SECTION 8: MOMENT GENERATION FUNCTIONS EXERCISES

Exercise 1: A discrete random variable X has pmf that is of the form:

$$f(x) = \frac{x}{8}, \quad x = 1, 2, 5$$
$$= 0, \qquad otherwise$$

a) Find moment generation function (mgf) of X.
b) Find expected value of X using mgf of X.
c) Find the variance of X using mgf of X.
d) Find the mgf of $3X$+2?

Solution:

a) $M_X(t) = \sum_{Rx} e^{tx} p(X = x) = \dfrac{e^t + 2e^{2t} + 5e^{5t}}{8}$ $\qquad M_X(0) = \dfrac{e^0 + 2e^0 + 5e^0}{8} = 1$

b) $E(X) = \dfrac{d}{dt} M_X(t)\Big|_{t=0} = \dfrac{1}{8}\left(e^t + 4e^{2t} + 25e^{5t}\right)\Big|_{t=0} = \dfrac{30}{8} = \dfrac{15}{4}$

c) For variance of X, first we need to find $E(X^2)$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \frac{1}{8}\left(e^t + 8e^{2t} + 125e^{5t}\right)\Big|_{t=0} = \frac{134}{8} = \frac{67}{4}$$

$$V(X) = \frac{67}{4} - \frac{15}{4^2}^2 = \frac{268 - 225}{16} = \frac{43}{16}$$

d) First way: Y=3X+2

$$M_Y(t) = E(e^{tY}) = E(e^{3tX+2t}) = e^{2t}E(e^{3tX}) = e^{2t}M_X(3t)$$
$$= \frac{e^{2t}\left(e^{3t} + 2e^{6t} + 5e^{15t}\right)}{8} = \frac{e^{5t} + 2e^{8t} + 5e^{17t}}{8}$$

Second way: First the pmf of Y is found and then the mgf of Y is found.

$$p(y) = P(Y = y) = P(3X + 2 = y) = P\left(X = \frac{y-2}{3}\right) = p_X\left(\underbrace{\frac{y-2}{3}}_{g^{-1}(y)}\right)$$

$$p(y) = \frac{1}{8}\left(\frac{y-2}{3}\right) = \frac{y-2}{24}, \quad y = 5, 8, 17$$
$$= 0, \qquad otherwise$$

$$M_Y(t)=\sum_{Ry} e^{ty}P(Y=y)=\frac{3e^{5t}+6e^{8t}+15e^{17t}}{24}=\frac{e^{5t}+2e^{8t}+5e^{17t}}{8}$$

Exercise 2: Suppose that X has the pmf below:

x	-1	0	1	2
$P_X(x)$	0.2	0.1	0.3	0.4

a) Find the mgf of $Y = X^2$.
b) Find the expected value of *Y* using mgf of *Y*.
c) Find $E(Y^3)$ using mgf of *Y*.

Solution:

a)

$$P(Y=0)=P(X=0)=0.1$$
$$P(Y=1)=P(X=-1)+P(X=1)=0.5$$
$$P(Y=4)=P(X=2)=0.4$$

y	0	1	4
$P_Y(y)$	0.1	0.5	0.4

$$M_Y(t)=E(e^{tY})=0.1+e^t(0.5)+e^{4t}(0.4)$$

b) $$E(Y)=\frac{d}{dt}M_Y(t)\bigg|_{t=0}=\frac{d}{dt}\left(0.1+0.5e^t+0.4e^{4t}\right)=\left(0.5e^t+1.6e^{4t}\right)\bigg|_{t=0}=0.5+1.6=2.1$$

c) For $E(Y^3)$, 3th derivation of mgf of *Y* is found and put 0 where *t* is:

$$E(Y^3)=\frac{d^3}{dt^3}M_Y(t)\bigg|_{t=0}=\frac{d^2}{dt^2}\left(0.5e^t+1.6e^{4t}\right)=\frac{d}{dt}\left(0.5e^t+6.4e^{4t}\right)$$
$$=\left(0.5e^t+25.6e^{4t}\right)\bigg|_{t=0}=0.5+25.6=26.1$$

Exercise 3: The pdf of continuous random variable *X* is given in below:

$$f(x)=\begin{cases} ke^x, & 0<x<2 \\ 0, & otherwise \end{cases}$$

a) Find *k* value.
b) Find the mgf of *X*.
c) Find the characteristic function of X
d) Find the characteristic function of X/2.

Solution:

a) $\int_0^2 f(x)dx = 1 \Rightarrow \int_0^2 ke^x dx = ke^x\Big|_0^2 = k(e^2-1) = 1 \Rightarrow k = \dfrac{1}{(e^2-1)}$

b) $M_X(t) = \int_0^2 e^{tx} f(x)dx = 1 \Rightarrow \int_0^2 e^{tx}\dfrac{e^x}{(e^2-1)}dx = \dfrac{e^{x(t+1)}}{(t+1)(e^2-1)}\Bigg|_0^2 = \dfrac{e^{2(t+1)}-1}{(t+1)(e^2-1)}$

c) $\varphi_X(t) = E(e^{itX}) \Rightarrow \varphi_X(t) = M_X(it) = \dfrac{e^{2(\text{it}+1)}-1}{(it+1)(e^2-1)}$

d) $\varphi_{X/2}(t) = E(e^{it(X/2)}) = E(e^{(it/2)X}) = \varphi_X\left(\dfrac{it}{2}\right) = \dfrac{2e^{\text{it}+2}-2}{(it+2)(e^2-1)}$

Exercise 4: The pmf of discrete random variable X is given in below:

$$p(x) = \begin{cases} \dfrac{1}{3}\left(\dfrac{2}{3}\right)^{x-1}, & x = 1,2,3,\cdots \\ 0, & otherwise \end{cases}$$

a) Find the mgf of *X*.
b) Find the factorial moment generation function of X.
c) Find E(X(X-1)).
d) Find the pmf of $Y = X + 1$.

