Advanced Algorithms Analysis and Design

By

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Lecture No 7 & 8

Time Complexity of Algorithms

(Asymptotic Notations)

Today Covered

- Major Factors in Algorithms Design
- Complexity Analysis
- Growth of Functions
- Asymptotic Notations
- Usefulness of Notations
- Reflexivity, Symmetry, Transitivity Relations over Θ, Ω, O, ω and o
- Relation between Θ, Ω and O
- Various Examples Explaining each concept

What is Complexity?

- The level in difficulty in solving mathematically posed problems as measured by
  - The time

    (time complexity)
  - number of steps or arithmetic operations

    (computational complexity)
  - memory space required
  - (space complexity)

Major Factors in Algorithms Design

1. Correctness

An algorithm is said to be correct if

- For every input, it halts with correct output.
- An incorrect algorithm might not halt at all OR
- It might halt with an answer other than desired one.
- Correct algorithm solves a computational problem

2. Algorithm Efficiency

Measuring efficiency of an algorithm,

- do its analysis i.e. growth rate.
- Compare efficiencies of different algorithms for the same problem.

Algorithms Growth Rate

Algorithm Growth Rates

- It measures algorithm efficiency

What means by efficient?

- If running time is bounded by polynomial in the input

Notations for Asymptotic performance

- How running time increases with input size
- O, Omega, Theta, etc. for asymptotic running time
- These notations defined in terms of functions whose domains are natural numbers
- convenient for worst case running time
- Algorithms, asymptotically efficient best choice

Complexity Analysis

- Algorithm analysis means predicting resources such as
  - computational time
  - memory
  - computer hardware etc
- Worst case analysis
  - Provides an upper bound on running time
  - An absolute guarantee
- Average case analysis
  - Provides the expected running time
  - Very useful, but treat with care: what is “average”?
    - Random (equally likely) inputs
    - Real-life inputs

Asymptotic Notations Properties

- Categorize algorithms based on asymptotic growth rate e.g. linear, quadratic, polynomial, exponential
- Ignore small constant and small inputs
- Estimate upper bound and lower bound on growth rate of time complexity function
- Describe running time of algorithm as n grows to ∞.
- Describes behavior of function within the limit.

Limitations

- not always useful for analysis on fixed-size inputs.
- All results are for *sufficiently large* inputs.

Asymptotic Notations

Asymptotic Notations Θ, O, Ω, ***o***, ω

- We use Θ to mean "order exactly",
- O to mean "order at most",
- Ω to mean "order at least",
- o to mean "tight upper bound",
- ω to mean "tight lower bound",

Define a ***set*** of functions: which is in practice used to compare two function sizes.

Big-Oh Notation (O)

If f, g: N $\rightarrow$ R$^+$, then we can define Big-Oh as

For a given function $g(n) \geq 0$, denoted by $O(g(n))$ the set of functions,

$O(g(n)) = \{f(n)$: there exist positive constants c and n_o such that

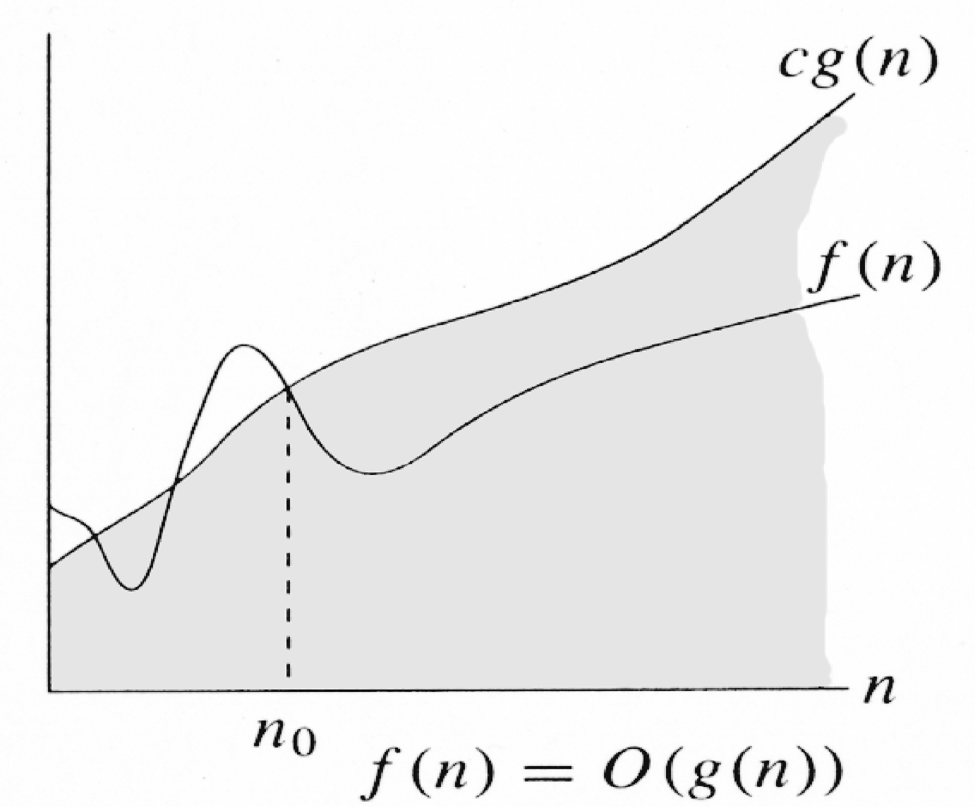
$0 \leq f(n) \leq cg(n)$, for all $n \geq n_o\}$

