İST 292 STATISTICS

Sections: 05-06

For Department of Computer Engineering

LESSON 6 HYPOTHESIS TESTS-PART II

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Test Concering Comparison of Two Variances

If σ_1^2 and σ_2^2 are the variances of independent random samples of size n_1 and n_2 from normal populations, then, $\frac{S_1^2\sigma_2^2}{S_1^2\sigma_1^2}$ is a random variable having an f distribution with $(n_1\text{-}1)$ and $(n_2\text{-}1)$ degrees of freedom. Therefore, the hypotheses tests concerning the comparison of two variances are based on f distribution.

$$\frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} \sim f_{(n_1-1),(n_2-1)}$$

Hypotheses	$H_0: \sigma_1^2 = \sigma_2^2 = \sigma^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$	$H_0: \sigma_1^2 = \sigma_2^2$ $(\sigma_1^2 \geq \sigma_2^2)$ $H_1: \sigma_1^2 < \sigma_2^2$	$H_0: \sigma_1^2 = \sigma_2^2$ $(\sigma_1^2 \leq \sigma_2^2)$ $H_1: \sigma_1^2 > \sigma_2^2$
Test Statistic	If $S_1^2 > S_2^2 \Rightarrow f = \frac{S_1^2}{S_2^2}$ If $S_2^2 > S_1^2 \Rightarrow f = \frac{S_2^2}{S_1^2}$	$f = \frac{S_2^2}{S_1^2}$	$f = \frac{S_1^2}{S_2^2}$
Decision	If $S_1^2 > S_2^2$, $f = \frac{S_1^2}{S_2^2}$ $f \geq f_{\frac{\alpha}{2},(n_1-1),(n_2-1)}$ If $S_2^2 > S_1^2$, $f = \frac{S_2^2}{S_1^2}$ $f \geq f_{\frac{\alpha}{2},(n_2-1),(n_1-1)}$ H_0 is rejected.	If $f \geq f_{\alpha,(n_2-1),(n_1-1)}$ H_0 is rejected.	If $f \geq f_{\alpha,(n_1-1),(n_2-1)}$, H_0 is rejected.

Test Concerning Difference Between Two Means ($\mu_1 - \mu_2$)

We have two samples from two normal populations:

$$\underbrace{N(\mu_1, \sigma_1^2)}_{X_{11}, X_{12}, \ldots X_{1n_1}} \qquad \underbrace{N(\mu_2, \sigma_2^2)}_{X_{21}, X_{22}, \ldots X_{2n_2}}$$

These are independent samples

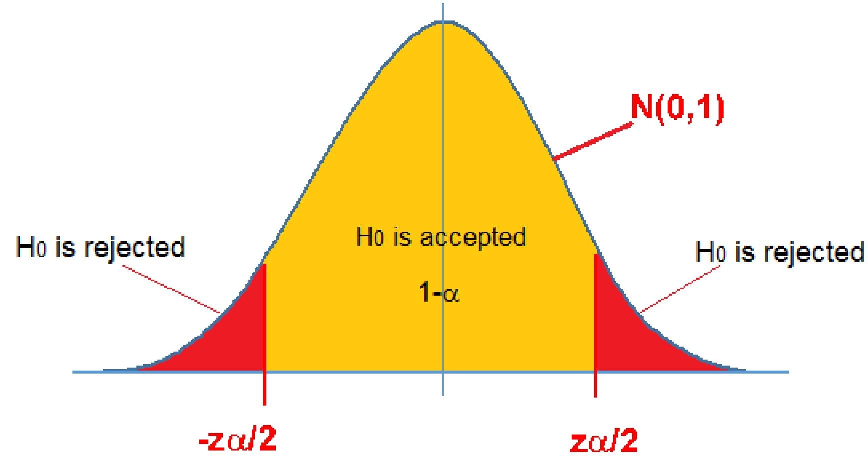
$$\bar{X}_1 \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right) \qquad \bar{X}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

The distribution of $\bar{X}_1 - \bar{X}_2$:

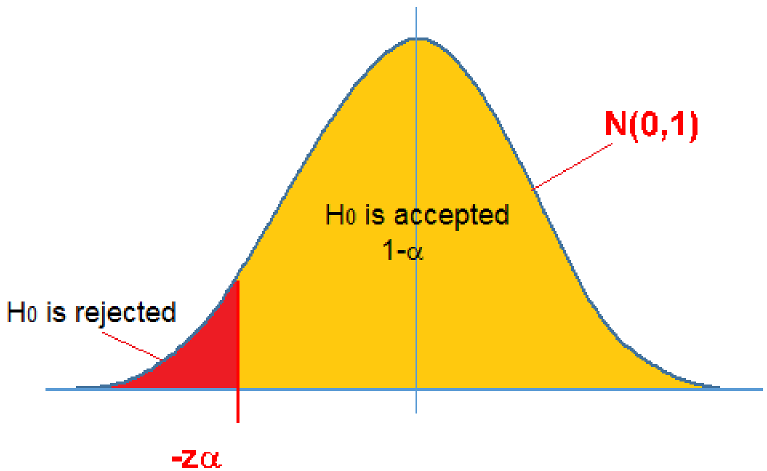
$$\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

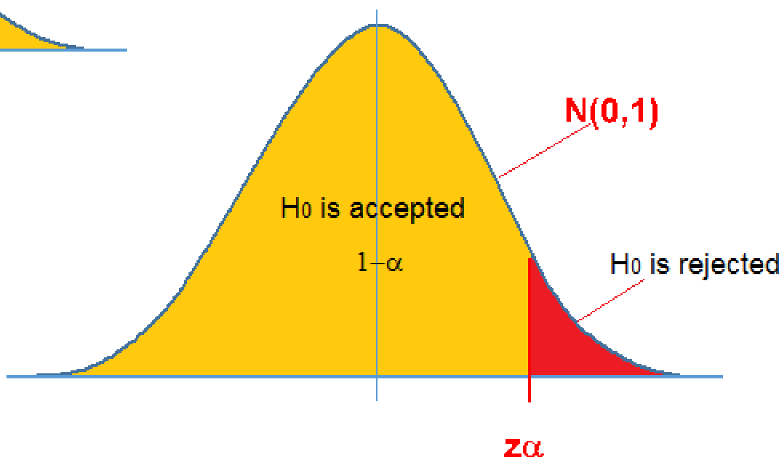
$$H_0 : \mu_1 - \mu_2 = \delta$$
$$H_1 : \mu_1 - \mu_2 \neq \delta$$



$$H_0 : \mu_1 - \mu_2 = \delta\,(\mu_1 - \mu_2 \geq \delta)$$
$$H_1 : \mu_1 - \mu_2 < \delta$$



$$H_0 : \mu_1 - \mu_2 = \delta\,(\mu_1 - \mu_2 \leq \delta)$$
$$H_1 : \mu_1 - \mu_2 > \delta$$



When the Population Variance σ_1^2 and σ_2^2 are Known

Hypotheses	$H_0: \mu_1 - \mu_2 = \delta$ $H_1: \mu_1 - \mu_2 \neq \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \geq \delta)$ $H_1: \mu_1 - \mu_2 < \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \leq \delta)$ $H_1: \mu_1 - \mu_2 > \delta$
Test Statistic	$$z = \frac{(\bar{x}_1 - \bar{x}_1) - \delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$		
Decision	If $z \geq z_{\frac{\alpha}{2}}$ or $z \leq -z_{\frac{\alpha}{2}}$ H_0 is rejected. (If $\|z\| \geq z_{\frac{\alpha}{2}}$, H_0 is rejected.)	If $z \leq -z_{\alpha}$, H_0 is rejected. (If $\|z\| \geq z_{\alpha}$, H_0 is rejected.)	If $z \geq z_{\alpha}$, H_0 is rejected.

When the Population Variance σ_1^2 *and* σ_2^2 are unknown, but n_1 and $n_2 \geq 30$

Hypotheses	$H_0: \mu_1 - \mu_2 = \delta$ $H_1: \mu_1 - \mu_2 \neq \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \geq \delta)$ $H_1: \mu_1 - \mu_2 < \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \leq \delta)$ $H_1: \mu_1 - \mu_2 > \delta$
Test Statistic	$z = \dfrac{(\bar{x}_1 - \bar{x}_1) - \delta}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$		
Decision	If $z \geq z_{\frac{\alpha}{2}}$ or $z \leq -z_{\frac{\alpha}{2}}$, H_0 is rejected. (If $\|z\| \geq z_{\frac{\alpha}{2}}$, H_0 is rejected.)	If $z \leq -z_{\alpha}$, H_0 is rejected. (If $\|z\| \geq z_{\alpha}$, H_0 is rejected.)	If $z \geq z_{\alpha}$, H_0 is rejected.

