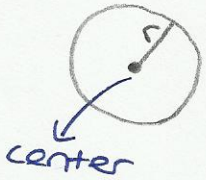
Conic Sections and Quadratic Equations $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

(A, C cannot both equal 0)

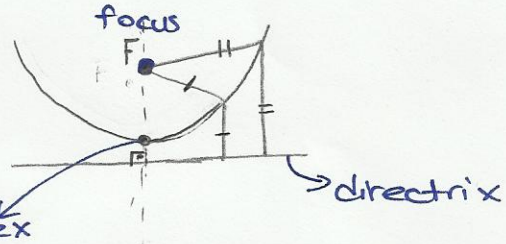
Here, we give geometric definitions and the standard definitions of circles, parabolas, ellipses, and hyperbolas. They are called conic sections or conics, since they result from intersecting a double cone with a plane.

Circle: set of points in a plane whose distance from a given fixed point in the plane is constant.



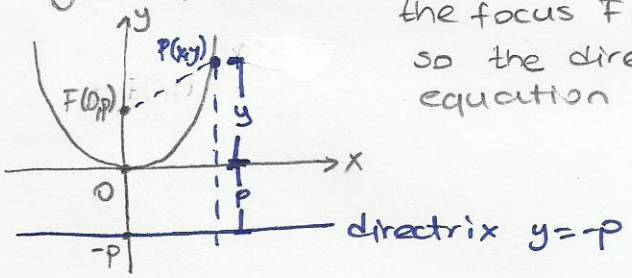
Circle of radius r centered at the point (a,b): $(x-a)^2+(y-b)^2=r^2$.

parabola: set of points in a plane that are equidistant from a fixed point F (called the focus) and a fixed line (called a directrix).



axis is perpendicular to directrix.

Example: Consider the parabola whose vertex is at the origin $(0,0)$.



the focus $F=F(0,p)$, so the directrix has the equation $y=-p$.

$|p|$: focal length

- Distance from the focus $F(0,p)$ to any point $P(x,y)$ on the parabola $|PF| = \sqrt{(x-0)^2+(y-p)^2} = \sqrt{x^2+(y-p)^2}$
- Distance from $P(x,y)$ to the directrix is $|y+p|$.

By the definition of a parabola these distances are equal; $\sqrt{x^2+(y-p)^2} = |y+p|$

$$\Rightarrow \quad x^2+(y-p)^2 = (y+p)^2 \Rightarrow x^2+y^2-2yp+p^2 = y^2+2yp+p^2$$

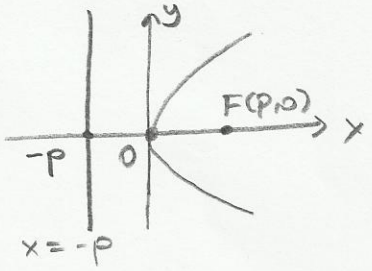
$$\Rightarrow \boxed{x^2=4yp}$$

an equation of the parabola with focus $(0,p)$ and directrix $y=-p$.

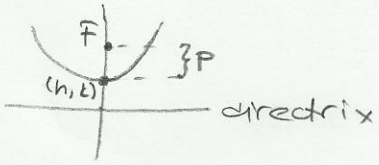
If we write $a = 1/4p$ then the standard equation of the above parabola becomes $y = ax^2$.

- The parabola opens upward if $p>0$ and downward if $p<0$.
- Since $y = ax^2 = a(-x)^2$, it is symmetric with respect to y-axis.
- If we interchange x and y in $x^2 = 4yp$ we get $y^2 = 4px$ which is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$.

The parabola opens right if $p>0$, left if $p<0$.



Let us write the general form of the equation: *for parabolas open up



Coordinates of the vertex: (h, k)

Coordinates of the focus: $(h, k+p)$

Equation of the directrix: $y = k - p$

Standard form: $(x-h)^2 = 4p(y-k)$.

Example: State the vertex and focus of the following parabolas.

1) $(x+3)^2 = -20(y-1)$ $\quad h = -3,\ k = 1,\ 4p = -20 \Rightarrow p = -5$

vertex: $(-3, 1)$ $\quad$ directrix: $y = 6$

focus: $(-3, -4)$

focal length: $|p| = 5$

2) $(y-3)^2 = 8(x-5)$ $\quad h = 5,\ k = 3$ (h always goes with the x and the k always goes with the y)

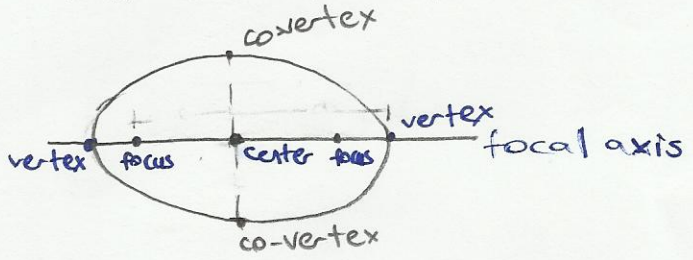
vertex: $(5, 3)$

focal length: $|p| = 2$ $\quad 4p = 8 \Rightarrow p = 2$

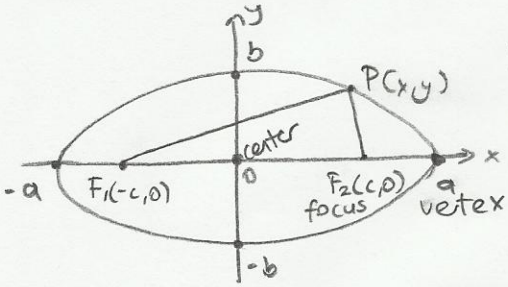
focus: $(7, 3)$

ellipse set of points in a plane whose distances from two fixed points in the plane have a constant sum.

The two fixed points are the **foci** of the ellipse.



Let us consider an ellipse centered at the origin.



If the foci are $F_1(-c,0)$ and $F_2(c,0)$ and $|PF_1|+|PF_2|$ is denoted by $2a$, then we have

$$|PF_1|+|PF_2|=2a$$

$$\sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a.$$

$$\Rightarrow \sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow (x+c)^2+y^2=4a^2+(x-c)^2+y^2-4a\sqrt{(x-c)^2+y^2}$$

$$\Rightarrow 4a\sqrt{(x-c)^2+y^2}=4a^2+\cancel{x^2}-2xc+\cancel{c^2}+\cancel{y^2}-\cancel{x^2}-2xc-\cancel{c^2}-\cancel{y^2}=4a^2-4xc$$

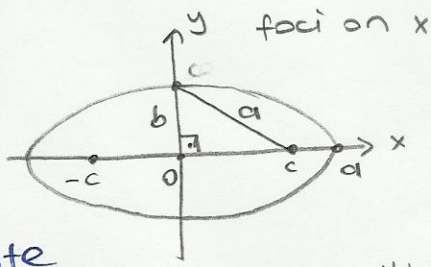
$$\sqrt{(x-c)^2+y^2}=a-\frac{xc}{a}$$

$$\Rightarrow x^2-\cancel{2xc}+c^2+y^2=a^2-\cancel{2xc}+\frac{x^2c^2}{a^2} \Rightarrow x^2\frac{(a^2-c^2)}{a^2}+y^2=a^2-c^2$$

$a>c$

$$\Rightarrow \boxed{\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1}$$

If we denote $b=\sqrt{a^2-c^2} \Rightarrow a^2-c^2=b^2 \Rightarrow$ $(*)$ $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$, $b=\sqrt{a^2-c^2}$, $a\geq b>0$



Note
If the foci of an ellipse are located on the y-axis, interchange x and y in $(*)$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\quad c^2=a^2-b^2.$$

major axis of this ellipse is the line segment of length $2a$ joining the points $(a,0)$, $(-a,0)$.

minor axis of this ellipse is the line segment of length $2b$ joining the points $(0,b)$, $(0,-b)$.

center-to-focus distance $c=\sqrt{a^2-b^2}$

The number a: **semimajor axis**

The number b: **semiminor axis**.

Example: Consider the ellipse $25x^2+4y^2+100x-40y+100=0$.

$$25x^2+4y^2+100x-40y=-100$$

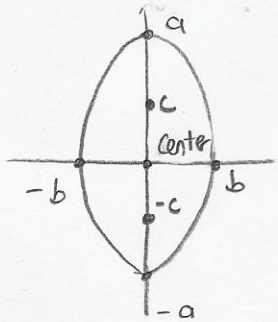
$$25(x+2)^2-100+4(y-5)^2-100=-100$$

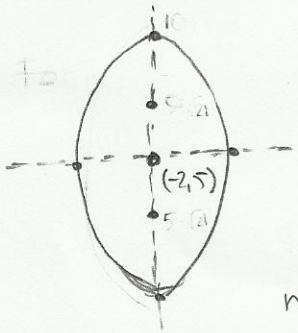
$$\Rightarrow 25(x+2)^2+4(y-5)^2=100 \Rightarrow \frac{(x+2)^2}{2^2}+\frac{(y-5)^2}{5^2}=1$$

center $(-2,5)$

The larger denominator is a^2 and the y part of the equation has the larger denominator, so this ellipse will be taller than wide.

So we take $a^2=5^2 \Rightarrow a=5$, $b^2=2^2 \Rightarrow b=2$ $\Big\}$ $c^2=a^2-b^2 \Rightarrow c^2=21 \Rightarrow c=\pm\sqrt{21}$





foci: $(-2,5+\sqrt{21})$ and $(-2,5-\sqrt{21})$

vertices: $(-2,10)$ and $(-2,0)$

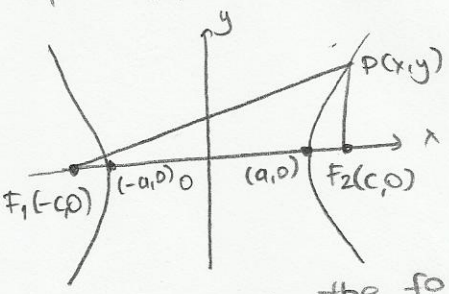
co-vertices: $(0,5)$ and $(-4,5)$

major axis length: 10

minor axis length: 4

hyperbola set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the foci of the hyperbola.

The definition of hyperbola is similar to that of an ellipse, the only change is that the sum of distances has become a difference of distances.



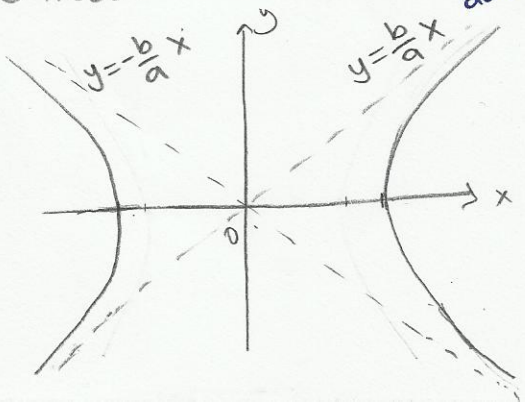
In this figure, the foci are on the x-axis.

$$|PF_1|-|PF_2|=\pm 2a \Rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ c^2=a^2+b^2.$$

Foci: $(-c,0),(c,0)$

Vertices: $(-a,0),(a,0)$

asymptotes



Example: Find the center, vertices, and asymptotes of the hyperbola with equation $4x^2-5y^2+40x-30y-45=0$.

$$4x^2-5y^2+40x-30y=45$$

$$\Rightarrow 4(x+5)^2-100-5(y+3)^2+45=45$$

$$\Rightarrow \frac{(x+5)^2}{25}-\frac{(y+3)^2}{20}=1$$

$a^2=25$, $b^2=20$

$a=5$, $b=2\sqrt{5}$

$c^2=a^2+b^2=45 \Rightarrow c=3\sqrt{5}$

Center: $(-5,-3)$

vertices: $(-10,-3)$, $(0,-3)$

foci: $(-5-3\sqrt{5},-3)$, $(-5+3\sqrt{5},-3)$

asymptotes: $y+3=\pm\frac{2\sqrt{5}}{5}(x+5)$

Classifying Conic Sections

Consider the quadratic equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

By rotating the coordinate axes, we can eliminate the term with the product of x and y. (The details will not be given)

Assume that we are studying such an equation.

$$Ax^2+Cy^2+Dx+Ey+F=0. \quad (*)$$

- a circle if $A=C\neq 0$ (special cases: the graph is a point or there is no graph at all)
- a parabola if $(*)$ is quadratic in one variable and linear in the other.
- an ellipse if A and C are both positive or both negative (special cases: circles, a single point or no graph at all)
- a hyperbola if A and C have opposite signs (special case: a pair of intersecting lines)
- a straight line if A and C are zero and at least one of D and E is different from zero;
- one or two straight lines if the left hand side of $(*)$ can be factored into the product of two linear factors.

* The Discriminant Test

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$ (be careful about the special cases written on page 5)

a) a parabola if $B^2-4AC=0$

b) an ellipse if $B^2-4AC<0$

c) a hyperbola if $B^2-4AC>0$.

Example:

1) $2x^2-xy+5y^2+4x+7=0$

$A=2,\ B=-1,\ C=5 \qquad B^2-4AC=1-4.2.5=-39<0$

So by the discriminant test, the equation represents an ellipse.

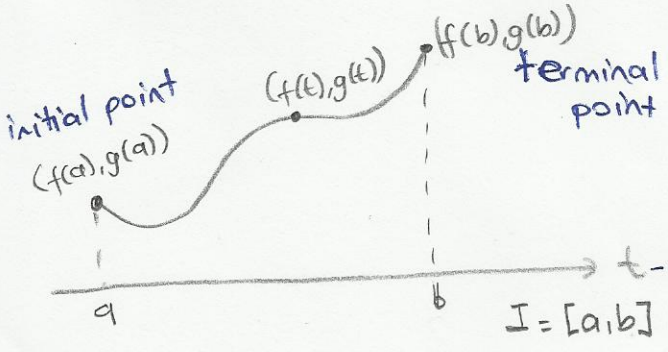
2) $xy-5y^2-7x+5y+1=0$

$A=0,\ B=1,\ C=-5 \qquad B^2-4AC=1>0$

So by the discriminant test, the equation represents a hyperbola.

Parametrization of Plane Curves

Suppose we want to plot the path of a particle moving in a plane. This path looks like a curve, but we cannot plot it like we would plot any other type of curve in the cartesian plane. The reason for this is the fact that we cannot express y directly in terms of x or x in terms of y. To get around this problem, we can describe the path of the particle with a pair of equations, $x=f(t)$ and $y=g(t)$.

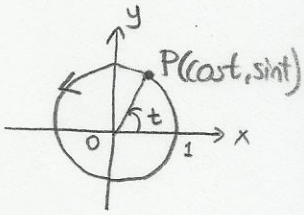


If x and y are given as continuous functions, $x=f(t)$, $y=g(t)$ over an interval of t-values, then the set of points $(x,y)=(f(t),g(t))$ is a curve in the coordinate plane

$x=f(t)$, $y=g(t)$ are called parametric equations for the curve.

Examples: 1) $x=\cos t$, $y=\sin t$ $\quad 0\le t\le 2\pi$

describe the position $P(x,y)$ of a particle that moves counterclockwise around the circle $x^2+y^2=1$ as t increases.



The point lies on this circle for every t since

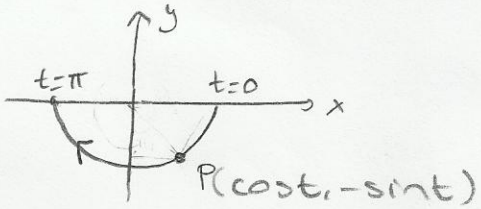
$$x^2+y^2=\cos^2 t+\sin^2 t=1.$$

$(\cos t, \sin t)$ is the parametrization of the circle. (counterclockwise)

2) Parametrization of an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

$x=a\cos t$
$y=b\sin t$
$0\le t\le 2\pi$ a particle traces the ellipse exactly once counterclockwise.

3) $x=\cos t$, $y=-\sin t$, $0\le t\le\pi$ describe the position $P(x,y)$ of a particle moves clockwise around the circle $x^2+y^2=1$ as t increases from 0 to π.



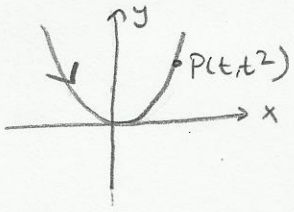
$(\cos t, -\sin t)$, $t\in[0,\pi]$ parametrization of a semicircle.

4) The position $P(x,y)$ of a particle moving in the xy-plane is given by the equations,

$$x=t,\quad y=t^2,\quad -\infty<t<\infty.$$

Identify the particle's path and describe the motion.

— Firstly, eliminate t. $y=t^2=x^2 \Rightarrow y=x^2$ this is a parabola so the particle moves along a parabola. as t increases from $-\infty$ to ∞

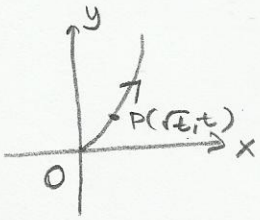


5) $x=\sqrt{t}$, $y=t$, $t \geq 0$.

Identify the path traced by the particle.

Note that we again have $y=x^2$. But now we do not have an entire parabola since x-coordinate is never negative, $x=\sqrt{t}$.

Hence we have <u>half a parabola.</u>

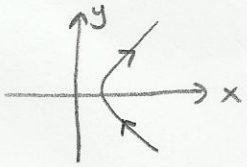


6) Find the parametric equation for <u>right-hand branch of the hyperbola</u> $x^2-y^2=1$.

Here we cannot use $x=\cos t$, $y=\sin t$ because $\cos^2 t+\sin^2 t=1$.

But we know that $\sec^2 t=\tan^2 t+1 \Rightarrow \sec^2 t-\tan^2 t=1$.

So let $x=\sec t$, $y=\tan t$, $-\pi/2 < t < \pi/2$



7) Parametrize $x=y^2-4y+3$.

Let $y=t$ then $x=t^2-4t+3$.

So (t^2-4t+3, t) , $-\infty < t < \infty$ is a parametrization of this <u>parabola.</u>

8) Find the Cartesian equation for $x=3-3t$, $y=2t$, $0 \leq t \leq 1$.

$y=2t \Rightarrow t=y/2$. So $x=3-3(y/2) \Rightarrow 2x+3y=6$. which is a <u>straight line.</u>

9) For a general form of hyperbola, $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$,

the parametric equations can be choosen as $x=h+a\sec t$

$y=k+b\tan t$.

Calculus with Parametrized Curves

Definition: A parametrized curve $x=f(t), y=g(t)$ is differentiable at $t=t_0$ if f and g are differentiable at $t=t_0$. The curve is differentiable if it is differentiable at every parameter value. The curve is smooth if f' and g' are continuous and not simultaneously zero.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad dx/dt \neq 0$$ (Explanation on page 13)

Example: Find the tangent lines to the parametric curve given by $x=t^5-4t^3$, $y=t^2$ at $(0,4)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{5t^4-12t^2}.$$

The point $(0,4)$ is given on xy-coordinate plane. So we have to find t-values which will give this point.

$(x,y)=(0,4)$ $\qquad t^5-4t^3=0 \Rightarrow t^3(t^2-4)=0 \Rightarrow t=0,\ t=\pm 2$

$t^2=4 \Rightarrow t=\pm 2$

$t=\pm 2$ appear in both lists. This means that there are two tangent lines at $(0,4)$.

For $t=-2$, $m=\left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$ so equation of the tangent line

$$y-4=-\frac{1}{8}(x-0)$$

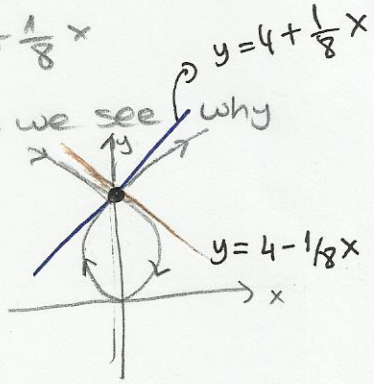
$$y=4-\frac{1}{8}x.$$

For $t=2$, $m=\left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$ so equation of the tangent line

$$y=4+\frac{1}{8}x$$

If we draw the graph of the parametric curve, we see why we have two tangent lines at this point.

The parametric curve crosses itself.



Horizontal tangent for parametric equations:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dy}{dt} = 0 \text{ provided } \frac{dx}{dt} \neq 0.$$

Vertical tangent for parametric equations:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} = 0 \text{ provided } \frac{dy}{dt} \neq 0.$$

Example: Determine the xy-coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$$x = t^3 - 3t, \quad y = 3t^2 - 9.$$

$\frac{dy}{dt} = 6t = 0$ for $t = 0$. So the xy-coordinates of the points where we have horizontal tangent is $(x,y) = (t^3 - 3t, 3t^2 - 9) \underset{t=0}{=} (0,-9)$.
$\frac{dx}{dt} \neq 0$

$\frac{dx}{dt} = 3t^2 - 3 = 0$ for $t = \pm 1$. So two vertical tangents will occur at the points $(x,y) = (t^3 - 3t, 3t^2 - 9) \underset{t=1}{=} (-2,-6)$
$\frac{dy}{dt} \neq 0$

$$(x,y) = (t^3 - 3t, 3t^2 - 9) \underset{t=-1}{=} (2,-6).$$

Second derivative for parametric equations

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

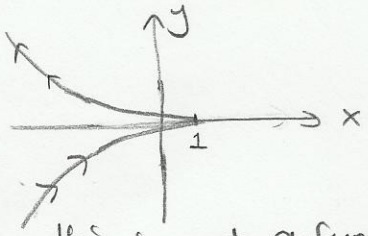
Example: Determine the values of t for which the parametric curve given by the following set of parametric equations is concave up and concave down. $x = 1 - t^2$, $y = t^7 + t^5$

For concavity, we have to deal with $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{7t^6 + 5t^4}{-2t} \underset{t \neq 0}{=} -\frac{1}{2}(7t^5 + 5t^3)$$

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{dy}{dx}\right) &= -\frac{1}{2}(35t^4 + 15t^2) \\ \frac{dx}{dt} &= -2t \end{aligned}\right\} \quad \frac{d^2y}{dx^2} \underset{t \neq 0}{=} \frac{1}{4}(35t^3 + 15t)$$

The parametric curve is increasing if $t<0$ and decreasing if $t>0$ since $\frac{dy}{dx} = -\frac{1}{2}(7t^5+5t^3)$. The second derivative



this is not a function!

$$\frac{d^2y}{dx^2} = \frac{1}{4}(35t^3+15t)$$ is negative if $t<0$ and positive if $t>0$. Hence the parametric curve is concave down for $t<0$ and concave up for $t>0$.

Lengths of parametrized curves.

We've learned in last semester that arclength of a function (length of a continuous function $y=h(x)$ on the interval $[a,b]$) is

$$L = \int_a^b \sqrt{1+h'(x)^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

$f'(x)$ is also continuous on $[a,b]$.

$$= \int_a^b \sqrt{(dx)^2+(dy)^2} = \int_a^b ds, \qquad ds = \sqrt{(dx)^2+(dy)^2}$$

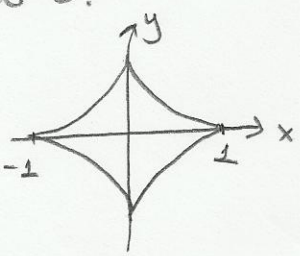
So the length of a smooth curve $x=f(t),\ y=g(t),\ a\le t\le b$,

$$L = \int_a^b \sqrt{(dx)^2+(dy)^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Note: The only requirement besides the continuity of the integrand is that the point $P(x,y)=P(f(t),g(t))$ not trace any portion of the curve more than once as t moves from a to b.

Example: Find the length of the astroid

$x=\cos^3 t,\ y=\sin^3 t,\ 0\le t\le 2\pi$.



It is enough to find the length of the first-quadrant portion then we will multiply it by 4.

$$\frac{dx}{dt} = -3\cos^2 t\sin t, \quad \frac{dy}{dt} = 3\sin^2 t\cos t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} = \sqrt{9\cos^2 t\sin^2 t\,\underbrace{(\cos^2 t+\sin^2 t)}_{1}}$$

$$= 3|\cos t\sin t|$$

$$= 3\cos t\sin t \qquad t\in[0,\pi/2] \text{ first quadrant}$$

$$L = \int_0^{\pi/2} 3\cos t \sin t \, dt = \frac{3}{2}\int_0^{\pi/2} 2\cos t \sin t \, dt = \frac{3}{2}\int_0^{\pi/2} \sin 2t \, dt$$

$$= -\frac{3}{4}\cos(2t)\Big|_0^{\pi/2} = \frac{3}{2}.$$

So length of the astroid $4 \times L = 4 \cdot \frac{3}{2} = 6$.

The Area of a Surface of Revolution

Similar to our last semester, we can write the following identities.

If a smooth curve $x=f(t), y=g(t), a \le t \le b$ is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

Revolution about the x-axis $(y \geq 0)$ $\quad S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

Revolution about the y-axis $(x \geq 0)$ $\quad S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

Area with parametric equation

Recall that we find the area under $y=F(x)$ on $a \le x \le b$ by

$$A = \int_a^b F(x)\,dx.$$

Now think of the parametric equation $x=f(t)$ as a substitution in the integral. Also assume that $a=f(\alpha)$ and $b=f(\beta)$.

$$\left.\begin{array}{l} x = f(t) \\ dx = f'(t)\,dt \end{array}\right) \quad A = \int_a^b F(x)\,dx = \int_\alpha^\beta F(f(t))\, f'(t)\,dt$$

$$y = F(x) = F(f(t)) = g(t) \qquad = \int_\alpha^\beta g(t)\, f'(t)\,dt$$

Example: Determine the area under the parametric curve given by the following parametric equations.

$$x = 6(\theta - \sin\theta) \quad , \quad y = 6(1-\cos\theta) \quad , \quad 0 \le \theta \le 2\pi .$$
$$= f(\theta) \qquad\qquad = g(\theta)$$

$$f'(\theta) = 6(1-\cos\theta)$$

$$A = \int_0^{2\pi} 36(1-\cos\theta)^2 d\theta = 36\int_0^{2\pi} (1 - 2\cos\theta + \underbrace{\cos^2\theta}_{\frac{\cos 2\theta}{2}+\frac{1}{2}})d\theta$$

$$= 36\int_0^{2\pi} \left(\frac{3}{2} - 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta = 36\left\{\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{2}\right\}\Big|_0^{2\pi}$$

$$= 108\pi .$$

Explanation for page 9: Let $x = f(t)$ and $y = g(t)$. Substitute them in the equation $y = F(x)$, we get

$$g(t) = F(f(t)).$$

If g, F, and f are differentiable, the Chain Rule gives

$$g'(t) = F'(f(t))\, f'(t) = F'(x) f'(t).$$

If $f'(t) \neq 0$, we can solve for $F'(x)$: $F'(x) = \dfrac{g'(t)}{f'(t)}$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if } \frac{dx}{dt} \neq 0.$$

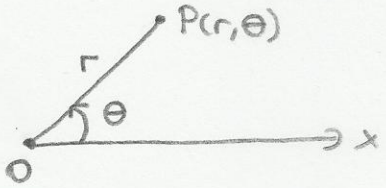
Note that we can write $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ if y is a <u>function</u> of t and t is a <u>function</u> of x, But this is not always the case. Hence it is not always correct to write $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$.

For instance, $x = t^2 \Rightarrow t = \pm\sqrt{x} \Rightarrow t$ is not a function of x.

Polar Coordinates

Usually we use cartesian (rectangular) coordinates, which are directed distances from two perpendicular axes. Here we describe a new coordinate system called the polar coordinate system.

- choose a point in the plane called the pole O (or origin).
- Draw a ray starting at O called the polar axis. This axis is usually drawn horizontally to the right corresponds to the positive x-axis in Cartesian coordinates.



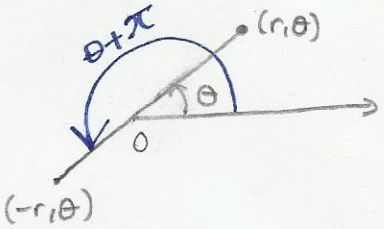
- P is any other point in the plane
- $r = |OP|$

$2\pi \text{ rad} = 360°$

- θ: angle (usually measured in radians) between the polar axis and the line OP.
- the ordered pair (r, θ) and r, θ are called polar coordinates of P

Here, we will use the convention that an angle is positive (negative) if measured in the counterclockwise (clockwise) direction.

If $P = O$, then $r = 0$ and $(0, \theta)$ represents the pole for any value of θ

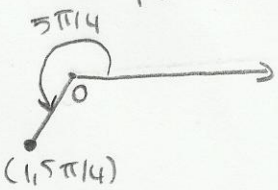


The point (r, θ) and $(-r, \theta)$ line on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.

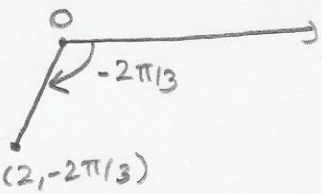
$(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Example Plot the points whose polar coordinates are given.

