Research Proposal

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1 Introduction

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2.2 Spin Hamiltonians

Spin Hamiltonians refer to a type of mathematical model used to describe the behavior of interacting spin systems. These Hamiltonians typically consist of sums of a string of spin operators.

$$H = -\sum_i \left[S_i^x + \bar{\lambda} S_i^z S_{i+1}^z\right]; \qquad \text{The Ising model} \tag{1}$$

$$H = \sum_i \left[(1+\gamma)\, S_i^x S_{i+1}^x + (1-\gamma)\, S_i^y S_{i+1}^y\right]; \qquad \text{The XY model} \tag{2}$$

$$H = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y - J_z \sum_{z-links} \sigma_j^z \sigma_k^z; \qquad \text{Kitaev Honeycomb Model} \tag{3}$$

S^i, σ^i are spin operators, which are represented by Pauli matrices. $\bar{\lambda}$, γ, J_i are model parameters, which are typically related to magnetic fields.

2.3 Jordan-Wigner Transformation

The Jordan-Wigner (JW) transformation is a unitary transformation used to map a system of interacting spins to a system of non-interacting fermions. This transformation allows for the use of fermionic statistics, which could improve the solvability of the system, and its correlation functions.

Jordan-Wigner definition consists of a string part made out of σ^z's, which must thread all the sites once, before the site of transformation. The objective is to define a proper path for this string of σ^z's that makes the model easily solvable.

For the Ising and XY models, or any 1-D chain model, the JW transformation is defined as the following:

$$S_i^+ = \prod_{j<i} \left[-S_j^z\right] c_i^\dagger \qquad S_i^- = c_i \prod_{j<i} \left[-S_j^z\right] \tag{4}$$

$$S_i^z = 2c_i^\dagger c_i - 1 \tag{5}$$

Where $c^\dagger$ and c are fermionic creation and annihilation operators.

For Kitaev honeycomb model, which is two dimensional, the string of sigma σ^z's is defined differently while maintaining the same condition: threading the whole lattice:

$$\sigma_{ij}^+ = 2\left[\prod_{j'<j}\prod_{i'} \sigma_{i'j'}^z\right]\left[\prod_{i'<i} \sigma_{i'j}^z\right] c_{ij}^\dagger \qquad \sigma_{ij}^- = 2c_{ij}\left[\prod_{j'<j}\prod_{i'} \sigma_{i'j'}^z\right]\left[\prod_{i'<i} \sigma_{i'j}^z\right] \tag{6}$$

$$\sigma_{ij}^z = 2c_{ij}^\dagger c_{ij} - 1 \tag{7}$$

2.4 Majorana Fermions

Majorana fermions are particles that are their own anti-particle, that can be described by a linear combination between creation and annihilation fermionic operators. This way of introducing these quasi-particles can be useful in finding conserved quantities in the Hamiltonian, which lead to a better solvability of the model. For example, in Kitaev honeycomb model, the Hamiltonian written in terms of Majorana fermions has a conserved quantity which allowed for the usage of Fourier transformation:

$$H = -iJ_x \sum_{x-links} A_w A_b + iJ_y \sum_{y-links} A_b A_w - iJ_z \sum_{z-links} \underbrace{iB_b B_w}_{conserved} A_b A_w \tag{8}$$

Where A_i's and B_i's are Majorana operators given as:

$$A_w \equiv \frac{\left(c - c^\dagger\right)_w}{i}; \quad B_w \equiv \left(c^\dagger + c\right)_w \tag{9}$$

$$A_b \equiv \left(c^\dagger + c\right)_b; \quad B_b \equiv \frac{\left(c - c^\dagger\right)_b}{i} \tag{10}$$

2.5 Fourier Transformation

Using Fourier transformation, which transforms the Hamiltonian into momentum space, is proven to be useful. After transforming the Hamiltonian into momentum space, one can find simplification that leads to easier computations. The transformation is defined as the following:

$$c_j^\dagger = \frac{1}{\sqrt{N}} \sum_q c_q^\dagger e^{iqj} \qquad\qquad c_j = \frac{1}{\sqrt{N}} \sum_q c_q e^{-iqj} \tag{11}$$

Where $c_q^\dagger$ and c_q are fermionic creation and annihilation operators in momentum space.

2.6 Exact Diagonalization

Solving the Hamiltonian is to diagonalize it. Some types of Hamiltonian are exactly diagonalizable, Bogoliubov types as an example. If the Hamiltonian can be written in this form:

$$H = \begin{bmatrix} \vec{c}_1^{\,\dagger}, \vec{c}_2 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} \vec{c}_1 \\ \vec{c}_2^{\,\dagger} \end{bmatrix} \tag{12}$$

Then it is diagonalized in this way:

$$H = \underbrace{\begin{bmatrix} \vec{c}_q^{\,\dagger} & \vec{c}_{-q} \end{bmatrix} U^\dagger}_{\begin{bmatrix} \vec{\eta}_q^{\,\dagger} & \vec{\eta}_{-q} \end{bmatrix}} \underbrace{U h U^\dagger}_{D} \underbrace{U \begin{bmatrix} \vec{c}_q \\ \vec{c}_{-q}^{\,\dagger} \end{bmatrix}}_{\begin{bmatrix} \vec{\eta}_q & \vec{\eta}_{-q}^{\,\dagger} \end{bmatrix}^T} = 2 \sum_q w_q \eta_q^\dagger \eta_q + E_0 \tag{13}$$

Where $D = \begin{bmatrix} E_q & 0 \\ 0 & E_{-q} \end{bmatrix}$. Then, the Hamiltonian in diagonalized form has the shape:

$$H = \sum_q E_q \eta_q^\dagger \eta_q + E_{-q} \eta_{-q} \eta_{-q}^\dagger \tag{14}$$

Where we may interpret E_q as particle energies and E_{-q} as hole energies.

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