BBM 205 Discrete Mathematics
Hacettepe University
http://web.cs.hacettepe.edu.tr/∼bbm205

Lecture 8: Connectivity, Euler Walk, Hamilton Paths/Cycles, Graph Coloring, Planar Graphs

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Resources:
Kenneth Rosen, "Discrete Mathematics and App."
http://www.inf.ed.ac.uk/teaching/courses/dmmr
http://www.cs.nthu.edu.tw/ wkhon/math16.html

Graph Colouring

Suppose we have k distinct colours with which to colour the vertices of a graph. Let $[k] = \{1, \ldots, k\}$. For an undirected graph, $G = (V, E)$, an admissible vertex **k-colouring** of G is a function $c : V \rightarrow [k]$, such that for all $u, v \in V$, if $\{u, v\} \in E$ then $\quad c(u) \neq c(v)$.

For an integer $k \geq 1$, we say an undirected graph $G = (V, E)$ is **k-colourable** if there exists a k-colouring of G.

The **chromatic number** of G, denoted $\chi(G)$, is the *smallest positive integer* k, such that G is k-colourable.

Some observations about Graph colouring

- Note that any graph G with n vertices in n-colourable.
- The ***n*-Clique**, K_n, i.e., the complete graph on n vertices, has chromatic number $\chi(K_n) = n$. All its vertices must get assigned different colours in any admissible colouring.
- The clique number, $\omega(G)$, of a graph G is the maximum positive integer $r \geq 1$, such that K_r is a subgraph of G.
- Note that for all graphs G, $\omega(G) \leq \chi(G)$: if G has an r-clique then it is not $(r-1)$-colorable.
- However, in general, $\omega(G) \neq \chi(G)$. For instance, The 5-cycle, C_5, has $\omega(C_5) = 2 < \chi(C_5) = 3$.

More observations about colouring

- As already mentioned, any bipartite graph is 2-colourable. Indeed, that is an equivalent definition of being bipartite.
- More generally, a graph G is k-colourable precisely if it is k-partite, meaning its vertices can be partitioned into k disjoint sets such that all edges of the graph are between nodes in different parts.

Algorithms/complexity of colouring graphs

To determine whether a n-vertex graph $G = (V, E)$ is k-colourable by "*brute force*", we could try all possible colourings of n nodes with k colours.

Difficulty: There are k^n such k-colouring functions $c : V \to [k]$.

Question: Is there an efficient (polynomial time) algorithm for determining whether a given graph G is k-colourable?

Algorithms/complexity of colouring graphs

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Question: Is there an efficient (polynomial time) algorithm for determining whether a given graph G is k-colourable?

Answer: No, no generally efficient (polynomial time) algorithm is known, and even the problem of determining whether a given graph is 3-colourable is **NP-complete**. (Even approximating the chromatic number of a given graph is NP-hard.)

In practice, there are hueristic algorithms that do obtain good colourings for many classes of graphs.

Applications of Graph Colouring (many)

Final Exam Scheduling

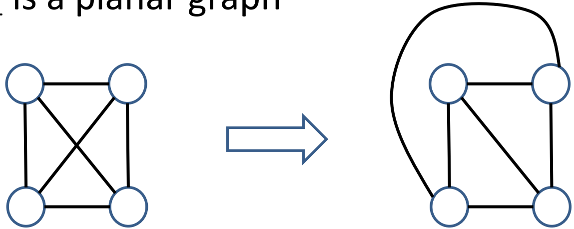
- There are n courses, $\{1, \ldots, n\}$.
- Some courses have the same students registered for both, so their exams can't be scheduled at the same time.
- Let $G = (\{1, \ldots, n\}, E)$ be a graph such that $\{i, j\} \in E$ if and only if $i \neq j$ and courses i and j have a student in common.
- **Question:** What is the minimum number of exam time slots needed to schedule all n exams?
- **Answer:** This is precisely the chromatic number $\chi(G)$ of G.

  Furthermore, a k-colouring of G yields an *admissible schedule* of exams into k time slots, allowing all students to attend all their exams, as long as different "colors" are scheduled in disjoint time slots.

What is a Planar Graph ?

