Problem 1

$$A = \begin{bmatrix} 2 & -4 & 0 & 1 & 7 & 11 \\ 1 & -2 & -1 & 1 & 9 & 12 \\ -1 & 2 & 1 & 3 & -5 & -16 \\ 4 & -8 & 1 & -1 & 6 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A, B row equivalent.

(a) Basis for RS(A) is the same as basis for RS(B)

Basis RS(A) $= \{(1,-2,0,0,3,2), (0,0,1,0,-5,-3), (0,0,0,1,1,7)\}$

(b) $-2C_{1,A} = C_{2,A}$ $(a_1 \cdot C_{1,A} + a_2 \cdot C_{2,A} + a_3 \cdot C_{3,A} = 0$ when $a_1 = -2, a_2 = 1, a_3 = 0$

different from trivial solution)

$\therefore C_{1,A}, C_{2,A}, C_{3,A}$ linearly dependent.

Also, $\dim CS(A) = \dim RS(A) = 3$ → NOT linearly independent

(c) basis for $CS(A) = \{(2,1,-1,4), (0,-1,1,1), (1,1,3,-1)\}$

does NOT span $\mathbb{R}^4$.

(d) $2C_{1,B} + (-3)C_{3,B} + 7C_{4,B} = C_{5,B}$

$\therefore C_{5,A} \in \text{span}\{C_{1,A}, C_{3,A}, C_{4,A}\}$

(e) Nullspace(A) = Nullspace(B) $= \{\vec{v} \in \mathbb{R}^6 \mid B\vec{v} = 0\}$

$$\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 & -5 & -3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 - 2v_2 + 3v_5 + 2v_6 = 0$

$v_3 - 5v_5 - 3v_6 = 0$

$v_4 + v_5 + 7v_6 = 0$

Let $v_2 = t, v_5 = s, v_6 = r$

$$\begin{cases} \boxed{v_4 = -v_5 - 7v_6} \\ \boxed{v_3 = 5v_5 + 3v_6} \\ \boxed{v_1 = 2v_2 - 3v_5 - 2v_6} \end{cases}$$

$$\vec{v} = \begin{bmatrix} 2t - 3s - 2r \\ t \\ 5s + 3r \\ -s - 7r \\ s \\ r \end{bmatrix}$$

$$\rightarrow \vec{v} = t\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - s\begin{bmatrix} 3 \\ 0 \\ 5 \\ -1 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -2 \\ 0 \\ 3 \\ -7 \\ 0 \\ 1 \end{bmatrix} = \text{span}\left\{ \begin{array}{l} (2,1,0,0,0,0), \\ (3,0,5,-1,1,0), \\ (-2,0,3,-7,0,1) \end{array} \right\}$$

(f) Rank(A) + Nullity(A) = 6 → $\therefore$ Nullity(A) = 3

Rank(A) = dim RS(A) = 3 → $\therefore$ Rank(A) = 3

Problem 2:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix}$$

(a) $\det(\lambda I - A) = 0 \rightarrow \begin{vmatrix} \lambda-1 & 0 & 0 \\ 1 & \lambda-1 & -1 \\ 1 & 2 & \lambda-4 \end{vmatrix} = 0$

$$\det(\lambda I - A) = (\lambda-1)\left[(\lambda-1)(\lambda-4)+2\right] = (\lambda-1)(\lambda^2-5\lambda+4)+2\lambda-2$$
$$= \lambda^3 - 5\lambda^2 + 4\lambda - \lambda^2 + 5\lambda - 4 + 2\lambda - 2$$
$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-1)(\lambda^2-5\lambda+6) = (\lambda-1)(\lambda-2)(\lambda-3)$$

$\rightarrow \therefore \lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

① $\lambda_1 = 1$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right]$$

$x_1 = x_3$

$x_1 + 2x_2 - 3x_3 = 0 \rightarrow x_2 = x_3$

let $x_3 = t \in \mathbb{R} - \{0\}$

(t can equal 0 for eigenspace)

$$\vec{x}_{\lambda_1=1} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$(\text{Eigenspace})_{\lambda=1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$

② $\lambda_2 = 2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right]$$

$x_1 = 0$

$x_2 = x_3$

let $x_3 = t \in \mathbb{R} - \{0\}$

$\vec{x}_{\lambda_2=2} = t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow (\text{Eigenspace})_{\lambda_2=2} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$

