

## SECTION 7: TRANSFORMATIONS OF RANDOM VARIABLES

### EXERCISES

**Exercise 1:** (Binomial transformation) A discrete random variable  $X$  has a binomial distribution if its pmf is of the form

$$P_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n,$$

where  $n$  is a positive integer and  $0 \leq p \leq 1$ . Consider the random variable  $Y = g(X)$ , where  $g(x) = n - x$ . Thus,  $g^{-1}(y)$  is the single point  $x = n - y$ , and

$$\begin{aligned} P_Y(y) &= \sum_{x \in g^{-1}(y)} P_X(x) = P_X(n - y) \\ &= \binom{n}{n-y} p^{n-y} (1-p)^{n-(n-y)} \\ &= \binom{n}{y} (1-p)^y p^{n-y} \end{aligned}$$

Thus, we see that  $Y$  also has a binomial distribution, but with parameters  $n$  and  $1 - p$ .

**Exercise 2:** Suppose that  $X$  has the pmf below:

$x$	-1	0	1	2
$P_X(x)$	0.2	0.1	0.3	0.4

Then, find the pmf of  $Y = X^2$ .

**Solution:**

$$P(Y = 0) = P(X = 0) = 0.1$$

$$P(Y = 1) = P(X = -1) + P(X = 1) = 0.5$$

$$P(Y = 4) = P(X = 2) = 0.4$$

$y$	0	1	4
$P_Y(y)$	0.1	0.5	0.4

**Exercise 3:** (Uniform-exponential relationship-I) Suppose  $X \sim f_X(x) = 1$  if  $0 < x < 1$  and 0 otherwise, the Uniform(0,1) distribution. It is straightforward to check that  $F_X(x) = x, 0 < x < 1$ . We now make the transformation  $Y = g(X) = -\ln(X)$ . Find the cdf of  $Y$ .

**Solution:**

Since  $\frac{d}{dx} g(x) = -\frac{1}{x} < 0$ , for  $0 < x < 1$ ,  $g(x)$  is a decreasing function. Therefore, for  $y > 0$ ,

$$F_Y(y) = 1 - F_X(g^{-1}(y)) = 1 - F_X(e^{-y}) = 1 - e^{-y}$$

Of course,  $F(y) = 0$  for  $y \leq 0$ .

If the pdf of  $Y$  is continuous, it can be obtained by differentiating the cdf.

**Exercise 4:** (Inverted gamma pdf) Let  $f_X(x)$  be the gamma pdf

$$f(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta}, 0 < x < \infty,$$

where  $\beta$  is a positive constant and  $n$  is a positive integer. If we let  $y = g(x) = 1/x$ , find the pdf of  $Y$ .

**Solution:**

$y = g(x) = 1/x$ , then  $g^{-1}(y) = 1/y$  and  $\frac{d}{dy} g^{-1}(y) = -1/y^2$ . For  $0 < y < 1$ , we get

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\ &= \frac{1}{(n-1)!\beta^n} \left( \frac{1}{y} \right)^{n-1} e^{-1/(\beta y)} \frac{1}{y^2} \\ &= \frac{1}{(n-1)!\beta^n} \left( \frac{1}{y} \right)^{n+1} e^{-1/(\beta y)} \end{aligned}$$

a special case of a pdf known as the inverted gamma pdf.

**Exercise 5:** The pdf of continuous random variable  $X$  is given in below:

$$f(x) = \begin{cases} \frac{1}{56}(x+3), & 0 \leq x \leq 8 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf and cdf of  $Y = -4X - 1$ .

