 & 0 & c \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+a \\ 2+b \\ 3+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This holds iff $a = -1$, $b = -2$, $c = -3$, so $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$.

(iii) (2 points) For A and B as given above, are ColSpace(A) and ColSpace(B) equal? Justify.

(iii) (2 points) For A and B as given above, are ColSpace(A) and ColSpace(B) equal? Justify.

No, they are not equal.

To see that, note that

$$\text{Colspace}(B) = \text{span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\1\\0\end{pmatrix}\right\}\right)$$

$$= \left\{\begin{pmatrix}s\\t\\0\end{pmatrix} \,\middle|\, s, t \in \mathbb{R}\right\}.$$

Now clearly, the first column of A, $\begin{pmatrix}1\\1\\1\end{pmatrix}$, which is in Colspace($A$), is <u>not</u> in Colspace(B), as the last component is not 0. So,

$$\text{Colspace}(A) \neq \text{Colspace}(B).$$

Problem 5. (4 points each) True or False? Circle one and prove your answer.

(i) The map $\langle\, ,\, \rangle$ given by

$$\langle (x, y, z)\,,\, (x', y', z') \rangle = x\,x' + y\,y' + (z + z')^2$$

is an inner product on $\mathbb{R}^3$.

Reason: True False

It is not compatible with scalar multiplication:
Consider $(0,0,1),(0,0,0) \in \mathbb{R}^3$ and $2 \in \mathbb{R}$. Then

$$\begin{aligned} \langle 2\cdot(0,0,1),(0,0,0)\rangle &= \langle (0,0,2),(0,0,0)\rangle \\ &= 4 \\ &\neq 2 \\ &= 2\cdot\langle (0,0,1),(0,0,0)\rangle \end{aligned}$$

(ii) Let $(V, +, \cdot)$ be a vector space such that $\dim(V) = n < \infty$. For any set S containing exactly n many vectors, if S is linearly independent, then S must span V.

Reason: True False

We do a proof by contradiction:
Assume $\dim(V)=n$, $S \subseteq V$ is linearly independent, $|S|=n$, but S does not span V. Then ex. $v \in V$ with $v \notin \text{span}(S)$.
Then $S \cup \{v\}$ is still lin. independent.
But $|S \cup \{v\}| = n+1 > \dim(V)$, whence by a Lemma

But $|S \cup \{v\}| = n+1 > \dim(V)$, whence by a Lemma from class implies that S is lin. dependent. This contradicts our assumptions. Hence, S must span V.

(iii) Let $A \in M_{m\times n}$. If $m > n$, then the m-many rows of A cannot form a basis for $\mathbb{R}^m$.

Reason: True False

Way 1): If $A \in M_{m\times n}$, then the rows have n-many entries, i.e. are elements of $\mathbb{R}^n$. As $m > n$, the set of rows is no subset of $\mathbb{R}^m$, hence cannot be a basis for it.

Way 2): The m-many rows cannot be linearly independent: They are lin. ind. iff $\operatorname{rank}(A) = m$. But $\operatorname{rank}(A) \leq n < m$, hence they cannot be lin. independent.

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