③ $\lambda_3 = 3$:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

$x_1 = 0$

$x_2 = \frac{1}{2}x_3$ let $x_3 = t \in \mathbb{R} - \{0\}$

$\vec{x}_{\lambda_3=3} = t\begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} \rightarrow (\text{Eigenspace})_{\lambda_3=3} = \text{span}\left\{\begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}\right\}$

(b) Basis for $\mathbb{R}^3$: $\{(1,1,1),\ (0,1,1),\ (0,1/2,1)\}$

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(2) c) $\vec{w}_1 = \vec{v}_1 = (1,1,1)$

$$\vec{w}_2 = \vec{v}_2 - \frac{\langle \vec{w}_1, \vec{v}_2 \rangle}{|\vec{w}_1|^2}\vec{w}_1 = (0,1,1) - \frac{2}{\sqrt{3}}(1,1,1)$$

$$\therefore \vec{w}_2 = \left(-\frac{2}{3}, \frac{1}{3}, 1/3\right), \quad |\vec{w}_2|^2 = \frac{2}{3}$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\langle \vec{w}_1, \vec{v}_3 \rangle}{|\vec{w}_1|^2}(\vec{w}_1) - \frac{\langle \vec{w}_2, \vec{v}_3 \rangle}{|\vec{w}_2|^2}(\vec{w}_2)$$

$$= (0, 1/2, 1) - \frac{1}{2}(1,1,1) - \frac{1/2}{2/4}(-2/3, 1/3, 1/3)$$

$$= (0, 1/2, 1) - (1/2, 1/2, 1/2) - (-1/2, 1/4, 1/4)$$

$$\therefore \vec{w}_3 = (0, -1/4, 1/4), \quad |\vec{w}_3|^2 = 1/8$$

$$\Big(\underbrace{\frac{\vec{w}_1}{|\vec{w}_1|}}_{\vec{e}_1}, \underbrace{\frac{\vec{w}_2}{|\vec{w}_2|}}_{\vec{e}_2}, \underbrace{\frac{\vec{w}_3}{|\vec{w}_3|}}_{\vec{e}_3}\Big) \text{ orthonormal basis}$$

$$\vec{e}_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), \quad \vec{e}_2 = \frac{(-2/3, 1/3, 1/3)}{\sqrt{2/3}} = \left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}\right)$$

$$\vec{e}_3 = \frac{(0, -1/4, 1/4)}{1/\sqrt{8}} = \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$\therefore$ ORTHONORMAL BASIS:

$$\boxed{B = \left\{ (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}), \left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}\right), \left(0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \right\}}$$

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Problem 3: $T(f) = \int_{-1}^{2} f(x)\,dx \qquad T: C[-1,2] \longrightarrow \mathbb{R}$

(a) $T(f+g) = \int_{-1}^{2} [f(x)+g(x)]\,dx = \int_{-1}^{2} f(x)\,dx + \int_{-1}^{2} g(x)\,dx = T(f)+T(g)$

$T(cf) = \int_{-1}^{2} cf(x)\,dx = c\int_{-1}^{2} f(x)\,dx = cT(f)$

$\therefore$ T linear transformation.

(b) $T(f) = \int_{-1}^{2} \cos\pi x\,dx = \frac{\sin\pi x}{\pi}\Big|_{-1}^{2} = 0 \quad (\because \frac{\sin 2\pi}{\pi} = 0, \frac{\sin(-\pi)}{\pi} = 0)$

$T(g) = \int_{-1}^{2} x^3\,dx = \frac{1}{4}x^4\Big|_{-1}^{2} = 4 - \frac{1}{4} = \frac{15}{4}$

$\therefore T(f) = 0$, $T(g) = \frac{15}{4}$

(c) Under T: one to one?

$\left.\begin{array}{l} T(0) = 0, \\ T(\cos\pi x) = 0 \end{array}\right\} \rightarrow 0, \cos\pi x \in \ker(T) \rightarrow \ker(T) \neq \{0\}$

$\rightarrow$ T NOT one-to-one.

onto?

from B, $T(x^3) \neq 0$, so $\text{Range}(T) \subseteq \mathbb{R}$ (codomain of T), Range(T) contains an element other than 0, $\dim(\text{Range}(T)) \geq 1$

$\dim(\text{Range}(T))$ cannot be > 1 $\because \dim(\mathbb{R}) = 1$

$\therefore \dim(\text{Range}(T)) = 1$

$\rightarrow (\text{Range}(T)) = \mathbb{R}$

$\therefore$ T onto.

