1 Introduction

Condensed matter physics is an area of physics that studies the physical properties of materials and their collective phenomena, such as magnetism, superconductivity. Of particular interest is the development of theoretical models, such as the Jordan-Wigner language, to study these phenomena. The Jordan-Wigner language is a mathematical formalism that provides a powerful tool for exploring quantum mechanical properties of many-body systems. In this research proposal, we will study three solved quantum models: Ising, XY, and Kitaev honeycomb models. Then we will propose a study of the extended Kiteav honeycomb model.

2 Background

2.1 Second Quantization

Second quantization is a formalism that is developed to describe and analyze quantum many-body systems. It enforces identical particle's statistics in the form of creation and annihilation operators. Any state can be generated by acting the creation and annihilation operators on a many-body vacuum state $|0\rangle$.

Fermionic operators must satisfy this set of anti-commutation relations:

$$\{c_i, c_j^\dagger\} = \delta_{ij}; \quad \{c_i^\dagger, c_j^\dagger\} = 0; \quad \{c_i, c_j\} = 0 \tag{1}$$

The action of fermionic operators is given as:

$$c_i\,|0\rangle = 0; \quad \langle 0|\, c_i^\dagger = 0 \tag{2}$$

For bosonic operators:

$$[a_i, a_j^\dagger] = \delta_{ij}; \quad [a_i^\dagger, a_j^\dagger] = 0; \quad [a_i, a_j] = 0 \tag{3}$$

The action of bosonic operators is given as:

$$a_i\,|0\rangle = 0; \quad \langle 0|\, a_i^\dagger = 0 \tag{4}$$

2.2 Spin Hamiltonians

Spin Hamiltonians refer to a type of mathematical model used to describe the behavior of interacting spin systems. These Hamiltonians typically consist of sums of spin operators.

Spin operators are operators that satisfy this commutation relation:

$$[S^i, S^j] = i\hbar\varepsilon_{ijk}\, S^k; \quad [\sigma^i, \sigma^j] = 2i\varepsilon_{ijk}\,\sigma^k \tag{5}$$

Where i, j and k can be x, y and z. Moreover, they fulfill this anti-commutation relation:

$$\{S^i, S^j\} = \hbar\delta_{ij}\, I; \quad \{\sigma^i, \sigma^j\} = 2\delta_{ij}\, I \tag{6}$$

Where σ_i's are spin 1/2 operators, and S_i's are general spin operators. They are related by the following:

$$S^i = \frac{\hbar}{2}\sigma^i \tag{7}$$

Examples of three spin Hamiltonians:

$$H = -\sum_i \left[S_i^x + \bar{\lambda} S_i^z S_{i+1}^z\right]; \qquad \text{The Ising model} \tag{8}$$

$$H = \sum_i \left[(1+\gamma)\, S_i^x S_{i+1}^x + (1-\gamma)\, S_i^y S_{i+1}^y\right]; \qquad \text{The XY model} \tag{9}$$

$$H = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y - J_z \sum_{z-links} \sigma_j^z \sigma_k^z; \qquad \text{Kitaev Honeycomb Model} \tag{10}$$

Where $\bar{\lambda}$, γ, J_i are model parameters, which are typically related to magnetic fields.

2.3 Jordan-Wigner Transformation

The Jordan-Wigner (JW) transformation is a unitary transformation used to map a system of interacting spins to a system of non-interacting fermions. This transformation allows for the use of fermionic statistics, which could improve the solvability of the system, and its correlation functions.
Jordan-Wigner definition includes a string of σ^z's, which must thread all the sites once, before the site of transformation. The objective is to define a convenient path for this string of σ^z's that makes the model easily solvable.
For the Ising and XY models, or any 1-D chain model, the JW transformation is defined as the following:

$$S_i^+ = \prod_{j<i} \left[-S_j^z\right] c_i^\dagger; \quad S_i^- = c_i \prod_{j<i} \left[-S_j^z\right] \tag{11}$$

$$S_i^x = \frac{1}{2}\left(S_i^+ + S_i^-\right); \quad S_i^y = \frac{i}{2}\left(S_i^- - S_i^+\right) \tag{12}$$

$$S_i^z = 2c_i^\dagger c_i - 1 \tag{13}$$

Where $c^\dagger$ and c are fermionic creation and annihilation operators.

