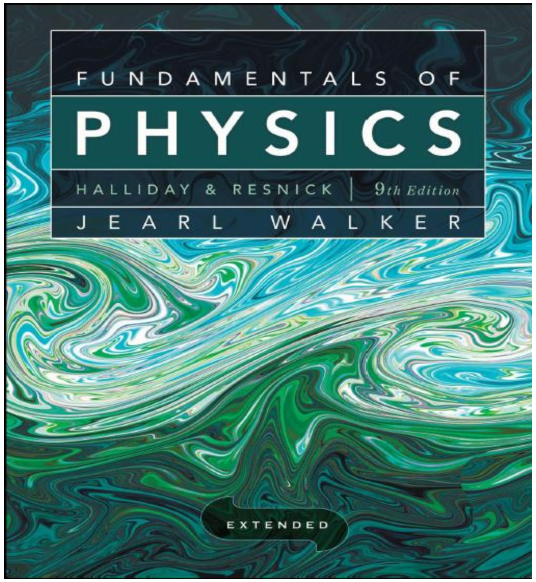
FİZ 137 – 25
CHAPTER 10
ROTATION



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FALL SEMESTER

ROTATIONAL KINEMATICS

In this chapter we will study the rotational motion of rigid bodies about a fixed axis.

Related Topics

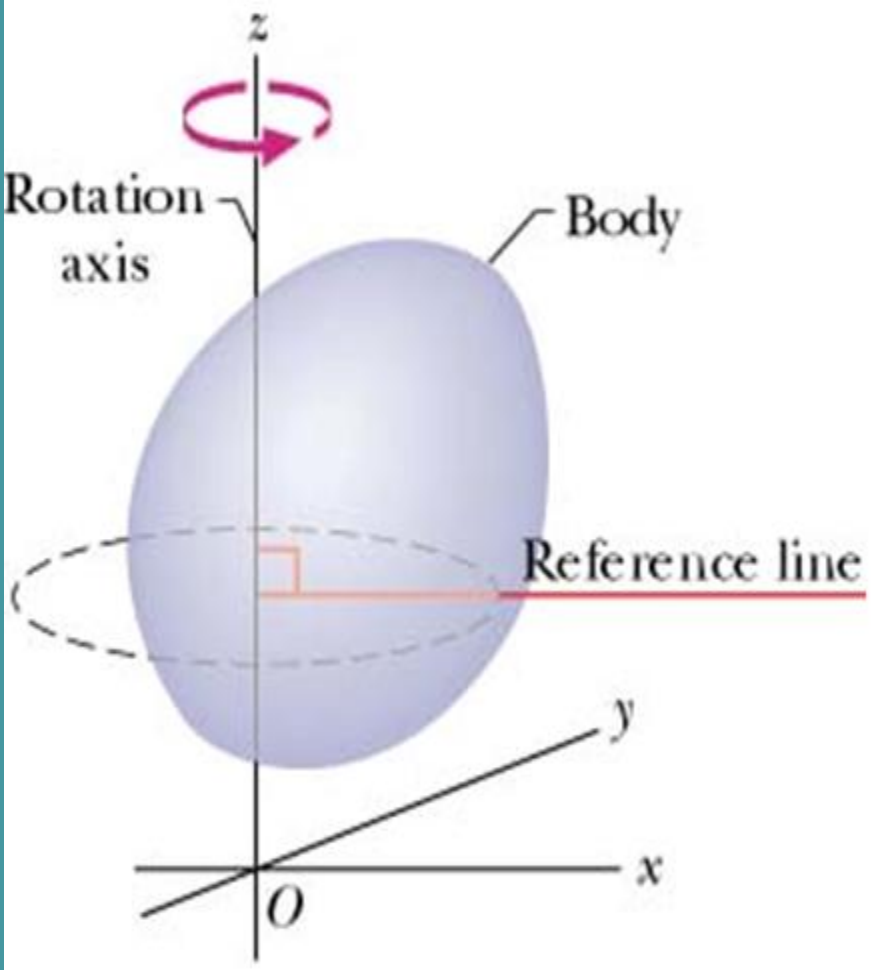
- **Angular Displacement (symbol: θ)**
- **Average and Instantaneous Angular Velocity (symbol: ω)**
- **Average and Instantaneous Angular Acceleration (symbol: α)**
- **Rotational Inertia or Moment of Inertia (symbol: I)**
- **Torque (symbol: τ)**
- **Kinetic Energy Associated with Rotation**
- **Newton's Second Law for Rotational Motion**
- **Work - Kinetic Energy for Rotational Motion**

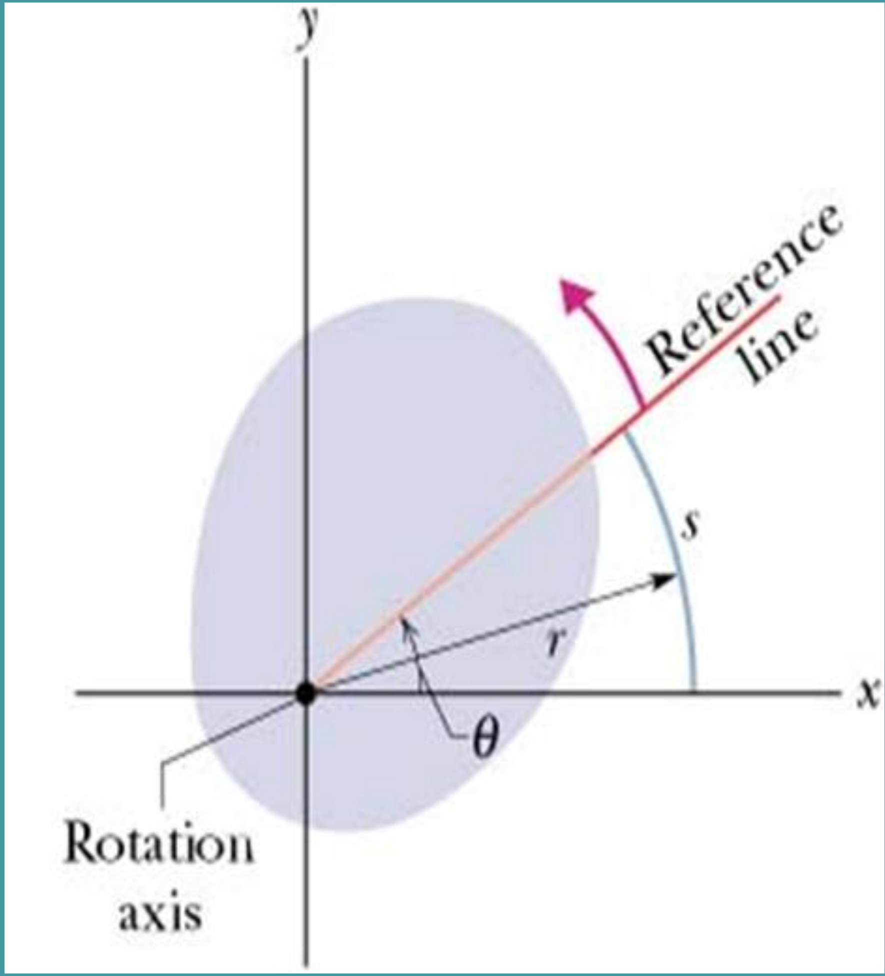
The Rotational Variables

- In this chapter we will study ***the rotational motion of rigid bodies about a fixed axes.***

- A ***rigid body*** is defined as one that can rotate with all its parts locked together and without any change of its shape.

Fixed axis means that the object rotates about an axis that does not move.

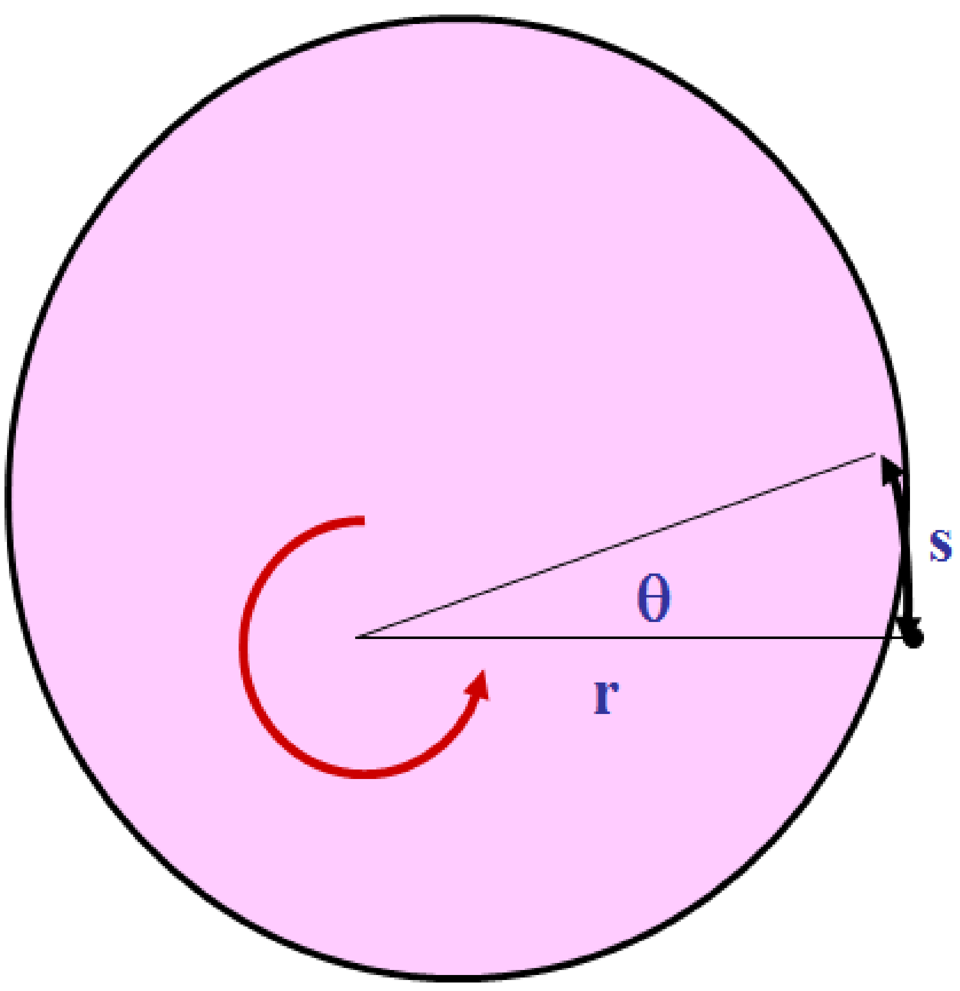




Rotational motion of rigid bodies about fixed axes.

