SP 2019

Q 1

(a) By a solution of a system of linear eqns. in $x_1, \ldots, x_n$ we mean a sequence of real numbers $(s_1, \ldots, s_n)$ that satisfies each eqn. in the system when we substitute s_1 for x_1, s_2 for x_2, ..., s_n for x_n.

(b) Two systems are equivalent if they have the same solns. set. This happens iff they can be obtained from each other using EROs.

(c) An elementary $n \times n$ matrix is this obtained from the identity matrix I_n using a single elementary row operation.

Q 2

(a)
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{array}\right] \xrightarrow[R_3 - 2R_1 \to R_3]{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{array}\right]$$

augmented matrix

$$\xrightarrow{R_3 - 2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{array}\right] \xrightarrow{-\frac{1}{7}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_1 + 3R_3 \to R_1]{R_2 - 7R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

R_3 : $z = 2$

R_2 : $y = -3$ unique soln

R : $x = 4$

(b) Matrix notation $\underline{A}\,\underline{X} = \underline{B}$

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}, \quad \underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

3×3 Coeff. matrix; 3×1 Variables column vector; 3×1 Constants column vector.

(c)

$$|A| = \begin{array}{ccccc} & & -6 & -4 & 0 \\ 1 & 0 & -3 & 1 & 0 \\ 3 & 1 & -2 & 3 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ & & 1 & 0 & -18 \end{array}$$

$$= 1 + 0 + (-18) - (-6) - (-4) - 0$$
$$= 1 - 18 + 6 + 4 = -7 \neq 0$$

A invertible iff $|A| \neq 0$

(d) The unique soln. of the system is given by

$$\underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\underline{B} = \frac{1}{7}\begin{bmatrix} -5 & 6 & -3 \\ 7 & -7 & 7 \\ -4 & 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 10 + 30 - 12 \\ -14 - 35 + 28 \\ 8 + 10 - 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 28 \\ -21 \\ 14 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 4$, $y = -3$, $z = 2$

Q3 The system is homogenuous, hence is always consistent because $(0,0,0)$ is a soln.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

R_2 : $y + 2z = 0$ $\therefore$ $\boxed{y = -2z}$

R_1 : $x - z = 0$ $\therefore$ $\boxed{x = z}$

$\boxed{z \text{ free}}$

$\therefore$ the system has infinitely many solns.

Q4 $\begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & K \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q5 $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$

(a) $\underset{3\times 3}{C = AB} = (c_{ij})_{i,j}$

$c_{11} = R_{1,A} \cdot C_{1,B} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} = (1)(3) + (2)(4) + (3)(3)$

$= 3 + 8 + 9 = 20$

$c_{12} = R_{1,A} \cdot C_{2,B} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = (1)(2) + (2)(5) + (3)($

$= 2 + 10 + 6 = 18$

$$c_{13} = R_{1,A} \cdot C_{3,B} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = (1)(1) + (2)(4) + (3)(1)$$

$$= 1 + 8 + 3 = 12$$

(b) $D = BA = (d_{ij})_{i,j}$

$$d_{21} = R_{2,B} \cdot C_{1,A} = \begin{bmatrix} 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$= 4 + 20 + 28 = 52$$

(c) $R = AB^2 = (r_{ij})_{i,j}$

$R = (AB)B$

$$r_{13} = R_{1,AB} \cdot C_{3,B} = \begin{bmatrix} 20 & 18 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$= 20 + 72 + 12 = 104$$

(d) $M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6$

$\boxed{Q6}$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{array}\right]$$

$A \qquad I_2 \qquad\qquad E_1$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & 4 & -1 \\ 0 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$I_2 \qquad A^{-1}$

Elementary row operation	Elementary matrix
① $R_2 - 3R_1 \to R_2$	$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$
② $\frac{1}{2}R_2 \to R_2$	$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
③ $R_1 - 2R_2 \to R_1$	$E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

$$\therefore \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E_3 E_2 E_1 A$$

$$\therefore \quad A^{-1} = E_3 E_2 E_1$$

&

$$A = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Q7 Let B be the matrix obtained from A by applying the ERO $R_j - R_i \longrightarrow R_j$

$\therefore |A| = |B|$

However, the jth row of B consists entirely of zeros.

Hence, $|B| = 0$, which was to be shown.

Q8

(a) $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore B^t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\quad B^{-1} = \frac{1}{|B|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore B^t = B^{-1}$

$\therefore B$ orthogonal.

(b) $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$C^t = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $\quad C^{-1} = \frac{1}{|C|} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$= \frac{1}{1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq C^t$

$\therefore C$ Not orthogonal

(C) Suppose that A is orthogonal

$\therefore A^{-1} = A^t$

$\therefore A A^t = I$

$\therefore |A A^t| = |I| = 1$

$\therefore |A| |A^t| = 1$

$\therefore |A| |A| = 1$

$\therefore |A|^2 = 1$

$\therefore |A| = \pm 1$