Example 1: Lesley E. Tan investigated the relationship between handedness (tek elini kullanabilme) and motor competence (kabiliyet) in preschool children. Random samples of 41 right-handers and 41 left-handers were administered several tests of motor skills, Is there evidence of a difference between the average motor skill scores of left- and right- handed preschoolers base on this experiment? Use α=0.10 (source: Tan, L. E. "Laterality and motor skills in four –years-olds", Child Development, 56.) Scores for each group has normal distribution.

($\bar{x}_1 = 97.5 \quad s_1 = 17.5 \quad \bar{x}_2 = 98.1 \quad s_2 = 19.2 \quad n_1 = n_2 = 41$)

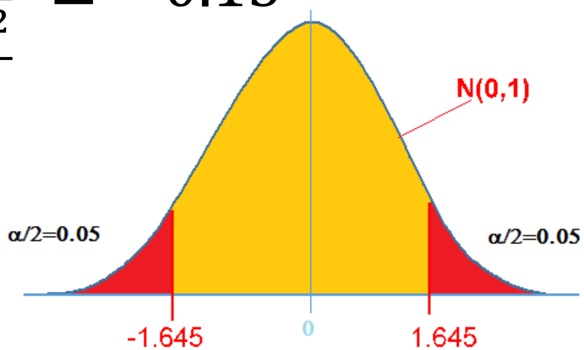
$H_0: \mu_1 - \mu_2 = 0$

$H_1: \mu_1 - \mu_2 \neq 0$

$$z = \frac{(\bar{x}_1 - \bar{x}_1) - \delta}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(97.5 - 98.1) - 0}{\sqrt{\frac{17.5^2}{41} + \frac{19.2^2}{41}}} \cong -0.15$$

table values: $z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$

$\underbrace{|z| = 0.15}_{test\ value} < \underbrace{z_{0.05} = 1.645}_{table\ value}$, H_0 can not be rejected.



We conclude that, there is no evidence of a difference between the average motor skill scores of left- and right- handed preschoolers at the level α =0.10.

NOTE

$\mathbf{H_0: \mu_1 - \mu_2 = \delta}$
$\mathbf{H_1: \mu_1 - \mu_2 \neq \delta}$
OR
$\mathbf{H_1: \mu_1 - \mu_2 < \delta}$
OR
$\mathbf{H_1: \mu_1 - \mu_2 > \delta}$

$$z = \frac{(\bar{x}_1 - \bar{x}_1) - \delta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$z = \frac{(\bar{x}_1 - \bar{x}_1) - \delta}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

By virtue of the central limit theorem, these two test statistics above are also used for independent random samples from non-normal populations when n_1 and n_2 are large, that is $n_1 \geq 30$ and $n_2 \geq 30$. If population variances are not known, we may also substitute for σ_1 and σ_2 the values of standard deviations s_1 and s_2 obtained from samples.

Example 2: A professor at a large community college wanted to determine whether there is a difference in the means of final exam scores between students who took his statistics course online and the students who took his face-to-face statistics class. He believed that the mean of the final exam scores for the online class would be lower than that of the face-to-face class. Was the professor correct? The randomly selected 30 final exam scores from each group are listed in the two tables below:

Online class

67.6	41.2	85.3	55.9	82.4	91.2	73.5	94.1	64.7	64.7
70.6	38.2	61.8	88.2	70.6	58.8	91.2	73.5	82.4	35.5
94.1	88.2	64.7	55.9	88.2	97.1	85.3	61.8	79.4	79.4

Face to face

77.9	95.3	81.2	74.1	98.8	88.2	85.9	92.9	87.1	88.2
69.4	57.6	69.4	67.1	97.6	85.9	88.2	91.8	78.8	71.8
98.8	61.2	92.9	90.6	97.6	100	95.3	83.5	92.9	89.4

Is the mean of the Final Exam scores of the online class lower than the mean of the Final Exam scores of the face-to-face class? Test at a 5% significance level.

