BBM 205

Spring 2015 Final Exam

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Name: SOLUTIONS

1. (3 points) Use **pigeonhole principle** to show that in any simple connected graph, there are two vertices that have the same degree.

Since the graph is connected, the degrees may vary from 1 to n-1.

Because the graph is simple, the degree is at most n-1 for each vertex.

n pigeons

n-1 pigeonholes (degrees)

$\left\lceil \frac{n}{n-1} \right\rceil = 2$ means there are two vertices of the same degree.

2. (2 points) Use a **proof by contraposition** to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

$$\underbrace{x+y \geq 2}_{p} \rightarrow \underbrace{x \geq 1 \text{ or } y \geq 1}_{q \vee r}$$

Contrapositive: $\overline{q \vee r} \rightarrow \overline{p}$ that is $\overline{q} \wedge \overline{r} \rightarrow \overline{p}$

It is same to prove the contrapositive as proving $p \rightarrow q \vee r$.

If $\overline{q}$: $x < 1$

AND

$\overline{r}$: $+ \, y < 1$

then $x + y < 2$ which is $\overline{p}$. Done.

3. (7 points) (a) (1 point) How many license plates can be made using either three letters followed by three digits or four letters followed by two digits? 10 digits, 29 letters: $29^3 \cdot 10^3 + 29^4 \cdot 10^2$

(b) (.5 points) How many different functions are there from a set with 10 elements to a set with 5 elements? 5^{10}

(c) (1.5 points) How many permutations of the letters ABCDEFG contain

a) the string BCD? A, BCD, E, F, G $\boxed{5!}$

b) the strings ABC and CDE? ABCDE, F, G $\boxed{3!}$

c) the strings CBA and BED? CBA CBED not possible $\boxed{0}$

(d) (1 point) Show that if n and k are integers with $1 \le k \le n$, then $\binom{n}{k} \le n^k/2^{k-1}$.

$$\binom{n}{k} = \frac{\overbrace{n(n-1)\cdots(n-k+1)}^{k \text{ of them}} \le n^k}{k\cdot(k-1)\cdots 3\cdot 2\cdot 1 \ge 2^{k-1}}$$ Therefore, $\binom{n}{k} \le \frac{n^k}{2^{k-1}}$

(e) (1 point) How many different ways are there to choose 6 donuts from the 21 varieties at a donut shop? $\binom{21+6-1}{6-1} = \binom{26}{5}$

(f) (1 point) How many different strings can be made from the letters in ABRACADABRA, using all letters?

5 A's, 2 B's, 2 R's, 1 C, 1 D — 11 letters

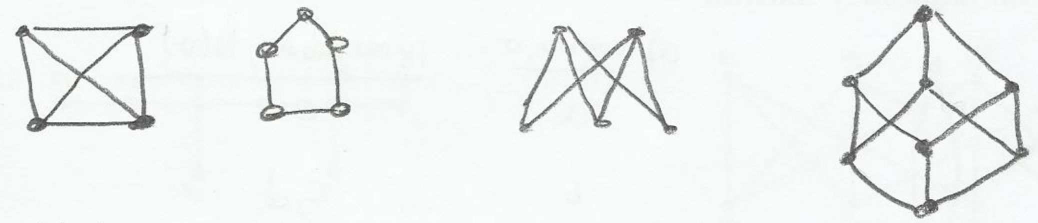
$$\boxed{\frac{11!}{5!\,2!\,2!\,1!\,1!}}$$

(g) (1 point) A bowl contains 10 red balls and 10 blue balls. A person selects balls at random without looking at them. How many balls must be selected to be sure of having at least three balls of the same color?

By pigeonhole principle, to have at least 3 pigeons in the same pigeonhole (color), the smallest n that satisfies $\left\lceil \frac{n}{2} \right\rceil = 3$ is $\boxed{n=5}$.

4. (8 points) (a) (2 points) Draw these graphs: K_4, C_5, $K_{2,3}$, Q_3.

K_4: C_5: $K_{2,3}$: Q_3



(b) (1.5 points) For which values of n are these graphs bipartite?
a) K_n b) C_n c) Q_n

K_n is bipartite if $n = 1, 2$
C_n is bip. if n, even
Q_n is bip. for all $n \geq 1$.

