SECTION 1: PERMUTATION AND COMBINATION

1.1. Permutation and Combination

As known that probability theory started with finding the solution of gamble in literature. In these games, the mathematicians were interested in determining how many different possibilities there were, so then the basic principle of counting and the bases of probability have been come forward.

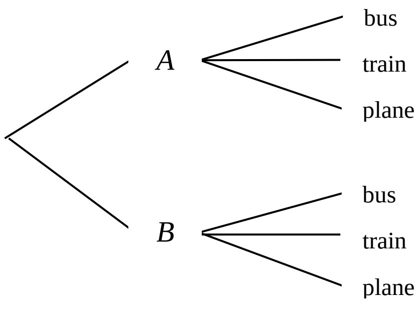
Let us start with the basic principle of counting.

Theorem 1.1: If an operation consists of two steps, of which the first can be done in n_1 ways and for each of these the second can be done in n_2 ways, then the whole operation can be done in $n_1.n_2$ ways. The theorem is called as "*a multiplication rule*".

Proof: We define the ordered pair (x_i, y_j) for each possibility of the first step x_i, the possibilities of second step y_j are arisen. Then the set of all possible of outcomes is composed of the following $n_1.n_2$ pairs.

$$
\begin{gathered}
(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_{n_2}) \\
(x_2, y_1), (x_2, y_2), \ldots, (x_2, y_{n_2}) \\
\cdots \\
(x_{n_1}, y_1), (x_{n_1}, y_2), \ldots, (x_{n_1}, y_{n_2})
\end{gathered}
$$

Example 1. 1: Suppose that someone wants to go to one of the two cities *A*, *B* using by bus, by train, or by plane. Find the number of different ways in which this can be done.
Using tree diagram,



This can be done by $2.3 = 6$ different ways.

Example 1.2: How many possible outcomes are there when we roll a coin and a dice?

$(H,1), (H,2), (H,3), (H,4), (H,5), (H,6)$
$(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)$
In the expression in parenthesis, first one shows *H* for "*Head*" or *T* for "*Tail*" on the coin; second one shows the number on the dice. So 2.6=12 possible outcomes are occurred.

Theorem 1.2: If an operation consists of k steps, of which the first can be done in n_1 ways, for each of these the second step can be done in n_2 ways, for each of these the third step can be done in n_3 ways, and so forth, then the whole operation can be done in $n_1 \bullet n_2 \bullet ... \bullet n_k$ ways.

Example 1. 3: In how many different ways can one answer all the questions of true-false test consisting of 20 questions?

Altogether there are

$$2 \bullet 2 \bullet ... \bullet 2 = 2^{20} = 1,048,576$$

different ways in which one can answer all the questions; only one of these corresponds to the case where all questions are answered right.

Theorem 1.3: The numbers of permutations of *n* distinct objects is *n*!

Example 1. 4: How many permutations are there of the letters *a, b, c, d, e, f, g, i*?

There are 8.7.6.5.4.3.2.1=8!=40,320 ways in which they are arranged.

How many of them are meaningful in English?

Example 1. 5: How many permutations of two random numbers are there of 1, 2, 3, 4? (each number cannot be repeated)

There are two positions to fill, with four choices for the first position and then three choices for the second position. 4.3=12

Example 1. 6: How many permutations of three random numbers are there of 1, 2, 3, 4? (each number can be repeated)

Under replacement condition, the solution is 4.4.4=64.

Theorem 1.4: The numbers of permutations of *n* distinct objects taken *r* at a time is:

$$_{n}P_{r} = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!} \tag{1}$$

for $r = 0,1,...,n$

Proof: The formula in Eq.(1),

$$_{n}P_{r} = n(n-1)(n-2)...(n-r+1)\frac{(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

Example 1. 7: Art Gallery owner exhibits the 4 ones of 6 paintings by Spanish painter “Frida” on its wall. How many exhibitions can be done using Frida’s paintings?

From Eq.(1), the solution is $_{6}P_{4} = 6 \bullet 5 \bullet 4 \bullet 3 = 360$.

Theorem 1.5: The numbers of permutations of n distinct objects arranged in a circle is $(n-1)!$

Example 1. 8: How many circular permutations are there of four persons (gamblers) playing bridge?

From Theorem 1.5, (4-1)!=3!=6.

Theorem 1.6: The number of permutations of n objects of which n_1 are of one kind, n_2 are of second kind,…, n_k are of a k^{th} kind, and $n_1 + n_2 + ... + n_k = n$ is:

$$\frac{n!}{n_1!n_2!...n_k!}. \tag{2}$$

Example 1. 9: In how many ways can two paintings by Monet, three paintings by Renoir, and two paintings by Degas be hung side by side on a museum wall if we do not distinguish between the paintings by the same artists?