Online class	Face to face class
$\bar{x}_1 = 72.85,\ s_1 = 16.9,\ n_1 = 30$	$\bar{x}_2 = 84.98,\ s_2 = 11.7,\ n_2 = 30$

$H_0: \mu_1 - \mu_2 = 0$
$H_1: \mu_1 - \mu_2 \neq 0$

$$z = \frac{(\bar{x}_1 - \bar{x}_1) - \delta}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{(72.85 - 84.98) - 0}{\sqrt{\frac{16.9^2}{30} + \frac{11.7^2}{30}}} \cong -3.23$$

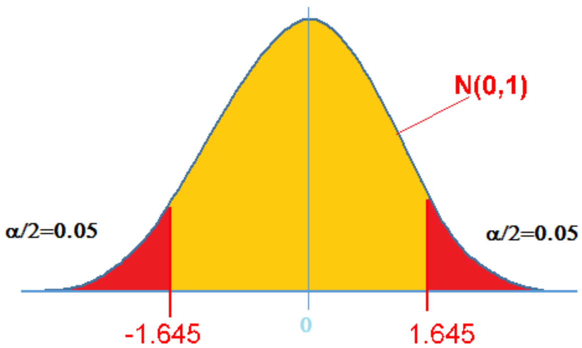


table values: $z_{\frac{\alpha}{2}} = z_{0.05} = 1.645$

$\underbrace{|z| = 3.23}_{test\ value} \geq \underbrace{z_{0.05} = 1.645}_{table\ value}$, H_0 is rejected.

We conclude that, the difference between the average score in online class and the average score in face to face class is significant at the level α =0.10. ***Face to face education is prefered to online education.***

When the Population Variance σ_1^2 *and* σ_2^2 are unknown, but n_1 and n_2<30

We have two samples from two normal populations:

$$\underbrace{N(\mu_1, \sigma_1^2)}_{X_{11}, X_{12}, \ldots X_{1n_1}} \qquad \underbrace{N(\mu_2, \sigma_2^2)}_{X_{21}, X_{22}, \ldots X_{2n_2}}$$

These are independent samples

σ_1^2 and σ_2^2 *are unknown*

When $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{(n_1 + n_2 - 2)}$$

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

pooled variance

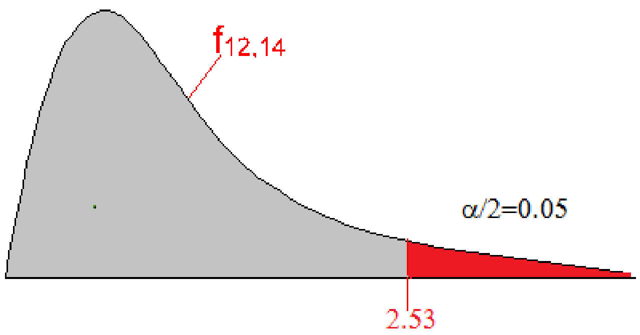
H_0: $\sigma_1^2 = \sigma_2^2 = \sigma^2$
H_1: $\sigma_1^2 \neq \sigma_2^2$

If H_0 can not be rejected (H_0 is accepted):

Hypotheses	$H_0: \mu_1 - \mu_2 = \delta$ $H_1: \mu_1 - \mu_2 \neq \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \geq \delta)$ $H_1: \mu_1 - \mu_2 < \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \leq \delta)$ $H_1: \mu_1 - \mu_2 > \delta$
Test Statistic	$t = \dfrac{(\bar{x}_1 - \bar{x}_1) - \delta}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$	$s_p = \sqrt{\dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$	
Decision	If $t \geq t_{\frac{\alpha}{2},(n-1)}$ or $t \leq -t_{\frac{\alpha}{2},(n-1)}$ H_0 is rejected. (If $\lvert t \rvert \geq t_{\frac{\alpha}{2},(n-1)}$ H_0 is rejected.)	If $t \leq -t_{\alpha,(n-1)}$ H_0 is rejected. (If $\lvert t \rvert \geq t_{\alpha,(n-1)}$ H_0 is rejected.)	If $t \geq t_{\alpha,(n-1)}$ H_0 is rejected.

