

PHYS210

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Chapter 1

Series

1.1 Infinite Series

- A *sequence* is simply a set of quantities, one for each n , represented with a_n .
- A *series* is an indicated sum of such represented with ' \sum ' symbol.
- In general **infinite series** can be written as $\Rightarrow a_1 + a_2 + a_3 + \dots + a_n + \dots$
- The geomtric series are written as $\Rightarrow a + ar + ar^2 + \dots + ar^{n-1} + \dots$
- or in short notation: $\Rightarrow \sum_{n=1}^{\infty} a_n$
- The sum first of first: n term is $S_n = \frac{a(1-r^n)}{1-r}$
 - ★ The sum of an infinite series is the limit of the sum of n terms as $n \rightarrow \infty$. The *sum of series* is $S = \lim_{n \rightarrow \infty} S_n = \frac{a}{1-r}$
 - ★ The geomtric series have a finite sum if and only if $|r| < 1$.
Then the series is called *convergent*, Otherwise it is called *divergent*.
- the *remainder* (or the remainder after n terms) is $R_n = S - S_n$.
Thus, $\lim_{n \rightarrow \infty} R_n = 0$.

1.2 Series Tests

- If the terms of an infinite series do not tend to zero (that is, if a $\lim_{n \rightarrow \infty} a_n \neq 0$), the series diverges. If $\lim_{n \rightarrow \infty} a_n = 0$, we must test further (Except the alternating series).
 - An alternating series is an example of a *conditional series*.
 - Its positive or negative terms (alone) *diverges*, thus we can control to which number (the sum, S) such series approaches.
 - Physically, We cannot stop at some point and say that the rest of the series is negligible as we could in the bouncing ball problem in Section 1.
 - But if we specify the order in which the charges are to be placed, then the sum S of the series is determined (S is probably different from F in (8.1) unless the charges are placed alternately).
- ★ The convergence or divergence of a series is not affected by multiplying every term of the series by the same nonzero constant. Neither is it affected by changing a finite number of terms (for example, omitting the first few terms).
- ★ Two *convergent* series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ may be added (or subtracted) term by term ($a_n + b_n$). The resulting series is *convergent*, and its sum is obtained by adding (subtracting) the sums of the two given series.

★ The terms of an *absolutely convergent series* may be rearranged in any order without affecting either the *convergence* or the *sum*. This is not true of *conditionally convergent series* as we have seen in Section 8.

1.3 Power Series

It is distinguished by having a variable x^n in its terms multiply by a constant.

By definition:

$$\sum_{n=0}^{\infty} a_n(x-a) = a_0 + a_1(x-a)^1 + a_2(x-a)^2 + \dots \quad (1.1)$$

- The *radius of convergence* R of Eq:1.1 depends on x values .
- We find R by the *ratio test* so that $L < 1$.

★ We see then that a power series (within its interval of convergence) defines a function of x , namely $S(x)$.

Theorems

1. A power series may be differentiated or integrated term by term; the resulting series converges to the derivative or integral of the function represented by the original series *within* the same interval of convergence as the original series (that is, not necessarily at the endpoints of the interval).
2. Two power series may be added, subtracted, or multiplied; the resultant series converges at least in the common interval of convergence. You may divide two series if the denominator series is not zero at $x = 0$, or if it is and the zero is canceled by the numerator [as, for example, in $(\sin x)/x$; see (13.1)]. The resulting series will have *some* interval of convergence (which can be found by the ratio test or more simply by complex variable theory—see Chapter 2, Section 7).
3. One series may be substituted in another provided that the values of the substituted series are in the interval of convergence of the other series.
4. The power series of a function is unique, that is, there is just one power series of the form $\sum_{n=0}^{\infty} a_n x^n$ which converges to a given function.

★ Expanding functions can be done using *Taylor series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad (1.2)$$

- *Maclaurin series* for $f(x)$ is a special case of Eq(1.2) where $a = 0$.

Chapter 2

Complex Numbers

2.1 Introduction

The *imaginary* number is written as $i = \sqrt{-1}$. We use the term *complex number* to mean any one of the whole set of numbers, *real*, *imaginary*, or combinations of the two

- The complex number can arise from the quadratic equation

$$az^2 + bz + c = 0 \quad (2.1)$$

- Where z is a unknown variable and its solution is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (2.2)$$

if the *discriminant* $d = b^2 - 4ac < 0$, z will be a complex number.

