Problem 1. Consider the following system of linear equations:

$$3x - 2y = b_1$$

$$6x - 4y = b_2$$

a) (3 pts) Determine the possible values of b_1 and b_2 that will make the system solvable.

$$\left[\begin{array}{cc|c} 3 & -2 & b_1 \\ 6 & -4 & b_2 \end{array}\right] \xrightarrow{R2-2R1} \left[\begin{array}{cc|c} 3 & -2 & b_1 \\ 0 & 0 & b_2-2b_1 \end{array}\right]$$

for the system to be consistent, we need $b_2 - 2b_1 = 0$, we need $b_2 = 2b_1$.

b) (2 pts) Solve the system above for $b_1 = 0$ and $b_2 = 0$.

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $y = t$, $t \in \mathbb{R}$

$3x - 2y = 0$

$3x = 2y$

$x = \frac{2}{3}y$

$x = \frac{2}{3}t$

Solution $\begin{pmatrix} 2/3 \\ 1 \end{pmatrix} t$, $t \in \mathbb{R}$

Problem 2. Consider the matrix:

$$A = \begin{pmatrix} a & b & b \\ a & a & b \\ a & a & a \end{pmatrix}$$

a) (4 pts) Determine the conditions that make A invertible.

$$|A| = a(a^2-ab) - b(a^2-ab) + b(\underbrace{a^2-a^2}_{0})$$

$$= a^3 - a^2b - a^2b + ab^2 = a^3 - 2a^2b + ab^2$$

$$= a(a^2-2ab+b^2) = a(a-b)^2$$

We need $|A| \neq 0$ $\therefore$ $a \neq 0$ and $a-b \neq 0$

$\therefore$ $a \neq 0$ and $a \neq b$

b) (6 pts) Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3-R1]{R2-R1} \left(\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & a-b & a-b & -1 & 0 & 1 \end{array}\right) \xrightarrow{R3-R2}$$

$$\left(\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right) \xrightarrow{R1-\frac{b}{a-b}R2} \left(\begin{array}{ccc|ccc} a & 0 & b & \frac{a-b+b}{a-b} & \frac{-b}{a-b} & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right) \xrightarrow{R1-\frac{b}{a-b}R3}$$

$$\left(\begin{array}{ccc|ccc} a & 0 & 0 & \frac{a}{a-b} & 0 & \frac{-b}{a-b} \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right) \xrightarrow[R3/(a-b)]{R1/a,\ R2/(a-b)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{pmatrix}$$

Problem 3. Consider the matrix:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

a) (5 pts) What three elementary matrices put A into its upper triangular form.

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow{R2-2R1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix} \xrightarrow{R3-3R1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{R3-2R2} *$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$* \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = U$$

b) (5 pts) Find the LU factorization of A.

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{U}$$

Problem 4. Let the matrices $X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $Y = \begin{pmatrix} -3 & 5 \\ 1 & 2 \end{pmatrix}$.

a) (3 pts) Compute X^{-1}.

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R2-R1} \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R1-R2} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$X^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

b) (3 pts) Compute XYX^{-1}.

$$XYX^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ -1 & 9 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -11 & 9 \\ -11 & 10 \end{pmatrix}$$

$-4-7$
$2+7$
$-2-9$
$1+9$

c) (3 pts) What is the relation between $det(Y)$ and $det(XYX^{-1})$? Justify your answer.

$\det(Y) = -3 \times 2 - 5 \times 1 = -6 - 5 = -11$

$\det(XYX^{-1}) = -11 \times 10 - (-11 \times 9) = -11$

$\det(Y) = \det(XYX^{-1})$

$$|XYX^{-1}| = |X|\,|Y|\,|X^{-1}| = |X|\,|Y|\,\frac{1}{|X|} = |Y|$$

Problem 4. (2 pts each) Assume that A is a 3×3 matrix with the property that $A^2 = A$. Determine whether the following statements about the matrix A are true or false. Justify your answer.

a) A is invertible. FALSE

$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $A^2 = A$ but A is not inveritble.

b) $det(A) = 0$ FALSE

$|A| = |A^2| = |A|^2 \Rightarrow |A|^2 - |A| = 0 \Rightarrow |A|(|A| - 1) = 0$

$\therefore |A| = 0$ or 1.

c) $det(A^5) = det(A)$. TRUE

$A^5 = A^2 . A^2 . A = A.A.A = A^2.A = A.A = A^2 = A$

$\therefore$ $det(A^5) = det(A)$ because $A^5 = A$.