Exam 2, Fall 2018

Problem 1 Any vector in W has the form

$(2s-t+4r, s+r, t-2r)$

$= (2s, s, 0) + (-t, 0, t) + (4r, r, -2r)$

$= s(2,1,0) + t(-1,0,1) + r(4,1,-2)$

$= s \cdot \underline{v}_1 + t \cdot \underline{v}_2 + r \cdot \underline{v}_3$

$\therefore W = \text{Span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

$\therefore W$ is a subspace of $\mathbb{R}^3$ being the span of a finite set of vectors in $\mathbb{R}^3$.

Second, we need to check the linear independence of $\underline{v}_1 = (2,1,0)$, $\underline{v}_2 = (-1,0,1)$, $\underline{v}_3 = (4,1,-2)$ to find the exact dimension of W.

$$A = \begin{array}{c} \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ \underline{v}_1 \;\; \underline{v}_2 \;\; \underline{v}_3 \end{array} \xRightarrow{\text{ERO's}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \underline{v}_1, \underline{v}_2$ are pivot columns in A, so they form a basis for W. Therefore, $\dim W = 2$.

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Problem 2 :

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 0 & -12 \\ 2 & 1 & 1 & -3 \end{bmatrix} \xrightarrow{R_4 - 2R_1 \to R_4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 3 & 0 & -12 \\ 0 & 1 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow[R_4 + R_2 \to R_4]{R_3 + 3R_2 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{3}R_3 \to R_3]{-R_2 \to R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 + R_3 \to R_2]{R_1 - R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = REF.$$

(a) Non-Zero rows in REF form a basis for $RS(A)$:

$\therefore \{(1,0,0,1), (0,1,0,-4), (0,0,1,-1)\}$ basis for $RS(A)$.

(b) the system $A\underline{x} = \underline{b}$ is consistent iff $\underline{b}$ belongs to the Column space $CS(A)$. Because

$\underline{b} = \begin{bmatrix} 0 \\ 3 \\ -12 \\ -3 \end{bmatrix}$ is an obvious memeber of $CS(A)$ (it is the 4^{th} column in A)

, then the system is Consistent.

(c) to find basis for $NS(A)$, we solve the system

$$A\underline{x} = \underline{0} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Row 4 : $0 = 0$ ✓

Row 3 : $x_3 = x_4$, x_4 free

Row 2 : $x_2 = 4x_4$

Row 1 : $x_1 = -x_4$

$$\therefore \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ 4x_4 \\ x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}, \quad x_4 \text{ free}$$

$$\therefore NS(A) = \text{Span}\Big\{ \underbrace{\begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}}_{\text{basis.}} \Big\}$$

(d) $\text{Rank}(A) = \dim(RS(A)) = \underline{\underline{3}}$

$\text{Nullity}(A) = \dim(NS(A)) = \underline{\underline{1}}$

• Problem 3

[a] $$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^t$$

$C_{11} = (-1)^{1+1} \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = 20$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 0 \\ -2 & 4 \end{vmatrix} = 0$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 5 \\ -2 & 0 \end{vmatrix} = 10$

$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & -2 \\ 0 & 4 \end{vmatrix} = 0$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ -2 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & -2 \\ 5 & 0 \end{vmatrix} = 10$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} = 0$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} = 5$

$$\therefore \operatorname{adj}(A) = \begin{bmatrix} 20 & 0 & 10 \\ 0 & 0 & 0 \\ 10 & 0 & 5 \end{bmatrix}^t = \begin{bmatrix} 20 & 0 & 10 \\ 0 & 0 & 0 \\ 10 & 0 & 5 \end{bmatrix}$$

[b] $$0 = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & 2 \\ 0 & \lambda-5 & 0 \\ 2 & 0 & \lambda-4 \end{vmatrix} = (\lambda-5) \begin{vmatrix} \lambda-1 & 2 \\ 2 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-5)\left[(\lambda-1)(\lambda-4) - 4\right] = (\lambda-5)(\lambda^2 - 5\lambda)$$

$$= (\lambda-5)\lambda(\lambda-5)$$

$\therefore$ eigenvalues are $\lambda = 0, 5, 5$

[c] $\lambda = 0$ is an eigenvalue for A, then A is Singular

$\therefore$ Cramer's rule doesn't apply.

- Problem 4 :

(a) True

$$\begin{aligned} Rank(A) &= \dim\left(Rowspace(A)\right) \\ &= \dim\left(\left(Columnspace(A^t)\right)\right) \\ &= Rank(A^t) \end{aligned}$$

(b) False

$\underline{u} = x^3+1$ $\quad$ 3^{rd} degree Polynomial

$\underline{v} = -x^3+1$ $\quad$ 3^{rd} degree Polynomial

But $\underline{u}+\underline{v} = 2$ not 3^{rd} degree Polynomial

$\therefore$ this set of vectors is not closed under Add.

$\therefore$ Not a Subspace

(c) False $\quad$ For example, the three vectors

$\underline{v}_1 = (1,0)$, $\underline{v}_2 = (0,0)$, $\underline{v}_3 = (0,-1)$

are linearly dependent (one of them is $\underline{0}$)

However, the two vectors

$\underline{v}_1 + \underline{v}_2 = (1,0)$, $\quad \underline{v}_2 - \underline{v}_3 = (0,1)$

are linearly independent (the standard basis of $\mathbb{R}^2$)

(d) True $\quad$ $\therefore \lambda = 0$ is an eigenvalue

$\therefore |0 \cdot I - A| = 0$

$\therefore |-A| = 0$

$\therefore \pm|A| = 0$

$\therefore |A| = 0$

$\therefore A$ Singular.
