KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DEPARTMENT OF PHYSICS
PHYS 422-222
Nuclear and Partcile Physics
HW-1 Solutions
Due on Friday 27 Jan. 2022
Problems are from Ch.2 of Textbook and are equally weighted (25 pts each)
Attempt the problems by yourself first, and then seek help if needed.
If you use a reference/solution manual, mention it and you will get full credit for a correct answer.
Please submit good PDF copy by email to khiari@kfupm.edu.sa

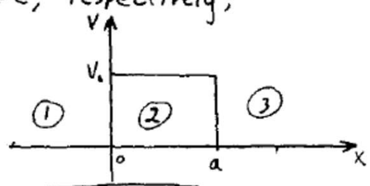
Q1. Pb # 1.

1. Derive Equation 2.37 and plot the transmission coefficient as a function of the energy E of the incident particle. Comment on the behavior of T.

Solution

$\boxed{2-1}$ For $E > V_0$, the solutions to the Schrödinger equation in the three regions are, respectively,

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}, \quad \text{for } x<0$$
$$\psi_2 = Ce^{ik_2x} + De^{-ik_2x}, \quad \text{for } 0<x<a$$
$$\psi_3 = Fe^{ik_3x}, \quad \text{for } x>a$$



where $k_1 = k_3 = \sqrt{2mE/\hbar^2}$ and $k_2 = \sqrt{2m(E-V_0)/\hbar^2}$

Using the continuity conditions at $x=0$ and $x=a$, we have

$$\psi_1|_{x=0} = \psi_2|_{x=0}: \qquad A+B = C+D$$

$$\psi_1'|_{x=0} = \psi_2'|_{x=0}: \qquad A-B = \left(\frac{k_2}{k_1}\right)(C-D)$$

$$\psi_2|_{x=a} = \psi_3|_{x=a}: \quad Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_3a}$$

$$\psi_2'|_{x=a} = \psi_3'|_{x=a}: \quad Ce^{ik_2a} - De^{-ik_2a} = \left(\frac{k_3}{k_2}\right)Fe^{ik_3a}$$

Rearranging the above four equations and taking A as a parameter in terms of which we solve for B, C, D and F, we have,

$$\begin{aligned}
-B + C + D &= A \\
B + \left(\tfrac{k_2}{k_1}\right)C - \left(\tfrac{k_2}{k_1}\right)D &= A \\
Ce^{ik_2a} + De^{-ik_2a} - Fe^{ik_1a} &= 0 \\
Ce^{ik_2a} - De^{-ik_2a} - \left(\tfrac{k_1}{k_2}\right)Fe^{ik_1a} &= 0
\end{aligned}$$

The above four equations are linear equations for B, C, D and F. We use the determinant method to solve them. For our purpose, we need to solve only for F.

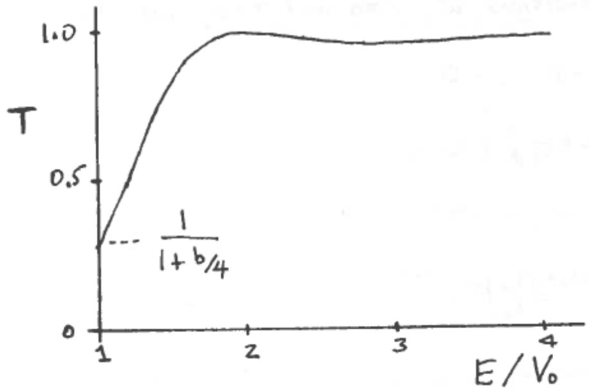
$$\Delta = \begin{vmatrix} -1 & 1 & 1 & 0 \\ 1 & \left(\frac{k_2}{k_1}\right) & \left(\frac{-k_2}{k_1}\right) & 0 \\ 0 & e^{ik_2a} & e^{-ik_2a} & -e^{ik_1a} \\ 0 & e^{ik_2a} & -e^{-ik_2a} & -\left(\frac{k_1}{k_2}\right)e^{ik_1a} \end{vmatrix} = e^{ik_1a}\left[4\cos(k_2a) - \frac{2i(k_1^2+k_2^2)}{k_1k_2}\sin(k_2a)\right]$$

$$\Delta_F = \begin{vmatrix} -1 & 1 & 1 & A \\ 1 & \left(\frac{k_2}{k_1}\right) & \left(-\frac{k_2}{k_1}\right) & A \\ 0 & e^{ik_2a} & e^{-ik_2a} & 0 \\ 0 & e^{ik_2a} & -e^{-ik_2a} & 0 \end{vmatrix} = 4A$$

