Midterm 2, SP 2019

Question 1

- $\underline{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\underline{A\,\underline{x}} = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+4 \\ 0-6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = -3\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$= -3\,\underline{x}$ $\quad\therefore$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector for A relative to the eigenvalue $\lambda = -3$

- $\underline{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\underline{A\,\underline{x}} = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+4 \\ 0-6 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} = -3\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$\neq -3\,\underline{x}$

$\therefore$ $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ not an eigenvector for A relative to $\lambda = -3$.

Question 2: NO, it is not a subspace. For instance, it is not closed under scaler multiplication.

Example: $\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the set because the two components $x = 1 \geq 0$, $y = 1 \geq 0$

But $-2 \cdot \underline{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is not in the set because $x = -2 \geq 0$ (False)

• Question 3   1. $0 \cdot \underline{v} = \underline{0}$

Proof let $\underline{u} = 0 \cdot \underline{v}$

$\therefore \underline{u} = 0 \cdot \underline{v} = (0+0) \cdot \underline{v}$

$= 0 \cdot \underline{v} + 0 \cdot \underline{v}$ ← Distributivity

$= \underline{u} + \underline{u}$ ← Definition

$= 2 \cdot \underline{u}$ ← Distributivity

$\therefore \underline{u} + (-\underline{u}) = 2 \cdot \underline{u} + (-\underline{u})$

$\therefore \underline{0} = (2-1) \cdot \underline{u}$ ← Distributivity

Additive inverse

$= 1 \cdot \underline{u}$

$= \underline{u}$

$\therefore$ multiplicative identity

$\therefore \underline{u} = \underline{0}$ $\therefore 0 \cdot \underline{v} = \underline{0}$

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2. $\underline{v} + (-1) \cdot \underline{v} = \underline{0}$

Proof $\underline{v} + (-1) \cdot \underline{v} = 1 \cdot \underline{v} + (-1) \cdot \underline{v}$

multiplicative identity

Distributivity $= (1 + (-1)) \cdot \underline{v}$

$= 0 \cdot \underline{v}$

Fact 1 above $= \underline{0}$

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3. The inverse of $\underline{v}$ is unique

<u>Proof:</u> Let $\underline{u}$, $\underline{w}$ be inverses of $\underline{v}$

$\therefore$ $\underline{u} + \underline{v} = \underline{0}$ and $\underline{w} + \underline{v} = \underline{0}$

$\therefore$ $\underline{u} = \underline{u} + \underline{0} = \underline{u} + (\underline{w} + \underline{v})$

Additive identity Assumption

$= \underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$

Commutativity Associativity

$= \underline{0} + \underline{w} = \underline{w}$

Assumption Additive identity

$\therefore$ $\underline{u} = \underline{w}$ "the inverse is unique"

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4. $(-1) \cdot \underline{v} = -\underline{v}$

<u>Proof:</u> let $\underline{u} = (-1) \cdot \underline{v}$

$\therefore$ $\underline{u} + \underline{v} = (-1) \cdot \underline{v} + \underline{v} = (-1 + 1) \cdot \underline{v}$

Assumption Distributivity

$= 0 \cdot \underline{v} = \underline{0}$

Fact ① above

$\therefore$ $\underline{u}$ is an additive inverse for $\underline{v}$, but the additive inverse is unique. So $\underline{u} = -\underline{v}$.

Question 4:

One way is to show that this set of vectors is the span of a finite set of vectors, and we're done because "Span" gives us a Subspace (Fact!)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a-b-c \end{bmatrix}$$

$a+b+c+d=0$

$\therefore d=-a-b-c$

$$= \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & -b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & -c \end{bmatrix}$$

$$= a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\underline{v}_1} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}}_{\underline{v}_2} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{\underline{v}_3}$$

$\therefore$ Any vector in the prescribed set is a linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

$\therefore$ it is the Span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

$\therefore$ it is a Subspace in $M_{2\times 2}$

Question $\underline{5}$:

