The American University in Cairo
Mathematics and Actuarial Science
Linear Algebra
October 8, 2019

MACT 2132
Fall 2019
Exam 1
Time Limit: 75 Minutes

Name: ______________________________ UID: __________

- This exam contains 6 pages (including this cover page).
- Answer all the problems (total of points is 45).
- Unsupported answers are considered miracles and will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.

Problem	Score	Points
1		10
2		10
3		10
4		15
Total		45

Problem 1. (10 pts) Find the value(s) of k such that the associated system of linear equations

$$\begin{bmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

has **a)** exactly one solution **b)** an infinite number of solutions **c)** no solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & K & 3 \\ 1 & K & 1 & 2 \\ K & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 - KR_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & K & 3 \\ 0 & K-1 & 1-K & -1 \\ 0 & 1-K & 1-K^2 & 1-3K \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & K & 3 \\ 0 & K-1 & 1-K & -1 \\ 0 & 0 & 2-K-K^2 & -3K \end{array}\right] \quad 3$$

From R_3: $(2-K-K^2) Z = -3K$

1 — if $2-K-K^2 \neq 0$

$\therefore Z = \dfrac{-3K}{2-K-K^2}$

$\therefore$ From R_2, we get one value for y (3)

From R_1, we get one value for x

$\therefore$ unique soln.

1 — if $2-K-K^2 = 0$

$\therefore K = 1$ or -2

when $K = 1$: $0 = -3$!

$\therefore$ inconsistent (1)

when $K = -2$: $0 = 6$!

$\therefore$ inconsistent. (1)

Conclusion: For any value of $K \neq 1, -2$, the system has a unique soln.

Otherwise (i.e $K = -2$ or 1), the system is inconsistent.

Problem 2. (10 pts) Let A be the 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 5 & -2 & 9 & 0 \\ 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

a) Find an LU-Factorization for A.

$$\xrightarrow{R_2 - 5R_1 \to R_2} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & -2 & 24 & 0 \\ 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore m_{21} = -5$

$$\xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & -2 & 24 & 0 \\ 0 & 0 & 31 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$\therefore m_{32} = 1$

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Now, $$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -m_{21} & 1 & 0 & 0 \\ -m_{31} & -m_{32} & 1 & 0 \\ -m_{41} & -m_{42} & -m_{43} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3

b) Does A have a unique LU-Factorization? Justify your answer.

First, an LU-factorization exists for A

Second, A is invertible since

$$|A| = |LU| = |L|\,|U| = 1 \cdot (-62) \neq 0$$

$\therefore$ A has a unique LU-factorization.

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Problem 3. (5 pts each)

a) Show that the matrix equation has no solution.

$$\underbrace{\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}}_{B} A = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{C}.$$

$BA = C$

$\therefore |BA| = |C|$

$\therefore |B||A| = |C|$ 2

$\therefore 0 \cdot |A| = -1$

$\therefore 0 = -1$ 2

$\therefore$ No soln. for the matrix eqn. 1

b) Let A be a square matrix such that $A^2 - 2A + I = O$. Show that A is invertible and then find A^{-1}.

1 $A^2 - 2A = -I \longrightarrow \boxed{A(A-2I) = -I}$

1 $\Rightarrow |A||A-2I| = |-I| = (-1)^n \neq 0$ $\longrightarrow$ (*)

$\therefore$ 1 $|A| \neq 0 \therefore A$ invertible

From (*), we've 2 $A^{-1} = 2I - A$

Problem 4. (5 pts each) True or False (Circle one and state your reason):

a) A system of two linear equations in three variables always has infinitely many solutions.

Reason: True **False** 2

Ex. $x_1 + x_2 + x_3 = 1$

$x_1 + x_2 + x_3 = -1$

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is inconsistent system of two eqns. in 3 var.

b) An $n \times n$ matrix can have only one eigenvalue.

Reason: 2 **True** False

Ex. $0_{3\times 3}$ has only one eigenvalue namely, $\lambda = 0$

Indeed,

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$$|\lambda I_{3\times3} - 0_{3\times3}| = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3 = 0$$

$$\longrightarrow \lambda = 0$$

Also, I_n has only one eigenvalue

c) If A is an invertible matrix with $A^3 = A$, then $\det(A^8) = 1$.

Reason: **True** False 2

$A^3 = A$ & A^{-1} exists

so $A^{-1}A^3 = A^{-1}A$

$\therefore A^2 = I$

3

$\therefore A^8 = (A^2)^4 = I$

$\therefore |A^8| = |I| = 1$

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