**Solution:**

$$F(x) = \int_0^x \frac{1}{56}(t+3) dt = \frac{1}{56} \left( \frac{t^2}{2} + 3t \right) \Bigg|_0^x = \frac{1}{56} \left( \frac{x^2}{2} + 3x \right)$$

$$F(x) = \begin{cases} \frac{(x^2 + 6x)}{112}, & 0 \leq x \leq 8 \\ 0, & x < 0 \\ 1, & x \geq 8 \end{cases}$$

**Method 1:** Finding pdf of random variable Y by using pdf of random variable X.

$$y = -4x - 1 \Rightarrow x = -\frac{(y+1)}{4} = g^{-1}(y) \quad \left. \begin{array}{l} x=0 \Rightarrow y=-1 \\ x=8 \Rightarrow y=-33 \end{array} \right\} -33 \leq y \leq -1$$

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d g^{-1}(y)}{dy} \right| \\ &= f_X\left(-\left(\frac{y+1}{4}\right)\right) \left| \frac{d}{dy} -\left(\frac{y+1}{4}\right) \right| \\ &= \frac{1}{56} \left[ -\left(\frac{y+1}{4}\right) + 3 \right] \left| -\frac{1}{4} \right| \\ &= \frac{11-y}{896} \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{11-y}{896}, & -33 \leq y \leq -1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = \int_{-33}^y \frac{11-t}{896} dt = \frac{1}{896} \left( 11t - \frac{t^2}{2} \right) \Big|_{-33}^y = \frac{-y^2 + 22y + 1815}{1792}$$

$$F_Y(y) = \begin{cases} \frac{-y^2 + 22y + 1815}{1792}, & -33 \leq y \leq -1 \\ 0, & y < -33 \\ 1, & y \geq -1 \end{cases}$$

**Method 2:** By using cdf of random variable X;

$Y = g(X)$  is monotonic increasing or decreasing?

$$\left. \begin{array}{l} x_1 = 1 \Rightarrow y_1 = g(x_1) = -4.1 - 1 = -5 \\ x_2 = 2 \Rightarrow y_2 = g(x_2) = -4.2 - 1 = -9 \end{array} \right\} x_1 = 1 < x_2 = 2 \Rightarrow g(x_1) = -5 > g(x_2) = -9 \text{ monotonic decreasing}$$

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(-4X - 1 \leq y) = P\left(X > -\frac{(y+1)}{4}\right) \\
&= 1 - P\left(X \leq -\frac{(y+1)}{4}\right) = 1 - F_X\left(-\frac{(y+1)}{4}\right) \\
&= 1 - \frac{1}{112} \left[ \left(\frac{y+1}{4}\right)^2 - 6\left(\frac{y+1}{4}\right) \right] = \frac{-y^2 + 22y + 1815}{1792}
\end{aligned}$$

$$F_Y(y) = \begin{cases} \frac{-y^2 + 22y + 1815}{1792}, & -33 \leq y \leq -1 \\ 0, & y < -33 \\ 1, & y \geq -1 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{-2y + 22}{1792} = \frac{-y + 11}{896}$$

$$f_Y(y) = \begin{cases} \frac{11 - y}{896}, & -33 \leq y \leq -1 \\ 0, & \text{otherwise} \end{cases}$$

**Exercise 6:** The pdf of continuous random variable X is given in below:

$$f(x) = \begin{cases} \frac{1}{6}, & -4 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf and cdf of  $Y = X^2$ .

**Solution:**

$$F(x) = P(X \leq x) = \int_{-4}^x \frac{1}{6} dt = \frac{1}{6} t \Big|_{-4}^x = \frac{x+4}{6}$$

$$F(x) = \begin{cases} \frac{x+4}{6}, & -4 < x < 2 \\ 0, & x < -4 \\ 1, & x \geq 2 \end{cases}$$

$$\begin{aligned}
y = x^2 &\Rightarrow x_1 = -\sqrt{y} = g_1^{-1}(y) \\
&\quad x_2 = \sqrt{y} = g_2^{-1}(y)
\end{aligned}$$