Example 3: Suppose a television network want to determine whether major sports events or first-run movies attract more viewers in the prime time hours. It selects 28 prime-time evenings; of these, 13 have programs devoted to major sports events and remaining 15 have first-run movies. The number of viewers (estimated by a television viewer rating firm) is recorded for each program. The television network's samples produced the following results:

Sport (n_1=13)	Movies (n_2=15)
$\bar{x}_1 = 6.8\ million$ $s_1 = 1.8\ million$	$\bar{x}_2 = 5.3\ million$ $s_2 = 1.6\ million$



Step 1: $H_0: \sigma_1^2 = \sigma_2^2 = \sigma^2$ (α=0.10)
$H_0: \sigma_1^2 \neq \sigma_2^2$

Step 2: test statistic's value is $f = \frac{s_1^2}{s_2^2} = \frac{(1.8)^2}{(1.6)^2} = 1.2656$

Step 3: table value $f_{\frac{\alpha}{2},(n_1-1)(n_2-1)} = f_{0.05,12,14} = 2.53$

Step 4: *When we compare the test value with the critical value, we say that the null hypothesis cannot be rejected since f=1.2656 does not exceed 2.53 and so we conclude that* $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

$$H_0: \mu_1 - \mu_2 = 0$$
$$H_1: \mu_1 - \mu_2 \neq 0$$

$$s_p = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(13 - 1)(1.8)^2 + (15 - 1)(1.6)^2}{13 + 15 - 2} = 2.87$$

$$t = \frac{(\bar{x}_1 - \bar{x}_1) - \delta}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(6.8 - 5.3) - 0}{\sqrt{2.87\left(\frac{1}{13} + \frac{1}{15}\right)}} = 2.3368$$

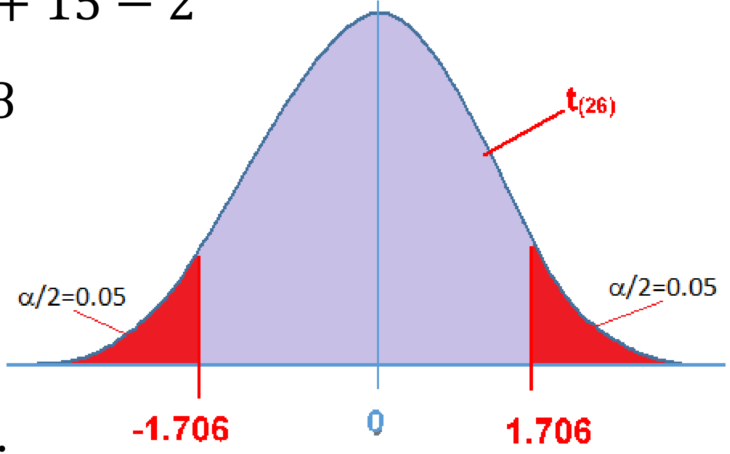


table values: $t_{\frac{\alpha}{2},(n_1+n_2-2)} = t_{0.05,26} = 1.706$

$\underbrace{|t| = 2.3368}_{test\ value} \geq \underbrace{t_{0.05,26} = 1.706}_{table\ value}$, H_0 is rejected.

We conclude that the mean numbers of viewers of sport events and is larger than the mean numbers of viewers of first-time movies at the significance level 0.10. ***(Note: Since*** **H_0 is rejected, we make these comments by looking sample means).**

H_0: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

H_1: $\sigma_1^2 \neq \sigma_2^2$ If H_0 is rejected (H_1 is accepted):

Hypotheses	$H_0: \mu_1 - \mu_2 = \delta$ $H_1: \mu_1 - \mu_2 \neq \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \geq \delta)$ $H_1: \mu_1 - \mu_2 < \delta$	$H_0: \mu_1 - \mu_2 = \delta$ $(\mu_1 - \mu_2 \leq \delta)$ $H_1: \mu_1 - \mu_2 > \delta$
Test Statistic	$t = \dfrac{(\bar{x}_1 - \bar{x}_1) - \delta}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$	$v \approx \dfrac{\left(\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_1}\right)^2}{\left(\dfrac{s_1^2}{n_1}\right)^2 \left(\dfrac{1}{n_1 - 1}\right) + \left(\dfrac{s_2^2}{n_2}\right)^2 \left(\dfrac{1}{n_2 - 1}\right)}$	
Decision	If $t \geq t_{\frac{\alpha}{2},(n-1)}$ or $t \leq -t_{\frac{\alpha}{2},(n-1)}$ H_0 is rejected. (If $\lvert t \rvert \geq t_{\frac{\alpha}{2},(n-1)}$ H_0 is rejected.)	If $t \leq -t_{\alpha,(n-1)}$ H_0 is rejected. (If $\lvert t \rvert \geq t_{\alpha,(n-1)}$ H_0 is rejected.)	If $t \geq t_{\alpha,(n-1)}$ H_0 is rejected.