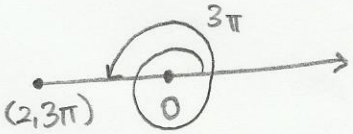
a) $(1, 5\pi/4)$



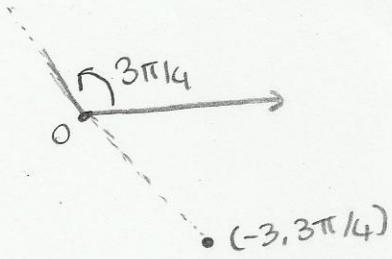
c) $(2, -2\pi/3)$



b) $(2, 3\pi)$



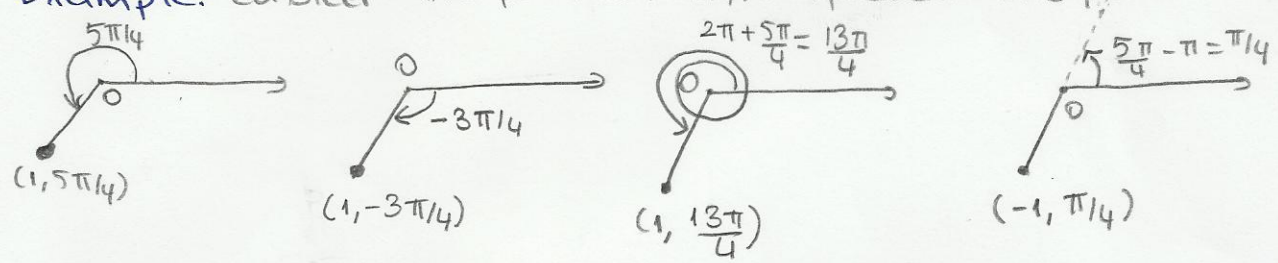
d) $(-3, 3\pi/4)$



$r = -3 < 0$

Note: In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations:

Example: Consider the point $(1, 5\pi/4)$. Represent this point in different ways.

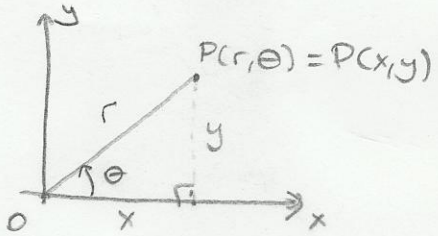


Note: Since a complete counterclockwise rotation is given by an angle 2π, the point (r, θ) is also represented by

$$(r, \theta + 2n\pi) \quad \text{or} \quad (-r, \theta + (2n+1)\pi), \quad n \text{ is any integer.}$$

connection between polar and Cartesian coordinates:

The pole corresponds to the origin and the the polar axis coincides with the positive x-axis.



$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}$$

$$\Rightarrow \boxed{x = r\cos\theta, \quad y = r\sin\theta} \quad \left(\tan\theta = \frac{y}{x}\right)$$

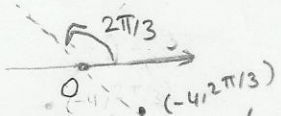
$$\theta = \tan^{-1}(y/x), \ -\pi/2 < \theta < \pi/2$$

Note: Although these equations are derived for $r > 0$ and $0 < \theta < \pi/2$, they are valid for all values of r and θ.

From these equations we can also deduce that $\boxed{x^2 + y^2 = r^2}$

Example: Convert each of the following points into the given coordinate system.

a) $(-4, 2\pi/3)$ into Cartesian coordinates.



Here $r = -4$ and $\theta = \frac{2\pi}{3}$. $\quad x = r\cos\theta = -4\cos(2\pi/3) = -4(-1/2) = 2$

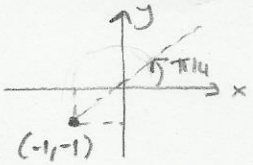
$$y = r\sin\theta = -4\sin(2\pi/3) = -4 \cdot \frac{\sqrt{3}}{2} = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $(2, -2\sqrt{3})$.

b) $(-1, -1)$ into polar coordinates. $\quad x = -1, \ y = -1$

$\tan\theta = \frac{y}{x} = 1 \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4} \in (-\pi/2, \pi/2)$ But this is not the correct angle. Since this value of θ is in the first quadrant,

but the point $(-1,-1)$ is in the third quadrant



So, the actual angle is

$$\theta = \pi/4 + \pi = 5\pi/4.$$

$r = \sqrt{x^2+y^2} = \sqrt{2}$. Therefore, the point is $(\sqrt{2}, 5\pi/4)$ in polar coordinates.

Example: a) Convert $2x - 7x^3 = 1 + xy$ into polar coordinates.

Substitute $x = r\cos\theta$, $y = r\sin\theta$.

$$2r\cos\theta - 7r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

b) Convert $r = -6\cos\theta$ into cartesian coordinates.

Recall that $r^2 = x^2+y^2$, $x = r\cos\theta$, $y = r\sin\theta$.

$$r = -6\cos\theta \underset{\text{multiply by } r}{\Rightarrow} r^2 = -6r\cos\theta \Rightarrow x^2+y^2 = -6x$$

<u>polar curves</u>: The graph of a polar equation $F(r,\theta) = 0$ consists of all points P that have at least one polar representation (r,θ) whose coordinates satisfy the equation.

Examples: 1) $r = a$

No matter what angle, we have the distance from the origin must be $|a|$. This is exactly the definition of a circle of radius $|a|$ centered at the origin.

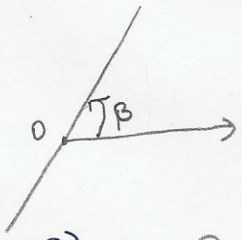
2) $\theta = \beta$

β constant in radian

$$\theta = \beta$$

$$\tan^{-1}(y/x) = \beta \Rightarrow \frac{y}{x} = \tan(\beta) \Rightarrow y = \tan(\beta)\,x$$

This is a line that goes through the origin and makes an angle of β with the positive x-axis.



3) $r\cos\theta = a$. Indeed, we have $x = a$ which is a vertical line.

4) $r\sin\theta = b$. This converts to $y = b$ and so it is a horizontal line.

5) $r = 2a\cos\theta$ (check example b) on page 16)

$r = 2a\cos\theta \quad r^2 = 2ar\cos\theta \Rightarrow x^2+y^2 = 2ax$

$$x^2 - 2ax + y^2 = 0$$

$$(x-a)^2 + (y-0)^2 = a^2,$$

Hence $r = 2a\cos\theta$ defines a circle of radius $|a|$ and center $(a,0)$.

6) $r = 2b\sin\theta$.

$r = 2b\sin\theta \Rightarrow r^2 = 2br\sin\theta \Rightarrow x^2+y^2 = 2by$

$$\Rightarrow (x-0)^2 + (y-b)^2 = b^2$$

Hence $r = 2b\sin\theta$ defines a circle of radius $|b|$ and center $(0,b)$.

7) $r = 2a\cos\theta + 2b\sin\theta$.

$r = 2a\cos\theta + 2b\sin\theta \Rightarrow r^2 = 2ar\cos\theta + 2br\sin\theta$

$$x^2+y^2 = 2ax + 2by$$

$$\Rightarrow (x-a)^2 + (y-b)^2 = a^2+b^2$$

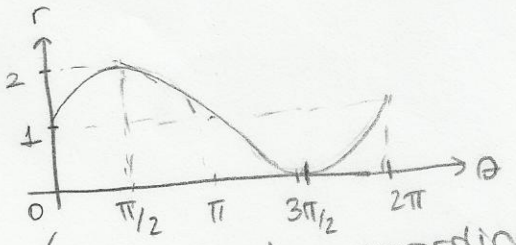
Hence $r = 2a\cos\theta + 2b\sin\theta$ defines a circle of radius $\sqrt{a^2+b^2}$ and center (a,b).

Exercise: Graph $r=7$, $r=4\cos\theta$, and $r=-7\sin\theta$ on the same axis system.

8) **Cardioids:** $r = a \pm a\cos\theta$ and $r = a \pm a\sin\theta$.

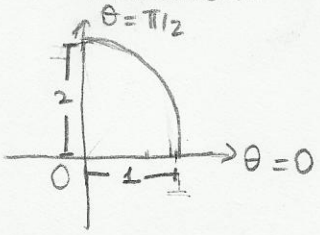
These have a graph that resembles to a heart and always contain the origin.

For instance, consider $r = 1+\sin\theta$.

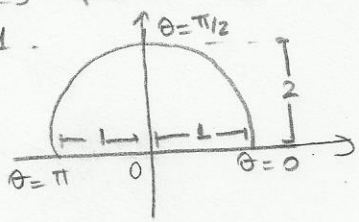


(In cartesian coordinates)

- As θ increases from 0 to $\pi/2$, r increases from 1 to 2



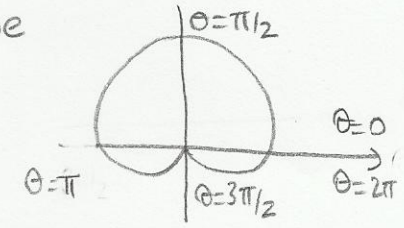
- As θ increases from $\pi/2$ to π, r decreases from 2 to 1.



- As θ increases from π to $3\pi/2$, r decreases from 1 to 0.

Also as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1.

Thus we have



9) Limacons: with an inner loop $r=a\pm b\cos\theta$ and $r=a\pm b\sin\theta$ $a<b$

These have an inner loop and always contain the origin.

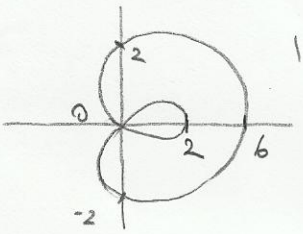
Limacons without an inner loop $r=a\pm b\cos\theta$ and $r=a\pm b\sin\theta$ $a>b$

These do not have an inner loop and do not contain the origin.

$r=2+4\cos\theta$

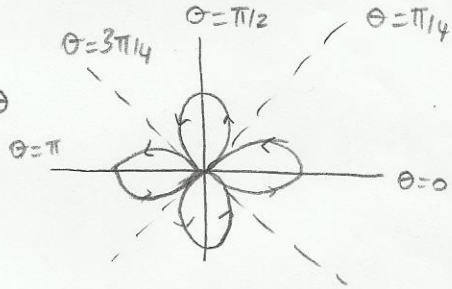
$a=2,\ b=4$

$a<b$



limacon with an inner loop

10) Four-leaved rose $r=\cos 2\theta$



Tangents to polar curves:

Assume that the equation is in the form $r=f(\theta)$.

Recall the conversion equation for going from polar coordinates to Cartesian coordinates.

$$x=r\cos\theta \quad , y=r\sin\theta.$$

Since we assume that $r=f(\theta)$ we have $x=f(\theta)\cos\theta$, $y=f(\theta)\sin\theta$. So,

$$\frac{dx}{d\theta}=f'(\theta)\cos\theta-f(\theta)\sin\theta=\frac{dr}{d\theta}\cos\theta-r\sin\theta$$

$$\frac{dy}{d\theta}=f'(\theta)\sin\theta+f(\theta)\cos\theta=\frac{dr}{d\theta}\sin\theta+r\cos\theta,$$

Therefore we have $\boxed{\dfrac{dy}{dx}=\dfrac{dy/d\theta}{dx/d\theta}=\dfrac{dr/d\theta\,\sin\theta+r\cos\theta}{dr/d\theta\,\cos\theta-r\sin\theta}}$ $dx/d\theta\neq 0$

horizontal tangent at the points where $dy/d\theta = 0$ (provided that $dx/d\theta \neq 0$)

vertical tangent at the points where $dx/d\theta = 0$ (provided that $dy/d\theta \neq 0$)

<u>Note</u>: If we are looking for tangent lines at the pole, then $r=0$ and we have

$$\frac{dy}{dx} = \tan\theta \quad \text{if} \quad \frac{dr}{d\theta} \neq 0.$$

Example: a) For the cardioid $r = 1+\sin\theta$, find the slope of the tangent line when $\theta = \pi/3$.

b) Find the points on the cardioid where the tangent line is horizontal or vertical.

- $r = 1+\sin\theta$. $\quad \frac{dr}{d\theta} = \cos\theta$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + \cos\theta + \sin\theta\cos\theta}{\underbrace{\cos^2\theta}_{1-\sin^2\theta} - \sin\theta - \sin^2\theta}$$

$$= \frac{\cos\theta\,(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

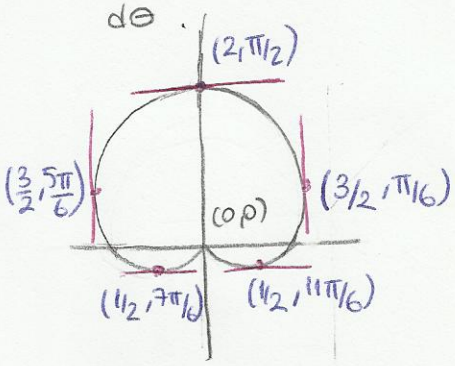
when $\theta = \pi/3$, $\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{1/2\,(1+2\cdot\sqrt{3}/2)}{(1+\sqrt{3}/2)(1-2\frac{\sqrt{3}}{2})} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} = \frac{1+\sqrt{3}}{2-2\sqrt{3}+\sqrt{3}-3}$

$$= \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1.$$

So the slope of the tangent line when $\theta = \pi/3$ is -1.

- $\frac{dy}{d\theta} = \cos\theta\,(1+2\sin\theta) = 0$ when $\theta = \underbrace{\pi/2, 3\pi/2}_{\cos\theta=0}, \underbrace{7\pi/6, 11\pi/6}_{\sin\theta=-1/2}$

$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) = 0$ when $\theta = \underbrace{\frac{3\pi}{2}}_{\sin\theta=-1}, \underbrace{\frac{\pi}{6}, \frac{5\pi}{6}}_{\sin\theta=1/2}$



Therefore, there are horizontal tangents at the points $(2, \pi/2)$, $(1/2, 7\pi/6)$, $(1/2, 11\pi/6)$ and vertical tangents at $(3/2, \pi/6)$ and $(3/2, 5\pi/6)$.

Be careful about $\theta = 3\pi/2$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0. Use L'Hopital's Rule.

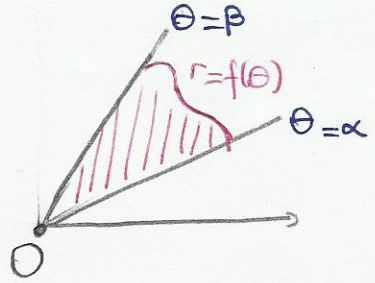
$$\lim_{\theta \to (3\pi/2)^-} \frac{dy}{dx} = \left(\lim_{\theta \to \frac{3\pi}{2}^-} \frac{1+2\sin\theta}{1-2\sin\theta} \right) \underbrace{\left(\lim_{\theta \to \frac{3\pi}{2}^-} \frac{\cos\theta}{1+\sin\theta} \right)}_{L'R}$$

$$= -\frac{1}{3} \cdot \lim_{\theta \to \frac{3\pi}{2}^-} \frac{-\sin\theta}{\cos\theta} = +\infty.$$

Similarly, $\lim_{\theta \to \frac{3\pi}{2}^+} \frac{dy}{dx} = -\infty$.

Thus, there is a vertical tangent line at the pole.

Areas and lengths in polar coordinates



The formula for finding this shaded area is

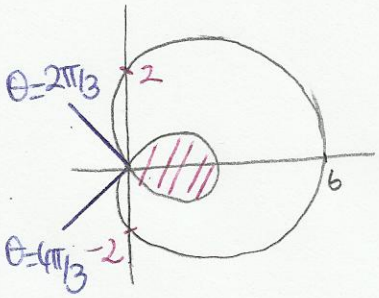
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

(Read the textbook for the proof)

Example: Determine the area of the <u>inner</u> loop of $r = 2 + 4\cos\theta$.

$a = 2,\ b = 4$
$a < b$

This is a limacon with an inner loop. The graph is drawn on page 18



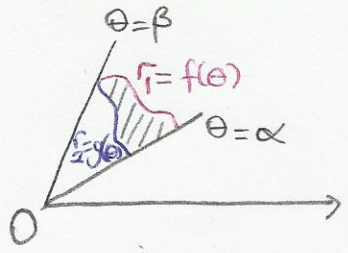
We need to know the values of θ where the curve goes through the origin. We get these values by setting the equation equal to zero.

$$0 = 2 + 4\cos\theta \Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

These points define where the inner loop starts and end and hence are also the limits of integration in the formula. So, the shaded area is

$$A = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2+4\cos\theta)^2 \, d\theta = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}\Big(4 + 16\cos\theta + 16\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\Big) d\theta = 4\pi - 6\sqrt{3}$$

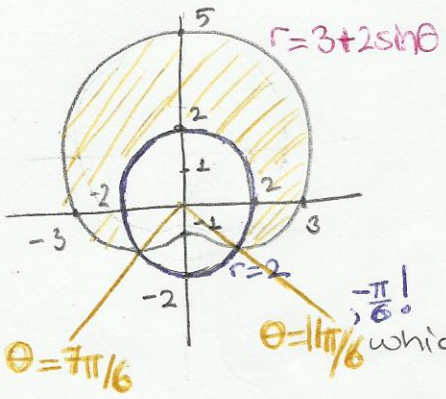
Area between two curves



Area between two curves $r_1=f(\theta)$ and $r_2=g(\theta)$

$$A=\int_{\alpha}^{\beta}\frac{1}{2}\left(r_1^2-r_2^2\right)d\theta.$$

Example: Determine the area that lies inside $r=3+2\sin\theta$ and outside $r=2$.



$r=3+2\sin\theta$ Limacon without an inner loop.
$a=3,\ b=2$
$a>b$

$r=2$ circle with radius 2.

We first determine the values of θ for which the two curves intersect.

$$3+2\sin\theta=2 \Rightarrow \sin\theta=-1/2 \Rightarrow \theta=7\pi/6,\ 11\pi/6$$

Another representation for the angle $11\pi/6$ is $-\pi/6$. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However if we use the angles $-\pi/6$ to $7\pi/6$ we will enclose the shaded area that we are asked. So the area $A=\int_{-\pi/6}^{7\pi/6}\frac{1}{2}\left((3+2\sin\theta)^2-2^2\right)d\theta=\frac{11\sqrt{3}}{2}+\frac{14\pi}{3}$

Arclength with polar coordinates

Arc length of the curve given by $r=f(\theta)$, $\alpha \le \theta \le \beta$

We assume that the curve is traced out exactly once.

$$x = r\cos\theta = f(\theta)\cos\theta \quad , \quad y = r\sin\theta = f(\theta)\sin\theta .$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad , \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta .$$

$$\Rightarrow \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - \cancel{2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\sin^2\theta \qquad (\sin^2\theta + \cos^2\theta = 1)$$

$$+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + \cancel{2r\frac{dr}{d\theta}\sin\theta\cos\theta} + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Hence the arc length formula for polar coordinates is $L = \int ds$

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example: Determine the length of $r=\theta$ for $0 \le \theta \le 1$.

$$\frac{dr}{d\theta} = 1 \qquad L = \int_0^1 \sqrt{\theta^2+1}\, d\theta = \int_0^{\pi/4} \sec^3 x\, dx = \int_0^{\pi/4} \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\, dx}_{dv}$$

$\theta = \tan x \quad d\theta = \sec^2 x\, dx$

$\theta = 0 \Rightarrow x = 0$

$\theta = 1 \Rightarrow x = \pi/4$

$\sqrt{\theta^2+1} = \sqrt{\tan^2 x + 1} = |\sec x| = \sec x$

$u = \sec x \qquad \sec^2 x\, dx = dv$

$du = \sec x \tan x\, dx \qquad \tan x = v$

$$= \sec x \tan x \Big|_0^{\pi/4} - \int_0^{\pi/4} \sec x \underbrace{\tan^2 x}_{\sec^2 x - 1} dx$$

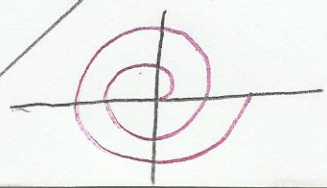
$$\Rightarrow L = \int_0^{\pi/4} \sec^3 x\, dx = \sec x \tan x \Big|_0^{\pi/4} - \int_0^{\pi/4} \sec^3 x\, dx + \int_0^{\pi/4} \sec x\, dx \frac{(\sec x + \tan x)}{(\sec x + \tan x)}$$

$$\Rightarrow 2\int_0^{\pi/4} \sec^3 x\, dx = \frac{2}{\sqrt{2}} + \ln|\sec x + \tan x| \Big|_0^{\pi/4}$$

$$\Rightarrow L = \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left|\frac{2}{\sqrt{2}} + 1\right|$$

The polar equation $r=\theta$ is the equation of a spiral.

$r = \theta$ for $0 \le \theta \le 4\pi$



Surface Area with Polar Coordinates

The surface area of the region found by rotating $r=f(\theta)$, $\alpha \le \theta \le \beta$ is

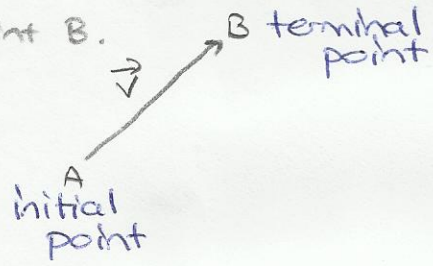
$S = \int 2\pi y \, ds$ rotation about x-axis where $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

$S = \int 2\pi x \, ds$ rotation about y-axis $r=f(\theta)$, $\alpha \le \theta \le \beta$

Vectors and the geometry of space

Definition: A vector in the plane is a directed line segment. Two vectors are equal or the same if they have the same length and direction.

For instance, suppose a particle moves along a line segment from point A to point B.

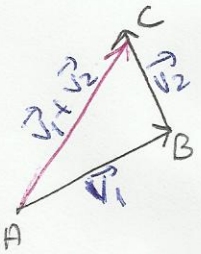


$\vec{v} = \overrightarrow{AB}$

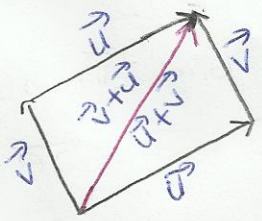
vector $\vec{v}$ directed from point A to point B.

Vector addition

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$



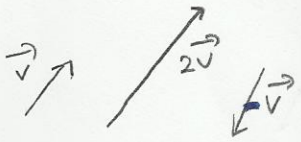
Triangle Law



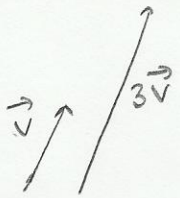
Parallelogram Law

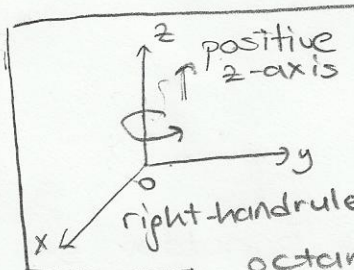
Scalar multiples $c\vec{v}$ If c is a nonzero real number, the direction of $c\vec{v}$ is same with $\vec{v}$ if c is positive and is opposite to that of $\vec{v}$ if c is negative.

If $c=0$ or $\vec{v}=0$ (zero vector: it has no direction) $c\vec{v}=0$.



Two vectors are said to be parallel if they are nonzero scalar multiples of one another.

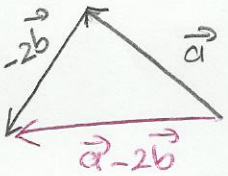




The xy-plane, yz-plane and xz-plane divide space into the eight parts, called octants. The first octant, is determined by the positive axes.

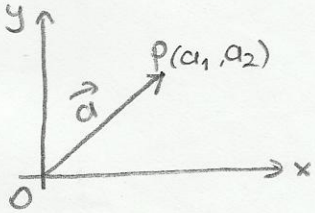
Equation of a sphere: An equation of a sphere with center $C(h,k,l)$ and radius r is
$(x-h)^2+(y-k)^2+(z-l)^2=r^2$.

Example: Draw $\vec{a}-2\vec{b}$ if the vectors are



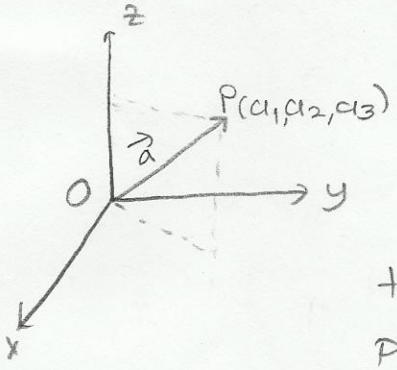
Components

Place the initial point of a vector $\vec{a}$ at the origin of a cartesian coordinate system.



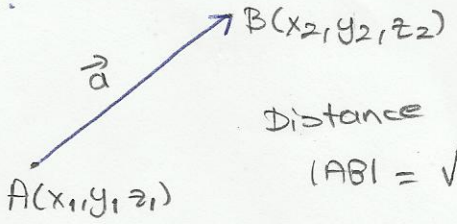
The terminal point P of $\vec{a}$ has coordinates of the form (a_1,a_2) or (a_1,a_2,a_3) depending on whether our coordinate system is two- or three-dimensional. These coordinates are called the components of $\vec{a}$ and we write

$$\vec{a}=\langle a_1,a_2\rangle \quad \text{or} \quad \vec{a}=\langle a_1,a_2,a_3\rangle.$$



The particular representation $\vec{OP}=\vec{a}$ from the origin to the point P is called the position vector of the point.

Note:

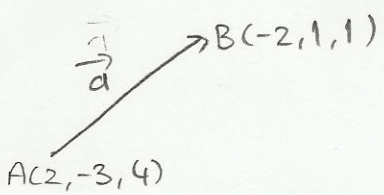


$$\vec{a}=\langle x_2-x_1,\ y_2-y_1,\ z_2-z_1\rangle$$

Distance from $A(x_1,y_1,z_1)$ to $B(x_2,y_2,z_2)$ is

$$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Example: Find the vector represented by the directed line segment with initial point $A(2,-3,4)$ and terminal point $B(-2,1,1)$.



$$\vec{a}=\langle -2-2,\ 1-(-3),\ 1-4\rangle=\langle -4,4,-3\rangle$$

Magnitude or length of a vector $\vec{v}$ denoted by $|\vec{v}|$ or $\|\vec{v}\|$.

The length of the two dimensional vector $\vec{a} = \langle a_1, a_2 \rangle$ is

$$|a| = \sqrt{a_1^2 + a_2^2}$$

The length of the three dimensional vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Unit vector: Any vector with magnitude of 1 is called a unit vector.

Algebraic operations with the components $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$

- Addition $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$
- Subtraction $\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$
- Scalar multiplication $c\vec{a} = \langle ca_1, ca_2 \rangle$

(similar for three dimensional vectors)

Example: If $\vec{a} = \langle 4, 0, 3 \rangle$ and $\vec{b} = \langle -2, 1, 5 \rangle$, find $|\vec{a}|$ and the vectors $\vec{a} - \vec{b}$, $3\vec{b}$, and $2\vec{a} + 4\vec{b}$.

- $|\vec{a}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5$
- $\vec{a} - \vec{b} = \langle 6, -1, -2 \rangle$
- $3\vec{b} = \langle -6, 3, 15 \rangle$
- $2\vec{a} + 4\vec{b} = \langle 8, 0, 6 \rangle + \langle -8, 4, 20 \rangle$
  $= \langle 0, 4, 26 \rangle$

Properties of Vectors $\vec{a}, \vec{b}, \vec{c}$ are vectors in V_n ; c, d are scalars

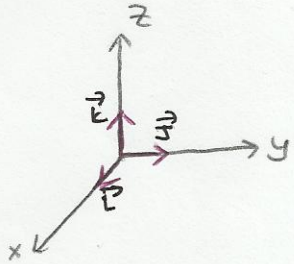
$\vec{a} + \vec{b} = \vec{b} + \vec{a}$ $\qquad$ $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

$\vec{a} + \vec{0} = \vec{a}$ $\qquad$ $\vec{a} + (-\vec{a}) = \vec{0}$

$c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$ $\qquad$ $(c + d)\vec{a} = c\vec{a} + d\vec{a}$

$(cd)\vec{a} = c(d\vec{a})$ $\qquad$ $1\vec{a} = \vec{a}$

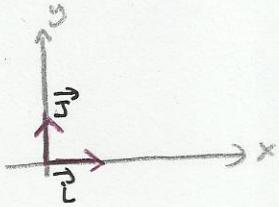
Standard Basis Vectors



In three-dimensional space V_3 there are three standard basis vectors

$\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$

They are unit vectors since $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$



In two-dimensional space V_2 there are two standard basis vectors

$\vec{i} = \langle 1, 0 \rangle$, $\vec{j} = \langle 0, 1 \rangle$; $|\vec{i}| = |\vec{j}| = 1$.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, then we can write

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = \langle a_1, 0, 0 \rangle + \langle 0, a_2, 0 \rangle + \langle 0, 0, a_3 \rangle$$
$$= a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$$
$$= a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}.$$

Thus any vectors in V_3 can be expressed in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

Similarly, in two dimesions $\vec{a} = \langle a_1, a_2 \rangle = a_1 \vec{i} + a_2 \vec{j}$.

