The American University in Cairo
Mathematics and Actuarial Science
Linear Algebra
March 19, 2020

MACT 2132
Spring 2020
Exam 1, V1
Time Limit: 75 Minutes

Name: ____________________ UID:__________

- This exam contains 8 pages (including this cover page and the draft page).
- Answer all the problems.
- Unsupported answers are considered miracles and will receive little or no credit.
- Anyone caught writing after time has expired will be given a mark of zero.

Problem	Score	Points
1		7
2		11
3		14
4		20
Total		52

Key

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & k \\ 1 & 0 & 1 \end{bmatrix}$$

a) (1 pts) Determine the number of equations and the number of variables.

b) (6 pts) Find the value(s) of k that make the system consistent, and then solve the system.

(a) Homogeneous system $\rightarrow A\vec{x} = \vec{b} = \vec{0}$

$$\Rightarrow \quad \begin{aligned} x_1 + x_2 + 0x_3 &= 0 \\ 0x_1 + x_2 + x_3 &= 0 \\ 3x_1 + x_2 + kx_3 &= 0 \\ x_1 + 0x_2 + x_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} &3 \text{ variables} \\ &4 \text{ equations} \end{aligned}$$

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 1 & k & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

HOMOGENEOUS SYSTEMS ARE ALWAYS CONSISTENT $\therefore k \in \mathbb{R}$

$R_3 - 3R_1 \Rightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & k & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

② $R_3 + 2R_2 \Rightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k+2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

$R_4 - R_1 \Rightarrow R_4$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k+2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$

④ $R_4 + R_2 \Rightarrow R_4$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k+2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$

$R_4 - \frac{2}{k+2} R_3 \Rightarrow R_4$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & k+2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

① $x_1 + x_2 = 0$

② $x_2 + x_3 = 0$

③ $(k+2)x_3 = 0 \rightarrow$ if $k = -2$, $0 = 0 \rightarrow$ infinitely many soln

$\hookrightarrow x_2 = -x_3$

$x_1 = x_3$, x_3 free

if $k \neq 2$, $x_3 = 0$ from ③

in ② $x_2 = 0$

in ① $x_1 = 0$ $\rightarrow (0, 0, 0)$ unique solution.

$\therefore \forall k \in \mathbb{R}$, system is consistent

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Problem 2. Let s and t be fixed real numbers. Consider the matrix

$$B = \begin{bmatrix} s+t & s & s \\ s & s+t & s \\ s & s & s+t \end{bmatrix}.$$

a) (4 pts) Show that $\det(B) = t^2(3s+t)$.

b) (5 pts) Find the inverse of B when $s = 1$ and $t = -1$ or show that it does not exist.

c) (2 pts) Determine the condition(s) on s and t for which B is invertible.

(a) $\det(B) = (s+t)\begin{vmatrix} s+t & s \\ s & s+t \end{vmatrix} - s\begin{vmatrix} s & s \\ s & s+t \end{vmatrix} + s\begin{vmatrix} s & s+t \\ s & s \end{vmatrix}$

$= (s+t)[(s+t)^2 - s^2] - s[s(s+t) - s^2] + s[s^2 - s(s+t)]$

$= (s+t)[\cancel{s^2} + 2st + t^2 - \cancel{s^2}] - s[\cancel{s^2} + st - \cancel{s^2}] + s[\cancel{s^2} - \cancel{s^2} - st]$

$= (s+t)(2st + t^2) - s(st) + s(-st)$

$= 2\cancel{s^2}t + t^2 s + 2st^2 + t^3 - 2\cancel{s^2}t$

$= t^2 s + 2st^2 + t^3 = t^2(s + 2s + t)$

$\therefore \det(B) = t^2(3s+t)$

(b) $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\det(B) = (-1)^2(3(1) - 1) = 2$

we compute $\text{cof}(A)$: $\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$C_{11} = (-1)^2\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$, $C_{12} = (-1)^3\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$, $C_{13} = (-1)^4\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

$C_{21} = (-1)^3\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$, $C_{22} = (-1)^4\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$, $C_{23} = (-1)^5\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$

$C_{31} = (-1)^4\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$, $C_{32} = (-1)^5\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1$, $C_{33} = (-1)^6\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$\Rightarrow \text{cof}(A) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\text{adj}(A) = [\text{cof}(A)]^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore B^{-1} = \frac{1}{2}\text{adj}(A)$

$\Rightarrow B^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/1 \end{bmatrix}$

(c) B is invertible of $\det(A) \neq 0$

$\Rightarrow t^2(3s+t) \neq 0 \rightarrow t^2 \neq 0 \rightarrow \boxed{t \neq 0}$

and $3s + t \neq 0 \rightarrow \boxed{s \neq -t/3}$

Problem 3. Let A be the 3×3 matrix

$$\begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

a) (8 pts) Determine the eigenvalues for A and the corresponding eigenvectors.

