SECTION 8: MOMENT GENERATION FUNCTIONS

The moments of distribution can be calculated by the evaluating integrals or sums, the moment generation functions provide the moment by simply ways.

Definition 8.1. The moment generation function of the random variable *X*, where it exists, is given by

$$M_X(t) = E\left(e^{tX}\right)$$

For *X* random variable is discrete, $M_X(t) = E\left(e^{tX}\right) = \sum_{\mathbb{R}_x} e^{tx} P(X = x) = \sum_{\mathbb{R}_x} e^{tx}\, p(x)$.

For *X* random variable is continuous, $M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx}\, f(x)\,dx$.

Where *t* is continuous variable. For *t*=0, $M_X(0) = E\left(e^{0}\right) = 1$

When the Maclaurin's series expansion for e^{tX} is taken,

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots$$

then the expected value of this expression is taken by:

$$\begin{aligned} E\left(e^{tX}\right) &= E\left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots\right) \\ &= 1 + tE(X) + \frac{t^2E(X^2)}{2!} + \frac{t^3E(X^3)}{3!} + \ldots \end{aligned} \qquad (1)$$

Thus we get the moment generation function of *X* and the function provides the moments $E(X^k) = \mu_k'$ with the coefficient of $\frac{t^k}{k!}$ about the origin (zero).

Theorem 8.1. If the moment generation function of the random variable *X*, $M_X(t)$, the moments is found as follows:

$$\mu_k' = E(X^k) = \left.\frac{d^kM_X(t)}{dt^k}\right|_{t=0} \qquad (2)$$

Example 1: Given that *X* has the probability mass function $p(x) = \frac{1}{8}\binom{3}{x},\ for\ x = 0,1,2,3$ find the moment generation function of this random variable and determine $\mu_1',\ \mu_2'$.

Solution:

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{3} e^{tx} \frac{1}{8}\binom{3}{x} = \frac{1}{8}\left(1+3e^t+3e^{2t}+e^{3t}\right)$$
$$= \frac{1}{8}\left(1+e^t\right)^3$$

From Theorem 8.1, the μ_1', μ_2' are found respectively:

$$\mu_1' = \left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{3}{8}\left(1+e^t\right)^2 e^t\right|_{t=0} = \frac{3}{2}$$

$$\mu_2'' = \left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = \left.\left(\frac{3}{4}\left(1+e^t\right)e^{2t} + \frac{3}{8}\left(1+e^t\right)^2 e^t\right)\right|_{t=0} = 3$$

Example 2: Let *X* be continuous random variable with probability density function given by:

$$\begin{aligned} f(x) &= e^{-x}, \quad x > 0 \\ &= 0, \qquad x \le 0. \end{aligned}$$

find the moment generation function of this random variable and determine moments of the random variable.

Solution:

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}e^{-x}dx = \int_0^\infty e^{-x(1-t)}dx = \frac{1}{1-t}, \quad t<1$$

Maclaurin's series of the moment generation function is:

$$\begin{aligned} \frac{1}{1-t} &= 1+t+t^2+t^3+\ldots \\ &= 1+t+\frac{2!t^2}{2!}+\frac{3!t^3}{3!}+\ldots \end{aligned}$$

From Equation (1), the moments of X is $\mu_k' = k!$.

Theorem 8.2. If the moment generation function of the random variable *X*, $M_X(t)$, the moment generation function of a*X*+b , for constant *a* and *b*,

$$M_{aX+b}(t) = e^{bt}M_X(at).$$

Proof:

$$M_{aX+b}(t) = E\left[e^{t(aX+b)}\right] = E\left[e^{atX} \underbrace{e^{bt}}_{constant}\right] = e^{bt}E\left[e^{atX}\right] = e^{bt}M_X(at). \tag{3}$$

Theorem 8.3. If the moment generation function of the random variable *X*, $M_X(t)$, the moment generation function of $\dfrac{X+a}{b}$, for constant *a* and *b*,

$$M_{\frac{X+a}{b}}(t) = e^{\frac{a}{b}t} M_X\left(\frac{t}{b}\right). \tag{4}$$

Proof:

$$M_{\frac{X+a}{b}}(t) = E\left[e^{t\left(\frac{X+a}{b}\right)}\right] = E\left[e^{\left(\frac{t}{b}\right)X} \underbrace{e^{\left(\frac{a}{b}\right)t}}_{\text{constant}}\right] = e^{\left(\frac{a}{b}\right)t} E\left[e^{\left(\frac{t}{b}\right)X}\right] = e^{\left(\frac{a}{b}\right)t} M_X\left(\frac{t}{b}\right).$$

Moment generation function of distribution could not be found for many cases. But characteristic function is always found since it is complex number.