We take the <u>*z*-axis to be the fixed axis of rotation</u>. We define a <u>reference line</u> which is fixed in the rigid body and is perpendicular to the rotational axis.

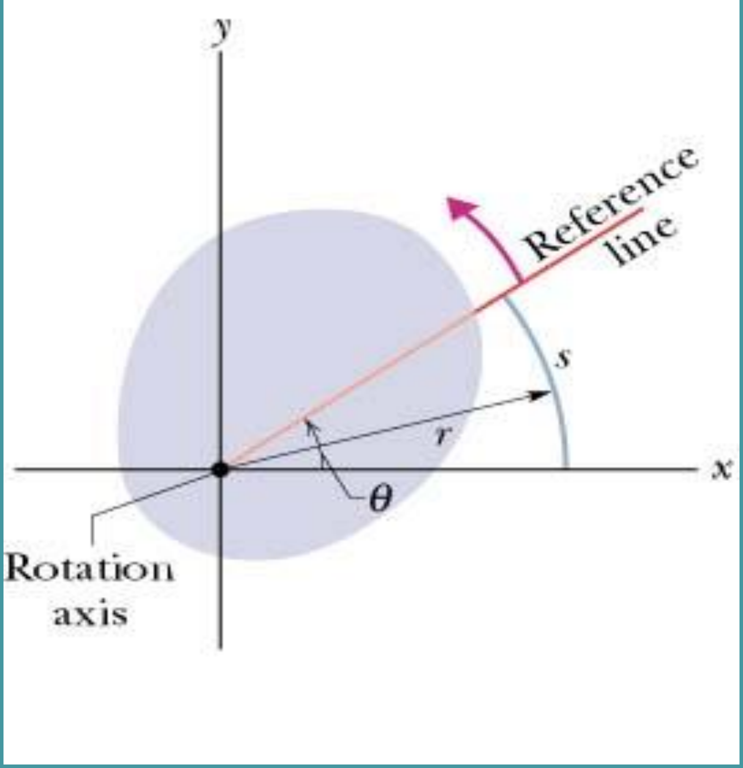
Linear and Angular Variables



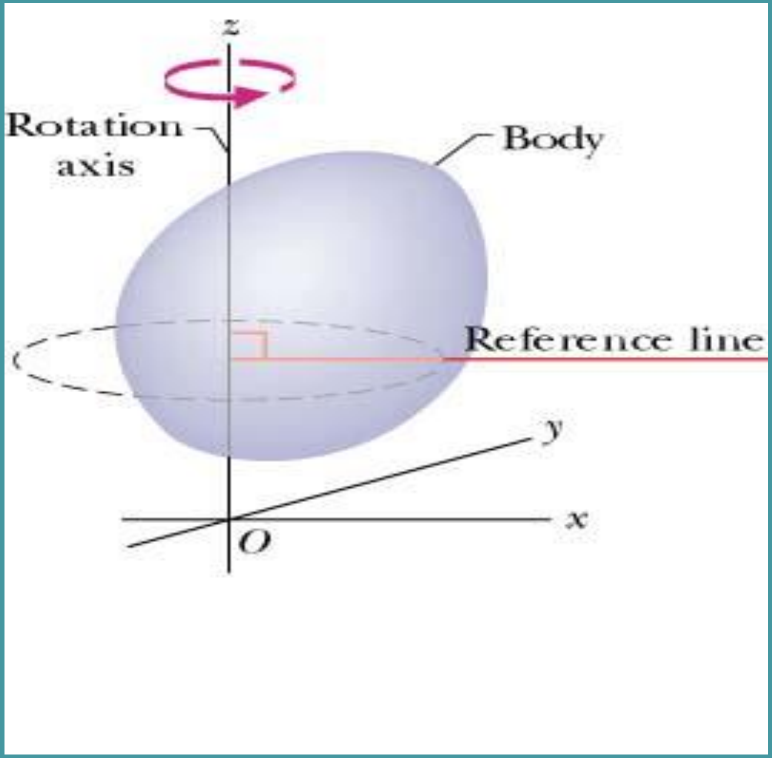
The **angle *θ*** is related to the **arc length s** traveled by a point at a distance **r** from the axis.

The angle *θ* is measured in **radians.**

$$s = \theta r$$



$$\theta = \frac{s}{r}$$

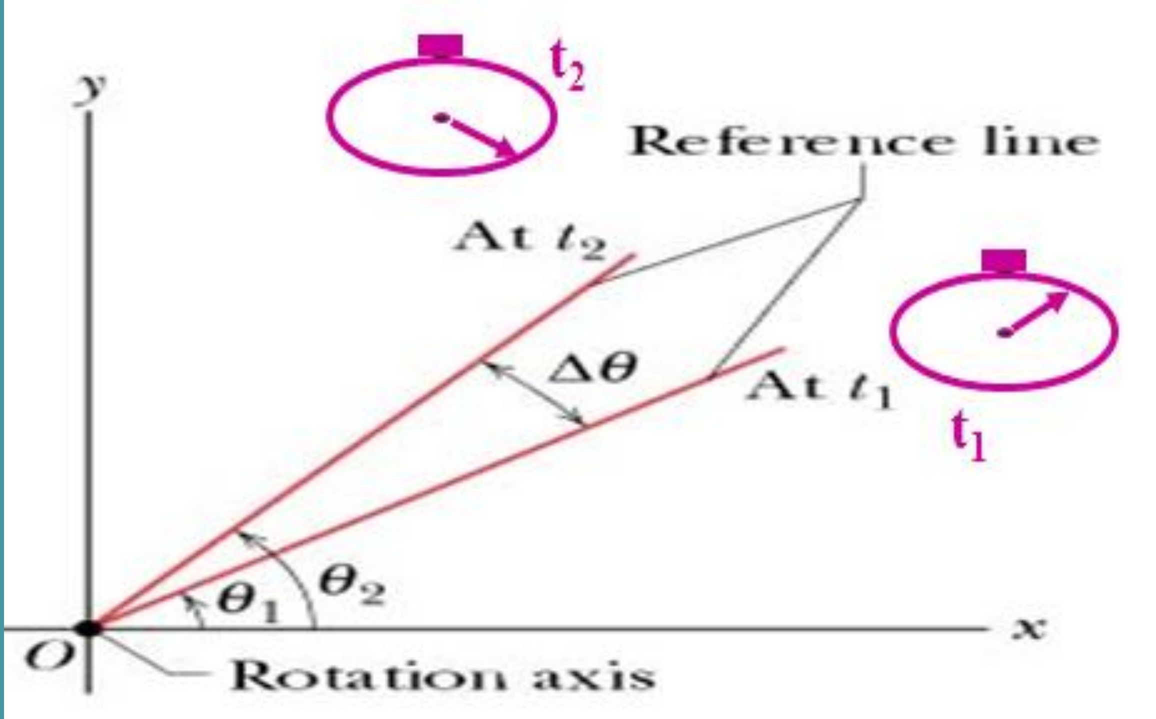


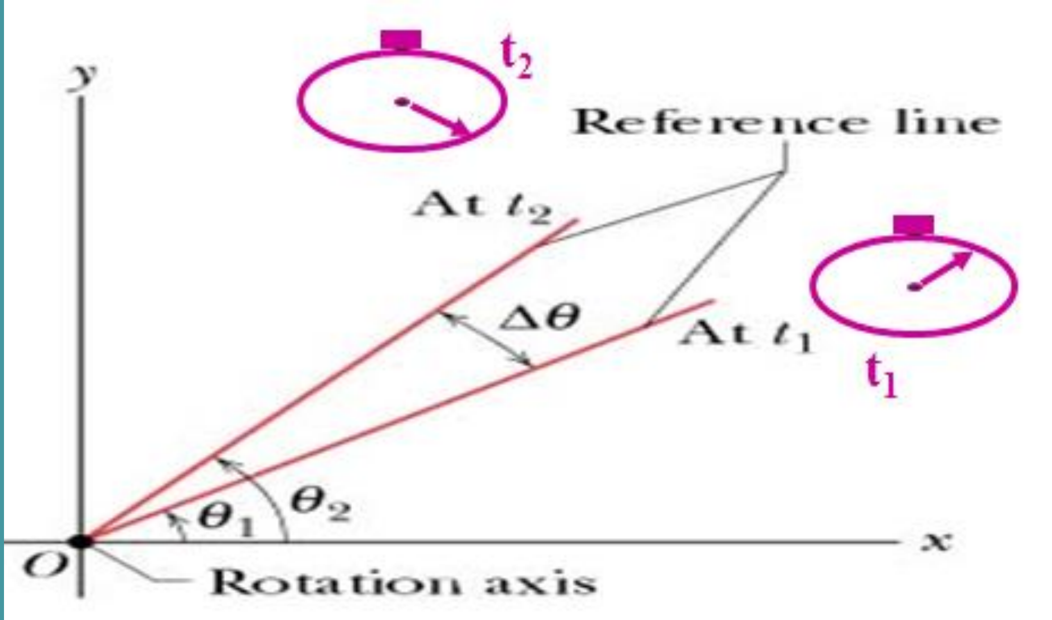
The "**angular position of the reference line" at any time *t* is defined by the angle *θ(t)*** that the reference lines makes with the position at $t = 0$.

