Assignment #02

Data Structure and Algorithms.

Topic: Complexity Analysis.

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CIIT/FA21-BCS-020/SWL (4-A)

Submitted To: Ms. Tahreem Saeed.

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QUESTION#01

Part-a

Big-Oh Notation (O)

Definition:-

If $f, g: N \rightarrow \mathbb{R}^+$, then we can define Big-Oh as.

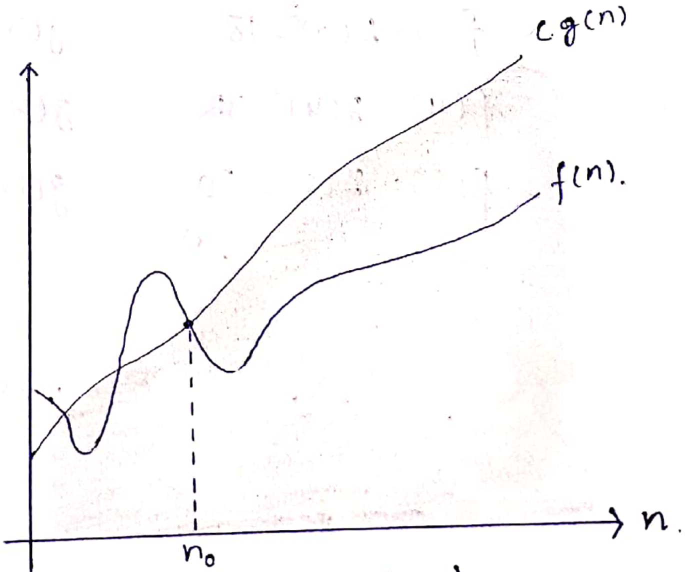
For a given function $g(n) \geq 0$, denoted by $O(g(n))$ the set of functions,

$O(g(n)) = \{f(n):$ there exist positive constant c and n_0 such that $0 \leq f(n) \leq cg(n)$, for all $n \geq n_0\}$

$f(n) = O(g(n))$ means function $g(n)$ is an asymptotically upper bound for $f(n)$.

We may write $f(n) = O(g(n))$ OR $f(n) \in O(g(n))$.

Graph:-



$f(n) \in O(g(n))$

$f(n) = O(g(n))$

$\exists\, c > 0,\ \exists\, n_0 \geq 0$, and $\forall n \geq n_0,\ 0 \leq f(n) \leq c \cdot g(n)$

$g(n)$ is an asymptotic upper bound for $f(n)$.

Example #01:-

Prove that $2n^2 \in O(n^3)$

Proof:-

Assume that $f(n) = 2n^2$

$g(n) = n^3$.

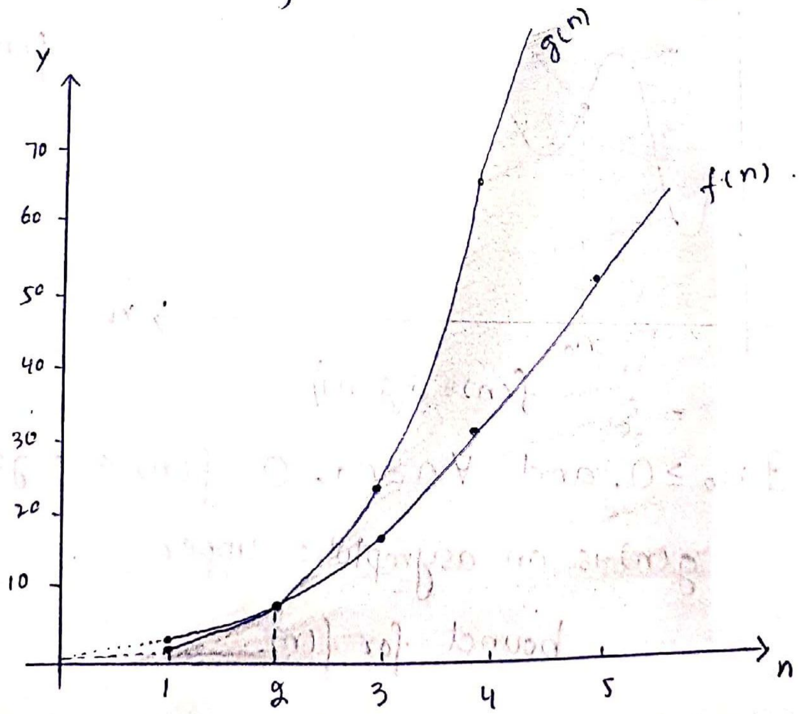
Now we have to find the existance of c and no.

