Q 1:

$$\psi(x) = \frac{1}{\sqrt{L}}e^{ikx}$$

a):

$$\langle x \rangle = \langle \psi | x | \psi \rangle = \frac{1}{L} \int_0^L e^{-ikx} x e^{ikx} dx = \frac{1}{L} \int_0^L x \ dx \ = \frac{L}{2}$$

We can interpret this as "The most probable position is located at L/2"

b):

$$\langle p_x \rangle = \langle \psi | p_x | \psi \rangle = \frac{\hbar}{iL} \int_0^L e^{-ikx} \frac{d}{dx} e^{ikx} dx = \frac{\hbar ki}{iL} \int_0^L dx = \hbar k$$

c):

$$\langle p_x^2 \rangle = \langle \psi | p_x^2 | \psi \rangle = \frac{-\hbar^2}{L} \int_0^L e^{-ikx} \frac{d^2}{dx^2} e^{ikx} dx = \frac{\hbar^2 k^2}{L} \int_0^L dx = \hbar^2 k^2$$

d):

$$\begin{split} \frac{\langle p_x^2 \rangle}{2m_0} &= \frac{\hbar^2 k^2}{2m_0}; \quad knowing \ k^2 = \frac{2m_0(E - U_0)}{\hbar^2} \\ &= \frac{\hbar^2 2m_0(E - U_0)}{2m_0\hbar^2} = E - U_0 \checkmark \end{split}$$

Q 2:

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin(\frac{\pi x}{a})$$

a):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi; \qquad k^2 \equiv \frac{2mE}{\hbar^2}$$
$$\frac{d^2\psi}{dx^2} + k^2\psi = 0 \implies \psi(x) = A\sin(kx) + B\cos(kx)$$

After applying $\psi(0) = \psi(L) = 0 \& \psi'(0) = \psi'(L) = 0$:

$$\psi_n(x) = A \sin \frac{n\pi x}{a}$$

$$\langle \psi_n | \psi_m \rangle = \delta_{nm} \implies A = \sqrt{\frac{2}{a}}$$

$$\therefore \psi_n(x) = \sqrt{\frac{2}{a}} \sin(\frac{n\pi x}{a}) \qquad \Box$$

b):

$$\langle x \rangle = \langle \psi_1 | x | \psi_1 \rangle = \frac{2}{a} \int_0^a x \sin^2(\frac{\pi x}{a}) dx = \frac{1}{a} \int_0^a x - x \cos(\frac{2\pi x}{a}) dx = \frac{1}{2} a$$

We can interpret this as "The most probable position is located at a/2"

c):

$$\langle p_x \rangle = \langle \psi_1 | p_x | \psi_1 \rangle = \frac{2\hbar}{ia} \int_0^a \sin(\frac{\pi x}{a}) \frac{d}{dx} \sin(\frac{\pi x}{a}) dx = \frac{2\hbar \pi}{a^2 i} \int_0^a \sin(\frac{\pi x}{a}) \cos(\frac{\pi x}{a}) dx = 0$$

This can be interpreted as this particle is a wave-like and has zero momentum.

d):

$$\langle p_x^2 \rangle = \langle \psi_1 | p_x^2 | \psi_1 \rangle = -\frac{2\hbar^2}{a} \int_0^a \sin(\frac{\pi x}{a}) \frac{d^2}{dx^2} \sin(\frac{\pi x}{a}) dx = \frac{2\pi^2 \hbar^2}{a^3} \int_0^a \sin^2(\frac{\pi x}{a}) dx$$
$$= \frac{\pi^2 \hbar^2}{a^3} \int_0^a 1 - \cos(\frac{2\pi x}{a}) dx = \frac{\pi^2 \hbar^2}{a^2}$$

e):

$$\frac{\langle p_x^2 \rangle}{2m_0} = \frac{\pi^2 \hbar^2}{2m_0 a^2} = E_1 \checkmark; \qquad E_n = \frac{n^2 \pi^2 \hbar^2}{2m_0 a^2}$$