For Kitaev honeycomb model, which is two dimensional, the string of sigma σ^z's is defined differently while maintaining the same condition: threading the whole lattice:

$$\sigma_{ij}^+ = 2\left[\prod_{j'<j}\prod_{i'} \sigma_{i'j'}^z\right]\left[\prod_{i'<i} \sigma_{i'j}^z\right] c_{ij}^\dagger; \quad \sigma_{ij}^- = 2c_{ij}\left[\prod_{j'<j}\prod_{i'} \sigma_{i'j'}^z\right]\left[\prod_{i'<i} \sigma_{i'j}^z\right] \tag{14}$$

$$\sigma_{ij}^x = \frac{1}{2}\left(\sigma_{ij}^+ + \sigma_{ij}^-\right); \quad \sigma_{ij}^y = \frac{i}{2}\left(\sigma_{ij}^- - \sigma_{ij}^+\right) \tag{15}$$

$$\sigma_{ij}^z = 2c_{ij}^\dagger c_{ij} - 1 \tag{16}$$

2.4 Majorana Fermions

Majorana fermions are particles that are their own anti-particle, that can be described by a linear combination between creation and annihilation fermionic operators.

Majorana operators have these relations satisfied:

$$\{A_i, A_j\} = \delta_{ij}; \quad A^\dagger = A; \quad A^2 = 1 \tag{17}$$

This way of introducing these quasi-particles can sometimes be useful in computing correlation functions easier as I will show in Section 3.1.3, and in finding conserved quantities in Kitaev type Hamiltonians, leading to easier computations.

2.5 Fourier Transformation of Fermionic Operators

Using Fourier transformation, which transforms the Hamiltonian into momentum space, is proven to be useful. After transforming the Hamiltonian into momentum space, due to transnational symmetry, one can find simplification that leads to easier computations. The transformation is defined as the following:

$$c_j^\dagger = \frac{1}{\sqrt{N}}\sum_q c_q^\dagger e^{iqj}; \quad c_j = \frac{1}{\sqrt{N}}\sum_q c_q e^{-iqj} \tag{18}$$

Where $c_q^\dagger$ and c_q are fermionic creation and annihilation operators in momentum space, and N is the total number of sites.
One important identity in dealing with Fourier transform is the following:

$$\sum_j e^{i(q-q')j} = N\delta_{qq'} \tag{19}$$

2.6 Bogoliubov Diagonalization

Solving the Hamiltonian is to diagonalize it. Some types of Hamiltonian are exactly diagonalizable, we consider Bogoliubov type Hamiltonians in momentum space as an example. If the Hamiltonian can be written in this form:

$$H = \sum_q \begin{bmatrix} \vec{c}_q^\dagger & \vec{c}_{-q} \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \vec{c}_q \\ \vec{c}_{-q}^\dagger \end{bmatrix} \tag{20}$$

Where h_{ij}'s are matrix blocks of the same size, then the Hamiltonian is diagonalized in this way:

$$H = \sum_q \underbrace{\begin{bmatrix} \vec{c}_q^\dagger & \vec{c}_{-q} \end{bmatrix} U^\dagger}_{\begin{bmatrix} \vec{\eta}_q^\dagger & \vec{\eta}_{-q} \end{bmatrix}} \underbrace{UhU^\dagger}_{D} \underbrace{U \begin{bmatrix} \vec{c}_q \\ \vec{c}_{-q}^\dagger \end{bmatrix}}_{\begin{bmatrix} \vec{\eta}_q & \vec{\eta}_{-q}^\dagger \end{bmatrix}^T} \tag{21}$$

Where $D = \begin{bmatrix} E_q & 0 \\ 0 & E_{-q} \end{bmatrix}$. Then, the Hamiltonian in diagonalized form has the shape:

$$H = \sum_q E_q \eta_q^\dagger \eta_q + E_{-q} \eta_{-q} \eta_{-q}^\dagger \tag{22}$$

Where we may interpret E_q as particle energies and E_{-q} as hole energies.

2.6.1 Specific Form of a Hamiltonian

If the Hamiltonian has this form:

$$H = \sum_q \begin{bmatrix} c_q^\dagger & c_{-q} \end{bmatrix} \underbrace{\begin{bmatrix} \alpha & -i\beta \\ i\beta & -\alpha \end{bmatrix}}_{2\times 2} \begin{bmatrix} c_q \\ c_{-q}^\dagger \end{bmatrix} \tag{23}$$

Where α and β are real, and are elements of the 2×2 matrix, then it has these eigenvalues:

$$\left| H - \omega_q \mathbb{I} \right| = \begin{vmatrix} \alpha - \omega_q & -i\beta \\ i\beta & -\alpha - \omega_q \end{vmatrix} = 0 \implies \omega_q = \pm\sqrt{\alpha^2 + \beta^2} \tag{24}$$

Then, the unitary matrix U in (24) is:

$$U = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} u_q & iv_q \\ iv_q & u_q \end{bmatrix}; \quad u_q = \frac{\alpha + \omega_q}{\sqrt{(\alpha + \omega_q)^2 + \beta^2}}; \quad v_q = \frac{\beta}{\sqrt{(\alpha + \omega_q)^2 + \beta^2}} \tag{25}$$

$$\begin{bmatrix} \eta_q^\dagger \\ \eta_{-q} \end{bmatrix} = \begin{bmatrix} u_q & iv_q \\ iv_q & u_q \end{bmatrix} \begin{bmatrix} c_q^\dagger \\ c_{-q} \end{bmatrix}; \quad \begin{bmatrix} \eta_q \\ \eta_{-q}^\dagger \end{bmatrix} = \begin{bmatrix} u_q & -iv_q \\ -iv_q & u_q \end{bmatrix} \begin{bmatrix} c_q \\ c_{-q}^\dagger \end{bmatrix} \tag{26}$$

Where V_1 & V_2 are the first and second eigenvectors of the matrix.

