Solutions

Name:
Student ID:
Section:

1	2	3	4	5	6	T

MAT 254 -01-02 Fundamentals of Linear Algebra Final
June 13, 2019

Note: You have 120 minutes.
1-) Let $W = span\{(1,-1,4), (3,-1,4), (1,1,-4), (4,-2,8)\}$.
a) Find $\dim W$. **(10 pt.)**
b) Find an orthogonal basis for the subspace W. **(10 pt.)**

First way

a) $\begin{bmatrix} 1 & -1 & 4 \\ 3 & -1 & 4 \\ 1 & 1 & -4 \\ 4 & -2 & 8 \end{bmatrix} \xrightarrow[\text{operation}]{\text{some row}} \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$W = span\{(1,-1,4),(0,2,-8)\}$

so a basis is $\{(1,-1,4),(0,2,-8)\}$

Second way

$\begin{bmatrix} 1 & 3 & 1 & 4 \\ -1 & -1 & 1 & -2 \\ 4 & 4 & -4 & 8 \end{bmatrix} \xrightarrow[\text{operation}]{\text{Some row}} \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$W = span\{(1,-1,4),(3,-1,4)\}$

so a basis $\{(1,-1,4),(3,-1,4)\}$

b) Let's take $\{(1,-1,4),(0,2,-8)\}$
By using Gram-Schmidt ortogonalization process to ortogonalize this set

$\alpha_1 = (1,-1,4)$

$\alpha_2 = (0,2,-8) - \frac{(0,2,-8).(1,-1,4)}{(1,-1,4).(1,-1,4)}(1,-1,4) = (0,2,-8) + \frac{34}{18}(1,-1,4)$

$= \left(\frac{17}{9}, \frac{1}{9}, \frac{-4}{9}\right)$

So $\left\{(1,-1,4), \left(\frac{17}{9}, \frac{1}{9}, \frac{-4}{9}\right)\right\}$ is an orthogonal basis of W.

2-) Find the inverse of the matrix (You can use any method) (**10 pt.**)

$$\begin{bmatrix} 0 & -3 & -2 \\ 1 & -4 & -2 \\ -3 & 4 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 0 & -3 & -2 & 1 & 0 & 0 \\ 1 & -4 & -2 & 0 & 1 & 0 \\ -3 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_1} \left[\begin{array}{ccc|ccc} 1 & -4 & -2 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & 0 & 0 \\ -3 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 \to r_3 + 3r_1} \left[\begin{array}{ccc|ccc} 1 & -4 & -2 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & 0 & 0 \\ 0 & -8 & -5 & 0 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 \to r_3 - 3r_2} \left[\begin{array}{ccc|ccc} 1 & -4 & -2 & 0 & 1 & 0 \\ 0 & -3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -3 & 3 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & -4 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -3 & 3 & 1 \\ 0 & -3 & -2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{r_3 \to r_3 + 3r_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -4 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -3 & 3 & 1 \\ 0 & 0 & 1 & -8 & 9 & 3 \end{array}\right] \xrightarrow[r_1 \to r_1 + 2r_3]{r_2 \to r_2 - r_3} \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & -16 & 19 & 6 \\ 0 & 1 & 0 & 5 & -6 & -2 \\ 0 & 0 & 1 & -8 & 9 & 3 \end{array}\right]$$

$$\xrightarrow{r_1 \to r_1 + 4r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 5 & 2 \\ 0 & 1 & 0 & 5 & -6 & -2 \\ 0 & 0 & 1 & \underbrace{-8}_{} & \underbrace{9}_{A^{-1}} & \underbrace{3}_{} \end{array}\right]$$

You can use $A^{-1} = \frac{1}{\det A} \operatorname{Adj} A$ or Cayley Hamilton Theorem, as well.

3-) Let $A = \begin{bmatrix} 0 & -1 & 2 \\ -2 & 1 & 3 \\ 1 & 2 & -3 \end{bmatrix}$ be the matrix represantation of the linear transformation $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ relative to the standart bases. Find $L(2,-3,1)$. **(10 pt.)**

$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

$L(1,0,0) = 0.e_1 + (-2)e_2 + 1.e_3 = (0,-2,1)$

$L(0,1,0) = (-1)e_1 + 1e_2 + 2e_3 = (-1,1,2)$

$L(0,0,1) = 2e_1 + 3e_2 - 3e_3 = (2,3,-3)$

$(2,-3,1) = 2.e_1 + (-3)e_2 + 1e_3 \Rightarrow$

$L(2,-3,1) = L(2e_1 + (-3)e_2 + e_3) = 2.L(1,0,0) - 3L(0,1,0) + L(0,0,1)$

$= 2.(0,-2,1) - 3(-1,1,2) + (2,3,-3) = (5,-4,-7)$

4-) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that $T(a,b,c) = (a+c, b-c)$.
a) Find a bases for $\ker T$. **(10 pt.)**
b) Find a generator set for imT. **(10 pt.)**

a) $\ker T = \{ (a,b,c) : T(a,b,c) = 0 \}$

$T(a,b,c) = 0 \Rightarrow (a+c, b-c) = (0,0) \Rightarrow$

$a+c=0$
$b-c=0$

$c=t=b$

$a=-t$

Then

$\ker T = \{(-t,t,t) : t \in \mathbb{R}\} = \langle(-1,1,1)\rangle$ so $\{(-1,1,1)\}$ is a basis for $\ker T$.