$f(n) = O(g(n))$ means function $g(n)$ is an asymptotically upper bound for $f(n)$.

We may write f(n) = O(g(n)) OR f(n) $\in$ O(g(n))

Intuitively:
Set of all functions whose *rate of growth* is the same as or lower than that of *g*(*n*).

Big-Oh Notation



$f(n) \in O(g(n))$

$\exists\, c > 0,\ \exists\, n_0 \geq 0$ and $\forall n \geq n_0,\ 0 \leq f(n) \leq c.g(n)$

***g*(*n*) is an *asymptotic upper bound* for *f*(*n*).**

Examples

Example 1: Prove that $2n^2 \in O(n^3)$

Proof:

Assume that $f(n) = 2n^2$, and $g(n) = n^3$

$f(n) \in O(g(n))$?

Now we have to find the existence of c and n_0

$f(n) \leq c.g(n) \Leftrightarrow 2n^2 \leq c.n^3 \Leftrightarrow 2 \leq c.n$

if we take, $c = 1$ and $n_0 = 2$ OR

$c = 2$ and $n_0 = 1$ then

$2n^2 \leq c.n^3$

Hence $f(n) \in O(g(n))$, $c = 1$ and $n_0 = 2$

Examples

Example 2: Prove that $n^2 \in O(n^2)$

Proof:

Assume that $f(n) = n^2$, and $g(n) = n^2$

Now we have to show that $f(n) \in O(g(n))$

Since

$f(n) \leq c.g(n) \Leftrightarrow n^2 \leq c.n^2 \Leftrightarrow 1 \leq c$, take, $c = 1$, $n_0= 1$

Then

$n^2 \leq c.n^2$ for $c = 1$ and $n \geq 1$

Hence, $2n^2 \in O(n^2)$, where $c = 1$ and $n_0= 1$

Examples

Example 3: Prove that $n^3 \notin O(n^2)$

Proof:

On contrary we assume that there exist some positive constants c and n_0 such that

$$0 \leq n^3 \leq c.n^2 \qquad \forall\, n \geq n_0$$

$$0 \leq n^3 \leq c.n^2 \Leftrightarrow n \leq c$$

Since c is any fixed number and n is any arbitrary constant, therefore $n \leq c$ is not possible in general.

Hence our supposition is wrong and $n^3 \leq c.n^2$, $\forall\, n \geq n_0$ is not true for any combination of c and n_0.

And hence, $n^3 \notin O(n^2)$

Big-Omega Notation (Ω)

If f, g: $N \to R^+$, then we can define Big-Omega as

For a given function $g(n)$ denote by $\Omega(g(n))$ the set of functions,

$$\Omega(g(n)) = \{f(n): \text{there exist positive constants } c \text{ and } n_o \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_o\}$$