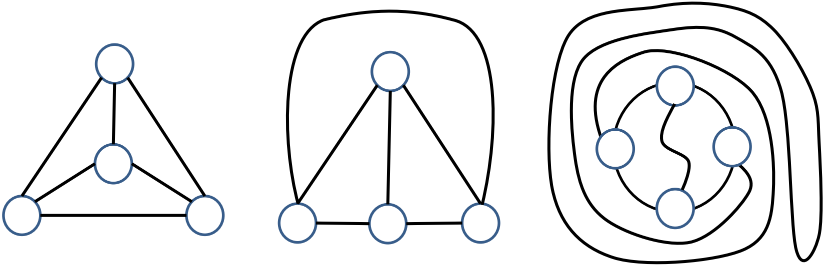
Definition : A planar graph is an undirected graph that can be drawn on a plane without any edges crossing. Such a drawing is called a planar representation of the graph in the plane.

- Ex : K_4 is a planar graph



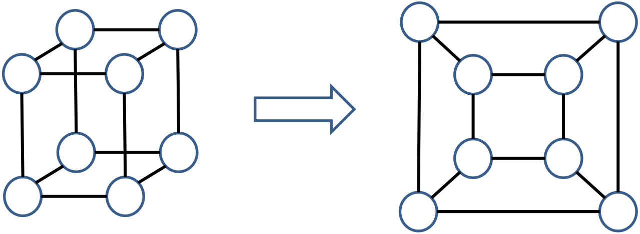
Examples of Planar Graphs

- Ex : Other planar representations of K_4



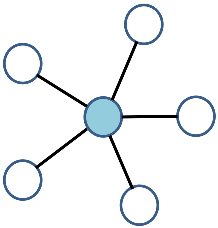
Examples of Planar Graphs

- Ex : Q_3 is a planar graph

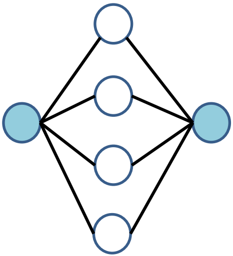


Examples of Planar Graphs

- Ex : $K_{1,n}$ and $K_{2,n}$ are planar graphs for all n



$K_{1,5}$

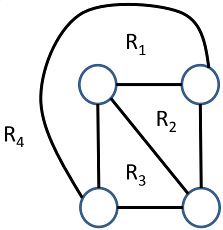


$K_{2,4}$

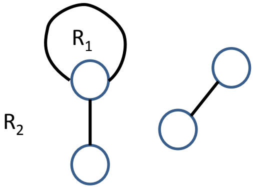
Euler's Planar Formula

Definition : A planar representation of a graph splits the plane into regions, where one of them has infinite area and is called the infinite region.

- Ex :



4 regions
(R_4 = infinite region)



2 regions
(R_2 = infinite region)

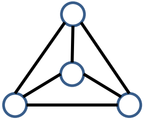
Euler's Planar Formula

- Let G be a connected planar graph, and consider a planar representation of G. Let

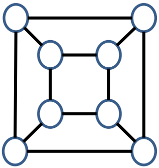
  V = # vertices, E = # edges, F = # regions.

Theorem : $V + F = E + 2$.

- Ex :



V = 4, F = 4, E = 6



V = 8, F = 6, E = 12

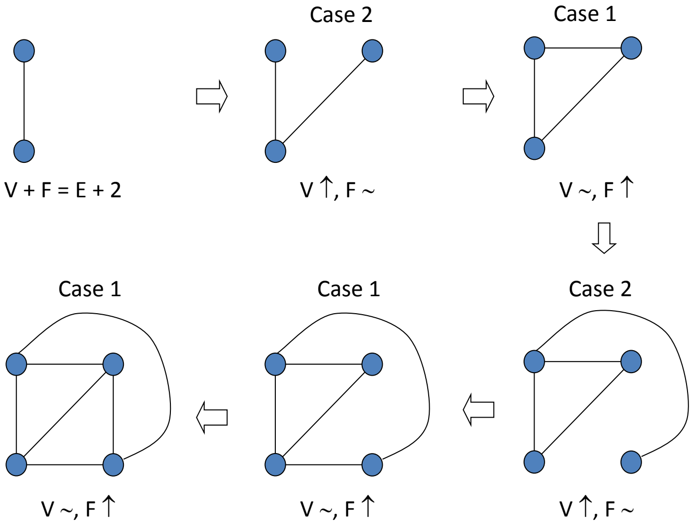
Euler's Planar Formula

- Proof Idea :
  - Add edges one by one, so that in each step, the subgraph is always connected
  - Use induction to show that the formula is always satisfied for each subgraph
  - For the new edge that is added, it either joins :

    (1) two existing vertices ➔ $V\sim$, $F\uparrow$

    (2) one existing + one new vertex ➔ $V\sim$, $F\uparrow$

Euler's Planar Formula



Euler's Planar Formula

- Let G be a connected simple planar graph with

  V = # vertices, E = # edges.