$$0 \le f(n) \le c(g(n)).$$

$$2\cancel{n^2} \le c \cdot n^{\cancel{3}}$$

$$2 \le c \cdot n$$

When	$f(n) = 2n^2$	$g(n) = n^3$
$n=1$	$f(1) = 2(1)^2 = 2$	$g(1) = 1^3 = 1$
$n=2$	$f(2) = 2(2)^2 = 8$	$g(2) = 2^3 = 8$
$n=3$	$f(3) = 2(3)^2 = 18$	$g(3) = 3^3 = 27$
$n=4$	$f(4) = 2(4)^2 = 32$	$g(4) = 4^3 = 64$
$n=5$	$f(5) = 2(5)^2 = 50$	$g(5) = 5^3 = 125$.



$n_0 = 2$.

if we take $c=2$ and $n_0=1$ or $c=1$ and $n_0=2$.

Hence $f(n) \in O(g(n))$, $c=1$ and $n_0=2$.

Example #02.

Prove that: $n^3 \notin O(n^2)$.

Proof:-

Assume that $f(n) = n^3$

$g(n) = n^2$.

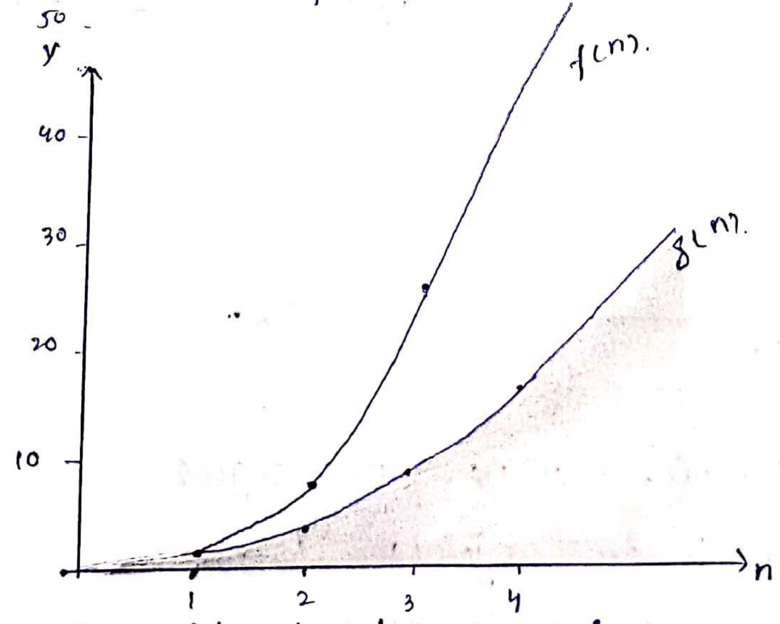
Now we have to find the existence of c and n_0.

$$0 \leq f(n) \leq c \cdot g(n)$$

$$n^{\cancel{3}} \leq c \cdot \cancel{n^2}$$

$$n \leq c.$$

When	$f(n) = n^3$	$g(n) = n^2$
$n=1$	$f(1) = 1^3 = 1$	$g(1) = 1^2 = 1$
$n=2$	$f(2) = 2^3 = 8$	$g(2) = 2^2 = 4$
$n=3$	$f(3) = 3^3 = 27$	$g(3) = 3^2 = 9$
$n=4$	$f(4) = 4^3 = 64$	$g(4) = 4^2 = 16$



Since According to behaviour of upper graph our supposition is true that

$$n^3 \notin O(n^2).$$

Part-b.