3 Previous Work

3.1 1-D Chains: Ising and XY Models

3.1.1 Ising Model

The Ising model defined in (8) will lead to quadratic terms when transforming $S_i^z S_{i+1}^z$. So, a canonical transformation is used:

$$S^x \to S^z; \quad S^z \to -S^x$$

After doing so, the Hamiltonian now reads:

$$H = -\sum_i \left[S_i^z - \bar{\lambda} S_i^x S_{i+1}^x \right] \tag{27}$$

Using (12) to rewrite spin operators as raising and lowering spin operators, we can rewrite the Hamiltonian as:

$$H = N - 2\sum_i S_i^+ S_i^- - \bar{\lambda}\sum_i \left[S_i^+ S_{i+1}^+ + S_i^+ S_{i+1}^- + S_{i+1}^+ S_i^- + S_i^- S_{i+1}^-\right] \tag{28}$$

Now, we employ JW transformation in (11) to obtain this fermionic Hamiltonian:

$$H = N - 2\sum_i c_i^\dagger c_i - \bar{\lambda}\sum_i \left[c_i^\dagger c_{i+1}^\dagger + c_i^\dagger c_{i+1} - c_i c_{i+1}^\dagger - c_i c_{i+1}\right] \tag{29}$$

The next step now is to apply Fourier transform in (18) and (19), and summing over positive modes, we obtain the following Hamiltonian:

$$H = -2\sum_{q>0}(1+\bar{\lambda}\cos q)(c_q^\dagger c_q - c_{-q}^\dagger c_{-q}) + 2i\bar{\lambda}\sum_{q>0}\sin q(c_q^\dagger c_{-q}^\dagger - c_q c_{-q}) \tag{30}$$

$$= -2\sum_{q>0}\begin{bmatrix} c_q^\dagger & c_{-q}\end{bmatrix}\begin{bmatrix} 1+\bar{\lambda}\cos q & -i\bar{\lambda}\sin q \\ i\bar{\lambda}\sin q & -1-\bar{\lambda}\cos q\end{bmatrix}\begin{bmatrix} c_q \\ c_{-q}^\dagger\end{bmatrix} \tag{31}$$

We can see that this Hamiltonian has a similar form to (23), thus, we can diagonalize it using (24). The result is this following Hamiltonian in its eigenspace:

$$H = 2\sum_q \omega_q \eta_q^\dagger \eta_q + E_0 \tag{32}$$

$$E_0 = -\sum_q \omega_q; \quad \omega_q = \sqrt{1 + 2\bar{\lambda}\cos q + \bar{\lambda}^2} \tag{33}$$

The ground state energy is calculated by computing the continuum limit of the summations:

$$E_0 = -\sum_q \omega_q \rightarrow \frac{E_0}{N} = -\int_{-\pi}^{\pi}\frac{dq}{2\pi}\omega_q \tag{34}$$

$$\frac{E_0}{N} = -\int_{-\pi}^{\pi}\frac{dq}{2\pi}\omega_q = -\frac{1}{\pi}\int_0^{\pi}\sqrt{1+2\bar{\lambda}\cos q + \bar{\lambda}^2}dq = -\frac{2}{\pi}(1+\bar{\lambda})\mathbf{E}\left(\frac{\pi}{2}, \sqrt{\frac{4\bar{\lambda}}{(1+\bar{\lambda})^2}}\right) \tag{35}$$

Where $\mathbf{E}(\frac{\pi}{2}, k)$ is the complete elliptic integral of the second kind.

