Math 3071

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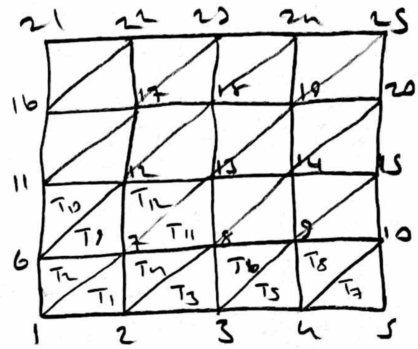
Project - 3

1) Some Remarks On the Code

- In my submission the file project3code.m contains the code that implements FEM. Also the files u.m, g.m, a.m, b.m, f.m, ux.m and uy.m contains the corresponding functions given in the question.
- In the first part of the code, we create the triangle mesh given below for $h = \frac{1}{4}$



- In the code,
  - The matrix TR is a 3-column matrix which contains the nodes of the vertices. In this case it is:

$$TR = \begin{bmatrix} 1 & 2 & 7 \\ 1 & 6 & 7 \\ 2 & 3 & 8 \\ & \vdots & \end{bmatrix}$$

  - The matrix CR is a 2-column matrix which contains x and y coordinates of the nodes. In this case it is:

$$CR = \begin{bmatrix} 0 & 0 \\ 0.25 & 0 \\ 0.50 & 0 \\ \vdots & \\ 0.75 & 1 \\ 1 & 1 \end{bmatrix}$$

  - The matrix NC is a 1-column matrix which contains the constrained nodes. In this case it is:

$$NC = [1\ 2\ 3\ 4\ 5\ 6\ 10\ 11\ \ldots\ 24\ 25]^T$$

- In the second part of the code, we determine the constrained nodes and store them in NC.
- Next, we implement the combined algorithm to form A and F at the same time
- After that we change rows of A and elements of F corresponding to constrained nodes.
- Then, the code finds the approximate and exact solutions

- Finally, the code finds L_2 error, H_1 error and plots the solution. Here, I want to note that, since I used counterclockwise orientation for odd numbered triangles and clockwise orientation for even numbered triangles, I needed to pay attention to this when calculating H_1 semi norm of the error. (I took the area of each triangle negative while calculating the gradients) This was not a problem in calculation of A and F because we were taking square of the area. However, when I code an FEM algorithm again, I will pay more attention to orientation.

2) H^1 and L^2 Errors

a) Question A, Case 1

- In this case the functions are:
  - $u(x,y) = \sin(\pi x)\sin(\pi y)$
  - $g(x,y) = u(x,y)$
  - $a(x,y) = 1$
  - $b(x,y) = 1$
  - $f(x,y) = (2\pi^2+1)\sin(\pi x)\sin(\pi y)$
  - $u_x(x,y) = \pi\cos(\pi x)\sin(\pi y)$
  - $u_y(x,y) = \pi\sin(\pi x)\cos(\pi y)$

- L^2 error is obtained as:

h	L^2 Err	p
$\frac{1}{4}$	0.0252	
$\frac{1}{8}$	0.0062	2.0231
$\frac{1}{16}$	0.0015	2.0473
$\frac{1}{32}$	3.824×10^{-4}	1.9718

- H^1 error is obtained as:

h	H^1 Err	p
$\frac{1}{4}$	1.2875	
$\frac{1}{8}$	0.6618	0.9713
$\frac{1}{16}$	0.3326	0.9926
$\frac{1}{32}$	0.1665	0.9983

b) Question A, Case 2

- In this case the functions are:

  * $u(x,y) = \sin(\pi x)\sin(\pi y)$
  * $g(x,y) = u(x,y)$
  * $a(x,y) = \dfrac{1}{1+10(x^2+y^2)}$
  * $b(x,y) = 0$
  * $f(x,y) = -(a_x u_x + a u_{xx}) - (a_y u_y + a u_{yy})$

  where,

  $a_x = -(1+10(x^2+y^2))^{-2}(20x)$

  $a_y = -(1+10(x^2+y^2))^{-2}(20y)$

  $u_x = \pi\cos(\pi x)\sin(\pi y)$

  $u_y = \pi\sin(\pi x)\cos(\pi y)$

  $u_{xx} = -\pi^2\sin(\pi x)\sin(\pi y)$

  $u_{yy} = -\pi^2\sin(\pi x)\sin(\pi y)$

- L^2 error is obtained as:

h	L^2 Er.	p
$\frac{1}{4}$	0.0130	
$\frac{1}{8}$	0.0038	1.7744
$\frac{1}{16}$	9.679×10^{-4}	1.9731
$\frac{1}{32}$	2.4356×10^{-4}	1.9906

- H^1 error is obtained as:

h	H^1 Er.	p
$\frac{1}{4}$	1.2590	
$\frac{1}{8}$	0.6564	0.9396
$\frac{1}{16}$	0.3319	0.9838
$\frac{1}{32}$	0.1664	0.9961

c) Question B

- In this case the functions are:

$$* \; u(x,y) = \begin{cases} x^2y^3 + \cos(xy) & , \quad 0 \le x \le 1/2 \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\left(\frac{2x+9}{20}\right)y\right) & , \quad \frac{1}{2} < x \le 1 \end{cases}$$

$$* \; g(x,y) = u(x,y)$$

$$* \; a(x,y) = \begin{cases} 1 & , \quad 0 \le x \le 1/2 \\ 10 & , \quad 1/2 < x \le 1 \end{cases}$$

$$* \; b(x,y) = 0$$

$$* \; f(x,y) = \begin{cases} y^2\cos(xy) - 2y^3 + x^2\cos(xy) - 6x^2y & , \quad 0 \le x \le 1/2 \\ 10\left[\cos\left(\left(\frac{2x+9}{20}\right)y\right)\frac{y^2}{100} - \frac{y^3}{50} + \left(\frac{2x+9}{20}\right)^2\left(\cos\left(\left(\frac{2x+9}{20}\right)y\right) - 6y\right)\right] & , \quad \frac{1}{2} < x \le 1 \end{cases}$$

$$* \; u_x = \begin{cases} \frac{y^3}{5}\left(\frac{2x+9}{20}\right) - \sin\left(\left(\frac{2x+9}{20}\right)y\right)\frac{y}{10} & , \quad \frac{1}{2} < x \le 1 \\ 2xy^3 - y\sin(xy) & , \quad 0 \le x \le \frac{1}{2} \end{cases}$$

$$* \; u_y = \begin{cases} 3x^2y^2 - x\sin(xy) & , \quad 0 \le x \le 1/2 \\ 3\left(\frac{2x+9}{20}\right)^2 y^2 - \sin\left(\left(\frac{2x+9}{20}\right)y\right)\left(\frac{2x+9}{20}\right) & , \quad \frac{1}{2} < x \le 1 \end{cases}$$

- L^2 error is obtained as:

h	L^2 Er	p
$\frac{1}{4}$	0.0022	
$\frac{1}{8}$	4.4282×10^{-4}	2.3127
$\frac{1}{16}$	1.0960×10^{-4}	2.0145
$\frac{1}{32}$	7.1407×10^{-5}	0.6181

- H^1 error is obtained as:

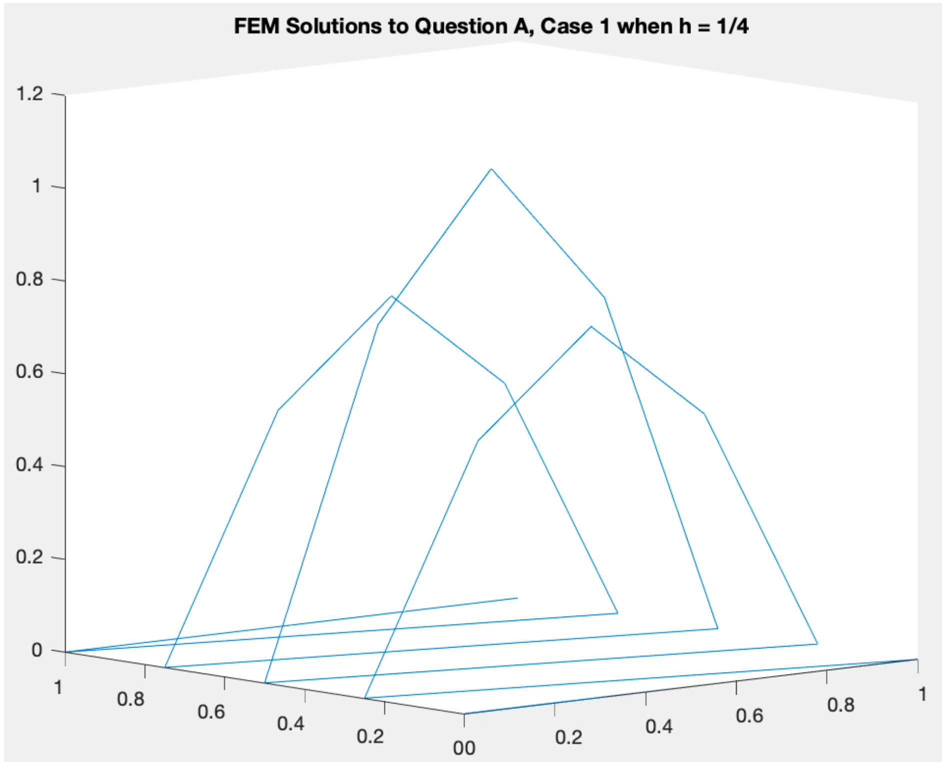
h	H^1 Er	p
$\frac{1}{4}$	0.1308	
$\frac{1}{8}$	0.0696	0.9102
$\frac{1}{16}$	0.0390	0.8356
$\frac{1}{32}$	0.0231	0.7556

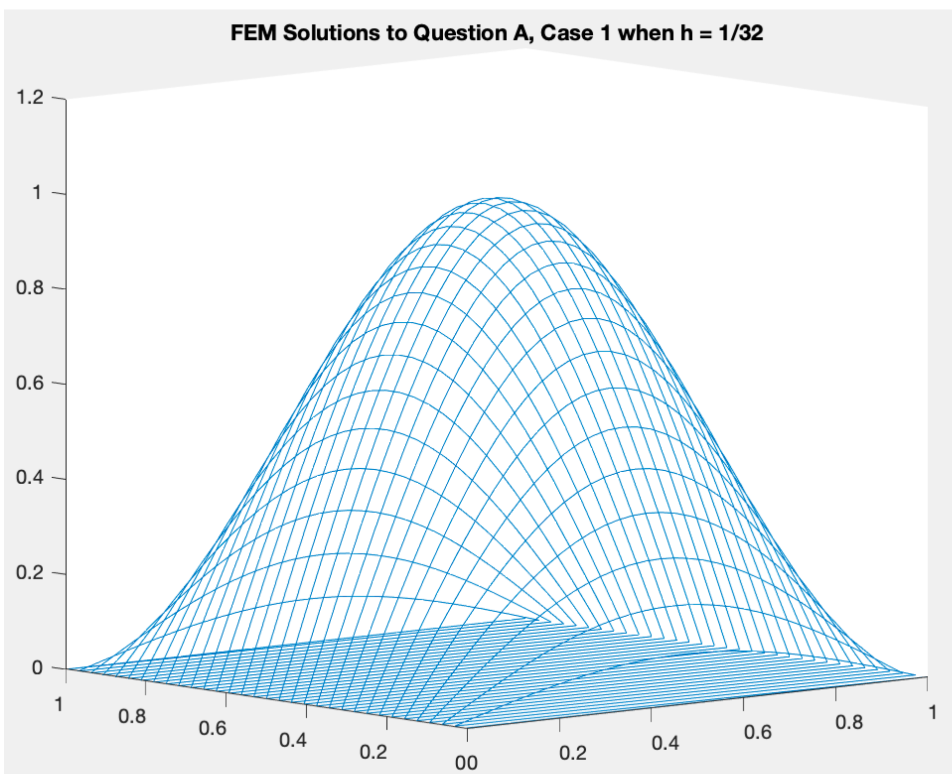
<u>Remark</u>: Here, for all cases we can estimate the order to be $p=2$ for L^2 error and $p=1$ for H^1 error, which is consistent with the theoretical results.