Big-Omega Notation(Ω)

Definition:-

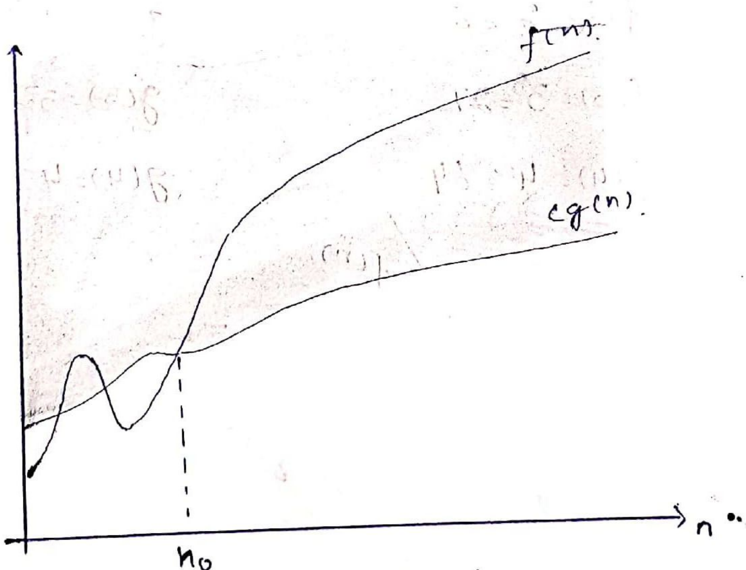
If $f, g: N \to \mathbb{R}^+$, then we can define Big-Omega as

For a given function $g(n)$ denote by $\Omega(g(n))$ the set of functions,

$$\Omega(g(n)) = \{f(n): \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le c g(n) \le f(n) \text{ for all } n \ge n_0\}$$

$f(n) = \Omega(g(n))$, means that function $g(n)$ is a asymptotically lower bound for $f(n)$.

We may write $f(n) = \Omega(g(n))$ -OR $f(n) \in \Omega(g(n))$.



$f(n) \in \Omega(g(n))$.

$f(n) = \Omega(g(n))$.

$\exists\, c > 0,\ \exists\, n_0 \ge 0,\ \forall\, n \ge n_0,\ f(n) \ge c \cdot g(n)$.

$g(n)$ is an asymptotic lower bound for $f(n)$.

Example #01.

Prove that $5 \cdot n^2 \in \Omega(n)$.

Proof:-

Assume that $f(n) = 5 \cdot n^2$

$g(n) = n$.

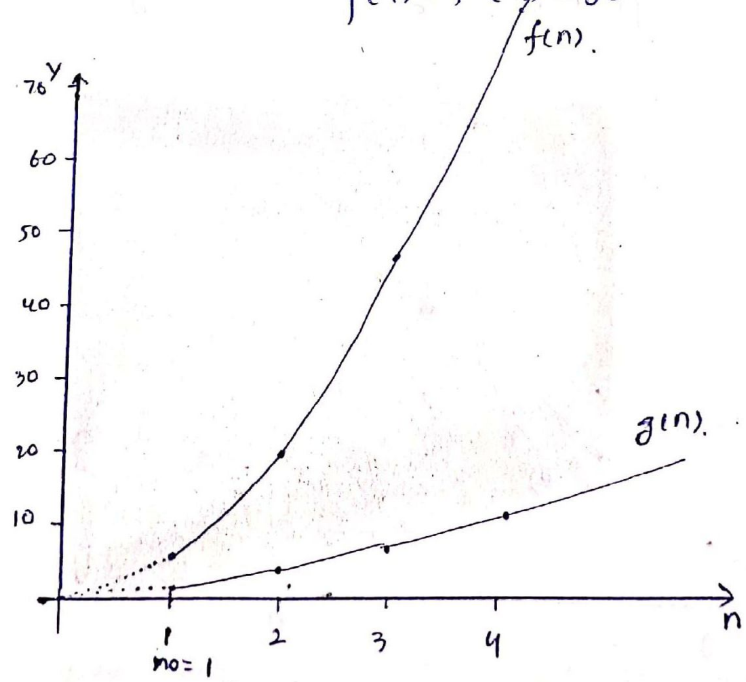
we have to find the existence of c and n_0.

$$0 \leq c \cdot g(n) \leq f(n).$$

$$c \cdot \cancel{n} \leq 5 \cdot n^{\cancel{2}}$$

$$c \leq 5 \cdot n.$$

Where	$f(n) = 5 \cdot n^2$	$g(n) = n$
$n=1$	$f(1) = 5 \cdot (1)^2 = 5$	$g(1) = 1$
$n=2$	$f(2) = 5 \cdot (2)^2 = 20$	$g(2) = 2$
$n=3$	$f(3) = 5 \cdot (3)^2 = 45$	$g(3) = 3$
$n=4$	$f(4) = 5 \cdot (4)^2 = 80$	$g(4) = 4$.