Example: If $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{b} = 4\vec{i} + 7\vec{k}$, express the vector $2\vec{a} + 4\vec{b}$ in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

$$2\vec{a} + 4\vec{b} = 2(\vec{i} + 2\vec{j} - 3\vec{k}) + 4(4\vec{i} + 7\vec{k}) = 18\vec{i} + 4\vec{j} + 22\vec{k} = \langle 18, 4, 22 \rangle$$

Example: Determine if the vectors $\vec{a} = 2\vec{i} - 4\vec{j} + \vec{k}$ and $\vec{b} = -6\vec{i} + 12\vec{j} - 3\vec{k}$ are parallel or not.

Notice that $\vec{b} = -3\vec{a}$ hence they are parallel.

Example: Find a unit vector that points in the same direction as $\vec{w} = \langle -5, 2, 1 \rangle$.

Here, we are asked for a new parallel vector (points in the same direction) that happens to be a unit vector.

First, find $|\vec{w}| = \sqrt{(-5)^2 + 2^2 + 1^2} = \sqrt{30}$.

Let us form the following new vector.

$$\vec{u} = \frac{\vec{w}}{|\vec{w}|} = \frac{1}{\sqrt{30}} \langle -5, 2, 1 \rangle = \left\langle -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right\rangle$$

$$\Rightarrow |\vec{u}| = \sqrt{\frac{25}{30} + \frac{4}{30} + \frac{1}{30}} = 1.$$

THE DOT PRODUCT If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\vec{a}$ and $\vec{b}$ is the **number** $\vec{a} \cdot \vec{b}$ given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

known also as
- scalar product
- inner product

Properties of the dot product: If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors in V_3 and c is a scalar, then

$\vec{a}\cdot\vec{a} = |\vec{a}|^2$ $\qquad$ $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ $\qquad$ $\vec{0}\cdot\vec{a} = 0$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ $\qquad$ If $\vec{v}\cdot\vec{v} = 0$ then $\vec{v} = \vec{0}$.

Example. a) $\langle 2,4\rangle\cdot\langle -1,5\rangle = -2+20 = 18$

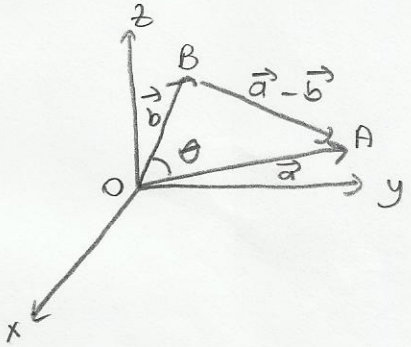
b) $\langle -1,7,4\rangle\cdot\langle 6,2,-\tfrac{1}{2}\rangle = -6+14-2 = 6$

c) $(\vec{i}+2\vec{j}-4\vec{k})\cdot(2\vec{j}-\vec{k}) = 1.0+2.2-4.(-1) = 8$

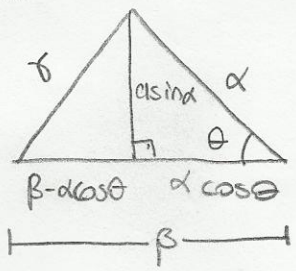
Theorem: If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ then

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$

$0 \le \theta \le \pi$



This relation is obtained by using the <u>Law of Cosines</u>.



$$\gamma^2 = (\alpha\sin\alpha)^2 + (\beta-\alpha\cos\theta)^2$$
$$= \alpha^2\sin^2\alpha + \beta^2 - 2\alpha\beta\cos\theta + \alpha^2\cos^2\theta$$
$$= \alpha^2 + \beta^2 - 2\alpha\beta\cos\theta$$

If we apply the Law of Cosines to triangle $\overset{\triangle}{OAB}$ we get

$$\underbrace{|\vec{a}-\vec{b}|^2} = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2\vec{a}\cdot\vec{b} = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta.$$

$$\Rightarrow \boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$$

Example: Determine the angle between $\vec{a} = \langle 3,-4,-1\rangle$ and $\vec{b} = \langle 0,5,2\rangle$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \qquad |\vec{a}| = \sqrt{3^2+(-4)^2+1^2} = \sqrt{26} \qquad |\vec{b}| = \sqrt{0^2+5^2+2^2} = \sqrt{29}$$

$$\vec{a}\cdot\vec{b} = 3.0+(-4).5+(-1).2 = -22$$

$$\cos\theta = \frac{-22}{\sqrt{26}\sqrt{29}} = -0.8011927 \Rightarrow \theta = \cos^{-1}(-0.8011927) = 2.5 \text{ radians}.$$

Remark: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called perpendicular or orthogonal if the angle between them is $\theta = \pi/2$. Hence

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\pi/2) = 0.$$

Conversely, if $\vec{a}\cdot\vec{b} = 0$ then $\cos\theta = 0$ so $\theta = \pi/2$.

$$\boxed{\text{Two vectors } \vec{a} \text{ and } \vec{b} \text{ are orthogonal if and only if } \vec{a}\cdot\vec{b} = 0.}$$

Example: Determine if the following vectors are parallel, orthogonal, or neither.

a) $\vec{a} = \langle 6,-2,-1\rangle$, $\vec{b} = \langle 2,5,2\rangle$.

$\vec{a}\cdot\vec{b} = 6.2 + (-2).5 + (-1).2 = 0$ hence $\vec{a}$ and $\vec{b}$ are orthogonal.

b) $\vec{u} = 2\vec{i} - \vec{j}$, $\vec{v} = -\frac{1}{2}\vec{i} + \frac{1}{4}\vec{j}$.

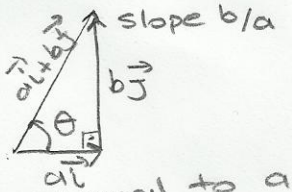
Since $\vec{u} = -4\vec{v}$, $\vec{u}$ and $\vec{v}$ are parallel. OR since

$\vec{u}\cdot\vec{v} = 2.(-\frac{1}{2}) + (-1).\frac{1}{4} = -\frac{5}{4}$, $|\vec{u}| = \sqrt{5}$, $|\vec{v}| = \frac{\sqrt{5}}{4}$ so

$-\frac{5}{4} = \frac{5}{4}\cos\theta \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$ the two vectors are parallel.

Slopes, Tangents, and Normals

The slope of a vector that is not vertical is the slope shared by the lines parallel to the vector. Thus if $a \neq 0$, the vector $\vec{v} = a\vec{i} + b\vec{j}$ has slope $\frac{b}{a} = \tan\theta$.



A vector is tangent or normal to a curve at a point if it is parallel or normal to the line that is tangent to the curve at that point.

Example: Find <u>unit</u> vectors tangent and normal to the curve $y = \frac{x^3}{2} + \frac{1}{2}$ at the point $(1,1)$.

$y = \frac{x^3}{2} + \frac{1}{2} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,1)} = \left.\frac{3x^2}{2}\right|_{(1,1)} = \frac{3}{2}$ the slope of the line tangent to the curve at $(1,1)$.

we'll find a <u>unit</u> vector with this slope. $\vec{v} = a\vec{i} + b\vec{j} = 2\vec{i} + 3\vec{j}$ and every nonzero multiple of $\vec{v}$ has slope $\frac{b}{a} = \frac{3}{2}$. But this vector is not a unit vector. Divide it by $|\vec{v}|$ to get the unit tangent vector.

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\vec{i} + 3\vec{j}}{\sqrt{4+9}} = \frac{2}{\sqrt{13}}\vec{i} + \frac{3}{\sqrt{13}}\vec{j}$$

Remark: $-\vec{u} = -\frac{2}{\sqrt{13}}\vec{i} - \frac{3}{\sqrt{13}}\vec{j}$ is also tangent to the curve at $(1,1)$ which points in the opposite direction.

To find unit vectors normal to the curve at $(1,1)$, we look for unit vectors whose slopes are the negative reciprocal of the slope of $\vec{u}$ that is $-\frac{2}{3}$. Hence

$$\vec{n} = -\frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} \quad \left(\text{or } -\vec{n} = \frac{3}{\sqrt{13}}\vec{i} - \frac{2}{\sqrt{13}}\vec{j}\right).$$

$\vec{u} \cdot \vec{n} = 0$ (they are perpendicular to eachother)

THE CROSS PRODUCT If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$ is the **vector**

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

Note: $\vec{a} \times \vec{b}$ is defined only when $\vec{a}$ and $\vec{b}$ are three-dimensional vectors.

To be more understandable use the following

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: If $\vec{a} = \langle 2,1,-1\rangle$ and $\vec{b} = \langle -3,4,1\rangle$ compute each of the following.

a) $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ -3 & 4 & 1 \end{vmatrix} = 5\vec{i} + \vec{j} + 11\vec{k} = \langle 5,1,11\rangle$

b) $\vec{b}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -5\vec{i} - \vec{j} - 11\vec{k} = \langle -5,-1,-11\rangle$

Example: Show that $\vec{a}\times\vec{a} = 0$ for any vector $\vec{a}$ in V_3.

Let $\vec{a} = \langle a_1, a_2, a_3\rangle$. $\quad \vec{a}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{i}(a_2a_3 - a_3a_2) - \vec{j}(a_1a_3 - a_3a_1) + \vec{k}(a_1a_2 - a_2a_1) = 0.$

Theorem: The vector $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Proof: Consider $(\vec{a}\times\vec{b})\cdot\vec{a}$.

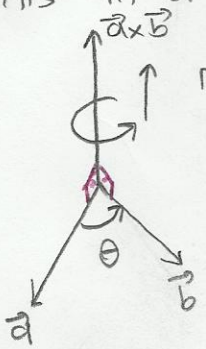
$$(\vec{a}\times\vec{b})\cdot\vec{a} = a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1)$$

$$= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1 = 0.$$

Similarly, $(\vec{a}\times\vec{b})\cdot\vec{b} = 0$.

Hence the vector $\vec{a}\times\vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Remark: This theorem says that if $\vec{a}$ and $\vec{b}$ are represented by directed line segments with the same initial point, the cross product $\vec{a}\times\vec{b}$ points in a direction perpendicular to the plane through $\vec{a}$ and $\vec{b}$.



right-hand rule: If the fingers of your right-hand curl in the direction of a rotation (through an angle less than π) from $\vec{a}$ to $\vec{b}$, then your thump points in the direction of $\vec{a}\times\vec{b}$.

Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ (so $0 \le \theta \le \pi$) then

$$\boxed{|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta}$$

Proof:

$$\begin{aligned}
|\vec{a}\times\vec{b}|^2 &= (a_2b_3-a_3b_2)^2 + (a_3b_1-a_1b_3)^2 + (a_1b_2-a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 \\
&\quad - 2a_2a_3b_2b_3 - 2a_1a_3b_1b_3 - 2a_1a_2b_1b_2 \\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2 \\
&= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \\
&= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2 \underbrace{\cos^2\theta}_{1-\sin^2\theta} = |\vec{a}|^2|\vec{b}|^2\sin^2\theta
\end{aligned}$$

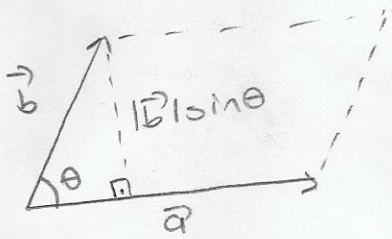
$\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ $\left(\sqrt{\sin^2\theta} = \sin\theta \text{ for } \theta\in[0,\pi]\right)$

Corollary: Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a}\times\vec{b}=0$.

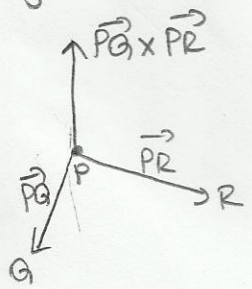
(Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\theta=0$ or π.
In either case $\sin\theta=0$ so $|\vec{a}\times\vec{b}|=0 \Rightarrow \vec{a}\times\vec{b}=0$)

Remark: If $\vec{a}$ and $\vec{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\vec{a}|$, altitude $|\vec{b}|\sin\theta$ and area

$$A = |\vec{a}|(|\vec{b}|\sin\theta) = |\vec{a}\times\vec{b}|.$$



Example: Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.



The vector $\vec{PQ}\times\vec{PR}$ is perpendicular to $\vec{PQ}$ and $\vec{PR}$ and is therefore perpendicular to the plane through P, Q, and R.

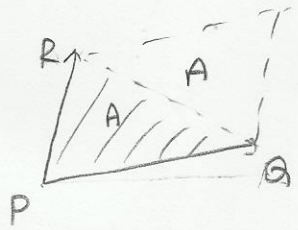
$\vec{PQ} = \langle -3, 1, -7 \rangle$, $\vec{PR} = \langle 0, -5, -5 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = (-5-35)\vec{i} - (15-0)\vec{j} + (15-0)\vec{k}$$

$$= \langle -40, -15, 15 \rangle.$$

is perpendicular to the given plane. Any nonzero scalar multiple of this vector is also perpendicular to the plane.

Example: Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

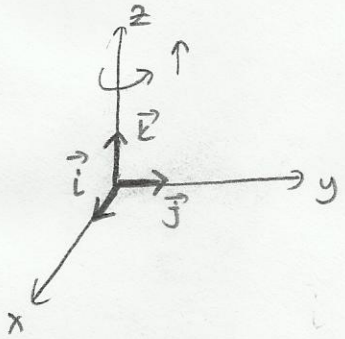


We know that $2A = |\vec{PQ} \times \vec{PR}| = \sqrt{(-40)^2 + (-15)^2 + 15^2}$

$\Rightarrow$ Hence area of the triangle

$A = \frac{1}{2} \cdot 5\sqrt{82}$.

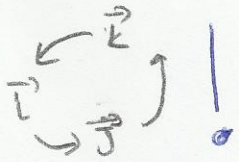
Cross product with standard basis vectors.



$\vec{i} \times \vec{j} = \vec{k}$ $\quad$ $\vec{j} \times \vec{i} = -\vec{k}$

$\vec{k} \times \vec{i} = \vec{j}$ $\quad$ $\vec{i} \times \vec{k} = -\vec{j}$

$\vec{j} \times \vec{k} = \vec{i}$ $\quad$ $\vec{k} \times \vec{j} = -\vec{i}$



!

$$\boxed{\vec{i} \times \vec{j} \neq \vec{j} \times \vec{i}}$$

The cross product is not <u>commutative</u>.

$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$ $\qquad$ $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$

Properties of the cross product $\vec{a}, \vec{b}$, and $\vec{c}$ are vectors, α is a scalar.

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2) $(\alpha\vec{a}) \times \vec{b} = \alpha(\vec{a} \times \vec{b}) = \vec{a} \times (\alpha\vec{b})$

3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

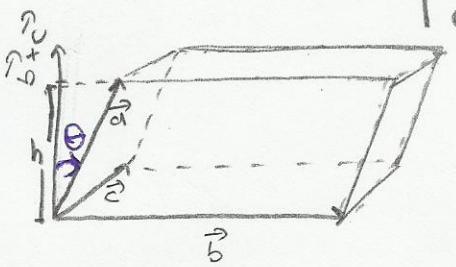
5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

) PROVE!

Triple products: $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$. $\vec{a}=\langle a_1,a_2,a_3\rangle$, $\vec{b}=\langle b_1,b_2,b_3\rangle$, $\vec{c}=\langle c_1,c_2,c_3\rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$



"parallelepiped"

The area of the base parallelogram is $A=|\vec{b}\times\vec{c}|$.

Height $h=|\vec{a}||\cos\theta|$. (we use $|\cos\theta|$ in case of $\theta>\pi/2$)

Hence volume of the parallelepiped is

$$\boxed{V=Ah=|\vec{b}\times\vec{c}|\,|\vec{a}||\cos\theta| = |\vec{a}\cdot(\vec{b}\times\vec{c})|.}$$

coplanar: If $V=|\vec{a}\cdot(\vec{b}\times\vec{c})|=0$ then the vectors must lie in the same plane; that is they are coplanar.

Example: Determine if the three vectors $\vec{a}=\langle 1,4,-7\rangle$, $\vec{b}=\langle 2,-1,4\rangle$, and $\vec{c}=\langle 0,-9,18\rangle$ lie in the same plane or not.

Check $\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$

To evaluate this determinant, you can use the Sarrus' rule.

$$\begin{array}{r} - \\ \\ 0 \\ -36 \\ +\ 144 \\ \hline 108 \end{array}\quad \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \\ 1 & 4 & -7 \\ 2 & -1 & 4 \end{vmatrix}\quad \begin{array}{l} + \\ \\ -18 \\ 126 \\ +\ 0 \\ \hline 108 \end{array}$$

write the first two rows under the 3x3 matrix.

$\vec{a}\cdot(\vec{b}\times\vec{c})=108-108=0$.

So they lie in the same plane i.e. they are coplanar.

2. way to evaluate the determinant

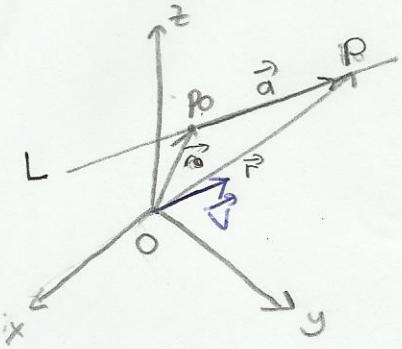
$$\vec{a}\cdot(\vec{b}\times\vec{c})=1\cdot\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix}-4\begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix}-7\begin{vmatrix} 2 & -1 \\ 0 & -9 \end{vmatrix}$$

$$=0$$

Equations of lines and planes

Equations of line

A line L in three-dimensional space is determined by a point $P_0(x_0, y_0, z_0)$ on L and the direction of L.



$\vec{v}$: a vector parallel to L, $\vec{v} = \langle a, b, c \rangle$

$P(x, y, z)$: any point on the line

$\vec{r_0} = \overrightarrow{OP_0}$: position vector of P_0

$\vec{r} = \overrightarrow{OP}$: position vector of P

$\vec{a} = \overrightarrow{P_0P}$

Hence by Triangle Law $\vec{r} = \vec{r_0} + \vec{a}$.

Note that $\vec{a}$ and $\vec{v}$ are parallel hence $\vec{a} = t\vec{v}$, t scalar.

So we have
$$\boxed{\begin{aligned}\vec{r} = \vec{r_0} + t\vec{v} &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \\ &= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle\end{aligned}}$$

vector form of the equation of a line

As t varies over all possible values, we will completely cover the line.

$$\vec{r} = \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$\boxed{\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\} \text{parametric form of the equation of a line}}$$

$\Rightarrow$ If we assume that a, b, and c are all nonzero numbers we have

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$ **symmetric equations of the line**

Remark: If, for instance $b = 0$, we only solve the parametric equations for x and z for t. In this case, we have

$$\frac{x - x_0}{a} = \frac{z - z_0}{c}, \quad y = y_0.$$

Example: a) Find a vector equation and parametric equations for the line that passes through the point (5, 1, 3) and is parallel to the vector $\vec{i} + 4\vec{j} - 2\vec{k}$.

b) Find two other points on the line.

a) $\vec{r_0} = \langle 5,1,3 \rangle$ position vector of the point $(5,1,3)$

$\vec{v} = \vec{i} + 4\vec{j} - 2\vec{k}$

$\vec{r} = (5\vec{i} + \vec{j} + 3\vec{k}) + t(\vec{i} + 4\vec{j} - 2\vec{k}) = (5+t)\vec{i} + (1+4t)\vec{j} + (3-2t)\vec{k}$

vector form

$$\left.\begin{aligned} x &= 5+t \\ y &= 1+4t \\ z &= 3-2t \end{aligned}\right\} \text{parametric form}$$

b) By choosing the parameter value t, we can find any points on that line.

Let $t=1 \Rightarrow (6,5,1)$ is a point on the line.

Let $t=-1 \Rightarrow (4,-3,5)$ is another point on the line.

Example: Determine if the line that passes through the point $(0,-3,8)$ is parallel to the line given by $x=10+3t$, $y=12t$, and $z=-3-t$ passes through the xz-plane. If it does give the coordinates of that point.

We first write the equation of the line. The point on the line is given. The line is parallel to $x=10+3t$, $y=12t$, and $z=-3-t$ that it is parallel to $\vec{v} = \langle 3, 12, -1 \rangle$.

Hence the equation of the line is,

$$\vec{r} = \langle 0,-3,8 \rangle + t\langle 3,12,-1 \rangle = \langle 3t, -3+12t, 8-t \rangle.$$

If this line passes through the xz-plane, the y-coordinate of that point must be zero; $-3+12t=0 \Rightarrow t = 1/4$.

So the line passes through the xz-plane at the point

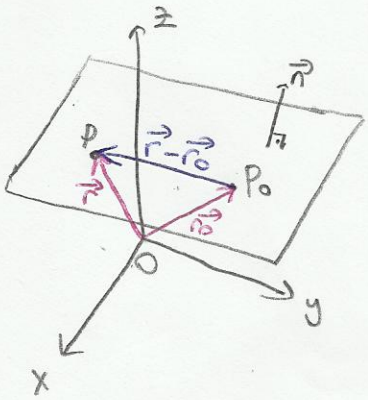
$$\left(3\cdot\frac{1}{4}, -3+12\cdot\frac{1}{4}, 8-\frac{1}{4}\right) = \left(\frac{3}{4}, 0, \frac{31}{4}\right).$$

Note: The vector equation of a line through the (tip of the) vector $\vec{r_0}$ in the direction of a vector $\vec{v}$ is $\vec{r} = \vec{r_0} + t\vec{v}$. If the line also passes through (the tip of r_1) then we can take $\vec{v} = \vec{r_1} - \vec{r_0}$ and so its vector equation is

$$\boxed{\vec{r} = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) = (1-t)\vec{r_0} + t\vec{r_1}} \quad 0 \le t \le 1$$

the line segment from $\vec{r_0}$ to $\vec{r_1}$

Equations of Planes



Assume that we know a point on the plane, $P_0 = (x_0, y_0, z_0)$.

$\vec{n} = \langle a, b, c \rangle$ normal vector to the plane

$P = (x, y, z)$ any point on the plane

$\vec{r_0}$ position vector of P_0

$\vec{r}$ position vector of P

Clearly, the vector $\vec{r} - \vec{r_0}$ lies on the plane. Because $\vec{n}$ is orthogonal to the plane, it is also orthogonal to any vector on plane. Hence it is orthogonal to $\vec{r} - \vec{r_0}$. Therefore we have

$$\boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \Rightarrow \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}}$$ **vector equation of the plane**

Since $\vec{n} = \langle a, b, c \rangle$, $\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, we have

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$= a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

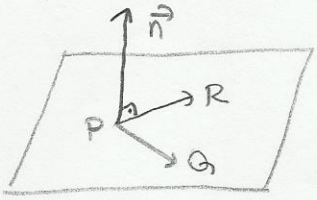
we get $$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$ **scalar equation of the plane**

Mostly, this equation is written as

$$ax + by + cz = d, \quad \text{where } d = ax_0 + by_0 + cz_0.$$

Example: Determine the equation of the plane that contains the points $P = (1, -2, 0)$, $Q = (3, 1, 4)$, $R = (0, -1, 2)$.

We first must find the normal vector.



$\vec{PQ} = \langle 2, 3, 4 \rangle$, $\vec{PR} = \langle -1, 1, 2 \rangle$

The normal vector can be chosen as

$$\vec{n} = \vec{PQ} \times \vec{PR}. \quad \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = 2\vec{i} - 8\vec{j} + 5\vec{k}$$

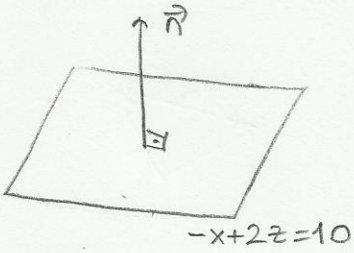
$$= \langle \underset{a}{2}, \underset{b}{-8}, \underset{c}{5} \rangle$$

You can use any of the points on the plane to write the equation of the plane. Let us chose the point $Q = (3, 1, 4)$.

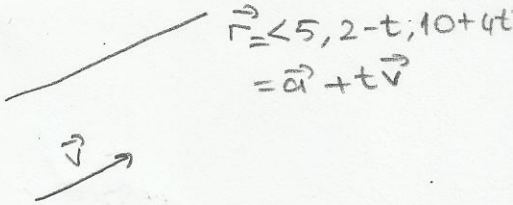
$$2(x - 3) - 8(y - 1) + 5(z - 4) = 0$$

$$2x - 8y + 5z = 18$$

Example: Determine if the plane given by $-x+2z=10$ and the line given by $\vec{r}=\langle 5, 2-t, 10+4t\rangle$ are orthogonal, parallel or neither.



$\vec{n}=\langle -1,0,2\rangle$ is a vector that is normal to the plane



$\vec{v}=\langle 0,-1,4\rangle$ is a vector that is parallel to the line.

If these two vectors are parallel then the line and the plane will be orthogonal.

Check $\vec{n}\times\vec{v}$.

$$\vec{n}\times\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 0 & 2\\ 0 & -1 & 4\end{vmatrix}=2\vec{i}+4\vec{j}+\vec{k}\neq 0.$$

Hence the vectors are not parallel and so the plane and the line are NOT ORTHOGONAL.

Check $\vec{n}\cdot\vec{v}$.

$\vec{n}\cdot\vec{v}=0+0+8=8\neq 0$. Hence the vectors are not orthogonal and so the plane and the line are NOT PARALLEL.

So, the line and the plane are neither orthogonal nor parallel

Quadric Surfaces

Quadric surfaces are the graphs of any equation that can be put into the general form

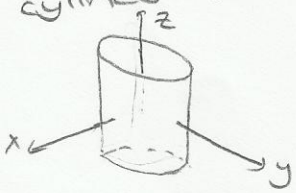
$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0,$$

where $A,\dots,J$ are constants.

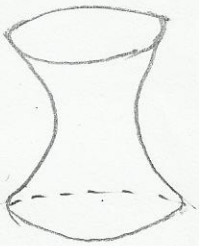
Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; If $a=b=c$ we have a Sphere.

Cone: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$

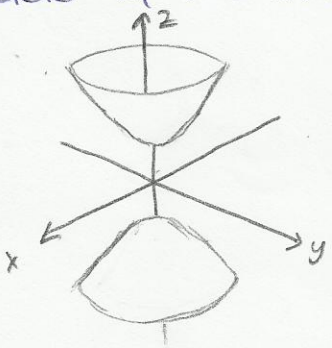
Cylinder: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ This is a cylinder whose cross-section is an ellipse. If $a=b$, we have a cylinder whose cross-section is a circle.



Hyperboloid of One sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

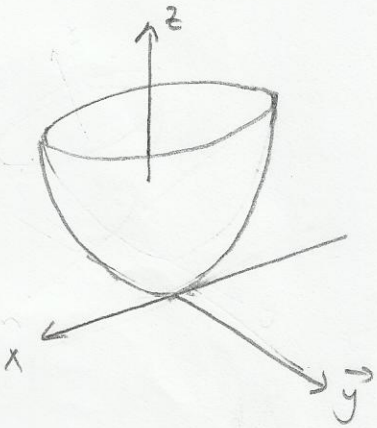


Hyperboloid of Two Sheets $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



Elliptic Paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

This has a cross-section of an ellipse, but if $a=b$, its cross-section is a circle.



Vector Functions and Space Curves

A vector function of a single variable in two and three-dimensional space (or vector-valued function) have the form

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad , \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

respectively, where $f(t)$, $g(t)$, and $h(t)$ are called the component functions.

Domain: The domain of a vector function is the set of all t's for which all the component functions are defined.

Example: Determine the domain of the following function,

$$\vec{r}(t) = \langle \sin t, \ln(5-t), \sqrt{t+2} \rangle$$

$f(t) = \sin t$ is defined for all t.

$g(t) = \ln(5-t)$ is defined for $t < 5$

$h(t) = \sqrt{t+2}$ is defined for $t \geq -2$

considering all of those together, we obtain the domain

$$[-2, 5)$$

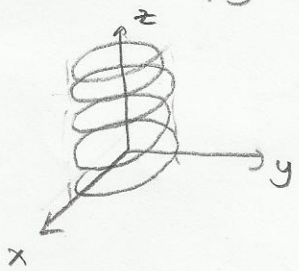
This is the largest possible interval for which all three components are defined.

Example: The vector function $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ is defined for all $t \in \mathbb{R}$.

The curve lies on the cylinder because the $\vec{i}$ and $\vec{j}$ components of $\vec{r}$ satisfy $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$.

The curve rises as the $\vec{k}$-component $z = t$ increases.

The equations $x = \cos t$, $y = \sin t$, $z = t$, $t \in (-\infty, \infty)$ parametrize the helix.



Limit: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Example: Compute $\lim_{t \to 1} \vec{r}(t)$: where $\vec{r}(t) = \left\langle t^3, \dfrac{\sin(3t-3)}{t-1}, e^{2t} \right\rangle$.

$f(t) = t^3$, $\lim_{t \to 1} t^3 = 1$; $g(t) = \dfrac{\sin(3t-3)}{t-1}$, $\lim_{t \to 1} \dfrac{\sin(3t-3)}{t-1} \underset{\text{LR}}{\overset{\frac{0}{0}}{=}} \lim_{t \to 1} \dfrac{3\cos(3t-3)}{1}$

$= 3$

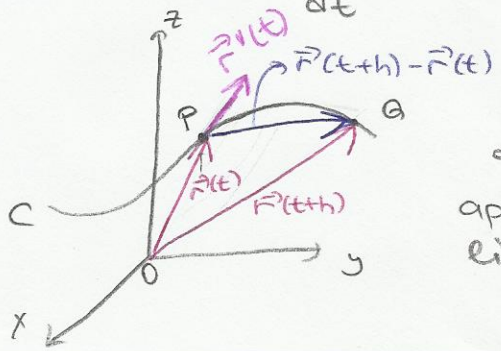
$h(t) = e^{2t}$, $\lim_{t \to 1} e^{2t} = e^2$.