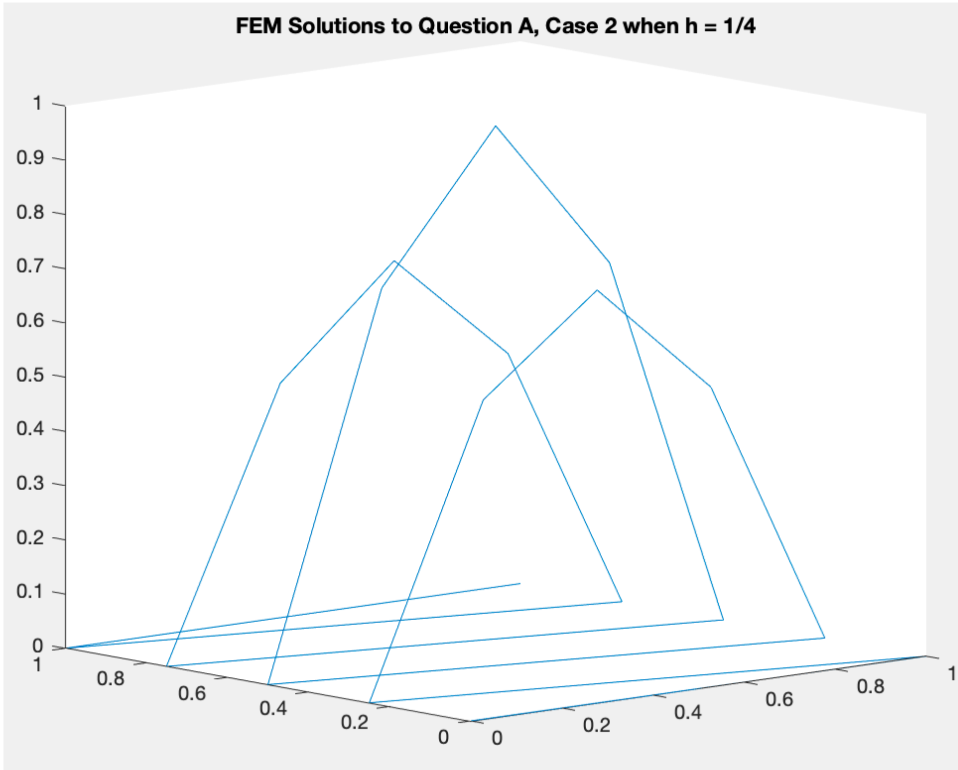
We can also note that, since we have discontinuous solution in Question B, the values of p in H^1 and L^2 errors are not as close to 1 and 2 as in the other (continuous) cases.

