Problem 1. Consider the set of symmetric matrices:

$$W = \{A \in M_{2\times 2} : A = A^T\}.$$

a) (4 pts) Show that W is a subspace of $M_{2\times 2}$.

1. $0 = 0^T \quad \therefore 0 \in W$ and W is nonempty

2. Let $A, B \in W$, then $A = A^T$ and $B = B^T$. So
$(A+B)^T = A^T + B^T = A + B$ and $A + B \in W$ closed under addition.

3. Let $A \in W$, c a scalar
$(cA)^T = cA^T = cA$, thus $cA \in W$ closed under scalar mult.

From 1–3, W is a subspace of $M_{2\times 2}$

b) (3 pts) Find a basis for W. What is the dimension of W?

A basis for $W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ we must have $b = c$

$\dim W = 3$

Problem 2. Consider the following homogeneous system of linear equations:

$$x_1 + 3x_2 + 2x_3 + 22x_4 + 13x_5 = 0$$
$$x_1 + x_3 - 2x_4 + x_5 = 0$$
$$3x_1 + 6x_2 + 5x_3 + 42x_4 + 27x_5 = 0$$

a) (1 pt) Write down the coefficient matrix A corresponding to the system.

$$\begin{pmatrix} 1 & 3 & 2 & 22 & 13 \\ 1 & 0 & 1 & -2 & 1 \\ 3 & 6 & 5 & 42 & 27 \end{pmatrix}$$

a) (3 pts) Find a basis for the solution space of the homogeneous system.

$$\begin{pmatrix} 1 & 3 & 2 & 22 & 13 \\ 1 & 0 & 1 & -2 & 1 \\ 3 & 6 & 5 & 42 & 27 \end{pmatrix} \xrightarrow[R3-3R1]{R2-R1} \begin{pmatrix} 1 & 3 & 2 & 22 & 13 \\ 0 & -3 & -1 & -24 & -12 \\ 0 & -3 & -1 & -24 & -12 \end{pmatrix} \xrightarrow{R3-R2}$$

$$\begin{pmatrix} 1 & 3 & 2 & 22 & 13 \\ 0 & -3 & -1 & -24 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x_3 &= t \\ x_4 &= s \\ x_5 &= r \end{aligned}$$

$$-3x_2 = x_3 + 24x_4 + 12x_5 \rightarrow x_2 = \tfrac{-1}{3}t - 8s - 4r$$

$$\begin{aligned} x_1 &= -3x_2 - 2x_3 - 22x_4 - 13x_5 \\ &= t + 24s + 12r - 2t - 22s - 13r \\ &= -t + 2s - r \end{aligned}$$

$$x = \begin{pmatrix} -3 \\ -1 \\ 3 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 2 \\ -8 \\ 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} r$$

Basis for solution space $= \{(-3,-1,3,0,0), (2,-8,0,1,0), (-1,-4,0,0,1)\}$

b) (1 pts) What is the dimension of the solution space?

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c) (3 pts) What is the rank of A? Justify your answer.

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$$\begin{aligned} \text{rank}(A) &= \text{no. of columns} - \text{nullity} \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

d) (2 pts) Is the vector $\begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$ in the column space of A? Justify your answer.

Basis for $CS(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix} \right\}$ $\quad c_1\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + c_2\begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 1 & 0 & 3 \\ 3 & 6 & 7 \end{array}\right] \xrightarrow[R3-3R1]{R2-R1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -3 & 1 \\ 0 & -3 & 1 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$c_2 = -1/3 \qquad c_1 = 3$

$\because$ the system is consistent $\therefore \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \in CS(A)$

Problem 4. (5 pts each) Determine whether the following statements are true or false. Justify your answer.

a) If V is an inner product space, then

$$< v, w >= \frac{||v+w||^2 - ||v-w||^2}{4},$$

for all $v, w \in V$.

$\boxed{\text{True}}$

$||v+w||^2 = ||v||^2 + ||w||^2 + 2||v|| \, ||w|| \cos\theta = ||v||^2 + ||w||^2 + 2<v,w>$

$||v-w||^2 = ||v||^2 + ||w||^2 - 2||v|| \, ||w|| \cos\theta = ||v||^2 + ||w||^2 - 2<v,w>$

$||v+w||^2 - ||v-w||^2 = 4<v,w>$

$\therefore <v,w> = \dfrac{||v+w||^2 - ||v-w||^2}{4}$

b) Let A be an $m \times n$ matrix, then $\text{rank}(A) = \text{rank}(A^T)$.

$\boxed{\text{True}}$

$\text{rank}(A) = \dim RS(A)$

$\text{rank}(A^T) = \dim CS(A) = \dim RS(A) = \text{rank}(A)$

c) There exists an inner product on $\mathbb{R}^2$ such that $||(x, y)|| = |x| + |y|$, for all $(x, y) \in \mathbb{R}^2$.

$\boxed{\text{FALSE}}$

parallelogram law: $||\bar{u}+\bar{v}||^2 + ||\bar{u}-\bar{v}||^2 = 2(||\bar{u}||^2 + ||\bar{v}||^2)$

Let $\bar{u} = (1,1)$, $\bar{v} = (1,-1)$ $\qquad \bar{u}+\bar{v} = (2,0)$, $\bar{u}-\bar{v} = (0,2)$

$||\bar{u}+\bar{v}||^2 = 2^2 = 4$

$||\bar{u}-\bar{v}||^2 = 2^2 = 4$ $\qquad ||\bar{u}+\bar{v}||^2 + ||\bar{u}-\bar{v}||^2 = 4+4 = 8$

$||\bar{u}||^2 = (1+1)^2 = 2^2 = 4$ $\qquad 2(||\bar{u}||^2 + ||\bar{v}||^2) = 2(4+4) = 16 \neq 8$

$||\bar{v}||^2 = (1+|-1|)^2 = 2^2 = 4$