Hence $\lim_{t \to 1} \vec{r}(t) = \langle 1, 3, e^2 \rangle$.

Continuity: $\vec{r}(t)$ is continuous at a if and only if its component functions are continuous at a.

(or $\vec{r}(t)$ is continuous at a if $\lim_{t \to a} r(t) = r(a)$)

Derivative: $\dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$ if this limit exists



If $h > 0$, $\frac{1}{h}(\vec{r}(t+h) - \vec{r}(t))$ has the same direction as $\vec{r}(t+h) - \vec{r}(t)$. As $h \to 0$, this vector approaches a vector that lies on the tangent line.

Hence the vector $\vec{r}'(t)$ is called the <u>tangent vector</u> to the curve defined by $\vec{r}(t)$ at the point P, provided that $\vec{r}'(t)$ exists and $\vec{r}'(t) \neq 0$.

unit tangent vector: $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$.

Theorem: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions then

$$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

Example: a) Find the derivative of $\vec{r}(t) = (1+t^3)\vec{i} + te^{-t}\vec{j} + \sin 2t\,\vec{k}$.

b) Find the unit tangent vector at the point where $t=0$.

(a) $\vec{r}'(t) = 3t^2\vec{i} + (e^{-t} - te^{-t})\vec{j} + 2\cos(2t)\vec{k}$

(b) at $t=0$, $\vec{r}(0) = \vec{i} + 0\vec{j} + 0\vec{k} = \vec{i} = \langle 1, 0, 0 \rangle$

$\vec{r}'(0) = \vec{j} + 2\vec{k} = \langle 0, 1, 2 \rangle$.

$\vec{T}(t) = \dfrac{\vec{r}'(0)}{|\vec{r}'(0)|} = \dfrac{\langle 0,1,2 \rangle}{\sqrt{5}} = \langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$ unit tangent vector at the point $(1, 0, 0)$

Differentiation Rules

$\dfrac{d}{dt}(\vec{u} + \vec{v}) = \dfrac{d}{dt}\vec{u} + \dfrac{d}{dt}\vec{v}$

$\dfrac{d}{dt}(\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$

$(c\vec{u})' = c\vec{u}'$

$\dfrac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

$\dfrac{d}{dt}\left(f(t)\vec{u}(t)\right) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

$\dfrac{d}{dt}\left(\vec{u}(f(t))\right) = f'(t)\,\vec{u}'(f(t))$

smooth curve: A smooth curve is any curve for which $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq 0$ for any t except possibly at the endpoints, Example: A helix is a smooth curve.

Example: Show that if $|\vec{r}(t)| = c$ (a constant), then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all t.

since $\vec{r} \cdot \vec{r} = |\vec{r}|^2 = c^2$ $\quad \frac{d}{dt}c^2 = 0$

$\dfrac{d}{dt}(\vec{r} \cdot \vec{r}) = 2\vec{r}' \cdot \vec{r} = 0 \Rightarrow \vec{r}'$ is orthogonal to $\vec{r}$.

Integrals: The definite integral of a continuous vector function $\vec{r}(t)$.

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Example: If $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$ then

$$\int \vec{r}(t)\,dt = \left(\int 2\cos t\,dt\right)\vec{i} + \left(\int \sin t\,dt\right)\vec{j} + \left(\int 2t\,dt\right)\vec{k}$$

$$= 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k} + c \quad ; \quad c \text{ is an integration constant.}$$

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \left(2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\vec{k}\right)\Big|_0^{\pi/2}$$

$$= 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}$$

Arclength: Recall that we defined the length of a curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$ as

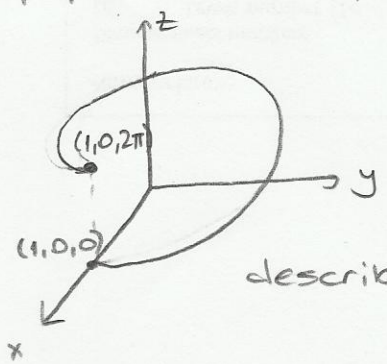
$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt .$$

Similarly, if a curve has the vector equation $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$, its length (traversed exactly once)

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt = \int_a^b \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}\,dt$$

$$= \int_a^b |\vec{r}\,'(t)|\,dt .$$

Example: Find the length of the arc of the circular helix with vector equation $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ from the point $(1,0,0)$ to the point $(1,0,2\pi)$.



$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

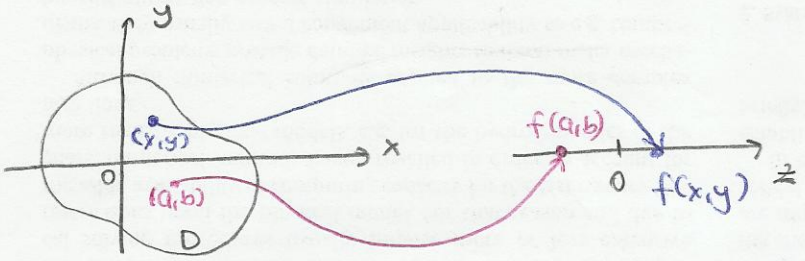
$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

The arc from $(1,0,0)$ to $(1,0,2\pi)$ is described by the parameter interval $0 \le t \le 2\pi$.

PARTIAL DERIVATIVES

- **Functions of two variables** A function f of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number denoted by $f(x,y)$.

D: domain of f Range of $f = \{f(x,y) \mid (x,y) \in D\}$



We often write as

$z = f(x,y)$

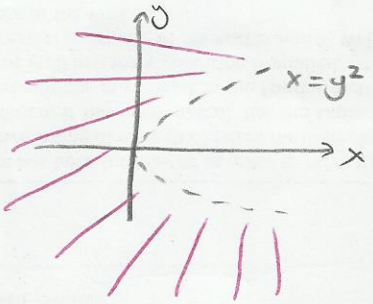
x, y independent variables

z dependent variable

Example: Let $f(x,y) = x\ln(y^2 - x)$. Evaluate $f(1,2)$ and find the domain.

$f(1,2) = \ln(3)$.

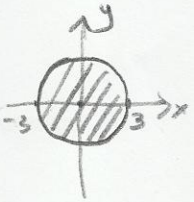
$\ln(y^2 - x)$ is defined only when $y^2 - x > 0 \Rightarrow x < y^2$. The domain of f is $D = \{(x,y) \mid x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$.



Example: Find the domain and range of $f(x,y) = \sqrt{9 - x^2 - y^2}$.

$D = \{(x,y) \mid 9 - x^2 - y^2 \geq 0\} = \{(x,y) \mid x^2 + y^2 \leq 9\}$ disk centered at the origin with radius 3

domain



$R = \{z \mid z = \sqrt{9 - x^2 - y^2},\ (x,y) \in D\}$

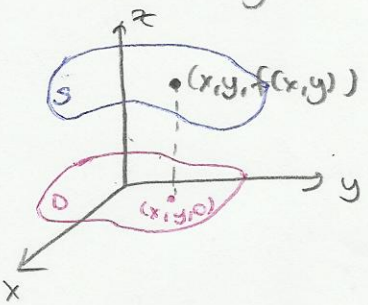
range

z is a positive square root $z \geq 0$.

Also, $9 - x^2 - y^2 \leq 9 \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$

So the range is $R = \{z \mid 0 \leq z \leq 3\} = [0,3]$.

Graphs: If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ s.t. $z=f(x,y)$ and (x,y) is in D.



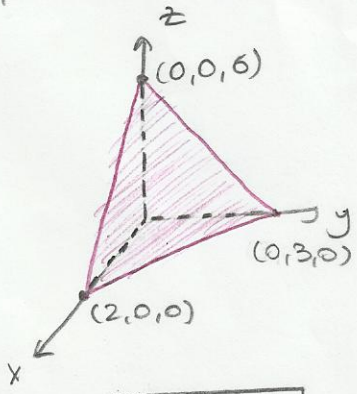
Example: Sketch the graph of the function $f(x,y)=6-3x-2y$.

$z=6-3x-2y \Rightarrow 3x+2y+z=6$ represents a plane.

Put $y=z=0 \Rightarrow x=2$ x-intercept

Put $x=z=0 \Rightarrow y=3$ y-intercept

Put $x=y=0 \Rightarrow z=6$ z-intercept

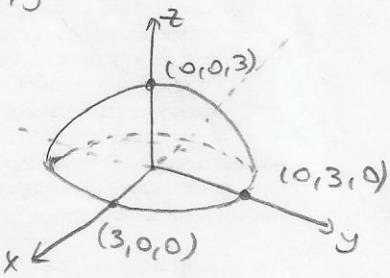


$f(x,y)=ax+by+c$ **(linear function)**

Example: Sketch the graph of the function $f(x,y)=\sqrt{9-x^2-y^2}$.

$z=\sqrt{9-x^2-y^2} \Rightarrow x^2+y^2+z^2=9$ the equation of the sphere centered at the origin with radius 3.

BUT, note that $z \geq 0$. Hence the graph of $f(x,y)$ is the top half of this sphere (upper hemisphere)

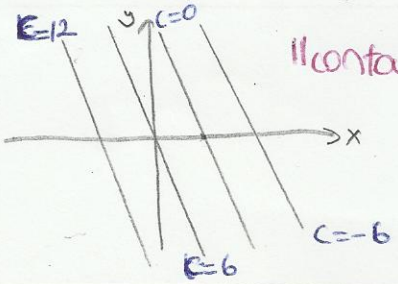


Level curves: The level curves of a function $f(x,y)$ are the curves with equations $f(x,y)=c$, c is a constant (in the range of f). It shows where the graph of f has height c.

Example: Sketch the level curves of the function $f(x,y)=6-3x-2y$ for the values $c=-6,0,6,12$.

$6-3x-2y=c \Rightarrow 3x+2y+(c-6)=0$

This is the family of lines with slope $-3/2$.



"contour map"

Functions of three or more variables:

A function of three variables f, is a rule that assigns to each ordered triple $(x,y,z) \in D \subset \mathbb{R}^3$ a unique real number denoted by $f(x,y,z)$.

Example: Find the domain of f if $f(x,y,z) = \underline{\ln(z-y)} + xy\sin z$.

$f(x,y,z)$ is defined for $\underline{z-y>0}$, so the domain of f

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid z > y\}.$$

This is a half-space consisting of all points that lie above the plane $z=y$.

Level surface: Level surfaces are the surfaces with equations $f(x,y,z)=c$, c is a constant.

Example: Find the level surfaces of the function

$$f(x,y,z) = x^2+y^2+z^2.$$

$x^2+y^2+z^2=c$, $c \geq 0$. These form a family of sphered centered at the origin with radius $\sqrt{c}$.

Limits and Continuity

Definition: Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x,y) \in D$ and $0 < \underbrace{\sqrt{(x-a)^2+(y-b)^2}}_{\text{distance between the point } (x,y) \text{ and the point } (a,b)} < \delta$ then $\underbrace{|f(x,y)-L|}_{\text{distance between the numbers } f(x,y) \text{ and } L} < \varepsilon$

Remark: The limit definition says that the distance between $f(x,y)$ and L can be made arbitrarily small by making the distance from (x,y) to (a,b) sufficiently small. The definition refers only to the distance between (x,y) and (a,b). It DOES NOT refer to the direction of approach. Therefore, if the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b).

* If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1, and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example: Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

$f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$. First, let us approach $(0,0)$ along the x-axis then $y=0$. This gives $f(x,0) = \dfrac{x^2}{x^2} = 1$ for all $x \neq 0$, so

$f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the x-axis.

Approach along the y-axis by putting $x=0$.

$f(0,y) = \dfrac{-y^2}{y^2} = -1$, for all $y \neq 0$. So

$f(x,y) \to -1$ as $(x,y) \to (0,0)$ along the y-axis.

Since f has two different limits along two different lines, the given limit does not exist.

Example: $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = ?$, $f(x,y) = \dfrac{xy}{x^2+y^2}$

Approach along x-axis; $f(x,0) = \dfrac{0}{x^2} = 0$, $x \neq 0$

Approach along y-axis; $f(0,y) = \dfrac{0}{y^2} = 0$, $x \neq 0$.

we have identical limits along the axes, BUT that does not show that the given limit is 0.

Approach along $y=x$; $f(x,x) = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

Since, we have obtained different limits along different paths, the given limit does not exist.

Example: 1) $\lim\limits_{(x,y)\to(0,2)} \dfrac{x - xy + 3}{x^2y - y^2} = \dfrac{0 - 0.2 + 3}{0.2 - 2^2} = -\dfrac{3}{4}$.

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = ?$

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{(\sqrt{x} - \sqrt{y})} \frac{(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})} = \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{(x-y)}} = 0.$$

Example: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4} = ?$, $f(x,y) = \dfrac{xy^2}{x^2+y^4}$

By keeping in mind the second example on page 45, let us consider $(x,y) \to (0,0)$ along any nonvertical line through the origin. Let $y = mx$.

$$f(x,y) = f(x, mx) = \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2x}{1 + m^4x^2}, \quad x \neq 0$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along $y = mx$.

Thus f has the same limiting value along every nonvertical line through the origin. BUT that does not show that the give limit is 0. For instance, if we let $(x,y) \to (0,0)$ along $x = y^2$, we have

$$f(y^2, y) = \frac{y^4}{y^4 + y^4} = \frac{1}{2}. \qquad f(x,y) \to \frac{1}{2} \text{ as } (x,y) \to (0,0) \text{ along } x = y^2$$

Since different paths lead to different limiting values, the given limit does not exist.

Sertöz Theorem: Let $a_1, \dots, a_N$ be nonnegative integers, $m_1, \dots, m_N$ be positive integers, and $c_1, \dots, c_N$ be positive real numbers, where $N > 1$. Then $\lim\limits_{(x_1,\dots,x_N)\to(0,\dots,0)} \dfrac{x_1^{a_1} \cdots x_N^{a_N}}{c_1 x_1^{2m_1} + \dots + c_N x_N^{2m_N}}$ exists iff $\sum\limits_{i=1}^{N} \dfrac{a_i}{2m_i} > 1$

Moreover, when the limit exists, then it is zero.

$\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x^3y^2z^2}{x^4 + y^{12} + z^{14}}$; $\dfrac{3}{4} + \dfrac{2}{12} + \dfrac{2}{14} = \dfrac{89}{84} > 1$, so the limit exist and is zero.

Continuity

A function $f(x,y)$ is called continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

We say that f is continuous on D if f is continuous at every point (a,b) in D.

(A surface that is the graph of a continuous function has no hole or break.)

Note: All polynomials are continuous on $\mathbb{R}^2$.

Any rational function $\left(\text{polynomial}/\text{polynomial}\right)$ is continuous on ITS DOMAIN.

Example. 1) $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 11.$

$f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, continuous everywhere, the limit can be found by direct substitution.

2) where is the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

The function is not continuous at $(0,0)$, because it is not defined at $(0,0)$. $f(x,y)$ is continuous at its domain that is $D = \{(x,y) \mid (x,y) \neq (0,0)\}$.

3) $h(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ $h(x,y)$ is defined at $(0,0)$ but it is not continuous since

we proved that $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist on page 45.

4) $f(x,y) = \dfrac{xy}{x+y}$ is a rational function. It is not defined for $y=-x$. It is continuous at its domain that is $D = \{(x,y) \mid y \neq -x\}$.

5) $\lim_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z) = 3.4.(-1) + 1.2.\cos(2\pi + \pi) = -14$

x^2z : polynomial, continuous
yx : polynomial, continuous
$\cos(\pi x - \pi z)$: continuous

6) Show that $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

is continuous at every point except the origin.

The function is continous for any $(x,y) \neq (0,0)$, what about at $(x,y) = (0,0)$?

Approach along $y = mx$; $f(x, mx) = \dfrac{2mx^2}{x^2 + m^2x^2} = \dfrac{2m}{1+m^2}$

Hence $f(x,y) \to \dfrac{2m}{1+m^2}$ as $(x,y) \to (0,0)$ along $y = mx$.

This limit changes with m. Therefore, there is no single number for the limit of $f(x,y)$ as $(x,y) \to (0,0)$.

Hence the function is not continuous at $(0,0)$.

Partial Derivative

The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim\limits_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h} = \left. f_x \right|_{(x_0,y_0)}$

partial derivative of $f(x,y)$ with respect to y at (x_0, y_0):

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h} = \left. f_y \right|_{(x_0,y_0)}.$$

- To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
- To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Example: 1) $w(x,y,z) = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

$w_x = 2xy + 43$, $w_y = x^2 - 20yz^3 - 28\sec^2(4y)$, $w_z = -30y^2z^2$

2) $h(s,t) = t^7 \ln(s^2) + \dfrac{9}{t^3} - \sqrt[7]{s^4}$

$h_t = 7t^6 \ln(s^2) - \dfrac{27}{t^4}$, $h_s = t^7 \cdot \dfrac{2s}{s^2} - \dfrac{4}{7} s^{-3/7}$.

3) Find z_x and z_y if $x^2\sin(2y-5z) = 1 + y\cos(6zx)$, $z = z(x,y)$

$$x^2\sin(2y-5z) = 1 + y\cos(6zx) \quad (*)$$

Differentiate $(*)$ wrt x

$$2x\sin(2y-5z) - 5x^2\cos(2y-5z)\,z_x = -y\,(6z + 6x z_x)\sin(6zx)$$

$$z_x\{-5x^2\cos(2y-5z) + 6xy\sin(6zx)\} = -2x\sin(2y-5z) - 6zy\sin(6z$$

$$z_x = \frac{2x\sin(2y-5z) + 6zy\sin(6zx)}{5x^2\cos(2y-5z) - 6xy\sin(6zx)}.$$

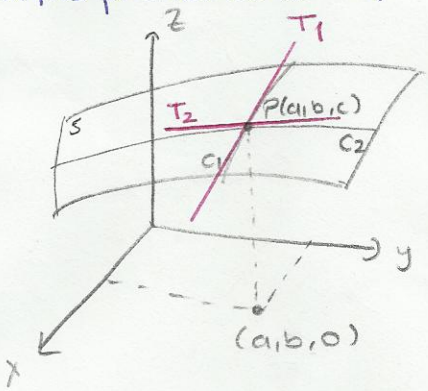
Differentiate $(*)$ wrt y

$$(2-5z_y)x^2\cos(2y-5z) = \cos(6zx) - 6xyz_y\sin(6zx)$$

$$z_y\{-5x^2\cos(2y-5z) + 6xy\sin(6zx)\} = \cos(6zx) - 2x^2\cos(2y-5z)$$

$$z_y = \frac{\cos(6zx) - 2x^2\cos(2y-5z)}{6xy\sin(6zx) - 5x^2\cos(2y-5z)}.$$

Interpretations of Partial Derivatives



$z = f(x,y)$ represents a surface S.
If $f(a,b) = c$, then the point $P(a,b,c)$ lies on S.

Fix $y = b$, so we restrict our attention to the vertical plane $y = b$ curve C_1 in which the intersects S.

Similarly, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P.

C_1: graph of the function $g(x) = f(x,b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a,b)$.

C_2: graph of the function $h(y) = f(a,y)$, so the slope of its tangent T_2 at P is $h'(b) = f_y(a,b)$.

Thus the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

Partial derivatives can also be interpreted as rates of change.

$f_x(x,y)$ represents the rate of change of the function $f(x,y)$ as we change x and hold y fixed. Similar for $f_y(x,y)$.

Example: Determine if $f(x,y) = \frac{x^2}{y^3}$ is increasing or decreasing at $(2,5)$,

a) if we allow x to vary and hold y fixed.

b) if we allow y to vary and hold x fixed.

(a) $f_x(x,y)\Big|_{(2,5)} = \frac{2x}{y^3}\Big|_{(2,5)} = \frac{4}{125} > 0$ the function is increasing at $(2,5)$ as we vary x, and hold y fixed

(b) $f_y(x,y)\Big|_{(2,5)} = -\frac{3x^2}{y^4}\Big|_{(2,5)} = -\frac{12}{625} < 0$ the function is decreasing at $(2,5)$ as we vary y, and hold x fixed.

Example: Find the slopes of the traces to $z = 10 - 4x^2 - y^2$ at the point $(1,2)$.

As we mentioned on page 49, partial derivatives are the slopes of the traces C_1, C_2.

C_2 → trace of $f(x,y)$ for the plane $x=a$

C_1 → trace of $f(x,y)$ for the plane $y=b$

So we need the two partial derivatives so we can get the slopes.

$f_x(x,y) = -8x$, $f_y(x,y) = -2y$.

$f_x(x,y)\Big|_{(1,2)} = -8$, $f_y(x,y)\Big|_{(1,2)} = -4$

So the tangent line at $(1,2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $y=2$ has a slope of -8. The tangent line at $(1,2)$ for the trace to $z = 10 - 4x^2 - y^2$ for the plane $x=1$ has a slope of -4.

Higher Derivatives

$$(f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad (f_x)_y = \frac{\partial^2 f}{\partial x \partial y}, \quad (f_y)_x = \frac{\partial^2 f}{\partial y \partial x}, \quad (f_y)_y = \frac{\partial^2 f}{\partial y^2}$$

Example: Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2.$$

$f_x = 3x^2 + 2xy^3$, $\quad f_y = 3x^2y^2 - 4y$

$f_{xx} = 6x + 2y^3$ $\quad f_{yy} = 6x^2y - 4$

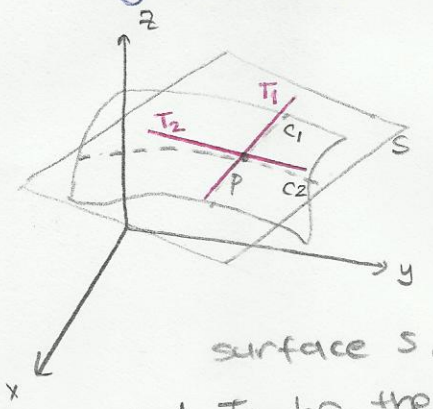
$f_{xy} = 6xy^2$ $\quad f_{yx} = 6xy^2$

Notice that $f_{xy} = f_{yx}$. This is true for most of the functions but there are conditions for this equality to happen.

Clairaut's Theorem: (Known also as Euler's Theorem) Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both <u>continuous</u> on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Tangent Planes and Linear Approximations



Suppose a surface S has equation $z = f(x,y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the tangent plane to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.

Recall that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Divide by c.

$$z - z_0 = -\frac{a}{c}(x-x_0) - \frac{b}{c}(y-y_0) = A(x-x_0) + B(y-y_0)$$

$$A = -a/c, \quad B = -b/c$$

If this equation represents the tangent plane at P, then its intersection with the plane $y=y_0$ must be the tangent line T_1. Set $y=y_0$, $z - z_0 = A(x-x_0)$. A is the slope of this line, and from page 49 we know that $A = f_x(x_0, y_0)$.

By similar argument, $B = f_y(x_0, y_0)$.

Hence the equation of the tangent plane to the surface $z = f(x,y)$ at (x_0, y_0) is then

$$\boxed{z - z_0 = f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

Since $z_0 = f(x_0, y_0)$ we can write the above equation as

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

Example: Find the equation of the tangent plane to $z = \ln(2x+y)$ at $(-1,3)$.

$$f(x,y) = \ln(2x+y) \Rightarrow f_x = \frac{2}{2x+y}, \quad f_y = \frac{1}{2x+y}$$

$$f_x(-1,3) = 2, \quad f_y(-1,3) = 1.$$

so the equation of the tangent plane at $(-1,3)$ is

$$z - 0 = 2(x+1) + 1(y-3) \Rightarrow z = 2x + y - 1.$$

- One use of tangent planes is they give us a way to approximate a surface near a point. As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point. Because of this we define the linear approximation (or tangent plane approximation) to be

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

and as long as we are near (x_0, y_0) we have

$$f(x,y) \approx L(x,y).$$

Example: Find the linear approximation to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4,3)$.

$f(x,y) = 3 + \frac{x^2}{16} + \frac{y^2}{9}$, $f_x(x,y) = \frac{x}{8}$ $\Rightarrow f_x(-4,3) = -1/2$

$f(-4,3) = 5$ $\qquad f_y(x,y) = \frac{2y}{9}$ $\Rightarrow f_y(-4,3) = 2/3$

The tangent plane or linear approximation is then

$$L(x,y) = 5 - \frac{1}{2}(x+4) + \frac{2}{3}(y-3).$$

Theorem: If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Example: Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

$f_x = e^{xy} + xye^{xy}$, $f_y = x^2 e^{xy}$ are continuous functions so f is differentiable.

$f_x(1,0) = 1$, $f_y(1,0) = 1$.

$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = x + y$.

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

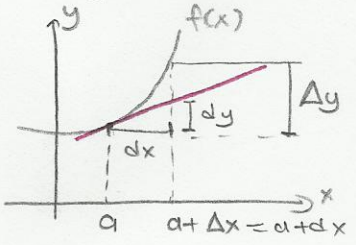
so $\quad f(1.1, -0.1) \approx 1.1 - 0.1 = 1$.

Indeed, the actual value of $f(1.1, -0.1) = 1.1\, e^{-0.11} \approx 0.98542$.

Differentials

For a differentiable function of one variable, $y = f(x)$, we define the differential dx to be an independent variable. The differential of y is defined as

$$dy = f'(x)\,dx.$$



Δy represents the change in height of the curve $y = f(x)$

dy represents the change in height of the tangent line when x changes by an amount $dx = \Delta x$.

For a differentiable function of two variables, $z = f(x,y)$, we define the differentials dx and dy to be independent variables. Then the differential dz (also called the total differential) is defined by,

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$, then

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

So $f(x,y) \approx L(x,y) = f(a,b) + dz$.

- dz represents the change in height of the tangent plane when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.
- Δz represents the change in height of the surface $z = f(x,y)$ when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.

Example: a) If $z = f(x,y) = x^2 + 3xy - y^2$, find dz.

b) If x changes from 2 to 2.05 and y changes from 3 to 2.96 compare the values of Δz and dz.

(a) $dz = (2x + 3y)\,dx + (3x - 2y)\,dy$.

(b) $\left.\begin{array}{l} x = 2,\ dx = \Delta x = 2.05 - 2 = 0.05 \\ y = 3,\ dy = \Delta y = 2.96 - 3 = -0.04 \end{array}\right\} \Rightarrow dz = 0.65$.

The increment of $z = \Delta z$ is

$$\Delta z = f(2.05, 2.96) - f(2,3) = 0.6449.$$

Note that $\Delta z \approx dz$ but dz is easier to compute.

Example: The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Let the dimensions of the box are x, y, z. So the volume is $V = xyz$.

$$dV = yz\,dx + xz\,dy + xy\,dz$$

We are given that $|\Delta x| \le 0.2$, $|\Delta y| \le 0.2$, $|\Delta z| \le 0.2$. To find the largest error in the volume, we take $dx = 0.2$, $dy = 0.2$, $dz = 0.2$, with $x = 75$, $y = 60$, $z = 40$.

Hence $\Delta V \approx dV = 1980$.

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of as much as $1980\ cm^3$ in the calculated volume.

The Chain Rule

I Suppose that $z = f(x,y)$ is differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example: If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

By the chain rule, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

$$= (2\sin 2t \cos t + 3\cos^4 t)(2\cos 2t) - (\sin^2 2t + 12\sin 2t \cos^3 t)\sin t$$

$$\left.\frac{dz}{dt}\right|_{t=0} = 6.$$

This derivative can be interpreted as the rate of change of z with respect to t as the point (x,y) moves along the curve C with parametric equations $x = \sin 2t$, $y = \cos t$.

II Suppose that $z=f(x,y)$ is a differentiable function of x and y, where $x=g(s,t)$ and $y=h(s,t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Example: If $g(s,t)=f(s^2-t^2,\ t^2-s^2)$ and f is differentiable, show that g satisfies the equation $t\frac{\partial g}{\partial s}+s\frac{\partial g}{\partial t}=0$.

Let $x=s^2-t^2$, $y=t^2-s^2$. Then $g(s,t)=f(x,y)$.

$$\frac{\partial g}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}=2s\frac{\partial f}{\partial x}-2s\frac{\partial f}{\partial y} \quad / \text{ multiply by } t$$

$$\frac{\partial g}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}=-2t\frac{\partial f}{\partial x}-2t\frac{\partial f}{\partial y} \quad / \text{ multiply by } s$$

Add

$$t\frac{\partial g}{\partial s}+s\frac{\partial g}{\partial t}=0.$$

Example: If $z=f(x,y)$ has continuous second-order partial derivatives and $x=r^2+s^2$ and $y=rs$, find $\frac{\partial^2 z}{\partial r^2}$.

