BBM 205

Spring 2015 Butunleme Exam

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Name: SOLUTIONS

1. (3 points) Solve the recurrence relation with the given initial condition below. $a_n = 2a_{n-1} + 8a_{n-2}$; $a_0 = 4$, $a_1 = 10$.

Let $a_n = t^n$ for $n \geq 0$

$t^n - 2t^{n-1} - 8t^{n-2} = 0$

$t^2 - 2t - 8 = 0$

$(t-4)(t+2) = 0$

$t_1 = 4$, $t_2 = -2$

Solution: $S_n = A t_1^n + B t_2^n$

where $S_0 = 4 = A \cdot 4^0 + B \cdot (-2)^0 = A + B$

$S_1 = 10 = A \cdot 4 - 2B$

$\times 2$

$+ \; 8 = 2A + 2B$

$18 = 6A$

$\boxed{3 = A}$

$B = \frac{10 - 4A}{-2} = \frac{10-12}{-2} = \boxed{1 = B}$

Solution: $S_n = 3 \cdot 4^n + (-2)^n$

2. (3 points) (a) (1 point) How many bit strings of length seven either begin with two 0's or end with three 1's?

$A = \{\text{strings begin with } 00\}$, $|A| = 2^5$
$B = \{\text{strings end with } 111\}$, $|B| = 2^4$
$A \cap B = \{\text{strings as } 00\ldots111\}$, $|A \cap B| = 2^2$

$|A \cup B| = |A| + |B| - |A \cap B| = 2^5 + 2^4 - 2^2$

(b) (1 point) How many subsets with more than two elements does a set with 100 elements have?

$$2^{100} - \underbrace{1}_{\binom{100}{0}} - \binom{100}{1} - \binom{100}{2}$$

(c) (1 point) How many ways are there to select three **unordered** elements from a set with five (different) elements when **repetition is allowed**? Same as finding the number of solutions

$x_1, x_2, x_3, x_4, x_5 \geq 0$ with $x_1 + x_2 + x_3 + x_4 + x_5 = 3$

$$\binom{3+5-1}{5-1} = \binom{7}{4}$$

3. (3 points) Suppose that there are nine students in a discrete mathematics class at a small college.

(a) (1.5 points) Show that the class must have at least five male students or at least five female students.

By pigeonhole principle, assuming the boxes are box-1 $\cong$ male and box 2 $\cong$ female, one box must contain at least $\left\lceil \frac{9}{2} \right\rceil = 5$ students.

(b) (1.5 points) Show that the class must have at least three male students or at least seven female students.

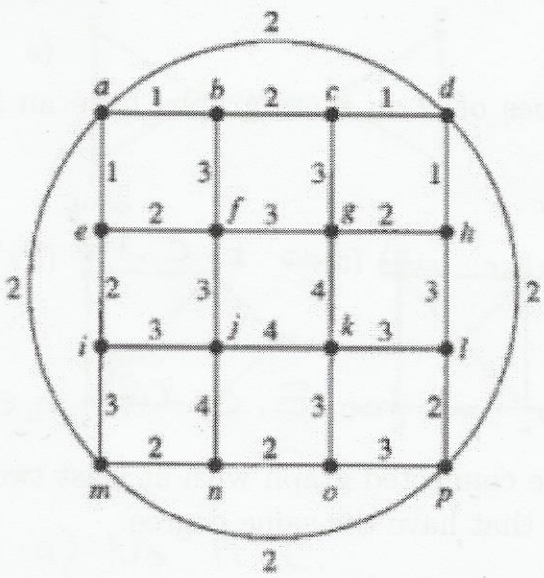
Proof by contradiction: Assume not. Then, the class has at most 2 male and at most 6 female students, adding up to at most 8 students, contradiction.

4. (4 points) Use

(a) (2 points) Kruskal's algorithm

(b) (2 points) Prim's algorithm

to find a minimum spanning tree for the weighted graph below.