The angle ***θ(t)*** *also* defines the position of all the points on the rigid body because all the points are locked as they rotate.

Angular Displacement

In the picture we show the reference line at a time t_1 and at a later time t_2. Between t_1 and t_2 the body undergoes an angular displacement $\boxed{\Delta\theta = \theta_2 - \theta_1}$. All the points of the rigid body have the same angular displacement because they rotate locked together.





$$\omega = \frac{d\theta}{dt}$$

Angular Velocity

We define as average angular velocity for the time interval (t_1, t_2) the ratio:

$$\boxed{\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}}$$

The SI unit for angular velocity is radians/second

We define as the instantaneous angular velocity the limit of $\dfrac{\Delta\theta}{\Delta t}$ as $\Delta t \to 0$

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

This is the definition of the first derivative with t

Algerbraic sign of angular frequency: If a rigid body rotates counterclockwise (CCW) ω has a positive sign. If on the other hand the rotation is clockwise (CW) ω has a negative sign

We can actually use the vector notation to describe rotational motion which is more complicated. The angular velocity vector is defined as follows:

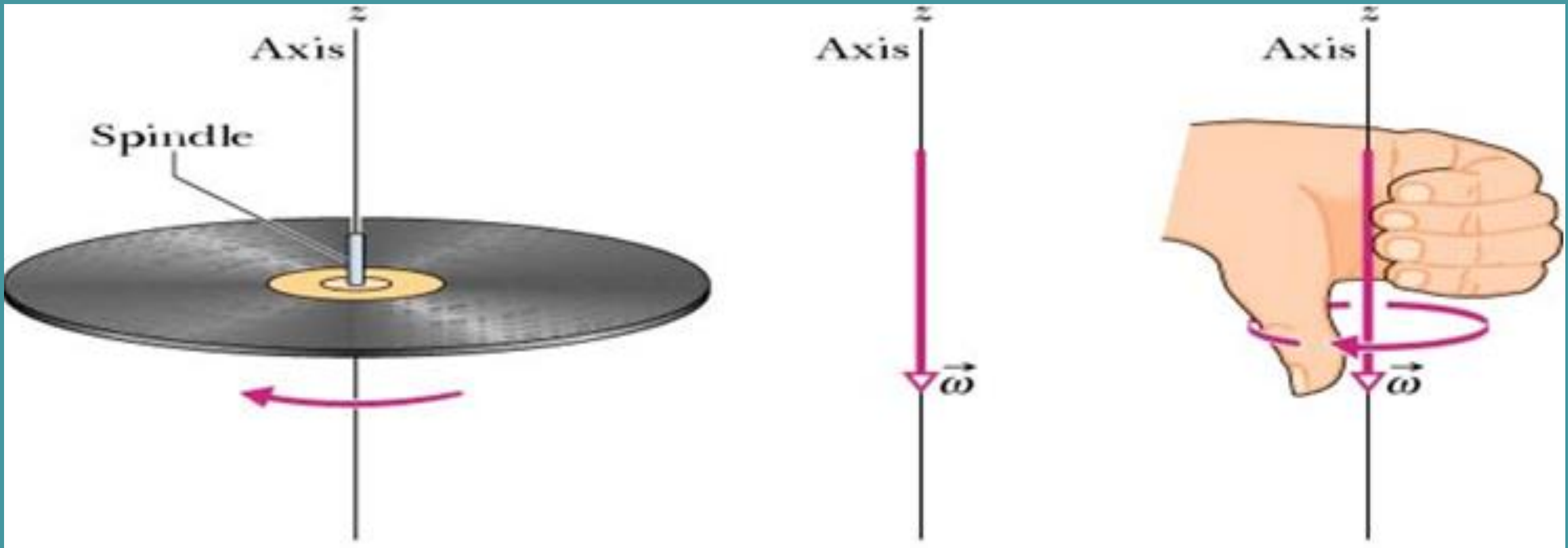
The direction of $\vec{\omega}$ is along the rotation axis.

The sense of $\vec{\omega}$ is defined by the right hand rule (RHL)

Right hand rule: Curl the right hand so that the fingers point in the direction of the rotation. The thumb of the right hand gives the sense of $\vec{\omega}$

Angular Velocity Vector

For rotations of rigid bodies about a fixed axis we can describe accurately the angular velocity by asigning an algebraic sigh. Positive for counterclockwise rotation and negative for clockwise rotation

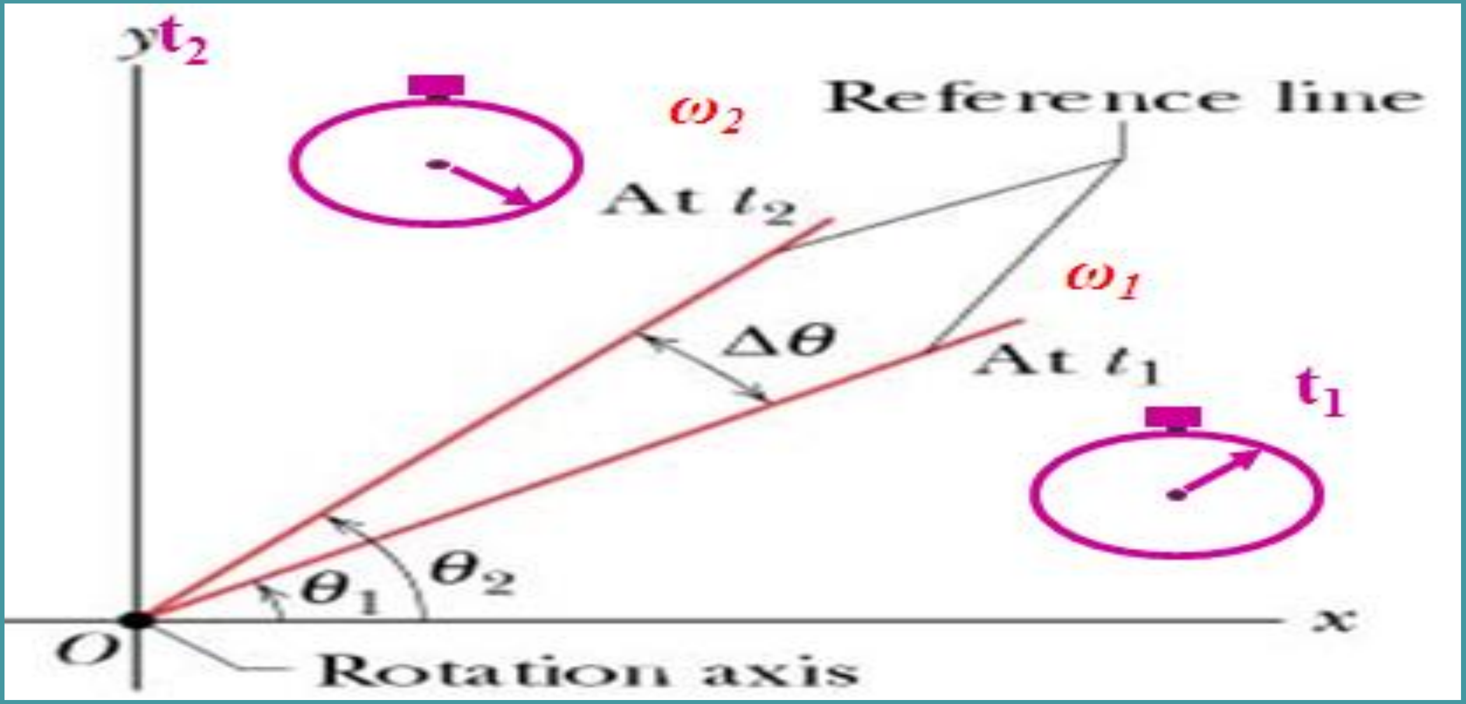


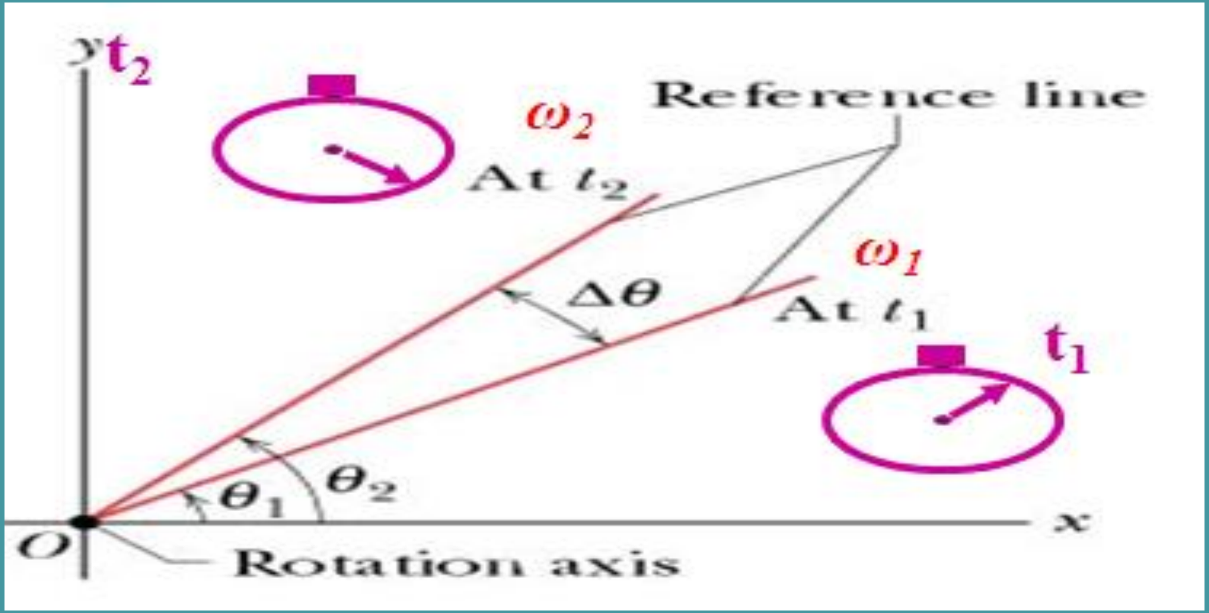
(a) (b) (c)

Angular Acceleration

If the angular velocity of a rotating rigid object changes with time we can describe the time rate of change of ω by defining the angular aceleration

$$\alpha = \frac{d\omega}{dt}$$





$$\alpha = \frac{d\omega}{dt}$$

The angular velocity of the rotating body is equal to ω_1 at t_1 and ω_2 at t_2.

