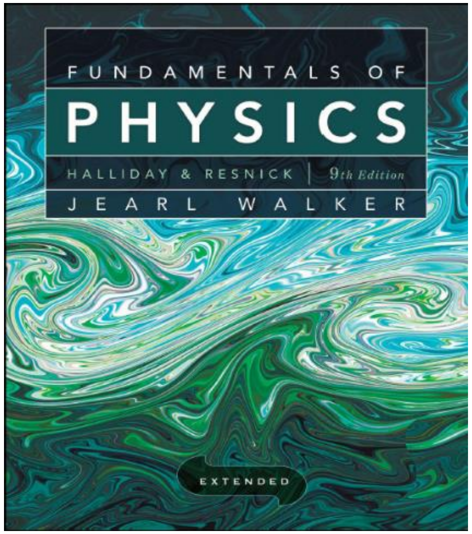
FİZ 137- 25
CHAPTER 4
MOTION IN TWO AND THREE DIMENSIONS



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2018 - 2019
FALL SEMESTER

In this chapter we will continue to study the motion of objects without the restriction to move along a straight line.

Instead we will consider motion is in a plane (two dimensional motion) and motion in space (three dimensional motion).

The following physical quantities will be defined for two and three dimensional motion:

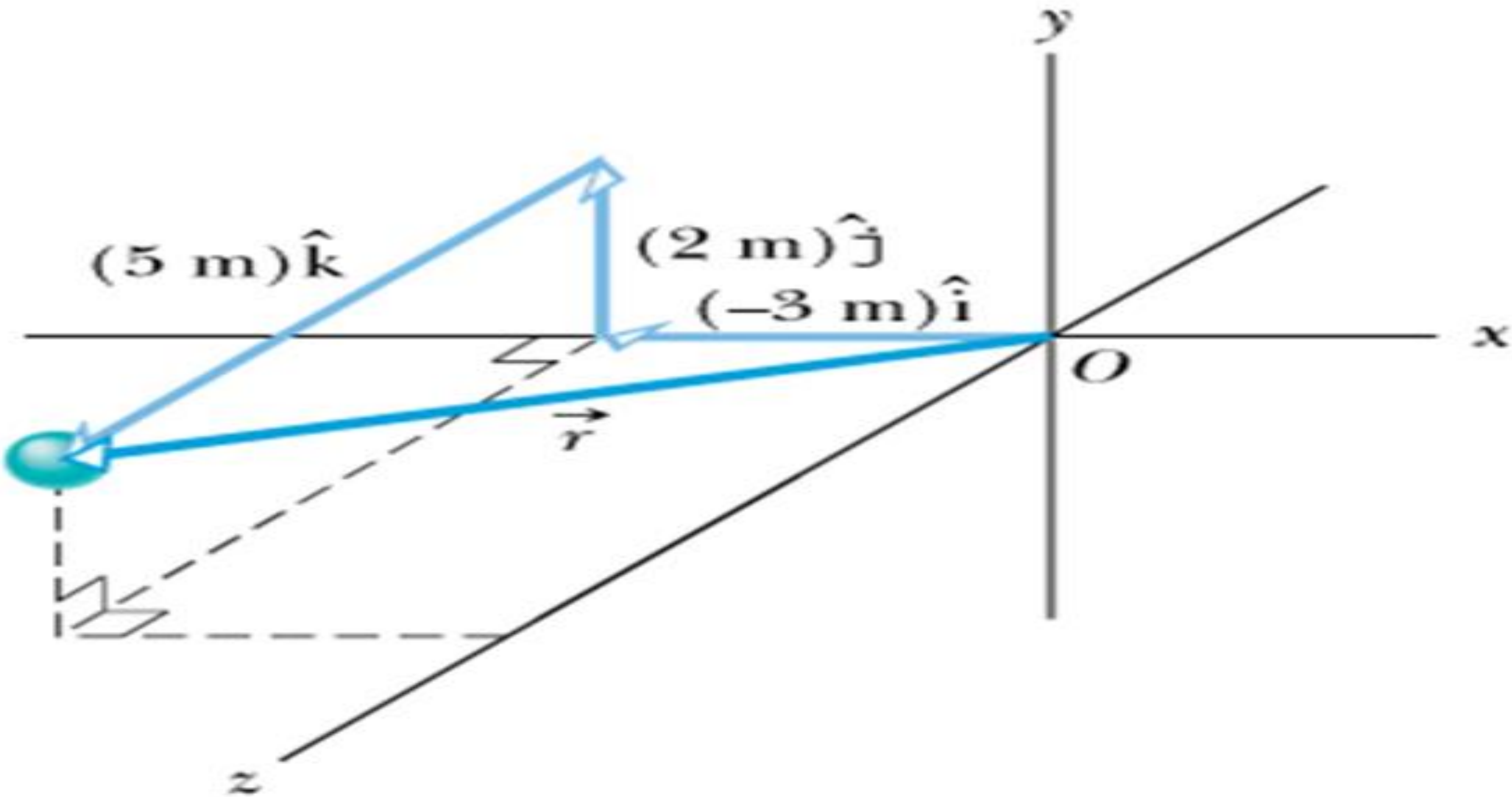
- ➔**Displacement**
- ➔**Average and instantaneous velocity**
- ➔**Average and instantaneous acceleration**
- ➔**Projectile motion**
- ➔**Uniform circular motion**
- ➔**Relative motion** (the transformation of velocities between two reference systems which move with respect to each other with constant velocity)