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Problem 4 :

$$\begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 + 8 \\ z_1 + z_2 - 3 & w_1 + w_2 \end{bmatrix}$$

$$\lambda \odot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \lambda x & \lambda y + \lambda(8) - 8 \\ \lambda z - 3\lambda + 3 & \lambda w \end{bmatrix}$$

(a) $\forall$ (for all) $\vec{u} = \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 & (-8) \\ (+3) & 0 \end{bmatrix}$

Such that $\vec{u} + \vec{v} = \begin{bmatrix} x_1 & y_1 + (-8) + 8 \\ z_1 + 3 - 3 & w_1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}$

$$\therefore \vec{v} = \begin{bmatrix} 0 & -8 \\ 3 & 0 \end{bmatrix} = \vec{0}$$

(b) $\vec{v} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, we want $\vec{v} + (-\vec{v}) = 0$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} -x & -y-16 \\ -z+6 & -w \end{bmatrix}$$

$$= \begin{bmatrix} x - x & y - y - 16 + 8 \\ z - z + 6 - 3 & w - w \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 3 & 0 \end{bmatrix}$$

$$\therefore -\vec{v} = \begin{bmatrix} -x & -y-16 \\ -z+6 & -w \end{bmatrix}$$

(c) $\because \lambda \odot (\vec{u} \oplus \vec{v})$

$$= \lambda \odot \left(\begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} \right)$$

$$= \lambda \odot \left(\begin{bmatrix} x_1 + x_2 & y_1 + y_2 + 8 \\ z_1 + z_2 - 3 & w_1 + w_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \lambda(x_1 + x_2) & \lambda(y_1 + y_2 + 8) + 8\lambda - 8 \\ \lambda(z_1 + z_2 - 3) - 3\lambda + 3 & \lambda(w_1 + w_2) \end{bmatrix}$$

$$(\lambda \odot \vec{u}) \oplus (\lambda \odot \vec{v}) = \begin{bmatrix} \lambda x_1 & \lambda y_1 + 8\lambda - 8 \\ \lambda z_1 - 3\lambda + 3 & \lambda w_1 \end{bmatrix} \oplus \begin{bmatrix} \lambda x_2 & \lambda y_2 + 8\lambda - 8 \\ \lambda z_2 - 3\lambda + 3 & \lambda w_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda x_1 + \lambda x_2 & \lambda y_1 + 8\lambda - 8 + \lambda y_2 + 8\lambda - \cancel{8} + \cancel{8} \\ \lambda z_1 - 3\lambda + 3 + \lambda z_2 - 3\lambda + 3 - 3 & \lambda w_1 + \lambda w_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda(x_1 + x_2) & \lambda(y_1 + y_2 + 8) + 8\lambda - 8 \\ \lambda(z_1 + z_2 - 3) - 3\lambda + 3 & \lambda(w_1 + w_2) \end{bmatrix}$$

$$\therefore \lambda \odot (\vec{u} \oplus \vec{v}) = (\lambda \odot \vec{u}) \oplus (\lambda \odot \vec{v})$$

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(ii) $(\lambda\mu) \odot \vec{u} = \begin{bmatrix} (\lambda\mu)x & (\lambda\mu)y + 8(\lambda\mu) - 8 \\ (\lambda\mu)z - 3(\lambda\mu) + 3 & (\lambda\mu)w \end{bmatrix}$

$\lambda \odot (\mu \odot \vec{u})$

$= \lambda \odot \begin{bmatrix} \mu x & \mu y + 8\mu - 8 \\ \mu z - 3\mu + 3 & \mu w \end{bmatrix}$

$= \begin{bmatrix} \lambda\mu x & \lambda(\mu y + 8\mu - 8) + 8\lambda - 8 \\ \lambda(\mu z - 3\mu + 3) - 3\lambda + 3 & \lambda w \end{bmatrix}$

$= \begin{bmatrix} \lambda\mu x & \lambda\mu y + 8\lambda\mu - \cancel{8\lambda} + \cancel{8\lambda} - 8 \\ \lambda\mu z - 3\lambda\mu + \cancel{3\lambda} - \cancel{3\lambda} + 3 & \lambda w \end{bmatrix}$

$\therefore (\lambda\mu) \odot \vec{u} = \lambda \odot (\mu \odot \vec{u})$

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Problem 5.

