SECTION 7: TRANSFORMATIONS OF RANDOM VARIABLES

7.1. Functions of Random Variables

If X is a random variable (rv) with sample space $X \subset \mathbb{R}$ and cumulative distribution function (cdf) $F_X(x)$ then any function of X, say $Y = g(X)$ is also a random variable. The new random variable Y has a new sample space $Y = g(X) \subset \mathbb{R}$. The objective is to find the cdf $F_Y(y)$ of Y.

7.2. Transformations of Discrete Random Variables

Suppose that X is a discrete random variable with probability mass function $p(x) = P(X = x)$. Then the probability mass function (pmf) of a 1-1 transformation $Y = g(X)$ is given by

$$P_Y(y) = P(Y = y) = P(g(X) = y) = P\left(X = g^{-1}(y)\right) \qquad (1)$$

In practice, one never sees many general results about transformations of discrete random variables because the results are so simple.

Example 1: Toss a fair coin 3 times. Let X be the random variable representing the number of heads tossed. Support set: $X \in \{0,1,2,3\}$

x	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

A game is played where a player has an entry fee of \$15 and gets \$10 for every head. Let Y represent the gain of the player. Then, $Y = 10X - 15$. Find the pmf of Y or $P(Y = y)$. This is a one-to-one mapping from X-space to Y -space and $Y \in \{-15, -5, 5, 15\}$

Solution:

$$P_Y(y) = P(Y = y) = P(10X - 15 = y) = P_X\left(X = \frac{y+15}{10}\right)$$

y	-15	-5	5	15
$P_Y(y)$	1/8	3/8	3/8	1/8

7.3. Transformations of Continuous Random Variables

A) Consider the transformation $Y = g(X)$ where $g(X)$ is ***strictly increasing*** (consequently a one-to-one transformation), and suppose g is differentiable. This means that we can also define the ***inverse function***, $g^{-1}(y)$.

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(g(X) \le y) \\ &= P\left(X \le g^{-1}(y)\right) = F_X\left(g^{-1}(y)\right) \end{aligned}$$

The *probability distribution function (pdf) of Y* is thus,

** $f_Y(y) = \frac{d}{dy} F_Y(y) = F_X'\left[g^{-1}(y)\right] \frac{d\, g^{-1}(y)}{dy} = f_X(x) \frac{dx}{dy}$ (2)

Since $x = g^{-1}(y)$, *so that* $\frac{dx}{dy} = \frac{d\, g^{-1}(y)}{dy}$

B) Suppose $Y = g(X)$ is still one-to-one, but ***decreasing*** instead of increasing:

$$F_Y(y) = P\left(g(X) \le y\right) = P\left[X \ge g^{-1}(y)\right] = 1 - F_X\left(g^{-1}(y)\right)$$

$$\text{and } f_Y(y) = -F_X'\left(g^{-1}(y)\right) \frac{d\, g^{-1}(y)}{dy} = -f_X\left(g^{-1}(y)\right) \frac{d\, g^{-1}(y)}{dy}$$

$$= f_X\left(\underbrace{g^{-1}(y)}_{x}\right) \left| \frac{d\, \overbrace{g^{-1}(y)}^{x}}{dy} \right| \quad (3)$$

$$= f_X(x) \left| \frac{dx}{dy} \right|$$

The last step follows because $\frac{dx}{dy}$ is negative.

Therefore, ***regardless of whether* $Y = g(X)$ ***is increasing or decreasing***, so long as it is monotonic, we have $f_Y(y) = f_X(x) \left\| \frac{dx}{dy} \right\|$ (4)

Example 2. Linear Transformation:

Given *X* with pdf $f_X(x)$, let $Y = aX + b$, $\frac{dx}{dy} = \frac{1}{a}$

Then $f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{dg^{-1}(y)}{dy} \right| = f_X(x) \left| \frac{dx}{dy} \right| = f_X\left(\frac{y-b}{a}\right) \left| \frac{1}{a} \right|$

This transformation is often used when *X* has mean 0 and standard deviation 1. The linear transformation above creates a rv *Y* with a distribution that has the same shape as that of *X* but has mean *b* and standard deviation *a*. Conversely, if *Y* has mean *b* and standard deviation *a*, then *X*=(*Y*−*b*)/*a* has mean 0 and standard deviation 1. This is called sometimes the *"Studentized"* transform.

