Relaxed-inertial derivative-free algorithm for systems of nonlinear pseudo-monotone equations

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1. Numerical results

Table 1: *Numerical result for Problem 1-3*

		Number of function evaluations				Number of iterations				CPU time			
Problem (Dim)	Starting point	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM
prblem 1 in [1](5000)	1	26	284	50	15	10	32	16	6	0.017	0.160	0.057	0.012
	2	30	44	27	50	11	11	10	17	0.015	0.036	0.019	0.031
	3	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	4	39	161	37	62	14	24	13	21	0.03485	0.08489	0.02815	0.0394
	5	39	153	37	65	14	22	13	22	0.02335	0.07452	0.02363	0.03983
	6	32	43	29	56	12	11	10	19	0.01784	0.02478	0.02065	0.03824
	7	48	297	50	43	16	35	16	15	0.03248	0.14126	0.03178	0.02635
prblem 1 in [1](10000)	1	19	415	58	15	8	41	18	6	0.02204	0.35326	0.05832	0.01745
	2	30	44	27	50	11	11	10	17	0.03062	0.04178	0.02864	0.05034
	3	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	4	39	230	46	65	14	29	15	22	0.0662	0.20844	0.05066	0.06972
	5	39	226	46	65	14	28	15	22	0.04116	0.19595	0.04722	0.06655
	6	38	48	29	56	14	12	10	19	0.04	0.04857	0.03004	0.05751
	7	43	427	58	35	15	44	18	13	0.04725	0.35469	0.06017	0.03778
prblem 2 in [1](5000)	1	37	237	49	32	11	27	15	9	0.01973	0.09281	0.02886	0.01982
	2	37	49	28	63	10	13	10	20	0.01967	0.02534	0.01666	0.03536
	3	48	620	92	100	14	56	21	26	0.02588	0.26298	0.04787	0.05221
	4	52	215	41	83	14	45	13	22	0.0275	0.09388	0.02481	0.04305
	5	52	215	41	83	14	45	13	22	0.02442	0.10038	0.02325	0.04284
	6	38	94	27	46	11	30	9	15	0.01774	0.04789	0.01695	0.02707
	7	59	280	49	115	16	41	15	29	0.02591	0.11484	0.0292	0.05689
prblem 2 in [1](10000)	1	37	368	59	32	11	37	17	9	0.02828	0.22575	0.04872	0.02594
	2	37	49	28	63	10	13	10	20	0.02791	0.03816	0.02516	0.0516
	3	86	937	123	95	22	77	26	25	0.05856	0.60854	0.09346	0.07665
	4	52	281	45	95	14	51	14	25	0.03604	0.18745	0.04075	0.07294
	5	52	281	45	87	14	51	14	23	0.0412	0.19043	0.0392	0.06897
	6	38	94	27	46	11	30	9	15	0.02964	0.07294	0.02572	0.0428
	7	50	415	59	69	14	52	17	20	0.04016	0.3011	0.05383	0.05614
prblem 3 in [1](5000)	1	33	482	68	33	12	44	19	11	0.02609	0.27766	0.05653	0.02597
	2	35	859	93	36	12	72	24	12	0.02887	0.47489	0.08137	0.04343
	3	42	1639	139	40	14	122	33	13	0.03191	0.91858	0.09522	0.03424
	4	33	672	83	33	12	59	22	11	0.02689	0.3989	0.06599	0.02934
	5	33	664	83	33	12	57	22	11	0.02803	0.38067	0.06002	0.02524
	6	35	859	93	36	12	71	24	12	0.03992	0.49148	0.06789	0.03006
	7	33	486	68	33	12	45	19	11	0.02523	0.27913	0.05199	0.02684
prblem 3 in [1](10000)	1	33	735	85	34	12	62	23	12	0.04171	0.70765	0.09943	0.0425
	2	42	1295	113	38	14	99	28	13	0.05175	1.25861	0.13077	0.04891
	3	62	2474	211	46	18	169	45	14	0.07996	2.39376	0.23022	0.05595
	4	49	1012	100	38	15	80	26	13	0.05648	0.9731	0.117	0.048
	5	49	1016	100	38	15	81	26	13	0.05624	1.08548	0.11673	0.04746
	6	42	1295	113	38	14	99	28	13	0.05531	1.19577	0.12308	0.05226
	7	33	748	85	34	12	65	23	12	0.04031	0.70774	0.10127	0.04029

