Elementary Matrices

Any matrix obtained from the identity matrix by applying one single elementary $\underbrace{\text{row operation}}_{\text{OR (column)}}$ is called an elementary matrix.

For example

$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is an elementary matrix; $E_1 = I_{3\,R_1 \leftrightarrow R_3}$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix; $E_2 = I_{3\,(-2)R_2}$

$E_3 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix; $E_3 = I_{3\,2R_2 + R_1 \to R_1}$

$E_4 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix; $E_4 = I_{3\,3C_1 + C_3 \to C_3}$

Theorem: Let A be an $m \times n$ matrix and let an elementary row (column) operation be performed on A to yield matrix B.

Let E be the elementary matrix obtained from I_m (I_n) by performing the same elementary row (column) operation as we performed on A.

Then $B = EA$ ($B = AE$).

For example let $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix}$

let $B = A_{(-2)R_3 + R_1 \to R_1}$; then

$$B = \begin{bmatrix} -5 & 3 & 0 & -3 \\ -1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix}$$

Now let $E = I_{3\,(-2)R_3 + R_1 \to R_1}$; then

$$E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can readily verify that $B = EA$.

Theorem: If the matrix B is obtained from A by applying elementary row operations $E_1, E_2, \ldots, E_r$, then $B = PA$ where

$$P = E_r E_{r-1} \cdots E_1(I) = E_r(I)\, E_{r-1}(I) \cdots E_1(I)$$

is a product of elementary matrices.

$$[A \mid I] \longrightarrow [B \mid P] \Rightarrow B = PA$$

A matrix A is said to have a left inverse if there exists a matrix A' such that $A'A = I$.

It is said to have a right inverse if there exists a matrix A' such that $AA' = I$.

A matrix A is called an invertible (a nonsingular) matrix if there exists a matrix A' such that

$$AA' = A'A = I.$$

- An invertible matrix A must be a square matrix because of the equality $AA' = A'A$.

- Let A'' be another matrix satisfying

$$AA'' = A''A = I$$

Then

$$\left.\begin{aligned} A'AA'' &= (A'A)A'' = IA'' = A'' \\ A'AA'' &= A'(AA'') = A'I = A' \end{aligned}\right\} \Rightarrow A' = A''.$$

Thus the inverse of an invertible matrix is uniquely determined.

- The inverse of A is denoted by A^{-1}.

Theorem: If $A_1, A_2, \ldots, A_r$ are invertible matrices matrices with inverses $A_1^{-1}, A_2^{-1}, \ldots, A_r^{-1}$ respectively, then the product $A_1 A_2 \cdots A_r$ is also invertible and its inverse is

$$(A_1 A_2 \cdots A_r)^{-1} = A_r^{-1} A_{r-1}^{-1} \cdots A_1^{-1}$$

* Every elementary matrix is invertible.

* Every product of elementary matrices is also invertible.

Theorem: For an $n \times n$ matrix A the following are equivalent:

(1) A is a product of elementary matrices.

(2) A is invertible.

(3) A is <u>not</u> row equivalent to a matrix with a zero row.

(4) A is row equivalent to the identity matrix.

Symbolically

We have

either $[A \mid I] \longrightarrow [\text{a zero row} \mid *]$

which means A is noninvertible (singular)

or $[A \mid I] \longrightarrow [I \mid A^{-1}]$

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We can characterize invertible matrices conveniently by means of systems of linear equations.

For a square matrix A, the following are equivalent

(1) A is invertible.

(2) $AX = B$ has a unique solution.

(3) $AX = 0$ has only the trivial solution.

Example: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-5)R_1 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \xrightarrow{-R_2 + R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{4}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right] \xrightarrow[\left(\frac{1}{2}\right)R_3 + R_1 \to R_1]{\left(-\frac{3}{2}\right)R_3 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/8 & -1/2 & -1/8 \\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8 \\ 0 & 0 & 1 & 5/4 & 0 & -1/4 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 13/8 & -1/2 & -1/8 \\ -15/8 & 1/2 & 3/8 \\ 5/4 & 0 & -1/4 \end{bmatrix}$

Example: Let $A=\begin{bmatrix}1 & 2 & -3\\ 1 & -2 & 1\\ 5 & -2 & -3\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1 & 2 & -3 & 1 & 0 & 0\\ 1 & -2 & 1 & 0 & 1 & 0\\ 5 & -2 & -3 & 0 & 0 & 1\end{array}\right] \xrightarrow[(-5)R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc}1 & 2 & -3 & 1 & 0 & 0\\ 0 & -4 & 4 & -1 & 1 & 0\\ 0 & -12 & 12 & -5 & 0 & 1\end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|ccc}1 & 2 & -3 & 1 & 0 & 0\\ 0 & 1 & -1 & 1/4 & -1/4 & 0\\ 0 & -12 & 12 & -5 & 0 & 1\end{array}\right] \xrightarrow[12R_2+R_3\to R_3]{-2R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc}1 & 0 & -1 & 1/2 & 1/2 & 0\\ 0 & 1 & -1 & 1/4 & -1/4 & 0\\ 0 & 0 & 0 & -2 & -3 & 1\end{array}\right]$$

A is not invertible!

Equivalen Matrices

If A and B are two $m \times n$ matrices, then A is equivalent to B if we obtain B from A by applying a finite sequence of elementary row or elementary column operations.

* Every matrix is equivalent to itself.

* If B is equivalent to A, then A is equivalent to B

* C is equivalent to B and B is equivalent to A

  $\Rightarrow$ C is equivalent to A.

* A and B are equivalent $\Longleftrightarrow$ $B = PAQ$ where P and Q are invertible.

* A matrix is equivalent to a uniquely determined row reduced echelon matrix.