جامعة الملك فهد للبترول والمعادن
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Intro. to JW Solution to Kitaev Honeycomb Model

A Summary of PHYS497 Progress

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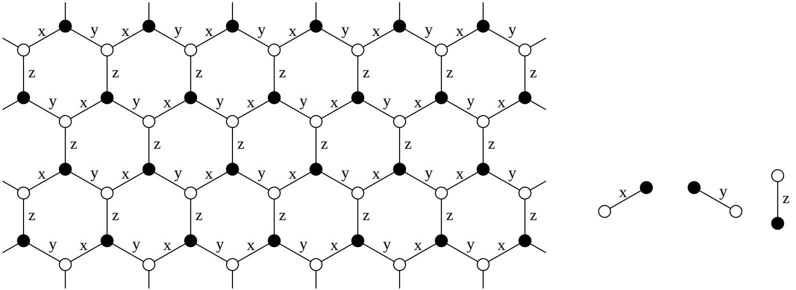
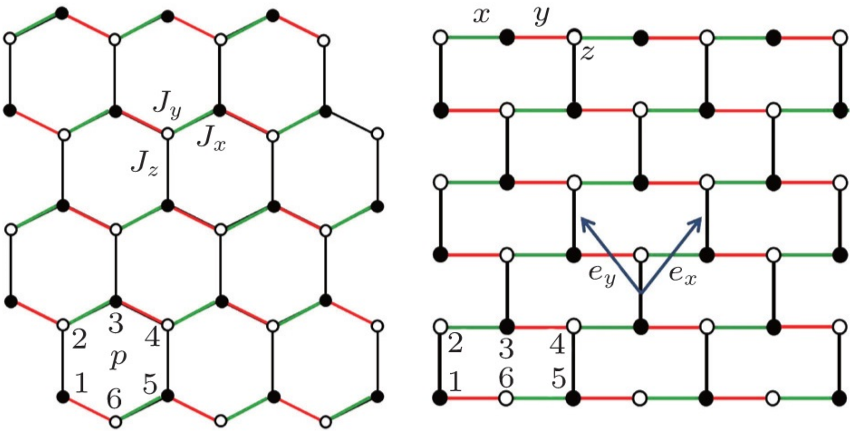
Kitaev's Honeycomb Hamiltonian

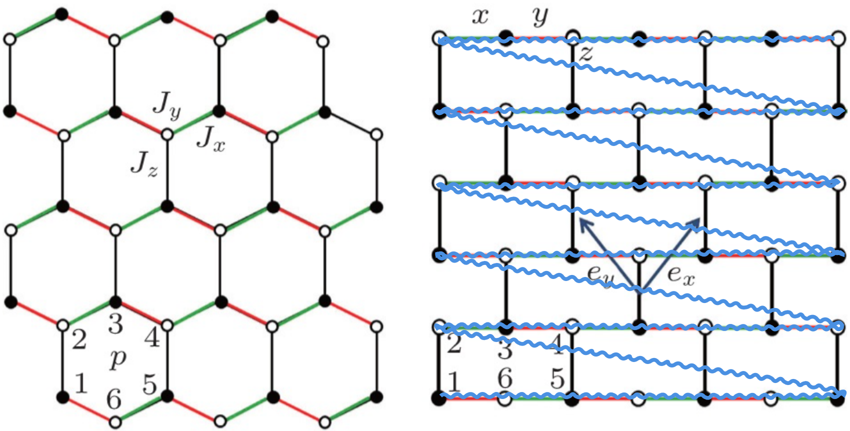
$$H = -\left(J_x \sum_{x-links} \sigma_j^x \sigma_k^x + J_y \sum_{y-links} \sigma_j^y \sigma_k^y + J_z \sum_{z-links} \sigma_j^z \sigma_k^z \right)$$