Test Concerning Differences between Means: Paired Samples

In some studies or researches, paired measurements of a variable for each unit, person, event or observation are obtained such as before/after, first/second, etc. In general, these measurements cannot be assumed independent, because these units (persons, events or observations) are same and thus these leads to dependence between the measurements.

a difference between the pairs of measurements

$$D_i = X_{1i} - X_{2i}, \quad i=1,2,\ldots n$$

$$D_i \sim N(\mu_1 - \mu_2, \sigma_D^2), \quad i=1,2,\ldots n$$

$H_0: \mu_1 - \mu_2 = \delta$ $H_1: \mu_1 - \mu_2 \neq \delta$	$H_0: \mu_1 - \mu_2 = \delta \ (\mu_1 - \mu_2 \geq \delta)$ $H_1: \mu_1 - \mu_2 < \delta$	$H_0: \mu_1 - \mu_2 = \delta \ (\mu_1 - \mu_2 \leq \delta)$ $H_1: \mu_1 - \mu_2 > \delta$

If σ_D^2 is known, the test statistic is based on the z distribution.

$$z = \frac{\overbrace{(\bar{x}_1 - \bar{x}_2)}^{\bar{d}} - \delta}{\sigma_D / \sqrt{n}}$$

If σ_D^2 is unknown and also n≥30, the test statistic is based on approximately the z distribution.

$$z = \frac{\overbrace{(\bar{x}_1 - \bar{x}_2)}^{\bar{d}} - \delta}{s_D / \sqrt{n}}$$

If σ_D^2 is unknown and also n<30, the test statistic is based on the t distribution with n-1 degrees of freedom:

$$t = \frac{\overbrace{(\bar{x}_1 - \bar{x}_2)}^{\bar{d}} - \delta}{s_D / \sqrt{n}}$$

$$\bar{d} = \frac{\sum_{i=1}^{n} (x_{1i} - x_{2i})}{n}$$

$$s_D^2 = \frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{n-1}$$

Example 4: The data shown in the table provide information on the average weekly losses of work hours due to accidents in 10 industrial plants before and after a certain safety program was put into operation. At the 0.05 level significance, test whether the safety program is effective.

before	45	73	46	124	33	57	83	34	26	17
after	36	60	44	119	35	51	77	29	24	11
d_i	**9**	**13**	**2**	**5**	**-2**	**6**	**6**	**5**	**2**	**6**

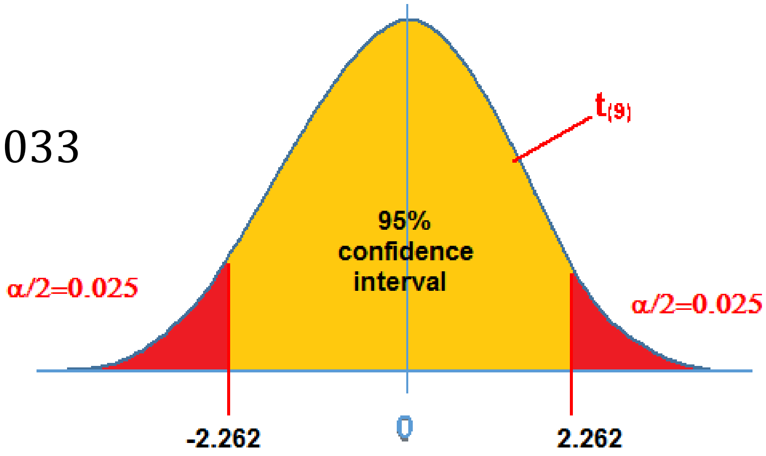
$\bar{d} = 5.2 \quad s_D^2 = 16.622$

$H_0: \mu_1 - \mu_2 = 0$
$H_1: \mu_1 - \mu_2 \neq 0$

$$t = \frac{\bar{d} - \delta}{s_D/\sqrt{n}} = \frac{5.2 - 0}{4.077/\sqrt{10}} = 4.033$$

table values: $t_{\frac{\alpha}{2},(n-1)} = t_{0.025,9} = 2.262$

$\underbrace{t = 4.033}_{test\ value} \geq \underbrace{t_{0.025,9} = 2.262}_{table\ value}$, H_0 is rejected.