Example 3. Square Root of an Exponential Random Variable:

We have already seen that a constant times an exponential random variable leads to another exponential random variable. Suppose $X \sim \exp(\lambda)$, *so that*

$$f_X(x) = \lambda e^{-\lambda x}, \quad (x \geq 0)$$

and consider the distribution of $Y = \sqrt{X}$. The transformation $y = g(x) = \sqrt{x}, x \geq 0$ is one-to-one and has an inverse $x = y^2$ with $dx/dy = 2y$. Thus

$$f_Y(y) = f_X(y^2) 2y = 2\lambda y e^{-\lambda y^2}, \; y \geq 0$$

7.4. Non-monotone Transformations

What if the transformation is not 1-1? The trick is to start with the cdf of the transformed random variable.

Example 4: Let $Y = |X|$, and assume X is continuous.

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y) \\ f_Y(y) &= F_X'(y) - F_X'(-y)(-1) = f_X(y) + f_X(-y) \end{aligned} \tag{5}$$

Suppose $X \sim N(0,1), \quad \in f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, -\infty < x < +\infty$

Then $f_Y(y) = \frac{2}{\sqrt{2\pi}} e^{-y^2/2}, 0 < y < \infty$

7.5. Quadratic Transformation

Let $Y = X^2, \frac{dy}{dx} = 2x, \left|\frac{dy}{dx}\right| = 2\sqrt{y}$

Then
$$\begin{aligned} F(y) &= P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} < X < \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned} \tag{6}$$

$$\begin{aligned} f_Y(y) &= F_X'(\sqrt{y})\left(\frac{1}{2} y^{-\frac{1}{2}}\right) - F_X'(-\sqrt{y})\left(-\frac{1}{2} y^{-\frac{1}{2}}\right) \\ &= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right], y > 0 \end{aligned} \tag{7}$$

Example 5: Suppose $X \sim N(0,1), Y = X^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad -\infty < x < \infty$$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left[\frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}y}\right] = \frac{y^{-\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{1}{2}y}, \; y > 0$$

EXERCISES

Exercise 1: (Binomial transformation) A discrete random variable X has a binomial distribution if its pmf is of the form

$$P_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \; x = 0, 1, \cdots, n,$$

where n is a positive integer and $0 \le p \le 1$. Consider the random variable $Y = g(X)$, where $g(x) = n - x$. Thus, $g^{-1}(y)$ is the single point x = n - y, and

$$\begin{aligned} P_Y(y) &= \sum_{x \in g^{-1}(y)} P_X(x) = P_X(n-y) \\ &= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)} \\ &= \binom{n}{y} (1-p)^y \, p^{n-y} \end{aligned}$$

Thus, we see that Y also has a binomial distribution, but with parameters n and 1 - p.

Exercise 2: Suppose that X has the pmf below:

x	-1	0	1	2
$P_X(x)$	0.2	0.1	0.3	0.4

Then, find the pmf of $Y = X^2$.

Solution:

$$P(Y = 0) = P(X = 0) = 0.1$$
$$P(Y = 1) = P(X = -1) + P(X = 1) = 0.5$$
$$P(Y = 4) = P(X = 2) = 0.4$$

y	0	1	4
$P_Y(y)$	0.1	0.5	0.4

Exercise 3: (Uniform-exponential relationship-I) Suppose $X \sim f_X(x) = 1$ if $0 < x < 1$ and 0 otherwise, the Uniform(0,1) distribution. It is straightforward to check that $F_X(x) = x, 0 < x < 1$. We now make the transformation $Y = g(X) = -\ln(X)$. Find the cdf of Y.

Solution:

Since $\frac{d}{dx} g(x) = -\frac{1}{x} < 0, \; \mathit{for} \; 0 < x < 1,$ g(x) is a decreasing function. Therefore, for y > 0,

$$F_Y(y) = 1 - F_X\left(g^{-1}(y)\right) = 1 - F_X\left(e^{-y}\right) = 1 - e^{-y}$$

Of course, $F(y) = 0 \ for \ y \leq 0.$

If the pdf of Y is continuous, it can be obtained by differentiating the cdf.

Exercise 4: (Inverted gamma pdf) Let $f_X(x)$ be the gamma pdf

$$f(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta}, 0 < x < \infty,$$

where β is a positive constant and n is a positive integer. If we let $y = g(x) = 1/x,$ find the pdf of Y.