We define as average angular acceleration for the time interval (t_1, t_2) the ratio:

$$\boxed{\alpha_{avg} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}}$$

The SI unit for angular velocity is radians/second^2

We define as the instantaneous angular acceleration the limit of $\frac{\Delta\omega}{\Delta t}$ as $\Delta t \to 0$

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

This is the definition of the first derivative with t

$$\boxed{\alpha = \frac{d\omega}{dt}}$$

Rotation with Constant Angular Acceleration

When the angular acceleration α is constant we can derive simple expressions that give us the angular velocity ω and the angular position θ as function of time We could derive these equations in the same way we did in chapter 2. Instead we will simply write the solutions by exploiting the analogy between translational and rotational motion using the following correspondance between the two motions

Translational Motion   Rotational Motion

$$x \leftrightarrow \theta$$

$$v \leftrightarrow \omega$$

$$a \leftrightarrow \alpha$$

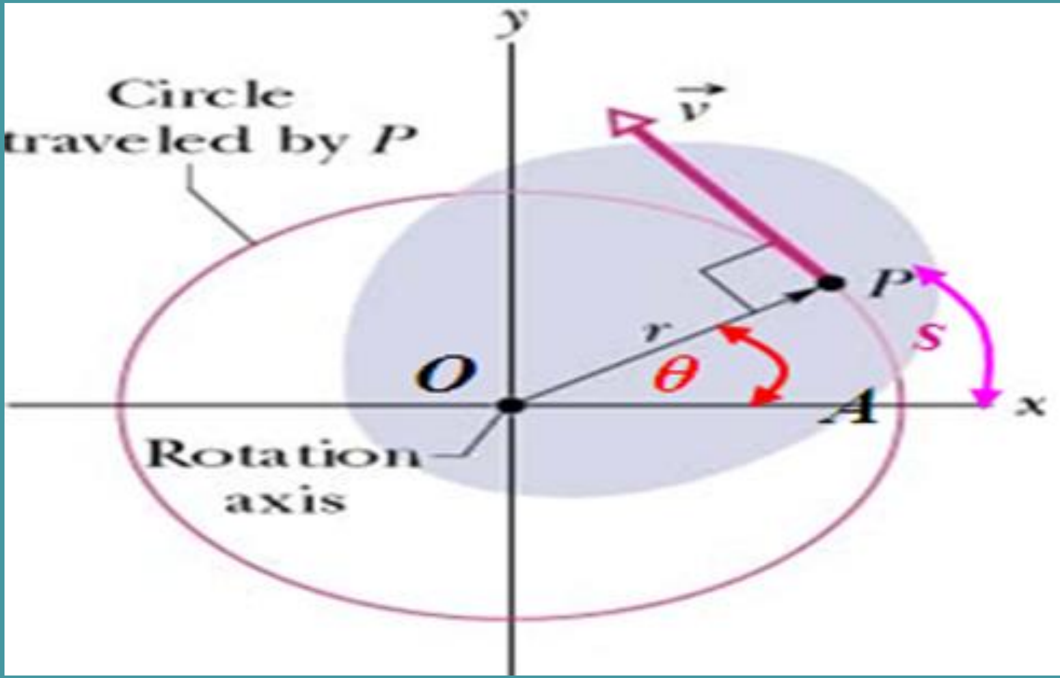
$$v = v_0 + at \leftrightarrow \omega = \omega_0 + \alpha t \qquad \text{(eqs.1)}$$

$$x = x_o + v_o t + \frac{at^2}{2} \leftrightarrow \theta = \theta + \omega_o t + \frac{\alpha t^2}{2} \qquad \text{(eqs.2)}$$

$$v^2 - v_o^2 = 2a\left(x - x_o\right) \leftrightarrow \omega^2 - \omega_o^2 = 2\alpha\left(\theta - \theta_o\right) \qquad \text{(eqs.3)}$$

Equations of Motion for Constant Linear Acceleration and for Constant Angular Acceleration

Linear Equation	Angular Equation
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x - x_0 = v_0 t + \frac{1}{2}at^2$	$\theta - \theta_0 = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_0 + v)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$
$x - x_0 = vt - \frac{1}{2}at^2$	$\theta - \theta_0 = \omega t - \frac{1}{2}\alpha t^2$



Relating the Linear and Angular Variables

Consider a point P on a rigid body rotating about a fixed axis. At $t = 0$ the reference line which connects the origin O with point P is on the x-axis (point A)

During the time interval t point P moves along arc $\overset{\frown}{AP}$ and covers a distance s. At the same time the reference line OP rotates by an angle θ.

Relation between angular velocity and speed

$$\boxed{s = \theta r}$$

$$v = \frac{ds}{dt}$$

$$v = \frac{d}{dt}(\theta r)$$

$$v = \frac{d\theta}{dt} r$$

$$\boxed{v = r\omega}$$

The period T of revolution is given by: $T = \dfrac{\text{circumference}}{\text{speed}} = \dfrac{2\pi r}{v} = \dfrac{2\pi r}{\omega r} = \dfrac{2\pi}{\omega}$

$$\boxed{v = r\omega}$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

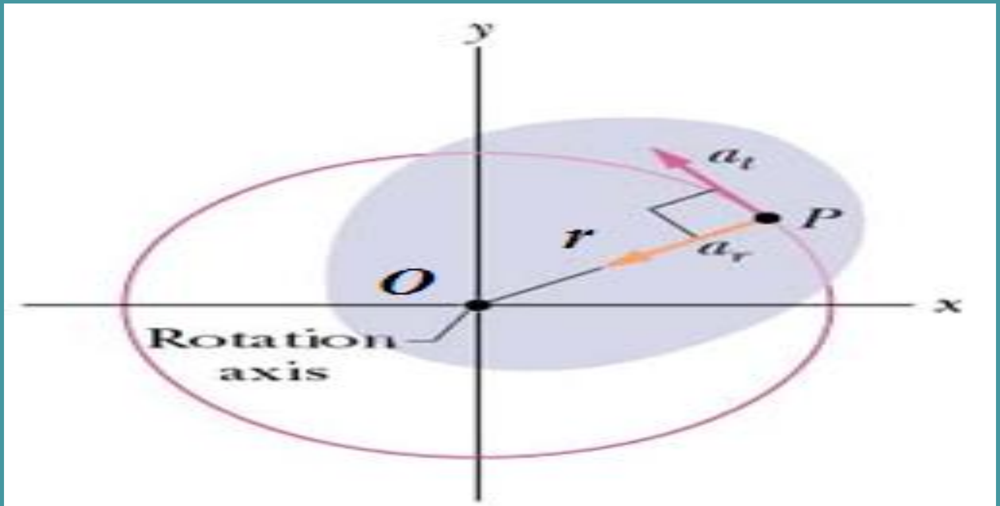
$$\boxed{T = \frac{1}{f}}$$

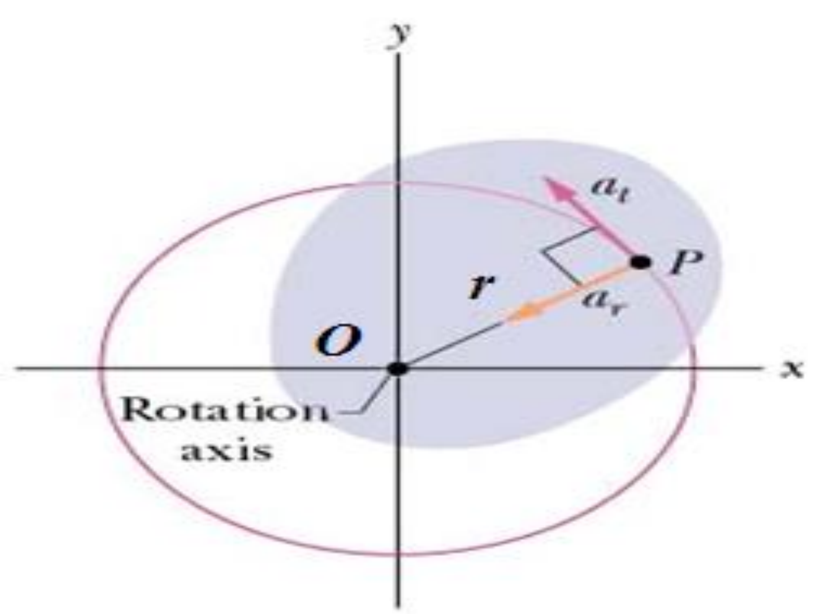
$$\boxed{\omega = 2\pi f}$$

The Acceleration

The acceleration of point P is a vector that has two components. A "radial" componet along the radius and pointing towards point O. We have enountered this component in chapter 4 where we called it "centripetal" acceleration. Its magnitude is:

$$a_r = \frac{v^2}{r} = \omega^2 r$$





$$\boxed{a_t = r\alpha}$$

The second component is along the tangent to the circular path of P and is thus known as the "tangential" component. Its magnitude is:

$$a_t = \frac{dv}{dt} = \frac{d(\omega r)}{dt} = r\frac{d\omega}{dt} = r\alpha$$

$$\boxed{a_t = r\alpha}$$

The magnitude of the acceleration vector is: $\boxed{a = \sqrt{a_t^2 + a_r^2}}$

Kinetic Energy of Rotation

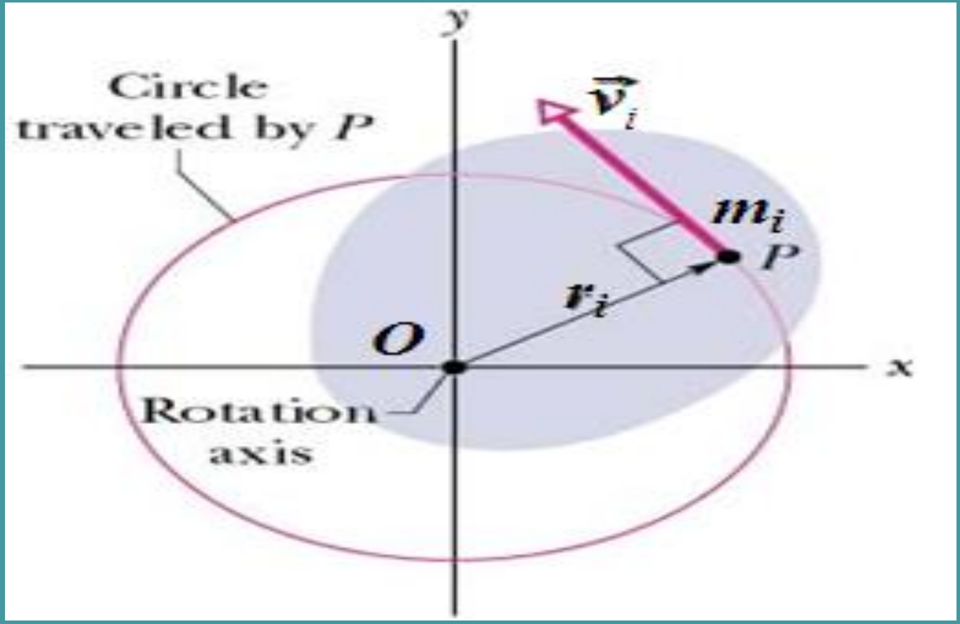
Consider the rotating rigid body shown in the figure.

We divide the body into parts of masses $m_1, m_2, m_3, ..., m_i, ...$

The part (or "element") at P has an index i and mass m_i

The kinetic energy of rotation is the sum if the kinetic

energies of the parts $K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + ...$



$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots$$

$K = \sum_i \frac{1}{2} m_i v_i^2$ The speed of the i-th element $v_i = \omega r_i \rightarrow K = \sum_i \frac{1}{2} m_i (\omega r_i)^2$

$$K = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I \omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

rotational inertia or moment of inertia about the axis of rotation.

$$I = \sum_i m_i r_i^2$$

The rotational inertia of an object describes how the mass is distributed about the rotation axis $I = \sum_i m_i r_i^2$

The axis of rotation must be specified because the value of I for a rigid body depends on its mass, its shape as well as on the position of the rotation axis.

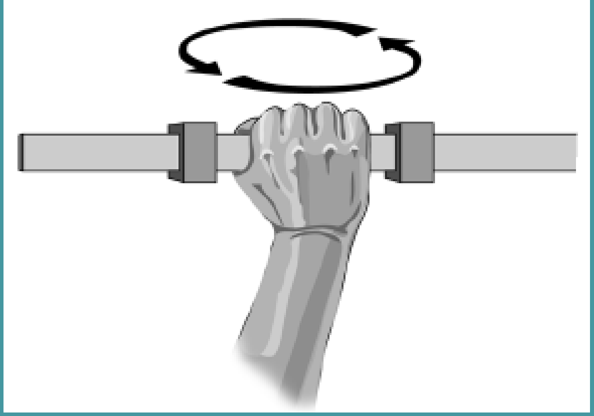
Calculating the Rotational Inertia

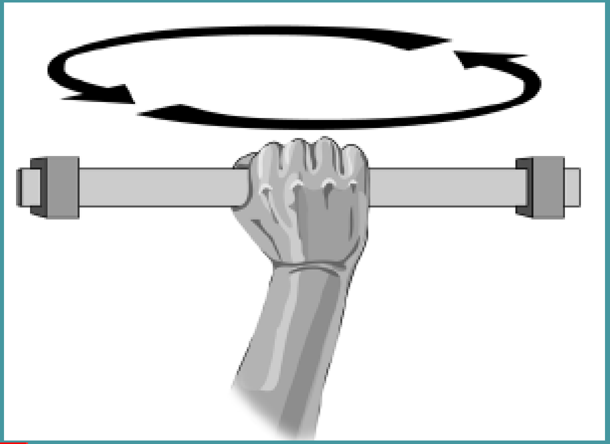
The rotational inertia $I = \sum_i m_i r_i^2$ This expression is useful for a rigid body that has a discreet disstribution of mass. For a continuous distribution of mass the sum becomes an integral $I = \int r^2 dm$

$$\boxed{I = \sum_i m_i r_i^2} \qquad \boxed{I = \int r^2 dm}$$

Rotational inertia

It is the term used to describe an <u>object's resistance to a change in its rotational motion</u>.



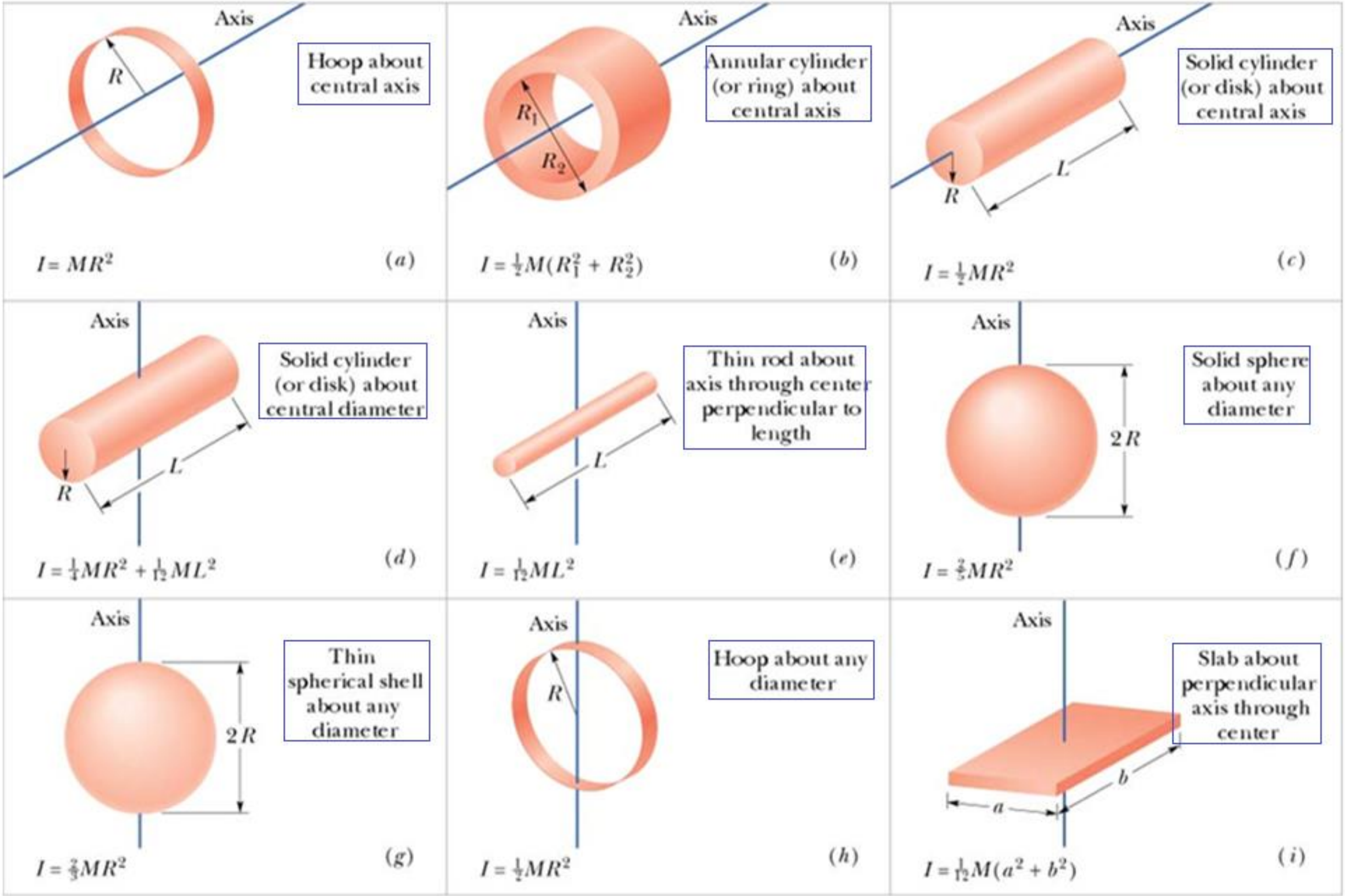


An object's rotational inertia depends not only its **<u>total mass</u>**, but also on **<u>the way that how mass is distributed</u>**.