**Method 1:**

$$f_Y(y) = f_X(-\sqrt{y}) \left| \frac{d}{dy}(-\sqrt{y}) \right| + f_X(\sqrt{y}) \left| \frac{d}{dy}(\sqrt{y}) \right|$$

$$0 < y < 4 \Rightarrow -\sqrt{y}, \sqrt{y} \in R_X$$

$$4 < y < 16 \Rightarrow -\sqrt{y} \in R_X$$

$$\begin{aligned} f_Y(y) &= f_X(-\sqrt{y}) \left| \frac{d}{dy}(-\sqrt{y}) \right| + f_X(\sqrt{y}) \left| \frac{d}{dy}(\sqrt{y}) \right| \\ &= \frac{1}{6} \left| -\frac{1}{2\sqrt{y}} \right| + \frac{1}{6} \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{6\sqrt{y}}, 0 < y < 4 \end{aligned}$$

$$f_Y(y) = f_X(-\sqrt{y}) \left| \frac{d}{dy}(-\sqrt{y}) \right| + \underbrace{f_X(\sqrt{y}) \left| \frac{d}{dy}(\sqrt{y}) \right|}_0 = \frac{1}{6} \left| -\frac{1}{2\sqrt{y}} \right| = \frac{1}{12\sqrt{y}}, 4 < y < 16$$

$$f_Y(y) = \begin{cases} \frac{1}{6\sqrt{y}}, & 0 < y < 4 \\ \frac{1}{12\sqrt{y}}, & 4 < y < 16 \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = \begin{cases} \int_0^y \frac{1}{6\sqrt{t}} dt = \frac{\sqrt{y}}{3}, & 0 < y < 4 \\ \int_0^4 \frac{1}{6\sqrt{y}} dy + \int_4^y \frac{1}{12\sqrt{t}} dt = \frac{\sqrt{y}+2}{6}, & 4 < y < 16 \\ 0, & y < 0 \\ 1, & y \geq 16 \end{cases}$$

**Method 2 for finding  $F_Y(y)$ :**

For  $-2 \leq x \leq 2 \Rightarrow 0 < y < 4$  so that  $x = \pm\sqrt{y} \Rightarrow y = x^2$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{6} dx = \frac{1}{6} x \Big|_{-\sqrt{y}}^{+\sqrt{y}} = \frac{2\sqrt{y}}{6} = \frac{\sqrt{y}}{3}, \quad 0 < y < 4 \end{aligned}$$

For  $x = -\sqrt{y}, y = 4 \Rightarrow x = -2, y = 16 \Rightarrow x = -4$   
 $4 < 16 \Rightarrow -2 > -4$  monotonic decreasing

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) \\
&= P(X \geq -\sqrt{y}) = 1 - P(X \leq -\sqrt{y}) = 1 - F_X(-\sqrt{y}) \\
&= 1 - \left( \frac{(-\sqrt{y} + 4)}{6} \right) = \frac{\sqrt{y} + 2}{6}, \quad 4 < y < 16
\end{aligned}$$

$$\text{or } 1 - P(X \leq -\sqrt{y}) = 1 - \int_{-4}^{-\sqrt{y}} \frac{1}{6} dx = \frac{\sqrt{y} + 2}{6}, \quad 4 < y < 16$$

$$F_Y(y) = \begin{cases} \frac{\sqrt{y}}{3}, & 0 < y < 4 \\ \frac{\sqrt{y} + 2}{6}, & 4 < y < 16 \\ 0, & y < 0 \\ 1, & y \geq 16 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{6\sqrt{y}}, & 0 < y < 4 \\ \frac{1}{12\sqrt{y}}, & 4 < y < 16 \\ 0, & \text{otherwise} \end{cases}$$

**Exercise 7:** The pmf of discrete random variable X is given in below:

$$p(x) = \begin{cases} \frac{1}{3} \left( \frac{2}{3} \right)^{x-1}, & x = 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$$

Find the pmf of  $Y = X - 4$ .