1. Spanning set of V : any vector in V can be written as a linear combination of the vectors in S

2. S linearly independent : No vector in S is a linear combination of the other vectors in S.

3. S basis for V : S is a Spanning set for V and linearly independent.

4. The dimension of V : the number of vectors in any basis.

Question $\underline{\underline{6}}$: $S=\{\underset{\underline{v}_1}{(4,3,2)},\underset{\underline{v}_2}{(0,3,2)},\underset{\underline{v}_3}{(0,0,2)}\}$

1. let $A=[\underline{v}_1 : \underline{v}_2 : \underline{v}_3]=\begin{bmatrix}4&0&0\\3&3&0\\2&2&2\end{bmatrix}$

We've $\det(A)=(4)(3)(2)\neq 0$

$\therefore$ A invertible, in particular $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are linearly independent.

Second, we know that any set of vectors in $\mathbb{R}^3$, that is linearly indep. and contains $\underline{3}$ vectors is a spanning set for $\mathbb{R}^3$. Therefore, S is a basis for $\mathbb{R}^3$.

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2. First, we need to write $\underline{u} = (8,3,8)$ as a linear combination of $\underline{v}_1, \underline{v}_2, \underline{v}_3$:

$$\underline{u} = a \cdot \underline{v}_1 + b \cdot \underline{v}_2 + c \cdot \underline{v}_3$$

$$(8,3,8) = a \cdot (4,3,2) + b \cdot (0,3,2) + c \cdot (0,0,2)$$

$\therefore$ $4a = 8 \longrightarrow a = 2$

$3a + 3b = 3 \longrightarrow 6 + 3b = 3 \longrightarrow b = -1$

$2a + 2b + 2c = 8 \longrightarrow 4 - 2 + 2c = 8 \longrightarrow c = 3$

$\therefore$ the coordinate vector $[\underline{u}]_S$ relative to S is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

Question I :

1. let W be any subspace in V.

if $\dim(W) > \dim(V)$, then we've a basis for W containing more than n vectors say $\underline{w}_1, \dots, \underline{w}_m$

Here $m = \dim(W) > n = \dim(V)$

By the fact, the vectors $\underline{w}_1, \dots, \underline{w}_m$ must be linearly dependent. But this contradicts the fact that they form a basis for W.

$\therefore$ $\dim(W) \leq \dim(V)$.

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2. We've S containing 4 vectors in $\mathbb{R}^3$.
Since S contains more vectors than the dimension of $\mathbb{R}^3$, then it is linearly dependent.

• Question 8 : 1. Any row-vector in A does.
Take $(1, 2, -3)$ for example.

2. $$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 4 \\ 4 & 3 & -2 \end{bmatrix} \xrightarrow[R_3 - 4R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & -5 & 10 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2 \to R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 - 2R_2 \to R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = REF$$

Non-zero rows in REF form a basis for $RS(A)$
∴ Basis $= \{ (1, 0, 1), (0, 1, -2) \}$

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3. Pivot columns in A form a basis for $CS(A)$.
We've pivots in 1st, 2nd columns, then

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$$

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4. $Rank(A) = \dim(RS(A)) = \dim((CS(A))$
$= \text{no. of Pivots} = \underline{\underline{2}}$

5. To find the nullspace $NS(A)$, we solve the system :

$$\underset{3\times3}{A}\ \underset{3\times1}{\underline{x}} = \underset{3\times1}{\underline{0}} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & -1 & 4 & 0 \\ 4 & 3 & -2 & 0 \end{array}\right] \xrightarrow{ERO's} \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right. & & & \left|\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] \end{array}$$

Row3 : $0 = 0$ ✓

Row 2 : $x_2 = 2x_3$ $\qquad x_3$ free

Row 1 : $x_1 = -x_3$

$$\therefore\ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore\ NS(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

6. From (5) $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ spans $NS(A)$

but also linearly indep. (only one non-zero vector)

$\therefore\ S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$ basis for $NS(A)$.

7. $Nullity(A) = \dim(NS(A)) = 1$.