Problem 1. Consider the matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

(i) (12 pts) Find the eigenvalues and the corresponding eigenspaces of A.

$$\det(\lambda I - A) = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right| = 0$$

$$= \left| \begin{bmatrix} \lambda-1 & -1 & 1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{bmatrix} \right| = (\lambda-1) \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -1 & \lambda-1 \end{vmatrix} + \begin{vmatrix} -1 & \lambda-1 \\ -1 & -1 \end{vmatrix}$$

$$= (\lambda-1)[(\lambda-1)^2 - 1] + (1-\lambda-1) + (1-(1-\lambda))$$

$$= (\lambda-1)[\lambda^2 - 2\lambda + \cancel{1} - \cancel{1}] + (-\cancel{\lambda}) + \cancel{\lambda} = (\lambda-1)(\lambda(\lambda-2)) = 0$$

$\rightarrow$ $\lambda = 0,\ \lambda = 1,\ \lambda = 2$

For $\lambda = 0$: $\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{matrix} -R_1 \to R_1 \\ -R_2 \to R_2 \\ -R_3 \to R_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

we see that $x_3 = 0$, $x_1 + x_2 = 0 \to x_1 = -x_2$

Let $x_2 = t \in \mathbb{R} \to x_1 = -t \to$ $\boxed{\vec{x}_{\lambda=0} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$

For $\lambda = 1$: $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{matrix} -R_1 \to R_1 \\ -R_2 \to R_2 \\ -R_3 \to R_3 \end{matrix} \Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3}$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ we see that $x_2 = x_3$, $x_1 = -x_3$; Let $x_3 = t \in \mathbb{R}$ $\boxed{\vec{x}_{\lambda=1} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$

(ii) (3 pts) Is A diagonalizable? why?

For all eigenvalues, the multiplicity of the eigenvalues = 1 = dimension of the eigenspace

$\therefore$ A diagonalizable.

For $\lambda = 2$: $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

$\xrightarrow[R_3 + R_1 \to R_3]{R_2 + R_1 \to R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -2 & 2 \end{bmatrix} \left(\begin{matrix} x_1 - x_2 + x_3 = 0 & (1) \\ -2x_2 + 2x_3 = 0 & (2) \end{matrix} \right.$

we see that $x_2 = x_3$ from (2)

Plugging into (1) we get $x_1 = 0$, let $x_3 = t \in \mathbb{R}$

$\boxed{\therefore \vec{x}_{\lambda=2} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$

$(\text{Eigenspace})_{\lambda=0} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$

$(\text{Eigenspace})_{\lambda=1} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$

$(\text{Eigenspace})_{\lambda=3} = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

Problem 2. Let $\mathcal{P}_3$ be the space of all polynomials in x of degree ≤ 3. Define a linear transformation $T : \mathcal{P}_3 \to \mathbb{R}^3$ by

$$T(p(x)) = (p(-1), p(0), p(1)).$$

(i) (5 pts) Determine a basis for the **kernel** of T.

$\ker(T) = \{ p(x) \in P_3 \mid T(p(x)) = \vec{0} \} = \{ p(x) \in P_3 \mid (p(-1), p(0), p(1)) = (0, 0, 0) \}$

$\Rightarrow p(-1) = 0, \quad p(0) = 0, \quad p(1) = 0$

So $p(x) = k(x-1)x(x+1)$ for $k \in \mathbb{R}$

$\Rightarrow \ker(T)$ is spanned by $p(x) = k(x-1)x(x+1)$

$\therefore B_{\ker(T)} = \{ (x-1)x(x+1) \}$

(ii) (3 pts) Is T one-to-one? onto? why?

T is not one to one because $x(x-1)(x+1) \in \ker(T)$ so there is a vector different from the zero vector in the kernel. $[\dim(\ker(T)) \neq 0]$

$\text{Rank}(T) = \dim(\text{Range})$, $\quad \dim(\text{Range}) + \dim(\ker) = \dim(\text{Domain})$

$\rightarrow \dim(\text{Range}) + 1 = 4$

$\rightarrow \dim(\text{Range}) = 3 = \dim(\mathbb{R}^3)$

$\therefore$ T onto

(iii) (2 pts) Describe the **range** of T.