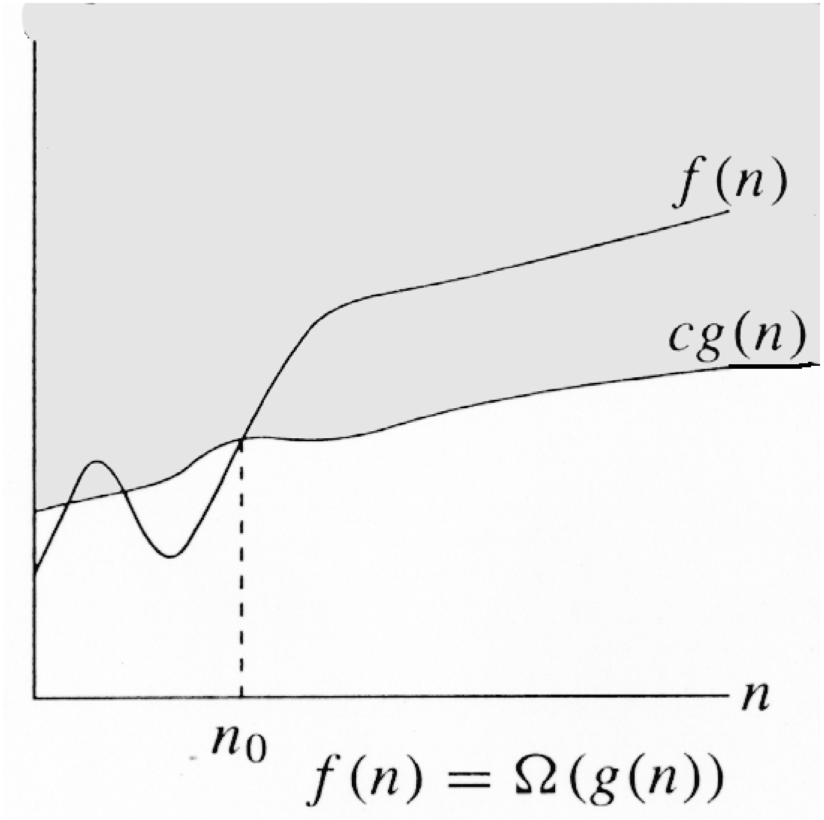
$f(n) = \Omega(g(n))$, means that function $g(n)$ is an asymptotically lower bound for $f(n)$.

We may write $f(n) = \Omega(g(n))$ OR $f(n) \in \Omega(g(n))$

Intuitively:

Set of all functions whose *rate of growth* is the same as or higher than that of $g(n)$.

Big-Omega Notation



$f(n) \in \Omega(g(n))$

$\exists\, c > 0, \exists\, n_0 \geq 0\ , \forall n \geq n_0, f(n) \geq c.g(n)$

***g*(*n*) is an *asymptotically lower bound* for *f*(*n*).**

Examples

Example 1: Prove that $5.n^2 \in \Omega(n)$

Proof:

Assume that $f(n) = 5.n^2$, and $g(n) = n$

$f(n) \in \Omega(g(n))$?

We have to find the existence of c and n_0 s.t.

$c.g(n) \le f(n) \qquad \forall\ n \ge n_0$

$c.n \le 5.n^2 \Leftrightarrow c \le 5.n$

if we take, $c = 5$ and $n_0 = 1$ then

$c.n \le 5.n^2 \qquad \forall\ n \ge n_0$

And hence $f(n) \in \Omega(g(n))$, for $c = 5$ and $n_0 = 1$

Theta Notation (Θ)