We conclude that, there is evidence of the effectiveness of the safety program on the average weekly losses of work hours at the level α=0.05.

Test Concerning Differences between Proportions (p_1-p_2)

In many problems in applied research, we must decide whether observed differences of two or more sample proportions, or percentages, are significant or whether they can be attributed to chance. For solving these problems, we use approximation of binomial distribution to normal distribution by virtue of Central Limit Theorem.

$$X_1 \sim Binomial(n_1, p_1) \qquad X_2 \sim Binomial(n_2, p_2)$$

$$\hat{p}_1 = \frac{X_1}{n_1} \qquad \hat{p}_2 = \frac{X_2}{n_2}$$

According to Centeral Limit Theorem:

$$Z = \frac{\left(\frac{X_1}{n_1} - \frac{X_2}{n_2}\right) - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} \sim N(0,1)$$

Hypotheses	$H_0: p_1 - p_2 = 0$ $H_1: p_1 - p_2 \neq 0$	$H_0: p_1 - p_2 = 0$ $(p_1 - p_2 \geq 0)$ $H_1: p_1 - p_2 < 0$	$H_0: p_1 - p_2 = 0$ $(p_1 - p_2 \leq 0)$ $H_1: p_1 - p_2 > 0$
Test Statistic	$z = \dfrac{\frac{x_1}{n_1} - \frac{x_2}{n_2}}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$		
Decision	If $z \geq z_{\frac{\alpha}{2}}$ or $z \leq -z_{\frac{\alpha}{2}}$ H_0 is rejected. (If $\lvert z \rvert \geq z_{\frac{\alpha}{2}}$ H_0 is rejected.)	If $z \leq -z_\alpha$ H_0 is rejected. (If $\lvert z \rvert \geq z_\alpha$ H_0 is rejected.)	If $z \geq z_\alpha$ H_0 is rejected.

Example 5: In random samples of 250 person with low incomes, 200 person with average incomes, and 150 person with high incomes, there were, respectively, 155, 118 and 87 who favor a certain piece of legislation (yasa). Use the 0.05 level of significance to test the null hypothesis $p_{low} = p_{high}$ against the alternative hypothesis $p_{low} \neq p_{high}$

$H_0: p_{low} = p_{high}$
$H_1: p_{low} \neq p_{high}$

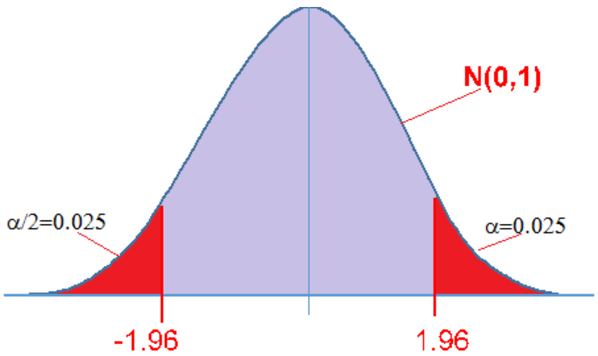
$$\hat{p}_{low} = \hat{p}_1 = \frac{155}{250} = 0.62 \qquad \hat{p}_{high} = \hat{p}_2 = \frac{87}{150} = 0.58$$

$$\hat{p} = \frac{x_1+x_2}{n_1+n_2} = \frac{155+87}{250+150} = \frac{242}{400} = 0.605$$

$$z = \frac{\frac{x_1}{n_1} - \frac{x_2}{n_2}}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{0.62-058}{\sqrt{0.605 \times 0.395\left(\frac{1}{250}+\frac{1}{150}\right)}} = 0.7923$$

table values: $z_\alpha = z_{0.025} = 1.96$

$\underbrace{z = 0.7923}_{test\ value} < \underbrace{z_{0.025} = 1.96}_{table\ value}$, H_0 cannot be rejected.



We conclude that, there is no evidence of a difference between the proportion of persons favoring the legislation is the same for two groups with lower and higher incomes at the level α=0.05.