- Notice the pattern if i to power of some numbers

$$i^2 = -1, i^3 = -i, i^{4n} = 1 \quad (2.3)$$

- A *complex* number such $3 + 5i$ has two parts a *real part* (here, 3) and an *imaginary part* (5).

2.2 The Complex Plane

- The rectangular coordinates representation for a complex number in the form $x + yi$ is (x, y) .
- In the polar coordinates recall that

$$x = r \cos \theta, \quad (2.4)$$

$$y = r \sin \theta \quad (2.5)$$

- Then we have (by euler formula) assigned to z . All θ are in *radian*.

$$z = x + yi = r(\cos \theta + i \sin \theta) = re^{i\theta} \quad (2.6)$$

- The *modulus* or *absolute value* of z is

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \quad (2.7)$$

2.3 Complex Algebra

- Note that in Eq:(2.7) ≥ 0 , always real.
- The *conjugate* of z is found by substituting $\theta = -\theta$ in Eq:(2.6)

$$\bar{z} = r(\cos \theta - i \sin \theta) = re^{-i\theta} \quad (2.8)$$

- Inotherwords, any equation involving complex numbers is really two equations involving real numbers.

Complex Equations In other words, any equation involving complex numbers is really two equations involving real numbers.

2.4 Complex Infinite Series

- The partial sums of a series of complex numbers will be complex numbers.
- It can be written as $S_n = X_n + Y_n i$ where X_n and Y_n are real.
- *Convergence* is defined just as real series: if S_n approaches a limit ($S = X + Yi$) as $n \rightarrow \infty$, we call the series convergent and call S its sum.
- This means that $X_n \rightarrow X$ and $Y_n \rightarrow Y$.
- It can be proved that an *absolutely convergent series* converges, recall that Eq:(2.7) is positive term.
- $\sum_{n=0}^{\infty} z^n$ is a geometric series, with ratio $= z$ and convergent when $|z| < 1$.

★ Thus any of the tests given in Chapter:1 for convergence of series of positive terms may be used here to test a *complex series* for *absolute convergence*.

2.5 Complex Power Series

They are in form of

$$\sum a_n z^n \quad (2.9)$$

where $z = x + yi$ and a_n are complex numbers.

- Note that Eq:(2.9) includes the *real series* as a special case when $y = 0$.

2.6 Euler's Formula

To derive Euler's Formula, for real θ we have from Chapter:1

$$\sin \theta = \dots \quad (2.10)$$

$$\cos \theta = \dots \quad (2.11)$$

thus we have

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (2.12)$$

2.7 Powers and Roots of Complex Numbers

- The n th power (and n th root) of z is given by

$$z^n = (re^{i\theta})^n = r^n e^{ni\theta} \quad (2.13)$$

- When $r = 1$ Eq:(2.13) becomes **DeMoivre's theorem**

$$(e^{i\theta})^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad (2.14)$$

2.8 The Exponential and The Trigonometric

- Although we have already defined e^z by a power series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, it is worth while to write it in another form.

$$e^z = e^{x+yi} = e^x e^{yi} = e^x (\cos y + i \sin y) \quad (2.15)$$

- We have already seen that there is a close relationship [Euler's formula (2.12)] between complex exponentials and trigonometric functions of real angles. It is useful to write this relation in another form.
- from Eq(2.12) we can rearrange it to

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad (2.16)$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (2.17)$$

- It can be shown that in Eq(2.16) θ can be z a complex number.
- The rest of the trigonometric functions of z are defined in the usual way in terms of these.
- ★ If z is a complex number, $\sin z$ and $\cos z$ can have *any* value we like.

2.9 Hyperbolic Functions

- Let us look at $\sin z$ and $\cos z$ for pure imaginary z in Eq(2.16), that is, $z = iy$:

$$\sin yi = \frac{e^{-y} - e^y}{2i} = i \frac{e^y - e^{-y}}{2} \quad (2.18)$$

$$\cos yi = \frac{e^{-y} + e^y}{2} = \frac{e^y + e^{-y}}{2} \quad (2.19)$$

- The real functions on the right have special names because these particular combinations of exponentials arise frequently in problems. They are called the hyperbolic. Their definitions for all z :

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad (2.20)$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad (2.21)$$

- The other hyperbolic functions are defined as in normal trig. functions.
- Thus from Eq(2.18) we have:

$$\sin yi = i \sinh y \quad (2.22)$$

$$\cos yi = \cosh y \quad (2.23)$$

Chapter 3

Linear Algebra

3.1 Introduction

- problems in various fields of science and mathematics involve the solution of sets of linear equations. Suppose you have solved two simultaneous linear equations and have found $x = 2$ and $y = -3$. We can think of $x = 2$, $y = -3$ as the point $(2, -3)$ in the (x, y) plane. Since two linear equations represent two straight lines, the *solution* is then the *point* of intersection of the lines.
- The language of vectors is very useful in studying sets of simultaneous equations. *Quantities* such as the velocity of an object, which have both *magnitude* and *direction*. Such quantities are called **vectors**; contrast them with such quantities as mass, which have *magnitude* only and are called **scalars**.
- Vector formulas are independent of the choice of coordinate system.
- A vector equation in two dimensions is equivalent to two component equations.