7 paintings are ordered by 7!=5040 ways. But we do not see two or more paintings successively. So we need to "subtract or other words divide" the successive paintings of any painter.

$\frac{7!}{2!3!2!} = 210$ arrangements can be done.

Combination means the same as "subset" the number of combinations of r objects selected from a set of n distinct objects. In the combination, putting objects in order is not asked. In general there are r! permutations of objects in a subset of r objects, so that the ${}_nP_r$ permutations of r objects selected from a set of n distinct objects contain each subset r! times. Dividing ${}_nP_r$ by r! and then combination formula is shown as $\binom{n}{r}$.

Theorem 1.7: The number of combinations of n distinct objects taken r at a time is:

$${}_nC_r = \binom{n}{r} = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}$$

for $r = 0,1,...,n$. In general, combination formula is written as:

$${}_nC_r = \binom{n}{r} = \frac{n(n-1)...(n-r+1)}{r!}.$$

Example 1. 10: In how many different groups with 3 persons from 10 people can be constituted?

$\binom{10}{3} = \frac{10!}{3!7!} = \frac{8 \times 9 \times 10}{3!} = 120$ different groups can be constituted.

Example 1. 11: In how many different ways can six tosses of a coin yield two heads and four tails?

HHTTTT, HTHTTT, THHTTT,… $\frac{6!}{2!4!}=\frac{6.5}{2!}=15$

Theorem 1.8: The number of ways in which a set of n distinct objects can be partitioned in to k subsets with n_1 objects in the first subset, n_2 objects in the second subset,…, and n_k objects in the k^{th} subset is:

$$\binom{n}{n_1,n_2,\ldots,n_k}=\frac{n!}{n_1!\bullet n_2!\bullet\ldots\bullet n_k!}. \tag{3}$$

Proof:

$$\begin{aligned}\binom{n}{n_1,n_2,\ldots,n_k}&=\binom{n}{n_1}\binom{n-n_1}{n_2}\bullet\ldots\bullet\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}\\&=\frac{n!}{n_1!(n-n_1)!}\bullet\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\\&\qquad\bullet\ldots\bullet\frac{(n-n_1-n_2-\ldots-n_{k-1})!}{n_k!0!}\\&=\frac{n!}{n_1!\bullet n_2!\bullet\ldots\bullet n_k!}\end{aligned}$$

1.2. Binomial Coefficients

When we multiply out $(x+y)^n$ term by term for n is positive integer, each term will be the product of x's and y's. For example,

$$\begin{aligned}(x+y)^3&=(x+y)(x+y)(x+y)\\&=x^3+3x^2y+3xy^2+y^3\end{aligned}$$

The terms x^3, x^2y, xy^2, y^3 has the components 1, 3, 3 and 1 respectively. These components can be written as ***binomial coefficients*** $\binom{3}{0}=1, \binom{3}{1}=3, \binom{3}{2}=3, \binom{3}{3}=1$.

Theorem 1.9: For any positive integer n,

$$(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r. \tag{4}$$

Theorem 1.10: For any positive integer n and for $r=0,1,\ldots,n$,

$$\binom{n}{r}=\binom{n}{n-r} \tag{5}$$

Proof: $\binom{n}{n-r} = \dfrac{n!}{(n-r)!r!} = \binom{n}{r}$

Theorem 1.11: For any positive integer n and for $r = 0,1,\ldots,n$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

Proof: Substituting x=1 into $(x+y)^n$, let us write

$(1+y)^n = (1+y)(1+y)^{n-1} = (1+y)^{n-1} + y(1+y)^{n-1}$ and the coefficient of the y^r in $(1+y)^n$ is $\binom{n}{r}$ which is equal to sum up the coefficient $\binom{n-1}{r}$ of the y^r in $(1+y)^{n-1}$ and the coefficient $\binom{n-1}{r-1}$ of the y^{r-1} in $(1+y)^{n-1}$. So the proof is completed.