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=2r\frac{\partial z}{\partial x}+s\frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial r^2}=\frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x}+s\frac{\partial z}{\partial y}\right)=2\frac{\partial z}{\partial x}+2r\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial x\partial y}\frac{\partial y}{\partial r}\right)$$

$$+s\left(\frac{\partial^2 z}{\partial y\partial x}\frac{\partial x}{\partial r}+\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\right) \quad \boxed{z_{xy}=z_{yx}}$$

$$=2\frac{\partial z}{\partial x}+4r^2\frac{\partial^2 z}{\partial x^2}+4rs\frac{\partial^2 z}{\partial x\partial y}+s^2\frac{\partial^2 z}{\partial y^2}$$

Implicit Differentiation

Suppose that an equation of the form $F(x,y)=0$ defines y implicitly as a differentiable function of x, $y=f(x)$.

If F is differentiable, we can use the Chain Rule to differentiate both sides of $F(x,y)=0$, $y=f(x)$.

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}}$$

If we have an equation $F(x,y,z)=0$ where $z=f(x,y)$, F and f are differentiable then

$$\frac{\partial F}{\partial x}\underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}}$$

Similarly, $\boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$.

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz=1$.

$F(x,y,z) = x^3+y^3+z^3+6xyz-1 = 0$, $z=f(x,y)$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2+6yz)}{3z^2+6xy}, \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(3y^2+6xz)}{3z^2+6xy}.$$

Directional Derivatives and the Gradient vector

Recall that if $z=f(x,y)$

$$f_x(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}, \quad f_y(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

represent the rates of change of z in the x- and y-directions that is in the directions of the unit vectors $\vec{i}$ and $\vec{j}$.

To find the rate of change of z at (x_0,y_0) in the direction of an arbitrary unit vector $\vec{u} = \langle a,b \rangle$, we introduce the directional derivative which is defined as

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+ha,\, y_0+hb) - f(x_0,y_0)}{h} \quad , \vec{u} = \langle a,b \rangle$$

$$|\vec{u}| = 1 .$$

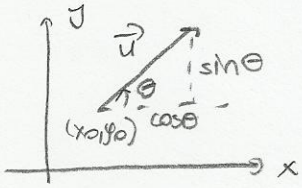
Remark: If $\vec{u} = \vec{i} = \langle 1,0 \rangle$, then $D_{\vec{i}} f = f_x$, and if $\vec{u} = \vec{j} = \langle 0,1 \rangle$, the $D_{\vec{j}} f = f_y$.

Theorem: If f is a differentiable function of x and y and f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a,b \rangle$ and

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b.$$

Remark: If the unit vector $\vec{u}$ makes an angle θ with the positive x-axis, then we can write $\vec{u} = \langle \cos\theta, \sin\theta \rangle$, so we have

$$D_{\vec{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$



Example: Find the directional derivative $D_{\vec{u}} f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

and $\vec{u}$ is the unit vector given by angle $\theta = \pi/6$. What is $D_{\vec{u}} f(1,2)$?

Use the above formula. $\vec{u} = \langle \cos \pi/6, \sin \pi/6 \rangle$

$f_x = 3x^2 - 3y$, $f_y = -3x + 8y$

$$D_{\vec{u}} f(x,y) = (3x^2 - 3y)\cos \pi/6 + (-3x + 8y)\sin \pi/6$$

$$= (3x^2 - 3y)\,\sqrt{3}/2 + (-3x + 8y)\,1/2$$

$$D_{\vec{u}} f(1,2) = -3\sqrt{3}/2 + 13/2 .$$

The Gradient Vector

$$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle$$

$$= \underbrace{\langle f_x(x,y), f_y(x,y)\rangle}_{\text{gradient of } f} \cdot \vec{u}.$$

Notation

$$\text{grad} f = \nabla f = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \qquad (\nabla : \text{nabla})$$

If $f = f(x,y,z)$, $\text{grad} f = \nabla f = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$.

Example: Find the directional derivative of $f(x,y,z) = \sin(yz) + \ln(x^2)$ at $(1,1,\pi)$ in the direction of $\vec{v} = \langle 1,1,-1\rangle$.

$$\text{grad} f = \nabla f = \langle f_x, f_y, f_z\rangle = \langle \frac{2x}{x^2}, z\cos(yz), y\cos(yz)\rangle$$

$$\text{grad} f \Big|_{(1,1,\pi)} = \nabla f(1,1,\pi) = \langle 2, -\pi, -1\rangle.$$

We need the unit vector for the direction:

$$|\vec{v}| = \sqrt{3} \Rightarrow \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\rangle$$

$$D_{\vec{u}} f(1,1,\pi) = \langle 2, -\pi, -1\rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\rangle = \frac{3-\pi}{\sqrt{3}}.$$

$$\mathbf{D_{\vec{u}} f = \text{grad} f \cdot \vec{u} = |\text{grad} f| \cos\theta, \ \theta \in [0,\pi]}$$

$(|\vec{u}| = 1)$

θ is the angle between the vectors grad f and $\vec{u}$.

- The function f increases most rapidly when $\cos\theta = 1$ (or when $\vec{u}$ is the direction of grad f, $\theta = 0$).

  $D_{\vec{u}} f = |\text{grad} f| \cos(0) = |\text{grad} f|$ determines how fast the function increases in the direction of grad f.

- Similarly, f decreases most rapidly when $\cos\theta = -1$ (or when $\vec{u}$ is the direction of $-$grad f, $\theta = \pi$)

  $D_{\vec{u}} f = |\text{grad} f| \cos(\pi) = -|\text{grad} f|$

- Any direction $\vec{u}$ orthogonal to a gradient grad $f \neq 0$ is a direction of zero change in f ($\theta = \pi/2$)

  $D_{\vec{u}} f = |\text{grad} f| \cos(\pi/2) = 0.$

Example: Suppose that the height of a hill above sea level is given by $z = 1000 - 0.01x^2 - 0.02y^2$. If you are at the point $(60, 100)$ in what direction is the elevation changing fastest? What is the maximum rate of change of the elevation at this point?

$$z = f(x,y) = 1000 - 0.01x^2 - 0.02y^2$$

$$\text{grad} f = \nabla f = \langle -0.02x, -0.04y \rangle$$

The maximum rate of change of the elevation will then occur in the direction of $\text{grad} f(60,100) = \nabla f(60,100) = \langle -1.2, -4 \rangle$ and the maximum rate $|\text{grad} f| = \sqrt{(-1.2)^2 + (-4)^2} = \sqrt{17.44}$.

Gradient as Surface Normal Vector

Theorem: Let f be a differentiable function in space. Let $f(x,y,z) = c$ represent a surface S. Then if $\text{grad} f \neq 0$ at a point P of S, it is a normal vector of S at P.

$f(x,y,z) = c$ level surface of f, c constant.

Proof: Let C be a curve parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ on S through a point P of S. Since C is on S, we have

$$f(x(t), y(t), z(t)) = c$$

A tangent vector of C is $\vec{r}\,'(t) = \langle x(t), y(t), z(t) \rangle$.

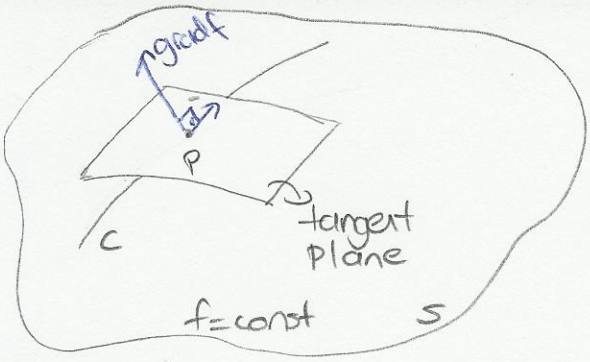
Tangent vectors of all curves on S passing through P form a tangent plane.

$$f(x(t), y(t), z(t)) = c$$

Take derivative wrt t.

$$\frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) + \frac{\partial f}{\partial z} z'(t) = 0 \qquad \left(\frac{dc}{dt} = 0 \right)$$

$$\Rightarrow \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle}_{\text{grad} f} \cdot \underbrace{\langle x'(t), y'(t), z'(t) \rangle}_{\vec{r}\,'(t)} = 0$$



Hence grad f and $\vec{r}'$ are perpendicular to each other. So it is a normal vector of S at the point P.

[Equation of the tangent plane at $P:(x_0,y_0,z_0)$

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$]

for a special case $z=F(x,y) \Rightarrow \underbrace{F(x,y)-z}_{f(x,y,z)}=0 \Rightarrow f_z=-1$

Check the page 52

The normal line to S at P is the line passing through P and perpendicular to the tangent plane. So the direction of the normal line is given by grad $f(x_0,y_0,z_0)$ and so the equation of the line is

$$\frac{x-x_0}{f_x(x_0,y_0,z_0)}=\frac{y-y_0}{f_y(x_0,y_0,z_0)}=\frac{z-z_0}{f_z(x_0,y_0,z_0)}$$

Example: Find the equation of the tangent plane to the surface $x^2-y^2+z^2=1$ at the point $P=(2,2,1)$.

$f(x,y,z)=x^2-y^2+z^2-1=0$, grad $f=\langle 2x,-2y,2z\rangle$

grad $f(P)\cdot\langle x-2,y-2,z-1\rangle=0$

$\langle 4,-4,2\rangle\cdot\langle x-2,y-2,z-1\rangle=4(x-2)-4(y-2)+2(z-1)=0.$

Example: Find a unit normal vector $\vec{u}$ of $z^2=y^2-x^2$ at the point $(2,2,1)$.

$f(x,y,z)=z^2-y^2+x^2$, $f(2,2,1)=1$.

So the point $(2,2,1)$ is on the level surface S given by

$f(x,y,z)=z^2-y^2+x^2=1$.

Since grad f is a normal vector of S at point $(2,2,1)$, find

grad $f(P)=\langle 4,-4,2\rangle \Rightarrow$ unit vector $\vec{u}=\dfrac{\text{grad} f(P)}{|\text{grad} f(P)|}=\langle 2/3,-2/3,1/3\rangle$

Maximum and Minimum Values

Definition:

$f(x,y)$ has a local maximum (relative) at (a,b) if $f(x,y) \leq \underbrace{f(a,b)}_{\text{local maximum value}}$ when (x,y) is near (a,b)

$f(x,y)$ has a local minimum (relative) at (a,b) if $f(x,y) \geq \underbrace{f(a,b)}_{\text{local minimum value}}$ when (x,y) is near (a,b)

If the inequalities hold for ALL points (x,y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a,b).

Theorem: If f has a local maximum or minimum at (a,b) and the first order partial derivatives of f exist there, then

$f_x(a,b)=0$ and $f_y(a,b)=0$. $(\text{grad} f(a,b) = \nabla f(a,b) = 0)$.

Definition: An interior point (a,b) of the domain of $f(x,y)$ is called a critical point (or stationary point) of f if both $f_x(a,b)=0$ and $f_y(a,b)=0$, or if one of these partial derivatives does not exist.

Warning! Not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or NEITHER.

Example: $f(x,y) = x^2+y^2-2x-6y+14$

$$\left.\begin{aligned} f_x &= 2x-2=0 \Rightarrow x=1 \\ f_y &= 2y-6=0 \Rightarrow y=3 \end{aligned}\right\} \text{Only critical point is } (1,3).$$

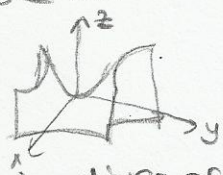
$f(x,y) = (x-1)^2+(y-3)^2+4$. Since $(x-1)^2 \geq 0$ and $(y-3)^2 \geq 0$ we have $f(x,y) \geq 4$ for all values of x and y. Then $f(1,3)=4$ is a local minimum, and in fact it is the absolute minimum of f.

Example: $f(x,y) = y^2-x^2$.

$$\left.\begin{aligned} f_x &= -2x=0 \Rightarrow x=0 \\ f_y &= 2y=0 \Rightarrow y=0 \end{aligned}\right) \text{Only critical point is } (0,0).$$

Notice that on the x-axis, we have $y=0$ so $f(x,y)=-x^2<0$ (if $x\neq 0$). On the y-axis, we have $x=0$ so $f(x,y)=y^2>0$ (if $y\neq 0$)

Hence every disk with center $(0,0)$ contains where f takes positive and also negative values. Therefore $f(0,0)=0$ cannot be an extreme value for f, so f has NO extreme value.



Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that (a,b) is a critical point of $f(x,y)$. Define

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-f_{xy}^2(a,b), \quad D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

Then,

1) If $D>0$ and $f_{xx}(a,b)>0$ then there is a local (relative) min. at (a,b)
2) If $D>0$ and $f_{xx}(a,b)<0$ then there is a local (relative) max. at (a,b)
3) If $D<0$ then (a,b) is a saddle point (in every open disk centered at (a,b) there are domain points where $f(x,y)>f(a,b)$ and domain points (x,y) where $f(x,y)<f(a,b)$)
4) If $D=0$, inconclusive.

In 1) and 2), we can replace $f_{yy}(a,b)$ instead of $f_{xx}(a,b)$.

Example: Find and classify all the critical points of $f(x,y)=4+x^3+y^3-3xy$

$f_x=3x^2-3y \Rightarrow y=x^2$

$f_y=3y^2-3x$

$3x^4-3x=3x(x^3-1)=3x(x-1)(x^2+x+1)=0$

$\begin{pmatrix} x=0 \\ y=0 \end{pmatrix} \begin{pmatrix} x=1 \\ y=1 \end{pmatrix}$

Hence $(0,0)$ and $(1,1)$ are critical points.

$f_{xx}=6x$, $f_{yy}=6y$, $f_{xy}=-3$

$D(x,y)=36xy-9$

At $(0,0)$, $D=D(0,0)=-9<0$ so $(0,0)$ is a saddle point.

At $(1,1)$, $D=D(1,1)=27>0$ and $f_{xx}(1,1)=6>0$, hence at $(1,1)$, $f(x,y)$ has a local minimum and local minimum value of $f(x,y)$ is $f(1,1)=3$.

Exercise: Find and classify all the critical points for

$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

Example: Determine the point on the plane $4x - 2y + z = 1$ that is closest to the point $(-2,-1,5)$.

The distance between the desired point and $(-2,-1,5)$ is given by

$$d = \sqrt{(x+2)^2 + (y+1)^2 + (z-5)^2}.$$

Since $z = 1 - 4x + 2y$, we have

$$d(x,y) = \sqrt{(x+2)^2 + (y+1)^2 + (-4x+2y-4)^2}$$

Finding the minimum value of $d(x,y)$ is equivalent to finding the minimum value of $d^2(x,y)$. So consider the function

$$f(x,y) = d^2(x,y) = (x+2)^2 + (y+1)^2 + (-4x+2y-4)^2$$

$$\left.\begin{aligned} f_x &= 2(x+2) - 8(-4x+2y-4) = 34x - 16y + 36 = 0 \\ f_y &= 2(y+1) + 4(-4x+2y-4) = 10y - 16x - 14 = 0 \end{aligned}\right\} \Rightarrow \left(\frac{-34}{21}, \frac{-25}{21}\right)$$

is the only critical point.

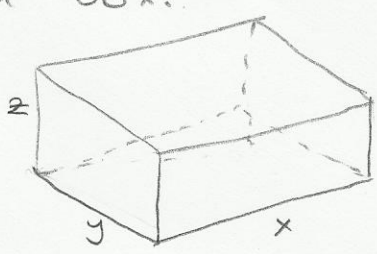
$$\left.\begin{aligned} f_{xx} &= 34 \\ f_{yy} &= 10 \\ f_{xy} &= -16 \end{aligned}\right\} D = 34 \cdot 10 - (-16)^2 = 84 > 0$$

Since $D > 0$ and $f_{xx} > 0$, we know that this critical point is a local minimum. The z-coordinate is $z = 1 - 4\left(-\frac{34}{21}\right) + 2\left(-\frac{25}{21}\right) = \frac{107}{21}$.

So the point on the plane that is closest to $(-2,-1,5)$ is $\left(\frac{-34}{21}, \frac{-25}{21}, \frac{107}{21}\right)$.

Exercise: A rectangular box without a lid is to be made from $12m^2$ of cardboard. Find the maximum volume of such a box.

$V = xyz$.



Absolute Maximum and Minimum Values

Definitions:

- A region in $\mathbb{R}^2$ is called closed if it includes its boundary. A region is called open if it does not include any of its boundary points.
- A region in $\mathbb{R}^2$ is called bounded if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

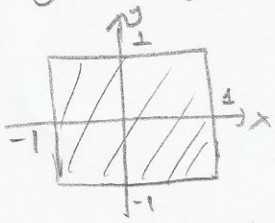
Closed sets sets that are not closed

Extreme Value Theorem: If $f(x,y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1,y_1) and (x_2,y_2) so that $f(x_1,y_1)$ is the absolute maximum and $f(x_2,y_2)$ is the absolute minimum of the function in D.

Finding Absolute Extrema

1) Find all the critical points of the function that lie in the region D and determine the function value at each of these points.
2) Find all extrema of the function on the boundary.
3) The largest and smallest values found in the first two steps are the absolute minimum and the absolute maximum of the function.

Example: Find the absolute minimum and absolute maximum of $f(x,y)=x^2+4y^2-2x^2y+4$ on the rectangle given by $-1\le x\le 1$ and $-1\le y\le 1$.



$f(x,y)=x^2+4y^2-2x^2y+4$

$f_x = 2x-4xy=0$

$f_y = 8y-2x^2=0$

$y=x^2/4$

$2x-x^3=0$

$x(2-x^2)=0$

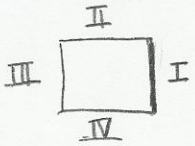
$x=0$, $x=\pm\sqrt{2}$

these points are not in the region.

Hence the critical point is $(0,0)$.

$$\boxed{f(0,0)=4.}$$

Now, we need to find the absolute extrema of the function along the boundary of the rectangle.



I : $x=1$, $-1 \le y \le 1$

$g(y) = f(1,y) = 1+4y^2-2y+4 = 4y^2-2y+5$

$g'(y) = 8y-2 = 0 \Rightarrow y = 1/4$ critical point

$g(1) = f(1,1) = 7$, $g(-1) = f(1,-1) = 11$, $g(1/4) = f(1, 1/4) = 19/4$

II : $y=1$, $-1 \le x \le 1$

$h(x) = f(x,1) = x^2+4-2x^2 = 8-x^2$

$h'(x) = -2x = 0 \Rightarrow x=0$. critical point

$h(-1) = f(-1,1) = 7$, $h(1) = f(1,1) = 7$, $h(0) = f(0,1) = 8$

III : $x=-1$, $-1 \le y \le 1$

$g(y) = f(-1,y) = 4y^2-2y+5$, $g'(y) = 8y-2 \Rightarrow y = 1/4$ critical point

$g(-1) = f(-1,-1) = 11$, $g(1) = f(-1,1) = 7$, $g(1/4) = f(-1, 1/4) = 19/4$

IV : $y=-1$, $-1 \le x \le 1$

$h(x) = f(x,-1) = x^2+4+2x^2+4 = 3x^2+8$

$h'(x) = 6x = 0 \Rightarrow x=0$ critical point.

$h(-1) = f(-1,-1) = 11$, $h(1) = f(1,-1) = 11$, $h(0) = f(0,-1) = 8$,

We have

$$\left.\begin{array}{l} f(0,0)=4 \\ f(1,1/4) = f(-1,1/4) = 19/4 \\ f(1,-1) = f(-1,-1) = 11 \\ f(-1,1) = f(1,1) = 7 \\ f(0,1) = f(0,-1) = 8 \end{array}\right\}$$

The absolute minimum is at $(0,0)$ since it gives the smallest function value and the absolute maximum occurs at $(1,-1)$ and $(-1,-1)$ since these two points give the largest function value.

Exercise: Find the absolute minimum and absolute maximum of $f(x,y) = 2x^2-y^2+6y$ on $x^2+y^2 \le 16$.

Lagrange Multipliers

We want to optimize a function $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$. The constraint may be the equation that describes the boundary of a region or it may not be.

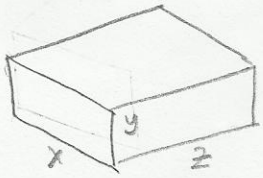
Method 1) Solve the following system of equations

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = k$$

2) Plug in all solutions (x,y,z) from the first step into $f(x,y,z)$ and identify the minimum and maximum values, if they exist

λ: constant called Lagrange multiplier

Example: Find the dimensions of the box with largest volume if the total surface area is $64\ cm^2$.



$V(x,y,z) = xyz$ volume of the box.

The constraint is given as the total surface area that is

$$2xy + 2yz + 2xz = 64 \Rightarrow \underbrace{xy+yz+xz}_{g(x,y,z)} = 32.$$

$$\nabla f = \lambda \nabla g$$
$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle$$

$$\left.\begin{array}{ll} \Rightarrow f_x = \lambda g_x & yz = \lambda(y+z) \\ \Rightarrow f_y = \lambda g_y & xz = \lambda(x+z) \\ \Rightarrow f_z = \lambda g_z & xy = \lambda(y+x) \end{array}\right\} \quad \begin{array}{l} xyz = \lambda x(y+z) \\ xyz = \lambda y(x+z) \\ xyz = \lambda z(y+x) \end{array} \begin{array}{l} \text{I} \\ \text{II} \end{array}$$

I: $\lambda x(y+z) = \lambda y(x+z)$

$\lambda(xz - yz) = 0$

$\boxed{\lambda = 0 \text{ or } xz = yz}$

$\lambda = 0$ is not possible since if this was the case we would have $yz = 0 \Rightarrow y = 0$ or $z = 0$.

So $xz = yz \Rightarrow x = y$ since $z \neq 0$.

II $\lambda y(x+z) = \lambda z(y+x) \Rightarrow \lambda(yx - zx) = 0 \underset{\substack{\lambda \neq 0 \\ x \neq 0}}{\Rightarrow} y = z.$

Hence $x = y = z$.

By the constraint, $y^2 + y^2 + y^2 = 32 \Rightarrow y = \pm \frac{\sqrt{32}}{\sqrt{3}}$

y defines a length so we take $y = \frac{\sqrt{32}}{\sqrt{3}} = x = z$.

Example: Find the maximum and minimum of $f(x,y)=5x-3y$ subject to constraint $x^2+y^2=136$.

Here we have a closed, bounded region. By the Extreme Value Theorem we know that a minimum and maximum value must exist.

$g(x,y)=x^2+y^2=136 \qquad \nabla f = \lambda \nabla g$

$f_x = \lambda g_x \Rightarrow 5 = \lambda 2x \Rightarrow x = 5/2\lambda$

$f_y = \lambda g_y \Rightarrow -3 = \lambda 2y \Rightarrow y = -3/2\lambda$

$x^2+y^2=136$

$\left(\frac{5}{2\lambda}\right)^2+\left(\frac{-3}{2\lambda}\right)^2 = \frac{34}{4\lambda^2} = 136 \Rightarrow \lambda^2 = 1/16 \Rightarrow \lambda = \pm 1/4$

If $\lambda = -1/4$, $x=-10$, $y=6$ $\Rightarrow f(-10,6) = -68$ Minimum at $(-10,6)$

If $\lambda = 1/4$, $x=10$, $y=-6$ $\Rightarrow f(10,-6) = 68$ maximum at $(10,-6)$

Example: Find the maximum and minimum values of $f(x,y)=4x^2+10y^2$ on the disk $x^2+y^2 \le 4$.

The constraint is the inequality for the disk. This is a closed-bounded region, by the Extreme Value Theorem we know that a minimum and maximum value must exist.

First find all the critical points that are in the disk.

$f_x = 8x = 0 \Rightarrow x=0$, $f_y = 20y = 0 \Rightarrow y=0$.

The only critical point is $(0,0)$ and it satisfies the constraint.

From this point, use Lagrange Multipliers. Treat the constraint as an equality instead of the inequality.

$f_x = \lambda g_x \Rightarrow 8x = 2\lambda x$; $f_y = \lambda g_y \Rightarrow 20y = 2\lambda y$

$2x(4-\lambda)=0$ $\qquad$ $2y(10-\lambda)=0$

$x=0$ or $\lambda=4$ $\qquad$ $y=0$ or $\lambda=10$

$\Downarrow$ $\qquad$ $\Downarrow$

$y=\pm 2$ $\qquad$ $x=\pm 2$

So we have four points to check: $(0,2)$, $(0,-2)$, $(2,0)$, $(-2,0)$

At critical point $f(0,0)=0$ minimum

$f(2,0)=f(-2,0)=16$

$f(0,2)=f(0,-2)=40$ maximum

Multiple Integrals

Here we extend the idea of a definite integral to double and triple integrals of functions of two or three variables.

- compute volumes, masses, and centroids of more general regions

To compute the double integrals we will use polar coordinates.

We also introduce two new coordinate systems in three-dimensional space
- cylindrical coordinates
- spherical coordinates.

Note: For the explanation of double integrals given by double Riemann sums, check your textbook.

A volume can be written as a double integral:

If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z=f(x,y)$ is

$$\boxed{V = \iint_R f(x,y)\,dA}$$

, $dA = dx\,dy$ or $dA = dy\,dx$

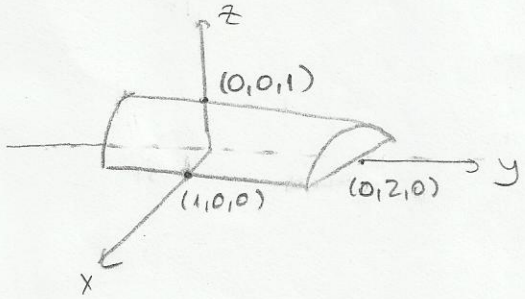
$R = [a,b] \times [c,d]$ i.e. $a \leq x \leq b$, $c \leq y \leq d$

Example: If $R = \{(x,y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1-x^2}\,dA.$$

$z = f(x,y) = \sqrt{1-x^2}$

$\Rightarrow x^2 + z^2 = 1$, and $z \geq 0$



It would be difficult to evaluate this integral directly. But because $\sqrt{1-x^2} \geq 0$ we can compute the integral by interpreting it as a volume.

So the given integral is the volume of the solid in figure that lies below the circular cylinder $x^2+z^2=1$ and above the rectangle R.

The volume of the solid = area of the semicircle $\times$ height

Hence

$$\iint_R \sqrt{1-x^2}\,dA = \frac{1}{2}\pi(1)^2 \times 4 = 2\pi.$$

Fubini's Theorem If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$ then

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy}$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

These integrals are called iterated integrals.

Properties of double integrals

1) $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

2) $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$, c a constant

3) If $f(x,y) \geq g(x,y)$, $\forall (x,y)\in R$, then $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

Examples Compute each of the following double integrals over the indicated rectangles.

1) $\iint_R [x^2y^2+\cos(\pi x)+\sin(\pi y)]\,dA$, $R=[-2,-1]\times[0,1]$

$$\iint_R [x^2y^2+\cos(\pi x)+\sin(\pi y)]\,dA = \int_0^1\int_{-2}^{-1}[x^2y^2+\cos(\pi x)+\sin(\pi y)]\,dx\,dy$$

$$=\int_0^1\left\{\left[\frac{x^3y^2}{3}+\frac{\sin(\pi x)}{\pi}+x\sin(\pi y)\right]_{-2}^{-1}\right\}dy$$

$$=\int_0^1\left\{\left(-\frac{1}{3}+\frac{8}{3}\right)y^2+\sin(\pi y)\right\}dy = \left(\frac{7}{3}\frac{y^3}{3}-\frac{\cos(\pi y)}{\pi}\right)\Big|_0^1 = \frac{7}{9}+\frac{1}{\pi}+\frac{1}{\pi}$$

$$=7/9+2/\pi$$

2) $\iint_R xe^{xy}\,dA$, $R=[-1,2]\times[0,1]$ It is easier to evaluate first with respect to y.

$$\iint_R xe^{xy}\,dA = \int_{-1}^{2}\int_0^1 xe^{xy}\,dy\,dx = \int_{-1}^{2}\left[\cancel{x}\frac{e^{xy}}{\cancel{x}}\right]_0^1 dx = \int_{-1}^{2}(e^x-1)\,dx$$

$$=(e^x-x)\Big|_{-1}^{2} = e^2-2-e^{-1}-1 = e^2-e^{-1}-3$$

3) Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$ and $y=2$, and the three coordinate planes.