3.2 Matrices: Row Reduction

- A matrix (plural: matrices) is just a rectangular array of quantities, such as

$$A = \begin{bmatrix} 1 & 5 & -2 \\ -3 & 0 & 6 \end{bmatrix} \quad (3.1)$$

- The A letter does not have a numerical value; it simply stands for the array.
- To indicate a number in the array, we will write A_{ij} where i is the *row* number and j is the *column* number.
- We will call a matrix with m rows and n columns as *m by n matrix*.
- **Transpose of a Matrix** We write:

$$A = \begin{bmatrix} 1 & -3 \\ 5 & 0 \\ -2 & 6 \end{bmatrix} \quad (3.2)$$

and call A^T the transpose of the matrix A in Matrix(3.1).

- To transpose a matrix, we simply write the rows as columns.
- Not that, $(A^T)_{ij} = A_{ji}$
- Consider the set of equations of:

$$\begin{cases} 2x - z = 2 \\ 6x + 5y + 3z = 7 \\ 2x - y = 4 \end{cases} \quad (3.3)$$

- Let's agree in organizing them in *standard form*, each column be for a specific variable.
- There are three markable matrices to extract from (3.3); first one is the *matrix of the coefficients*

$$M = \begin{bmatrix} 2 & 0 & -1 \\ 6 & 5 & 3 \\ 2 & -1 & 0 \end{bmatrix} \quad (3.4)$$

Also, there are 3×1 matrices, r and k .

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad k = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \quad (3.5)$$

- The Eqs(3.3) can be written in matrix as $Mr = k$.
- Now we want to write Eqs(3.3) in an *augmented matrix*

$$A = \begin{bmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{bmatrix} \quad (3.6)$$

and then solve it by the *row reduction* with the following *elementary row operations*:

1. Interchange two rows.
 2. Multiply (or divide) a row by a (nonzero) constant.
 3. Add a multiple of one row to another; this includes subtracting, that is, using a negative multiple.
- (3.7)

- If the last row in a reduced augmented matrix is like $0 \times z = 5$, which cannot be true for any finite number on z , then it is called *inconsistent*.
- **Rank of a Matrix** The number of nonzero rows remaining when a matrix has been row reduced is called the rank of the matrix.

1. If $(\text{rank } M) < (\text{rank } A)$, the equations are inconsistent and there is no solution.
 2. If $(\text{rank } M) = (\text{rank } A) = n$ (# of unknowns), there is one solution.
 3. If $(\text{rank } M) = (\text{rank } A) = R < n$, then R unknowns can be found in terms of the remaining $n - R$ unknowns.
- (3.8)

3.3 Determinants; Carmer's Rule

For a square matrix, however, there is a useful number called the determinant of the matrix.

- For a 2×2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (3.9)$$

- If we remove one row and one column from a determinant of order n , we have a determinant of order $n - 1$.
- When removing the *row* and *column* containing the element a_{ij} and call the remaining determinant M_{ij} , which is called the **minor** of a_{ij} .
- The **cofactor** of a_{ij} is:

$$C_{ij} = (-1)^{i+j} M_{ij} \quad (3.10)$$

- In general, for $n \times n$ matrix we have,

$$\det A = \sum_{j=1}^n a_{ij} C_{ij}, \quad \text{where } j \text{ is fixed} \quad (3.11)$$

- The signs goes like $\begin{vmatrix} + & - \\ - & + \\ & & \ddots \end{vmatrix}$.

★ The value of a *determinant*: Multiply each element of one row (or one column) by its cofactor and add the results.

Useful Fact About Determinants:

1. If each element of one row (or one column) of a determinant is multiplied by a number k , the value of the determinant is multiplied by k .
2. The value of a determinant is zero if
 - (a) all elements of one row (or column) are zero; or if
 - (b) two rows (or two columns) are identical; or if
 - (c) two rows (or two columns) are proportional.
3. If two rows (or two columns) of a determinant are interchanged, the value of the determinant changes sign.
4. The value of a determinant is unchanged if
 - (a) rows are written as columns and columns as rows; or if
 - (b) we add to each element of one row, k times the corresponding element of another row, where k is any number (and a similar statement for columns).