(c) (2 points) How many vertices and how many edges do these graphs have?
a) K_n b) C_n c) $K_{m,n}$ d) Q_n

a) n vxs, $\binom{n}{2}$ edges
b) n vxs, n edges
c) $m+n$ vxs, $m \cdot n$ edges
d) 2^n vxs, $n \cdot 2^{n-1}$ edges

(d) (1.5 points) Find the degree sequence of each of the following graphs:
a) K_4 b) C_5 c) $K_{2,3}$

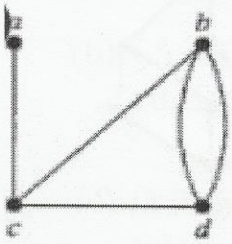
a) 3, 3, 3, 3
b) 2, 2, 2, 2, 2
c) 2, 2, 2, 3, 3

(e) (1 point) Determine whether each of these sequences is the degree sequence of a graph. For those that are, draw a graph having the given degree sequence.
a) 5,4,3,2,1,0
b) 1,1,1,1,1,1

a) $5+4+3+2+1 = 15$, not even. Since degree sum is even for all graphs, this deg. seq. is <u>NOT</u> graphical.

b)  is an example.

5. (2 points) Represent the graph below using
a) an adjacency list,
b) an adjacency matrix.

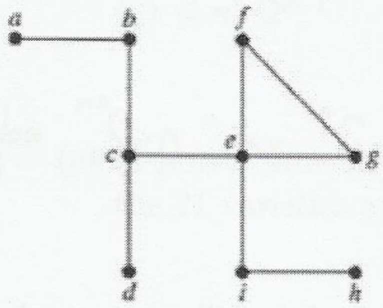


a)

vertex, v	its neighbors, $N(v)$
a	c
b	c, d
c	a, b, d
d	b, c

b)
$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 1 & 2 \\ c & 1 & 1 & 0 & 1 \\ d & 0 & 2 & 1 & 0 \end{array}$$

6. (1 point) Find all cut-vertices of the graph below.



Cut vertices are b, c, e, i.

7. (1 point) A simple graph is called k-regular if every vertex has degree k. Show that if a bipartite graph $G = (V, E)$ is k-regular for some positive integer k and (V_1, V_2) is a bipartition of V, then $|V_1| = |V_2|$.

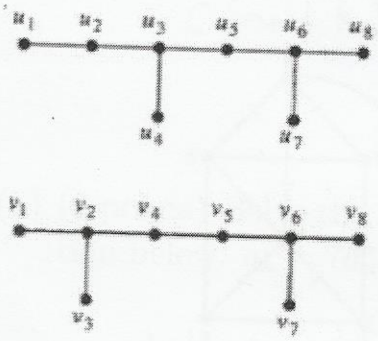
(1). The number of edges is the degree sum of the vertices in V_1, therefore $k \cdot |V_1|$.

(2). The number of edges is also the degree sum of the vertices in V_2, therefore $k \cdot |V_2|$.

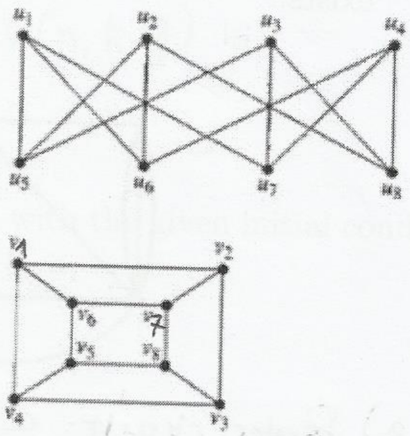
By (1) and (2) number of edges $= k \cdot |V_1| = k \cdot |V_2|$.
Therefore, $|V_1| = |V_2|$.

8. (4 points) Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

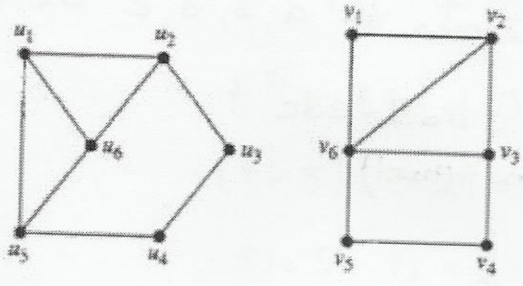
a)



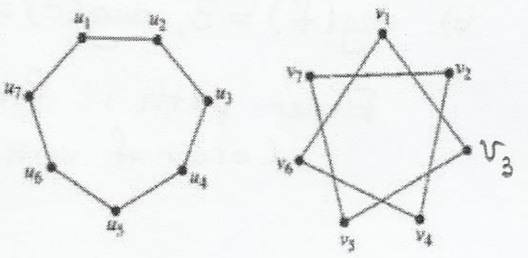
b)



c)



d)



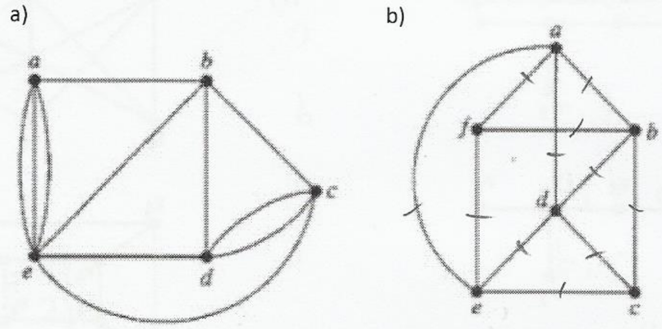
a) NOT isomorphic.
One reason is, on graph-I, there are two paths of length 5. On graph-II, there are four paths of length 5.