If f, g: N $\rightarrow$ R$^+$, then we can define Big-Theta as

For a given function $g(n)$ denoted by $\Theta(g(n))$ the set of functions,

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2 and n_o such that

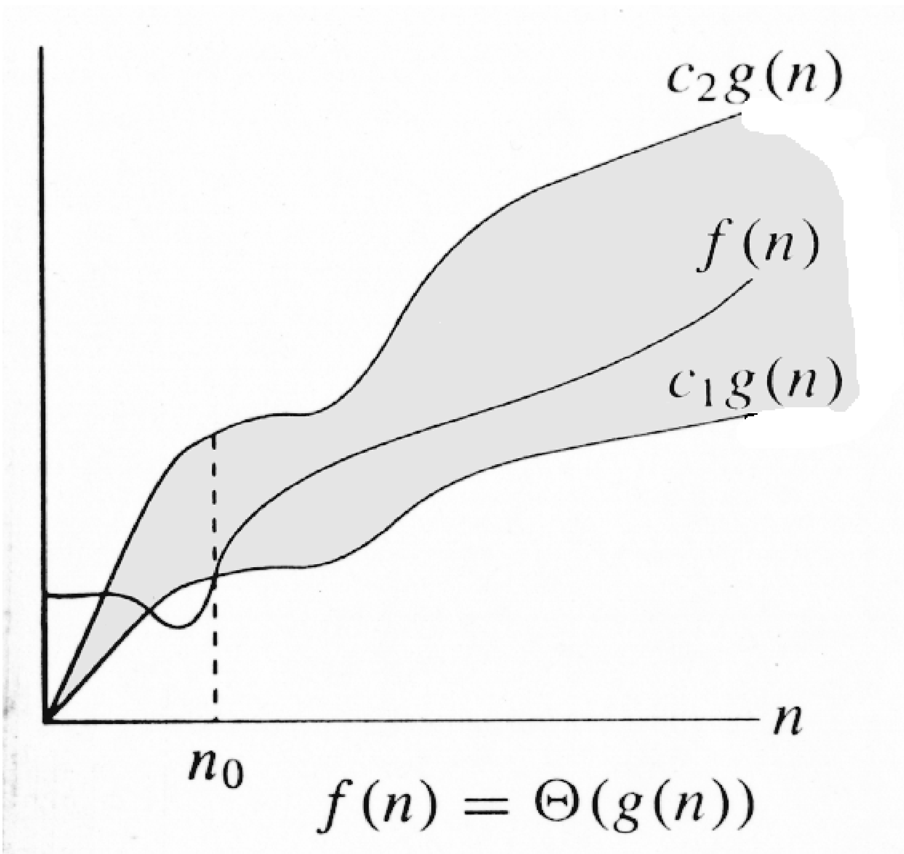
$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_o \}$

$f(n) = \Theta(g(n))$ means function $f(n)$ is equal to $g(n)$ to within a constant factor, and $g(n)$ is an asymptotically tight bound for $f(n)$.

We may write f(n) = Θ(g(n)) OR f(n) $\in$ Θ(g(n))

Intuitively: Set of all functions that have same *rate of growth* as *g*(*n*).

Theta Notation



$f(n) \in \Theta(g(n))$

$\exists\, c_1 > 0,\ c_2 > 0,\ \exists\, n_0 \geq 0,\ \forall\, n \geq n_0,\ c_2.g(n) \leq f(n) \leq c_1.g(n)$

***We say that g*(*n*) is an *asymptotically tight bound* for *f*(*n*).**

Little-Oh Notation

o-notation is used to denote a upper bound that is not asymptotically tight.

For a given function $g(n) \geq 0$, denoted by $o(g(n))$ the set of functions,

$$o(g(n)) = \left\{ \begin{array}{l} f(n): \text{for any positive constants } c, \text{ there exists a constant } n_o \\ \text{such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_o \end{array} \right\}$$

f(n) becomes insignificant relative to g(n) as n approaches infinity

$$\text{e.g., } 2\text{n} = o(n^2) \text{ but } 2n^2 \neq o(n^2)..$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

g(n) is an upper bound for f(n), not asymptotically tight

Examples

Example 1: Prove that $2n^2 \in o(n^3)$

Proof:

Assume that $f(n) = 2n^2$, and $g(n) = n^3$

$f(n) \in o(g(n))$?

Now we have to find the existence n_0 for any c

$f(n) < c.g(n)$ this is true

$\Leftrightarrow 2n^2 < c.n^3 \Leftrightarrow 2 < c.n$

This is true for any c, because for any arbitrary c we can choose n_0 such that the above inequality holds.

Hence $f(n) \in o(g(n))$

Examples

Example 2: Prove that $n^2 \notin o(n^2)$

Proof:

Assume that $f(n) = n^2$, and $g(n) = n^2$

Now we have to show that $f(n) \notin o(g(n))$

Since

$f(n) < c.g(n) \Leftrightarrow n^2 < c.n^2 \Leftrightarrow 1 \leq c$,

In our definition of small o, it was required to prove for any c but here there is a constraint over c .

Hence, $n^2 \notin o(n^2)$, where $c = 1$ and $n_0 = 1$

Little-Omega Notation

Little-ω notation is used to denote a lower bound that is not asymptotically tight.

For a given function $g(n)$, denote by $\omega(g(n))$ the set of all functions.