(3.12)

Cramer's Rule This is a formula in terms of determinants for the solution of n linear equations in n unknowns when there is exactly one solution.

Let's start with the following equations:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \quad (3.13)$$

If we multiply the first equation by b_2 , the second by b_1 , and then subtract the results and solve for x , we get if $(a_1b_2 - a_2b_1 \neq 0)$

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \quad (3.14)$$

Using the definition (3.9) we can write Eq.(3.14)

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{1}{D} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{1}{D} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad D \neq 0 \quad (3.15)$$

★ To remember it, the denominator for both x and y is the *determinant of the coefficients*, for the numerator for x replace its column with the k matrix (constants column), and do so for y .

This method of solution of a set of linear equations is called Cramer's rule. It may be used to solve n equations in n unknowns if $D \neq 0$; the solution then consists of one value for each unknown. The denominator determinant D is the n by n determinant of the *coefficients* when the equations are arranged in standard form. The numerator determinant for each unknown is the determinant obtained by replacing the column of coefficients of that unknown in D by the constant terms from the right-hand sides of the equations. Then to find the unknowns, we must evaluate each of the determinants and divide. (3.16)

Rank of Matrix Here is another way to find the rank of a matrix. A submatrix means a matrix remaining if we remove some rows and/or remove some columns from the original matrix. To find the rank of a matrix, we look at all the square submatrices and find their determinants. The *order of the largest nonzero determinant* is the rank of the matrix.

3.4 Vectors

Notation We shall indicate a vector by a boldface letter (for example, \mathbf{A}) and a component of a vector by a subscript (for example A_x is the x component of \mathbf{A}). For handwriting you should write a vector with an arrow (like, \vec{A}).

Magnitude of a Vector The *length* of the arrow representing a vector \mathbf{A} is called the *length* or the *magnitude* of \mathbf{A} (written $|\mathbf{A}|$ or A) or the *norm* of \mathbf{A} (written $||\mathbf{A}||$). Note the use of A to mean the magnitude of \mathbf{A} .

- By the Pythagorean theorem, we find

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (3.17)$$

Addition of Vectors There are two ways to add vectors: Addition of Vectors by the parallelogram law: To find $\vec{A} + \vec{B}$, place the tail of \vec{B} at the head of \vec{A} and draw the vector from the tail of \vec{A} to the head of \vec{B} . Or by adding their components together, like $A_x + B_x$ and $A_y + B_y$. They follow:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}, \quad (\text{commutative law for addition}); \quad (3.18)$$

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}), \quad (\text{associative law for addition}). \quad (3.19)$$

- In other words, vectors may be added together by the usual laws of algebra. They can be multiplied by a number. And we can define vectors subtraction so:

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

- The **zero vector** is a vector of zero magnitude; its components are all zero and it does not have a direction.

A vector of length or magnitude 1 is called a **unit vector**. Then for any $\vec{A} \neq 0$, the vector $\vec{A}/|\vec{A}|$ is a unit vector.

Vectors in Terms of Components We consider a set of rectangular axes as in Fig.(3.1) let \hat{i} be a unit vector in the positive x direction; \hat{j} and \hat{k} in y and z positive direction. For \vec{A}

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad (3.20)$$

The vectors $\hat{i}, \hat{j}, \hat{k}$ are called *unit basis vectors*.

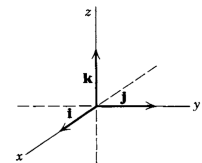


Figure 3.1: The unit basis vectors in a rectangular system.

Multiplication of Vectors There are two kinds of product of two vectors. One, called the scalar product (or **dot product** or **inner product**), gives a result which is a scalar; the other, called the *vector product* (or **cross product**), gives a vector answer.

Scalar Product For vectors \vec{A} , and \vec{B} , and the angle $\theta (\leq 180^\circ)$ between them, we have:

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) \cdot (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) \\ \vec{A} \cdot \vec{B} &= |\vec{A}||\vec{B}| \cos \theta = A_x B_x + A_y B_y + A_z B_z\end{aligned}\quad (3.21)$$

The dot product in Eq.(3.21) holds the commutative law. And the distributive law.