Definition 8.2. The characteristic function of the random variable *X*, is given by

$$\varphi_X(t) = E\left(e^{itX}\right) \tag{5}$$

For *X* random variable is discrete, $\varphi_X(t) = E\left(e^{itX}\right) = \sum_{\mathbb{R}_x} e^{itx} P(X = x) = \sum_{\mathbb{R}_x} e^{itx} p(x)$.

For *X* random variable is continuous, $\varphi_X(t) = E\left(e^{itX}\right) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$.

Where *t* is continuous variable and imaginary unit $i = \sqrt{-1}$. For *t*=0, $\varphi_X(0) = E\left(e^0\right) = 1$.

Note: The imaginary unit or unit imaginary number ($i = \sqrt{-1}$) is a solution to the quadratic equation $x^2 + 1 = 0$. Since there is no real number with this property, it extends the real numbers, and under the assumption that the familiar properties of addition and multiplication (namely closure, associativity, commutativity and distributivity) continue to hold for this extension, the complex numbers are generated by including it. https://en.wikipedia.org/wiki/Imaginary_unit

Theorem 8.4. If the characteristic function of the random variable *X*, $\varphi_X(t)$, the moments is found as follows:

$$\mu_k' = E(X^k) = \frac{\left.\dfrac{d^k \varphi_X(t)}{dt^k}\right|_{t=0}}{i^k} \tag{6}$$

Example 3: Let *X* be continuous random variable with probability density function given by:

$$\begin{aligned} f(x) &= e^{-x}, \quad x > 0 \\ &= 0, \qquad x \le 0. \end{aligned}$$

the moment generation function of this random variable is found as $M_X(t) = \dfrac{1}{1-t}$. Find characteristic function of *X*.

Solution:

Using $M_X(t) = \dfrac{1}{1-t}$, to get the characteristic function of *X*, substituting *it* for *t* into the moment generation function $\varphi_X(t) = M_X(it)$ and so $\varphi_X(t) = \dfrac{1}{1-it}$.

Example 4: A discrete random variable *X* and its probability mass function is of the form

$$
\begin{aligned}
p_X(x) &= \binom{n}{x} p^x (1-p)^{n-x}, \quad && x = 0,1,\ldots,n \\
&= && otherwise
\end{aligned},
$$

where *n* is a positive integer and $0 \le p \le 1$.

a) Find characteristic function of *X*.
b) Find *E*(*X*) and *V*(*X*) using its characteristic function.

Solution:

a)

$$
\begin{aligned}
\varphi_X(t) &= E(e^{itX}) = \sum_{x=0}^{n} e^{itX} p_X(x) = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} \left(pe^{it}\right)^x (1-p)^{n-x} = \left[1-p+pe^{it}\right]^n
\end{aligned}
$$

b) From Equation(6) $E(X) = \dfrac{\left.\dfrac{d\varphi_X(t)}{dt}\right|_{t=0}}{i}$ and the first derivation of $\varphi_X(t)$ is:

$$\frac{d\varphi_X(t)}{dt} = \left.\left(n\left[1-p+pe^{it}\right]^{n-1} pie^{it}\right)\right|_{t=0} = npi$$ hence *E*(*X*)=*np*

From the second derivation of $\varphi_X(t)$ is:

$$
\begin{aligned}
\frac{d^2\varphi_X(t)}{dt^2} &= \left.\left(n(n-1)\left[1-p+pe^{it}\right]^{n-2} p^2 i^2 e^{it} + n\left[1-p+pe^{it}\right]^{n-1} pi^2 e^{it}\right)\right|_{t=0} \\
&= n(n-1)p^2 i^2 + npi^2 = n^2 p^2 i^2 - np^2 i^2 + npi^2
\end{aligned}
$$

$$E(X^2) = \frac{n^2 p^2 i^2 - np^2 i^2 + npi^2}{i^2} = n^2 p^2 - np^2 + np$$ then the variance of *X*

$$V(X) = n^2 p^2 - np^2 + np - n^2 p^2 = np(1-p)$$

Definition 8.3. The factorial moment generation function of random variable *X* is defined by:

$$g_X(t) = E\left(t^X\right) \tag{7}$$

$$g_X(t) = E\left(t^X\right) = \sum_{R_x} t^x P(X = x) = \sum_{R_x} t^x p(x) \text{ for } X \text{ discrete random variable.}$$

$$g_X(t) = E\left(t^X\right) = \int_{-\infty}^{\infty} t^x f(x)\, dx \text{ for } X \text{ continuous random variable.}$$

Where *t* is continuous variable. For *t*=1, $g_X(1) = E\left(1\right) = 1$

Theorem 8.5. If the factorial moment generation function of the random variable *X*, $g_X(t)$, the k^{th} factorial moment is found as follows:

$$E\left[X(X-1)\ldots(X-k+1)\right] = \left.\frac{d^k g_X(t)}{dt^k}\right|_{t=1}. \tag{8}$$

Theorem 8.6. If *X* discrete random variable and it has positive integer values, and the factorial moment generation function of the random variable *X*, $g_X(t)$, the probabilities of *X* are found as follows:

$$\begin{aligned} &g_X(0) = P(X = 0) \\ &\left.\frac{dg_X(t)}{dt}\right|_{t=0} = P(X = 1) \\ &\left.\frac{d^2 g_X(t)}{dt^2}\right|_{t=0} = 2!P(X = 2) \\ &\vdots \\ &\left.\frac{d^k g_X(t)}{dt^k}\right|_{t=0} = k!P(X = k) \end{aligned} \tag{9}$$

g(*t*) is called as the probability generation function.