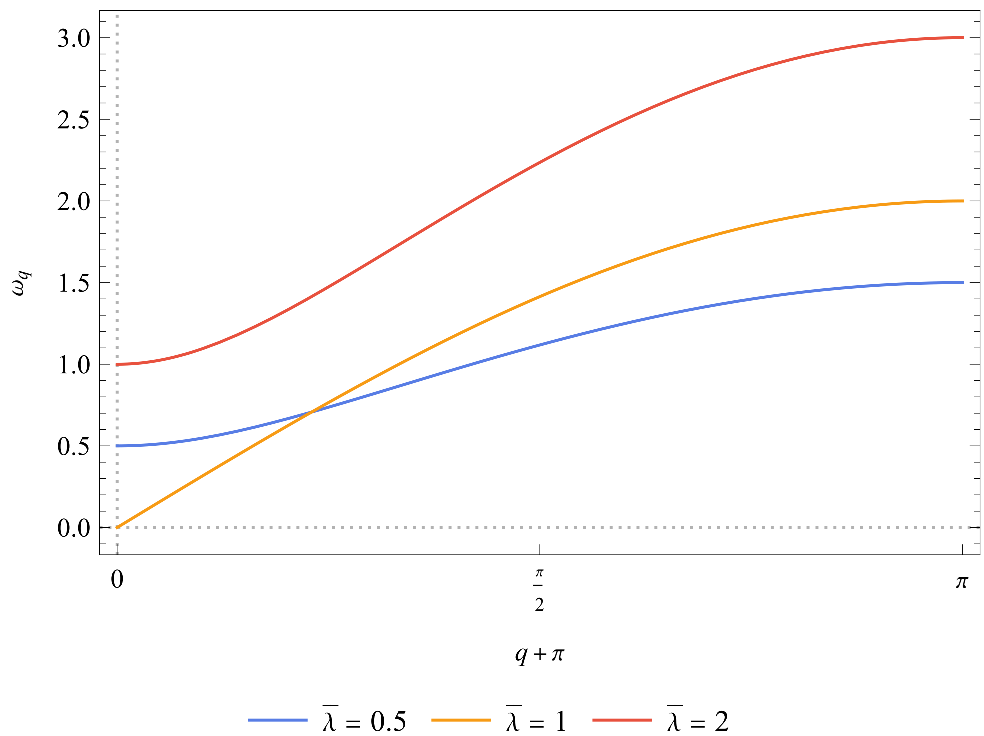

Figure 1: Elementary excitation energy for different $\bar{\lambda}$

3.1.2 XY Model

The XY model Hamiltonian (9) does not need any canonical transformations. Because when we rewrite it in fermionic language it is going to be quadratic on the spot. We start by using (12) to rewrite spin operators as raising and lowering spin operators, we can rewrite the Hamiltonian as:

$$H = 2\sum_i \left[S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + \gamma\left(S_i^+ S_{i+1}^+ + S_i^- S_{i+1}^-\right)\right] \tag{36}$$

Now, we employ JW transformation in (11) to obtain this fermionic Hamiltonian:

$$H = 2\sum_i \left[c_i^+ c_{i+1} - c_i c_{i+1}^+ + \gamma\left(c_i^+ c_{i+1}^+ - c_i c_{i+1}\right)\right] \tag{37}$$

The next step now is to apply Fourier transform in (18) and (19), and summing over positive modes, we obtain the following Hamiltonian:

$$H = 4\sum_{q>0} \left[\cos q\left(c_q^+ c_q - c_{-q} c_{-q}^+\right) + \gamma i \sin q\left(c_{-q} c_q - c_q^+ c_{-q}^+\right)\right] \tag{38}$$

$$= 4\sum_{q>0} \begin{bmatrix} c_q^\dagger & c_{-q} \end{bmatrix} \begin{bmatrix} \cos q & -i\gamma\sin q \\ i\gamma\sin q & -\cos q \end{bmatrix} \begin{bmatrix} c_q \\ c_{-q}^\dagger \end{bmatrix} \tag{39}$$

We can also see that this Hamiltonian has a similar form to (23), thus, we can diagonalize it using (24). The result is this following Hamiltonian in its eigenspace:

$$H = 4\sum_q \omega_q \eta_q^\dagger \eta_q + E_0 \tag{40}$$

$$E_0 = -2\sum_q \omega_q; \quad \omega_q = \sqrt{1-(1-\gamma^2)\sin^2 q} \tag{41}$$

Similar to (35), the ground state energy is:

$$\frac{E_0}{N} = -2\int_{-\pi}^{\pi} \frac{dq}{2\pi}\omega_q = -\frac{1}{2\pi}\int_0^{\pi/2} \sqrt{1-(1-\gamma^2)\sin^2 q}\,dq = -\frac{2}{\pi}\mathbf{E}\left(\frac{\pi}{2}, \sqrt{1-\gamma^2}\right) \tag{42}$$

Where $\mathbf{E}(\frac{\pi}{2}, k)$ is the complete elliptic integral of the second kind.