The range of T is all vectors in $\mathbb{R}^3$, (because T is onto)

or : $\{ (a, b, c) \mid a, b, c \in \mathbb{R} \}$

Problem 3. (10 pts) Determine whether the sets S_1 and S_2 span the same subspace of $\mathcal{P}_2$ (the space of all polynomials in x of degree ≤ 2).

$$S_1 = \{\mathbf{v}_1 = 1 + 3x - 2x^2,\ \mathbf{v}_2 = 2x + 2x^2,\ \mathbf{v}_3 = -2 + 10x^2\},$$
$$S_2 = \{\mathbf{v}_4 = 1 + 5x,\ \mathbf{v}_5 = -2 + x + 11x^2\}.$$

Notice that $\vec{v}_1 + \vec{v}_2 = (1+3x-2x^2) + (2x+2x^2)$
$= 1 + 5x = \vec{v}_4$

and $\frac{1}{2}\vec{v}_2 + \vec{v}_3 = x + x^2 + -2 + 10x^2$
$= -2 + x + 11x^2 = \vec{v}_5$

So span(S_2) is a 2-dimensional space contained in span(S_1)

Also, notice that for the vectors in S_1:

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 2 & 0 \\ -2 & 2 & 10 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 6 \\ 0 & 2 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \left.\begin{aligned} c_1 &= 2c_3 \\ c_2 &= -c_3 \\ 0 &= 0 \end{aligned}\right\}$$ v_1, v_2, v_3 are linearly dependent.

$\rightarrow$ S_1 spans a 2 dimensional space of P_2 as S_2

S_1 cannot span P_2 as it has a redundant vector, and vectors in S_1, S_2 are linear combinations of each other (can write vectors in S_2 as linear combination of vectors in S_1). $\therefore$ S_1, S_2 span same subspace of P_2

Problem 4. (4 pts each) Let V, $\langle , \rangle$ be an inner product space.

(i) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors in V such that $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all vectors $\mathbf{w} \in V$. Show that $\mathbf{u} = \mathbf{v}$.

$\langle \vec{u}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle = 0$

$= \langle \vec{u} - \vec{v}, w \rangle = 0$

Let $\vec{w} = \vec{u} - \vec{v}$ (we can assume this because $u, v \in V$ $\rightarrow u+v \in V$ $\rightarrow u-v \in V$ $\because V$ is an IPS)

$\rightarrow \langle \vec{u} - v, \vec{u} - \vec{v} \rangle = 0$

IFF $\vec{u} - \vec{v} = 0$

$\rightarrow \vec{u} = \vec{v}$

(ii) Suppose that $B = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis of V. Find the length of the vector $\mathbf{v}$ if $\mathbf{v} = 2\mathbf{q}_1 - 3\mathbf{q}_2 + 4\mathbf{q}_3$.

orthonormal Basis $\rightarrow |\vec{q}_1| = |\vec{q}_2| = |\vec{q}_3| = 1$, $\vec{q}_i \perp \vec{q}_j$ for $i \neq j$

we use inner product space axioms to determine the magnitude:

$\langle \vec{v}, \vec{v} \rangle = \langle 2\vec{q}_1 - 3\vec{q}_2 + 4\vec{q}_3, 2\vec{q}_1 - 3\vec{q}_2 + 4\vec{q}_3 \rangle$

using linearity

$= 2\langle \vec{q}_1, 2\vec{q}_1 - 3\vec{q}_2 + 4\vec{q}_3 \rangle - 3\langle \vec{q}_2, 2\vec{q}_1 - 3\vec{q}_2 + 4\vec{q}_3 \rangle + 4\langle \vec{q}_3, 2\vec{q}_1 - 3\vec{q}_2 + 4\vec{q}_3 \rangle$