And hence $f(n) \in \Omega(g(n))$, for $c=5$, and $n_0 = 1$.

Example #02

Prove that . $2n^2 \notin \Omega(n^3)$.

Proof :-

Assume that $f(n) = 2n^2$

$g(n) = n^3$

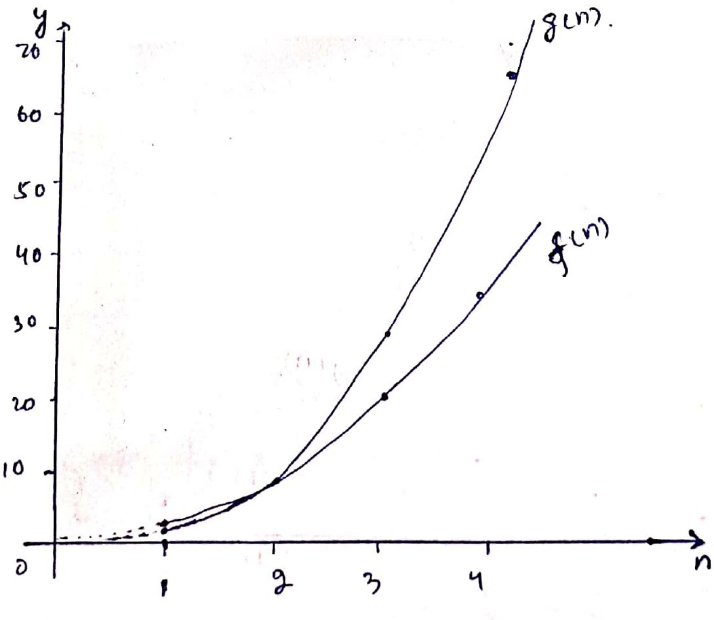
we have to find the existence of c and n_0,

$$0 \le c \cdot g(n) \le f(n).$$

$$c \cdot n^{\cancel{3}} \le 2\cancel{n^2}$$

$$c \cdot n \le 2.$$

When	$f(n) = 2n^2$	$g(n) = n^3$
$n=1$	$f(1) = 2(1)^2 = 2$	$g(1) = 1^3 = 1$
$n=2$	$f(2) = 2(2)^2 = 8$	$g(2) = 2^3 = 8$
$n=3$	$f(3) = 2(3)^2 = 18$	$g(3) = 3^3 = 27$
$n=4$	$f(4) = 2(4)^2 = 32$.	$g(4) = 4^3 = 64$.



Hence proved that $f(n) \notin \Omega(g(n))$ according to behaviour of upper graph.

Part-c

Theta Notation (Θ)

Definition:-

If $f, g : N \rightarrow \mathbb{R}^+$, then we can define Big-Theta as.

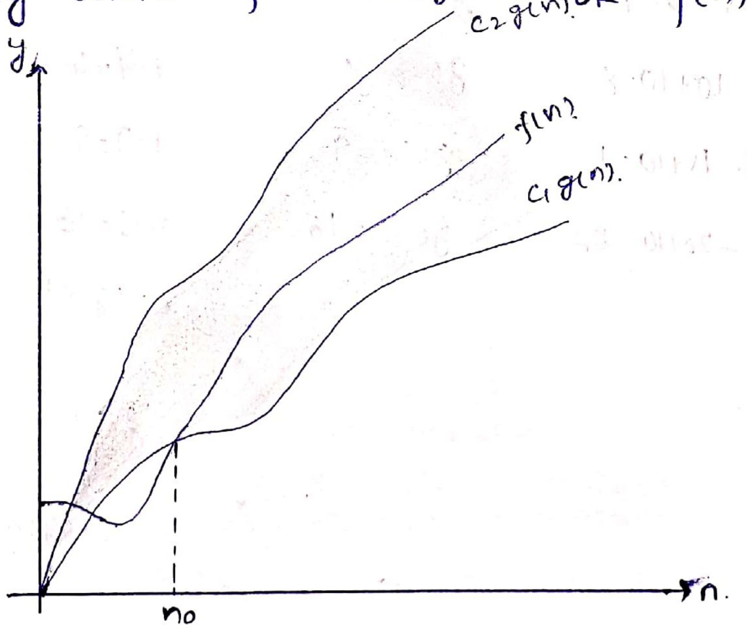
For a given function $g(n)$ denoted a $\Theta(g(n))$ the set of functions,

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2 and n_0 such that. $0 \leq c_1 g(n) \leq f(n) \leq c_2(g(n))$ for all $n \geq n_0 \}$.