- A vector dot itself gives its magnitude:

$$\vec{A} \cdot \vec{A} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2 = A^2$$

Perpendicular and Parallel Vectors If two vectors are perpendicular, then $\cos \theta = 0$; thus

$$\begin{aligned}\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z = 0, \quad \text{if } \vec{A} \text{ and } \vec{B} \text{ are perpendicular vectors.} \\ \frac{A_x}{B_x} &= \frac{A_y}{B_y} = \frac{A_z}{B_z}, \quad \text{if } \vec{A} \text{ and } \vec{B} \text{ are perpendicular vectors.}\end{aligned}\quad (3.22)$$

Vector Product The vector or cross product of \vec{A} and \vec{B} is written $\vec{A} \times \vec{B}$. By definition, $\vec{A} \times \vec{B}$ is a vector whose magnitude and direction are given as follows:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \theta, \quad (3.23)$$

The direction of $\vec{A} \times \vec{B}$ is perpendicular to the plane of \vec{A} and \vec{B} , using *right-hand rule*

Intresting properties:

$$\begin{aligned}\vec{A} \times \vec{B} &= -\vec{B} \times \vec{A} \quad \text{not commutative,} \\ \vec{A} \times \vec{B} &= 0 \quad \text{if } \vec{A} \text{ and } \vec{B} \text{ are parallel or antiparallel,} \\ \vec{A} \times \vec{A} &= 0 \quad \text{for any } \vec{A}\end{aligned}\quad (3.24)$$

★ A good way to remember the basis-unit vectors cross-product is to write them cyclically. Reading around the circle counterclockwise (positive θ direction), we get the positive products (for example, $\hat{i} \times \hat{j} = \hat{k}$); reading the other way we get the negative products (for example, $\hat{i} \times \hat{k} = -\hat{j}$). This works with the *right-haned systems*.

To write $\vec{A} \times \vec{B}$ in component form we need the distributive law, namely

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \quad (3.25)$$

Thus we have:

$$\begin{aligned}\vec{A} \times \vec{B} &= (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z) \times (\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) \\ &= \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned}\quad (3.26)$$

3.5 Lines and Planes

In analytic geometry a point is a set of three coordinates (x, y, z) ; we shall think of this point as the *head* of a vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ with *tail* at the **origin**. In two dimensions, we write the equation of a *straight line* through (x_0, y_0) with slope m as

$$\frac{y - y_0}{x - x_0} = m \quad (3.27)$$

- Suppose, instead of the slope, we are given a vector in the direction of the line, say $\vec{A} = \hat{i}a + \hat{j}b$. Then the line through (x_0, y_0) and parallel to \vec{A} we can write its equation. If we have two points on the line from (x_0, y_0) to any point (x, y) , then the vector $\vec{r} - \vec{r}_0$ with components $x - x_0$ and $y - y_0$:

$$\vec{r} - \vec{r}_0 = (x - x_0)\hat{i} + (y - y_0)\hat{j} \quad (3.28)$$

Since this vector is parallel to \vec{A} , then their components are proportional, so (for $a, b \neq 0$):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{or} \quad \frac{y - y_0}{x - x_0} = \frac{b}{a} \quad (3.29)$$

The equation is for a given **line**, see tht Eq.(3.27) and Eq.(3.29) are identical.

- Since $\vec{r} - \vec{r}_0$ and \vec{A} are parallel, thus they are only differen by a factor of t , so

$$\vec{r} - \vec{r}_0 = \vec{A}t \quad \text{or} \quad \vec{r} = \vec{r}_0 + \vec{A}t \quad (3.30)$$

Then their *components form*:

$$\begin{aligned} x - x_0 &= at & \text{or} & & x &= x_0 + at \\ y - y_0 &= bt & & & y &= y_0 + bt \end{aligned} \quad (3.31)$$

Eliminating t yields the equation of the line in Eq.(3.29)

- In three dimensions, we want the equations of a straight line through a given point (x_0, y_0, z_0) and parallel to a given vector $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$. If (x, y, z) is any point on the line, the vector joining (x_0, y_0, z_0) and (x, y, z) is parallel to \vec{A} . Then,

$$\begin{aligned} \frac{x - x_0}{a} &= \frac{y - y_0}{b} = \frac{z - z_0}{c} & a, b, c \neq 0 \\ \frac{x - x_0}{a} &= \frac{y - y_0}{b}, z = z_0 & \text{if say } c = 0 \end{aligned} \quad (3.32)$$