Position Vector

The position vector $\vec{r}$ of a particle is defined as a vector whose tail is at a reference point (usually the origin O) and its tip is at the particle at point P.

Example: The position vector in the figure is:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = \left(-3\hat{i} + 2\hat{j} + 5\hat{k}\right)m$$

Displacement Vector

For a particle that changes postion vector from $\vec{r}_1$ to $\vec{r}_2$ we define the displacement vector $\Delta\vec{r}$ as follows:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

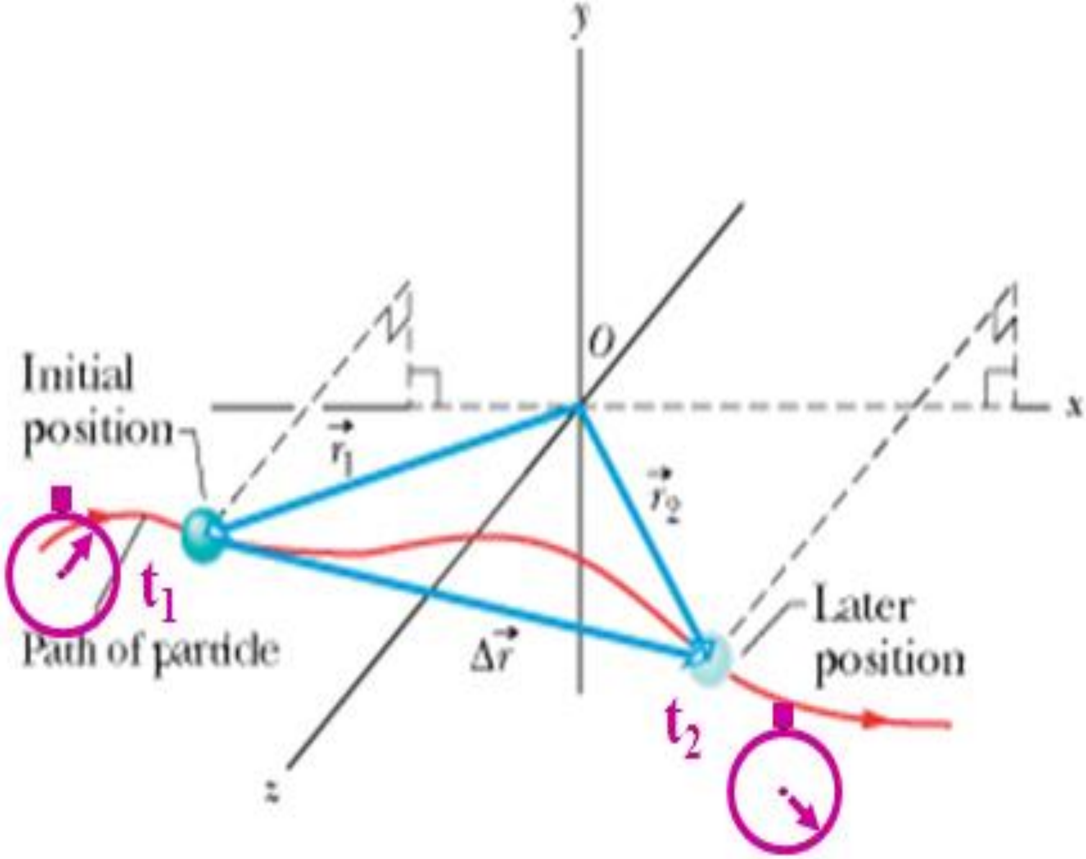
The position vectors $\vec{r}_1$ and $\vec{r}_2$ are written in terms of components as:

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

$$\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

The displacement $\Delta\vec{r}$ can then be written as:

$$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$$



$$\Delta x = x_2 - x_1$$

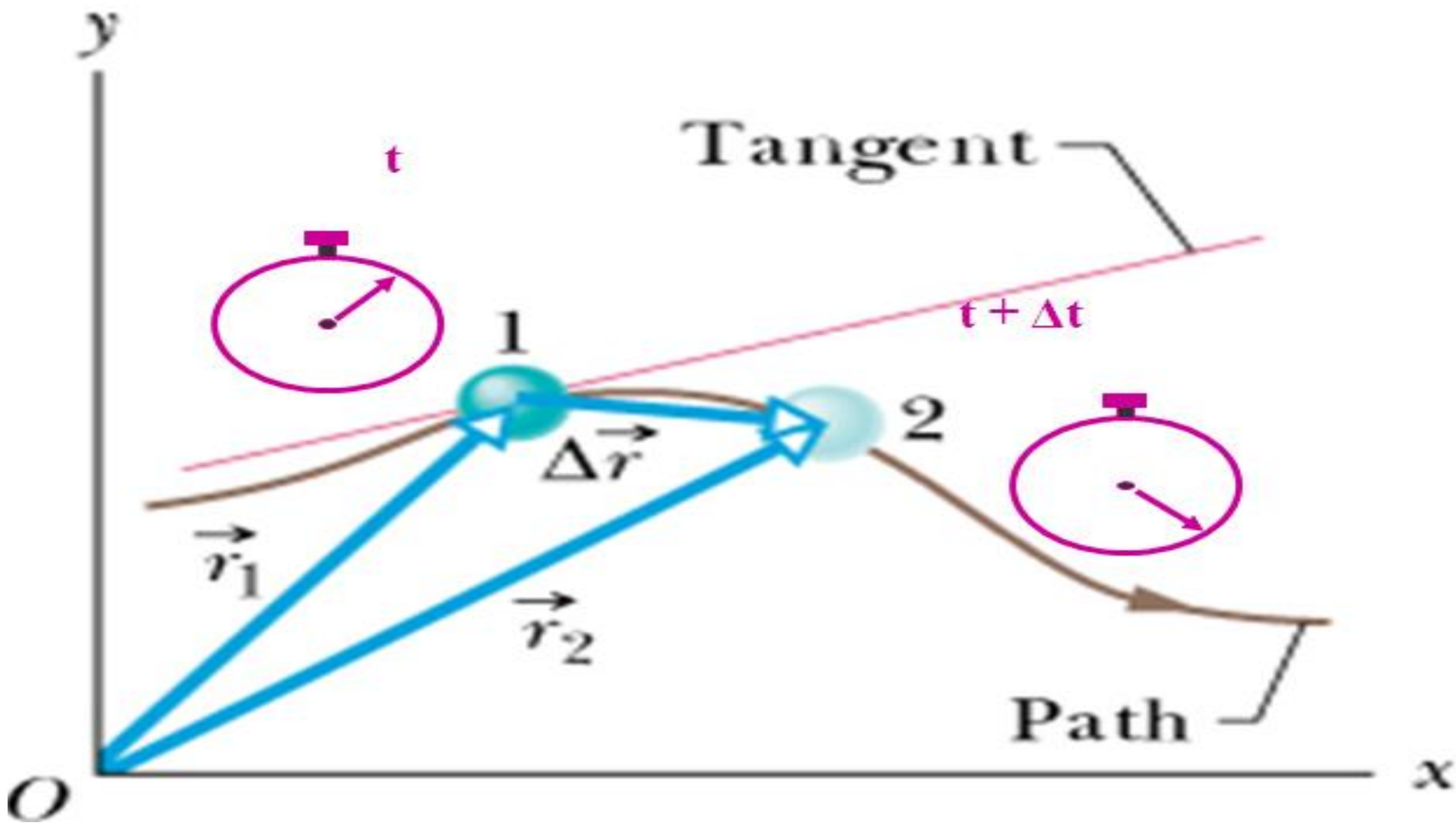
$$\Delta y = y_2 - y_1$$

$$\Delta z = z_2 - z_1$$

Average and Instantaneous Velocity

$$\text{average velocity} = \frac{\text{displacement}}{\text{time interval}}$$

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x\hat{i}}{\Delta t} + \frac{\Delta y\hat{j}}{\Delta t} + \frac{\Delta z\hat{k}}{\Delta t}$$



