Exam 1
October 23, 2021

Name: ____________________ UID: ____________________

- The exam consists of FOUR problems.
- Unsupported answers will receive little or no credit.
- Upload your answers to Gradescope as a pdf only. Make sure to allocate your work to the appropriate question.
- Missing or blank pages will result in an automatic zero for the question.
- Time: 75 minutes.

Problem	Score	Points
1		8
2		10
3		8
4		16
Total		42

Best wishes!

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Problem 1. (8 points) Let a, b, c be given constants, and consider the matrix

$$A = \begin{bmatrix} 1 & a & b \\ 1 & a+1 & b+c \\ 1 & a & b+1 \end{bmatrix}.$$

Determine the condition(s) on a, b, c for which A invertible? For those value(s) of a, b, c, find the inverse of A.

A invertible iff $|A| \neq 0$.

$$|A| = \begin{vmatrix} 1 & a & b \\ 1 & a+1 & b+c \\ 1 & a & b+1 \end{vmatrix} \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - R_1}{=}} \begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} = (1)(1)(1) \neq 0$$

upper Δ matrix

$\therefore$ A invertible for any a, b, c.

Now, we'll get A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 1 & a+1 & b+c & 0 & 1 & 0 \\ 1 & a & b+1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - aR_2} \left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & a+1 & -a & 0 \\ 0 & 1 & c & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - (b-ac)R_3]{R_2 \to R_2 - cR_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a+b-ac+1 & -a & ac-b \\ 0 & 1 & 0 & c-1 & 1 & -c \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$= [\, I \mid A^{-1} \,]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} a+b-ac+1 & -a & ac-b \\ c-1 & 1 & -c \\ -1 & 0 & 1 \end{bmatrix}.$$

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Problem 2 (10 points) Find a cubic polynomial in x say,

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

such that $P(1) = 1$, $P'(1) = 5$, $P(-1) = 3$, and $P'(-1) = 1$. Here $P'(x)$ denotes the 1st derivative of $P(x)$ with respect to x. $P'(x) = a_1 + 2a_2 x + 3a_3 x^2$

$P(1) = 1 \Rightarrow a_0 + a_1 + a_2 + a_3 = 1$

$P(-1) = 3 \Rightarrow a_0 - a_1 + a_2 - a_3 = 3$

$P'(1) = 5 \Rightarrow a_1 + 2a_2 + 3a_3 = 5$

$P'(-1) = 1 \Rightarrow a_1 - 2a_2 + 3a_3 = 1$

$\therefore$ Augmented matrix :

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 3 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & 1 & -2 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 2 \\ 0 & 1 & 2 & 3 & 5 \\ 0 & 1 & -2 & 3 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_4]{R_2 \to -\frac{1}{2} R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 1 & -2 & 3 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{4} R_3]{R_4 \to R_4 - R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 2 & 2 \end{array}\right] \xrightarrow{R_4 \to R_4 + 2R_3} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right] \\ \begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & \end{array} \end{array}$$

$\therefore \boxed{a_3 = 2}$

$\boxed{a_2 = 1}$

$\boxed{a_1} = -1 - a_3 = \boxed{-3}$

$\boxed{a_0} = 1 - a_1 - a_2 - a_3 = 1 + 3 - 1 - 2 = \boxed{1}$

$\therefore P(x) = 1 - 3x + x^2 + 2x^3$.

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Problem 4. (4 points each) True or False (Circle one and state your reason):

i) If A is an $n \times n$ matrix with n odd integer such that $A^t = -A$, then $|A| = 0$.

Reason: Suppose $\underset{n\times n}{A^t = -A}$, n odd | True | False

$\therefore |A^t| = |-A|$.

$\therefore |A| = (-1)^n |A| = -|A|$.

$\therefore 2|A| = 0$.

$\therefore |A| = 0$.

ii) For any square matrices A and B of the same size, the following is true:

$$(A+B)^2 = A^2 + 2AB + B^2.$$

Reason: True | False

$$(A+B)^2 = (A+B)(A+B)$$
$$= A(A+B) + B(A+B)$$
$$= A^2 + AB + BA + B^2.$$

$\therefore (A+B)^2 = A^2 + 2AB + B^2$.

iff

$AB = BA$

this is not always true. For example, take

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

here $AB \neq BA$.

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iii) Every invertible matrix has unique LU-Factorization.

Reason: True **False**

Counter example

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ invertible as $|A| = 1 \neq 0$

if A has an LU, then $A = LU = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & d \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac+d \end{bmatrix}$

$\therefore b = 0$

$\therefore 1 = ab = 0$ ↯ So, A doesn't have an LU.

iv) Two systems of linear equations $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ and $B\underline{\mathbf{x}} = \underline{\mathbf{c}}$ have the same solution set if their coefficient matrices A and B are row-equivalent.

Reason: True **False**

$[A : \underline{b}]$, $[B : \underline{c}]$ should be row-equivalent

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B$

Notice that A, B are row-equivalent.

However,

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has solns. $\{\underline{x} = \underline{0}\}$,

and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has solns. $\{\underline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$

that is, the two systems satisfy the condition but they've different solns.

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Problem 3. A matrix whose entries are all integer numbers is called an *integer matrix.*

For example, $\begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$ is an integer 2×2 matrix, but $\begin{bmatrix} 1 & -4 \\ -2 & 9/2 \end{bmatrix}$ is not.

i) (2 points) Give an example of an invertible integer matrix A whose inverse is NOT an integer matrix.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$ integer.

But, $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/7 \end{bmatrix}$ non-integer.

ii) (6 points) Suppose that A is an integer matrix, which is invertible. Use the fact that **"the determinant of an integer matrix is an integer number"** to show:

(a) If A^{-1} is an integer matrix, then $|A| = \pm 1$.

Suppose that A^{-1} integer. Then, $|A^{-1}|$, $|A|$ are both integers by the fact. Let $|A| = c$ (integer)

$\therefore$ c and $\frac{1}{c}$ are both integers.

This holds only if $c = \pm 1$. Equivalently, $|A| = \pm 1$.

(b) The converse is also true, that is, if $|A| = \pm 1$, then A^{-1} is an integer matrix.

Suppose that A is an integer matrix with $|A| = \pm 1$

$|A| = \pm 1 \neq 0 \Rightarrow A$ invertible, $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$

$\therefore A^{-1} = \frac{1}{\pm 1} \operatorname{adj}(A) = \pm \operatorname{adj}(A)$.

The elements of $\operatorname{adj}(A)$ are cofactors of A.

A integer $\Rightarrow$ cofactors are determinants of integer matrices, so they are integers by the fact.

$\therefore \operatorname{adj}(A)$ integer matrix.

$\therefore A^{-1} = \pm$ (integer matrix) is also integer.

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