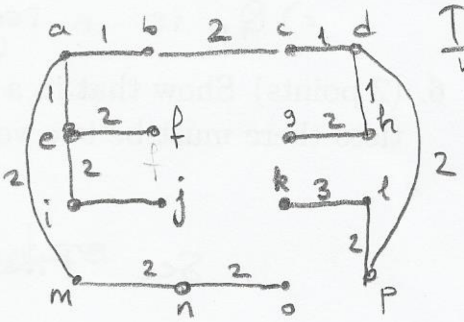


Note: The tree in (a),(b) is NOT unique.

a) Kruskal's algorithm

edges picked in order:

ab (1), ae (1), cd (1), dh (1)
bc (2), ef (2), gh (2), ei (2),
am (2), dp (2), lp (2), mn (2),
no (2), ij (3), kl (3)

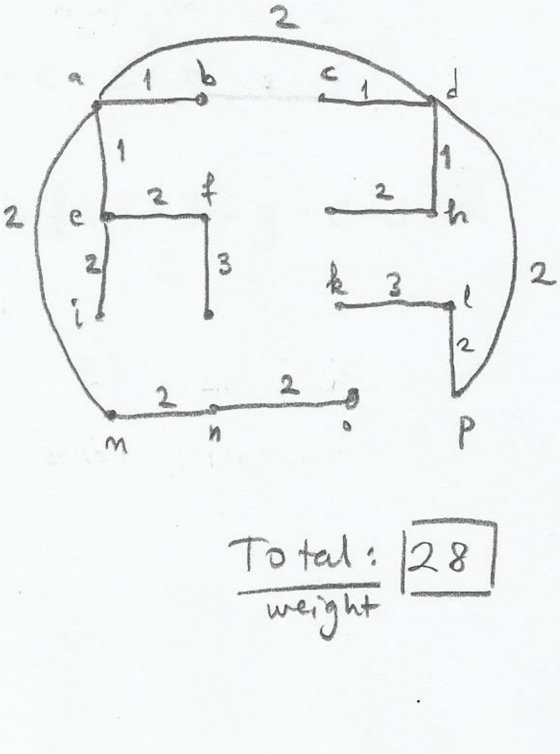


Total weight obtained using Kruskal's Algo.: $\boxed{28}$

b) Prim's algorithm:

edges picked in order: (if we start at the edge ab)

ab(1), ae (1), ad (2), cd (1),
dh (1), am (2), ei (2), ef (2),
hg (2), mn (2), dp (2), lp (2),
no (2), fj (3), kl (3)



Total weight: $\boxed{28}$

5. (3 points) (a) (1.5 points) Write the chromatic number of the graphs below depending on the values of m and n.
a) K_n b) C_n c) $K_{m,n}$

a) n b) 2 if n, even
3 if n, odd
c) Always 2 independent of $m \geq 1$ and $n \geq 1$.

(b) (1.5 points) For which values of n do these graphs have an Euler circuit?
a) K_n b) C_n c) Q_n

a) K_n is $(n-1)$-regular and has E. C. only if n, odd.
b) All n
c) Q_n is n-regular and has E. C. only if n, even.

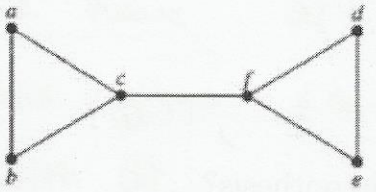
6. (2 points) Show that in a simple connected graph with at least two vertices there must be two vertices that have the same degree.

See Final Exam, question 1.

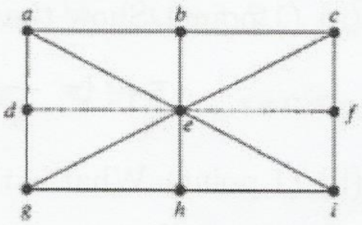
7. (4 points) (a) (2 points) Determine whether the given graph has a Hamilton cycle. Construct such a cycle when one exists.

(b) (2 points) If no Hamilton cycle exists, determine whether the graph has an Hamilton path and construct such a path if one exists.

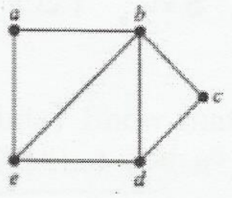
a)



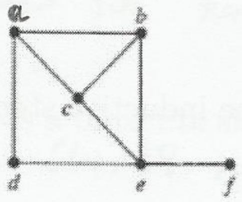
d)



b)



c)



a) No H. C.

H. path: a - b - c - f - d - e

b) H. cycle: a - b - c - d - e - a

c) No H. C.

H. path: d - a - b - c - e - f

d) H. cycle: a - b - c - f - i - h - g - e - d - a

8. (3 points) Let $P(n)$ be the statement that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n},$$

where n is an integer greater than 1. Show that $P(n)$ is true for all $n \geq 2$ using induction by following the steps below.