**Solution:**

$$P_Y(y) = P_X(g^{-1}(y)) = P_X(y + 4) = \frac{1}{3} \left( \frac{2}{3} \right)^{y+4-1} = \frac{1}{3} \left( \frac{2}{3} \right)^{y+3}$$

$$y = x - 4 \Rightarrow x = y + 4 = g^{-1}(y)$$

$$x = 1 \Rightarrow y = 1 - 4 = -3$$

$$x = 2 \Rightarrow y = 2 - 4 = -2 \quad \Rightarrow \quad P_Y(y) = \begin{cases} \frac{1}{3} \left( \frac{2}{3} \right)^{y+3}, & y = -3, -2, -1, 0, 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$x = 3 \Rightarrow y = 3 - 4 = -1$$

$$x = 4 \Rightarrow y = 4 - 4 = 0$$

$$F_Y(y) = \sum_{t=-3}^y \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^{t+3} = \frac{1}{3} \left[ 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{y+3} \right] = \frac{1}{3} \frac{1 - \left(\frac{2}{3}\right)^{y+4}}{1 - \frac{2}{3}} = 1 - \left(\frac{2}{3}\right)^{y+4}$$

$$F_Y(y) = \begin{cases} 1 - \left(\frac{2}{3}\right)^{y+4}, & y = -3, -2, -1, 0, 1, 2, 3, \dots \\ 0, & y < -3 \\ 1, & y \rightarrow +\infty \end{cases}$$

**Exercise 8:** The pmf of discrete random variable X is given in below:

$$p(x) = \begin{cases} \frac{1}{6}, & x = -3, -2, -1, 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the pmf of  $Y = X^2$ .

**Solution:**

$$y = x^2 \Rightarrow \left. \begin{matrix} x = -\sqrt{y} \\ x = \sqrt{y} \end{matrix} \right\} \Rightarrow P_Y(y) = P_X(-\sqrt{y}) + P_X(\sqrt{y}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$P_Y(y) = \begin{cases} \frac{1}{3}, & y = 1, 4, 9 \\ 0, & \text{otherwise} \end{cases}$$

**Exercise 9:** The pdf of continuous random variable X is given in below:

$$f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf and cdf of  $Y = 2X^2$ .

**Solution:**

$$-1 < x < 1 \Rightarrow \begin{cases} 0 < x^2 < 1 \\ 0 < 2x^2 < 2 \end{cases} \Rightarrow 0 < y < 2$$

$$1 < x < 2 \Rightarrow \begin{cases} 1 < x^2 < 4 \\ 2 < 2x^2 < 8 \end{cases} \Rightarrow 2 < y < 8$$

$$g_1^{-1}(y) = x_1 = -\sqrt{\frac{y}{2}} \quad g_2^{-1}(y) = x_2 = \sqrt{\frac{y}{2}}$$

**Probability Density Function (pdf):**

$$\begin{aligned} f_Y(y) &= f_X(g_1^{-1}(y)) \left| \frac{d(g_1^{-1}(y))}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{d(g_2^{-1}(y))}{dy} \right| \\ &= \frac{1}{3} \cdot \left| \frac{d}{dy} \left( -\sqrt{\frac{y}{2}} \right) \right| + \frac{1}{3} \cdot \left| \frac{d}{dy} \left( \sqrt{\frac{y}{2}} \right) \right| \quad \Rightarrow \quad f_Y(y) = \frac{\sqrt{2}}{6\sqrt{y}}, \quad 0 < y < 2 \\ &= \frac{1}{3} \cdot \frac{\sqrt{2}}{4\sqrt{y}} + \frac{1}{3} \cdot \frac{\sqrt{2}}{4\sqrt{y}} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= f_X(g_2^{-1}(y)) \left| \frac{d(g_2^{-1}(y))}{dy} \right| \\ &= \frac{1}{3} \cdot \left| \frac{d}{dy} \left( \sqrt{\frac{y}{2}} \right) \right| \quad \Rightarrow \quad f_Y(y) = \frac{\sqrt{2}}{12\sqrt{y}}, \quad 2 < y < 8 \\ &= \frac{1}{3\sqrt{2}} \cdot \frac{1}{2\sqrt{y}} \\ &= \frac{\sqrt{2}}{12\sqrt{y}} \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{\sqrt{2}}{6\sqrt{y}}, & 0 < y < 2 \\ \frac{\sqrt{2}}{12\sqrt{y}}, & 2 < y < 8 \\ 0, & \text{otherwise} \end{cases}$$