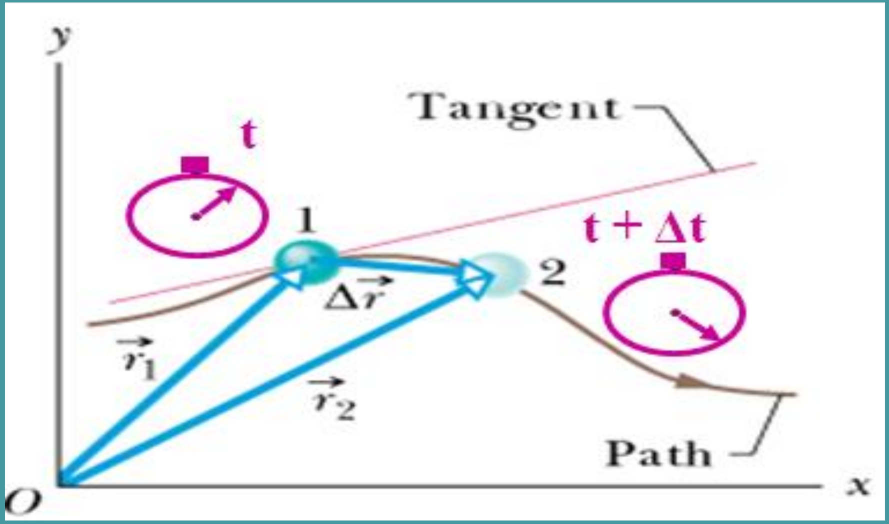
$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}}{\Delta t} = \frac{\Delta x\hat{i}}{\Delta t} + \frac{\Delta y\hat{j}}{\Delta t} + \frac{\Delta z\hat{k}}{\Delta t}$$

If we allow the time interval Δt to shrink to zero, the following things happen:

1. Vector $\vec{r}_2$ moves towards vector $\vec{r}_2$ and $\Delta\vec{r} \to 0$

2. The direction of the ratio $\dfrac{\Delta\vec{r}}{\Delta t}$ (and thus $\vec{v}_{avg}$)approaches the direction of the tangent to the path at position 1