Q 3:

$$E=100~keV$$

$$E=\frac{1}{2}mv^2\implies v=\sqrt{\frac{2E}{m}}$$

$$\lambda=\frac{h}{p}=\frac{h}{mv}=\frac{h}{m}\sqrt{\frac{m}{2E}}=3.88~pm=0.00388~nm<<193~nm$$

Q 4:

a):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - U_0)\psi = 0$$

$$\frac{d^2\psi}{dx^2} - \alpha^2\psi = 0; \quad since \ U_0 > E, \ \alpha = \frac{\sqrt{2m(U_0 - E)}}{\hbar}$$

b):

$$E = \frac{3k_BT}{2} = 0.0387 \ eV;$$
 $U_0 = 3.38 \ eV$
$$\frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = 0.107 \ nm$$

Q 5:

$$k_{1} = \zeta \sqrt{E}; \quad k_{2} = \zeta \sqrt{E - U_{0}}; \qquad \zeta = \frac{\sqrt{2m}}{\hbar}$$

$$\psi(0_{+}) = \psi(0_{-}) \implies 1 + r = t \qquad (1)$$

$$\psi'(0_{+}) = \psi'(0_{-}) \implies k_{1} - rk_{1} = tk_{2} \qquad (2)$$

(1) in (2)
$$\implies k_1 - rk_1 = k_2 + rk_2 \implies r = \frac{\zeta\sqrt{E} - \zeta\sqrt{E - U_0}}{\zeta\sqrt{E} + \zeta\sqrt{E - U_0}}$$
$$|r^2| = \frac{|2E - 2\sqrt{E^2 - EU_0} - U_0|}{|2E + 2\sqrt{E^2 - EU_0} - U_0|}$$

As the energy increases above U_0 , the reflection coefficient will become smaller and smaller rapidly, as we can see in Fig. 1:

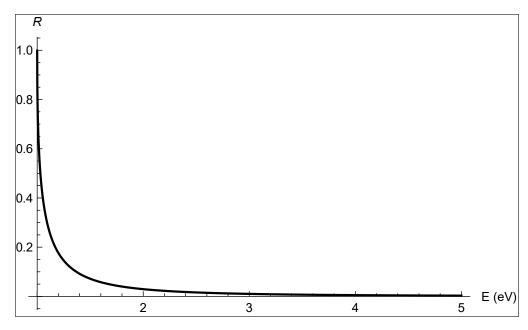


Figure 1: R Vs. E, for $U_0 = 1 \ eV$

Q 6:

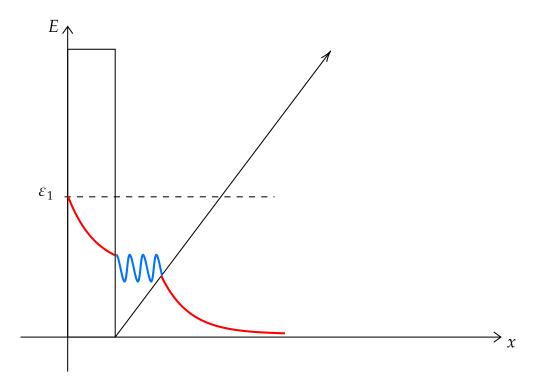


Figure 2: Sketch of $\psi(x)$

For this question, we only need to determine whether $\varepsilon_1 - V$ is positive or negative, is positive, then the wavefunction will be a traveling wave. If negative, the wavefunction will be a decaying wave.

Q 7:

- a) Only one band is visible, so I set it to $3 \, eV$ and the energy band is $2.86 0.93 = 1.93 \, eV$
- b) It increased by $\approx 0.5 \ eV$
- c) at $3 \ eV$ Band 1's width: 2.15, Band 2: 1.27
- at $4 \ eV$ Band 1's width: 2.475, Band 2: 1.834

Hence, bandwidths became wider.