$\omega(g(n)) = \{f(n)$: for any positive constants c, there exists a constant n_o such that $0 \le cg(n) < f(n)$ for all $n \ge n_o\}$

f(*n*) becomes arbitrarily large relative to *g*(*n*) as *n* approaches infinity

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

$$\text{e.g.,} \frac{n^2}{2} = \omega(n) \text{ but } \frac{n^2}{2} \ne \omega(n^2)..$$

Examples

Example 1: Prove that $5.n^2 \in \omega(n)$

Proof:

Assume that $f(n) = 5.n^2$, and $g(n) = n$

$f(n) \in \Omega(g(n))$?

We have to prove that for any c there exists n_0 s.t.,

$c.g(n) < f(n) \quad \forall\ n \geq n_0$

$c.n < 5.n^2 \Leftrightarrow c < 5.n$

This is true for any c, because for any arbitrary c e.g. c = 1000000, we can choose n_0 = 1000000/5 = 200000 and the above inequality does hold.

And hence $f(n) \in \omega(g(n))$,

Example 3: Prove that $100.n \notin \omega(n^2)$

Proof:

Let $f(n) = 100.n$, and $g(n) = n^2$

Assume that $f(n) \in \omega(g(n))$

Now if $f(n) \in \omega(g(n))$ then there n_0 for any c s.t.

$c.g(n) < f(n) \quad \forall\, n \geq n_0$ this is true

$\Leftrightarrow c.n^2 < 100.n \Leftrightarrow c.n < 100$

If we take $c = 100$, $n < 1$, not possible

Hence $f(n) \notin \omega(g(n))$ i.e. $100.n \notin \omega(n^2)$

Reflexive Relation

Definition:

- Let X be a non-empty set and R is a relation over X then R is said to be reflexive if

$$(a, a) \in R, \forall a \in X,$$

Example 1:

- Let P be a set of all persons, and S be a relation over P such that if $(x, y) \in S$ then x has same sign as y.
- Of course this relation is reflexive because

$$(x, x) \in S, \quad \forall a \in P,$$

Example 2:

- Let P be a set of all persons at KFU and let T be a relation over P such that if $(x, y) \in T$ then x is teacher of y. This relation is not reflexive because

  $(x, x) \notin T, \forall a \in X$

Reflexivity Relations over Θ, Ω, O

Example 1

Since, $0 \leq c_1 f(n) \leq f(n) \leq c_2 f(n) \quad \forall\ n \geq n_0 = 1$, if $c_1 = c_2 = 1$

Hence $f(n) = \Theta(f(n))$

Example 2

Since, $0 \leq f(n) \leq cf(n) \qquad \forall\ n \geq n_0 = 1, \qquad$ if $c = 1$

Hence $f(n) = O(f(n))$

Example 3

Since, $0 \leq cf(n) \leq f(n) \qquad \forall\ n \geq n_0 = 1, \qquad$ if $c = 1$

Hence $f(n) = \Omega(f(n))$

Note: All the relations, Θ, Ω, *O, are reflexive*

Reflexivity Relations over o, ω

Example

As we can not prove that f(n) < f(n), for any n, and for all c > 0

Therefore

1. $f(n) \neq o(f(n))$ and
2. $f(n) \neq \omega(f(n))$

Note :

Hence small o and small omega are not reflexive relations

Symmetry over Ω

Definition:

- Let X be a non-empty set and R is a relation over X then R is said to be symmetric if

  $\forall$ a, b $\in$ X, (a, b) $\in$ R $\Rightarrow$ (b, a) $\in$ R

Example 1:

- Let P be a set of persons, and S be a relation over P such that if (x, y) $\in$ S then x has the same sign as y.
- This relation is symmetric because

  (x, y) $\in$ S $\Rightarrow$ (y, x) $\in$ S

Example 2:

- Let P be a set of all persons, and B be a relation over P such that if (x, y) $\in$ B then x is brother of y.
- This relation is not symmetric because

  (Anwer, Sadia) $\in$ B $\Rightarrow$ (Saida, Brother) $\notin$ B

Symmetry over Θ

Property : prove that

$$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

Proof

- Since $f(n) = \Theta(g(n))$ i.e. $f(n) \in \Theta(g(n)) \Rightarrow$

  $\exists$ constants $c_1, c_2 > 0$ and $n_0 \in N$ such that

  $0 \le c_1 g(n) \le f(n) \le c_2 g(n) \quad \forall\ n \ge n_0 \qquad (1)$

$(1) \Rightarrow 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \Rightarrow 0 \le f(n) \le c_2 g(n)$

