Name:

BBM 205 . Spring 2015

Exam 2

SOLUTIONS

(4 points)

1. Let $P(n)$ be the statement that $1^3+2^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) Complete the inductive step.

a) $P(1):\ 1^3 = \left(\frac{1\cdot 2}{2}\right)^2 = 1$ True

b) See (a).

c) I. H.: For all $i \leq n$, assume that $P(i)$ is true.

d) By (c), we assume that $P(n)$ is true.

Show that $P(n+1)$ is true.

$$1^3+2^3+\cdots+n^3+(n+1)^3 \underset{\substack{\downarrow \\ \text{By (c)} \\ \text{(I. H.)}}}{=} \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 =$$

$$= (n+1)^2\left[\frac{n^2}{4}+(n+1)\right] = (n+1)^2\left[\frac{n^2+4n+4}{4}\right] = \left[\frac{(n+1)(n+2)}{2}\right]^2$$

So, $P(n+1)$ is true.

(2 points)

2. Prove that 2 divides n^2+n whenever n is a positive integer. (<u>using induction</u>)

n, even $\rightarrow n^2+n$, even; for $n \geq 1$.

Base step: $P(1)$: $1^2+1 = 2$ divisible by 2.

Ind. Hypo.: $P(i)$, true for all $i \leq n$.

Ind. Step: Show that $P(n+1)$ is true.

Since $P(n)$ is true $n^2+n = 2k$ for some integer k.

$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = n^2 + 3n + 2 = 2k + 2n + 2 = 2(k+n+1)$ divisible by 2.

(1 point)

3. What is the cardinality of each of these sets?

a) $\emptyset$ b) $\{\emptyset\}$ c) $\{\emptyset, \{\emptyset\}\}$ d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

a) 0 b) 1 c) 2 d) 3

(2 points)

4. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ b) $\{x\} \subseteq \{x\}$ c) $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$ e) $\emptyset \subseteq \{x\}$ f) $\emptyset \in \{x\}$

a) True b) True c) False

d) True e) True f) False

(2 points)

5. Use the Euclidean algorithm to find gcd(1529, 14039).

Step 1: $14039 \div 1529 = 9$

$$\begin{array}{r} 14039 \\ -13761 \\ \hline 278 \end{array}$$

Step 2: gcd(1529, 14039) = gcd(1529, 278)

$1529 \div 278 = 5$

$$\begin{array}{r} 1529 \\ -1390 \\ \hline 139 \end{array}$$

Answer:

gcd(1529, 14039) = 139

Step 3: gcd(1529, 278) = gcd(278, 139)

$278 \div 139 = 2$

$$\begin{array}{r} 278 \\ -278 \\ \hline 0 \end{array}$$

(3 points)

6. Solve the recurrence relation with the given initial conditions: $a_n = 2a_{n-1} + 8a_{n-2}$, $a_0 = 4$, $a_1 = 10$

$$a_n - 2a_{n-1} - 8a_{n-2} = 0$$

Let $a_n = t^n$

Solution:

$$\boxed{a_n = 3 \cdot 4^n + (-2)^n}$$

$$t^n - 2t^{n-1} - 8t^{n-2} = 0$$

$$t^2 - 2t - 8 = 0$$

$$(t-4)(t+2) = 0$$

$$t_1 = 4, \ t_2 = -2$$

Solution: $S_n = A \cdot t_1^n + B t_2^n$

where $a_0 = S_0$

$$4 = A \cdot 4^0 + B \cdot (-2)^0 = A + B \rightarrow 16 = 4A + 4B$$

$a_1 = S_1$

$$10 = A \cdot 4^1 + B \cdot (-2)^1 = 4A - 2B \quad - \quad 10 = 4A - 2B$$

$$6 = 6B \rightarrow B = 1$$

$A = 3$

(3 points)

7a) Find a recurrence relation and initial conditions for C_n, the minimum number of moves in which the n-disk Tower of Hanoi puzzle can be solved.

b) Solve this recurrence relation.

a) $C_{n+1} = 2C_n + 1$

b) Use substitution (or telescope) method:

$$C_{n+1} = 2C_n + 1 = 2(2C_{n-1}+1)+1 = 2(2(2C_{n-2}+1)+1)+1 =$$

$$= \overbrace{2(2(2(\ldots}^{n \text{ times}}(2C_1+1)+1)+1)\cdots+1 =$$

$$= 2^n + 2^{n-1} + 2^{n-2} + \cdots + 1 = 2^{n+1} - 1$$

use $C_1 = 1$

(3 points)

8. Let f_n be the n^{th} Fibonacci number. Show that $f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ when n is a positive integer.

Use induction:

Base step: for $n=1$ $\quad \overset{=0}{f_0} - \overset{=1}{f_1} + \overset{=1}{f_2} = \overset{=1}{f_1} - 1$ $\quad$ True for $n=1$

Ind. Hypo.: Assume true for all $i \le n$ that

$$f_0 - f_1 + f_2 - \cdots - f_{2i-1} + f_{2i} = f_{2i-1} - 1$$

Ind. Step: $\underbrace{f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n}} - f_{2n+1} + f_{2(n+1)} =$

$$= f_{2n-1} - 1 - f_{2n+1} + f_{2n+2} = f_{2n-1} - 1 + \underbrace{(f_{2n+2} - f_{2n+1})}_{= f_{2n}} =$$

By Ind. Hypo.

$$= f_{2n} + f_{2n-1} - 1 = f_{2n+1} - 1$$

true