**Cumulative Distribution Function (cdf):**

$$F(x) = P(X \leq x) = \int_{-1}^x \frac{1}{3} dt = \frac{t}{3} \Big|_{-1}^x = \frac{x+1}{3} \quad \Rightarrow \quad F(x) = \begin{cases} \frac{x+1}{3}, & -1 \leq x \leq 2 \\ 0, & x \leq -1 \\ 1, & x \geq 2 \end{cases}$$

$$P(Y \leq y) = F_X\left(\sqrt{\frac{y}{2}}\right) - F_X\left(-\sqrt{\frac{y}{2}}\right) = \frac{\sqrt{\frac{y}{2}} + 1}{3} - \frac{-\sqrt{\frac{y}{2}} + 1}{3} = \frac{2\sqrt{\frac{y}{2}}}{3} = \frac{\sqrt{2y}}{3}, \quad 0 \leq y \leq 2$$



$$P(Y \leq y) = F_x \left( \sqrt{\frac{y}{2}} \right) - \underbrace{F_x \left( -\sqrt{\frac{y}{2}} \right)}_0 = \frac{\sqrt{\frac{y}{2}} + 1}{3}, \quad 2 \leq y \leq 8$$

$$F(y) = \begin{cases} \frac{\sqrt{2y}}{3}, & 0 \leq y \leq 2 \\ \frac{\sqrt{\frac{y}{2}} + 1}{3}, & 2 < y \leq 8 \\ 0, & y < 0 \\ 1, & y \geq 8 \end{cases}$$

$$\text{pdf is also obtained as } f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{\sqrt{2}}{6\sqrt{y}}, & 0 < y < 2 \\ \frac{\sqrt{2}}{12\sqrt{y}}, & 2 < y < 8 \\ 0, & \text{otherwise} \end{cases}$$

**Exercise 10:** The pdf of continuous random variable X is given in below:

$$f(x) = \begin{cases} \frac{4+x}{20}, & -1 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf and cdf of  $Y = X^2$ .

**Solution:**

**Method 1:**

$$F(x) = P(X \leq x) = \int_{-1}^x \frac{4+t}{20} dt = \frac{8t+t^2}{40} \Big|_{-1}^x = \left[ \left( \frac{8x+x^2}{40} \right) - \left( \frac{8(-1)+(-1)^2}{40} \right) \right] = \frac{x^2+8x+7}{40}$$

$$F(x) = \begin{cases} \frac{x^2+8x+7}{40}, & -1 \leq x \leq 3 \\ 0, & x \leq -1 \\ 1, & x \geq 3 \end{cases}$$

$$-1 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq 1 \Rightarrow 0 \leq y \leq 1$$

$$1 < x \leq 3 \Rightarrow 1 < x^2 \leq 9 \Rightarrow 1 < y \leq 9$$

\*For  $-1 \leq x \leq 1$ ;  $y = x^2 \Rightarrow \begin{cases} x_1 = -\sqrt{y} \\ x_2 = \sqrt{y} \end{cases}$ ;  $P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y})$

$$P(Y \leq y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{(\sqrt{y})^2 + 8\sqrt{y} + 7}{40} - \frac{(-\sqrt{y})^2 - 8\sqrt{y} + 7}{40} = \frac{16\sqrt{y}}{40} = \frac{2\sqrt{y}}{5}, 0 \leq y \leq 1$$