S: the solid lies under the surface $z=16-x^2-2y^2$ and above the square $R=[0,2]\times[0,2]$. So

$$V=\iint_R (16-x^2-2y^2)\,dA=\int_0^2\int_0^2 (16-x^2-2y^2)\,dx\,dy=\int_0^2\left\{\left[16x-\frac{x^3}{3}-2y^2x\right]_0^2\right\}dy$$

$$=\int_0^2\left[\left(32-\frac{8}{3}\right)-4y^2\right]dy=\left(\frac{88}{3}y-\frac{4y^3}{3}\right)\Big|_0^2=\frac{176}{3}-\frac{32}{3}=48.$$

Special Case: If $f(x,y)=g(x)h(y)$ and $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA=\iint_R g(x)h(y)\,dA=\left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Example: Evaluate $\iint_R x\cos^2(y)\,dA$, $R=[-2,3]\times\left[0,\pi/2\right]$.

$$\iint_R x\cos^2(y)\,dA=\left(\int_{-2}^{3} x\,dx\right)\left(\int_0^{\pi/2}\cos^2(y)\,dy\right)$$

$\cos(2y)=2\cos^2(y)-1$

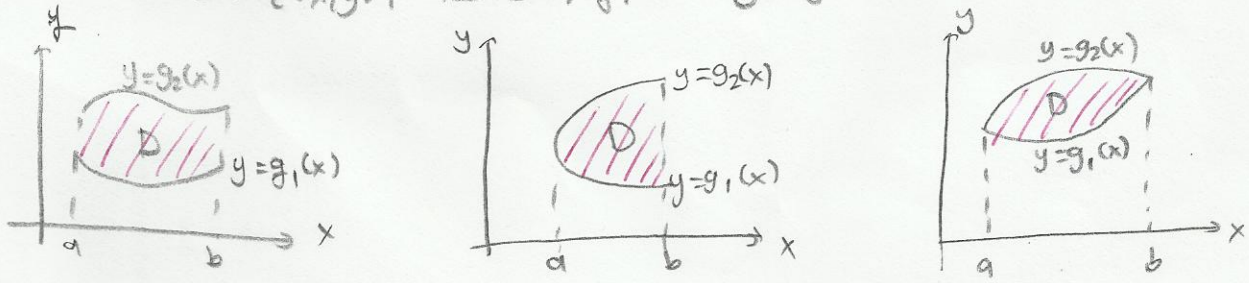
$$=\left(\frac{x^2}{2}\Big|_{-2}^{3}\right)\left(\int_0^{\pi/2}\left(\frac{\cos(2y)+1}{2}\right)dy\right)=\frac{5}{2}\left(\frac{\sin(2y)}{4}+\frac{y}{2}\right)\Big|_0^{\pi/2}=\frac{5\pi}{8}.$$

Double integrals over general regions

Most of the regions are not rectangular.

Type I Region: region that lies between the graphs of two continuous functions of x, that is

$$D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$$



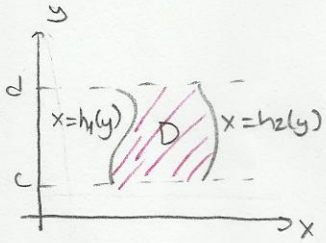
- If f is continuous on a type I region D s.t.

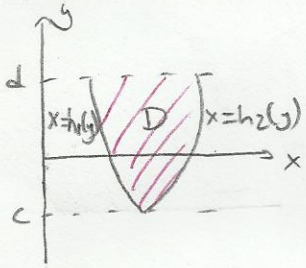
$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

then $$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Type II Region: region that lies between the graphs of two continuous function of y, that is

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$



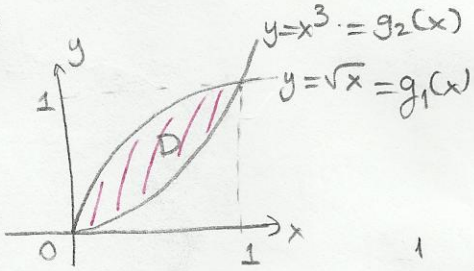


Then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Examples: Evaluate the following integrals over the given region D.

1) $\iint_D (4xy - y^3)\,dA$ D is the region bounded by $y=\sqrt{x}$ and $y=x^3$.



$0 \le x \le 1$, $x^3 \le y \le \sqrt{x}$

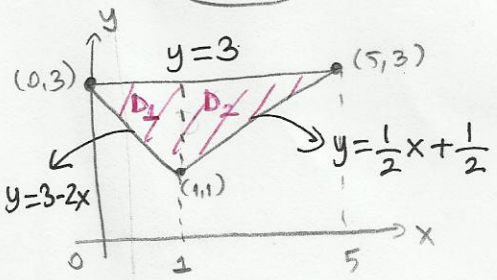
$$\iint_D (4xy - y^3)\,dA = \int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3)\,dy\,dx$$

$$= \int_0^1 \left\{ \left(2xy^2 - \frac{y^4}{4}\right)\Big|_{x^3}^{\sqrt{x}} \right\} dx = \int_0^1 \left(2x^2 - \frac{x^2}{4} - 2x^7 + \frac{x^{12}}{4}\right) dx$$

$$= \left(\frac{7x^3}{12} - \frac{x^8}{4} + \frac{x^{13}}{52}\right)\Big|_0^1 = \frac{7}{12} - \frac{1}{4} + \frac{1}{52} = \frac{55}{156}$$

2) $\iint_D (6x^2 - 40y)\,dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$

1. way



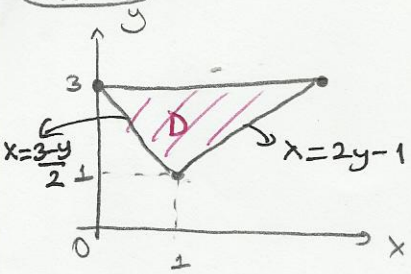
$D_1 = \{(x,y) \mid 0 \le x \le 1,\ 3-2x \le y \le 3\}$

$D_2 = \{(x,y) \mid 1 \le x \le 5,\ \frac{1}{2}x + \frac{1}{2} \le y \le 3\}$

$D = D_1 \cup D_2$

$$\iint_D (6x^2 - 40y)\,dA = \int_0^1 \int_{3-2x}^{3} (6x^2 - 40y)\,dy\,dx + \int_1^5 \int_{\frac{x+1}{2}}^{3} (6x^2 - 40y)\,dy\,dx$$

2. way



$y=3-2x \Rightarrow x=\frac{3-y}{2}$

$y=\frac{1}{2}x+\frac{1}{2} \Rightarrow x=2y-1$

$D=\{(x,y) \mid \frac{3-y}{2} \le x \le 2y-1,\ 1 \le y \le 3\}.$

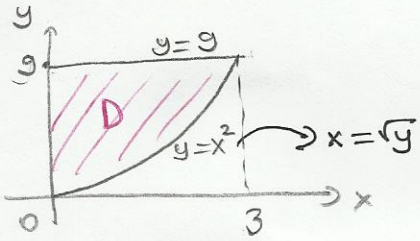
$$\iint_D (6x^2-40y)\,dA = \int_1^3 \int_{\frac{3-y}{2}}^{2y-1} (6x^2-40y)\,dx\,dy = \int_1^3 \left\{ \left[2x^3-40xy \right]_{(3-y)/2}^{2y-1} \right\} dy = -\frac{935}{3}$$

3) $\displaystyle\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\,dy\,dx$ If we try to integrate firstly with respect to y, we cannot evaluate it. $\boxed{\int e^{y^3}\,dy = ?}$

So we will reverse the order of integration. We have to change the limits of integration. First, sketch the region.

$0 \le x \le 3$

$x^2 \le y \le 9$



So define the region D as

$0 \le y \le 9$

$0 \le x \le \sqrt{y}$

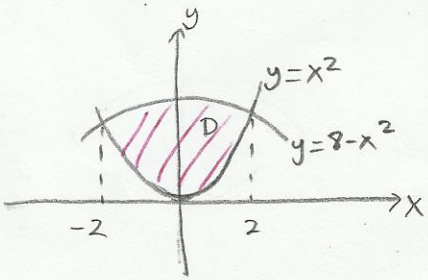
Therefore, $\displaystyle\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3}\,dx\,dy = \int_0^9 \left\{ \frac{x^4 e^{y^3}}{4} \Big|_0^{\sqrt{y}} \right\} dy$

$$= \frac{1}{4}\int_0^9 y^2 e^{y^3}\,dy = \frac{1}{12} e^{y^3} \Big|_0^9 = \frac{1}{12}\left(e^{729}-1\right).$$

Exercise: Evaluate $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\,dx\,dy$.

Example: Find the volume of the solid that lies below the surface given by $z=16xy+200$ and lies above the region xy-plane bounded by $y=x^2$ and $y=8-x^2$.

$y=x^2$
$y=8-x^2$) $8-x^2=x^2$
$2x^2=8 \Rightarrow x=\pm 2$
intersection points



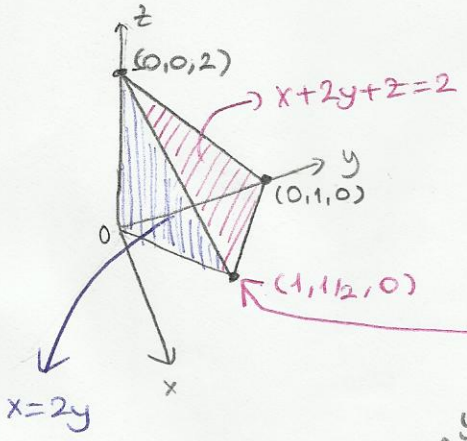
$-2 \le x \le 2$

$x^2 \le y \le 8-x^2$

$$V = \iint_D (16xy+200)\,dA = \int_{-2}^{2}\int_{x^2}^{8-x^2} (16xy+200)\,dy\,dx = \frac{12800}{3}.$$

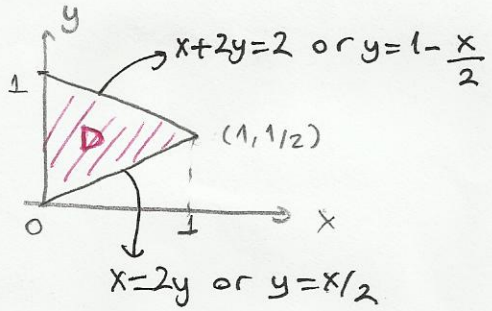
Example: Find the volume of the tetrahedron bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$.

Draw two diagrams: one of the three-dimensional solid and another of the plane region D over which it lies.



$x+2y+z=2$ intersect the xy-plane ($z=0$) in the line $\left.\begin{array}{l} x+2y=2 \\ x=2y \end{array}\right) \Rightarrow \underline{\underline{y=1/2 \Rightarrow x=1}}$

plane region D:



To sum up, the tetrahedron lies above the triangular region D in the xy-plane bounded by the lines $x=2y$, $x+2y=2$, and $x=0$.

$x+2y+z=2 \Rightarrow z=2-x-2y$. So the required volume lies under the graph of the function $z=2-x-2y$ and above

$D=\{(x,y) \mid 0\le x\le 1,\ x/2 \le y \le 1-x/2\}$.

Hence,

$$V = \iint_D (2-x-2y)\,dA = \int_0^1\int_{x/2}^{1-x/2} (2-x-2y)\,dy\,dx = 1/3$$

Remark:

If we integrate the constant function $f(x,y)=1$ over a region D, we get the area of D:

$$\boxed{\iint_D 1 \cdot dA = A(D)} = \text{area of } D.$$

- If $m \le f(x,y) \le M$ for all (x,y) in D, then

$$m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D)$$

Example: Estimate the integral $\iint_D e^{\sin x \cos y}\,dA$ where D is the disk with center origin and radius 2.

Since $-1 \le \sin x \le 1$, $-1 \le \cos y \le 1$, we have $-1 \le \sin x \cos y \le 1$.

Hence

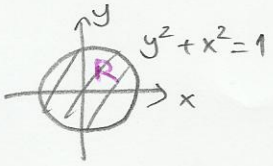
$$\underbrace{e^{-1}}_{m} \le e^{\sin x \cos y} \le \underbrace{e}_{M}$$

So by the above property,

$$m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D) \Rightarrow 4\pi e^{-1} \le \iint_D e^{\sin x \cos y}\,dA \le 4\pi e.$$

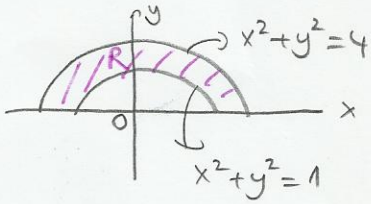
$A(D) = \pi r^2 = 4\pi$

Double integrals in polar coordinates Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$ where R is one of the regions as



$R = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

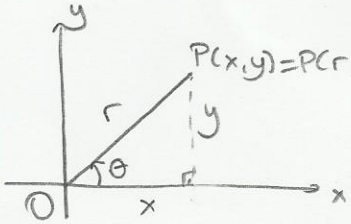
or



$R = \{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

In such cases it is rather complicated to define R by using rectangular coordinates but R is easily described by using polar coordinates.

Recall:



$r^2 = x^2+y^2$, $x = r\cos\theta$, $y = r\sin\theta$.

Change to polar coordinates in a double integral

If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta .$$

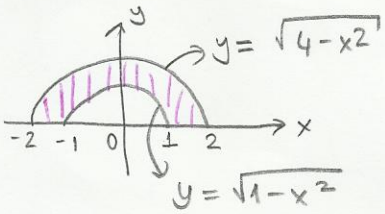
$\boxed{dA = r\,dr\,d\theta}$ $\quad x = r\cos\theta,\ y = r\sin\theta$ Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r .$$

$$dA = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta$$

For more information check the textbook.

Example: Evaluate $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$ $\quad x = r\cos\theta,\ y = r\sin\theta$

$$\iint_R (3x+4y^2)\,dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\,dr\,d\theta$$

$$= \int_0^\pi \left\{ \left[r^3\cos\theta + r^4\sin^2\theta \right]_1^2 \right\} d\theta = \int_0^\pi (7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$= \int_0^\pi \left[7\cos\theta + \frac{15}{2}(1-\cos 2\theta) \right] d\theta = \left(7\sin\theta + \frac{15}{2}\theta - \frac{15}{2}\frac{\sin 2\theta}{2} \right)\Bigg|_0^\pi$$

$$= 15\pi/2 .$$

Example: Evaluate $\iint_D e^{x^2+y^2}\,dA$, D is the unit circle centered at the origin.

$D=\{(r,\theta) \mid 0\le r\le 1,\ 0\le\theta\le 2\pi\}$, $x^2+y^2=r^2$

$$\iint_D e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{e^{r^2}}{2}\Big|_0^1\right)d\theta = \int_0^{2\pi}\left(\frac{e}{2}-\frac{1}{2}\right)d\theta$$

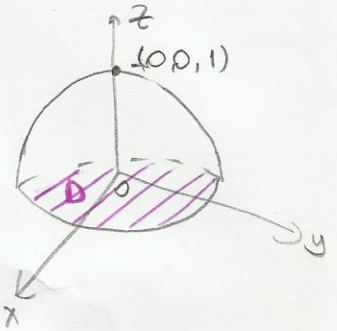
$r^2=u$
$2r\,dr=du$
$r\,dr=du/2$

$$=\left(\frac{e-1}{2}\right)\theta\Big|_0^{2\pi}$$

$$=(e-1)\pi.$$

Example: Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

$z=1-x^2-y^2=f(x,y)$



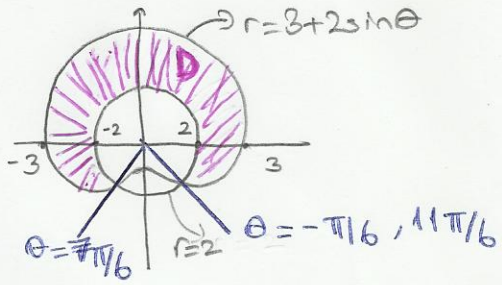
$D=\{(r,\theta) \mid 0\le r\le 1,\ 0\le\theta\le 2\pi\}$, $x^2+y^2=r^2$

$$V=\iint_D f(x,y)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \pi/2.$$

If you would use rectangular coordinates, you evaluate

$$\iint_D (1-x^2-y^2)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy\,dx.$$

Example: Remember the example on page 21.
Determine the area of the region lies inside $r=3+2\sin\theta$ and outside $r=2$.



Find the value of θ for which the two curves intersect.

$3+2\sin\theta=2 \Rightarrow \sin\theta=-1/2$

$\Downarrow$

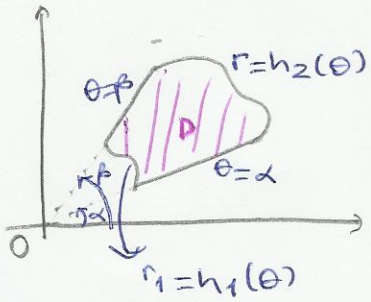
$\theta=7\pi/6,\ 11\pi/6$

To define the shaded region choose $-\pi/6$ instead of $11\pi/6$.

Hence the region is defined by, $-\pi/6\le\theta\le 7\pi/6$, $2\le r\le 3+2\sin\theta$

The area of the region D is then

$$A(D)=\iint_D dA = \int_{-\pi/6}^{7\pi/6}\int_2^{3+2\sin\theta} r\,dr\,d\theta = 11\sqrt{3}/2+14\pi/3.$$



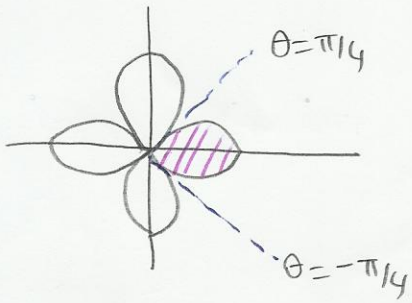
If f is continuous on a polar region of the form

$$D=\{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Example: Use a double integral to find the area enclosed by one loop of the four-leaved rose $r=\cos 2\theta$.



$$D=\{(r,\theta) \mid -\pi/4 \le \theta \le \pi/4,\ 0 \le r \le \cos 2\theta\}$$

$$A(D)=\iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{r^2}{2}\Big|_0^{\cos 2\theta} d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\,d\theta$$

$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4} (1+\cos 4\theta)\,d\theta = \frac{1}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]\Big|_{-\pi/4}^{\pi/4} = \pi/8.$$

The Triple Integrals

We define single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

$$\iiint_E f(x,y,z)\,dV$$

Start with the simple case. Consider the rectangular box

$$B=[a,b]\times[c,d]\times[r,s]$$

so the triple integral in this case is

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

Example: Evaluate the following integral

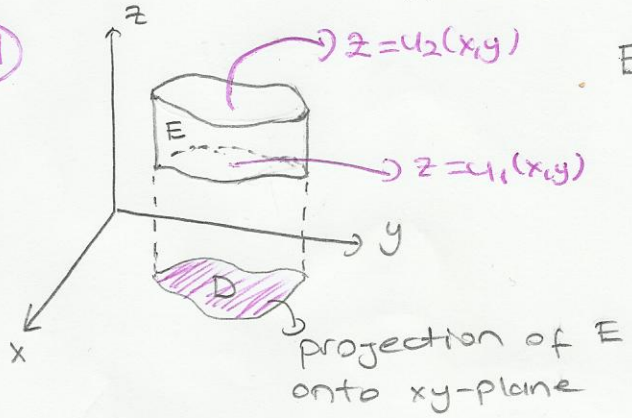
$$\iiint_B 8xyz\, dV, \quad B=[2,3]\times[1,2]\times[0,1].$$

$$\iiint_B 8xyz\, dV = \int_1^2\int_2^3\int_0^1 8xyz\, dz\,dx\,dy = \int_1^2\int_2^3 \left(4xyz^2\Big|_0^1\right) dx\,dy$$

$$= \int_1^2\int_2^3 4xy\, dx\,dy = \int_1^2 \left(2x^2y\Big|_2^3\right) dy = \int_1^2 10y\, dy = 5y^2\Big|_1^2 = 20-5=15.$$

We have three different possibilities for a general region.

1



$$E=\{(x,y,z) \mid (x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\}$$

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

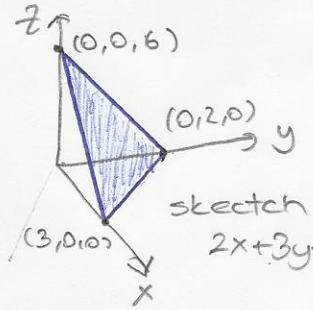
Example: Evaluate $\iiint_E 2x\, dV$ where E is the region under plane $2x+3y+z=6$ that lies in the first octant.

The first octant is the octant in which all three of the coordinat

are positive.

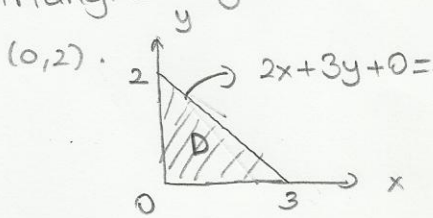
$2x+3y+z=6$

$x=y=0 \Rightarrow z=6$
$x=z=0 \Rightarrow y=2$
$y=z=0 \Rightarrow x=3$



sketch of the plane $2x+3y+z=6$.

We now determine the region D in the xy-plane. It is the triangle region in the xy-plane with vertices at (0,0), (3,0), and



$2x+3y+0=6 \Rightarrow x=\dfrac{6-3y}{2}$

or

$y=\dfrac{6-2x}{3}$

we need the limits of integration. Since we are under the plane and in the first octant (so we are above the plane $z=0$) we have $0 \le z \le 6-2x-3y$.

For x and y, we may use either $0 \le x \le 3$, $0 \le y \le \frac{6-2x}{3}$

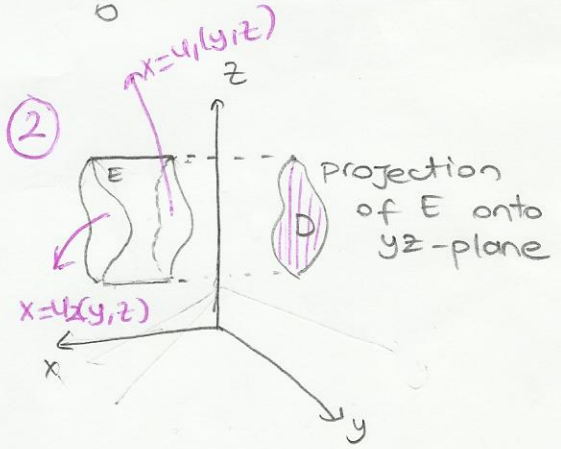
or $0 \le x \le \frac{6-3y}{2}$, $0 \le y \le 2$.

$$\iiint_E 2x\,dV = \iint_D \left[\int_0^{6-2x-3y} 2x\,dz \right] dA = \int_0^3 \int_0^{\frac{6-2x}{3}} \left(2xz \Big|_0^{6-2x-3y} \right) dy\,dx$$

$$= \int_0^3 \int_0^{\frac{6-2x}{3}} 2x(6-2x-3y)\,dy\,dx = \int_0^3 \left[(12xy - 4x^2y - 3xy^2) \Big|_0^{\frac{6-2x}{3}} \right] dx$$

$$= \int_0^3 \left(\frac{4}{3}x^3 - 8x^2 + 12x \right) dx = \left(\frac{x^4}{3} - \frac{8x^3}{3} + 6x^2 \right) \Big|_0^3 = 9.$$

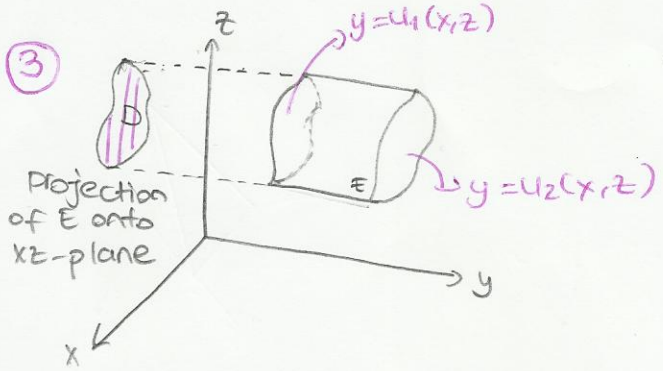
(2)



$$E = \{(x,y,z) \mid (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$$

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx \right] dA$$

(3)



$$E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$$

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \right] dA$$

Remark: Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y=f(x)$ from a to b; if $f(x,y) \geq 0$ then the double integral $\iint_D f(x,y)\,dA$ represents the volume under the surface $z=f(x,y)$ and above D.

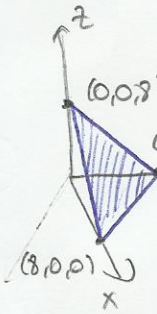
A triple integral $\iiint_E f(x,y,z)\,dV$, where $f(x,y,z) \geq 0$, is not very useful because it would be the hypervolume of a four-dimensional object and, of course, that is very difficult to visualize.

- For the special case where $f(x,y,z)=1$ for all points in D, the triple integral represents the volume of E:

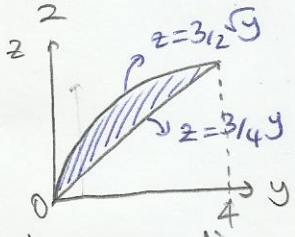
$$\boxed{V(E) = \iiint_E dV}$$

Examples

1) Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the yz-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$.



sketch of the plane $x+y+z=8$



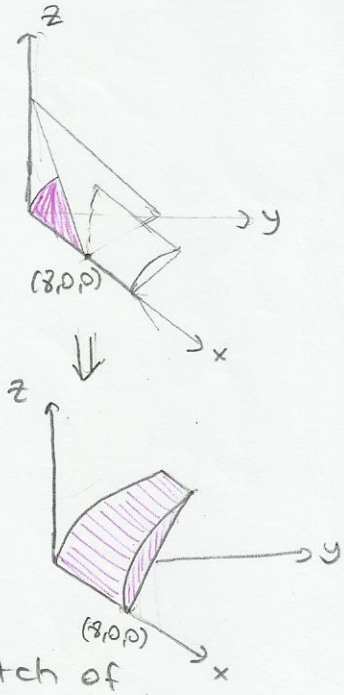
intersection points

$$\frac{3}{2}\sqrt{y} = \frac{3}{4}y$$

$$\Rightarrow y = 2\sqrt{y}$$

$$y^2 = 4y \Rightarrow y(y-4)=0$$

$$y=0,\ y=4.$$



sketch of the region.

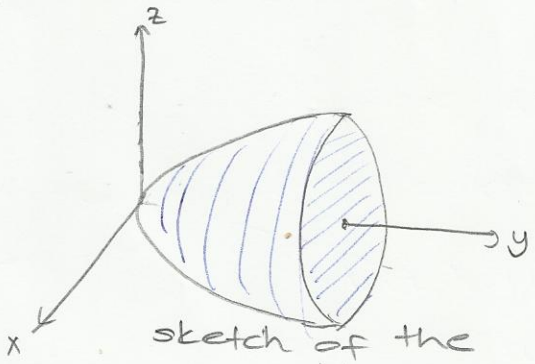
$$\Rightarrow 0 \leq y \leq 4$$

$$\frac{3}{4}y \leq z \leq \frac{3}{2}\sqrt{y}$$

$$0 \leq x \leq 8-y-z$$

$$\Rightarrow V = \iiint_E dV = \iint_D \left[\int_0^{8-y-z} dx\right] dA = \int_0^4 \int_{3y/4}^{3/2\sqrt{y}} (8-y-z)\,dz\,dy = 49/5.$$

2) Evaluate $\iiint_E \sqrt{3x^2+3z^2}\, dV$ where E is the solid bounded by $y = 2x^2+2z^2$ and the plane $y=8$.



sketch of the solid

The region D in the xz-plane can be found by "standing" in front of this solid and we can see that D is a disk in the xz-plane.

We can determine the equation of the disk by setting the elliptic paraboloid and the plane equal.

$$2x^2+2z^2=8 \Rightarrow x^2+z^2=4.$$

This region is a polar rectangle region hence it is better to define it by polar coordinates.

$$x^2+z^2=r^2=4 \Rightarrow \begin{array}{l} x=r\cos\theta \\ z=r\sin\theta \end{array} \qquad D=\{(r,\theta) \mid 0\le r\le 2,\ 0\le\theta\le 2\pi\}$$

So $E=\{(r,\theta,y) \mid (r,\theta)\in D,\ 2x^2+2z^2\le y\le 8\} = \{(r,\theta,y) \mid 0\le r\le 2,\ 0\le\theta\le 2\pi,\ 2x^2+2z^2\le y\le 8\}$.

$$\iiint_E \sqrt{3x^2+3z^2}\, dV = \iint_D \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\, dy \right] dA$$

$$= \iint_D \sqrt{3(x^2+z^2)}\,(8-(2x^2+2z^2))\, dA$$

$$\sqrt{3(x^2+z^2)}\,(8-(2x^2+2z^2)) = \sqrt{3}(8r-2r^3)$$

convert over the polar coordinates

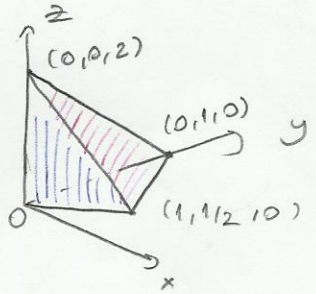
$$= \iint_D \sqrt{3}(8r-2r^3)\, dA$$

$$= \sqrt{3}\int_0^{2\pi}\int_0^2 (8r-2r^3)\, r\, dr\, d\theta = \frac{256\sqrt{3}\,\pi}{15}$$

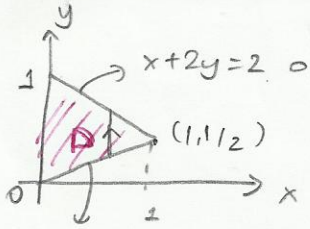
Example: Remember the example on page 74.

Now solve this question by using a triple integral.