$$I = \int r^2 dm$$

The Rotational Inertias For Some Rigid Bodies

$$I = \int r^2 dm$$



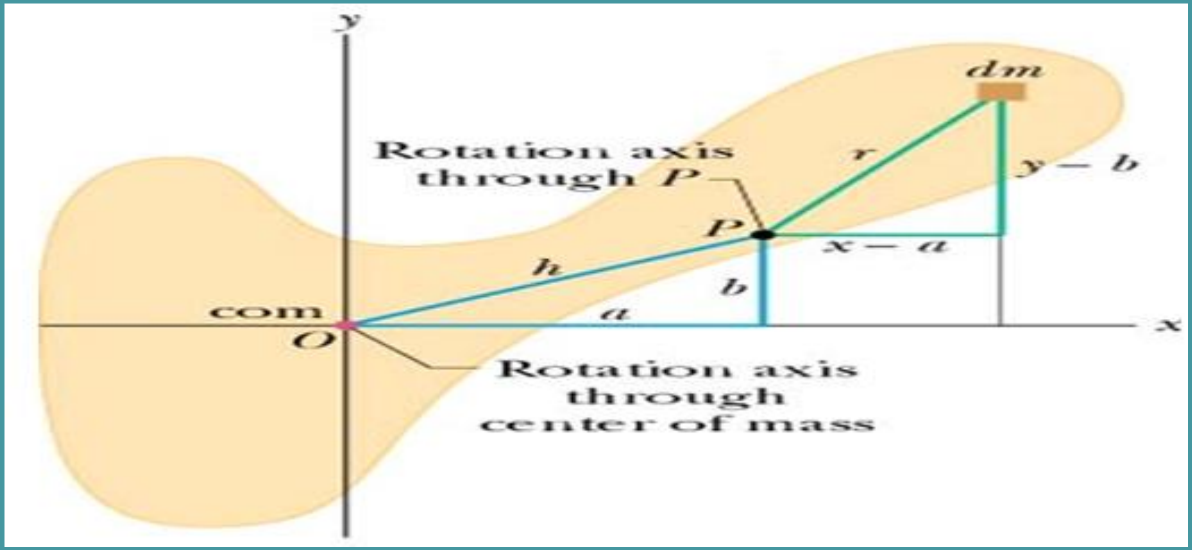
Parallel-Axis Theorem

We saw earlier that I depends on the position of the rotation axis

For a new axis we must recalculate the integral for I. A simpler method takes advantage of the parallel-axis theorem

Consider the rigid body of mass M shown in the figure.

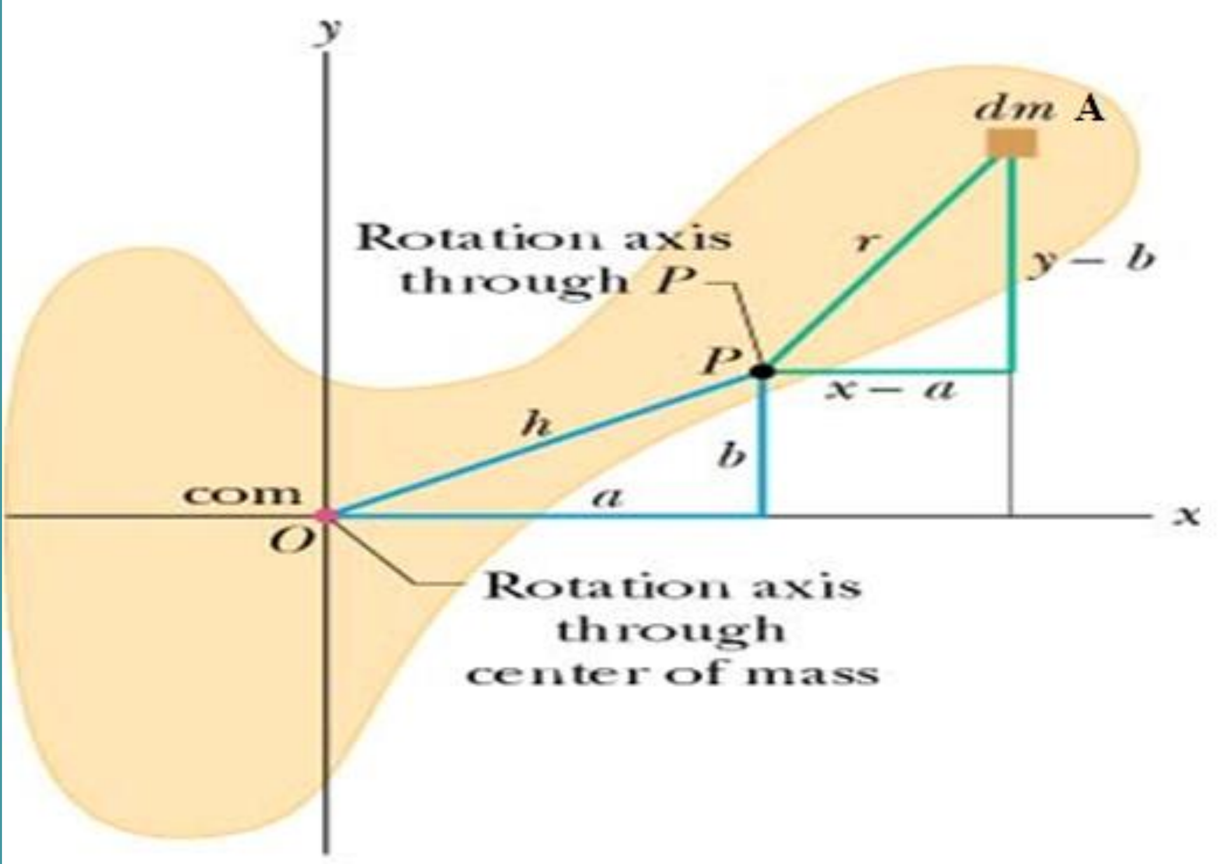
We assume that we know the rotational inertia I_{com} about a rotation axis that passes through the center of mass O and is perpendicular to the page.



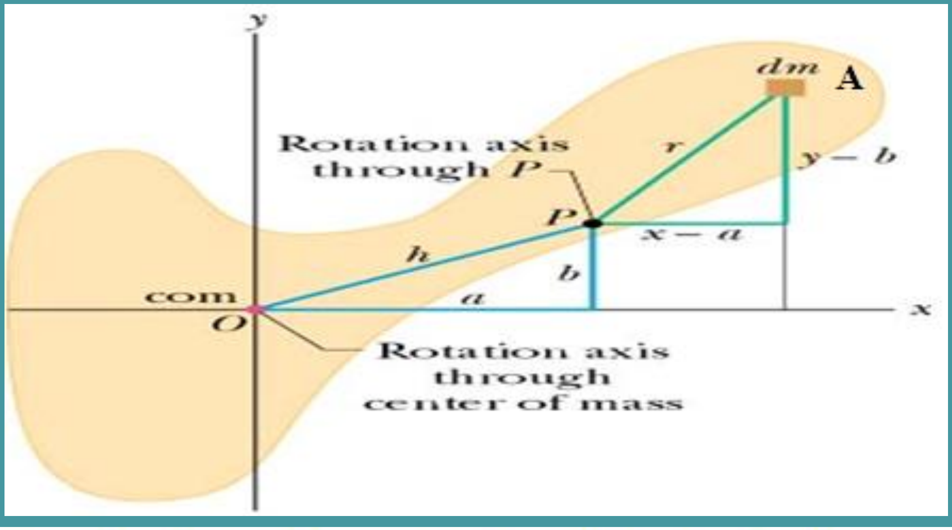
$$I = I_{com} + Mh^2$$

The rotational inertia I about an axis parallel to the axis through O that passes through point P, a distance h from O is given by the equation:

Proof of the Parallel-Axis Theorem We take the origin O to coincide with the center of mass of the rigid body shown in the figure. We assume that we know the rotational inertia I_{com} for an axis that is perpendicular to the page and passes through O.



$$I = I_{com} + Mh^2$$



$$\boxed{I = I_{com} + Mh^2}$$

We wish to calculate the rotational ineria I about a new axis perpendicular to the page and passes through point P with coodrinates (a, b). Consider an element of mass dm at point A with coordinates (x, y). The distance r between points A and P is: $\boxed{r = \sqrt{(x-a)^2 + (y-b)^2}}$

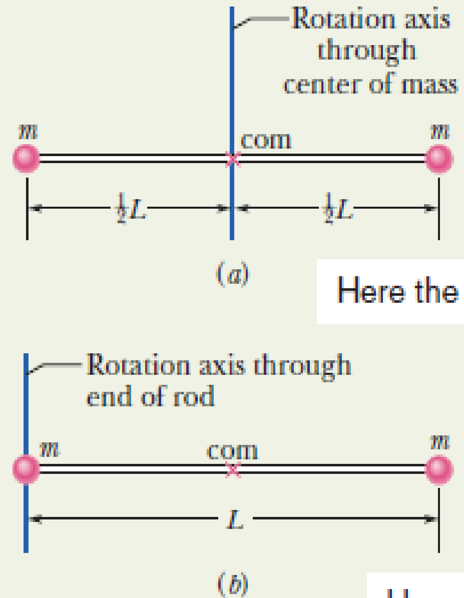
Rotational Inertia about P: $\boxed{I = \int r^2 dm = \int \left[(x-a)^2 + (y-b)^2\right] dm}$

$\boxed{I = \int \left(x^2 + y^2\right) dm - 2a \int x dm - 2b \int y dm - \int \left(a^2 + b^2\right) dm}$ The second and third integrals are zero. The first integral is I_{com}. The term $\left(a^2 + b^2\right) = h^2$

Thus the fourth integral is equal to $h^2 \int dm = Mh^2 \rightarrow$ $\boxed{I = I_{com} + Mh^2}$

Rotational inertia of a two-particle system

Figure 10-13a shows a rigid body consisting of two particles of mass m connected by a rod of length L and negligible mass.



Here the rotation axis is through the com.

Here it has been shifted from the com without changing the orientation. We can use the parallel-axis theorem.

Fig. 10-13 A rigid body consisting of two particles of mass m joined by a rod of negligible mass.

(a) What is the rotational inertia I_{com} about an axis through the center of mass, perpendicular to the rod as shown?

KEY IDEA

Because we have only two particles with mass, we can find the body's rotational inertia I_{com} by using Eq. 10-33 rather than by integration.

