Reflections (Easy)

Consider what happens at a reflection. Consider a ray XY that hits the mirror ZW at point Y. Suppose the reflected ray is YX'. Let X'' be the reflection of X'' across the line ZW. Then from the laws of reflection, we note that X, Y, X'' are collinear. Hence if we reflect the half-plane (of which the ray XY is a part) over the reflecting surface, then the image of XY gets mapped to a line collinear with XY.

Now create an infinite grid of which our original square is a part, and consider the quadrant which contains it and has A as a corner. Then we note that if we do the above transformation, we get a path from our initial point to some point on this quadrant. Hence, there is a bijective correspondence between a ray in this quadrant, and a ray emanating from A.

B is mapped to points whose x-coordinate is odd and y-coordinate is even, C is mapped to points both of whose coordinates are odd, D is mapped to points whose x-coordinate is even and y-coordinate is odd, and A is mapped to points both of whose coordinates are even. Correspondingly, by reflecting along the intersections of the gridlines with the line from the origin to a point on the quadrant, we can see that each point can be obtained in precisely one way.

Note that we can't ever exit through A, since that path in our infinite quadrant will always pass through a point to which one of B, C or D is mapped (as both coordinates are even).

Now consider any point (x, y) where both x, y are positive integers. The number of intersections the line segment (without its endpoints) joining it to A has with the gridlines is precisely equal to the number of reflections. But the number of such intersections is precisely $x + y - 2$.

Hence, the problem reduces to solving the linear equation $x + y = k + 2$ in positive integers, where x, y have specified parities and are coprime.

For the easier version, iterating over the possible values of x and counting valid solutions works.

This runs in $O(k \log k)$ time, since computing $\gcd(a, b)$ takes $O(\log \min(a, b))$ time.

However, for the harder version iteration will time out. Solving the by noting that $gcd(x, y) = gcd(x, x+y)$, we get the following answer:

1. If k is even, there are $s_C \times \phi(k + 2)$ solutions.
2. If k is odd, there are $(s_B + s_D) \times \phi(k + 2)/2$ solutions.

Here ϕ is the Euler totient function.