(a) (1 point) Show that $P(2)$ is true.

$\frac{5}{4} = 1 + \frac{1}{4} < 2 - \frac{1}{2} = \frac{3}{2}$, True.

(b) (1 point) What is the inductive hypothesis?

Assume that for all $i \leq n$, $P(i)$ is true.

(c) (1 point) Complete the inductive step.

Show that $P(n+1)$ is ~~in~~ true using (b);

$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} \underset{\text{by I.H. (b)}}{<} 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \underset{?}{<} 2 - \frac{1}{n+1}$

YES, because $\frac{1}{(n+1)^2} < \frac{1}{n} - \frac{1}{n+1}$ (arithmetic skipped)

9. (3 points) (a) (1 point) Show that $x^2 + 4x + 17$ is $O(x^3)$.

See final exam, question 13 (a), (b).

(b) (2 points) Show that x^3 is **not** $O(x^2 + 4x + 17)$.

10. (3 points) Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers.

Assume not. Then, if $\frac{a_1+a_2+\cdots+a_n}{n} = M$ ⊛,

we have $a_i < M$ for all $i = 1, 2, \ldots, n$.

This gives $a_1+a_2+\cdots+a_n < M \cdot n$, contradiction with ⊛.

11. (3 points) Show that if G is a bipartite simple graph with n vertices and e edges, then $e \leq n^2/4$.

$\boxed{\text{Solution 1}}$: Use induction on n.

Base step: True for $n = 2$ or $e \leq \frac{2^2}{4} = 1$

Inductive Hypothesis (I. H.): Assume true for all bipartite simple graphs with $i \leq n$ vertices.

Inductive Step: Show for all graphs with $n+1$ vertices.

- Let G be any bipartite, simple graph with $n+1$ vertices and e edges.
- Remove from G a vertex x with minimum degree $d(x)$.
- Clearly, $d(x) \leq \frac{n+1}{2}$ (if n, odd) or $d(x) \leq \frac{n}{2}$ (if n, even)
- By I. H., $e(G-x) \leq \frac{n^2}{4}$.

$$+ \quad d(x) \leq \frac{n+1}{2} \Big\} \text{ if } n, \text{odd}$$

$$e(G) \leq \left\lfloor \frac{n^2}{4} + \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n^2+2n+2}{4} \right\rfloor \leq \frac{(n+1)^2}{4}$$

(n, odd)

if n, even:

$$e(G-x) \leq n^2/4$$

$$+ \quad d(x) \leq n/2$$

$$e(G) \leq \left\lfloor \frac{n^2}{4} + \frac{n}{2} \right\rfloor = \left\lfloor \frac{n^2+2n}{4} \right\rfloor \leq \frac{(n+1)^2}{4}$$

$\boxed{\text{Solution 2}}$: (shorter)

If $G = (X, Y)$ with $|X| = x$, $|Y| = y$ and $x + y = n$, then

$$e(G) \leq x \cdot y = \underbrace{x \cdot (n-x)}_{}$$

maximize for x when $0 \leq x \leq n$ (by MAT123)

we observe $x(n-x) \leq \frac{n^2}{4}$

12. (3 points) Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

$\boxed{\Rightarrow}$ If v is a cut vertex, then $G_2 - \{v\}$ is not empty. Therefore, v is not pendant.

$\boxed{\Leftarrow}$

If v is not pendant, then G_2 is not empty in the figure and removing v disconnects u from $G_2 - \{v\}$. Therefore, v is a cut vertex.

13. (3 points) Let $S(n,k)$ denote the number of functions from $\{1,\ldots,n\}$ onto $\{1,\ldots,k\}$. Show that $S(n,k)$ satisfies the recurrence relation

$$S(n,k) = k^n - \sum_{i=1}^{k-1} C(k,i)S(n,i).$$

Let $A_i = \{$ functions that have <u>exactly</u> i numbers from $\{1,2,\ldots,k\}$ as its value $\}$

Therefore, $|A_i| = \binom{k}{i} \cdot S(n,i)$ and

$$S(n,k) = k^n - \sum_{i=1}^{k-1} |A_i| = k^n - \sum_{i=1}^{k-1} \binom{k}{i} S(n,i).$$

The number of all functions from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,k\}$.