Solution:

$y = g(x) = 1/x,$ then $g^{-1}(y) = 1/y$ and $\frac{d}{dy} g^{-1}(y) = -1/y^2$. For 0 < y < 1, we get

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy} g^{-1}(y)\right| \\ &= \frac{1}{(n-1)!\beta^n}\left(\frac{1}{y}\right)^{n-1} e^{-1/(\beta y)} \frac{1}{y^2} \\ &= \frac{1}{(n-1)!\beta^n}\left(\frac{1}{y}\right)^{n+1} e^{-1/(\beta y)} \end{aligned}$$

a special case of a pdf known as the inverted gamma pdf.

Exercise 5: The pdf of continuous random variable X is given in below:

$$f(x) = \begin{cases} \frac{1}{56}(x+3), & 0 \leq x \leq 8 \\ 0, & otherwise \end{cases}$$

Find the pdf and cdf of $Y = -4X - 1$.

Solution:

$$F(x) = \int_0^x \frac{1}{56}(t+3)\,dt = \frac{1}{56}\left(\frac{t^2}{2} + 3t\right)\Bigg|_0^x = \frac{1}{56}\left(\frac{x^2}{2} + 3x\right)$$

$$F(x) = \begin{cases} \frac{\left(x^2 + 6x\right)}{112}, & 0 \leq x \leq 8 \\ 0, & x < 0 \\ 1, & x \geq 8 \end{cases}$$

Method 1: Finding pdf of random variable Y by using pdf of random variable X.

$$y=-4x-1 \Rightarrow x=-\frac{(y+1)}{4}=g^{-1}(y) \qquad \left.\begin{array}{l} x=0 \quad \Rightarrow y=-1 \\ x=8 \quad \Rightarrow y=-33 \end{array}\right\} -33 \le y \le -1$$

$$\begin{aligned} f_Y(y) &= f_X\left(g^{-1}(y)\right)\left|\frac{d\, g^{-1}(y)}{dy}\right| \\ &= f_X\left(-\left(\frac{y+1}{4}\right)\right)\left|\frac{d}{dy}-\left(\frac{y+1}{4}\right)\right| \\ &= \frac{1}{56}\left[-\left(\frac{y+1}{4}\right)+3\right]\left|-\frac{1}{4}\right| \\ &= \frac{11-y}{896} \end{aligned}$$

$$f_Y(y)=\begin{cases} \dfrac{11-y}{896}, & -33 \le y \le -1 \\ 0, & otherwise \end{cases}$$

$$F_Y(y)=\int_{-33}^{y}\frac{11-t}{896}dt=\frac{1}{896}\left(11t-\frac{t^2}{2}\right)\Bigg|_{-33}^{y}=\frac{-y^2+22y+1815}{1792}$$

$$F_Y(y)=\begin{cases} \dfrac{-y^2+22y+1815}{1792}, & -33 \le y \le -1 \\ 0, & y<-33 \\ 1, & y \ge -1 \end{cases}$$

Method 2: By using cdf of random variable X;

$Y=g(X)$ *is monotonic increasing or decreasing*?

$$\left.\begin{array}{l} x_1=1 \Rightarrow y_1=g(x_1)=-4.1-1=-5 \\ x_2=2 \Rightarrow y_2=g(x_2)=-4.2-1=-9 \end{array}\right\} x_1=1<x_2=2 \Rightarrow g(x_1)=-5>g(x_2)=9 \; monotonic\ decreasing$$

$$\begin{aligned} F_Y(y) &= P(Y \le y)=P(-4X-1 \le y)=P\left(X>-\frac{(y+1)}{4}\right) \\ &= 1-P\left(X \le -\frac{(y+1)}{4}\right)=1-F_X\left(-\frac{(y+1)}{4}\right) \\ &= 1-\frac{1}{112}\left[\left(\frac{y+1}{4}\right)^2-6\left(\frac{y+1}{4}\right)\right]=\frac{-y^2+22y+1815}{1792} \end{aligned}$$

$$F_Y(y) = \begin{cases} \dfrac{-y^2+22y+1815}{1792}, & -33 \le y \le -1 \\ 0, & y < -33 \\ 1, & y \ge -1 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{-2y+22}{1792} = \frac{-y+11}{896}$$

$$f_Y(y) = \begin{cases} \dfrac{11-y}{896}, & -33 \le y \le -1 \\ 0, & otherwise \end{cases}$$

Exercise 6: The pdf of continuous random variable X is given in below:

$$f(x) = \begin{cases} \dfrac{1}{6}, & -4 < x < 2 \\ 0, & otherwise \end{cases}$$

Find the pdf and cdf of $Y = X^2$.