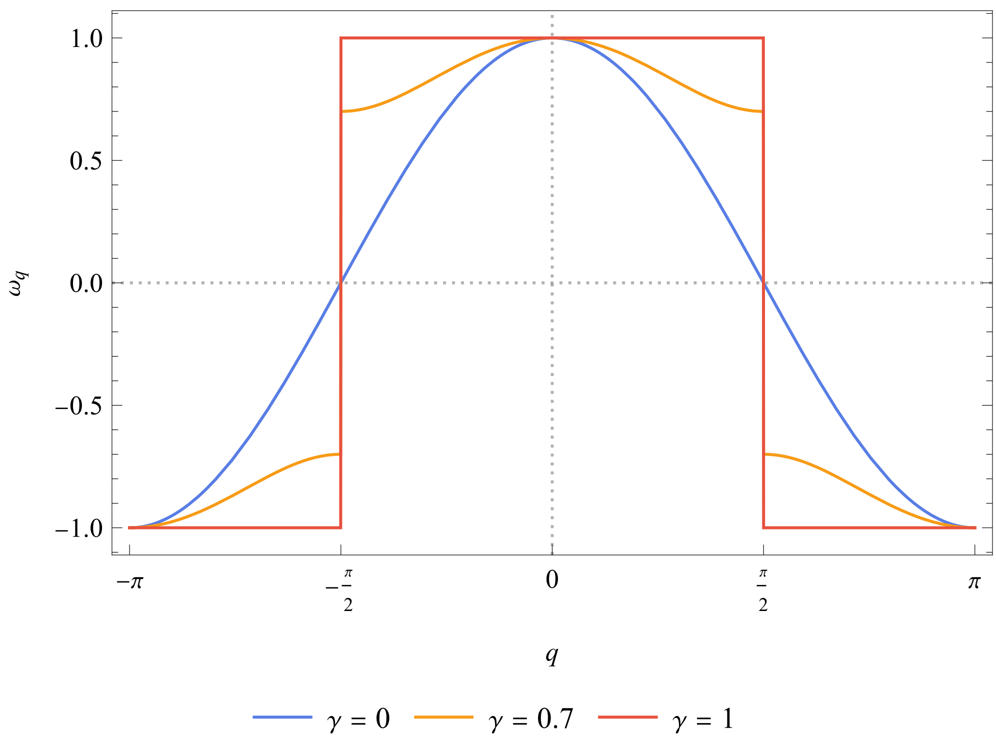

Figure 2: Elementary excitation energy for different γ

3.1.3 Correlation Functions

Correlation functions in condensed matter theory are related to physical observables that can be measured experimentally. Such as conductivity, magnetization, and spin-spin correlation functions. They can also be used to study the behavior of a system under external perturbations, such as an applied electric field or a magnetic field. The correlation functions we are interested in are defined as:

$$C^x_{ij} = \langle 0| S^x_i S^x_j |0\rangle ; \quad C^y_{ij} = \langle 0| S^y_i S^y_j |0\rangle ; \quad C^z_{ij} = \langle 0| S^z_i S^z_j |0\rangle \tag{43}$$

In order to calculate these correlation function we will use (12) as well as JW transformation (11). The next calculations are shown for C^x_{ij}:

$$C^x_{ij} = \langle 0| S^x_i S^x_j |0\rangle = \langle 0| \left(c^\dagger_i + c_i\right) \prod_{i\leq k<j} [-S^z_k] \left(c^\dagger_j + c_j\right) |0\rangle \tag{44}$$

However, we can simplify the string of $\prod_{i\leq k<j} [-S^z_k]$ by introducing Majorana fermions:

$$A_i \equiv c^\dagger_i + c_i; \quad B_i \equiv c^\dagger_i - c_i; \quad A^2_i = 1; \quad B^2_i = -1; \quad \{A_i, B_j\} = 0 \tag{45}$$

$$S^z_k = 2c^\dagger_k c_k - 1 = \left(c^\dagger_k + c_k\right)\left(c^\dagger_k - c_k\right) = A_k B_k \tag{46}$$

$$\therefore C^x_{ij} = \langle 0| S^x_i S^x_j |0\rangle = \langle 0| A_i \prod_{i\leq k<j} [A_k B_k] A_j |0\rangle = \langle 0| \prod_{i\leq k<j} [B_k A_{k+1}] |0\rangle \tag{47}$$

Therefore, the correlation functions will yield:

$$C^x_{ij} = \langle 0| \prod_{i\leq k<j} B_k A_{k+1} |0\rangle ; \quad C^y_{ij} = \langle 0| \prod_{i\leq k<j} B_{k+1} A_k |0\rangle ; \quad C^z_{ij} = \langle 0| B_i A_i B_j A_j |0\rangle ; \tag{48}$$

Now we will employ Wick's theorem to calculate the Vacuum Expectation Values (VEVs). For two operators $\hat{A}$ and $\hat{B}$, their contraction is defined as:

$$\langle AB\rangle \equiv \hat{A}\hat{B} - :\hat{A}\hat{B}: \tag{49}$$

Where $:\hat{O}:$ is the normal order which is defined with creation operators left of annihilation operators. The first simplification occur when considering Wick's theorem for VEVs for fermions: all terms involving normal orders vanish, leaving only full contractions:

$$\langle 0|ABCDEF\ldots|0\rangle = \sum_\sigma sgn(\sigma) \prod_{\text{all pairs}} \text{contraction pair}$$