Table 2: *Numerical result for Problem 4-6*

		Number of function evaluations				Number of iterations				CPU time			
Problem (Dim)	Starting point	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM
prblem 4 in [1](5000)	1	3149	898	564	1386	185	160	47	80	1.085773	0.359339	0.213584	0.521681
	2	4988	Failed	7439	2719	293	Failed	623	158	1.669196	Failed	2.66745	0.937132
	3	487	857	474	1078	29	88	40	63	0.193942	0.349756	0.200556	0.443797
	4	2900	3089	553	1649	170	919	46	95	0.996182	1.298784	0.209493	0.602408
	5	2543	Failed	9258	2355	149	Failed	774	137	0.860763	Failed	3.321901	0.843097
	6	3212	Failed	16316	2520	188	Failed	1357	147	1.079647	Failed	6.13913	0.874484
	7	2941	Failed	564	1293	173	Failed	47	75	1.02835	Failed	0.217214	0.469713
prblem 4 in [1](10000)	1	3836	1242	997	1433	213	212	79	78	2.17218	0.773562	0.60507	0.85021
	2	4427	Failed	Failed	2838	245	Failed	Failed	157	2.385417	Failed	Failed	1.615986
	3	711	1252	660	1228	40	111	52	68	0.464	0.820937	0.465186	0.825613
	4	1561	Failed	1077	1543	86	Failed	86	84	0.880758	Failed	0.651819	0.899035
	5	4990	Failed	Failed	4113	276	Failed	Failed	228	2.631542	Failed	Failed	2.249725
	6	9054	Failed	Failed	2580	501	Failed	Failed	143	4.69084	Failed	Failed	1.47119
	7	3426	Failed	1011	1481	189	Failed	81	82	1.929764	Failed	0.624986	0.871603
prblem 6 in [1](5000)	1	35	317	66	52	9	37	16	13	0.016811	0.117864	0.031656	0.028002
	2	917	177	55	110	162	46	15	27	0.355543	0.078243	0.027491	0.048457
	3	71	598	118	50	17	69	27	12	0.031104	0.223628	0.057724	0.024896
	4	Failed	Failed	Failed	260	Failed	Failed	Failed	57	Failed	Failed	Failed	0.127237
	5	38	169	36	83	13	24	12	28	0.02006	0.065951	0.019393	0.042394
	6	59	73	41	60	15	18	13	17	0.027853	0.031178	0.023869	0.031448
	7	141	379	84	193	31	52	22	41	0.058388	0.143569	0.040439	0.081846
prblem 6 in [1](10000)	1	35	466	78	52	9	48	19	13	0.025065	0.246329	0.050887	0.035522
	2	694	84	56	133	126	21	16	32	0.381931	0.05614	0.038128	0.086502
	3	78	882	132	56	18	90	28	13	0.049127	0.505522	0.0922	0.038184
	4	378	Failed	Failed	334	83	Failed	Failed	72	0.270817	Failed	Failed	0.242296
	5	37	247	46	86	13	31	15	29	0.029032	0.130988	0.032885	0.061532
	6	70	77	36	74	18	19	12	22	0.0448	0.049114	0.02752	0.050251
	7	110	530	95	203	25	65	24	44	0.064833	0.293818	0.06347	0.120078
prblem 7 in [1](5000)	1	1	1	1	1	1	1	1	1	1.73E-05	2.13E-05	0.00002	2.76E-05
	2	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	3	Failed	411	79	Failed	Failed	47	18	Failed	Failed	0.17903	0.040938	Failed
	4	Failed	444	82	251	Failed	52	21	53	Failed	0.177703	0.043942	0.12653
	5	Failed	440	79	179	Failed	48	20	35	Failed	0.181132	0.041344	0.084474
	6	75	153	53	134	16	23	14	28	0.034254	0.068306	0.031291	0.062045
	7	Failed	1624	88	Failed	Failed	422	22	Failed	Failed	0.758529	0.053823	Failed
prblem 7 in [1](10000)	1	1	1	1	1	1	1	1	1	2.38E-05	3.66E-05	1.92E-05	2.51E-05
	2	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	3	106	625	100	118	22	64	21	24	0.085257	0.383352	0.085001	0.098145
	4	Failed	640	84	243	Failed	60	22	51	Failed	0.418113	0.070217	0.194882
	5	Failed	1123	84	177	Failed	222	21	35	Failed	0.811575	0.07401	0.141265
	6	81	219	56	239	17	30	14	49	0.062938	0.149442	0.048894	0.182425
	7	Failed	8254	109	Failed	Failed	2567	27	Failed	Failed	6.890287	0.096526	Failed