"Find the volume of the tetrahedron T bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, and $z=0$."



on xy-plane



$x/2 \le y \le 1 - x/2$

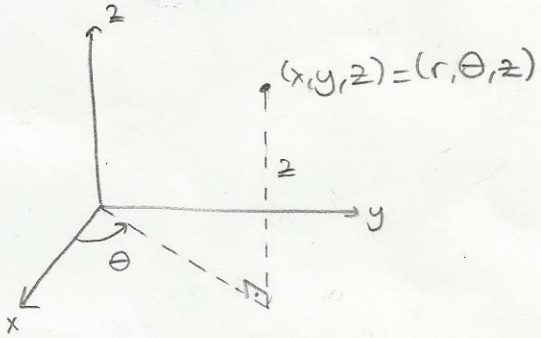
$x=2y$ or $y=x/2$

$0 \le x \le 1$

$x+2y+z=2 \Rightarrow 0 \le z \le 2-x-2y$

Hence $$V(T) = \iiint_T dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dV = 1/3$$

Triple integrals in cylindrical coordinates



Cylindrical coordinates are just an extension of polar coordinates into three dimensions.

The conversion formulas for cylindrical coordinates are

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}}$$

so we have $\boxed{dV = r\,dz\,dr\,d\theta}$

The region E over which we are integrating becomes,

$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$ (if D is on xy-plane)

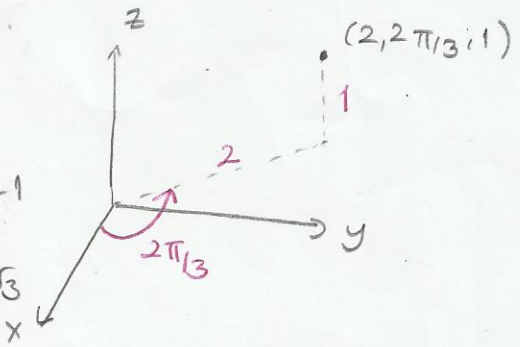
$= \{(r,\theta,z) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta),\ u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta)\}$

so a triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta.$$

Example:

a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.



$(r, \theta, z) = (2, 2\pi/3, 1)$

$x = r\cos\theta = 2\cos(2\pi/3) = 2\cdot(-1/2) = -1$

$y = r\sin\theta = 2\sin(2\pi/3) = 2(\sqrt{3}/2) = \sqrt{3}$

$z = 1$

$(-1, \sqrt{3}, 1)$ in rectangular coordinates.

b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

$(x, y, z) = (3, -3, -7)$ $\quad \left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right) \Rightarrow \tan\theta = y/x = -1$

$\theta = -\frac{\pi}{4} + 2n\pi \quad n \in \mathbb{Z}$

$x^2 + y^2 = r^2 \Rightarrow r^2 = 18 \Rightarrow r = 3\sqrt{2}$

Therefore one set of cylindrical coordinates is

$(r, \theta, z) = (3\sqrt{2}, -\pi/4, -7)$.

Example: Evaluate $\iiint_E y\,dV$ where E is the region that lies below the plane $z = x+2$, above the xy-plane, and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Use the cylindrical coordinates.

$0 \le z \le x+2 \Rightarrow 0 \le z \le r\cos\theta + 2$ (below the plane $z=x+2$, above the xy-plane)

$x^2+y^2=1,\ x^2+y^2=4 \Rightarrow 1 \le r \le 2,\ 0 \le \theta \le 2\pi$ (between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$)

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} \overset{y}{(r\sin\theta)}\, r\,dz\,dr\,d\theta = 0$$ (Make the calculations!)

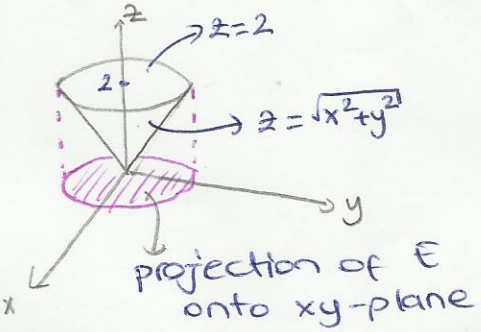
Example: Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$

This integral is a triple integral over the solid region

$$E=\{(x,y,z)\mid -2\le x\le 2,\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\ \sqrt{x^2+y^2}\le z\le 2\}$$

and the projection of E onto xy-plane is the disk $x^2+y^2\le 4$.

The lower surface of E is the cone $z=\sqrt{x^2+y^2}$ and its upper surface is the plane $z=2$.



To have a simpler triple integral we may use the cylindrical coordinates.

$x=r\cos\theta$
$y=r\sin\theta$

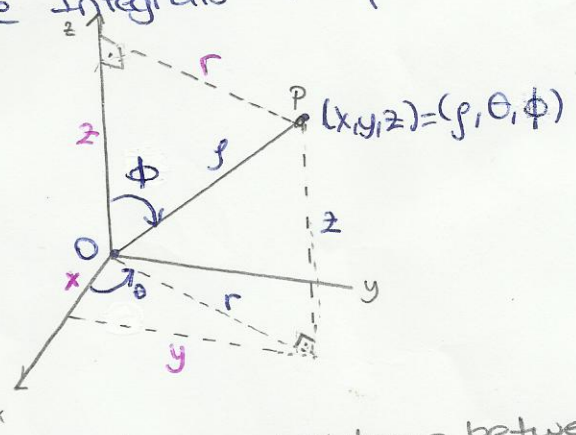
$\sqrt{x^2+y^2}\le z\le 2 \Rightarrow r\le z\le 2$

$\left.\begin{array}{l}0\le r\le 2\\ 0\le\theta\le 2\pi\end{array}\right)$ $\quad x^2+y^2=r^2$

Hence we have

$$\int_0^{2\pi}\int_0^{2}\int_r^{2} r^2\, r\,dz\,dr\,d\theta=\frac{16}{5}\pi.$$

Triple Integrals in Spherical Coordinates



- $\rho=|OP|$ distance from the origin to P
- θ same angle as in cylindrical coordinate from positive x-axis through the positive y-axis
- ϕ angle between the positive z-axis and the line segment OP

$$\rho\geq 0,\quad 0\le\phi\le\pi$$

For 1-to-1 correspondence between the rectangular coordinate system and spherical coordinate system, we take $0\le\phi\le\pi$ and $0\le\theta\le 2\pi$. These intervals can be taken differently.

$\dfrac{z}{\rho}=\cos\phi \Rightarrow z=\rho\cos\phi$; $\dfrac{r}{\rho}=\sin\phi \Rightarrow r=\rho\sin\phi$

We also have $x=r\cos\theta$, $y=r\sin\theta$. Hence to convert from spherical to rectangular coordinates we use the equations,

$$\boxed{x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi}\qquad \boxed{dV=\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

And we have

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

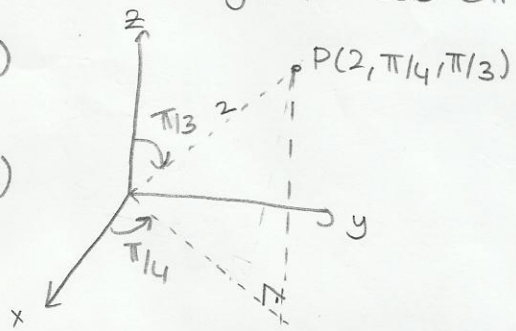
that is the equation of a sphere centered at the origin with radius ρ.

$$\left(x^2+y^2+z^2 = \underbrace{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta} + \rho^2 \cos^2\phi \right.$$

$$\left. = \rho^2 \sin^2\phi + \rho^2 \cos^2\phi = \rho^2 \right)$$

Example. a) The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

$$(\rho, \theta, \phi) = (2, \pi/4, \pi/3)$$



$$x = \rho \sin\phi \cos\theta = 2\sin(\pi/3)\cos(\pi/4)$$

$$= 2\,\frac{\sqrt{3}}{2}\,\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}$$

$$y = \rho \sin\phi \sin\theta = 2\sin(\pi/3)\sin(\pi/4) = \frac{\sqrt{6}}{2}$$

$$z = \rho\cos\phi = 2\cos(\pi/3) = 1$$

Thus the point $(2, \pi/4, \pi/3)$ is $(\sqrt{6}/2, \sqrt{6}/2, 1)$ in rectangular coordinates.

b) The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

$$(x, y, z) = (0, 2\sqrt{3}, -2) \Rightarrow \rho^2 = x^2 + y^2 + z^2 = 0 + 12 + 4 = 16$$

$$\Rightarrow \rho = 4$$

$$z = \rho\cos\phi \Rightarrow \cos\phi = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \phi = \frac{2\pi}{3}$$

$$x = \rho\sin\phi\cos\theta \Rightarrow \cos\theta = \frac{x}{\rho\sin\phi} = 0 \Rightarrow \theta = \pi/2$$

$(\theta \neq 3\pi/2$ since $y = r\sin\theta = 2\sqrt{3} > 0)$

Therefore spherical coordinates of the given point are $(\rho, \theta, \phi) = (4, \pi/2, 2\pi/3)$.

for 1-to-1 correspondence, we will not deal with $\theta = \pi/2 + 2k\pi$, $k \in \mathbb{Z}$

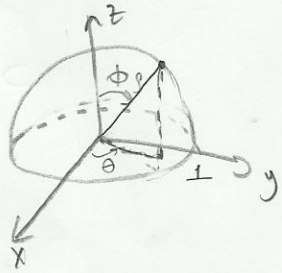
If E is a spherical wedge given by

$$E=\{(\rho,\theta,\phi) \mid a\le\rho\le b,\ \alpha\le\theta\le\beta,\ c\le\phi\le d\}$$

then

$$\iiint_E f(x,y,z)\,dV=\int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Example: Evaluate $\iiint_E 16z\,dV$ where E is the upper half of the sphere $x^2+y^2+z^2=1$.



$0\le\theta\le 2\pi$

$0\le\phi\le\pi/2$

$0\le\rho\le 1$

$z=\rho\cos\phi$

$$\iiint_E 16z\,dV=\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 16\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{2\pi}\int_0^1 8\rho^3\sin 2\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{2\pi} 2\sin 2\phi\,d\theta\,d\phi=\int_0^{\pi/2}4\pi\sin 2\phi\,d\phi=-2\pi\cos 2\phi\Big|_0^{\pi/2}$$

$$=2\pi(-1-1)=4\pi$$

Example: Convert $\displaystyle\int_0^3\int_0^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2+y^2+z^2\,dz\,dx\,dy$ into spherical coordinate

$$E=\{(x,y,z) \mid \underbrace{0\le y\le 3}, \underbrace{0\le x\le\sqrt{9-y^2}}, \sqrt{x^2+y^2}\le z\le\sqrt{18-x^2-y^2}\}$$

this says us that we have a portion of the right half of a disk of radius 3 centered at the origin. (quarter disk in the first quadrant) Hence we have $0\le\theta\le\pi/2$.

$z=\sqrt{18-x^2-y^2}\Rightarrow x^2+y^2+z^2=18=\rho^2\Rightarrow 0\le\rho\le\sqrt{18}=3\sqrt{2}$

The lower bound $z=\sqrt{x^2+y^2}$, is the upper half of a cone.

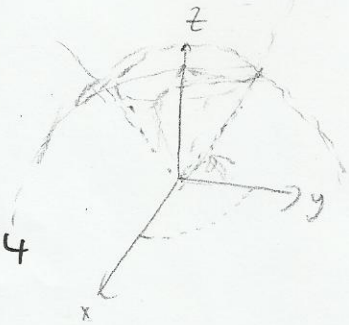
To find the range for ϕ, we'll use the intersection of the cone and the sphere.

$x^2+y^2+z^2=18\Rightarrow(\underbrace{\sqrt{x^2+y^2}}_{z})^2+z^2=18\Rightarrow 2z^2=18\Rightarrow z=3$ (z is positive since we have the upper half of the cone and/or sphere)

$z = \rho\cos\phi = 3$

$3\sqrt{2}\cos\phi = 3 \Rightarrow \cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \pi/4$

$0 \le \phi \le \pi/4$



So the integral becomes

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \underbrace{x^2+y^2+z^2}_{\rho^2} dz\,dx\,dy = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^{3\sqrt{2}} \rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

Change of Variables

Remember that in Calculus I, we had a substitution rule that is

$$\int_a^b f(g(x))\, g'(x)dx = \int_c^d f(u)du \quad , u = g(x).$$

Here we also want to approach double and triple integrals in a similar manner.

Recall that we did not mention how we obtained dA and dV in polar, cylindrical, or spherical coordinates.

Note that the reason for changing variables is to get an easier integral or convert the region into a nicer region to work with.

Before moving into changing variables with multiple integrals, let us first see how the region may change with a change of variables.

Example: 1) Determine the new region that we get by applying the given transformation to the region R.

a) R is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = u/2$, $y = 3v$.

$$x^2 + \frac{y^2}{36} = 1 \xrightarrow{x=u/2,\, y=3v} \frac{u^2}{4} + \frac{9v^2}{36} = 1 \Rightarrow u^2 + v^2 = 4$$

ellipse

circle of radius 2

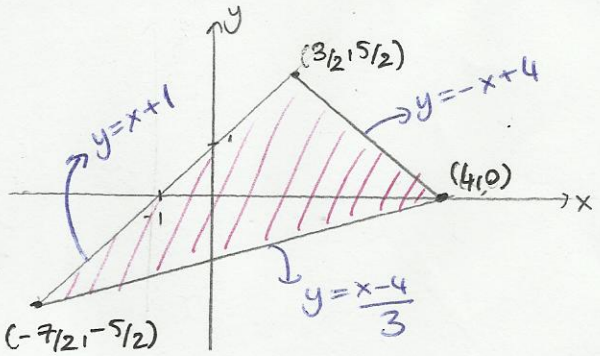
b) R is the region bounded by $y=-x+4$, $y=x+1$, and $y=\frac{x-4}{3}$ and the transformation is $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$.

intersection points of the lines

$-x+4=x+1 \Rightarrow x=3/2 \Rightarrow (3/2, 5/2)$

$-x+4=\frac{x-4}{3} \Rightarrow -3x+12=x-4 \Rightarrow x=4 \Rightarrow (4,0)$

$x+1=\frac{x-4}{3} \Rightarrow 3x+3=x-4 \Rightarrow x=-7/2 \Rightarrow (-7/2, -5/2)$



a triangle

Apply the transformation to each edge of the triangle.

- $y=x+1$; $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$

$\frac{1}{2}(u-v)=\frac{1}{2}(u+v)+1 \Rightarrow 2v=-2 \Rightarrow \boxed{v=-1}$

- $y=-x+4$; $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$

$\frac{1}{2}(u-v)=-\frac{1}{2}(u+v)+4 \Rightarrow \boxed{u=4}$

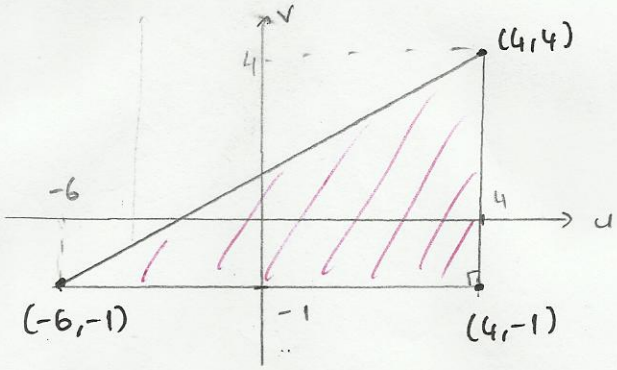
- $y=\frac{x-4}{3}$; $x=\frac{1}{2}(u+v)$, $y=\frac{1}{2}(u-v)$

$\frac{1}{2}(u-v)=\frac{1}{6}(u+v)-\frac{4}{3}$

$3u-3v=u+v-8 \Rightarrow 4v=2u+8$

$\boxed{v=\frac{1}{2}u+2}$

After this transformation, we get



a nicer triangle

$v=-1$, $u=4$, $v=\frac{1}{2}u+2$

intersection points

$-1=\frac{1}{2}u+2 \Rightarrow u=-6 \Rightarrow (-6,-1)$

$v=\frac{1}{2}\cdot 4+2 \Rightarrow v=4 \Rightarrow (4,4)$

Note that in these two examples, we take two dimensional regions and convert them into the regions that would be nicer in integrate over.

Now let us see how we do change of variables in the integral. start with the double integral. To do that we will give the definition of the Jacobian of the transformation.

Definition: The Jacobian of the transformation $x=g(u,v)$, $y=h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Change of Variables for a Double Integral

Suppose that we want to integrate $f(x,y)$ over the region R. Under the transformation $x=g(u,v)$, $y=h(u,v)$ the region becomes S and the integral becomes

$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(g(u,v),h(u,v)) \overbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}^{dA}.} \quad , \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \neq 0$$

Example: Show that when changing to polar coordinates we have $dA = r\,dr\,d\theta$.

$x=r\cos\theta$, $y=r\sin\theta$

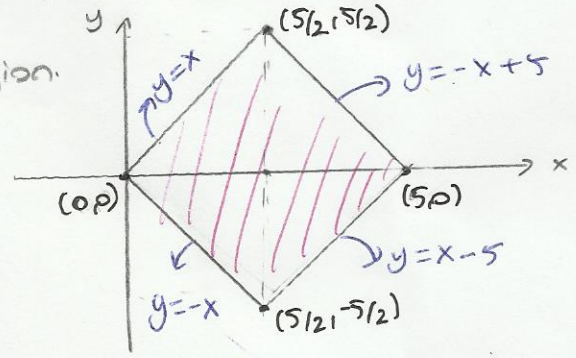
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

Hence $dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = |r|\,dr\,d\theta = r\,dr\,d\theta$.

Example: Evaluate $\iint_R (x+y)\,dA$ where R is the trapezoidal region (ikizkenar yamuk) with vertices given by $(0,0)$, $(5,0)$, $(5/2,5/2)$, and $(5/2,-5/2)$ using the transformation $x=2u+3v$ and $y=2u-3v$.

First, sketch the region.



Now apply the transformation.

$x=2u+3v$, $y=2u-3v$

- $y=x \Rightarrow 2u-3v=2u+3v$ $\boxed{v=0}$
- $y=-x \Rightarrow 2u-3v=-2u-3v$ $\boxed{u=0}$
- $y=-x+5 \Rightarrow 2u-3v=-2u-3v+5 \Rightarrow \boxed{u=5/4}$
- $y=x-5 \Rightarrow 2u-3v=2u+3v-5 \Rightarrow \boxed{v=5/6}$

So the region R becomes a rectangular region S defined by

$0 \le u \le 5/4$, $0 \le v \le 5/6$

What happens to dA? $\quad dA = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix} = -6-6 = -12$$

Then the double integral becomes

$$\iint_R (x+y)\,dA = \int_0^{5/6}\int_0^{5/4} (2u+3v+2u-3v)\cdot|-12|\,du\,dv = \int_0^{5/6}\int_0^{5/4} 48u\,du\,dv$$

$$= \int_0^{5/6}\left(24u^2\Big|_0^{5/4}\right)dv = 24\cdot\frac{25}{16}\,v\Big|_0^{5/6} = \cancel{24}^{\,4}\cdot\frac{25}{\cancel{16}_{\,4}}\cdot\frac{5}{\cancel{6}} = \frac{125}{4}.$$

Now we will look at triple integrals. Start with a region R and use the transformation $x=g(u,v,w)$, $y=h(u,v,w)$, and $z=k(u,v,w)$ to transform the region R into a new region S.

Here we will again use the Jacobian.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$\left|\dfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| \neq 0$

So the triple integral becomes

$$\boxed{\iiint_R f(x,y,z)\,dV = \iiint_S f(g(u,v,w),h(u,v,w),k(u,v,w))\underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}_{dV}.}$$

Example: Verify that $dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ when using spherical coordinates.

$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$.

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi\,[\underbrace{-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta}_{-\rho^2\sin\phi\cos\phi}] - \rho\sin\phi\,[\underbrace{\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta}_{\rho\sin^2\phi}$$

$$= -\rho^2\sin\phi$$

So we have

$$dV = |-\rho^2 \sin\phi|\, d\rho\, d\theta\, d\phi = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Remember that we take $0 \le \phi \le \pi$ hence $\sin\phi \geq 0$ so we do not need the absolute value.

Vector Calculus

Here we study the calculus of vector fields (These are functions that assign vectors to points in space.)

- line integrals (which can be used to find the work done by a force field in moving an object along a curve)
- surface integrals (which can be used to find the rate of fluid flow across a surface).

Definition: A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x,y) (or (x,y,z)) a two two (or three dimensional) vector given by $\vec{F}(x,y)$ (or $\vec{F}(x,y,z)$)

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} \quad \text{(in two dimensional space)}$$

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k} \quad \text{(in three dimensional space)}$$

P, Q, R component functions

Example: Velocity vector fields (wind, ocean currents)

Force fields (gravitational force)

Example: Sketch the vector field $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$.

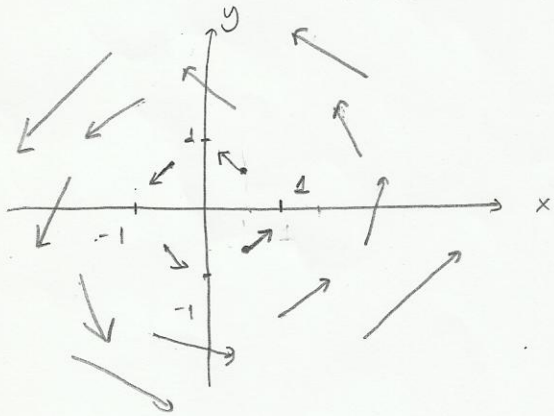
Let us get some "values" of the function.

$\vec{F}(1/2, 1/2) = -1/2\,\vec{i} + 1/2\,\vec{j}$ (at the point $(1/2, 1/2)$ plot the vector $-1/2\,\vec{i} + 1/2\,\vec{j}$. similar for the rest)

$\vec{F}(1/2, -1/2) = 1/2\,\vec{i} + 1/2\,\vec{j}$

$\vec{F}(3/2, 1/4) = -1/4\,\vec{i} + 3/2\,\vec{j}$

$\vdots$



Notice that we have already seen a vector field function.

$$\nabla f = \text{grad } f = \langle f_x, f_y, f_z \rangle$$ gradient vector field (in three-dim. space)

Example: Find the gradient vector field of $f(x,y,z) = z e^{-xy}$.

$$\nabla f = \text{grad } f = \langle -yze^{-xy}, -zye^{-xy}, e^{-xy} \rangle$$

Definition: A vector field $\vec{F}$ is called a conservative vector field if there exists a function f such that $\vec{F} = \nabla f$. If $\vec{F}$ is a conservative vector field then the function f is called a potential function for $\vec{F}$.

Example: Show that $\vec{F} = \langle 3x^2, 2yz, y^2 \rangle$ has a potential.

we have to show that there exists a function f such that

$\vec{F} = \nabla f = \text{grad } f$. $\quad \vec{F} = \langle 3x^2, 2yz, y^2 \rangle = \langle f_x, f_y, f_z \rangle$

$$\Rightarrow \quad f_x = 3x^2, \quad f_y = 2yz, \quad f_z = y^2$$

$$f_x = 3x^2 \Rightarrow f(x,y,z) = x^3 + h(y,z) \Rightarrow f_y = \frac{\partial h(y,z)}{\partial y} = 2yz$$

$$\Downarrow$$

$$h(y,z) = y^2 z + k(z)$$

Hence $f(x,y,z) = x^3 + y^2 z + k(z)$

$$f_z = y^2 + k'(z) = y^2 \Rightarrow k'(z) = 0 \Rightarrow k(z) = c \text{ constant}$$

we may choose $k(z) = 0$.

Hence $\vec{F} = \langle 3x^2, 2yz, y^2 \rangle$ has a potential $f(x,y,z) = x^3 + y^2 z$.

($\vec{F}$ is a conservative vector field)

Line Integrals

Here we will integrate the function $f(x,y)$ along a curve C which is smooth and given by the parametric equations $x = h(t)$, $y = g(t)$, $a \le t \le b$.

Generally, we use the parametrization of the curve as a vector function, $\vec{r}(t) = h(t)\vec{i} + g(t)\vec{j}$, $a \le t \le b$.

Definition: A curve parametrized by $\vec{r}(t)$ is called smooth if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \neq 0$ for all t.

The line integral of $f(x,y)$ along C is denoted by

$$\int_C f(x,y)\,ds$$

Remember that we used ds for the arclength of a curve $L = \int_a^b ds$, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$. Because of it the line integral is also called as a line integral of f with respect to arc length. And

$$\int_C f(x,y)\,ds = \int_a^b f(h(t), g(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

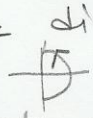
Parametrization of C: $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle, \quad |\vec{r}\,'(t)|^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\Rightarrow \boxed{\int_C f(x,y)\,ds = \int_a^b f(h(t), g(t))\,|\vec{r}\,'(t)|\,dt.}$$

Note: As long as the parametrization of the curve C is traced out exactly once as t increases from a to b the value of the line integral is independent of the parametrization of the curve.

Example: Evaluate $\int_C xy^4\,ds$ where C is the right half of the circle $x^2+y^2=16$ rotated in the counterclockwise direction.



$-\pi/2 \le t \le \pi/2$

- Parametrize the circle $x = 4\cos t$, $y = 4\sin t$ or $\vec{r}(t) = \langle 4\cos t, 4\sin t \rangle$

$\vec{r}\,'(t) = \langle -4\sin t, 4\cos t \rangle$

$$ds = |\vec{r}\,'(t)|\,dt = \sqrt{16\sin t + 16\cos^2 t}\,dt = 4\,dt$$

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} 4\cos t\,(4\sin t)^4\,dt = 4096 \int_{-\pi/2}^{\pi/2} \cos t\,(\sin t)^4\,dt$$

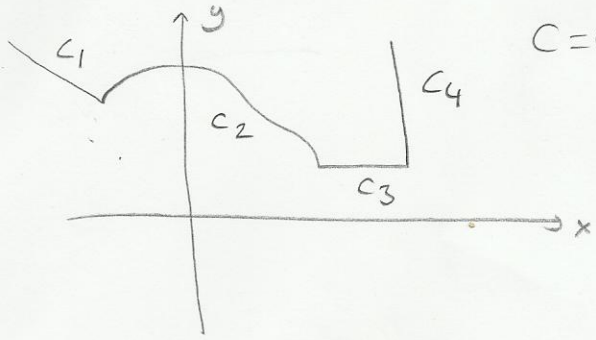
$\sin t = u$

$\cos t\,dt = du$

$$= 4096 \int_{-1}^{1} u^4\,du = \frac{4096}{5}(1-(-1)) = \frac{8192}{5}$$

line integrals over piecewise smooth curves

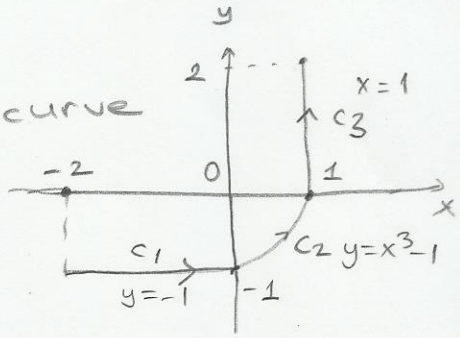
A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves.



$C = C_1 \cup C_2 \cup C_3 \cup C_4$

$$\int_C f(x,y)\,ds = \sum_{k=1}^{4} \int_{C_k} f(x,y)\,ds$$

Example: Evaluate $\int_C 4x^3\,ds$ where C is the curve



Parametrize the curves:

C_1: $x = t$, $y = -1$, $-2 \le t \le 0$

C_2: $x = t$, $y = t^3 - 1$, $0 \le t \le 1$

C_3: $x = 1$, $y = t$, $0 \le t \le 2$

$$\int_C 4x^3\,ds = \underbrace{\int_{-2}^{0} 4t^3 \cdot \sqrt{1^2+0^2}\,dt}_{C_1} + \underbrace{\int_0^1 4t^3 \cdot \sqrt{1^2+9t^4}\,dt}_{C_2} + \underbrace{\int_0^2 4 \cdot \sqrt{0^2+1^2}\,dt}_{C_3}$$

Example: Evaluate the same integral along the line segment from $(1,2)$ to $(-2,-1)$.

parametrization of the line segment:

$x = (1-t)x_0 + tx_1$, $y = (1-t)y_0 + ty_1$, $0 \le t \le 1$

$\vec{r}(t) = \langle (1-t)x_0 + tx_1, (1-t)y_0 + ty_1 \rangle = (1-t)\langle x_0, y_0 \rangle + t\langle x_1, y_1 \rangle$

$\vec{r}(t) = (1-t)\langle 1,2 \rangle + t\langle -2,-1 \rangle$

$= \langle 1-3t, 2-3t \rangle$, $0 \le t \le 1$

$\vec{r}\,'(t) = \langle -3, -3 \rangle$

$|\vec{r}\,'(t)| = \sqrt{9+0}$

$$\int_C 4x^3\,ds = \int_0^1 4(1-3t)\sqrt{18}\,dt = -15\sqrt{2}$$

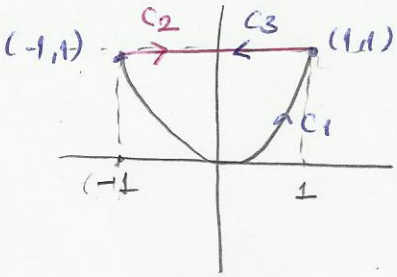
Property: 1) $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$ This true for line integrals with respect to arc length. We'll see that there are other kinds of line integrals in which this won't be the case.

