The American University in Cairo **MACT 2132**
Mathematics and Actuarial Science **Fall 2019**
Linear Algebra **Final Exam**
December 13, 2019 **Time Limit: 120 Minutes**

Name: ______________________ **UID:** ____________

- This exam contains 10 pages (including this cover page).
- Answer **all** the questions.
- Unsupported answers are considered miracles and will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.

Problem	Score	Points
1		20
2		20
3		14
4		14
5		24
Total		92

Problem 1. For $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, consider the following inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_2v_2 + 3u_3v_3.$$

Suppose that $\mathbf{u} = (1, 0, -1)$ and $\mathbf{v} = (2, -1, 3)$.

a) (6 pts) Compute $\langle \mathbf{u}, \mathbf{v} \rangle$, $||\mathbf{u}||$, $d(\mathbf{u}, \mathbf{v})$, and $\text{proj}_{\mathbf{u}}\, \mathbf{v}$.

b) (6 pts) Determine all vectors in $\mathbb{R}^3$ that are orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

c) (8 pts) Apply Gram-Schmidt algorithm to transform the basis $B = \{(0, 1, 2),\ (2, 0, 0),\ (1, 1, 1)\}$ into an orthonormal basis.

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Problem 2. Consider the following 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

a) (10 pts) Find the eigenvalues of A and the corresponding eigenspaces.

b) (4 pts) Is A diagonalizable? Justify your answer.

c) (6 pts) What is the rank of A? the nullity of A? Justify your answer.

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Problem 3. For any real numbers a and b, consider the homogeneous system of linear equations:

$$\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

a) (6 pts) Determine the condition(s) on a and b such that the corresponding system has

(i) exactly one solution **(ii)** an infinite number of solutions **(iii)** no solution.

b) (8 pts) Describe the set of solutions whenever the system is solvable.

Problem 4. Let $\mathcal{P}_2$ be the space of all polynomials $p(x)$ of degree at most two in x. Suppose that a transformation $T : \mathcal{P}_2 \to \mathbb{R}$ is given by

$$T(p) = \int_{-1}^{1} p(x)\, dx.$$

a) (4 pts) Show that T is a linear transformation.

b) (2 pts) Find the image of $p(x) = x$.

c) (4 pts) Is T one-to-one? Justify your answer.

d) (4 pts) Is T onto? Justify your answer.

Problem 5. (4 pts each) Prove or disprove **six** of the following.

a) The subset W of all continuous functions f such that $f(0) = 2$ is a subspace of $\mathcal{C}(-\infty, \infty)$.

b) If A and B are similar $n \times n$ matrices, then, A is singular if B does.

c) The vector space $\mathbb{R}^2$ contains infinitely many subspaces of dimension one.

d) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is never onto.

e) If A and B are 2×2 matrices such that $|A| = 3$ and $|B| = -2$, then, $|(A^2B)^T B^2 A^{-1}| = 24$.

f) If A, B, and C are $n \times n$ matrices such that $AB = AC$, then $B = C$.

g) If $\lambda = 0$ is an eigenvalue for a square matrix A, then A is singular.

h) For any linear transformation $T : V \to W$, $T(\mathbf{0}_V) = \mathbf{0}_W$.

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