\*\*For  $1 < x \leq 3$ ;  $y = x^2 \Rightarrow x = \sqrt{y}$  } *positive value*;

$$P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y})$$

$$P(Y \leq y) = \frac{(\sqrt{y})^2 + 8\sqrt{y} + 7}{40} = \frac{y + 8\sqrt{y} + 7}{40}, 1 < y \leq 9$$

$$F(y) = \begin{cases} \frac{2\sqrt{y}}{5}, & 0 \leq y \leq 1 \\ \frac{y + 8\sqrt{y} + 7}{40}, & 1 < y \leq 9 \\ 0, & y < 0 \\ 1, & y \geq 9 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} \Rightarrow f_Y(y) = \begin{cases} \frac{d}{dy} \left( \frac{2\sqrt{y}}{5} \right), & 0 \leq y \leq 1 \\ \frac{d}{dy} \left( \frac{y + 8\sqrt{y} + 7}{40} \right), & 1 < y \leq 9 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{5\sqrt{y}}, & 0 \leq y \leq 1 \\ \frac{4 + \sqrt{y}}{40\sqrt{y}}, & 1 < y \leq 9 \\ 0, & \text{otherwise} \end{cases}$$

## Method 2:

$$\begin{aligned} f_Y(y) &= f_X(x) \left| \frac{dx}{dy} \right| \\ &= f_X(g^{-1}(y)) \left| \frac{d(g^{-1}(y))}{dy} \right| \end{aligned}$$

$$f_Y(y) = f_X(-\sqrt{y}) \left| \frac{d(-\sqrt{y})}{dy} \right| + f_X(\sqrt{y}) \left| \frac{d(\sqrt{y})}{dy} \right|, 0 \leq y \leq 1$$

$$f_Y(y) = \frac{4-\sqrt{y}}{20} \left| -\frac{1}{2\sqrt{y}} \right| + \frac{4+\sqrt{y}}{20} \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{5\sqrt{y}}, 0 \leq y \leq 1$$

$$f_Y(y) = f_X(\sqrt{y}) \left| \frac{d(\sqrt{y})}{dy} \right|, 1 < y \leq 9$$

$$f_Y(y) = \frac{4+\sqrt{y}}{20} \left| \frac{1}{2\sqrt{y}} \right| = \frac{4+\sqrt{y}}{40\sqrt{y}}, 1 < y \leq 9$$

$$f_Y(y) = \begin{cases} \frac{1}{5\sqrt{y}}, & 0 \leq y \leq 1 \\ \frac{4+\sqrt{y}}{40\sqrt{y}}, & 1 < y \leq 9 \\ 0, & \text{otherwise} \end{cases}$$

$$F(y) = \begin{cases} \int_0^y \frac{1}{5\sqrt{t}} dt = \frac{2\sqrt{y}}{5}, & 0 \leq y \leq 1 \\ \int_0^1 \frac{1}{5\sqrt{t}} dy + \int_1^y \frac{4+\sqrt{t}}{40\sqrt{t}} dt = \frac{y+8\sqrt{y}+7}{40}, & 1 < y \leq 9 \\ 0, & y < 0 \\ 1, & y \geq 9 \end{cases}$$

**Exercise 11:** The pmf of discrete random variable X is given in below:

$$p(x) = \begin{cases} \frac{1}{30}x, & x=1,2,3 \\ \frac{1}{60}(1+2x), & x=4,5,6,7 \\ 0, & \text{otherwise} \end{cases}$$

Find the pmf and cdf of  $Y = 2X + 1$ .

