Final Exam
December 12, 2020

Name: ____________________ UID: ________________

- The exam consists of FIVE problems.
- Unsupported answers will receive little or no credit.
- Points will be deduced if you continue writing after time has expired.
- Time: 100 minutes.

Problem	Score	Points
1		10
2		13
3		10
4		10
5		20
Total		63

Problem 1. (5 pts each) Find a basis for each of the following vector spaces. **Justify your answer.**

(i) The subspace of all lower triangular 3×3 matrices.

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{v_2} + c\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{v_3} + d\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{v_4}$$

$$+ e\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{v_5} + f\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{v_6}$$ 2

$\therefore \{v_1, v_2, v_3, v_4, v_5, v_6\}$ spans 1 the vector space

Also, it is linearly indep. being part of the standard basis for $M_{3\times3}$. 2

So it is a basis for the given space.

(ii) The subspace of all polynomials $p(x)$ of degree ≤ 3 such that $p(2) = 0$.

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$P(2) = 0 \Rightarrow a_0 + 2a_1 + 4a_2 + 8a_3 = 0$

$\Rightarrow a_0 = -2a_1 - 4a_2 - 8a_3$, a_1, a_2, a_3 free 2

$\therefore P(x) = a_1(-2+x) + a_2(-4+x^2) + a_3(-8+x^3)$

So $\{\underbrace{-2+x}_{v_1}, \underbrace{-4+x^2}_{v_2}, \underbrace{-8+x^3}_{v_3}\}$ spans the space 2

Also, they're linearly indep. because none of them is a linear comb. 1 of the other two for degree issues.

Problem 2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation that satisfies

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix},\ T\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix},\ T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}.$$

(i) (5 pts) Find a matrix A such that $A\begin{pmatrix}x\\y\\z\end{pmatrix} = T\begin{pmatrix}x\\y\\z\end{pmatrix}$.

$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$

$T\begin{pmatrix}0\\1\\0\end{pmatrix} = T\left[\begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\right]$

$= T\begin{pmatrix}1\\1\\0\end{pmatrix} - T\begin{pmatrix}1\\0\\0\end{pmatrix}$

3

$= \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$

$\therefore A = \begin{bmatrix}1 & 0 & 1\\2 & -1 & -2\end{bmatrix}$ 2

the standard matrix representation for T

$T\begin{pmatrix}0\\0\\1\end{pmatrix} = T\left[\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}\right] = T\begin{pmatrix}1\\0\\1\end{pmatrix} - T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

(ii) (4 pts) Determine the **image** of T, and a basis for it.

$\begin{bmatrix}1 & 0 & 1\\2 & -1 & -2\end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2} \begin{bmatrix}1 & 0 & 1\\0 & -1 & -4\end{bmatrix} \xrightarrow{-R_2 \to R_2} \begin{bmatrix}1 & 0 & 1\\0 & 1 & 4\end{bmatrix}$ 1

Image $=$ Col(A) $=$ span $\underbrace{\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}\right\}}_{\text{basis}}$ 2 $= \mathbb{R}^2$ 1

(iii) (4 pts) Determine the **kernel** of T, and a basis for it.

Kernel $=$ Null(A) 1

$=$ span $\underbrace{\left\{\begin{bmatrix}-1\\-4\\1\end{bmatrix}\right\}}_{\text{basis}}$ 1

$\begin{bmatrix}1 & 0 & 1 & | & 0\\0 & 1 & 4 & | & 0\end{bmatrix}$

$x_1 \quad x_2 \quad x_3$

x_3 free

$x_2 = -4x_3$ 2

$x_1 = -x_3$

$\underline{x} = x_3\begin{bmatrix}-1\\-4\\1\end{bmatrix}$

Problem 3.

(i) (6 pts) Determine the eigenvectors for the matrix $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$, and determine the **angle** between linearly independent eigenvectors.

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -3 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2) = 0$$

Eigenvalues are $\lambda = 1, 2$

For $\lambda = 1$

$$\begin{bmatrix} 0 & -3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_2 = 0$, x_1 free

$\therefore$ Eigenvectors are 2

$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_1 \neq 0$

Length $= |x_1|$

For $\lambda = 2$

$$\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 = 3x_2$, x_2 free

$\therefore$ Eigenvectors are 2

$x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $x_2 \neq 0$

Length $= \sqrt{10}\,|x_2|$

$\cos(\theta) = \dfrac{x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}}{|x_1|\,|x_2|\sqrt{10}}$ 2

$= \dfrac{3x_1x_2}{\sqrt{10}\,|x_1x_2|} = \pm\dfrac{3}{\sqrt{10}}$

$\therefore \theta = \cos^{-1}\left(\pm \dfrac{3}{\sqrt{10}}\right)$

(ii) (4 pts) Find the characteristic polynomial and eigenvalues of the following matrix (**You don't need to find the Eigenvectors for** A**!**).