Table 3: *Numerical result for Problem 7-9*

		Number of function evaluations				Number of iterations				CPU time			
Problem (Dim)	Starting point	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM
prblem 8 in [1](5000)	1	318	383	83	269	65	65	22	54	0.191721	0.235493	0.055953	0.172228
	2	168	118	61	225	36	34	17	48	0.112114	0.073945	0.041813	0.136047
	3	413	810	104	299	77	92	28	63	0.247684	0.452597	0.074971	0.183792
	4	295	268	69	324	64	55	19	67	0.182231	0.157046	0.047446	0.194332
	5	295	291	69	341	64	63	19	71	0.181612	0.17922	0.046951	0.205215
	6	200	127	59	236	44	35	16	50	0.129075	0.079962	0.039258	0.147997
	7	213	384	84	317	38	65	23	64	0.132458	0.22051	0.06208	0.18963
prblem 8 in [1](10000)	1	325	529	91	286	67	82	24	59	0.323966	0.497983	0.092973	0.287296
	2	168	119	61	231	36	34	17	49	0.187147	0.121656	0.066201	0.230619
	3	431	1138	96	266	77	107	26	54	0.41163	1.065264	0.104235	0.257017
	4	303	339	72	317	64	64	20	66	0.301317	0.324424	0.082284	0.323183
	5	320	327	69	323	68	60	20	67	0.314181	0.307579	0.071801	0.304962
	6	200	125	59	236	44	34	16	50	0.196378	0.129972	0.062515	0.250961
	7	262	529	82	471	50	82	22	89	0.259643	0.510558	0.089811	0.459968
prblem 9 in [1](5000)	1	302	394	76	173	67	80	19	40	0.139525	0.17384	0.039631	0.085311
	2	195	308	104	231	43	82	28	52	0.094411	0.15261	0.056584	0.111687
	3	298	937	156	199	66	136	37	47	0.135067	0.379842	0.081641	0.097526
	4	200	281	106	194	44	66	29	44	0.084703	0.131396	0.054749	0.091129
	5	200	266	106	194	44	61	29	44	0.092611	0.128874	0.057483	0.088038
	6	191	330	97	222	42	90	27	50	0.084857	0.156564	0.052919	0.106974
	7	232	458	119	232	51	99	31	51	0.108876	0.215444	0.063856	0.106085
prblem 9 in [1](10000)	1	321	457	83	167	71	77	20	39	0.250235	0.356785	0.074692	0.136106
	2	219	331	99	225	48	86	27	52	0.176008	0.270181	0.085496	0.188311
	3	296	1294	187	220	65	161	42	50	0.238111	0.923154	0.153899	0.181418
	4	253	335	106	183	56	74	28	42	0.217198	0.26706	0.09485	0.149627
	5	253	309	106	193	56	65	28	44	0.210047	0.240532	0.088467	0.158738
	6	98	326	62	202	21	85	17	46	0.09309	0.269334	0.054649	0.169187
	7	271	517	131	199	60	95	33	46	0.233249	0.39352	0.12017	0.159983
prblem 10 in [1](5000)	1	30	232	49	24	11	27	15	9	0.015489	0.081754	0.025947	0.014498
	2	38	139	29	34	12	46	10	11	0.019095	0.064105	0.017273	0.017443
	3	54	675	70	43	16	57	21	15	0.024742	0.233222	0.041691	0.025955
	4	43	135	39	86	15	22	13	29	0.021023	0.050915	0.022663	0.042178
	5	43	131	39	86	15	21	13	29	0.021793	0.05277	0.022274	0.044692
	6	38	36	22	63	13	12	8	21	0.01982	0.017834	0.011059	0.034037
	7	73	286	49	93	20	45	15	31	0.032549	0.119565	0.024432	0.050881
prblem 10 in [1](10000)	1	30	357	58	24	11	37	17	9	0.024661	0.198309	0.04273	0.024083
	2	38	142	29	34	12	47	10	11	0.026541	0.093417	0.022118	0.029068
	3	58	1036	78	35	17	81	23	13	0.042041	0.568169	0.060478	0.030882
	4	43	199	43	89	15	28	14	30	0.032426	0.114139	0.034498	0.06621
	5	43	199	43	89	15	28	14	30	0.03207	0.11393	0.032108	0.06479
	6	38	36	22	63	13	12	8	21	0.028429	0.027324	0.017686	0.052231
	7	59	370	58	81	18	41	17	27	0.045153	0.206874	0.042074	0.069811