Solution:

$$F(x) = P(X \le x) = \int_{-4}^{x} \frac{1}{6} dt = \frac{1}{6} t \Big|_{-4}^{x} = \frac{x+4}{6}$$

$$F(x) = \begin{cases} \dfrac{x+4}{6}, & -4 < x < 2 \\ 0, & x < -4 \\ 1, & x \ge 2 \end{cases}$$

$$y = x^2 \Rightarrow x_1 = -\sqrt{y} = g_1^{-1}(y)$$
$$x_2 = \sqrt{y} = g_2^{-1}(y)$$

Method 1:

$$f_Y(y) = f_X\left(-\sqrt{y}\right)\left|\frac{d}{dy}\left(-\sqrt{y}\right)\right| + f_X\left(\sqrt{y}\right)\left|\frac{d}{dy}\left(\sqrt{y}\right)\right|$$

$$0 < y < 4 \Rightarrow -\sqrt{y}, \sqrt{y} \in R_x$$
$$4 < y < 16 \Rightarrow -\sqrt{y} \in R_X$$

$$f_Y(y) = f_X(-\sqrt{y})\left|\frac{d}{dy}(-\sqrt{y})\right| + f_X(\sqrt{y})\left|\frac{d}{dy}(\sqrt{y})\right|$$

$$= \frac{1}{6}\left|-\frac{1}{2\sqrt{y}}\right| + \frac{1}{6}\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{6\sqrt{y}}, 0 < y < 4$$

$$f_Y(y) = f_X(-\sqrt{y})\left|\frac{d}{dy}(-\sqrt{y})\right| + \underbrace{f_X(\sqrt{y})\left|\frac{d}{dy}(\sqrt{y})\right|}_{0} = \frac{1}{6}\left|-\frac{1}{2\sqrt{y}}\right| = \frac{1}{12\sqrt{y}}, 4 < y < 16$$

$$f_Y(y) = \begin{cases} \frac{1}{6\sqrt{y}}, & 0 < y < 4 \\ \frac{1}{12\sqrt{y}}, & 4 < y < 16 \\ 0, & otherwise \end{cases}$$

$$F_Y(y) = \begin{cases} \int_0^y \frac{1}{6\sqrt{t}}\,dt = \frac{\sqrt{y}}{3}, & 0 < y < 4 \\ \int_0^4 \frac{1}{6\sqrt{y}}\,\mathrm{dy} + \int_4^y \frac{1}{12\sqrt{t}}\,dt = \frac{\sqrt{y}+2}{6}, & 4 < y < 16 \\ 0, & y < 0 \\ 1, & y \ge 16 \end{cases}$$

Method 2 for finding $F_Y(y)$:

For $-2 \le x \le 2 \Rightarrow 0 < y < 4$ *so that* $x = \pm\sqrt{y} \Rightarrow y = x^2$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{6}\,dx = \frac{1}{6}x\Big|_{-\sqrt{y}}^{+\sqrt{y}} = \frac{2\sqrt{y}}{6} = \frac{\sqrt{y}}{3}, \quad 0 < y < 4$$

For $\begin{array}{l} x = -\sqrt{y},\ y = 4 \Rightarrow x = -2,\ y = 16 \Rightarrow x = -4 \\ 4 < 16 \Rightarrow -2 > -4\ monotonic\ dcrea\sin g \end{array}$

$$F_y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= P(X \ge -\sqrt{y}) = 1 - P(X \le -\sqrt{y}) = 1 - F_X(-\sqrt{y})$$

$$= 1 - \left(\frac{(-\sqrt{y}+4)}{6}\right) = \frac{\sqrt{y}+2}{6}, 4 < y < 16$$

or $1-P\left(X \le -\sqrt{y}\right)=1-\int_{-4}^{-\sqrt{y}} \frac{1}{6}dx=\frac{\sqrt{y}+2}{6}, 4<y<16$

$$F_Y(y)=\begin{cases} \frac{\sqrt{y}}{3}, & 0<y<4 \\ \frac{\sqrt{y}+2}{6}, & 4<y<16 \\ 0, & y<0 \\ 1, & y \ge 16 \end{cases}$$

$$f_Y(y)=\frac{d}{dy}F_Y(y)=\begin{cases} \frac{1}{6\sqrt{y}}, & 0<y<4 \\ \frac{1}{12\sqrt{y}}, & 4<y<16 \\ 0, & otherwise \end{cases}$$

Exercise 7: The pmf of discrete random variable X is given in below:

$$p(x)=\begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{x-1}, & x=1,2,3,\cdots \\ 0, & otherwise \end{cases}$$

Find the pmf of $Y=X-4$.