Calculations: For the two particles, each at perpendicular distance $\frac{1}{2}L$ from the rotation axis, we have

$$I = \sum m_i r_i^2 = (m)(\tfrac{1}{2}L)^2 + (m)(\tfrac{1}{2}L)^2$$
$$= \tfrac{1}{2}mL^2. \qquad \text{(Answer)}$$

(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis (Fig. 10-13b)?

KEY IDEAS

This situation is simple enough that we can find I using either of two techniques. The first is similar to the one used in part (a). The other, more powerful one is to apply the parallel-axis theorem.

First technique: We calculate I as in part (a), except here the perpendicular distance r_i is zero for the particle on the left and L for the particle on the right. Now Eq. 10-33 gives us

$$I = m(0)^2 + mL^2 = mL^2. \qquad \text{(Answer)}$$

Second technique: Because we already know I_{com} about an axis through the center of mass and because the axis here is parallel to that "com axis," we can apply the parallel-axis theorem (Eq. 10-36). We find

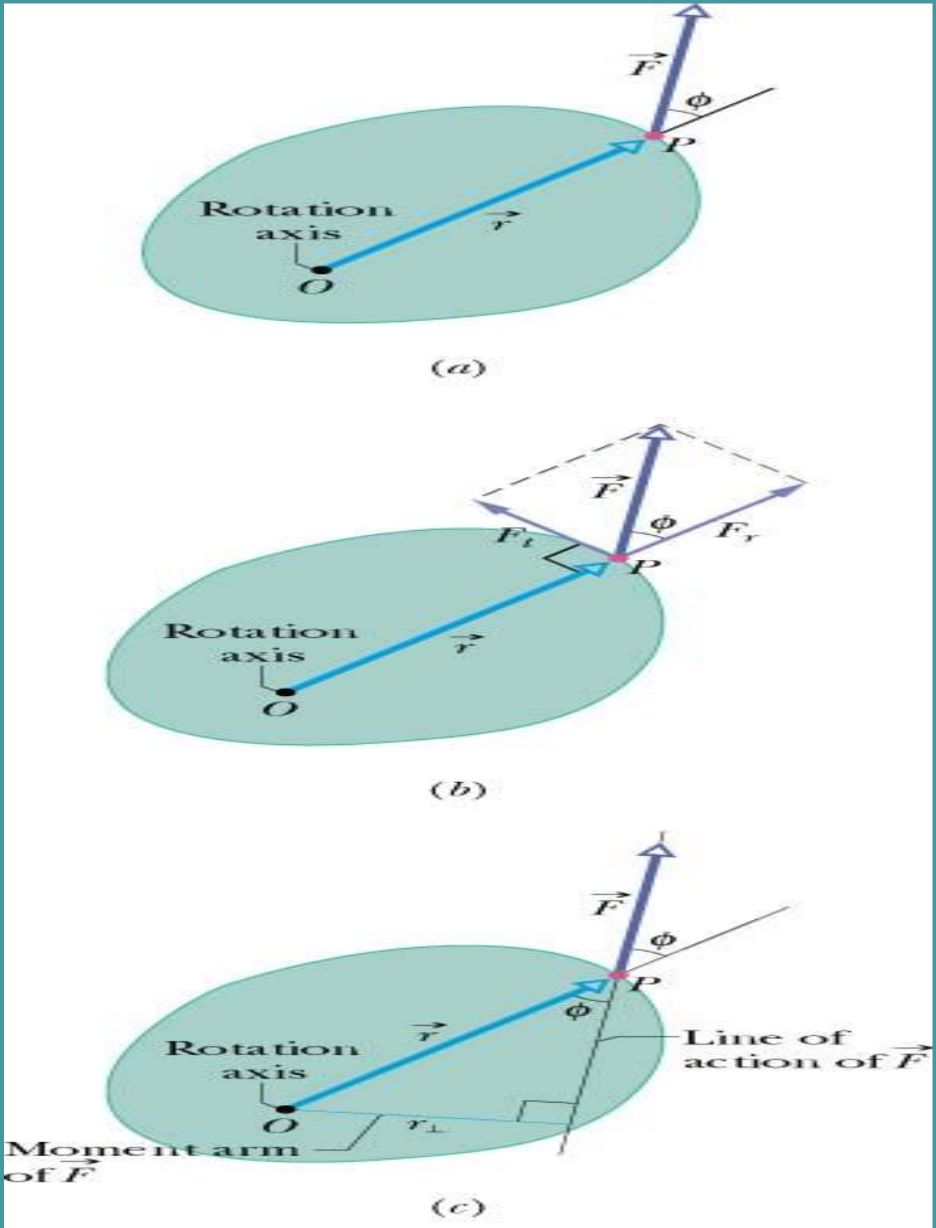
$$I = I_{com} + Mh^2 = \tfrac{1}{2}mL^2 + (2m)(\tfrac{1}{2}L)^2$$
$$= mL^2. \qquad \text{(Answer)}$$

TORQUE

Torque is an action that causes objects to rotate.

***Torque is <u>not</u> the same thing as force*.**

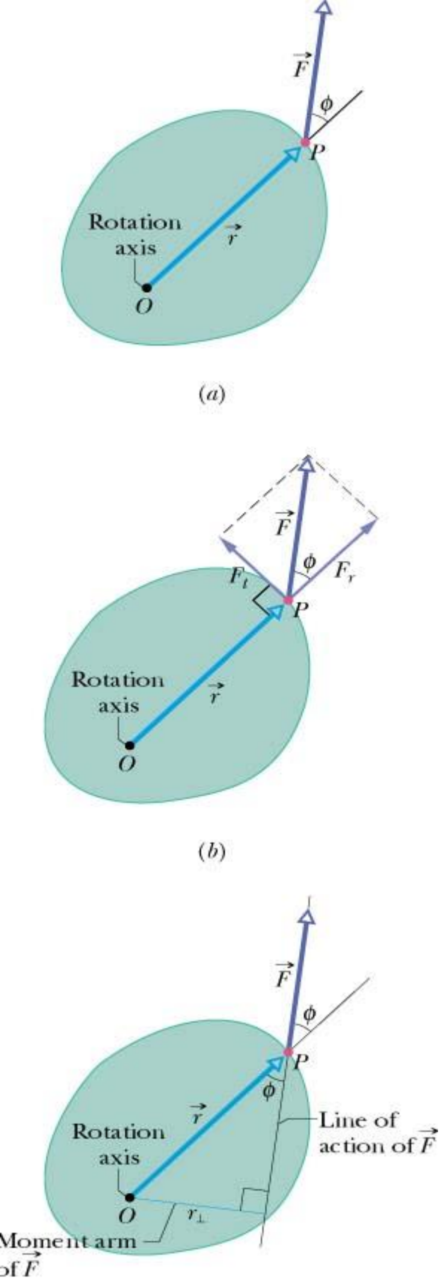
For rotational motion, the torque is what is most directly related to the motion, not the force.



$$\tau = r \times F$$

$$\tau = rF_t = rF\sin\phi = r_\perp F$$

$$\boxed{\tau = I\alpha}$$



$$\boxed{\tau = r_\perp F}$$

Torque

In fig.a we show a body which can rotate about an axis through point O under the action of a force $\vec{F}$ applied at point P a distance r from O. In fig.b we resolve $\vec{F}$ into two componets, radial and tangential. The radial component F_r cannot cause any rotation because it acts along a line that passes through O. The tangential component $F_t = F\sin\phi$ on the other hand causes the rotation of the object about O. The ability of $\vec{F}$ to rotate the body depends on the magnitude F_t and also on the distance r between points P and A.

Thus we define as torque $\tau = rF_t = rF\sin\phi = r_\perp F$

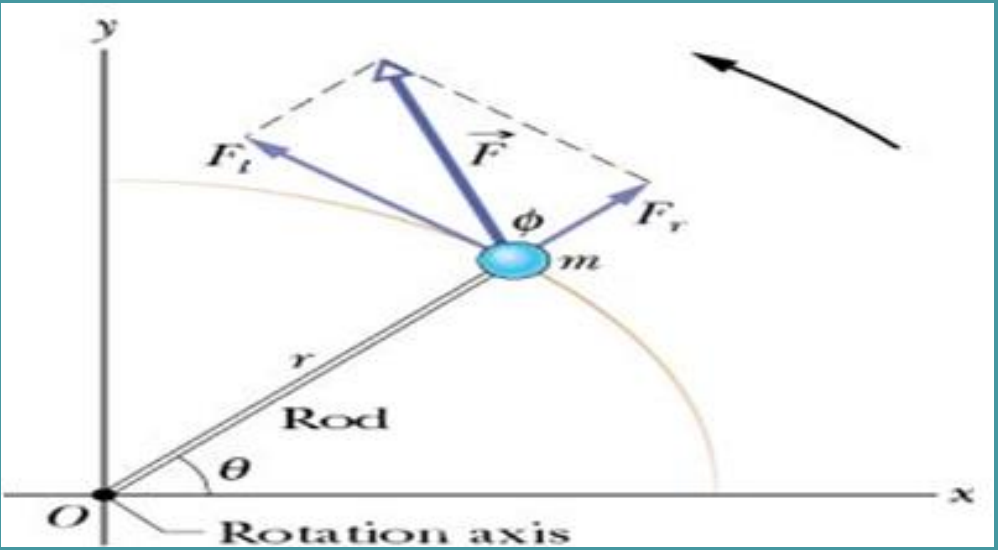
The distance $r_\perp$ is known as the moment arm and it is the perpendicular distance between point O and the vector $\vec{F}$

The algebraic sign of the torque is asigned as follows:

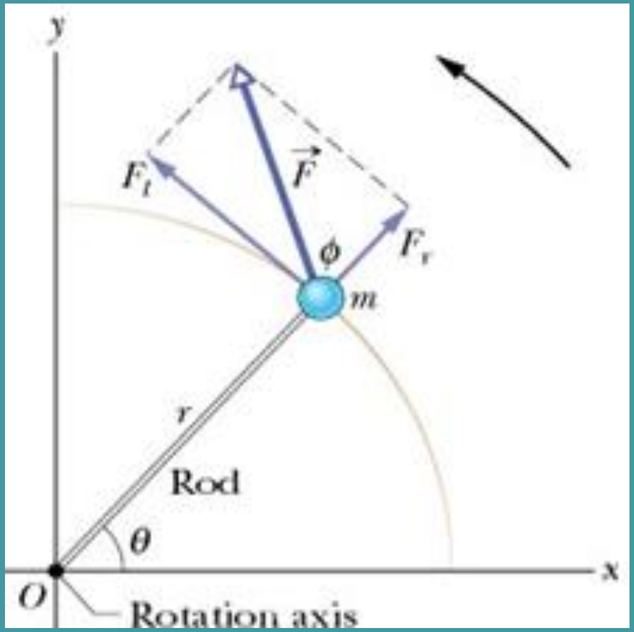
If a force $\vec{F}$ tends to rotate an object in the coubterclockwise direction the sign is positive. If a force $\vec{F}$ tends to rotate an object in the clockwise direction the sign is negative.