Proof: If *X* discrete random variable has positive integer values, the factorial moment generation function,

$$g_X(t) = E\left(t^X\right) = \sum_{x=0}^{\infty} t^x P(X = x) = t^0 P(X = 0) + t^1 P(X = 1) + \ldots + t^k P(X = k) + \ldots$$

$$\left. g_X(t)\right|_{t=0} = P(X = 0)$$

$$\left.\frac{dg_X(t)}{dt}\right|_{t=0} = \left. P(X = 1) + 2t^1 P(X = 2) + \ldots + kt^{k-1} P(X = k) + \ldots\right|_{t=0} = P(X = 1)$$

$$\left.\frac{d^2 g_X(t)}{dt^2}\right|_{t=0} = \left. 2P(X = 2) + 3 \bullet 2 \bullet t^1 P(X = 3) \ldots + k(k-1)t^{k-2} P(X = k) + \ldots\right|_{t=0} = 2!P(X = 2)$$

These results are generalized for the k^{th} derivation of the function and so the proof is end.

Example 5: A cyber gambler plays a computer game until he wins. From his previous experience, he wins the game with the probability 0.90. Let X random variable is the number of games he plays assuming that each game is independent form previous one.

a) Find factorial moment generation function of *X*.
b) Find *E*(*X*) and *V*(*X*) using factorial moment generation function of *X*.
c) Find moment generation function of X using the factorial moment generation function of *X*.

Solution:

a) For this random event, the probability mass function is given as follows:

$$p(x) = (0.1)^{x-1}(0.9), \quad x = 1, 2, \ldots$$
$$= 0, \qquad otherwise$$

The factorial moment generation function of X is,

$$g_X(t) = E(t^X) = \sum_{x=1}^{\infty} t^x (0.1)^{x-1}(0.9) = (0.9)t \sum_{x=1}^{\infty} (0.1 \times t)^{x-1} = \frac{0.9t}{1-0.1t}$$

b) *E*(*X*) and *V*(*X*) are respectively:

$$E(X) = \left.\frac{dg(t)}{dt}\right|_{t=1} = \frac{d}{dt}\left(\frac{0.9t}{1-0.1t}\right) = \left.\frac{0.9(1-0.1t)+0.1(0.9t)}{(1-0.1t)^2}\right|_{t=1}$$
$$= \frac{1}{0.9} = \frac{10}{9}$$

$$E\left[X(X-1)\right] = \left.\frac{d^2g(t)}{dt^2}\right|_{t=1} = \frac{d}{dt}\left[\frac{0.9}{(1-0.1t)^2}\right] = \left.\frac{2\times 0.9(1-0.1t)\times 0.1}{(1-0.1t)^4}\right|_{t=1}$$
$$= \frac{0.2}{0.81}$$

$$V(X) = E(X^2) - \left[E(X)\right]^2$$
$$= E\left[X(X-1)\right] + E(X) - \left[E(X)\right]^2$$
$$= \frac{0.2}{0.81} + \frac{1}{0.9} - \left(\frac{1}{0.9}\right)^2$$
$$= \frac{0.1}{0.81}$$

c) Put e^t where *t* is in $g_X(t) = \frac{0.9t}{1-0.1t}$ and so $M_X(t) = \frac{0.9e^t}{1-0.1e^t}$ is found easily.

Example 6: Let *X* be discrete random variable and the factorial moment generation function of *X* be $g(t) = \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3$. Find the probability mass function of *X*.

Solution: Using Theorem 8.6,

$$\left.\begin{aligned}
P(X=0) &= g(0) = \frac{1}{8} \\
P(X=1) &= g'(0) = \frac{3}{8} \\
P(X=2) &= \frac{g''(0)}{2!} = \frac{3}{8} \\
P(X=3) &= \frac{g'''(0)}{3!} = \frac{1}{8}
\end{aligned}\right\} \Rightarrow \quad
\begin{aligned}
p(x) &= \frac{1}{8}, \quad x = 0,3 \\
&= \frac{3}{8}, \quad x = 1,2 \\
&= 0, \quad \textit{otherwise}
\end{aligned}$$