$\Rightarrow 0 \le (1/c_2) f(n) \le g(n) \qquad (2)$

$(1) \Rightarrow 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \Rightarrow 0 \le c_1 g(n) \le f(n)$

$\Rightarrow 0 \le g(n) \le (1/c_1) f(n) \qquad (3)$

Symmetry over Θ

From (2),(3): $0 \le (1/c_2)f(n) \le g(n) \wedge 0 \le g(n) \le (1/c_1)f(n)$

$\Rightarrow\ 0 \le (1/c_2)f(n) \le g(n) \le (1/c_1)f(n)$

Suppose that $1/c_2 = c_3$, and $1/c_1 = c_4$,

Now the above equation implies that

$$0 \le c_3 f(n) \le g(n) \le c_4 f(n),\ \forall\ n \ge n_0$$

$\Rightarrow g(n) = \Theta(f(n)),\ \forall\ n \ge n_0$

Hence it proves that,

$$f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$$

Exercise:

prove that big O, big omega Ω, little ω, and little o, do not satisfy the symmetry property.

Transitivity

Definition:

- Let X be a non-empty set and R is a relation over X then R is said to be transitive if

  $\forall$ a, b, c $\in$ X, (a, b) $\in$ R $\wedge$ (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R

Example 1:

- Let P be a set of all persons, and B be a relation over P such that if (x, y) $\in$ B then x is brother of y.
- This relation is transitive this is because

  (x, y) $\in$ B $\wedge$ (y, z) $\in$ B $\Rightarrow$ (x, z) $\in$ B

Example 2:

- Let P be a set of all persons, and F be a relation over P such that if (x, y) $\in$ F then x is father of y.
- Of course this relation is not a transitive one this is because if (x, y) $\in$ F $\wedge$ (y, z) $\in$ F $\Rightarrow$ (x, z) $\notin$ F

Transitivity Relation over Θ, Ω, O, o and ω

Prove the following

1. $f(n) = \Theta(g(n))$ & $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
2. $f(n) = O(g(n))$ & $g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$
3. $f(n) = \Omega(g(n))$ & $g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$
4. $f(n) = o(g(n))$ & $g(n) = o(h(n)) \Rightarrow f(n) = o(h(n))$
5. $f(n) = \omega(g(n))$ & $g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$

Note

It is to be noted that all these algorithms complexity measuring notations are in fact relations which satisfy the transitive property.

Transitivity Relation over Θ

Property 1

$$f(n) = \Theta(g(n)) \;\&\; g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$$

Proof

1. Since $f(n) = \Theta(g(n))$ i.e. $f(n) \in \Theta(g(n)) \Rightarrow$

   $\exists$ constants $c_1, c_2 > 0$ and $n_{01} \in N$ such that

   $0 \le c_1 g(n) \le f(n) \le c_2 g(n) \quad \forall\, n \ge n_{01}$ (1)

2. Now since $g(n) = \Theta(h(n))$ i.e. $g(n) \in \Theta(h(n)) \Rightarrow$

   $\exists$ constants $c_3, c_4 > 0$ and $n_{02} \in N$ such that

   $0 \le c_3 h(n) \le g(n) \le c_4 h(n) \quad \forall\, n \ge n_{02}$ (2)

3. Now let us suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over Θ

4. Now we have to show that $f(n) = \Theta(h(n))$ i.e. we have to prove that

   $\exists$ constants $c_5, c_6 > 0$ and $n_0 \in N$ such that

   $0 \leq c_5 h(n) \leq f(n) \leq c_6 h(n)$?

$(2) \Rightarrow 0 \leq c_3 h(n) \leq g(n) \leq c_4 h(n)$

$\Rightarrow 0 \leq c_3 h(n) \leq g(n) \qquad (3)$

$(1) \Rightarrow 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$

$\Rightarrow 0 \leq c_1 g(n) \leq f(n)$

$\Rightarrow 0 \leq g(n) \leq (1/c_1) f(n) \qquad (4)$

From (3) and (4), $0 \leq c_3 h(n) \leq g(n) \leq (1/c_1) f(n)$

$\Rightarrow 0 \leq c_1 c_3 h(n) \leq f(n) \qquad (5)$

Transitivity Relation over Θ

$(1) \Rightarrow 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$

$\Rightarrow 0 \leq f(n) \leq c_2 g(n) \Rightarrow 0 \leq (1/c_2) f(n) \leq g(n) \qquad (6)$