Corollary : If $V \geq 3$, then $E \leq 3V - 6$.

- Proof : Each region is surrounded by at least 3 edges (how about the infinite region?)

  ➔ $3F \leq$ total edges $= 2E$

  ➔ $E + 2 = V + F \leq V + 2E/3$

  ➔ $E \leq 3V - 6$

Euler's Planar Formula

Theorem : K_5 and $K_{3,3}$ are non-planar.

- Proof :

(1) For K_5, $V = 5$ and $E = 10$

$\rightarrow$ $E > 3V - 6$ $\rightarrow$ non-planar

(2) For $K_{3,3}$, $V = 6$ and $E = 9$.

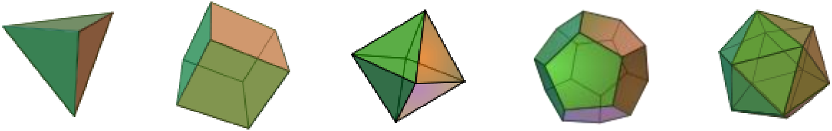
$\rightarrow$ If it is planar, each region is surrounded by at least 4 edges (why?)

$\rightarrow$ $F \leq \lfloor 2E/4 \rfloor = 4$

$\rightarrow$ $V + F \leq 10 < E + 2$ $\rightarrow$ non-planar

Platonic Solids

Definition : A Platonic solid is a convex 3D shape that all faces are the same, and each face is a regular polygon



Platonic Solids

Theorem: There are exactly 5 Platonic solids

- Proof:

  Let n = # vertices of each polygon

  m = degree of each vertex

  For a platonic solid, we must have

$$n F = 2E \qquad \text{and} \qquad V m = 2E$$

Platonic Solids

- Proof (continued):

  By Euler's planar formula,

  $$2E/m + 2E/n = V + F = E + 2$$

  $$\Rightarrow \quad 1/m + 1/n = 1/2 + 1/E \qquad \ldots\ldots (*)$$

  Also, we need to have

  $n \geq 3$ and $m \geq 3$ [from 3D shape]

  but one of them must be $= 3$ [from (*)]

Platonic Solids

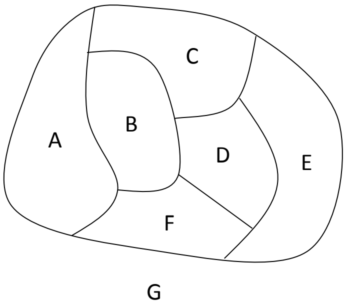
- Proof (continued):

  ➔ Either

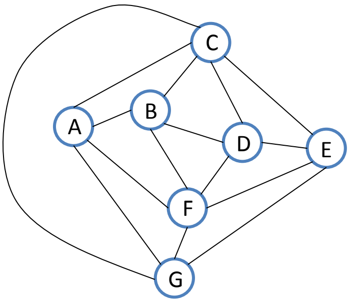
  (i) $n = 3$ (with $m = 3, 4,$ or 5)

  (ii) $m = 3$ (with $n = 3, 4,$ or 5)

Map Coloring and Dual Graph



A Map M

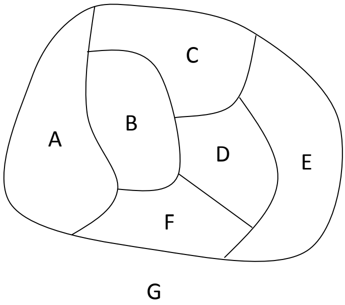


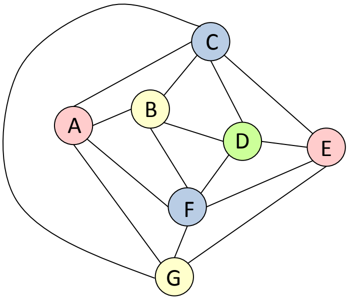
Dual Graph of M

Map Coloring and Dual Graph

Observation: A proper color of M

⇔ A proper vertex color the dual graph





Proper coloring : Adjacent regions (or vertices) have to be colored in different colors

Five Color Theorem

- Appel and Haken (1976) showed that every planar graph can be 4 colored

  (Proof is tedious, has 1955 cases and many subcases)

- Here, we shall show that :

Theorem : Every planar graph can be 5 colored.