$f(n) = \Theta(g(n))$ means function $f(n)$ is equal to $g(n)$ to within a constant factor, and $g(n)$ is an asymptotically tight bound for $f(n)$.

We may write $f(n) = \Theta(g(n))$ OR $f(n) \in \Theta(g(n))$.

Graph:-



$f(n) \in \Theta(g(n))$

$f(n) = \Theta(g(n))$.

$\exists\ c_1 > 0, c_2 > 0, \exists\ n_0 \geq 0, \forall\ n \geq n_0, c_2 \cdot g(n) \leq f(n) \leq c_1 g(n)$

We may say as $g(n)$ is an asymptotically tight bound for $f(n)$.

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Example#01.

Prove that $2n^2 - 5n + 10 \in \Theta(n^2)$.

Proof:-

Assume that $f(n) = 2n^2 - 5n + 10$

$g(n) = n^2$.

we have to find the value of c_1, c_2 and n_0.

$$c_1(g(n)) \le f(n) \le c_2(g(n)).$$

$$c_1 \cdot n^2 \le 2n^2 - 5n + 10 \le c_2 \cdot n^2$$

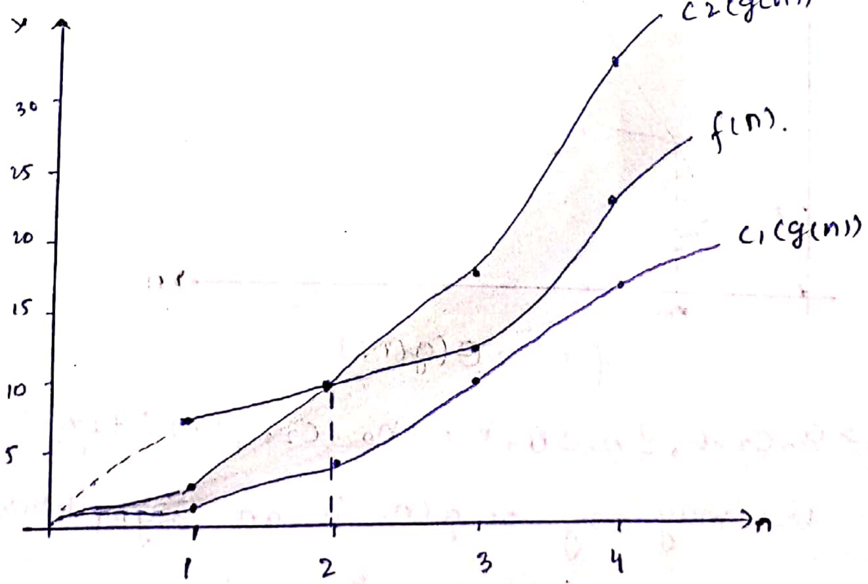
For lower bound.

$c_1 = 1$

For upper bound:

$c_2 = 2$.

When	$f(n) = 2n^2 - 5n + 10$	$g(n) = n^2$	$c_1 \cdot n^2 = 1 \cdot n^2$	$c_2 \cdot n^2 = 2 \cdot n^2$
$n = 1$	$f(1) = 2 - 5 + 10 = 7$	$g(1) = 1$	$1 \cdot 1^2 = 1$	$2 \cdot 1^2 = 2$
$n = 2$	$f(2) = 8 - 10 + 10 = 8$	$g(2) = 4$	$1 \cdot 4 = 4$	$2 \cdot 2^2 = 8$
$n = 3$	$f(3) = 18 - 15 + 10 = 13$	$g(3) = 9$	$1 \cdot 9 = 9$	$2 \cdot 9 = 18$
$n = 4$	$f(4) = 32 - 20 + 10 = 22$.	$g(4) = 16$.	$1 \cdot 16 = 16$	$2 \cdot 16 = 32$.