**Solution:**

$$y = 2x + 1 \Rightarrow x = \frac{y-1}{2} \quad \begin{array}{llll} x=1 \Rightarrow y=3 & x=2 \Rightarrow y=5 & x=3 \Rightarrow y=7 \\ x=4 \Rightarrow y=9 & x=5 \Rightarrow y=11 & x=6 \Rightarrow y=13 & x=7 \Rightarrow y=15 \end{array}$$

$$P_Y(y) = P_X(g^{-1}(y)) = P_X\left(\frac{y-1}{2}\right) = \begin{cases} \frac{1}{30}\left(\frac{y-1}{2}\right) = \frac{y-1}{60}, & y=3,5,7 \\ \frac{1}{60}\left[1+2\left(\frac{y-1}{2}\right)\right] = \frac{y}{60}, & y=9,11,13,15 \end{cases}$$

**Exercise 12:** The pdf of continuous random variable X is given in below:

$$f(x) = \begin{cases} \frac{2}{3}x, & 0 \leq x \leq 1 \\ \frac{1}{3}, & 1 < x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf of  $Y = X^2 + 1$ .

**Solution:**

$$y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow g_1^{-1}(y) = -\sqrt{y-1} = x_1, \\ g_2^{-1}(y) = \sqrt{y-1} = x_2,$$

$$\begin{aligned} P(Y \leq y) &= P(X^2 + 1 \leq y) = P(-y \leq X^2 + 1 \leq y) \\ &= P(-y-1 \leq X^2 \leq y-1) = P(-\sqrt{y-1} < X < \sqrt{y-1}) \\ &= F_X(\sqrt{y-1}) - \underbrace{F_X(-\sqrt{y-1})}_0 \end{aligned}$$

According to domain of random variable X,  $x_1 = -\sqrt{y-1}$  can not be used.

$$F_Y(y) = F_X(\sqrt{y-1})$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d(\sqrt{y-1})}{dy} \right|$$

$$\begin{aligned} \text{Domains } 0 \leq x \leq 1 &\Rightarrow 1 \leq y \leq 2 \\ 1 < x \leq 3 &\Rightarrow 2 < y \leq 10 \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{2}{3}\sqrt{y-1} \left( \frac{1}{2\sqrt{y-1}} \right) = \frac{1}{3}, & 1 \leq y \leq 2 \\ \frac{1}{3} \left( \frac{1}{2\sqrt{y-1}} \right) = \frac{1}{6\sqrt{y-1}}, & 2 \leq y \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

**Exercise 13:** Let  $X \sim U([-1,1])$ . Find the distribution of the random variable  $Y = X^2$ . The pdf of X is given by

$$f_X(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

**Solution:**

**Method 1:** Note that the range of random variable Y is  $[0,1]$ . There are two solutions to the equation  $y = x^2$ . Hence, the density of  $Y = X^2$  is given by

$$\begin{aligned} f_Y(y) &= \sum_{x^2=y} f_X(x) \left| \frac{dx}{dy} \right| = f_X(-\sqrt{y}) \left| \frac{d(-\sqrt{y})}{dy} \right| + f_X(\sqrt{y}) \left| \frac{d(\sqrt{y})}{dy} \right|, 0 \leq y \leq 1 \\ &= \frac{1}{2} \left| \frac{1}{-2\sqrt{y}} \right| + \frac{1}{2} \left| \frac{1}{2\sqrt{y}} \right| \\ &= \begin{cases} \frac{1}{2\sqrt{y}} & y \in [0,1] \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

**Method 2:** The cdf of X is  $F(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0, & x < -1 \\ 1, & x \geq 1 \end{cases}$

The cdf of Y is given by

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) \\ &= \begin{cases} 0 & y \in [-\infty, 0] \\ P(-\sqrt{y} \leq X \leq \sqrt{y}) & \text{if } y \in [0,1] \\ 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & y \in [-\infty, 0] \\ \frac{\sqrt{y}+1}{2} - \frac{(-\sqrt{y}+1)}{2} = \sqrt{y} & \text{if } y \in [0,1] \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

Hence, the density of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & y \in [0,1] \\ 0 & \textit{otherwise} \end{cases}$$