The paramtric equations for a line, see Fig.(3.5), from Eq.(3.32)

$$\vec{r} = \vec{r}_0 + \vec{A}t \quad \text{or} \quad \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases} \quad (3.33)$$

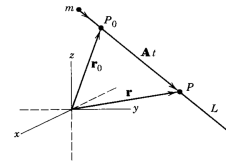


Figure 3.2:

- Suppose we want the equation of a straight line L through the point (x_0, y_0) and **perpendicular** to a given vector $\vec{N} = a\hat{i} + b\hat{j}$. Let the vector in Eq.(3.28) lies along the line but to be perpendicular to \vec{N} . Recall from (3.22), we have $(\vec{r} - \vec{r}_0) \cdot \vec{N} = 0$, thus their components

$$a(x - x_0) + b(y - y_0) = 0 \quad \text{or} \quad \frac{y - y_0}{x - x_0} = -\frac{a}{b} \quad (3.34)$$

This is the equation for a line that is perpendicular to \vec{N}

- In 3D, we can use this method to assign an equation for a plane. Suppose that two points on the plane, (x_0, y_0, z_0) and (x, y, z) is any point and represented by the vector in (3.28). If $\vec{N} = a\hat{i} + b\hat{j} + c\hat{k}$ is a normal vector to the plane, we have $(\vec{r} - \vec{r}_0) \cdot \vec{N} = 0$, so

$$\begin{aligned} a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0, \\ ax + by + cz &= d \quad \text{where} \quad d = ax_0 + by_0 + cz_0 \end{aligned} \quad (3.35)$$

3.6 Matrix Operations

In Section 2 we used matrices simply as arrays of numbers. Now we want to go farther into the subject and discuss the meaning and use of multiplying a matrix by a number and of combining matrices by addition, subtraction, multiplication, and even (in a sense) division.

Matrix Equations Two matrices are only equal if they are identical, for example,

$$\begin{bmatrix} w & m \\ r & k \end{bmatrix} = \begin{bmatrix} 4 & 5i \\ 5 & 0 \end{bmatrix} \quad (3.36)$$

then $w = 4$, $m = 5i$, $r = 5$, $k = 0$.

Remember this concept in the equation $z = x + iy = 2 - 3i$ is equivalent to the two real equations $x = 2$, $y = -3$;

a vector equation in three dimensions is equivalent to three component equations.

Multiplication of a Matrix by a Number We can write a vector $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ in matrix-like way,

$$\begin{aligned} A &= \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ a column matrix or column vector,} \\ A^T &= [a \quad b \quad c] \text{ called a row matrix or row vector.} \end{aligned} \quad (3.37)$$

The row matrix A^T is the transpose of the column matrix A

- Suppose we multiply the vector \vec{A} by a constant C , then $C\vec{A} = aC\hat{i} + bC\hat{j} + cC\hat{k}$ and its matrix becomes

$$A = [aC \quad bC \quad cC] \quad (3.38)$$

- ★ Thus when a matrix is multiplied by a number each element is multiplied by it.

Multiplying a matrix by a number k means multiplying every element by k ,
But multiplying just one *row* of a determinant by k multiplies the determinant by k . Thus $\det kA = k^2 \det A$ for a 2 by 2 matrix.

Addition of Matrices When we add vectors algebraically, we add them by components. Matrices are added in the same way, but by adding *corresponding elements*. If the two matrices have different m by n , we say that the sum is undefined or meaningless.

Multiplication of Matrices Let us start by defining the product of two matrices,

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = C \quad (3.39)$$

Each row for each column. :“row times column”.

$$\begin{aligned} \text{The element in row } i \text{ and column } j \text{ of the product matrix } AB \\ \text{is equal to row } i \text{ of } A \text{ times column } j \text{ of } B. \text{ In index notation} \end{aligned} \quad (AB)_{ij} = \sum_k A_{ik} B_{kj} \quad (3.40)$$

The product AB (in that order) can be found if and only if the number of elements in a *row* of A equals the number of elements in a *column* of B ; the matrices A, B in that order are then called **conformable**.

(Observe that the number of rows in A and of columns in B have nothing to do with the question of whether we can find AB or not.)

$$[A, B] = AB - BA = \text{commutator of } A \text{ and } B. \quad (3.41)$$

Zero Matrix The **zero** or **null** matrix means one with all its elements equal to zero. It is often abbreviated by 0, but we must be careful about this

Identity Matrix or Unit Matrix This is a square matrix with every element of the main diagonal equal to 1 and all other elements equal to zero. For example

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.42)$$

In multiplication, a unit matrix acts like the number 1, that is, if A is any matrix and I is the *unit* matrix *conformable* with A in the order in which we multiply, then $IA = AI = A$.