So $F = \frac{\Delta_F}{\Delta} = 4Ae^{-ik_1a} \Big/ \left[4\cos(k_2a) - \frac{2i(k_1^2+k_2^2)}{k_1k_2}\sin(k_2a)\right]$

Finally,

$$T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{(k_1^2-k_2^2)^2}{4k_1^2k_2^2}\sin^2(k_2a)} = \frac{1}{1 + \frac{V_0^2}{4E(E-V_0)}\sin^2(k_2a)}$$



where,
$k_2 = \sqrt{2m(E-V_0)/\hbar^2}$

Q2. Pb # 3.

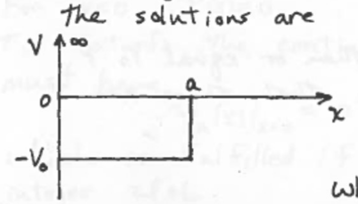
3. Solve the Schrödinger equation for the following potential:

$$V(x) = \infty \qquad x < 0$$
$$= -V_0 \qquad 0 < x < a$$
$$= 0 \qquad x > a$$

Here V_0 is positive and solutions are needed for energies $E > 0$. Evaluate all undetermined coefficients in terms of a single common coefficient, but do not attempt to normalize the wave function. Assume particles are incident from $x = -\infty$.

Solution

$\boxed{2-3}$ Assuming that the particles are incident from $x = +\infty$, the solutions are



$$\psi_1 = A e^{ikx} + B e^{-ikx}, \quad \text{for } x \geq a$$
$$\psi_2 = C e^{ik'x} + D e^{-ik'x}, \quad \text{for } 0 < x < a$$
$$\psi_3 = 0, \quad \text{for } x < 0$$

where $k = \sqrt{2mE/\hbar^2}$ and $k' = \sqrt{2m(E+V_0)/\hbar^2}$

The continuity conditions at $x=0$ and $x=a$ give us

$$C + D = 0$$
$$-B e^{-ika} + C e^{ik'a} + D e^{-ik'a} = A e^{ika}$$
$$B e^{-ika} + C\left(\frac{k'}{k}\right) e^{ik'a} - D\left(\frac{k'}{k}\right) e^{-ik'a} = A e^{ika}$$

Using the determinant method to solve for B, C and D in terms of A, we have

$$\Delta = \begin{vmatrix} 0 & 1 & 1 \\ -e^{-ika} & e^{ik'a} & e^{-ik'a} \\ e^{-ika} & \left(\frac{k'}{k}\right) e^{ik'a} & \left(-\frac{k'}{k}\right) e^{-ik'a} \end{vmatrix} = -2e^{-ika}\left[i \sin(k'a) + \left(\frac{k'}{k}\right)\cos(k'a)\right]$$

$$\Delta_B = \begin{vmatrix} 0 & 1 & 1 \\ Ae^{ika} & e^{ik'a} & e^{-ik'a} \\ Ae^{ika} & \left(\frac{k'}{k}\right) e^{ik'a} & \left(-\frac{k'}{k}\right) e^{-ik'a} \end{vmatrix} = -2Ae^{ika}\left[i \sin(k'a) - \left(\frac{k'}{k}\right)\cos(k'a)\right]$$

$$\Delta_C = \begin{vmatrix} 0 & 0 & 1 \\ -e^{-ika} & Ae^{ika} & e^{-ik'a} \\ e^{-ika} & Ae^{ika} & \left(-\frac{k'}{k}\right) e^{-ik'a} \end{vmatrix} = -2A$$

$$B = \frac{\Delta_B}{\Delta} = A e^{2ika}\left[-1 + \frac{ikk' \sin(2k'a)}{(k^2+k'^2)\cos^2(k'a) - k^2}\right]$$

$$C = \frac{\Delta_C}{\Delta} = -D = A e^{ika}\left[\frac{kk' \cos(k'a) - ik^2 \sin(k'a)}{(k^2+k'^2)\cos^2(k'a) - k^2}\right]$$

Q3. Pb # 7.