$n_0 = 2$

For $n_0 = 2$, $c_1 = 1$, $c_2 = 2$ Hence proved

$2n^2 - 5n + 10 \in \Theta(n^2)$

Example # 02.

Prove that $6n^3 \notin \Theta(n^2)$.

Proof:-

Assume that $f(n) = 6n^3$

$g(n) = n^2$.

We have to find the $c_1, c_2,$ and n_0.

$c_1 g(n) \leq f(n) \leq c_2 g(n)$.

$c_1 n^2 \leq 6n^3 \leq c_2 \cdot n^2$.

For lower bound.

$c_1 = 5$.

For Upper bound

$c_2 = ?$

because It always false.

Hence prove that

$6n^3 \notin \Theta(n^2)$ because $f(n)$ is not titly (gh) bound between $c_1(g(n))$ and $c_2(g(n))$.

Part-d.

Little-Oh Notation

o-notation is used to denote a upper bound that is not asymptotically tight.

For a given function $g(n) \geq 0$, denoted by $o(g(n))$, the set of functions,

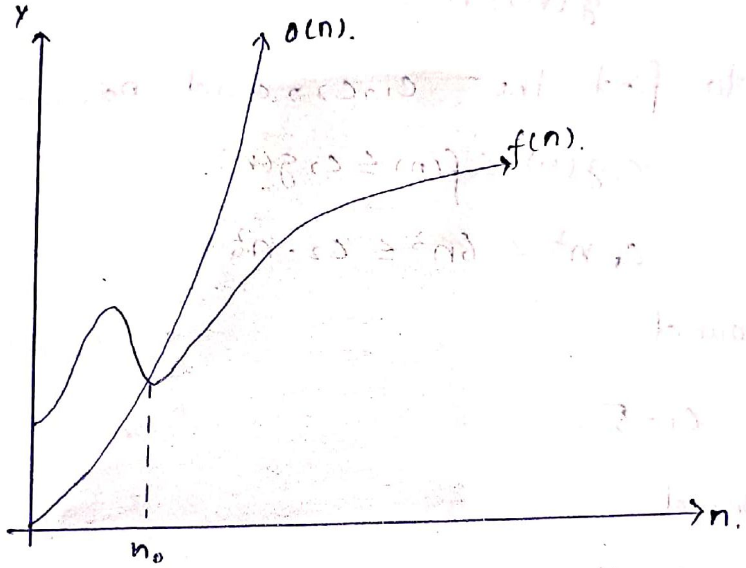
$o(g(n)) = \{ f(n)$: for any positive constants c, there exists a constant n_0 such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0 \}$

$f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity.

e.g., $2n = o(n^2)$ but $2n^2 \neq o(n^2)$... $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$g(n)$ is an upper bound for $f(n)$, not asymptotically tight.

Graph:-



$f(n) \in o(g(n))$

Example #01.

Prove that $2n^2 \in o(n^3)$.

Proof:-

Assume that $f(n) = 2n^2$

$g(n) = n^3$

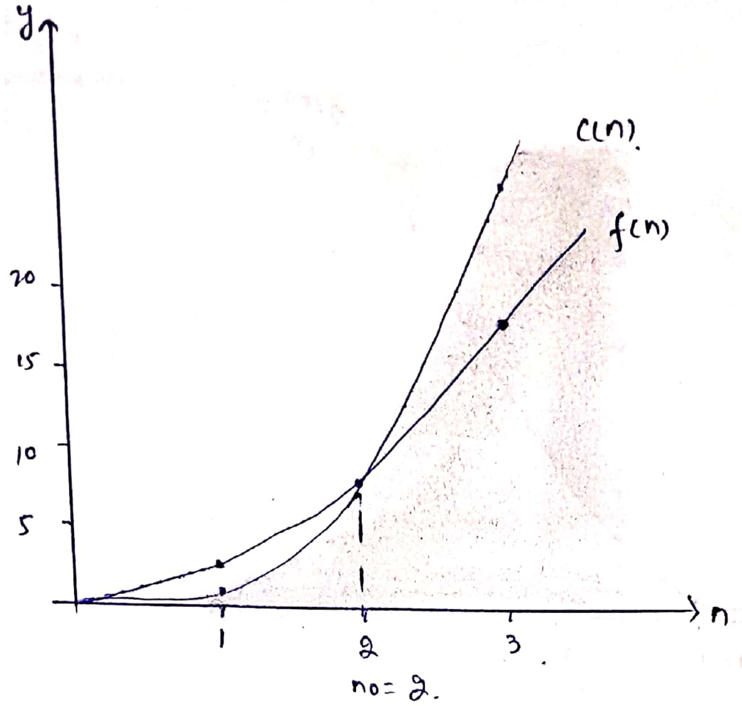
Now, we have to find the existance of n_0 for any c.