Operations with Determinants we multiply determinants the same way we multiply matrices.

$$\det AB = \det BA = \det A \cdot \det B \quad (3.43)$$

Applications of Matrix Multiplication We can now write sets of simultaneous linear equations in a very simple form using matrices. Consider the matrix equation

$$\begin{bmatrix} 2 & 6 & -3 \\ 7 & 4 & 4 \\ 2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \quad (3.44)$$

By matrices multiplication,

$$\begin{bmatrix} 2x + 6y - 3z \\ 7x + 4y + 4z \\ 2x + y + 8z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \quad (3.45)$$

Recall from (3.6), then we get

$$\begin{cases} 2x + 6y - 3z = 5, \\ 7x + 4y + 4z = 4, \\ 2x + y + 8z = 2 \end{cases} \quad (3.46)$$

Consequently (3.44) is the matrix form for the set of equations in (3.46), if

$$M = \begin{bmatrix} 2x + 6y - 3z \\ 7x + 4y + 4z \\ 2x + y + 8z \end{bmatrix}, \quad r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad k = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \quad (3.47)$$

then we can write the (3.44) as $Mr = k$ or $\sum_j M_{ij}r_j = k_i$.

Inverse of a Matrix *Inverse of a Matrix* The reciprocal or inverse of a number x is x^{-1} such that the product $xx^{-1} = 1$. We define the inverse of a matrix M (if it has one) as the matrix M^{-1} such that MM^{-1} and $M^{-1}M$ are both equal to a unit matrix I .

- Note that only square matrices can have inverses (otherwise we could not multiply both MM^{-1} and $M^{-1}M$). Actually, some square matrices do not have inverses either. You can see from (3.43) that if $M^{-1}M = I$, then $(\det M^{-1})(\det M) = \det I = 1$. If two numbers have product = 1, then neither of them is zero; thus $\det M \neq 0$ is a requirement for M to have an inverse.

- If a matrix has an inverse we say that it is **invertible**; if it doesn't have an inverse, it is called **singular**. We can find the inverse of a matrix as

$$M^{-1} = \frac{1}{\det M} C^T \quad (3.48)$$

3.7 Linear Combinations, Functions, and Operators

- Given two vectors \vec{A} and \vec{B} , the vector $3\vec{A} - 2\vec{B}$ is called a **linear combination** of \vec{A} and \vec{B} . In general, a linear combination of \vec{A} and \vec{B} means $a\vec{A} + b\vec{B}$ where a and b are *scalars*. Geometrically, if \vec{A} and \vec{B} have the same *tail* and do not lie along a line, then they determine a plane. All linear combinations of \vec{A} and \vec{B} lie in the plane.

- It is also true that every vector in the plane can be written as a linear combination of \vec{A} and \vec{B} . The vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with tail at the origin (which we used in writing equations of lines and planes) is a linear combination of the unit basis vectors $\hat{i}, \hat{j}, \hat{k}$.

A function of a vector, $f(\vec{r})$ is called linear if

$$f(\vec{r}_1 + \vec{r}_2) = f(\vec{r}_1) + f(\vec{r}_2), \quad \text{and} \quad f(a\vec{r}) = af(\vec{r}) \quad (3.49)$$

where a is a scalar.

- Note, $f(\vec{r}) = |\vec{r}|$ is not a linear function, because the length of the sum of two vectors is not in general the sum of their lengths. That is,

$$f(\vec{r}_1 + \vec{r}_2) = |\vec{r}_1 + \vec{r}_2| \neq |\vec{r}_1| + |\vec{r}_2| = f(\vec{r}_1) + f(\vec{r}_2)$$

as in Fig.(3.3). Also notice that we call $y = mx + b$ a linear equation, however the function $f(x) = mx + b$ is not linear by the test (3.49) (unless $b = 0$).

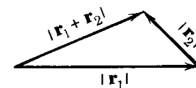


Figure 3.3:

- Now consider **vector function** of a vector \vec{r}

$\vec{F}(\vec{r})$ is a linear vector function if

$$\vec{F}(\vec{r}_1 + \vec{r}_2) = \vec{F}(\vec{r}_1) + \vec{F}(\vec{r}_2) \quad \text{and} \quad \vec{F}(a\vec{r}) = a\vec{F}(\vec{r}) \quad (3.50)$$

where a is a scalar.