7. (a) For the ground state of the one-dimensional simple harmonic oscillator, evaluate $\langle x \rangle$ and $\langle x^2 \rangle$.

(b) Find $\Delta x = [\langle x^2 \rangle - \langle x \rangle^2]^{1/2}$.

(c) Without carrying out any additional calculations, evaluate $\langle p_x \rangle$ and $\langle p_x^2 \rangle$. (*Hint:* Find $\langle p_x^2/2m \rangle$.)

(d) Evaluate Δp_x and the product $\Delta x \cdot \Delta p_x$. A wave packet with this shape (called a Gaussian shape) is known as a "minimum-uncertainty" wave packet. Why?

Solution

$\boxed{2-7}$ (a) $\psi_0(x) = \frac{\alpha^{1/2}}{\pi^{1/4}} e^{-\alpha^2 x^2/2}$

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\, x\, |\psi_0(x)|^2 = \frac{\alpha}{\pi^{1/2}} \int_{-\infty}^{\infty} dx\, x e^{-\alpha^2 x^2} = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\, x^2 |\psi_0(x)|^2 = \frac{\alpha}{\pi^{1/2}} \int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha^2 x^2} = \frac{2}{\pi^{1/2}\alpha^2} \int_0^{\infty} dy\, y^2 e^{-y^2}$$

$$= \frac{2}{\pi^{1/2}\alpha^2} \left(\frac{1}{4}\right) \pi^{1/2} = \frac{1}{2\alpha^2}$$

(b) $\Delta x = [\langle x^2 \rangle - \langle x \rangle^2]^{1/2} = \left(\frac{1}{2\alpha^2}\right)^{1/2} = \frac{1}{\sqrt{2}\,\alpha}$

(c) under $x \to -x$, $P_x \to -P_x$ so $\langle P_x \rangle = 0$

From $\langle H \rangle = \left\langle \frac{P_x^2}{2m} + V(x) \right\rangle = \frac{1}{2}\hbar\omega_0$

we have,

$$\langle P_x^2 \rangle = 2m\left[\frac{1}{2}\hbar\omega_0 - \frac{1}{2}m\omega_0^2 \langle x^2 \rangle\right] = 2m\left[\frac{1}{2}\frac{\hbar^2\alpha^2}{m} - \frac{1}{2}m\frac{\hbar^2\alpha^4}{m^2}\cdot\frac{1}{2\alpha^2}\right]$$

$$\langle P_x^2 \rangle = \frac{1}{2}\hbar^2\alpha^2$$

(d) $\Delta P_x = [\langle P_x^2 \rangle - \langle P_x \rangle^2]^{1/2} = \frac{\hbar\alpha}{\sqrt{2}}$; $\Delta X \cdot \Delta P_x = \frac{1}{\sqrt{2}\,\alpha} \cdot \frac{\hbar\alpha}{\sqrt{2}} = \frac{\hbar}{2}$

The wave packet with this shape has the minimum value of $\Delta X \cdot \Delta P_x$ permitted by the uncertainty principle ($\Delta X \cdot \Delta P_x \geq \hbar/2$).

Q4. Pb # 15.

15. (a) What are the possible values of j for f states?

(b) What are the corresponding m_j?

(c) How many total m_j states are there?

(d) How many states would there be if we instead used the labels m_ℓ and m_s?

Solution

$\boxed{2-15}$ (a) For f states, $\ell = 3$.

$s = \frac{1}{2}$

From $\vec{j} = \vec{\ell} + \vec{s}$, we know that the possible j values are,

$j = \ell \pm \frac{1}{2} = \frac{5}{2}, \frac{7}{2}$

(b) For $j = \frac{5}{2}$, $m_j = \pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$

For $j = \frac{7}{2}$, $m_j = \pm\frac{7}{2}, \pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$

(c) From part (b), we see that we have 14 m_j states. In fact This is just the number $(2j_1+1)+(2j_2+1) = 14$

(d) If we use the labels m_ℓ and m_s, the number of states is:

$(2s+1)(2\ell+1) = 2\times 7 = 14$, which is the same number as in (c).