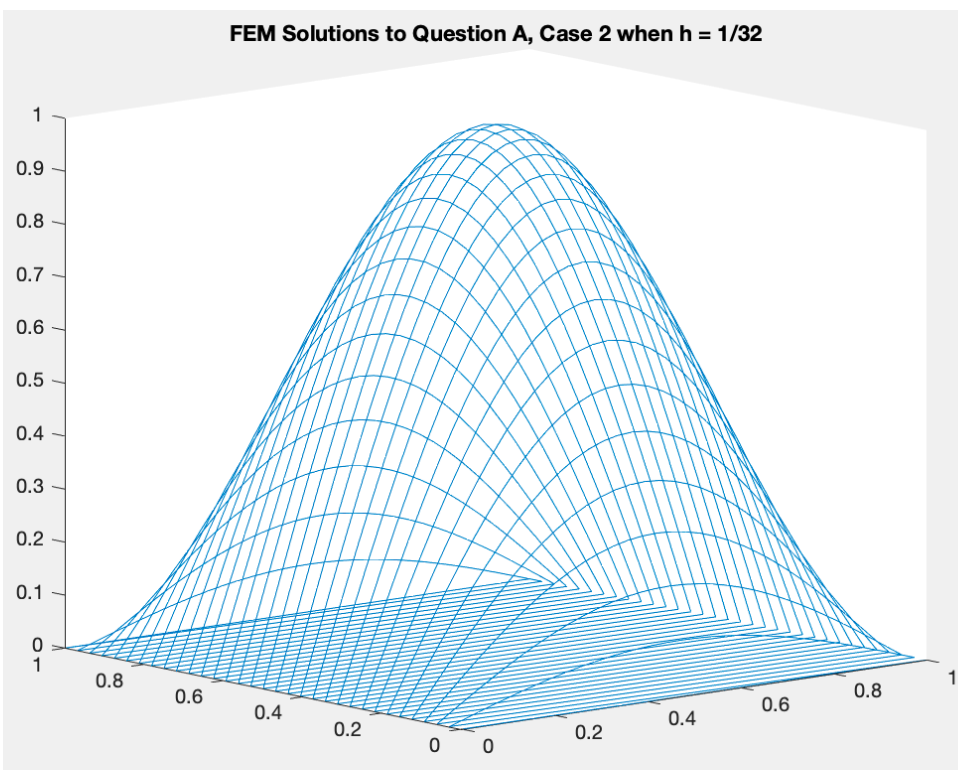
4) <u>The Graphs</u>

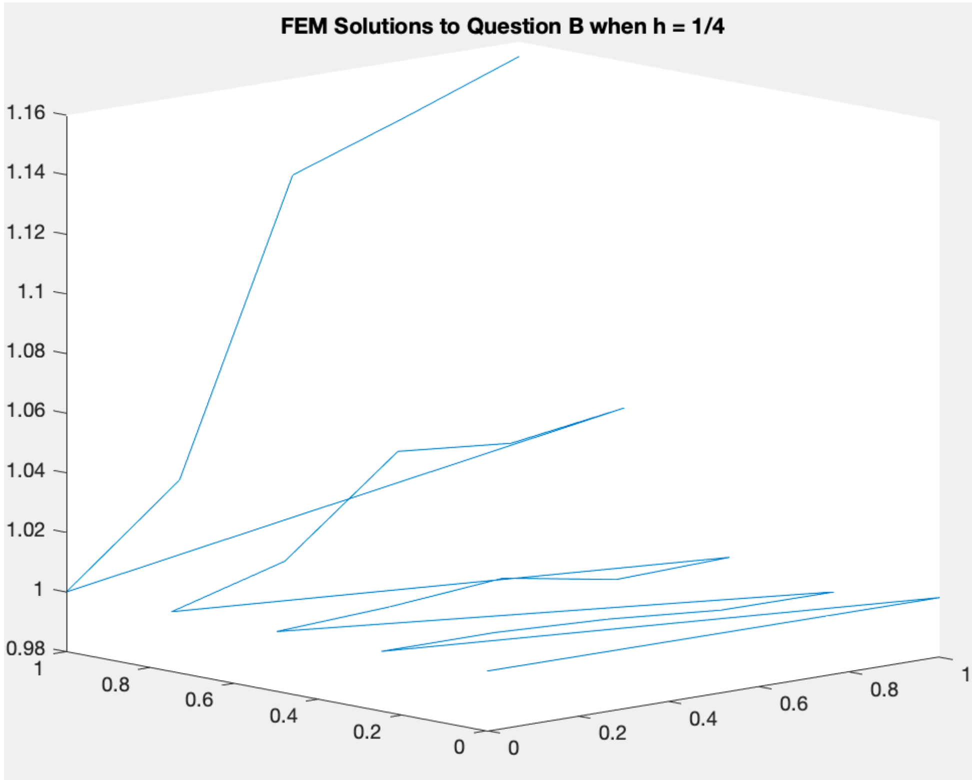
- The following 3 pages contains the graphs of all cases.