- Recall that from calculus:

$$\begin{aligned} \frac{d}{dx} [f(x) + g(x)] &= \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \quad \text{and} \\ \frac{d}{dx} [kf(x)] &= k \frac{d}{dx} f(x) \end{aligned} \quad (3.51)$$

where k is a constant

This is similar to (3.49), then we call d/dx a **linear operator**. An **operator** or **operation** simply means a rule or some kind of instruction telling us what to do with whatever follows it. In other words, a *linear operator* is a *linear function*, so

O is a linear operator if

$$O(A + B) = O(A) + O(B) \quad \text{and} \quad O(kA) = kO(A) \quad (3.52)$$

where k is a number

Matrix Operators, Linear Transformations Consider the following equation set

$$\begin{cases} X = ax + by \\ Y = cx + dy \end{cases} \quad \text{or} \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad R = Mr \quad (3.53)$$

where a , b , c , and d are constants. For every point (x, y) , these equations results a point (X, Y) .

If we think of each point of the (x, y) plane being moved to some other point (the origin not being moved), we can call this process a **mapping** or **transformation** of the plane into itself. All the information about this *transformation* is contained in the matrix M . We say that this matrix is an *operator* which maps the plane into itself.

- Any matrix can be thought of as an *operator* on (conformable) column matrices r . From (3.52), M is a linear operator.

- We can interpreted Eqs.(3.53) geometrically in two ways. In Fig.(3.4) The vector \vec{r} has been changed to the vector \vec{R} by the transformation (3.53).

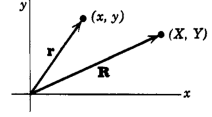


Figure 3.4: fixed co-ordinates axes

- However, in Fis.(3.5) **two** sets of coordinates axes (x, y) and (x', y') , and **one** vector $\vec{r} = \vec{r}'$. The transformation can ve written as

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \text{or} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{or} \quad r' = Mr \quad (3.54)$$

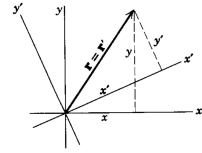


Figure 3.5: A fixed vector

Orthogonal Transformations This is a special case of a *linear transformation* which preserves the length of a vector. In (3.54) is an *orthogonal transformation* if

$$x'^2 + y'^2 = x^2 + y^2 \quad (3.55)$$

then either it is reflected or rotated. The matrix M of an orthogonal transformation is called an **orthogonal matrix**. The inverse of an orthogonal matrix equals its transpose.

$$M^{-1} = M^T, \quad M \text{ is orthogonal} \quad (3.56)$$

From Eq.(3.54) and (3.55)

$$\begin{aligned} x'^2 + y'^2 &= (ax + by)^2 + (cx + dy)^2 \\ &= (a^2 + c^2)x^2 + 2(ab + cd)xy + (b^2 + d^2)y^2 \equiv x^2 + y^2 \end{aligned}$$

Thus we must have $a^2 + c^2 = b^2 + d^2 = 1$, $ab + cd = 0$

$$MM^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let Eq.(3.56) be $MM^T = I$ and from Eq.(3.43) we have

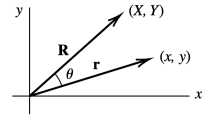
$$\begin{aligned} \det M^T M &= \det M^T \det M \\ \text{and } \det M^T &= \det M \\ \text{so } \det M^2 &= \det M^T M = \det I = 1 \end{aligned}$$

$$\det M = \pm 1 \quad (3.57)$$

★ $\det M = 1$ corresponds geometrically to a *rotation*, and $\det M = -1$ means that a *reflection* is involved.

Rotations in 2 Dimensions In Fig.(3.6) we have two vectors \vec{R} and \vec{r} which is rotated by θ . we can write in matrix their transformation as

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (3.58)$$

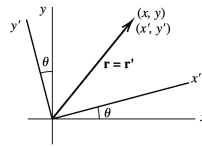


- However for axes rotation as in Fig.(3.7) we have

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (3.59)$$

- Both Eq.(3.58) and Eq.(3.59) are called **rotation equations** and the matrices that contain θ are called **rotation matrices**. Also notice that Eq.(3.58) and Eq.(3.59) are inverses of each other.

Figure 3.6: Vectors rotation with an angle θ



Rotations and Reflections in 3 Dimensions For a vector $\vec{r} = \langle x, y, z \rangle$. The the following matrix is a rotation about $z - axis$,

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.60)$$

And the following is a rotation and reflection about $xy - plane$

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (3.61)$$

Figure 3.7: Axes rotation