Newton's Second Law for Rotation

For translational motion Newton's second law connects the force acting on a particle with the resulting acceleration There is a similar relationship between the torque of a force applied on a rigid object and the resulting angular acceleration



We will explore a simple body which consists of a point mass *m* at the end of a massless rod of length *r*. A force $\vec{F}$ is applied on the particle and rotates the system about an axis at the origin.



we resolve F into a tangential and a radial component. The tangential component is responsible for the rotation.

$$F_t = ma_t \quad \text{(eqs.1)}$$

The torque τ acting on the particle is: $\tau = F_t r$ (eqs.2)

We eliminate F_t between equations 1 and 2:

$$\tau = ma_t r = m(\alpha r)r = (mr^2)\alpha = I\alpha$$

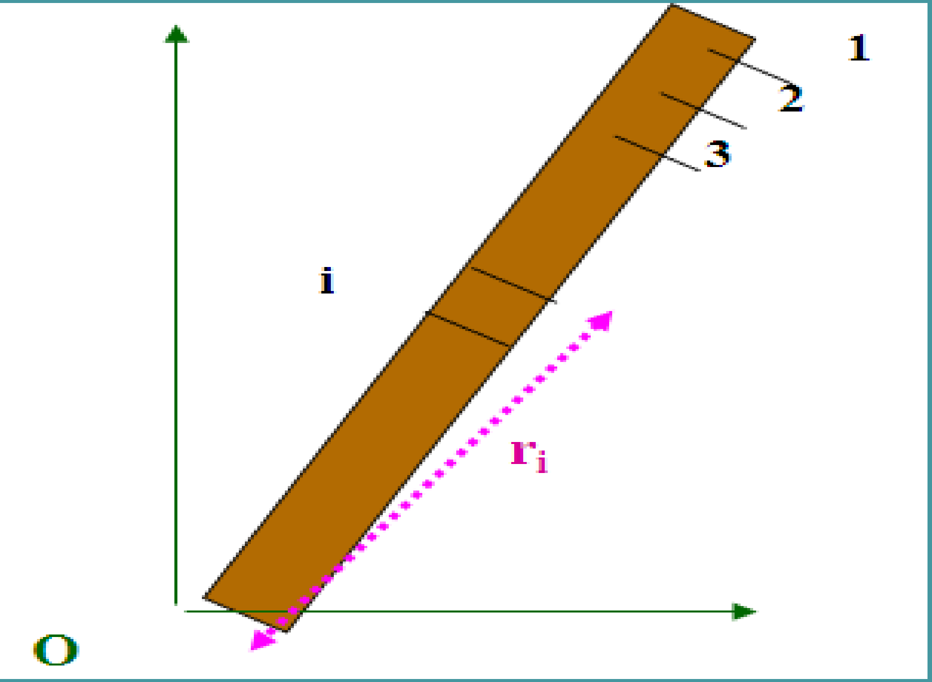
$$\boxed{\tau = I\alpha}$$

Newton's Second Law for Rotation

We have derived Newton's second law for rotation for a special case. A rigid body which consists of a point mass *m* at the end of a massless rod of length *r*. We will now derive the same equation for a general case.

$$\boxed{\tau = I\alpha}$$

Consider the rod-like object shown in the figure which can rotate about an axis through point O undet the action of a net torque τ_{net}. We divide the body into parts or "elements" and label them. The elements have masses $m_1, m_2, m_3, \ldots, m_n$ and they are located at distances $r_1, r_2, r_3, \ldots, r_n$ from O.



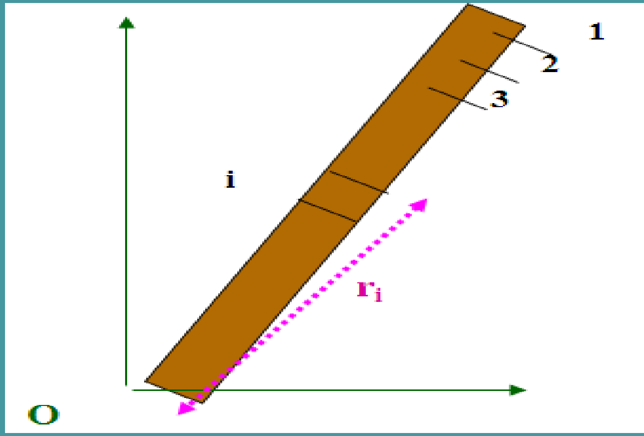
$$\tau_1 = I_1\alpha \quad \text{(eqs.1)},$$
$$\tau_2 = I_2\alpha \quad \text{(eqs.2)},$$
$$\tau_3 = I_3\alpha \quad \text{(eqs.3), etc.}$$

$$\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n = \left(I_1 + I_2 + I_3 + \ldots + I_n\right)\alpha.$$

Here $I_i = m_i r_i^2$ is the rotational inertia of the i-th element.

The sum $\tau_1 + \tau_2 + \tau_3 + \ldots + \tau_n$ is the net torque τ_{net} applied.

The sum $I_1 + I_2 + I_3 + \ldots + I_n$ is the rotational inertia I of the body.

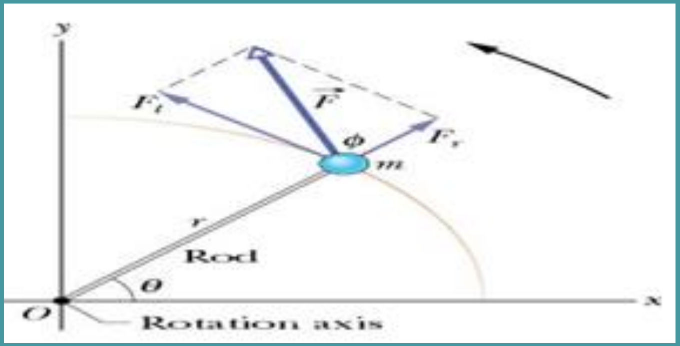


$$\boxed{\tau_{net} = I\alpha}$$

Work and Rotational Kinetic Energy

In chapter 7 we saw that if a force does work W on an object, this results in a change of its kinetic energy $\boxed{\Delta K = W}$. In a similar way, when a torque does work W on a rotating rigid body, it changes its rotational kinetic energy by the same amount

$$\boxed{W = \Delta K}$$



Consider the simple rigid body shown in the figure which consists of a mass *m* at the end of a massless rod of length *r*.

The force $\vec{F}$ does work $dW = F_t r d\theta = \tau d\theta$

The radial component F_r does zero work because it is at right angles to the motion.

$$W = \int F_t r d\theta = \int_{\theta_i}^{\theta_f} \tau d\theta$$

$$\Delta K = W = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \frac{1}{2} m r^2 \omega_f^2 - \frac{1}{2} m r^2 \omega_i^2$$

$$W = \Delta K$$

$$\Delta K = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$$

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

$\boxed{\text{Power}}$

Power has been defined as the rate at which work is done by a force and in the case of rotational motion by a torque We saw that a torque τ produces work $\boxed{dW = \tau d\theta}$ as it rotates an object by an angle $d\theta$.

$$P = \frac{dW}{dt} = \frac{d}{dt}\left(\tau d\theta\right) = \tau \frac{d\theta}{dt} = \tau\omega$$

(Compare with $P = Fv$)

Below we summarize the results of the work-rotational kinetic energy theorem

$$W = \int_{\theta_i}^{\theta_f} \tau d\theta$$

$$W = \tau\left(\theta_f - \theta_i\right)$$ For constant torque

$$W = \Delta K = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$ Work-Rotational Kinetic Energy Theorem

$$P = \tau\omega$$

Analogies between translational and rotational Motion

Translational Motion		Rotational Motion
x	$\leftrightarrow$	θ
v	$\leftrightarrow$	ω
a	$\leftrightarrow$	α
$v = v_0 + at$	$\leftrightarrow$	$\omega = \omega_0 + \alpha t$
$x = x_o + v_o t + \frac{at^2}{2}$	$\leftrightarrow$	$\theta = \theta_o + \omega_o t + \frac{\alpha t^2}{2}$
$v^2 - v_o^2 = 2a(x - x_o)$	$\leftrightarrow$	$\omega^2 - \omega_o^2 = 2\alpha(\theta - \theta_o)$
$K = \frac{mv^2}{2}$	$\leftrightarrow$	$K = \frac{I\omega^2}{2}$
m	$\leftrightarrow$	I
$F = ma$	$\leftrightarrow$	$\tau = I\alpha$
F	$\leftrightarrow$	τ
$P = Fv$	$\leftrightarrow$	$P = \tau\omega$