Alternatively, the proof of the theorem can be seen easily from Pascal triangle:

$$\begin{array}{lccccccccc}
n=0 \Rightarrow & & & & & 1 & & & & \\
n=1 \Rightarrow & & & & 1 & & 1 & & & \\
n=2 \Rightarrow & & & \boxed{1} & & \boxed{2} & & 1 & & \\
n=3 \Rightarrow & & 1 & & 3 & & 3 & & 1 & \\
n=4 \Rightarrow & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
\qquad
\binom{2}{0} + \binom{2}{1} = \binom{3}{1}$$

Theorem 1.12: For any positive integer n and for $r = 0,1,\ldots,n$,

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}.$$

Proof: Using the same technique as Theorem 1.11,

$(1+y)^{m+n} = (1+y)^m (1+y)^n$, and the coefficient of y^k in $(1+y)^{m+n}$ is $\binom{m+n}{k}$ which is equal to the multiplying of the coefficients of term $(1+y)^m (1+y)^n$.

Note: For many other properties, see Miller and Miller, 1999.

Example 1. 11: Let we show that,

a) $\sum_{r=0}^{n} \binom{n}{r} = 2^n$, when x=1 and y=1 are substituted in the term of $(x+y)^n$, we get this result.

b) $\sum_{r=0}^{n}(-1)^r\binom{n}{r}=0$, when y=-x and x=1 are substituted in the term of $(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$, we get this result.

c) $\sum_{r=0}^{n}\binom{n}{r}(a-1)^r=a^n$, when x=1, y=a-1 are substituted in the term of $(x+y)^n=\sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$, we get this result.

Example 1. 12: There are four routes, A, B, C, and D, between a person's home and the place where he works, but route B is one way, so he cannot take it on the way to work, and route C is one way, so he cannot take it on the way home.

a) Draw a tree diagram showing the various ways the person can go to and from work.
AA, AD, DA, DD, AB, DB, CA, CD, CB that is 3 ×3=9
First letter says us the route is used to go work from home,
Second letter says us the route is used to go back home from work.

b) Draw a tree diagram showing the various ways he can go to and from work without taking the same route both ways.
AD, DA, AB, DB, CA, CD, CB so that is 7.

Example 1. 13: A person with $2 in her pocket bets $1, even money, on the flip of coin, and she continuous to bet $1 as long as she has any money. Draw a tree diagram to show the various things that can happen during the first four flips of the coin. After the fourth flip of the coin, in how many of the cases will she be

a) Exactly no money,

b) Exactly $2

Example 1. 14: Seven people are in an elevator which stops at ten floors. No two people get off at the same floor. Before the elevator begins to travel, each of them pushes a button for his or her floor. In how many ways can the elevator buttons be lighted?

It doesn't matter what order the buttons were pressed, just which 7 of the 10 floors were selected. $\binom{10}{7}$

Example 1. 15:

a) How many positive integers less than 1000 using the digits (numbers) 2, 3, and 4?

$3+3^2+3^3=3+3.3+3.3.3=39$ (repeated)
$3+{}_3P_2+3!=3+3.2+3.2.1=15$ (unrepeated)

b) How many positive integers less than 1000 using the digits (numbers) 2, 3, and 4 in the order given?

$$\begin{array}{ccccccl} 3 & + & 3 & + & 1 & = 7 & \\ \{2,3,4\} & & \{(23),(24),(34)\} & & \{(234)\} & & \textit{possible results} \end{array}$$

Example 1. 16: If eight persons are having dinner together, in how many different ways can three order chickens, four order steaks, one order lobster? (Assume that they are sitting circle table)

$$\frac{(8-1)!}{3!4!} = \frac{4!5.6.7}{4!.6} = 35$$

Example 1.17: How many 5 card hands can be made if there must be 3 of one face value and 2 other cards with different face values?

$$\binom{3}{1}.\binom{4}{3}.\binom{4}{1}\binom{4}{1}$$

Example 1. 18: In selecting an ace, king, queen, and jack from an ordinary deck of 52 cards, how many ways may we choose if: (a) they must be of different suits? (b) they may or may not be of different suits? (c) they must be of the same suit?

There are 13 cards of each symbol.

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Diller	♥	♦	♣	♠
Türkçe	Kupa	Karo	Sinek	Maça
Fransızca	Cœur	Carreau	Trèfle	Pique
İngilizce	Hearts	Diamonds	Clubs	Spades

Example 1. 19: On a Friday morning, the pro shop of a tennis club has 14 identical cans of tennis balls. If they are all sold by Sunday night and we are interested only in how many were sold on each day, <u>in how many different ways</u> could the tennis balls have been sold on Friday, Saturday, and Sunday?
In each data from 0 to 14 cans of tennis balls could be sold.
Friday, Saturday, Sunday: x+y+z=14.

$$\binom{14+3-1}{3} = \binom{16}{3} \text{ are different ways}$$

Example 1. 20: Rework Example 1. 19 given that at least two of the cans of tennis balls were sold on each of three days

$\binom{8+3-1}{3}$ are different ways