$f(n) < c \cdot g(n)$

$2n^2 < c \cdot n^3$

$2 < c \cdot n$

When	$f(n) = 2n^2$	$g(n) = n^3$
$n=1$	$f(1) = 2$	$g(1) = 1$
$n=2$	$f(2) = 8$	$g(2) = 8$
$n=3$	$f(3) = 18$	$g(3) = 27$



Hence proved that $2n^2 \in o(n^3)$ where $n_0 = 2$, $c = 1$.

Example #02.

Prove that $n^2 \notin o(n^2)$.

Proof:-

Assume that $f(n) = n^2$

$g(n) = n^2$.

we have to find the existence of n_0 and c.

$$f(n) < c \cdot g(n)$$

$$\cancel{n^2} < c \cdot \cancel{n^2}$$

$$1 < c.$$

When.	$f(n) = n^2$	$g(n) = n^2$
$n=1$	$f(1) = 1$	$g(1) = 1^2 = 1$
$n=2$	$f(2) = 4$	$g(2) = 4$
$n=3$	$f(3) = 9$	$g(3) = 9$.

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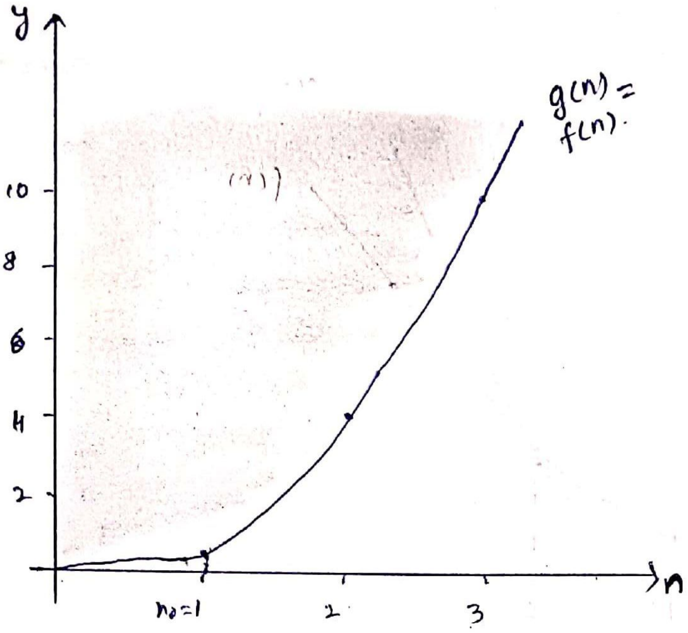


Hence proved that $n^2 \notin o(n^2)$ according to upper graph.

where $n_0 = 1$, $c = 1$

Part-e

Little Omega Notation

Definition:-

Little-ω notation is used to denote a lower bound that is not asymptotically tight.

For a given function $g(n)$, denote by $\omega(g(n))$ the set of functions,

$\omega(g(n)) = \{ f(n)$: for any positive constants c, there exits a constant n_0 such that

$0 \leq g(n) \leq f(n)$ for all $n \geq n_0 \}$

$f(n)$ become arbitrarily large relative to $g(n)$ as n approches infinity.

e.g., $\frac{n^2}{2} = \omega(n)$ but $\frac{n^2}{2} \neq \omega(n^2)$... $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$.

Example # 01.

Prove that $5 \cdot n^2 \in \omega(n)$.

Proof:-

Assume that $f(n) = 5 \cdot n^2$

$g(n) = n$.

