Lecture 5: Number Theory

BBM205

Exercises (from Rosen's book)

7. Convert $(ABCDEF)_{16}$ from its hexadecimal expansion to its binary expansion.
8. Convert each of these integers from binary notation to hexadecimal notation.
   a) 1111 0111
   b) 1010 1010 1010
   c) 111 0111 0111 0111
9. Convert $(1011\ 0111\ 1011)_2$ from its binary expansion to its hexadecimal expansion.
10. Convert $(1\ 1000\ 0110\ 0011)_2$ from its binary expansion to its hexadecimal expansion.
11. Show that the hexadecimal expansion of a positive integer can be obtained from its binary expansion by grouping together blocks of four binary digits, adding initial digits if necessary, and translating each block of four binary digits into a single hexadecimal digit.
12. Show that the binary expansion of a positive integer can be obtained from its hexadecimal expansion by translating each hexadecimal digit into a block of four binary digits.
13. Give a simple procedure for converting from the binary expansion of an integer to its octal expansion.
14. Give a simple procedure for converting from the octal expansion of an integer to its binary expansion.
15. Convert $(7345321)_8$ to its binary expansion and $(10\ 1011\ 1011)_2$ to its octal expansion.
16. Give a procedure for converting from the hexadecimal expansion of an integer to its octal expansion using binary notation as an intermediate step.
17. Give a procedure for converting from the octal expansion of an integer to its hexadecimal expansion using binary notation as an intermediate step.
18. Convert $(12345670)_8$ to its hexadecimal expansion and $(ABB093BABBA)_{16}$ to its octal expansion.
19. Use Algorithm 5 to find 7^{644} **mod** 645.
20. Use Algorithm 5 to find 11^{644} **mod** 645.
21. Use Algorithm 5 to find 3^{2003} **mod** 99.
22. Use Algorithm 5 to find 123^{1001} **mod** 101.
23. Use the Euclidean algorithm to find
   a) gcd(12, 18). b) gcd(111, 201).
   c) gcd(1001, 1331). d) gcd(12345, 54321).
   e) gcd(1000, 5040). f) gcd(9888, 6060).
24. Use the Euclidean algorithm to find
   a) gcd(1, 5). b) gcd(100, 101).
   c) gcd(123, 277). d) gcd(1529, 14039).
   e) gcd(1529, 14038). f) gcd(11111, 111111).
25. How many divisions are required to find gcd(21, 34) using the Euclidean algorithm?
26. How many divisions are required to find gcd(34, 55) using the Euclidean algorithm?
27. Show that every positive integer can be represented uniquely as the sum of distinct powers of 2. [*Hint:* Consider binary expansions of integers.]
28. It can be shown that every integer can be uniquely represented in the form

$$e_k 3^k + e_{k-1}3^{k-1} + \cdots + e_1 3 + e_0,$$

where $e_j = -1, 0,$ or 1 for $j = 0, 1, 2, \ldots, k$. Expansions of this type are called **balanced ternary expansions.** Find the balanced ternary expansions of
   a) 5. b) 13. c) 37. d) 79.
29. Show that a positive integer is divisible by 3 if and only if the sum of its decimal digits is divisible by 3.
30. Show that a positive integer is divisible by 11 if and only if the difference of the sum of its decimal digits in even-numbered positions and the sum of its decimal digits in odd-numbered positions is divisible by 11.
31. Show that a positive integer is divisible by 3 if and only if the difference of the sum of its binary digits in even-numbered positions and the sum of its binary digits in odd-numbered positions is divisible by 3.

One's complement representations of integers are used to simplify computer arithmetic. To represent positive and negative integers with absolute value less than 2^{n-1}, a total of n bits is used. The leftmost bit is used to represent the sign. A 0 bit in this position is used for positive integers, and a 1 bit in this position is used for negative integers. For positive integers, the remaining bits are identical to the binary expansion of the integer. For negative integers, the remaining bits are obtained by first finding the binary expansion of the absolute value of the integer, and then taking the complement of each of these bits, where the complement of a 1 is a 0 and the complement of a 0 is a 1.

32. Find the one's complement representations, using bit strings of length six, of the following integers.
   a) 22 b) 31 c) -7 d) -19
33. What integer does each of the following one's complement representations of length five represent?
   a) 11001 b) 01101
   c) 10001 d) 11111
34. If m is a positive integer less than 2^{n-1}, how is the one's complement representation of $-m$ obtained from the one's complement of m, when bit strings of length n are used?
35. How is the one's complement representation of the sum of two integers obtained from the one's complement representations of these integers?
36. How is the one's complement representation of the difference of two integers obtained from the one's complement representations of these integers?
37. Show that the integer m with one's complement representation $(a_{n-1}a_{n-2}\ldots a_1a_0)$ can be found using the equation $m = -a_{n-1}(2^{n-1} - 1) + a_{n-2}2^{n-2} + \cdots + a_1 \cdot 2 + a_0$.

Two's complement representations of integers are also used to simplify computer arithmetic and are used more commonly than one's complement representations. To represent an integer x with $-2^{n-1} \le x \le 2^{n-1} - 1$ for a specified positive integer n, a total of n bits is used. The leftmost bit is used to represent the sign. A 0 bit in this position is used for positive