Solution:

$$P_Y(y)=P_X\left(g^{-1}(y)\right)=P_X(y+4)=\frac{1}{3}\left(\frac{2}{3}\right)^{y+4-1}=\frac{1}{3}\left(\frac{2}{3}\right)^{y+3}$$

$y=x-4 \Rightarrow x=y+4=g^{-1}(y)$

$x=1 \Rightarrow y=1-4=-3$

$x=2 \Rightarrow y=2-4=-2$ $\qquad \Rightarrow \quad P_Y(y)=\begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{y+3}, & y=-3,-2,-1,0,1,2,3,\cdots \\ 0, & otherwise \end{cases}$

$x=3 \Rightarrow y=3-4=-1$

$x=4 \Rightarrow y=4-4=0$

$$F_Y(y)=\sum_{t=-3}^{y}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{t+3}=\frac{1}{3}\left[1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots+\left(\frac{2}{3}\right)^{y+3}\right]=\frac{1}{3}\frac{1-\left(\frac{2}{3}\right)^{y+4}}{1-\frac{2}{3}}=1-\left(\frac{2}{3}\right)^{y+4}$$

$$F_Y(y)=\begin{cases}1-\left(\frac{2}{3}\right)^{y+4}, & y=-3,-2,-1,0,1,2,3,\cdots\\ 0, & y<-3\\ 1, & y\to+\infty\end{cases}$$

Exercise 8: The pmf of discrete random variable X is given in below:

$$p(x)=\begin{cases}\frac{1}{6}, & x=-3,-2,-1,1,2,3\\ 0, & otherwise\end{cases}$$

Find the pmf of $Y=X^2$.

Solution:

$$y=x^2\Rightarrow \left.\begin{matrix}x=-\sqrt{y}\\ x=\sqrt{y}\end{matrix}\right\}\Rightarrow P_Y(y)=P_X\left(-\sqrt{y}\right)+P_X\left(\sqrt{y}\right)=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$$

$$P_Y(y)=\begin{cases}\frac{1}{3}, & y=1,4,9\\ 0, & otherwise\end{cases}$$

Exercise 9: The pdf of continuous random variable X is given in below:

$$f(x)=\begin{cases}\frac{1}{3}, & -1<x<2\\ 0, & otherwise\end{cases}$$

Find the pdf and cdf of $Y=2X^2$.

Solution:

$$-1<x<1 \quad\Rightarrow\begin{cases}0<x^2<1\\ 0<2x^2<2\end{cases}\quad\Rightarrow 0<y<2$$

$$1<x<2\Rightarrow\begin{cases}1<x^2<4\\ 2<2x^2<8\end{cases}\quad\Rightarrow 2<y<8$$

$$g_1^{-1}(y)=x_1=-\sqrt{\frac{y}{2}}\qquad g_2^{-1}(y)=x_2=\sqrt{\frac{y}{2}}$$

Probability Density Function (pdf):

$$f_Y(y) = f_X\left(g_1^{-1}(y)\right)\left|\frac{d\left(g_1^{-1}(y)\right)}{dy}\right| + f_X\left(g_2^{-1}(y)\right)\left|\frac{d\left(g_2^{-1}(y)\right)}{dy}\right|$$

$$= \frac{1}{3}.\left|\frac{d}{dy}\left(\frac{-\sqrt{y}}{\sqrt{2}}\right)\right| + \frac{1}{3}.\left|\frac{d}{dy}\left(\frac{\sqrt{y}}{\sqrt{2}}\right)\right| \qquad \Rightarrow \quad f_Y(y) = \frac{\sqrt{2}}{6\sqrt{y}}, \qquad 0 < y < 2$$

$$= \frac{1}{3}.\frac{\sqrt{2}}{4\sqrt{y}} + \frac{1}{3}.\frac{\sqrt{2}}{4\sqrt{y}}$$