- The above theorem implies that every map can be 5 colored (as its dual is planar)

Five Color Theorem

- Proof :

  We assume the graph has at least 5 vertices. Else, the theorem will immediately follow.

  Next, in a planar graph, we see that there must be a vertex with degree at most 5.

  Else,

  $$2E = \text{total degree} \geq 3V$$

  which contradicts with the fact $E \leq 3V - 6$.

Five Color Theorem

- Proof (continued) :

  Let v be a vertex whose degree is at most 5.

  Now, assume inductively that all planar graphs with n – 1 vertices can be colored in 5 colors

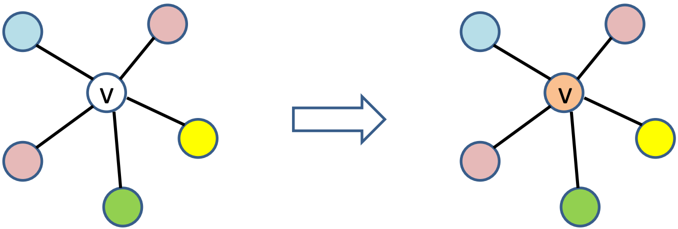
  ➔ Thus if v is removed, we can color the graph properly in 5 colors

  What if we add back v to the graph now ??

Five Color Theorem

- Proof (continued) :

  Case 1 : Neighbors of v uses at most 4 colors

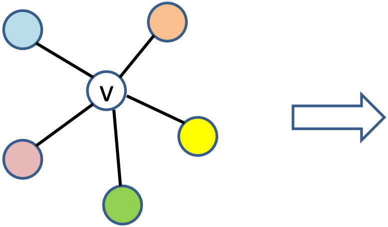


there is a 5th color for v

Five Color Theorem

- Proof (continued) :

  Case 2 : Neighbors of v uses up all 5 colors

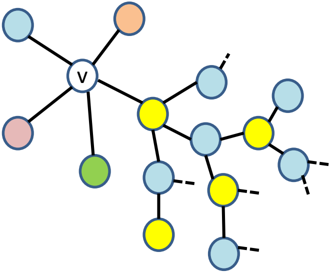


Can we save 1 color,
by coloring the yellow
neighbor in blue ?

Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?

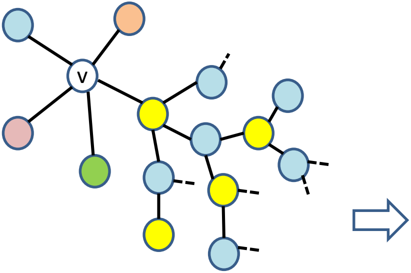


First, we check if the yellow neighbor can connect to the blue neighbor by a "switching" yellow-blue path

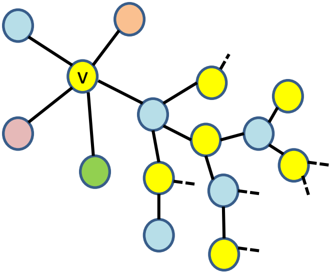
Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?



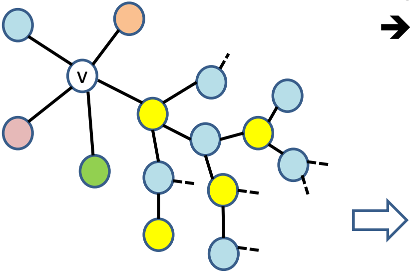
If not, we perform "switching" and thus save one color for v



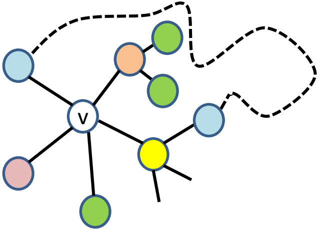
Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?