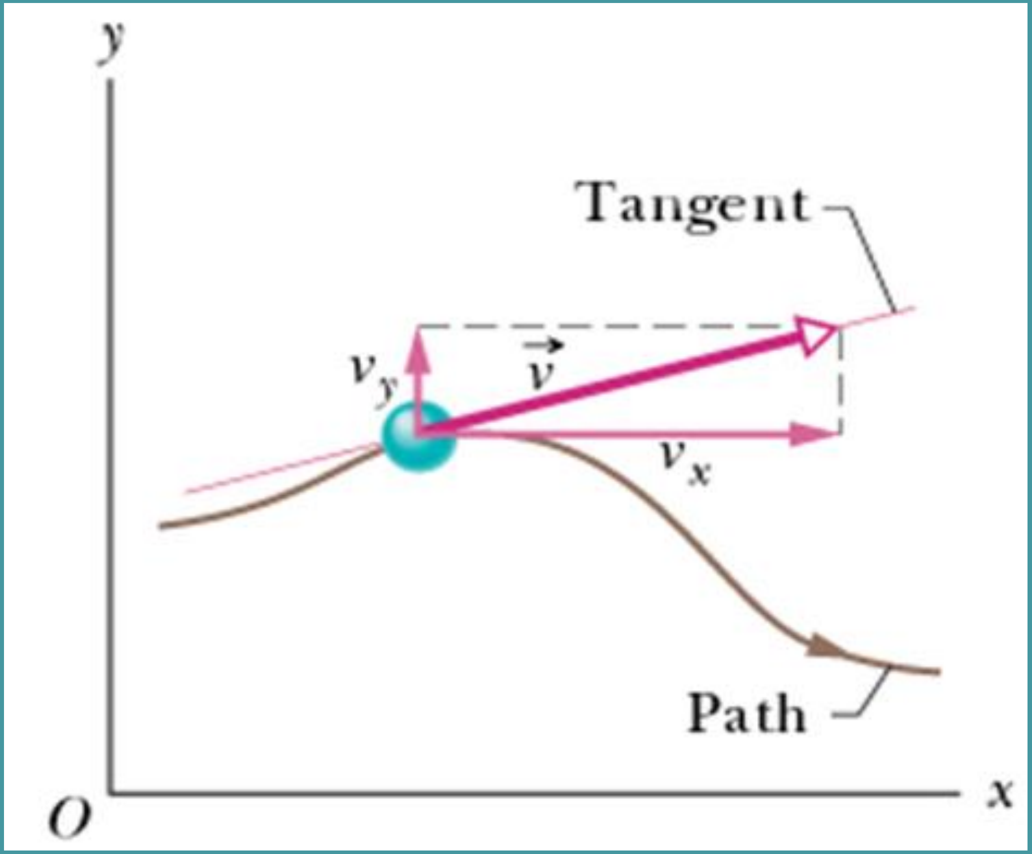
3. $\vec{v}_{avg} \to \vec{v}$



$$\vec{v} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

We define as the instantaneous velocity as the limit:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$



$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$v_x = \frac{dx}{dt}$$

$$v_y = \frac{dy}{dt}$$

$$v_z = \frac{dz}{dt}$$

Average Acceleration

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time interval}}$$

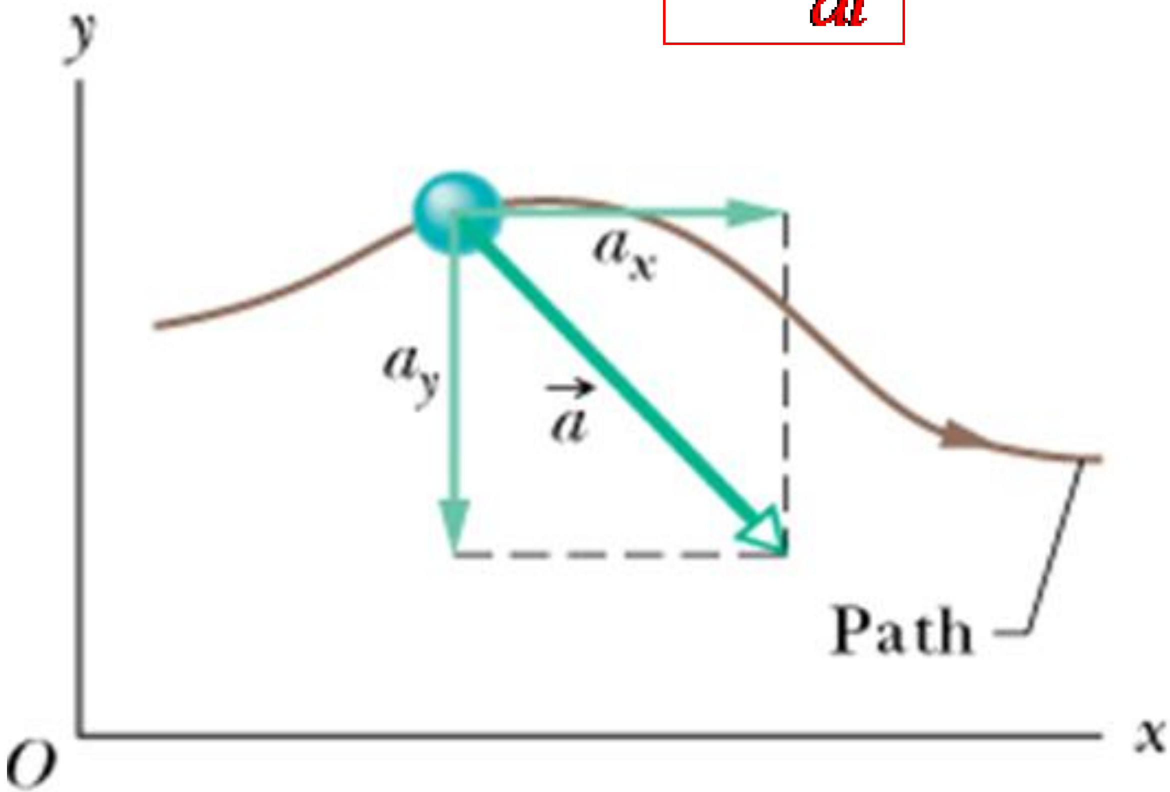
$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}$$