Deforming The Hamiltonian



Threading The Lattice



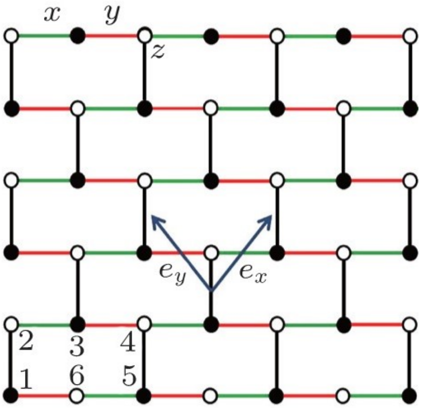
Jordan-Wigner Definition

$$\sigma_{ij}^{+} = 2\left[\prod_{j'<j}\prod_{i'}\sigma_{i'j'}^{z}\right]\underbrace{\left[\prod_{i'<i}\sigma_{i'j}^{z}\right]}_{1D\ String} c_{ij}^{\dagger}$$

$$\sigma_{ij}^{z} = 2c_{ij}^{\dagger}c_{ij} - 1$$

$$\sigma_{ij}^{x} = \frac{1}{2}\left(\sigma_{ij}^{+} + \sigma_{ij}^{-}\right)$$

$$\sigma_{ij}^{y} = \frac{i}{2}\left(\sigma_{ij}^{-} - \sigma_{ij}^{+}\right)$$



Example

We will now transform one part of the Hamiltonian as an example:
Using:

$$\sigma^x_{ij} = \frac{1}{2}\left(\sigma^+_{ij} + \sigma^-_{ij}\right)$$

$$\sigma^x_{i,j}\sigma^x_{i+1,j} \implies \frac{1}{4}\left(\sigma^+_{i,j}\sigma^+_{i+1,j} + \sigma^+_{i,j}\sigma^-_{i+1,j} + \sigma^-_{i,j}\sigma^+_{i+1,j} + \sigma^-_{i,j}\sigma^-_{i+1,j}\right)$$

Employing JW transformation:

$$\implies c^\dagger_{i,j}c^\dagger_{i+1,j} + c^\dagger_{i,j}c_{i+1,j} - c_{i,j}c^\dagger_{i+1,j} - c_{i,j}c_{i+1,j}$$

$$\implies \left(c^\dagger_{i,j} - c_{i,j}\right)\left(c^\dagger_{i+1,j} + c_{i+1,j}\right)$$

Majorana Fermions

Now, we will define new Majorana operators at each site, and we will distinguish between the two sub-lattices by the indices w & b:

$$A_w \equiv \frac{\left(c - c^\dagger\right)_w}{i}; \quad B_w \equiv \left(c^\dagger + c\right)_w$$

$$A_b \equiv \left(c^\dagger + c\right)_b; \quad B_b \equiv \frac{\left(c - c^\dagger\right)_b}{i}$$

Now, our Hamiltonian reads:

$$H = -iJ_x \sum_{x-links} A_w A_b + iJ_y \sum_{y-links} A_b A_w + J_z \sum_{z-links} B_b B_w A_b A_w$$

Conserved Quantity

$$H = -iJ_x \sum_{x-links} A_w A_b + iJ_y \sum_{y-links} A_b A_w + J_z \sum_{z-links} B_b B_w A_b A_w$$

Now, the term $B_b B_w A_b A_w$ is not quadratic, but luckily, there is a conserved quantity which will make it quadratic.

Since B^* is hermitian, and $B^2 = 1$, B will have eigenvalues of ± 1. Moreover, B operators ***anti-commute*** with A operators. Consequently, BB operators will ***commute*** with A operators.

$$\{B, A\} = 0; \qquad [BB, A] = 0$$

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*I dropped the w & b indices

New 1-Fermion Operator

Now that we identified the conserved quantity in our Hamiltonian, we will replace it by its eigenvalue, we will choose $+1$.

$$\alpha_r \equiv iB_b B_w$$

Now, we will define a new 1-fermionic operator which will live in the middle of z-bonds, as:

$$d \equiv \frac{A_w + iA_b}{2}; \qquad d^\dagger \equiv \frac{A_w - iA_b}{2}$$

$$H = J_x \sum_r \left(d_r^\dagger + d_r\right)\left(d_{r+\hat{e}_x}^\dagger + d_{r+\hat{e}_x}\right) + J_y \sum_r \left(d_r^\dagger + d_r\right)\left(d_{r+\hat{e}_y}^\dagger + d_{r+\hat{e}_y}\right)$$
$$+ J_z \sum_r \left(2d_r^\dagger d_r - 1\right)$$

Thank you!