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}.$$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 & 0 \\ -2 & \lambda-1 & 0 \\ -2 & -1 & \lambda-1 \end{vmatrix} \quad 1$$

$$\overset{\text{col } 3}{=} (\lambda-1)\begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} \quad 1$$

$$= (\lambda-1)\left[(\lambda-1)^2 - 4\right] \quad 1$$

$$= (\lambda-1)\,(\lambda-1-2)(\lambda-1+2)$$

$$= (\lambda-1)(\lambda-3)(\lambda+1) \quad 1$$ the c/c poly.

$\therefore$ Eigenvalues are $\lambda = 1, 3, -1$.

Problem 4. (5 pts each) Let $(V, \langle\, , \rangle)$ be an inner product space.

(i) For $\underline{u}, \underline{v} \in V$, show that $\underline{u} + \underline{v}$ is orthogonal to $\underline{u} - \underline{v}$ if and only if $\underline{u}$ and $\underline{v}$ have the same length.

$\underline{u}+\underline{v} \perp \underline{u}-\underline{v}$ iff $\langle \underline{u}+\underline{v}, \underline{u}-\underline{v} \rangle = 0$ 1

iff $\langle \underline{u}, \underline{u} \rangle - \cancel{\langle \underline{u}, \underline{v} \rangle} + \cancel{\langle \underline{v}, \underline{u} \rangle}$ 1 $- \langle \underline{v}, \underline{v}$

iff $\langle \underline{u}, \underline{u} \rangle - \langle \underline{v}, \underline{v} \rangle = 0$ 1

iff $\|\underline{u}\|^2 - \|\underline{v}\|^2 = 0$ 1

iff $\|\underline{u}\| = \|\underline{v}\|$ 1

(ii) Suppose that $B = \{\underline{u}_1, \underline{u}_2\}$ is an orthonormal basis of V, and that $\underline{w}$ is a vector in V satisfying $\langle \underline{w}, \underline{u}_1 \rangle = -2$ and $\langle \underline{w} + \underline{u}_1, \underline{u}_2 - \underline{u}_1 \rangle = 3$. Find $[\underline{w}]_B$ (the coordinates of $\underline{w}$ with respect to the basis B).

B orthonormal $\Rightarrow \langle \underline{u}_1, \underline{u}_1 \rangle = 1 = \langle \underline{u}_2, \underline{u}_2 \rangle$

$\langle \underline{u}_1, \underline{u}_2 \rangle = 0$

$3 = \langle \underline{w} + \underline{u}_1, \underline{u}_2 - \underline{u}_1 \rangle \overset{1}{=} \langle \underline{w}, \underline{u}_2 \rangle - \langle \underline{w}, \underline{u}_1 \rangle + \langle \underline{u}_1, \underline{u}_2 \rangle - \langle \underline{u}_1, \underline{u}_1 \rangle$

$\overset{1}{=} \langle \underline{w}, \underline{u}_2 \rangle - (-2) + 0 - 1$

so $\langle \underline{w}, \underline{u}_2 \rangle \overset{1}{=} 3 + 1 - 2 = 2$

Now, $[\underline{w}]_B \overset{2}{=} \begin{bmatrix} \langle \underline{w}, \underline{u}_1 \rangle \\ \langle \underline{w}, \underline{u}_2 \rangle \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Problem 5. (5 pts each) Prove or disprove the following statements.

(i) The set of all 2×2 orthogonal matrices is a subspace of $M_{2\times2}$.

Reason: True **False** 2

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ are orthogonal matrices (columns are orthonormal).
However, $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ not orthogonal because the column $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not a unit vector. 3
$\therefore$ The set is not closed under Addition.

(ii) If A is a symmetric matrix then all of its eigenvalues are distinct.

Reason: True **False** 2

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is symmetric and has only one repeated eigenvalue $\lambda = 1$
3

(iii) There are exactly two unit vectors in $\mathbb{R}^3$ which are orthogonal to the vector $(1, 2, 0)$.

Reason: True False ?

For a vector $\underline{v} = (a, b, c)$ to be $\perp$ $(1,2,0)$, we should have $a + 2b = 0$

so $\underline{v} = b(-2,1,0) + c(0,0,1)$ b, c free

thus there exist ∞-many vectors that are orthogonal to $(1,2,0)$. For instance,

$\underline{v}_1 = (-2,1,0)$ $b=1, c=0$ so $\underline{u}_1 = \frac{1}{\sqrt{5}} \underline{v}_1$ 3

$\underline{v}_2 = (0,0,1)$ $b=0, c=1$

$\underline{u}_2 = \frac{1}{1} \underline{v}_2$ are 3 unit vectors

$\underline{v}_3 = (-2,1,1)$ $b=1, c=1$

$\underline{u}_3 = \frac{1}{\sqrt{6}} \underline{v}_3$ $\perp$ $(1,2,1)$

(iv) If a square matrix A is invertible, then it is diagonalizable.

Reason: True False 2

For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible because $\det(A) = 1 \neq 0$.

$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2$

so $\lambda = 1, 1$ eigenvalues

For eigenvectors: $(\lambda I - A)\underline{x} = \underline{0}$ 3

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $x_2 = 0$, x_1 free

so the eigenspace has dimension one $\neq$ two

so Not diagonalizable.

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