Example: Evaluate $\int_C x\,ds$ for each of the following curves

(a) C_1 : $y=x^2$, $-1\le x\le 1$

(b) C_2 : the line segment from $(-1,1)$ to $(1,1)$

(c) C_3 : the line segment from $(1,1)$ to $(-1,1)$



(a) Parametrization of C_1 : $x=t$, $y=t^2$ $-1\le t\le 1$

$$\int_{C_1} x\,ds = \int_{-1}^{1} t\cdot\sqrt{(1)^2+(2t)^2}\,dt = 0$$

(b) Parametrization of C_2: $\vec{r}(t) = (1-t)\langle -1,1\rangle + t\langle 1,1\rangle$

$= \langle 2t-1, 1\rangle$, $0\le t\le 1$

$\vec{r}\,'(t) = \langle 2,0\rangle$

$|\vec{r}\,'(t)| = 2$

$$\int_{C_2} x\,ds = \int_{-1}^{1} t\sqrt{1^2+0}\,dt = 0$$

or

C_2: $x=t$, $y=1$
$-1\le t\le 1$
(easier)

(c) Parametrization of C_3: $\vec{r}(t) = (1-t)\langle 1,1\rangle + t\langle -1,1\rangle$

$= \langle 1-2t, 1\rangle$, $0\le t\le 1$

$\vec{r}\,'(t) = \langle -2,1\rangle$

$|\vec{r}\,'(t)| = 2$

$$\int_{C_3} x\,ds = \int_{0}^{1} (1-2t)\cdot 2\,dt = 0.$$

$C_3 = -C_2$

The line integral of f with respect to x or y

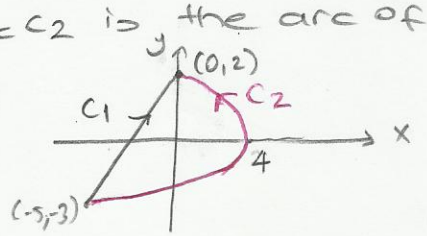
parametrization of a two dimensional curve C:

$x = x(t)$, $y = y(t)$, $a\le t\le b$

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

Example: Evaluate $\int_C y^2dx + x\,dy$ where a) $C=C_1$ is the line segment from $(-5,-3)$ to $(0,2)$ and b) $C=C_2$ is the arc of the parabola $x=4-y^2$ from $(-5,-3)$ to $(0,2)$



a) Parametric representation of C_1:

$$\vec{r}(t) = (1-t)\langle -5,-3\rangle + t\langle 0,2\rangle$$

$$= \langle \underbrace{5t-5}_{x(t)}, \underbrace{5t-3}_{y(t)}\rangle \quad 0\le t\le 1$$

$$\int_{C_1} y^2dx + xdy = \int_0^1 (5t-3)^2\, 5\,dt + (5t-5)5\,dt = -5/6$$

b) Parametric representation of C_2:

$x=4-y^2$; $y=t$, $x=4-t^2$ $\quad -3\le t\le 2$

$$\int_{C_2} y^2dx + xdy = \int_{-3}^{2} t^2\cdot(-2t)\,dt + (4-t^2).1.dt = \left[-\frac{t^4}{2} - \frac{t^3}{3} + 4t\right]_{-3}^{2}$$

$$= \left[\left(-\frac{16}{2} - \frac{8}{3} + 8\right) - \left(-\frac{81}{2} + \frac{27}{3} - 12\right)\right]$$

$$= -8/3 + 3 + 81/2 = \frac{-16+18+243}{6} = \frac{245}{6}$$

Note that even the initial and terminal points of C_1 and C_2 are same we get different values. In general, the value of a line integral depends on the path.

Here we have;

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \quad , \quad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy .$$

Line Integrals of Vector Fields

$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ vector field

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ $\quad a\le t\le b$ smooth curve.

The line integral of $\vec{F}$ along C is

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot\vec{T}\,ds}$$

dot product

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

unit tangent vector

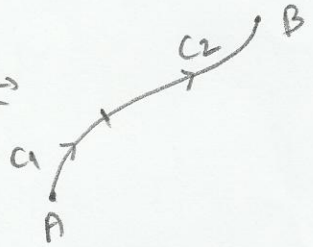
Note: If C is a closed curve we use the notation $\oint_C$

Properties

1) $\int_C k\vec{F}\cdot d\vec{r} = k\int_C \vec{F}\cdot d\vec{r}$ (k constant)

2) $\int_C (\vec{F}+\vec{G})\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_C \vec{G}\cdot d\vec{r}$

3) $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$



4) ANY representation of C that give the same positive direction on C also yield the same value of the line integral.

5) $\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$

Example: Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = xz\,\vec{i} - yz\,\vec{k}$ and C is the line segment from $(-1,2,0)$ and $(3,0,1)$.

Parametization of C: $\vec{r}(t) = (1-t)\langle -1,2,0\rangle + t\langle 3,0,1\rangle$

$= \langle 4t-1, 2-2t, t\rangle \quad 0\le t\le 1$

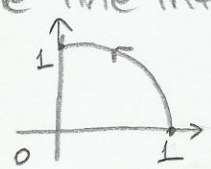
$\vec{F}(\vec{r}(t)) = (4t-1)(t)\vec{i} - (2-2t)(t)\vec{k} = (4t^2-t)\vec{i} - (2t-2t^2)\vec{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^1 \langle 4t^2-t, 0, -2t+2t^2\rangle \cdot \langle 4,-2,1\rangle\,dt$$

$$= \int_0^1 (16t^2-4t-2t+2t^2)\,dt = \int_0^1 (18t^2-6t)\,dt = 3.$$

Exercise: Find the value of the line integral when $\vec{F}(x,y) = -y\,\vec{i} - xy\,\vec{j}$ and C is the circular arc

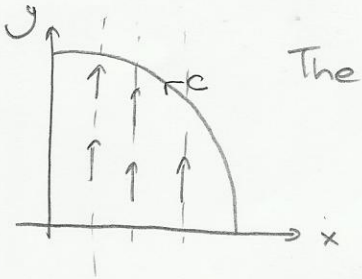


Remark: The work W done by a constant force $\vec{F}$ in the displacement along a straight segment $\vec{d}$, $W = \vec{F} \cdot \vec{d}$.

If the force is not constant and the displacement is along a curve C parametrized by $\vec{r}(t)$, we have

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r}}$$

Example: If a force is given by $\vec{F}(x,y) = \langle 0, x \rangle$, compute the work done by the force field on a particle that moves along the curve C that is the counterclockwise quarter unit circle in the first quadrant



The vector field $\vec{F}$ is represented by the vertical arrows

Parametrization of C: $\vec{r}_1(t) = \langle \cos t, \sin t \rangle$
$0 \le t \le \pi/2$

or we can take $\vec{r}_2(t) = \langle \sqrt{1-t^2}, t \rangle$, $0 \le t \le 1$

Take $\vec{r}_1(t)$. $\vec{r}_1{}'(t) = \langle -\sin t, \cos t \rangle$, $\vec{F}(\vec{r}_1(t)) = \langle 0, \cos t \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r}_1 = \int_0^{\pi/2} \langle 0, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle dt = \int_0^{\pi/2} \cos^2 t \, dt$$

$$= \int_0^{\pi/2} \frac{\cos 2t + 1}{2} dt = \frac{\sin 2t}{4} + \frac{t}{2} \Big|_0^{\pi/2}$$

$\cos 2t = 2\cos^2 t - 1$

$$= \pi/4$$

Take $\vec{r}_2(t)$. $\vec{r}_2{}'(t) = \langle \frac{-t}{\sqrt{1-t^2}}, 1 \rangle$

$\vec{F}(\vec{r}_2(t)) = \langle 0, \sqrt{1-t^2} \rangle$. $\quad W = \int_C \vec{F} \cdot d\vec{r}_2 = \int_0^1 \langle 0, \sqrt{1-t^2} \rangle \cdot \langle -\frac{t}{\sqrt{1-t^2}}, 1 \rangle dt$

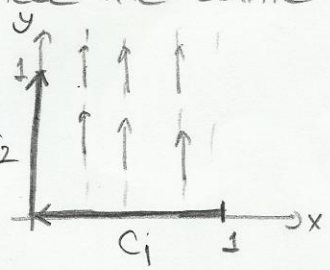
$$= \int_0^1 \sqrt{1-t^2} \, dt = \int_0^{\pi/2} \cos^2 u \, du = \pi/4$$

$t = \sin u$
$dt = \cos u \, du$

As it is seen in this example, the line integral does not depend on parametrization. (Be careful about the direction!)

But the line integral, in general, depend on the path.

Check the same example with another path C.



$C_1 \cup C_2 = C$

Along C_1: C_1 is orthogonal to the vector field hence the dot product gives zero.

Along C_2: $x=0$ so $\vec{F}(x,y) = \langle 0, x\rangle = \langle 0,0\rangle$; and so the line integral is zero along C_2.

Hence $W = \int_C \vec{F}\cdot d\vec{r} = 0$. "No work's done"

Path Independence of Line Integrals

Under which conditions, the line integral is path independent in a domain D in space?

① Line integral is path independent $\Leftrightarrow$ $\vec{F} = \langle P, Q, R\rangle$ is the gradient of some function f in D. (That is $\vec{F}$ is a conservative vector field.)

$\Leftarrow$ Assume that $\vec{F} = \nabla f = \langle f_x, f_y, f_z\rangle$

Let C be ANY path in D from A to B in D given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. $a \le t \le b$, $\vec{r}(a) = A$, $\vec{r}(b) = B$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = \int_a^b \nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt = \int_a^b \frac{d}{dt}\left[f(\vec{r}(t))\right]dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A).$$

Example: Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1, 1/2, 2)$ and ends at $(2,1,-1)$

Here we did not specify the path from $(1,1/2,2)$ to $(2,1,-1)$ since by the above consideration we do not need.

So let $\vec{r}(t)$, $a \le t \le b$ be ANY path that starts at $(1,1/2,2)$ and ends at $(2,1,-1)$. So $\vec{r}(a) = (1, 1/2, 2)$, $\vec{r}(b) = (2,1,-1)$

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = \left[\cos(2\pi) + \sin(\pi) - 2.1(-1)\right] - \left[\cos(\pi) + \sin(\pi/2) - 1.\frac{1}{2}\right]$$

$$= (1+0+2) - (-1+1-1) = 4$$

Theorem: If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

- Let $\vec{F}$ be a conservative vector $\Rightarrow \vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$
$= f_x\vec{i} + f_y\vec{j}$

$f_x = P$, $f_y = Q$ $\Rightarrow f_{xy} = P_y = Q_x = f_{yx}$ $\quad f_{xy} = f_{yx} \Leftarrow (f_x, f_y$ continuous$)$

For the converse of that theorem we need a stronger condition for D.

Theorem: Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D.

Then $\vec{F}$ is conservative.

Example: Determine whether or not the following vector fields are conservative.

a) $\vec{F}(x,y) = (3+2xy)\vec{i} + (x^2 - 3y^2)\vec{j}$ $\quad P = 3+2xy$, $Q = x^2 - 3y^2$
$P_y = 2x = Q_x$

Domain of $\vec{F}$ is $\mathbb{R}^2$ which is open and simply connected so $\vec{F}$ is conservative by the theorem.

b) $\vec{F}(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$ $\quad P = (x-y)$, $Q = (x-2)$
$P_y = -1 \neq 1 = Q_x$

Since $P_y \neq Q_x$, $\vec{F}$ is not conservative.

Example:- a) $\vec{F}(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$. Find a function f such that $\vec{F} = \nabla f$.

b) Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve given by $\vec{r}(t) = e^t \sin t\,\vec{i} + e^t \cos t\,\vec{j}$ $\quad 0 \le t \le \pi$.

a) $\vec{F} = \nabla f = \langle f_x, f_y \rangle$ $\quad \boxed{f_x = 3+2xy}$, $\quad f_y = x^2 - 3y^2$

$$\Downarrow$$

$$f(x,y) = 3x + x^2 y + k(y) \Longrightarrow f_y = 0 + x^2 + k'(y) = x^2 - 3y^2$$

$$k'(y) = -3y^2$$

$$k(y) = -y^3 + \alpha$$

α constant.

Hence $f(x,y) = 3x + x^2 y - y^3 + \alpha \Rightarrow \vec{F}$ is a conservative field.

potential function
choose $\alpha = 0$.

b) Since $\vec{F}$ is a conservative field, we can immediately use the identity at page 100. We need initial and terminal points of C. $\quad 0 \le t \le \pi \quad \vec{r}(0) = (0,1)$, $\vec{r}(\pi) = (0, -e^{\pi})$

$$\int_C \vec{F}\cdot d\vec{r} = f(0,-e^{\pi}) - f(0,1) = e^{3\pi} + 1.$$

Green's Theorem

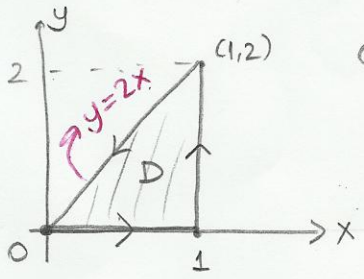
Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

positive orientation: counterclockwise direction.

GREEN'S THEOREM: Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \qquad \vec{F} = \langle P, Q \rangle$$

Example: Use Green's theorem to evaluate $\oint_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.



C: simple, closed, piecewise smooth curve

$P = xy$, $Q = x^2y^3$ polynomials, they have continuous partial derivatives everywhere.

Hence we can use the Green's theorem.

$0 \le x \le 1$, $0 \le y \le 2x$ $\quad \oint_C xy\,dx + x^2y^3\,dy = \int_0^1\int_0^{2x} (2xy^3 - x)\,dy\,dx$

$$= \int_0^1 \left\{ \left(\frac{xy^4}{2} - xy \right)\Big|_0^{2x} \right\} dx = \int_0^1 \left(\frac{\cancel{16}^{\,8} x^5}{\cancel{2}} - 2x^2 \right) dx = \left(\frac{8x^6}{6} - \frac{2x^3}{3} \right)\Big|_0^1 = \frac{2}{3}$$

Example: Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$ where C is the circle $x^2 + y^2 = 9$.

C: closed, simple, smooth curve

$P = 3y - e^{\sin x}$, $Q = 7x + \sqrt{y^4+1}$ have continuous partial derivatives.

we can use the Green's theorem. $\quad dA = dx\,dy = r\,dr\,d\theta$

$$\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy = \iint_D (7-3)\,dA = \int_0^{2\pi}\int_0^3 4\,r\,dr\,d\theta$$

$$= 36\pi.$$

Remark: If we know that $P(x,y) = Q(x,y) = 0$ on the curve C the

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy = 0$$ no matter what values

P and Q assume in the region.

Remark: Remember that $\iint_D 1 \cdot dA = \text{Area of } D$. Hence we choose P and Q so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. There are several possibilities

Hence by Green's theorem

$P(x,y)=0,\ Q(x,y)=x \Rightarrow A=\oint_C x\,dy$

$P(x,y)=-y,\ Q(x,y)=0 \Rightarrow A=\oint_C -y\,dx$

$P(x,y)=-\frac{1}{2}y,\ Q(x,y)=\frac{1}{2}x \Rightarrow A=\frac{1}{2}\oint_C x\,dy-y\,dx$

$$A=\oint_C x\,dy=\oint_C -y\,dx$$
$$=\frac{1}{2}\oint_C x\,dy-y\,dx$$

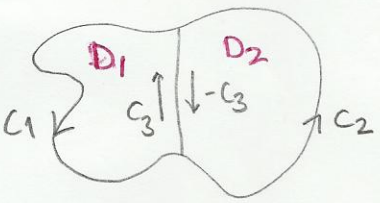
Example: Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The ellipse has parametric equations $x=a\cos t,\ y=b\sin t \quad 0\le t\le 2\pi$.

Hence $A=\frac{1}{2}\oint x\,dy-y\,dx=\frac{1}{2}\int_0^{2\pi} a\cos t(b\cos t)\,dt-b\sin t(-a\sin t)\,dt$

$$=\frac{ab}{2}\int_0^{2\pi} dt = ab\pi$$

Remark: We give the Green's theorem for D is simple. We can extend it to the case where D is a finite union of simple regions.



$D=D_1\cup D_2$ Boundary of D_1 $C_1\cup C_3$

$C=C_1\cup C_2$ Boundary of D_2 $C_2\cup(-C_3)$

Apply Green's theorem to D_1 and D_2 separately.

$$\int_{C_1\cup C_3} P\,dx+Q\,dy=\iint_{D_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

$$\int_{C_2\cup(-C_3)} P\,dx+Q\,dy=\iint_{D_2}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

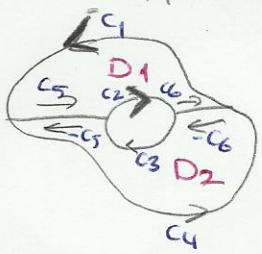
Add

$$\int_{C_1\cup C_3} P\,dx+Q\,dy+\int_{C_2\cup(-C_3)} P\,dx+Q\,dy=\int_{C_1} P\,dx+Q\,dy+\int_{C_3} P\,dx+Q\,dy+\int_{C_2} P\,dx+Q\,dy$$

$$+\underbrace{\int_{-C_3} P\,dx+Q\,dy}_{-\int_{C_3} P\,dx+Q\,dy}=\int_{C_1} P\,dx+Q\,dy+\int_{C_2} P\,dx+Q\,dy$$

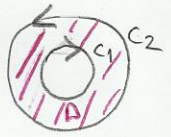
$$=\int_{\underbrace{C_1\cup C_2}_{C}} P\,dx+Q\,dy=\iint_{\underbrace{D_1\cup D_2}_{D}}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

we will approach similarly if we have region with a hole;



Here we have, $\iint_D (Q_x - P_y)\,dA = \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} P\,dx + Q\,dy$ (*)

Example: Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C are the two circles of radius 2 and radius 1 centered at the origin with positive and negative orientation respectively. Use Green's theorem.

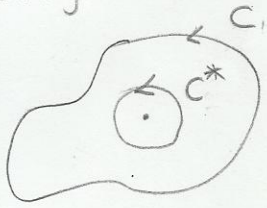


$C = C_1 \cup C_2$

$$\oint_C y^3\,dx - x^3\,dy = -3\iint_D x^2 + y^2\,dA$$

$$= -3\int_0^{2\pi}\int_1^2 r^2\,r\,dr\,d\theta = -45\pi/2$$

Example: If $\vec{F}(x,y) = \dfrac{(-y\vec{i} + x\vec{j})}{x^2+y^2}$ show that $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.



Let us consider a counterclockwise-oriented circle C^* with center the origin and radius a as in the figure.

By (*) we have $\int_{C \cup (-C^*)} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$

$$= \iint_D \left(\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} \right) dA = 0$$

Therefore $\int_{C \cup (-C^*)} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy + \underbrace{\int_{-C^*} P\,dx + Q\,dy}_{-\int_{C^*} P\,dx + Q\,dy} = 0$

$$\Rightarrow \int_C P\,dx + Q\,dy = \int_{C^*} P\,dx + Q\,dy.$$

$$\Downarrow$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C^*} \vec{F}\cdot d\vec{r}$$

Parametrization of C^*: $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$ $\quad 0 \le t \le 2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C^*} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi .$$

(2) $\int_C \vec{F}\cdot d\vec{r}$ is path independent in $D \Leftrightarrow \oint_C \vec{F}\cdot d\vec{r} = 0$ for every closed path C in D.

($\Rightarrow$) If $\int_C \vec{F}\cdot d\vec{r}$ is path independent in D, then $\vec{F} = P\vec{i} + Q\vec{j}$ conservative vector field and we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$\oint_C \vec{F}\cdot d\vec{r} \underset{\text{by Green's theorem}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\, dA = 0.$$

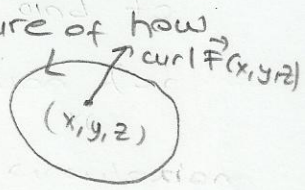
Curl and Divergence

<u>CURL</u> If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\vec{F}$ is the vector field on $\mathbb{R}^3$:

$$\boxed{\text{curl}\,\vec{F} = \nabla\times\vec{F}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Note: If $\vec{F}$ represents the velocity field in fluid flow: Particles near (x,y,z) in the fluid tend to rotate about the axis that points in the direction of curl $\vec{F}(x,y,z)$ and the length of this curl vector is a measure of how quickly the particles move around the axis.



If curl $\vec{F} = 0$ at a point P, then the fluid is free from rotations at P i.e. no whirlpool at P.

Note: If $\nabla \times \vec{F} = 0$, then the field is said to be an irrotational field.

Example: If $\vec{F}(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\,\vec{k}$, find curl $\vec{F}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = (-2y - xy)\,\vec{i} - (0 - x)\,\vec{j} + (yz - 0)\,\vec{k}$$

$$= \langle -2y - xy,\ x,\ yz \rangle$$

Theorem: If f is a function of three variables that has continuous second-order partial derivatives, then

$$\boxed{\text{curl}(\nabla f) = 0}$$

Proof: $\nabla f = \langle f_x, f_y, f_z \rangle$, $\text{curl}(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$

$$= (f_{zy} - f_{yz})\,\vec{i} - (f_{zx} - f_{xz})\,\vec{j} + (f_{yx} - f_{xy})\,\vec{k}$$

$= 0$ by Clairaut's theorem.

$(f_{zy} = f_{yz},\ f_{zx} = f_{xz},\ f_{yx} = f_{xy})$

Conclusion: A conservative vector field is one for which $\vec{F} = \nabla f$.

Hence by the above theorem;

$$\boxed{\text{If } \vec{F} \text{ is conservative then curl } \vec{F} = 0}$$

Example: Show that the vector field $\vec{F}(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\,\vec{k}$ is not conservative.

This is the same vector field in the previous example.

curl $\vec{F} = \langle -2y - xy,\ x,\ yz \rangle \neq 0$. So by the above conclusion, $\vec{F}$ is not conservative.

! The converse of the conclusion is NOT true in general.

Theorem: If $\vec{F}$ is a vector field DEFINED on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl}\,\vec{F}=0$, then $\vec{F}$ is a conservative vector field.

Example: a) Show that $\vec{F}(x,y,z)=\langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ is a conservative vector field.

b) Find a function f such that $\vec{F}=\nabla f$.

a) $$\text{curl}\,\vec{F}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2-6xyz^2)\vec{i}-(3y^2z^2-3y^2z^2)\vec{j}+(2yz^3-2yz^3)\vec{k}$$
$$=0$$

Since $\text{curl}\,\vec{F}=0$ and the domain of $\vec{F}$ is $\mathbb{R}^3$, F is a conservative vector field by the theorem.

b) $\vec{F}=\nabla f=\langle f_x, f_y, f_z\rangle$ $\quad \overset{①}{\boxed{f_x=y^2z^3}}$, $\overset{②}{f_y=2xyz^3}$, $\overset{③}{f_z=3xy^2z^2}$

$$f(x,y,z)=y^2z^3x+h(y,z) \Rightarrow f_y=2y\cancel{z^3}x+\frac{\partial h(y,z)}{\partial y}=2x\cancel{yz^3} \Rightarrow h(y,z)=k(z)$$

$$\Rightarrow f(x,y,z)=y^2z^3x+k(z) \Rightarrow f_z=3y^2z^2x+k'(z)=3xy^2z^2 \Rightarrow k'(z)=0$$
$$k(z)=c$$
constant

Hence $f(x,y,z)=y^2z^3x+c$.

DIVERGENCE If $\vec{F}=P\vec{i}+Q\vec{j}+R\vec{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist then the divergence of $\vec{F}$ is the function

$$\boxed{\text{div}\,\vec{F}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=\nabla\cdot\vec{F}}$$

Theorem: If $\vec{F}=P\vec{i}+Q\vec{j}+R\vec{k}$ is a vector field on $\mathbb{R}^3$, and P, Q, and R have continuous second-order partial derivatives then

$$\boxed{\text{div curl}\,\vec{F}=0}$$

Proof: $\vec{F} = \langle P, Q, R \rangle$, $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$

$$\text{div curl } \vec{F} = \nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0 \quad \text{by Clairaut's theorem}$$

Example: Show that the vector field $\vec{F}(x,y,z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$ cannot be written as the curl of another vector field, that is, $\vec{F} \neq \text{curl } \vec{G}$.

$P = xz, Q = xyz, R = -y^2$ $\quad$ $\text{div } \vec{F} = z + xz \Rightarrow \text{div } \vec{F} \neq 0$.

If it were true that $\vec{F} = \text{curl } \vec{G}$ then $\text{div } \vec{F} = \text{div curl } \vec{G} = 0$ which contradicts to $\text{div } \vec{F} \neq 0$. Therfore $\vec{F}$ is not the curl of another vector field.

Note: We can explain the divergence in the context of fluid flow. If $\vec{F}(x,y,z)$ is the velocity of a fluid (or gas), the $\text{div } \vec{F}(x,y,z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x,y,z) per unit volume i.e. $\text{div } \vec{F}(x,y,z)$ measures the tendency of the fluid to diverge from the point (x,y,z). If $\text{div } \vec{F} = 0$, then $\vec{F}$ is said to be incompressible.

Note: $\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ "Laplace's Equation"

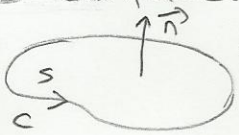
Stokes' Theorem

Let S be a piecewise smooth (orientable) oriented surface $f(x,y,z)=0$ that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also let $\vec{F}$ be a vector field then

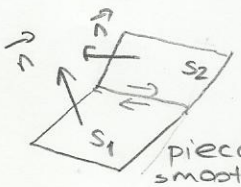
$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}}$$

$$d\vec{S} = \vec{n}\, dS = \frac{\nabla f}{|\nabla f|}\, dS = \frac{\nabla f}{|\nabla f|} \cdot |\nabla f|\, dA$$

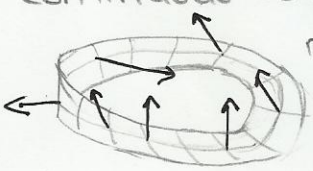
smooth oriented (orientable) surface: A smooth surface S is called orientable if the positive normal direction, when given at an arbitrary point P_0 of S, can be continued in a unique and continuous way to the entire surface.



smooth oriented surface



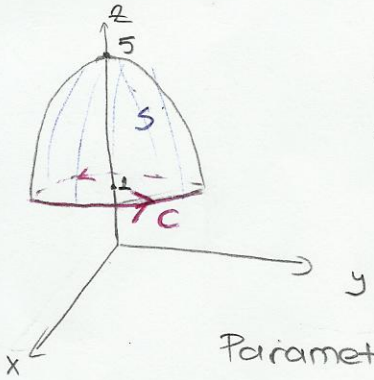
piecewise smooth oriented surface



möbius strip not orientable

Example: Use Stokes' theorem to evaluate $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ where $\vec{F} = z^2\vec{i} - 3xy\vec{j} + x^3y^3\vec{k}$ and S is the part of $z = 5 - x^2 - y^2$ above the plane $z = 1$. Assume that S is oriented upwards.

a paraboloid



The boundary curve C will be where the surface intersects the plane $z = 1$ and so it will be the curve

$$z = 5 - x^2 - y^2$$

$$1 = 5 - x^2 - y^2 \implies x^2 + y^2 = 4 \quad \text{at } z = 1.$$

Parametrization of the curve C:

$$C: \vec{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle \quad , \; 0 \le t \le 2\pi$$

$$\Rightarrow \vec{F}(\vec{r}(t)) = \langle 1, -12\cos t\sin t, 64\cos^3 t\sin^3 t\rangle$$

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

So by Stokes' theorem,

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \langle 1, -12\cos t\sin t, 64\cos^3 t\sin^3 t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\,dt$$

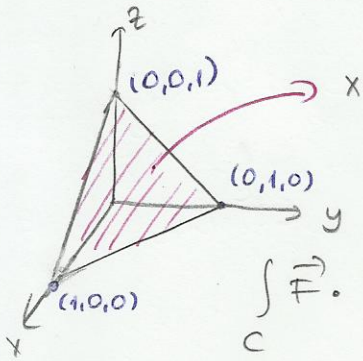
$$= \int_0^{2\pi} (-2\sin t - 24\cos^2 t\sin t)\,dt = -2\int_0^{2\pi}\sin t\,dt - 24\int_0^{2\pi}\cos^2 t\sin t\,dt$$

$u = \cos t$

$du = -\sin t\,dt$

$$= 2\cos t\Big|_0^{2\pi} + 24\,\frac{\cos^3 t}{3}\Big|_0^{2\pi} = 0.$$

Example: Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = z^2\vec{i} + y^2\vec{j} + x\vec{k}$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ with counter-clockwise rotation.



$x + y + z = 1$

$f(x,y,z) = z - 1 + x + y = 0$

$\nabla f = \langle 1, 1, 1\rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & y^2 & x \end{vmatrix} = 0\vec{i} - (1 - 2z)\vec{j} + 0\vec{k}$$

$$= (2z - 1)\vec{j}$$

$$= \langle 0, 2z - 1, 0\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \langle 0, 2z-1, 0\rangle\cdot\langle 1,1,1\rangle\,dA = \iint_D (2z-1)\,dA$$

$z = 1 - x - y$

$$= \int_0^1\int_0^{-x+1} (1 - 2x - 2y)\,dy\,dx = -1/6$$