For our strings in C^x, C^y, C^z described in A and B operators, only $\langle A_i B_j\rangle$, and $\langle B_i A_j\rangle$ are nonzero. $\langle A_i A_j\rangle = \delta_{ij}$ and $\langle B_i B_j\rangle = -\delta_{ij}$. Since $A's$ and $B's$ anti-commute, then $\langle B_i A_j\rangle = -\langle A_j B_i\rangle$. The correlation functions can be expressed as the following determinants:

$$G_{ij} \equiv \langle B_i A_j\rangle ; \quad G_r \equiv G_{ii+r} = \langle B_i A_{i+r}\rangle = -\langle A_{i+r} B_i\rangle = G_{-r} \tag{50}$$

$$C^x_r = \begin{vmatrix} G_1 & G_2 & \ldots & G_r \\ G_0 & G_1 & \ldots & G_{r-1} \\ \ldots & \ldots & \ldots & \ldots \\ G_{2-r} & G_2 & \ldots & G_1 \end{vmatrix} \quad C^y_r = \begin{vmatrix} G_{-1} & G_0 & \ldots & G_{r-2} \\ G_{-2} & G_{-1} & \ldots & G_{r-1} \\ \ldots & \ldots & \ldots & \ldots \\ G_{-r} & G_{1-r} & \ldots & G_{-1} \end{vmatrix} \quad C^z_r = \begin{vmatrix} G_0 & G_r \\ G_{-r} & G_0 \end{vmatrix} \tag{51}$$

$$\text{With transverse field: } C^z_r \equiv C^z_r - (m^z)^2 ; \quad m^z = \langle B_i A_i\rangle = G_0 \implies C^z_r = -G_r G_{-r} = -G^2_r \tag{52}$$

To evaluate these Green's functions, we need to evaluate the VEVs in terms of the diagonalized operators. Therefore we will apply Fourier transform (18), then Bogoliubov diagonalization by using the definition (26) for each model:

$$G_r = \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{u_q}{\omega_q} \cos qr - \frac{v_q}{\omega_q} \sin qr \tag{53}$$

For the transverse Ising model:

$$G_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda \cos[q(r+1)] + \cos qr}{\sqrt{1+\lambda^2+2\lambda\cos q}}\, dq \tag{54}$$

One can now evaluate the following values G_r for some special values of $\bar{\lambda}$:

$$G_r = \begin{cases} \frac{2}{\pi}\frac{(-1)^r}{2r+1} & \text{For } \bar{\lambda} = 1 \\ \frac{1}{\Gamma(-r)\Gamma(r+2)} \equiv \delta_{r,-1} & \text{For } \bar{\lambda} = \infty \\ \frac{1}{\Gamma(1-r)\Gamma(r+1)} \equiv \delta_{r,0} & \text{For } \bar{\lambda} = 0 \end{cases} \tag{55}$$

For the XY model:

$$G_r = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos q \cos qr - \gamma \sin q \sin qr}{\sqrt{1+(\gamma^2-1)\sin^2(q)}}\, dq & \text{For } r \textit{ odd} \\ 0 & \text{For } r \textit{ even} \end{cases} \tag{56}$$

One can now evaluate the following values G_r for some special values of γ:

$$G_r = \begin{cases} -\frac{1}{\pi}\frac{\sin \pi r}{r+1} \equiv \delta_{r,-1} & \text{For } \gamma = 1 \\ \frac{2(-1)^{1/2(r+1)}}{\pi r} & \text{For } \gamma = 0 \end{cases} \tag{57}$$

3.2 Kitaev Honeycomb Model

Kitaev honeycomb model is defined on a 2D honeycomb lattice by the Hamiltonian introduced in (10). The honeycomb lattice is defined by two triangular Bravais lattices, and consequently, we have two sub-lattices which we will denote by white (w) and black (b) as seen in Figure 3:

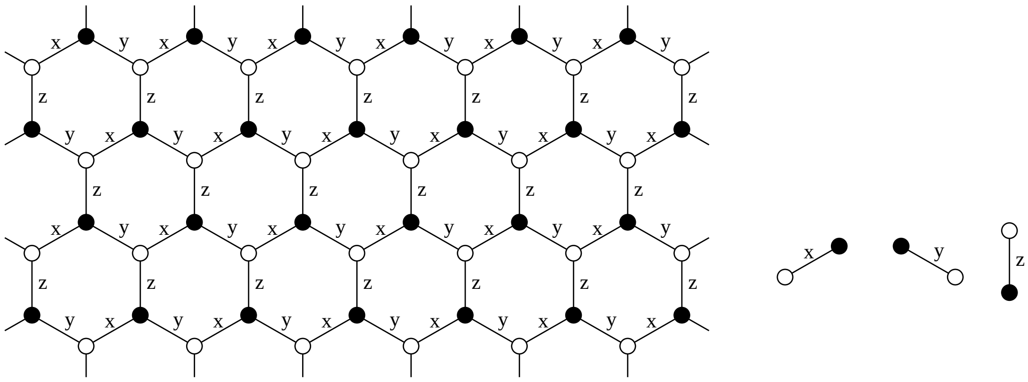

Figure 3: Kitaev's honeycomb lattice, with sub-lattices denoted by (w) & (b)