$$f_Y(y) = f_X\left(g_2^{-1}(y)\right)\left|\frac{d\left(g_2^{-1}(y)\right)}{dy}\right|$$

$$= \frac{1}{3}.\left|\frac{d}{dy}\left(\frac{\sqrt{y}}{\sqrt{2}}\right)\right| \qquad \Rightarrow \quad f_Y(y) = \frac{\sqrt{2}}{12\sqrt{y}}, \qquad 2 < y < 8$$

$$= \frac{1}{3\sqrt{2}}.\frac{1}{2\sqrt{y}}$$

$$= \frac{\sqrt{2}}{12\sqrt{y}}$$

$$f_Y(y) = \begin{cases} \dfrac{\sqrt{2}}{6\sqrt{y}}, & 0 < y < 2 \\ \dfrac{\sqrt{2}}{12\sqrt{y}}, & 2 < y < 8 \\ 0, & otherwise \end{cases}$$

Cumulative Distribution Function (cdf):

$$F(x) = P(X \le x) = \int_{-1}^{x} \frac{1}{3}\,dt = \left.\frac{t}{3}\right|_{-1}^{x} = \frac{x+1}{3} \quad \Rightarrow \quad F(x) = \begin{cases} \dfrac{x+1}{3}, & -1 \le x \le 2 \\ 0, & x \le -1 \\ 1, & x \ge 2 \end{cases}$$

$$P(Y \le y) = F_X\left(\sqrt{\frac{y}{2}}\right) - F_X\left(-\sqrt{\frac{y}{2}}\right) = \frac{\sqrt{\frac{y}{2}}+1}{3} - \frac{-\sqrt{\frac{y}{2}}+1}{3} = \frac{2\sqrt{\frac{y}{2}}}{3} = \frac{\sqrt{2y}}{3}, \quad 0 \le y \le 2$$

$$P(Y \le y) = F_X\left(\sqrt{\frac{y}{2}}\right) - \underbrace{F_X\left(-\sqrt{\frac{y}{2}}\right)}_{0} = \frac{\sqrt{\frac{y}{2}}+1}{3}, \quad 2 \le y \le 8$$

$$F(y)=\begin{cases} \dfrac{\sqrt{2y}}{3}, & 0\le y\le 2 \\ \dfrac{\sqrt{\dfrac{y}{2}}+1}{3}, & 2<y\le 8 \\ 0, & y<0 \\ 1, & y\ge 8 \end{cases}$$

pdf is also obtained as $f_Y(y)=\dfrac{d}{dy}F_Y(y)=\begin{cases} \dfrac{\sqrt{2}}{6\sqrt{y}}, & 0<y<2 \\ \dfrac{\sqrt{2}}{12\sqrt{y}}, & 2<y<8 \\ 0, & otherwise \end{cases}$

Exercise 10: The pdf of continuous random variable X is given in below:

$$f(x)=\begin{cases} \dfrac{4+x}{20}, & -1\le x\le 3 \\ 0, & otherwise \end{cases}$$

Find the pdf and cdf of $Y=X^2$.

Solution:

Method 1:

$$F(x)=P(X\le x)=\int_{-1}^{x}\frac{4+t}{20}dt=\frac{8t+t^2}{40}\Bigg|_{-1}^{x}=\left[\left(\frac{8x+x^2}{40}\right)-\left(\frac{8(-1)+(-1)^2}{40}\right)\right]=\frac{x^2+8x+7}{40}$$

$$F(x)=\begin{cases} \dfrac{x^2+8x+7}{40}, & -1\le x\le 3 \\ 0, & x\le -1 \\ 1, & x\ge 3 \end{cases}$$

$-1\le x\le 1 \quad \Rightarrow 0\le x^2\le 1 \;\Rightarrow\; 0\le y\le 1$

$1<x\le 3 \quad \Rightarrow 1<x^2\le 9 \;\Rightarrow\; 1<y\le 9$

*For $-1\le x\le 1; \quad y=x^2 \;\Rightarrow\; \left.\begin{matrix} x_1=-\sqrt{y} \\ x_2=\sqrt{y} \end{matrix}\right\}$; $P(Y\le y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})$

$$P(Y \le y) = F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right) = \frac{\left(\sqrt{y}\right)^2 + 8\sqrt{y} + 7}{40} - \frac{\left(-\sqrt{y}\right)^2 - 8\sqrt{y} + 7}{40} = \frac{16\sqrt{y}}{40} = \frac{2\sqrt{y}}{5}, 0 \le y \le 1$$

