SOLUTIONS OF THE EXAM I

1) a) $A = \underbrace{\begin{bmatrix} a & 0 & 0 \\ 2 & b & 0 \\ 3 & 1 & c \end{bmatrix}}_{\text{lower triang.}} \cdot \underbrace{\begin{bmatrix} d & -1 & 3 \\ 0 & e & 1 \\ 0 & 0 & f \end{bmatrix}}_{\text{upper triangular}} \Rightarrow |A| = (abc)(def)$
$= abcdef.$

If $AX=0$ has a nontrivial solution, then A cannot be invertible. Then $|A|=0$. So, $abcdef=0$. This means that at least one of a,b,c,d,e,f must be zero

b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a solution to

$(X-A)^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - A^2$ then $(B-A)^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - A^2$.

So, $(B-A)^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow |B-A||B-A| = \begin{vmatrix} -1 & 0 \\ 1 & 2 \end{vmatrix} = -2$

$\Rightarrow \underbrace{(|B-A|)^2}_{\geq 0} = -2$, a contradiction.

Therefore $(X-A)^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - A^2$ has no solutions among 2×2 matrices.

2) $A = \begin{bmatrix} 1 & 2 & 0 & 4 & 2 \\ 2 & 3 & -1 & 5 & 6 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 4 & 1 \end{bmatrix}$.

a) $|A| = (3)(-1)^{4+4} \begin{vmatrix} 1 & 2 & 0 & 2 \\ 2 & -3 & -1 & 6 \\ 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = 3\left[2(-1)^{3+2} \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 6 \\ 1 & 0 & 1 \end{vmatrix} + 1(-1)^{3+4} \begin{vmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 1 & 2 & 0 \end{vmatrix}\right]$

$$= 3\left[-2 \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 6 \\ 1 & 0 & 1 \end{vmatrix} - 1 \underbrace{\begin{vmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 1 & 2 & 0 \end{vmatrix}}_{0}\right] = -6 \begin{vmatrix} 1 & 0 & 2 \\ 2 & -1 & 6 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= -6\left[\underbrace{1.(-1)^{1+1}}_{1} \begin{vmatrix} -1 & 6 \\ 0 & 1 \end{vmatrix} + 2(-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}\right]$$

$$= -6\left[1(-1) + 2.(1).(1)\right] = -6(-1+2) = -6 \,//$$

b) $|2A^{-1}| = 2^5 |A^{-1}| = 32 \dfrac{1}{|A|} = 32.\dfrac{1}{-6} = -\dfrac{32}{6} = -\dfrac{16}{3} \,//$

c) $A \xrightarrow{R_2 \leftrightarrow R_3} B \Rightarrow |B| = -|A| = -(-6) = 6$

$B \xrightarrow{100R_3 + R_4} C \underset{\text{changed}}{\overset{\text{not}}{\Rightarrow}} |B| = |C| = 6$

$C \xrightarrow{-2R_1} D \Rightarrow |C| = \dfrac{1}{-2}|D| \Rightarrow -\dfrac{1}{2}|D| = 6 \Rightarrow |D| = -12 \,//$

d) $|B^T D^{-1}| = \underbrace{|B^T|}_{|B|}\underbrace{|D^{-1}|}_{\frac{1}{|D|}} = |B| \dfrac{1}{|D|} = \dfrac{6}{-12} = -\dfrac{1}{2} \,//$

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3 $x+y+7z=-7$ a) no solution

$2x+3y+17z=-16$ b) inf. many sol. $\to$ find

$x+2y+(a^2+1)z=3a$ c) unique sol. $\to$ find

$$\left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 2 & 3 & 17 & -16 \\ 1 & 2 & a^2+1 & 3a \end{array}\right] \xrightarrow[(-1)R_1+R_3]{(-2)R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 7 & -7 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & a^2-6 & 3a+7 \end{array}\right] \xrightarrow{(-1)R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & a^2-6 & 3a+7 \end{array}\right]$$

$$\xrightarrow{(-1)R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & a^2-9 & 3a+9 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & \frac{a^2-9}{3} & a+3 \end{array}\right]$$

a) If $a^2-9=0$, and $a+3\neq 0$ then the system is inconsistent, so there is no solution