$= 4\langle \vec{q}_1, \vec{q}_1 \rangle - 6\langle \vec{q}_1, \vec{q}_2 \rangle + 8\langle \vec{q}_1, \vec{q}_3 \rangle - 6\langle \vec{q}_2, \vec{q}_1 \rangle + 9\langle \vec{q}_2, \vec{q}_2 \rangle - 12\langle \vec{q}_2, \vec{q}_3 \rangle + 8\langle \vec{q}_3, \vec{q}_1 \rangle - 12\langle \vec{q}_3, \vec{q}_2 \rangle + 16\langle \vec{q}_3, \vec{q}_3 \rangle$

using symmetric property

$= 4\langle \vec{q}_1, \vec{q}_1 \rangle - \cancel{12\langle \vec{q}_1, \vec{q}_2 \rangle}^{0} + \cancel{16\langle \vec{q}_1, \vec{q}_3 \rangle}^{0} - \cancel{24\langle \vec{q}_2, \vec{q}_3 \rangle}^{0} + 9\langle \vec{q}_2, \vec{q}_2 \rangle + 16\langle \vec{q}_3, \vec{q}_3 \rangle$

Terms cancel out to 0 due to orthogonality

$= 4\langle \vec{q}_1, \vec{q}_1 \rangle + 9\langle \vec{q}_2, \vec{q}_2 \rangle + 16\langle \vec{q}_3, \vec{q}_3 \rangle$

($\because \vec{q}_i$'s are unit vectors) $= 4 + 9 + 16 = 29$

$\therefore |\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{29}$

Problem 5. (5 pts each) Prove or disprove the following statements.

(i) If A is a 2×2 matrix with eigenvalues $\lambda = -1, 3$, then $A^4 = 20A + 21I$.

Reason: $\boxed{\text{True}}$ False

we use the cayley-hamilton theorem:

$(\lambda+1)(\lambda-3) = 0 \to \lambda^2 - 2\lambda - 3 = 0$

$\to A^2 - 2A - 3I = 0$

$\to A^2 = 2A + 3I$, squaring both sides:

$\to A^4 = (2A+3I)(2A+3I)$

$= 4A^2 + 6A + 6A + 9I$

$= 4(2A+3I) + 12A + 9I$

$= 8A + 12I + 12A + 9I$

$= 20A + 21I$

$\therefore A^4 = 20A + 21I$

(ii) If two matrices A and B are row-equivalent, then they have the same eigenvalues.

Reason: True $\boxed{\text{False}}$

Counter ex/

$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{vmatrix} = (\lambda-2)^2$

$\to \lambda = 2$

$\det(\lambda I - B) = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2$

$\to \lambda = 1$

(iii) There are infinitely many unit vectors in $\mathbb{R}^3$ that are orthogonal to $(0, 2, -1)$.

Reason:

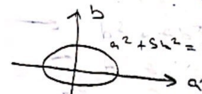
$\boxed{\text{True}}$ False

$(a, b, c) \cdot (0, 2, -1) = 0$

$= 2b - c = 0 \rightarrow 2b = c$

and $a^2 + b^2 + c^2 = 1$

$\rightarrow a^2 + 5b^2 = 1$



infinitely many points (a, b) satisfy this

(iv) A linear transformation $T : \mathbb{R}^3 \to \mathcal{P}_2$ is one-to-one if and only if it is onto.

Reason:

True False

Assume T is one-to-one

$\rightarrow \ker(T) = \{0\} \rightarrow \dim(\ker(T)) = 0$

$\dim(\text{Range}(T)) = 3 - \dim(\ker(T))$

$\dim(\text{Domain}) = 3 \rightarrow \dim(\text{Range}) = \dim(\text{Domain}) = \dim(\text{codomain})$

$\therefore$ T onto

Assume T is onto:

$\dim(\text{Range}(T)) = \dim(\underset{P_2}{\text{codomain}}) = 3$

$\dim(\text{Range}(T)) + \dim(\ker(T)) = \dim(\text{domain})$

$3 + \dim(\ker(T)) = 3$

$\rightarrow \dim(\ker(T)) = 0$

$\therefore$ T one-to-one