**For $1 < x \le 3; \quad y = x^2 \Rightarrow \quad x = \sqrt{y}\Big\}\, \textit{positive value};$

$$P(Y \le y) = P\left(X^2 \le y\right) = P\left(X \le \sqrt{y}\right) = F_X\left(\sqrt{y}\right)$$

$$P(Y \le y) = \frac{\left(\sqrt{y}\right)^2 + 8\sqrt{y} + 7}{40} = \frac{y + 8\sqrt{y} + 7}{40}, 1 < y \le 9$$

$$F(y) = \begin{cases} \dfrac{2\sqrt{y}}{5}, & 0 \le y \le 1 \\ \dfrac{y + 8\sqrt{y} + 7}{40}, & 1 < y \le 9 \\ 0, & y < 0 \\ 1, & y \ge 9 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} \Rightarrow f_Y(y) = \begin{cases} \dfrac{d}{dy}\left(\dfrac{2\sqrt{y}}{5}\right), & 0 \le y \le 1 \\ \dfrac{d}{dy}\left(\dfrac{y + 8\sqrt{y} + 7}{40}\right), & 1 < y \le 9 \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{5\sqrt{y}}, & 0 \le y \le 1 \\ \dfrac{4 + \sqrt{y}}{40\sqrt{y}}, & 1 < y \le 9 \\ 0, & \textit{otherwise} \end{cases}$$

Method 2:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$
$$= f_X\left(g^{-1}(y)\right)\left|\frac{d\left(g^{-1}(y)\right)}{dy}\right|$$

$$f_Y(y) = f_X\left(-\sqrt{y}\right)\left|\frac{d\left(-\sqrt{y}\right)}{dy}\right| + f_X\left(\sqrt{y}\right)\left|\frac{d\left(\sqrt{y}\right)}{dy}\right|, 0 \le y \le 1$$

$$f_Y(y) = \frac{4 - \sqrt{y}}{20}\left|-\frac{1}{2\sqrt{y}}\right| + \frac{4 + \sqrt{y}}{20}\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{5\sqrt{y}}, 0 \le y \le 1$$

$$f_Y(y) = f_X\left(\sqrt{y}\right)\left|\frac{d\left(\sqrt{y}\right)}{dy}\right|, 1 < y \le 9$$

$$f_Y(y) = \frac{4+\sqrt{y}}{20}\left|\frac{1}{2\sqrt{y}}\right| = \frac{4+\sqrt{y}}{40\sqrt{y}}, 1 < y \le 9$$

$$f_Y(y) = \begin{cases} \dfrac{1}{5\sqrt{y}}, & 0 \le y \le 1 \\ \dfrac{4+\sqrt{y}}{40\sqrt{y}}, & 1 < y \le 9 \\ 0, & otherwise \end{cases}$$

$$F(y) = \begin{cases} \displaystyle\int_0^y \frac{1}{5\sqrt{t}}\,dt = \frac{2\sqrt{y}}{5}, & 0 \le y \le 1 \\ \displaystyle\int_0^1 \frac{1}{5\sqrt{y}}\,dy + \int_1^y \frac{4+\sqrt{t}}{40\sqrt{t}}\,dt = \frac{y+8\sqrt{y}+7}{40}, & 1 < y \le 9 \\ 0, & y < 0 \\ 1, & y \ge 9 \end{cases}$$

Exercise 11: The pmf of discrete random variable X is given in below:

$$p(x) = \begin{cases} \dfrac{1}{30}x, & x = 1,2,3 \\ \dfrac{1}{60}(1+2x), & x = 4,5,6,7 \\ 0, & otherwise \end{cases}$$

Find the pmf and cdf of $Y = 2X + 1$.