(a) $\dim(V) = n \longrightarrow \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ must be linearly independent.

Let $\dim(V) = 3$, $V = \mathbb{R}^3$

$\{\underset{\vec{v}_1}{(1,2,3)}, \underset{\vec{v}_2}{(-1,-2,-3)}\}$ set of 2 vectors

but $\vec{v}_1 + \vec{v}_2 = 0$

$\therefore$ False.

(b) $U \leq V$, $W \leq V \longrightarrow U \cap W \leq V$ ($\leq$ means subspace)

Test for subspace:

① $U \leq V \to \vec{0} \in U$, $W \leq V \to \vec{0} \in W$

$\therefore 0 \in U \cap W$

② $u, v \in U \cap W \to \vec{u} + \vec{v} \in U \cap W$

$\to \vec{u}, \vec{v} \in U \to \vec{u} + \vec{v} \in U$

$\to \vec{u}, \vec{v} \in W \to \vec{u} + \vec{v} \in W$

$\to \therefore \vec{u} + \vec{v} \in U \cap W$

③ $\vec{u} \in U \cap W \to c\vec{u} \in U \cap W$

$\vec{u} \in U \to c\vec{u} \in U$

$\vec{u} \in W \to c\vec{u} \in W$ $\longrightarrow$ $c\vec{u} \in U \cap W$

$\therefore U \cap W \leq V$. TRUE

(c) $W = \{(x, xy, y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\} \leq \mathbb{R}^3$

$\to$ Test for subspace:

① $W \neq \emptyset \to \vec{0} \in W$ $\because x = 0, y = 0 \to (0,0,0) \in W$.

② $\vec{u}, \vec{v} \in W \to \vec{u} + \vec{v} \in W$

$(x_1, x_1y_1, y_1) + (x_2, x_2y_2, y_2)$

$= ((x_1+x_2), x_1y_1 + x_2y_2, (y_1+y_2))$

but if $x = x_1 + x_2$, $y = y_1 + y_2$

$\to (x_1+x_2, (x_1+x_2)(y_1+y_2), (y_1+y_2))$

$\to (x_1+x_2, (x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2), (y_1+y_2))$

$\overset{\parallel}{x_1y_1 + x_2y_2}$ iff $x_1y_2 = -x_2y_1$ $\not\Leftrightarrow$

$\therefore W \not\leq \mathbb{R}^3$. FALSE

easier method!

$\vec{u} = (1,1,1) \in W$

but $2\vec{u} = (2,2,2) \notin W$

$\therefore W$ not closed under scaler multiplication.

(d) $A^2 = 0$, $B \sim A \rightarrow B^2 = 0$

$B \sim A \rightarrow B = P^{-1}AP$

$\rightarrow B^2 = (P^{-1}AP)^2 = (P^{-1}AP)(P^{-1}AP)$

$= P^{-1}A(PP^{-1})AP$

$= P^{-1}A^2P$

$= P^{-1}(0)P$

$= 0$ TRUE

$\therefore B^2 = 0$

(f) $\|\vec{u}\| \leq 1, \|\vec{v}\| \leq 1 \longrightarrow |\langle\vec{u},\vec{v}\rangle| \leq 1$

$|\langle\vec{u},\vec{v}\rangle| \leq \|\vec{u}\|\,\|\vec{v}\|$ (cauchy Shwertz.)

$|\langle\vec{u},\vec{v}\rangle| \leq 1 \cdot 1$

$\therefore |\langle\vec{u},\vec{v}\rangle| \leq 1$

$\therefore$ TRUE

(g) Test for subspace : $\ker(T) \leq V$: ① $T(\vec{0}) = \vec{0} \rightarrow \vec{0} \in \ker(T)$.

② $\vec{a} \in \ker(T), \vec{b} \in \ker(T) \rightarrow T(\vec{a}) = \vec{0}, T(\vec{b}) = \vec{0}$

$T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{0} + \vec{0}$

$\therefore T(\vec{a}+\vec{b}) = \vec{0}$

$\therefore \vec{a}+\vec{b} \in \ker(T)$.

③ $\vec{a} \in \ker(T) \xrightarrow{\text{WTS}} c\vec{a} \in \ker(T)$

$T(c\vec{a}) = cT(\vec{a}) = c \cdot \vec{0} = \vec{0}$

$\therefore c\vec{a} \in \ker(T)$

$\therefore \ker(T) \leq V$.

TRUE

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