b) isomorphic. Isomorphic relation defined by the function f as: $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_7$, $f(u_5) = v_2$, $f(u_6) = v_4$, $f(u_7) = v_6$, $f(u_8) = v_8$

c) NOT isomorphic: One reason is, on graph-I there is no vertex of degree 4. On graph-II, there is a vertex of dgree 4.

d) isomorphic. Isomorphic relation defined by the function f as: $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_7$, $f(u_5) = v_2$, $f(u_6) = v_4$, $f(u_7) = v_6$.

9. (2 points) Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



a) Euler circuit: a b c e d c d b e a e a
(order of vertices to travel)

b) deg(f) = 3, deg(c) = 3, degrees of a, b, d, e are even.

Euler path : f b a e c b d a f e d c
(order of vertices to travel)

10. (2 points) (a) (1 point) For which values of m and n does the complete bipartite graph $K_{m,n}$ have a Hamilton circuit?

for all $m = n$, $m, n \geq 1$

(b) (1 point) Can you find a simple graph with n vertices (and $n \geq 3$) that does not have a Hamilton circuit, yet the degree of every vertex in the graph is at least $(n-1)/2$?

As observed in (a), $K_{m,r}$ does not have a Ham. circ. if $m = r+1$. This graph has $n = m + r = 2r+1$ vertices and each vertex has degree either $r = \frac{n-1}{2}$

or $m = \frac{n-1}{2} + 1$. In

11. (3 points) (a) (1 point) Derive a recurrence relation for $C(n,k) = \binom{n}{k}$, the number of k-element subsets of an n-element subset. Specifically, write $C(n+1,k)$ in terms of $C(n,i)$ for appropriate i.

$$C(n+1,k) = C(n,k) + C(n,k-1)$$

(b) (2 points) Solve the recurrence relation with the given initial condition below. $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 5$, $a_1 = 16$.

Let $a_n = t^n$ for $n \geq 0$

$$t^n - 7t^{n-1} + 10t^{n-2} = 0$$

$$t^2 - 7t + 10 = 0$$

$$(t-5)(t-2) = 0$$

$$t_1 = 5, \ t_2 = 2$$

Solution: $S_n = A t_1^n + B t_2^n$

where $S_0 = 5 = A \cdot 5^0 + B \cdot 2^0 = A + B$ (I)

$S_1 = 16 = A \cdot 5^1 + B \cdot 2^1 = 5A + 2B$ (II)

By (I) $\cdot$ 2 : $10 = 2A + 2B$

Subtract from (II): $16 = 5A + 2B$

$-10 = -2A + 2B$

$6 = 3A$

$\boxed{2 = A \\ 3 = B}$

$$\boxed{a_n = 2 \cdot 5^n + 3 \cdot 2^n}$$

12. (2 points) Let f_i be the ith Fibonacci number. **Use induction** to prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

Base step: $n=1$ $\quad f_1^2 = f_1 \cdot f_2$ True.
$\qquad 1 \quad\; 1 \quad 1$

Ind. Hypo.: For $i \leq n$, assume that $f_1^2 + f_2^2 + \cdots + f_i^2 = f_i f_{i+1}$

Ind. Step: For $n+1$, $f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 \overset{\text{By Ind. Hypo.}}{=} f_n \cdot f_{n+1} + f_{n+1}^2 =$

$= f_{n+1}(f_n + f_{n+1}) = f_{n+1} \cdot f_{n+2}$ True.

13. (3 points) (a) (1 point) Show that $x^2 + 4x + 17$ is $O(x^3)$.

$x^2 + 4x + 17 \leq 22x^3$ for all $x \geq 1$.

(b) (2 points) Show that x^3 is **not** $O(x^2 + 4x + 17)$.

Proof by contradiction:

Assume that $x^3 = O(x^2+4x+17)$. By definition of O-notation, there are constants $C>0$ and $k>1$ such that $x^3 \leq C \cdot (x^2+4x+17)$ for all $x \geq k$.

Divide both sides by x^3, we get

$1 \leq C \cdot \left(\frac{1}{x} + \frac{4}{x^2} + \frac{17}{x^3}\right) \leq C \cdot \frac{22}{x}$

But if $x = 23 \cdot C$, then $1 \leq C \cdot \frac{22}{23 \cdot C} = \frac{22}{23}$ is NOT true.

Contradiction.