$(2) \Rightarrow 0 \leq c_3 h(n) \leq g(n) \leq c_4 h(n)$

$\Rightarrow 0 \leq g(n) \leq c_4 h(n) \qquad (7)$

From (6) and (7), $0 \leq (1/c_2) f(n) \leq g(n) \leq (c_4) h(n)$

$\Rightarrow 0 \leq (1/c_2) f(n) \leq (c_4) h(n)$

$\Rightarrow 0 \leq f(n) \leq c_2 c_4 h(n) \qquad (8)$

From (5), (8), $0 \leq c_1 c_3 h(n) \leq f(n) \wedge 0 \leq f(n) \leq c_2 c_4 h(n)$

$0 \leq c_1 c_3 h(n) \leq f(n) \leq c_2 c_4 h(n)$

$0 \leq c_5 h(n) \leq f(n) \leq c_6 h(n)$

And hence $f(n) = \Theta(h(n)) \qquad \forall\, n \geq n_0$

Transitivity Relation over Big *O*

Property 2

$$f(n) = O(g(n)) \ \& \ g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$$

Proof

1. Since $f(n) = O(g(n))$ i.e. $f(n) \in O(g(n)) \Rightarrow$
   $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that
   $0 \leq f(n) \leq c_1 g(n) \qquad \forall \ n \geq n_{01}$ (1)
2. Now since $g(n) = O(h(n))$ i.e. $g(n) \in O(h(n)) \Rightarrow$
   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that
   $0 \leq g(n) \leq c_2 h(n) \qquad \forall \ n \geq n_{02}$ (2)
3. Now let us suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over Big *O*

Now we have to two equations

$$0 \le f(n) \le c_1 g(n) \qquad \forall\ n \ge n_{01} \qquad (1)$$

$$0 \le g(n) \le c_2 h(n) \qquad \forall\ n \ge n_{02} \qquad (2)$$

$$(2) \Rightarrow 0 \le c_1 g(n) \le c_1 c_2 h(n) \qquad \forall\ n \ge n_{02} \qquad (3)$$

From (1) and (3)

$$0 \le f(n) \le c_1 g(n) \le c_1 c_2 h(n)$$

Now suppose that $c_3 = c_1 c_2$

$$0 \le f(n) \le c_1 c_2 h(n)$$

And hence $f(n) = O(h(n)) \qquad \forall\ n \ge n_0$

Transitivity Relation over Big Ω

Property 3

$$f(n) = \Omega(g(n)) \;\&\; g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$$

Proof

1. Since $f(n) = \Omega(g(n)) \Rightarrow$

   $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that

   $0 \leq c_1 g(n) \leq f(n) \qquad \forall\, n \geq n_{01} \qquad (1)$

2. Now since $g(n) = \Omega(h(n)) \Rightarrow$

   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that

   $0 \leq c_2 h(n) \leq g(n) \qquad \forall\, n \geq n_{02} \qquad (2)$

3. Suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over Big Ω

4. We have to show that $f(n) = \Omega(h(n))$ i.e. we have to prove that

   $\exists$ constants $c_3 > 0$ and $n_0 \in N$ such that

   $0 \le c_3 h(n) \le f(n) \qquad \forall\, n \ge n_0 \qquad ?$

$(2) \Rightarrow 0 \le c_2 h(n) \le g(n)$

$(1) \Rightarrow 0 \le c_1 g(n) \le f(n)$

$\Rightarrow 0 \le g(n) \le (1/c_1) f(n) \qquad (3)$

From (2) and (3), $0 \le c_2 h(n) \le g(n) \le (1/c_1) f(n)$

$\Rightarrow 0 \le c_1 c_2 h(n) \le f(n)$ hence $f(n) = \Omega(h(n)), \forall\, n \ge n_0$