First, we will deform the honeycomb lattice into a topologically equivalent brick-wall lattice:

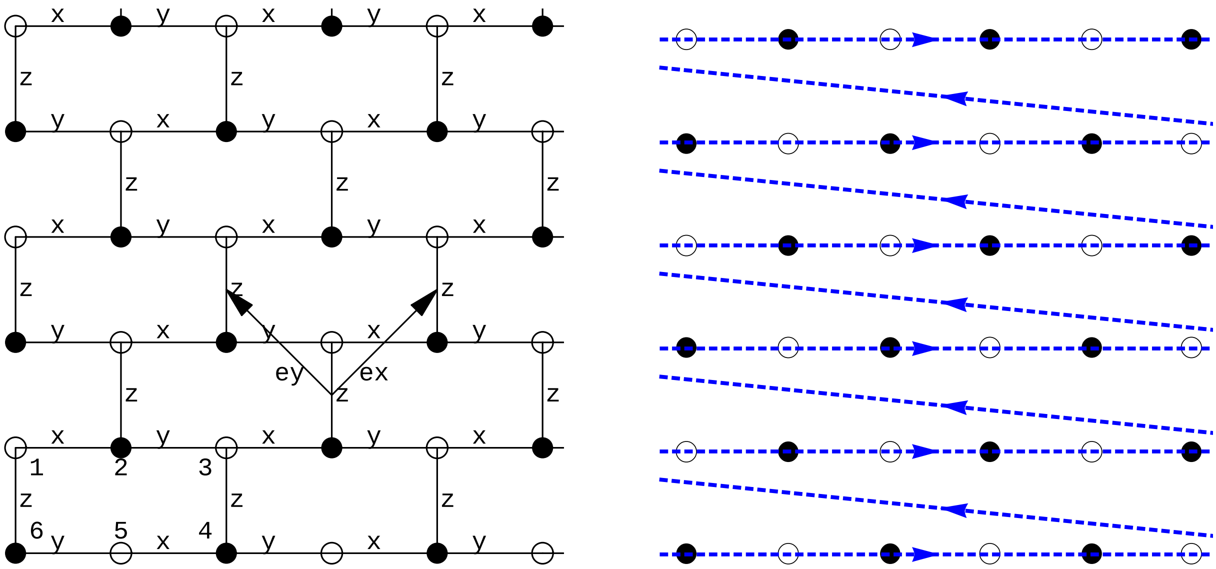

Figure 4: Honeycomb lattice after deformation, showing JW path

Then, it becomes more clear how to define a convenient path for JW transformation. Using JW transformation defined in (14) will thread the brick-wall lattice in a zig-zag fashion, as illustrated in Figure 4. The result of JW transformation is the following Hamiltonian:

$$H = J_x \sum_{x-links} \left(c - c^\dagger\right)_w \left(c^\dagger + c\right)_b - J_y \sum_{y-links} \left(c^\dagger + c\right)_b \left(c - c^\dagger\right)_w - J_z \sum_{z-links} \left(2c^\dagger c - 1\right)_b \left(2c^\dagger c - 1\right)_w \tag{58}$$

Where $w\&b$ denotes the two sub-lattices. Now, we will introduce Majorana fermions at each site, which are defined by:

$$A_w \equiv \frac{\left(c - c^\dagger\right)_w}{i}; \quad B_w \equiv \left(c^\dagger + c\right)_w; \quad A_b \equiv \left(c^\dagger + c\right)_b; \quad B_b \equiv \frac{\left(c - c^\dagger\right)_b}{i} \tag{59}$$

Now, the Hamiltonian will read:

$$H = -iJ_x \sum_{x-links} A_w A_b + iJ_y \sum_{y-links} A_b A_w + J_z \sum_{z-links} B_b B_w A_b A_w \tag{60}$$

Now, the term $B_b B_w A_b A_w$ is not quadratic, but luckily, there is a conserved quantity α_r which will make it quadratic:

$$\alpha_r \equiv iB_b B_w \tag{61}$$

$$\therefore H = -iJ_x \sum_{x-links} A_w A_b + iJ_y \sum_{y-links} A_b A_w - iJ_z \sum_{z-links} \alpha_r A_b A_w \tag{62}$$

Where r is the midpoint coordinate of the z-bonds.
Since $B_{b/w}$ is hermitian, and $B_{b/w}^2 = 1$ (17), then $B_{b/w}$ will have eigenvalues of ± 1. Moreover, $B_{b/w}$ operators ***anti-commute*** with $A_{b/w}$ operators, and consequently, $B_{b/w} B_{b/w}$ operators will ***commute*** with $A_{b/w}$ operators.