Solution:

a)

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx}\frac{1}{3}\left(\frac{2}{3}\right)^{x-1} = \frac{1}{2}\sum_{x=1}^{\infty}\left(\frac{2}{3}e^t\right)^x = \frac{1}{2}\left[\frac{2}{3}e^t + \left(\frac{2}{3}e^t\right)^2 + \left(\frac{2}{3}e^t\right)^3 + \ldots + \left(\frac{2}{3}e^t\right)^x + \ldots\right]$$

$$= \frac{e^t}{3}\left[1 + \left(\frac{2}{3}e^t\right) + \left(\frac{2}{3}e^t\right)^2 + \ldots + \left(\frac{2}{3}e^t\right)^{x-1} + \ldots\right] = \frac{\left(\dfrac{e^t}{3}\right)}{1-\left(\dfrac{2}{3}e^t\right)} = \frac{e^t}{3-2e^t}$$

b) $M_X(t) = \dfrac{e^t}{3-2e^t} \Rightarrow$ put $\ln(t)$ where t is in the mgf of *X*, the factorial moment generation function of *X*, $g_X(t) = M_X(\ln(t)) = \dfrac{t}{3-2t}$

c)

$$E(X(X-1)) = \frac{d^2}{dt^2}\left(g(t)\right)\Bigg|_{t=1} = \frac{d^2}{dt^2}\left(\frac{t}{3-2t}\right) = \frac{d}{dt}\left(\frac{3}{(3-2t)^2}\right) = \frac{12(3-2t)}{(3-2t)^4}$$

$$= \frac{12}{(3-2t)^3}\Bigg|_{t=1} = 4$$

d) Method 1. Since the random variable Y takes positive integer values, we can use factorial moment generation function to find probabilities of Y.

$$g_Y(t) = E(t^{X+1}) = tE(t^X) = tg_X(t) = \frac{t^2}{3-2t}$$

$$g_Y(0) = P(Y=0) = 0$$

$$g_Y'(0) = P(Y=1) = \frac{d}{dt}\left(\frac{t^2}{3-2t}\right) = \frac{6t-4t^2+2t^2}{(3-2t)^2} = \left.\frac{6t-2t^2}{(3-2t)^2}\right|_{t=0} = 0$$

$$g_Y''(0) = 2!P(Y=2) = \frac{d}{dt}\left(\frac{6t-2t^2}{(3-2t)^2}\right)$$

$$= \left.\frac{(6-4t)(3-2t)+4(6t-2t^2)}{(3-2t)^3}\right|_{t=0} = \frac{2}{3} \Rightarrow P(Y=2) = \frac{1}{3}$$

$$g_Y'''(0) = 3!P(Y=3) = \frac{d}{dt}\left(\frac{18}{(3-2t)^3}\right)$$

$$= \left.\frac{18\times 6}{(3-2t)^4}\right|_{t=0} = \frac{3^3\times 4}{3^4} \Rightarrow P(Y=2) = \frac{4}{3\times 3!} = \frac{1}{3}\left(\frac{2}{3}\right)$$

$\vdots$

Method 2.

$$y = x+1 \Rightarrow x = y-1 = g^{-1}(y)$$

$$x=1 \Rightarrow y=2$$
$$x=2 \Rightarrow y=3$$
$$x=3 \Rightarrow y=4$$
$$x=4 \Rightarrow y=5$$

$$p_Y(y) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^{y-2}, & y = 2,3,\cdots \\ 0, & otherwise \end{cases}$$

Exercise 5: Given the mgf $M_X(t) = \mathrm{e}^{3t+8t^2}$, find the mgf of the random variable $Z = \frac{X-3}{4}$ and use it to determine the mean and variance of Z.

Solution:

$$M_Z(t) = E(e^{tZ}) = E\left[e^{t\left(\frac{X-3}{4}\right)}\right] = e^{-\frac{3t}{4}}M_X\left(\frac{t}{4}\right) = e^{-\frac{3t}{4}}\,\mathrm{e}^{\frac{3t}{4}+\frac{t^2}{2}} = e^{\frac{t^2}{2}}$$

$$E(Z) = \left.\frac{d}{dt}M_Z(t)\right|_{t=0} = \frac{d}{dt}\left(e^{\frac{t^2}{2}}\right) = \left.te^{\frac{t^2}{2}}\right|_{t=0} = 0$$

$$E(Z^2) = \left.\frac{d^2}{dt^2}M_Z(t)\right|_{t=0} = \frac{d}{dt}\left(te^{\frac{t^2}{2}}\right) = \left.(e^{\frac{t^2}{2}} + t^2e^{\frac{t^2}{2}})\right|_{t=0} = 1$$

$$V(Z) = E(Z^2) - [E(Z)]^2 = 1 - 0^2 = 1$$