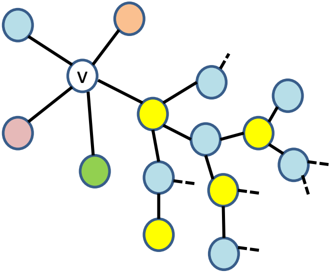
Else, they are connected
➔ orange and green cannot be
connected by "switching path



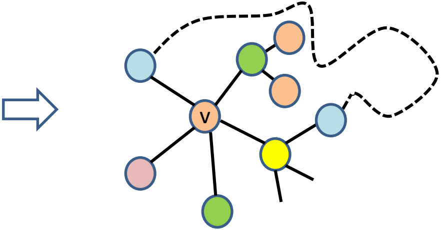
Five Color Theorem

- Proof ("Case 2" continued):

We color the orange neighbor in green !



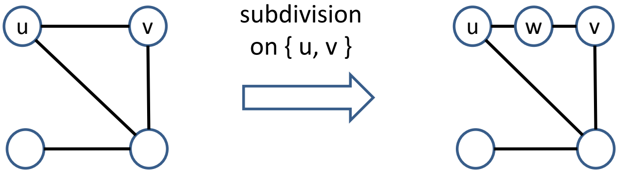
→ we can perform "switching" (orange and green) to save one color for v



Kuratowski's Theorem

Definition : A subdivision operation on an edge { u, v } is to create a new vertex w, and replace the edge by two new edges { u, w } and { w, v }.

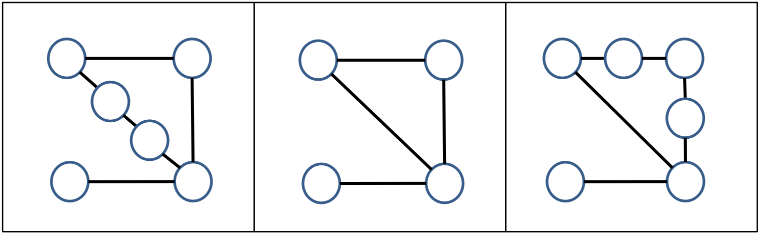
- Ex :



Kuratowski's Theorem

Definition : Graphs G and H are homeomorphic if both can be obtained from the same graph by a sequence of subdivision operations.

- Ex : The following graphs are all homeomorphic :



Kuratowski's Theorem

- In 1930, the Polish mathematician Kuratowski proved the following theorem :

Theorem :

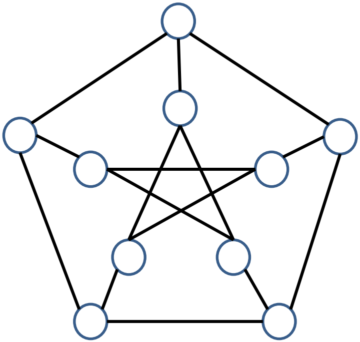
Graph G is non-planar

$\Leftrightarrow$ G has a subgraph homeomorphic to K_5 or $K_{3,3}$

- The "if" case is easy to show (how?)
- The "only if" case is hard (I don't know either …)

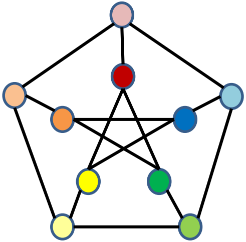
Kuratowski's Theorem

- Ex : Show that the Petersen graph is non-planar.

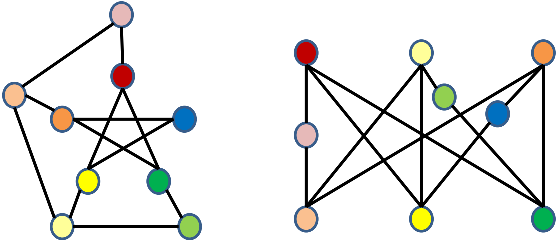


Kuratowski’s Theorem

- Proof :



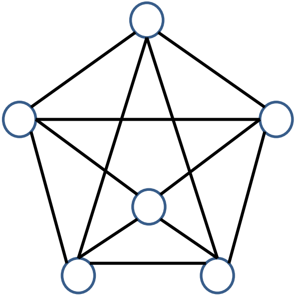
Petersen Graph



Subgraph homeomorphic to $K_{3,3}$

Kuratowski's Theorem

- Ex : Is the following graph planar or non-planar ?



Kuratowski's Theorem

- Ans : Planar