Table 4: *Numerical result for Problem 10-12*

		Number of function evaluations				Number of iterations				CPU time			
Problem (Dim)	Starting point	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM	IM3TFR2	M3TFR2	RIDFA	RTTCGPM
prblem 11 in [1](5000)	1	Failed	Failed	16042	Failed	Failed	Failed	4456	Failed	Failed	Failed	9.570082	Failed
	2	Failed	Failed	16069	Failed	Failed	Failed	4463	Failed	Failed	Failed	9.625643	Failed
	3	Failed	Failed	16111	Failed	Failed	Failed	4471	Failed	Failed	Failed	9.620829	Failed
	4	Failed	Failed	16053	Failed	Failed	Failed	4459	Failed	Failed	Failed	9.537039	Failed
	5	Failed	Failed	16053	Failed	Failed	Failed	4459	Failed	Failed	Failed	9.550588	Failed
	6	Failed	Failed	16068	Failed	Failed	Failed	4463	Failed	Failed	Failed	9.726171	Failed
	7	Failed	Failed	16042	Failed	Failed	Failed	4456	Failed	Failed	Failed	9.503744	Failed
prblem 11 in [1](10000)	1	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	2	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	3	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	4	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	5	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	6	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	7	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
problem 3 in [2](5000)	1	Failed	Failed	15314	Failed	Failed	Failed	3696	Failed	Failed	Failed	8.396181	Failed
	2	204	482	124	227	39	75	29	42	0.099056	0.235863	0.072115	0.119655
	3	226	386	148	Failed	43	68	34	Failed	0.115769	0.183322	0.083955	Failed
	4	Failed	690	160	Failed	Failed	75	38	Failed	Failed	0.313728	0.09852	Failed
	5	270	676	177	Failed	52	71	42	Failed	0.132842	0.305799	0.102299	Failed
	6	Failed	426	123	174	Failed	60	29	32	Failed	0.195832	0.071387	0.087551
	7	Failed	855	230	Failed	Failed	94	55	Failed	Failed	0.401125	0.130213	Failed
problem 3 in [2](10000)	1	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed	Failed
	2	209	632	113	213	40	90	26	40	0.172145	0.491073	0.107167	0.20182
	3	226	483	152	Failed	43	68	35	Failed	0.186609	0.394139	0.144373	Failed
	4	Failed	991	168	Failed	Failed	96	40	Failed	Failed	0.721092	0.153791	Failed
	5	Failed	969	163	Failed	Failed	89	39	Failed	Failed	0.742737	0.146118	Failed
	6	Failed	582	136	205	Failed	72	32	38	Failed	0.452735	0.12382	0.187393
	7	Failed	1235	243	Failed	Failed	121	58	Failed	Failed	0.942996	0.212947	Failed
problem 4 in [2](5000)	1	1	1	1	1	1	1	1	1	4.25E-05	2.81E-05	2.32E-05	1.65E-05
	2	171	324	69	221	26	43	13	32	0.170102	0.322257	0.076919	0.231139
	3	Failed	924	184	232	Failed	87	34	35	Failed	0.914949	0.204655	0.243218
	4	147	180	58	172	23	27	12	25	0.152233	0.18969	0.064358	0.208573
	5	859	194	58	137	142	32	12	20	0.900485	0.206558	0.070177	0.140858
	6	159	308	70	Failed	25	38	14	Failed	0.159051	0.313012	0.073943	Failed
	7	121	62	40	177	19	16	8	26	0.125972	0.067303	0.045914	0.183914
problem 4 in [2](10000)	1	1	1	1	1	1	1	1	1	3.04E-05	0.000024	2.34E-05	3.45E-05
	2	165	467	102	281	25	52	21	40	0.277108	0.822281	0.18845	0.49129
	3	349	1386	216	Failed	56	121	36	Failed	0.600965	2.375031	0.416585	Failed
	4	154	250	63	165	24	33	12	24	0.289921	0.440315	0.1188	0.292912
	5	762	250	64	179	126	33	13	27	1.303371	0.441502	0.121812	0.314487
	6	153	451	93	Failed	24	47	19	Failed	0.265006	0.758099	0.165211	Failed
	7	121	60	40	177	19	16	8	26	0.21933	0.110861	0.079555	0.315567

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