Solution:

$$y = 2x+1 \quad \Rightarrow \quad x = \frac{y-1}{2} \qquad \begin{array}{llll} x=1 \Rightarrow y=3 & x=2 \Rightarrow y=5 & x=3 \Rightarrow y=7 & \\ x=4 \Rightarrow y=9 & x=5 \Rightarrow y=11 & x=6 \Rightarrow y=13 & x=7 \Rightarrow y=15 \end{array}$$

$$P_Y(y) = P_X\left(g^{-1}(y)\right) = P_X\left(\frac{y-1}{2}\right) = \begin{cases} \dfrac{1}{30}\left(\dfrac{y-1}{2}\right) = \dfrac{y-1}{60}, & y = 3,5,7 \\ \dfrac{1}{60}\left[1+2\left(\dfrac{y-1}{2}\right)\right] = \dfrac{y}{60}, & y = 9,11,13,15 \end{cases}$$

Exercise 12: The pdf of continuous random variable X is given in below:

$$f(x)=\begin{cases} \frac{2}{3}x, & 0\le x\le 1 \\ \frac{1}{3}, & 1<x\le 3 \\ 0, & otherwise \end{cases}$$

Find the pdf of $Y=X^2+1$.

Solution:

$$y=x^2+1\Rightarrow x^2=y-1\Rightarrow \begin{array}{l} g_1^{-1}(y)=-\sqrt{y-1}=x_1 \\ g_2^{-1}(y)=\sqrt{y-1}=x_2 \end{array},$$

$$\begin{aligned} P(Y\le y)&=P\left(X^2+1\le y\right)=P\left(-y\le X^2+1\le y\right) \\ &=P\left(-y-1\le X^2\le y-1\right)=P\left(-\sqrt{y-1}<X<\sqrt{y-1}\right) \\ &=F_X\left(\sqrt{y-1}\right)-\underbrace{F_X\left(\sqrt{y}-1\right)}_{0} \end{aligned}$$

According to domain of random variable X, $x_1=-\sqrt{y-1}$ can not be used.

$$F_Y(y)=F_X\left(\sqrt{y-1}\right)$$

$$f_Y(y)=f_X\left(g^{-1}(y)\right)\left|\frac{d\left(\sqrt{y-1}\right)}{dy}\right|$$

Domains $\begin{array}{l} 0\le x\le 1\Rightarrow 1\le y\le 2 \\ 1<x\le 3\Rightarrow 2<y\le 10 \end{array}$

$$f_Y(y)=\begin{cases} \frac{2}{3}\sqrt{y-1}\left(\frac{1}{2\sqrt{y-1}}\right)=\frac{1}{3}, & 1\le y\le 2 \\ \frac{1}{3}\left(\frac{1}{2\sqrt{y-1}}\right)=\frac{1}{6\sqrt{y-1}}, & 2\le y\le 10 \\ 0, & otherwise \end{cases}$$

Exercise 13: Let $X \sim U([-1,1])$. Find the distribution of the random variable $Y = X^2$. The pdf of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2} & if \ x \in [-1,1] \\ 0 & otherwise \end{cases}$$

Solution:

Method 1: Note that the range of random variable Y is $[0,1]$. There are two solutions to the equation $y = x^2$. Hence, the density of $Y = X^2$ is given by

$$f_Y(y) = \sum_{x^2=y} f_X(x)\left|\frac{dx}{dy}\right| = f_X(-\sqrt{y})\left|\frac{d(-\sqrt{y})}{dy}\right| + f_X(\sqrt{y})\left|\frac{d(\sqrt{y})}{dy}\right|, 0 \le y \le 1$$

$$= \frac{1}{2}\left|\frac{1}{-2\sqrt{y}}\right| + \frac{1}{2}\left|\frac{1}{2\sqrt{y}}\right|$$

$$= \begin{cases} \frac{1}{2\sqrt{y}} & y \in [0,1] \\ 0 & otherwise \end{cases}$$

Method 2: The cdf of X is $F(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & x < -1 \\ 1, & x \ge 1 \end{cases}$

The cdf of Y is given by

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= \begin{cases} 0 & y \in [-\infty, 0] \\ P(-\sqrt{y} \le X \le \sqrt{y}) & if \ y \in [0,1] \\ 1 & otherwise \end{cases}$$

$$= \begin{cases} 0 & y \in [-\infty, 0] \\ \frac{\sqrt{y}+1}{2} - \frac{(-\sqrt{y}+1)}{2} = \sqrt{y} & if \ y \in [0,1] \\ 1 & otherwise \end{cases}$$

Hence, the density of Y is given by

$$f_Y\left(y\right)=\begin{cases}\dfrac{1}{2\sqrt{y}} & y\in\left[0,1\right]\\ 0 & otherwise\end{cases}$$