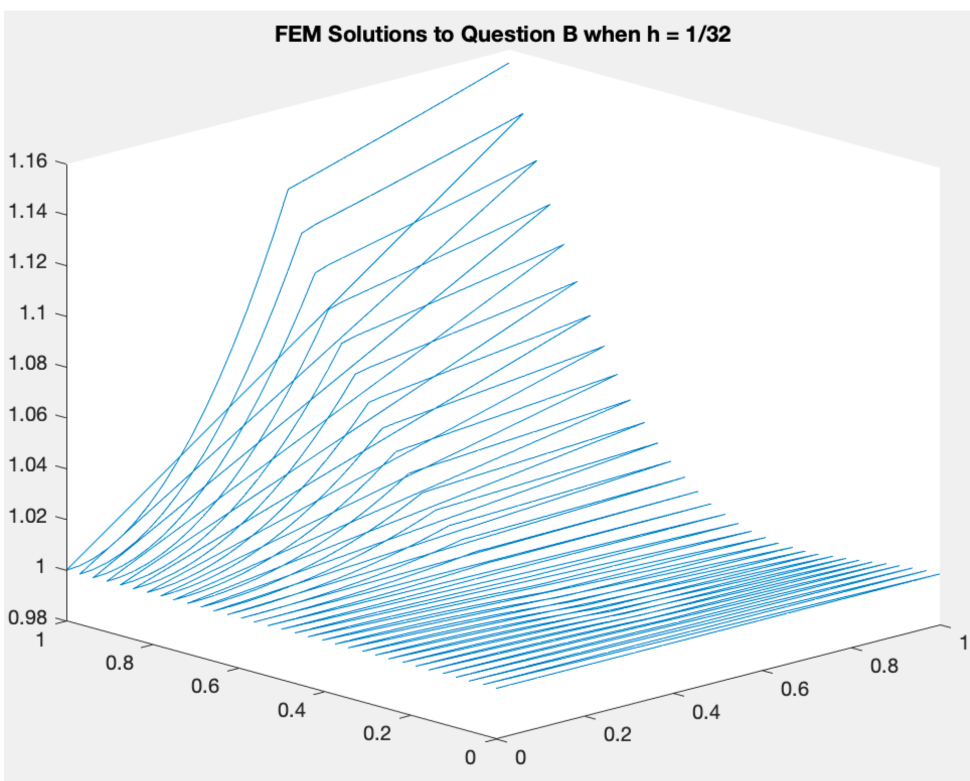

FEM Solutions to Question A, Case 1 when h = 1/4
1.2
1
0.8
0.6
0.4
0.2
0
1
0.8
0.6
0.4
0.2
00
0.2
0.4
0.6
0.8
1


FEM Solutions to Question A, Case 1 when h = 1/32
1.2
1
0.8
0.6
0.4
0.2
0
1
0.8
0.6
0.4
0.2
00
0.2
0.4
0.6
0.8
1


FEM Solutions to Question A, Case 2 when h = 1/4
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1


FEM Solutions to Question A, Case 2 when h = 1/32
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1


FEM Solutions to Question B when h = 1/4
1.16
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
0.98
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1


FEM Solutions to Question B when h = 1/32
1.16
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
0.98
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1