Namely $(a-3)(a+3)=0$ and $(a+3)\neq 0$ means that

$$a-3=0 \Rightarrow \boxed{a=3}$$

b) If $\underbrace{a^2-9=0=a+3}$, then the system has inf. many sol.

$$\Downarrow$$

$$(a-3)(a+3)=0 \text{ and } a+3=0 \Rightarrow \boxed{a=-3}.$$

Let us find the solutions: Then we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left.\begin{array}{l} x+4z=-5 \\ y+3z=-2 \end{array}\right\} \text{ Let } z=t. \text{ Then}$$

$$\boxed{\begin{array}{l} x=-5-4t \\ y=-2-3t \end{array} \quad t\in\mathbb{R}}$$

c) If $\underbrace{\overset{=(a-3)(a+3)}{a^2-9} \neq 0 \text{ and } a+3 \neq 0}$, then there is a uniq. sol.

$\Rightarrow a \neq -3$ $\Rightarrow a \in \mathbb{R} \setminus \{\mp 3\}$ //
$a \neq 3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & \underset{\neq 0}{\frac{a^2-9}{3}} & \underset{\neq 0}{a+3} \end{array}\right] \Rightarrow \begin{array}{l} x+4z=-5 \\ y+3z=-2 \\ \frac{a^2-9}{3} z = a+3 \Rightarrow z = \dfrac{a+3}{\frac{(a+3)(a-3)}{3}} \end{array}$$

$$\Rightarrow z = \frac{\cancel{(a+3)} \cdot 3}{\cancel{(a+3)}\underset{\neq 0}{(a-3)}} = \boxed{\frac{3}{a-3}}$$

Then $y+3z=-2 \Rightarrow y = -2-3z = -2-3 \cdot \frac{3}{a-3} = \boxed{-2-\frac{9}{a-3}}$

and $x+4z=-5 \Rightarrow x = -5-4z = -5-4 \cdot \frac{3}{a-3} = \boxed{-5-\frac{12}{a-3} = x}$

$x = -5 - \frac{12}{a-3}$, $y = -2 - \frac{9}{a-3}$, $z = \frac{3}{a-3}$ is the uniq. sol.

4) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

a) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-1)R_1+R_3]{(-1)R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right] \xrightarrow[(-1)R_3]{(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{(-1)R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \end{array}\right] \xrightarrow[(-3)R_3+R_1]{(-1)R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & -3/2 & 3/2 \\ 0 & 1 & 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \end{array}\right]$$

$$\xrightarrow{(-2)R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3/2 & 1/2 \\ 0 & 1 & 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 & -1/2 \end{array}\right]$$

$$\underbrace{\begin{matrix} -1 & 3/2 & 1/2 \\ 1 & -3/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{matrix}}_{A^{-1}}$$

b) $A\,adj(A) = |A| I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$

$$\Rightarrow adj(A) = A^{-1} \cdot \begin{bmatrix} \overset{2}{|A|} & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = \begin{bmatrix} -1 & 3/2 & 1/2 \\ 1 & -3/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 1 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad //$$

c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 0 & 2 \end{array}\right] \xrightarrow[(-1)R_1+R_3]{(-1)R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -4 \\ 0 & -1 & -3 & 0 \end{array}\right] \xrightarrow{(-1)R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & -2 & 4 \end{array}\right]$

$\Rightarrow x+2y+3z=2, \quad \underbrace{-y-z=-4}, \quad -2z=4 \Rightarrow \boxed{z=-2}$

$\Downarrow \qquad\qquad \Downarrow$

$-y+2=-4 \Rightarrow -y=-6 \Rightarrow \boxed{y=6}$

$x+2.6+3.(-2)=2$

$\Downarrow$

$x=2-12+6=2-6 \Rightarrow \boxed{-4=x}$