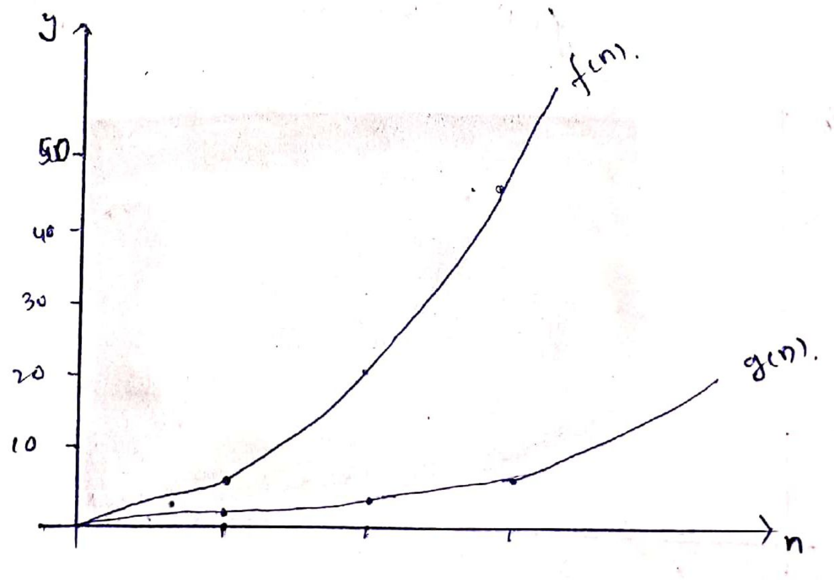
we have to find the existance of n_0, and c.

$$c \cdot g(n) < f(n) \quad \forall\, n \geq n_0.$$

$$c \cdot \cancel{n} < 5 \cdot n^{\cancel{2}}.$$

$$c < 5 \cdot n.$$

When	$f(n) = n^2 \cdot 5$	$g(n) = n$
$n = 1$	$f(1) = 5$	$g(1) = 1$
$n = 2$	$f(2) = 20$	$g(2) = 2$
$n = 3$	$f(3) = 45$	$g(3) = 3$



Hence proved that $5 \cdot n^2 \in \omega(n)$ for any n and c.

Example #02.

Prove that $100 \cdot n \notin \omega(n^2)$.

Proof:-

we assume that $f(n) = 10 \cdot n$

$g(n) = n^2$.

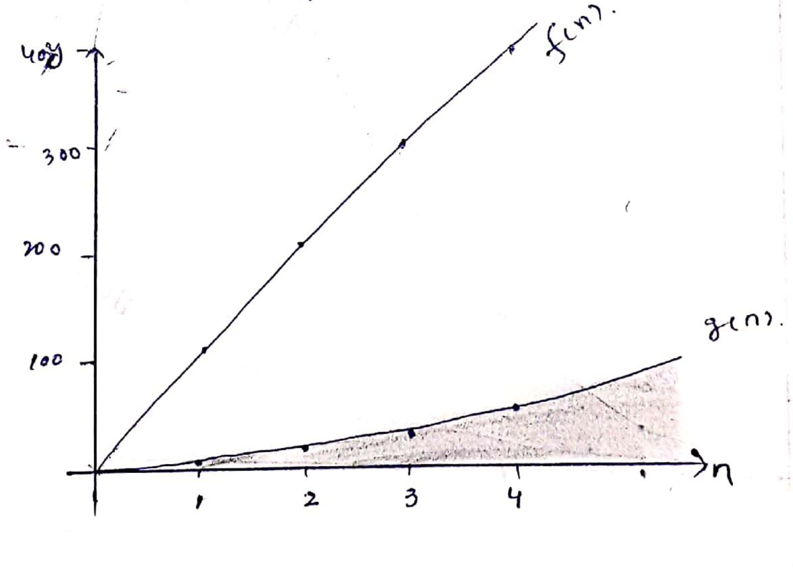
we have to find the existance of any n_0 and c.

$$c\,(g(n)) < 100n$$

$$c \cdot \cancel{n^2} < 100 \cdot \cancel{n}$$

$$c \cdot n < 100$$

when	$f(n) = 100 \cdot n$	$g(n) = n^2$
$n=1$	$f(1) = 100.1 = 100$	$g(1) = 1^2 = 1$
$n=2$	$f(2) = 100.2 = 200$	$g(2) = 2^2 = 4$
$n=3$	$f(3) = 100.3 = 300$	$g(3) = 3^2 = 9$
$n=4$	$f(4) = 100.4 = 400$.	$g(4) = 4^2 = 16$.



Hence proved that $100 \cdot n \notin \omega(n^2)$.

for any arbitartary value.