b) $im\,T = \{ T(a,b,c) : (a,b,c) \in \mathbb{R}^3 \}$

$= \{ (a+c, b-c) : a,b,c \in \mathbb{R} \}$

$= \{ a(1,0) + b(0,1) + c(1,-1) : a,b,c \in \mathbb{R} \}$

$= \langle (1,0), (0,1), (1,-1) \rangle$

so $\{(1,0), (0,1), (1,-1)\}$ is a generator set.

5-) Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$.

a) Find the characteristic polynomial of A. (**5 pt.**)

b) Find the eigenvalues and eigenvectors of A. (**15 pt.**)

c) Is A diagonalizable? If so, determine the invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. (**10 pt.**)

a) $|xI - A| = \begin{vmatrix} x-1 & -2 & 1 \\ -1 & x & -1 \\ -4 & 4 & x-5 \end{vmatrix} = (x-1)\begin{vmatrix} x & -1 \\ 4 & x-5 \end{vmatrix} + 2\begin{vmatrix} -1 & -1 \\ -4 & x-5 \end{vmatrix} + \begin{vmatrix} -1 & x \\ -4 & 4 \end{vmatrix}$

$= (x-1)(x^2-5x+4) + 2(1-x) + 4(x-1)$

$= (x-1)(x^2-5x+4-2+4) = (x-1)(x^2-5x+6)$

$= (x-1)(x-2)(x-3)$

b) $\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 3$

For $\lambda_1 = 1$ $\quad A - I = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix} \xrightarrow{r_3 \to r_3 - 4r_2} \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow (A-I)x = 0 \Rightarrow 2x_2 - x_3 = 0 \qquad x_2 = t \quad x_3 = 2t$

$x_1 - x_2 + x_3 = 0 \qquad x_1 = -t$

$\Rightarrow$ eigenvector $= t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda_2 = 2$

$A - 2I = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{bmatrix} \xrightarrow[r_3 \to r_3 + 4r_1]{r_2 \to r_2 - r_1} \begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 4 & -1 \end{bmatrix} \Rightarrow (A-2I)x = 0$

$4x_2 - x_3 = 0$

$-x_1 + 2x_1 - x_3 = 0$

$x_2 = t \quad x_3 = 4t$

$x_1 = -2t$

$\Rightarrow$ eigenvector $= t\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

For $\lambda_3 = 3$

$$(A-3I) = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix} \xrightarrow{r_3 \to r_3 + 2r_1} \begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \to r_1 + 2r_2} \begin{bmatrix} 0 & -4 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow (A-3I)x = 0 \Rightarrow$ $4x_2 - x_3 = 0$ $\quad x_2 = t \quad x_3 = 4t$

$x_1 - 3x_2 + x_3 = 0 \quad x_1 = -t$

$\Rightarrow$ eigenvector $= t\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

c) Each eigenvalue is distinct then A is diagonalizable

$$P = \begin{array}{c} \downarrow \quad \downarrow \quad \downarrow \text{ eigenvectors} \\ \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix} \end{array} \qquad D = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\ \uparrow \uparrow \uparrow \text{ eigen values with same order} \end{array}$$

6-) True or False. If true, you should prove the statement. If false, you should provide a counterexample (Undisclosed answers will not be evaluated).

a) The set $W = \{f \in P_3(x) : \deg f = 3\}$ is a subspace of $P_3(x)$. **(5 pt.)**

b) The set of $n \times n$ skew-symmetric matrices is closed under the matrix addition. **(5 pt.)**

c) Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$. Then $\det A = 4$. **(5 pt.)**

d) Let A be an $n \times n$ diagonalizable matrix with eigenvalues only 1's and -1's. Then $A^2 = I$. **(5 pt.)**

Note that a and b were also your midterm questions.

a) (F) $x_3 \in W$ $-x^3 \in W$ but $x_3 + (-x_3) = 0 \notin W$

so it is not closed under addition

b) (T) Let A, B be skew-symmetric matrices ($A^T = -A$, $B^T = -B$)

$A+B$ is skew-symmetric?

$(A+B)^T = A^T + B^T = -(A+B)$ so $A+B$ skew symetric

c) (F) determinants of nonsquare matrices are not defined.

d) (T) Since A is diagonalizable, we can write $A = PDP^{-1}$

where all diagonal entries of D consist 1's and (-1)'s

since $A^2 = PD^2P^{-1}$ and $1^2 = 1$, $(-1)^2 = 1$, $D^2 = I$ and $A^2 = PIP^{-1} = I$

GOOD LUCK

Talha Arıkan (01) - H. Melis Tekin Akçin (02)