Transitivity Relation over *little o*

Property 4

$$f(n) = o(g(n)) \;\&\; g(n) = o(h(n)) \Rightarrow f(n) = o(h(n))$$

Proof

- Since $f(n) = o(g(n))$ i.e. $f(n) \in o(g(n)) \Rightarrow$

  $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that

  $0 \leq f(n) < c_1 g(n) \qquad \forall\, n \geq n_{01} \qquad (1)$

2. Now since $g(n) = o(h(n))$ i.e. $g(n) \in o(h(n)) \Rightarrow$

   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that

   $0 \leq g(n) < c_2 h(n) \qquad \forall\, n \geq n_{02} \qquad (2)$

3. Now let us suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over *little o*

Now we have to two equations

$$0 \leq f(n) < c_1 g(n) \qquad \forall\ n \geq n_{01} \qquad (1)$$

$$0 \leq g(n) < c_2 h(n) \qquad \forall\ n \geq n_{01} \qquad (2)$$

$$(2) \Rightarrow 0 \leq c_1 g(n) < c_1 c_2 h(n) \qquad \forall\ n \geq n_{02} \qquad (3)$$

From (1) and (3)

$$0 \leq f(n) \leq c_1 g(n) < c_1 c_2 h(n)$$

Now suppose that $c_3 = c_1 c_2$

$$0 \leq f(n) < c_1 c_2 h(n)$$

And hence $f(n) = o(h(n)) \qquad \forall\ n \geq n_{01}$

Transitivity Relation over little ω

Property 5

$$f(n) = \omega(g(n)) \;\&\; g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$$

Proof

- Since $f(n) = \omega(g(n)) \Rightarrow$

  $\exists$ constants $c_1 > 0$ and $n_{01} \in N$ such that

  $0 \le c_1 g(n) < f(n) \qquad \forall\, n \ge n_{01} \qquad (1)$

2. Now since $g(n) = \omega(h(n)) \Rightarrow$

   $\exists$ constants $c_2 > 0$ and $n_{02} \in N$ such that

   $0 \le c_2 h(n) < g(n) \qquad \forall\, n \ge n_{02} \qquad (2)$

3. Suppose that $n_0 = \max(n_{01}, n_{02})$

Transitivity Relation over little ω

4. We have to show that $f(n) = \omega(h(n))$ i.e. we have to prove that

   $\exists$ constants $c_3 > 0$ and $n_0 \in N$ such that

   $0 \leq c_3 h(n) \leq f(n) \qquad \forall\, n \geq n_0 \qquad ?$

$(2) \Rightarrow 0 \leq c_2 h(n) < g(n)$

$(1) \Rightarrow 0 \leq c_1 g(n) < f(n)$

$\Rightarrow 0 \leq g(n) < (1/c_1) f(n) \qquad (3)$

From (2) and (3), $\quad 0 \leq c_2 h(n) \leq g(n) < (1/c_1) f(n)$

$\Rightarrow 0 \leq c_1 c_2 h(n) < f(n)$ hence $f(n) = \omega(h(n))$, $\forall\, n \geq n_0$

Transpose Symmetry

Property 1

Prove that $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$

Proof

Since $f(n) = O(g(n)) \Rightarrow$

$\exists$ constants $c > 0$ and $n_0 \in N$ such that

$0 \leq f(n) \leq cg(n) \quad \forall\, n \geq n_0$

Dividing both side by c

$0 \leq (1/c)f(n) \leq g(n) \;\forall\, n \geq n_0$

Put $1/c = c'$

$0 \leq c'f(n) \leq g(n) \quad \forall\, n \geq n_0$

Hence, $g(n) = \Omega(f(n))$

Transpose Symmetry

Property 2

Prove that $f(n) = o(g(n)) \Leftrightarrow g(n) = \omega f(n))$

Proof

Since $f(n) = o(g(n)) \Rightarrow$

$\exists$ constants $c > 0$ and $n_0 \in N$ such that

$0 \leq f(n) < cg(n) \qquad \forall\ n \geq n_0$

Dividing both side by c

$0 \leq (1/c)f(n) < g(n) \qquad \forall\ n \geq n_0$

Put $1/c = c'$

$0 \leq c'f(n) < g(n) \qquad \forall\ n \geq n_0$

Hence, $g(n) = \omega(f(n))$

Standard Logarithms Notations

Some Definitions

Exponent

- $x = \log_b a$ is the exponent for $a = b^x$.

Natural log

- $\ln a = \log_e a$

Binary log

- $\lg a = \log_2 a$

Square of log

- $\lg^2 a = (\lg a)^2$

Log of Log

- $\lg \lg a = \lg (\lg a)$

Standard Logarithms Notations

$$a = b^{\log_b a}$$

$$\log_c(ab) = \log_c a + \log_c b$$

$$\log_b a^n = n\log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b(1/a) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a}$$