$$\{B_i, A_j\} = 0; \qquad [B_i B_j, A_k] = 0; \quad ijk \in \{b, w\} \tag{63}$$

Now that we identified the conserved quantity in our Hamiltonian, we will replace it by its eigenvalue, we will choose +1 which will minimize the ground state energy. Then, we will introduce a new spinon excitation fermionic operator which will live in the middle of z-bonds, defined as:

$$d \equiv \frac{A_w + iA_b}{2}; \qquad d^\dagger \equiv \frac{A_w - iA_b}{2} \tag{64}$$

We can observe that

$$[\alpha_r, d_r] = \left[\alpha_r, d_r^\dagger\right] = 0 \tag{65}$$

Finally, the Hamiltonian now reads:

$$H = J_x \sum_r \left(d_r^\dagger + d_r\right) \left(d_{r+\hat{e}_x}^\dagger + d_{r+\hat{e}_x}\right) + J_y \sum_r \left(d_r^\dagger + d_r\right) \left(d_{r+\hat{e}_y}^\dagger + d_{r+\hat{e}_y}\right) + J_z \sum_r \left(2d_r^\dagger d_r - 1\right) \tag{66}$$

Where $\hat{e}_x \& \hat{e}_y$ are the basis vectors shown in Figure 4. Now we apply Fourier transform in 2-D, which is slightly different than (18):

$$d_{\mathbf{r}}^\dagger = \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} d_{\mathbf{q}}^\dagger e^{i\mathbf{q}\cdot\mathbf{r}}; \quad d_{\mathbf{r}} \quad = \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} d_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{r}} \tag{67}$$

And (19) becomes:

$$\sum_{\mathbf{r}} e^{i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{r}} = N\delta_{\mathbf{q}\mathbf{q}'} \tag{68}$$

Using (67) and (68), and summing over positive modes, the Hamiltonian will read:

$$H = \sum_{q>0} \left[\epsilon_q (d_q^\dagger d_q - d_{-q} d_{-q}^\dagger) + i\Delta_q (d_q^\dagger d_{-q}^\dagger - d_{-q} d_q)\right] \tag{69}$$

$$= \sum_{q>0} \begin{bmatrix} d_q^\dagger & d_{-q} \end{bmatrix} \begin{bmatrix} \epsilon_q & i\Delta_q \\ -i\Delta_q & -\epsilon_q \end{bmatrix} \begin{bmatrix} d_q \\ d_{-q}^\dagger \end{bmatrix} \tag{70}$$

$$\epsilon_q = 2J_z - 2J_x \cos q_x - 2J_y \cos q_y; \quad \Delta_q = 2J_x \sin q_x + 2J_y \sin q_y; \quad q_i \equiv \mathbf{q} \cdot \hat{e}_i; \quad i \in \{x, y\} \tag{71}$$

$$\sum_q \Longrightarrow \sum_{q_x} \sum_{q_y}; \quad \sum_{q>0} \Longrightarrow \sum_{q_x>0} \sum_{q_y>0} \tag{72}$$

Which now has a similar form to (23), thus, we can diagonalize it using (24). The result is this following Hamiltonian in its eigenspace:

$$H = \sum_q \omega_q \eta_q^\dagger \eta_q + E_0 \tag{73}$$

$$E_0 = -\frac{1}{2}\sum_q \omega_q; \quad \omega_q = \sqrt{\epsilon_q^2 + \Delta_q^2} \tag{74}$$

4 Research Questions

Kitaev's honeycomb model only encompasses nearest neighbor interactions. However, what physical properties can one study by including the next nearest neighbor interaction? Or perhaps the next next neighbor interaction? How this relates to thermal conductivity? Can we find Kitaev spin liquid candidate materials? How the magnetic field dependence on thermal conductivity change by including these interactions? Will the model still be exactly solvable? This research proposes a method to study a generalize Kitaev honeycomb model by extending it to encompass such interactions, and check how this affect the physical properties of the model.

5 Methodology

An extended Kitaev honeycomb model can be written as:

$$H = H_1 + H_2 + H_3 \tag{75}$$

Where H_1 is the original Kitaev honeycomb model, H_2 includes the next nearest neighbor interactions, H_3 includes the next next nearest neighbor interactions.
The way to approach such Hamiltonian, is to first study it up to H_2, checking the solvability of the model. Then attempt to include H_3.

The research scheme is to first write the Hamiltonian in fermionic language using (11). Then to introduce Majorana fermions to check what conserved quantities are present in the system. Then, to use such quantities to attempt performing a Fourier transform defined in (67). Finally, if the results have similar form to Bogoliubov Hamiltonians, we employ (26).

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