Characteristic eq: $\det[\lambda I - A] = 0$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} \lambda-4 & -1 & -3 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda-3 \end{bmatrix} \right| \rightarrow \left| \begin{matrix} \lambda-4 & -1 & -3 \\ 0 & \lambda-2 & -1 \\ 0 & 0 & \lambda-3 \end{matrix} \right| \begin{matrix} \lambda-4 & -1 \\ 0 & \lambda-2 \\ 0 & 0 \end{matrix}$$

$$= (\lambda-4)(\lambda-2)(\lambda-3) = 0$$

$$\therefore \lambda = \{2, 3, 4\}$$

for $\lambda = 2$:

$$\begin{bmatrix} -2 & -1 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} -2 & -1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2 \rightarrow R_3} \left[\begin{array}{ccc|c} -2 & -1 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-2x_1 - x_2 - 3x_3 = 0$$

$$-x_3 = 0 \rightarrow \boxed{x_3 = 0}$$

$$\Rightarrow -2x_1 = +x_2 \rightarrow x_1 = -\tfrac{1}{2} x_2, \text{ let } x_2 = t \in \mathbb{R} - \{0\}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2\, t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}}$$

for $\lambda = 3$:

$$\begin{bmatrix} -1 & -1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x_1 - x_2 - 3x_3 = 0 \\ x_2 - x_3 = 0 \\ \Rightarrow \boxed{x_2 = x_3} \end{matrix}$$

Let $x_3 = t \in \mathbb{R} - \{0\} \rightarrow -x_1 - t - 3t = 0 \rightarrow -x_1 = 4t \rightarrow \boxed{x_1 = -4t}$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}}$$

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for $\lambda = 4$:

$$\begin{bmatrix} 0 & -1 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R_2 + 2R_1 \to R_2} \left[\begin{array}{ccc|c} 0 & -1 & -3 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

① $R_2 \cdot (-1/7) \to R_2$
② $R_3 - R_2 \to R_3$

$$\longrightarrow \left[\begin{array}{ccc|c} 0 & -1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-x_2 - 3x_3 = 0$

$x_3 = 0 \qquad x_2 = 0 \quad , \quad x_1$ free

$x_1 = t \in \mathbb{R} - \{0\}$

$$\therefore \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

b) (6 pts) Compute the adjoint matrix adj(A) and show that $A \cdot \text{adj}(A) = \det(A)\, I_3$.

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

we calculate cof(A)

$C_{11} = (-1)^2 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$, $C_{12} = (-1)^3 \begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix} = 0$, $C_{13} = (-1)^4 \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$C_{21} = (-1)^3 \begin{vmatrix} -1 & 3 \\ 0 & 3 \end{vmatrix} = 3$, $C_{22} = (-1)^4 \begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix} = 12$, $C_{23} = (-1)^5 \begin{vmatrix} 4 & -1 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = (-1)^4 \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = -7$ $C_{32} = (-1)^5 \begin{vmatrix} 4 & 3 \\ 0 & 1 \end{vmatrix} = -4$, $C_{33} = (-1)^6 \begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} = 8$

$$\text{cof}(A) = \begin{bmatrix} 6 & 0 & 0 \\ 3 & 12 & 0 \\ -7 & -4 & 8 \end{bmatrix}$$

$$[\text{cof}(A)]^T = \text{adj}(A) = \begin{bmatrix} 6 & 3 & -7 \\ 0 & 12 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\det(A) = 4 \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + 1 \underbrace{\begin{vmatrix} 0 & 1 \\ 0 & 3 \end{vmatrix}}_{=0} + 3 \underbrace{\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix}}_{=0}$$

$$= 4(6) = 24$$

$$A \cdot \text{adj}(A) = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 6 & +3 & -7 \\ 0 & 12 & -4 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix} = 24 I_3$$

$\therefore A \cdot \text{adj}(A) = \det(A) I_3$ ✓

Problem 4. (5 pts each) Prove or disprove four of the following.

a) A system of linear equations with fewer equations than variables always has a solution.

Reason: True **False**

Counterexample

$x_1 + x_2 + x_3 = 0$

$x_1 + x_2 + x_3 = 1$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$0 = 1$ $\times$

$\therefore$ False

b) If A and B are $n \times n$ matrices such that A^3B is singular, then A or B is singular.

Reason: **True** False

Suppose A, B invertible

$\to$ $\det(A) \neq 0$, $\det(B) \neq 0$

$\det(A^3B) = [\det(A)]^3 \det(B) \neq 0$

$\Rightarrow$ A^3B invertible $\times$

$\therefore$ A or B must be singular.

c) A square matrix is invertible if it is row-equivalent to the identity matrix.

Reason: Assume A is row-equivalent to I_n **True** False

$\to A = E_1 E_2 \cdots E_k I_n$ for some elementary matrices $E_1, E_2, \cdots, E_k$ invertible.

$= E_1 E_2 \cdots E_k$

$\to A^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$ ✓

d) If A is a 2×3 matrix, then the only vector $\underline{\mathbf{u}}$ in $\mathbb{R}^3$ such that $A\underline{\mathbf{u}} = \underline{\mathbf{0}}$ is $\underline{\mathbf{u}} = \underline{\mathbf{0}}$.

Reason: True **False**

Consider $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$

$\rightarrow$ $x_1 + x_2 - x_3 = 0$

$x_1 - x_2 = 0$

try $(1, 1, 2)$ $\rightarrow$ $1 + 1 - 2 = 0$

$1 - 1 = 0$

Also, $A = O_{2\times 3}$

$\Rightarrow$ any $u \in \mathbb{R}^3$ satisfies the eq.n.

✓

e) If A is a square matrix such that $A^2 = A$, then $I - 2A = (I - 2A)^{-1}$.

Reason: **True** False

$(I-2A)(I-2A) = I - 2A - 2A + 4A^2$

$= I - 4A + 4A$

$= I$

$\rightarrow I - 2A = (I - 2A)^{-1}$

f) If A and B are $n \times n$ matrices such that $AB = O_{n\times n}$, then $A = O_{n\times n}$ or $B = O_{n\times n}$.

Reason: True **False**

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} (1)(1) + (1)(-1) & (1)(1) + (1)(-1) \\ (1)(1) + (1)(-1) & (1)(1) + (1)(-1) \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

However, the two matrices are non-zero

✗