Instantaneous Acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v_x \hat{i} + v_y \hat{j} + v_z \hat{k}\right) = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

The three acceleration components are given by the equations:

$$\boxed{\vec{a} = \frac{d\vec{v}}{dt}}$$



$$\boxed{a_x = \frac{dv_x}{dt}}$$

$$\boxed{a_y = \frac{dv_y}{dt}}$$

$$\boxed{a_z = \frac{dv_z}{dt}}$$

Non-Constant Accelaration

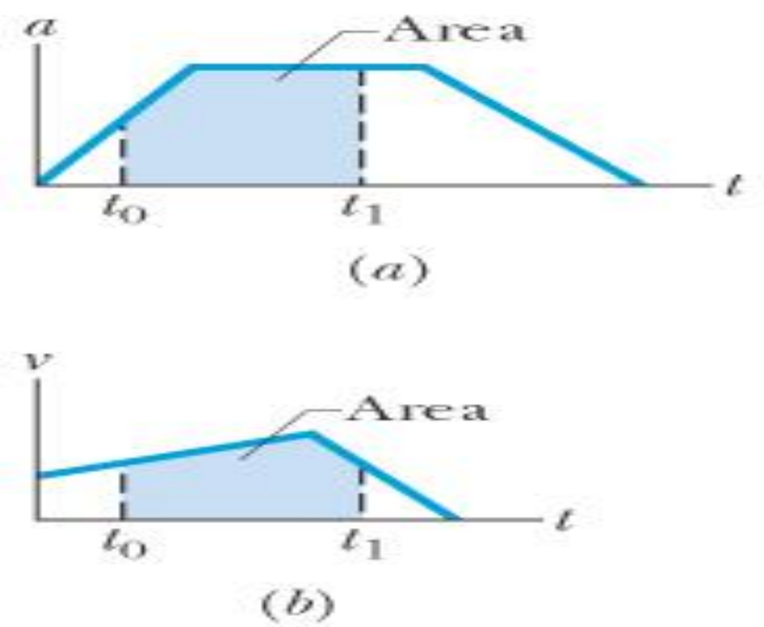
Graphical Integration in Motion Analysis (non-constant acceleration)

When the acceleration of a moving object is not constant we must use integration to determine the velocity v(t) and the position x(t) of the object.

The integration can be done either using the analytic or the graphical approach

$$a = \frac{dv}{dt} \rightarrow dv = adt \rightarrow \int_{t_o}^{t_1} dv = \int_{t_o}^{t_1} adt \rightarrow v_1 - v_o = \int_{t_o}^{t_1} adt \rightarrow v_1 = v_o + \int_{t_o}^{t_1} adt$$

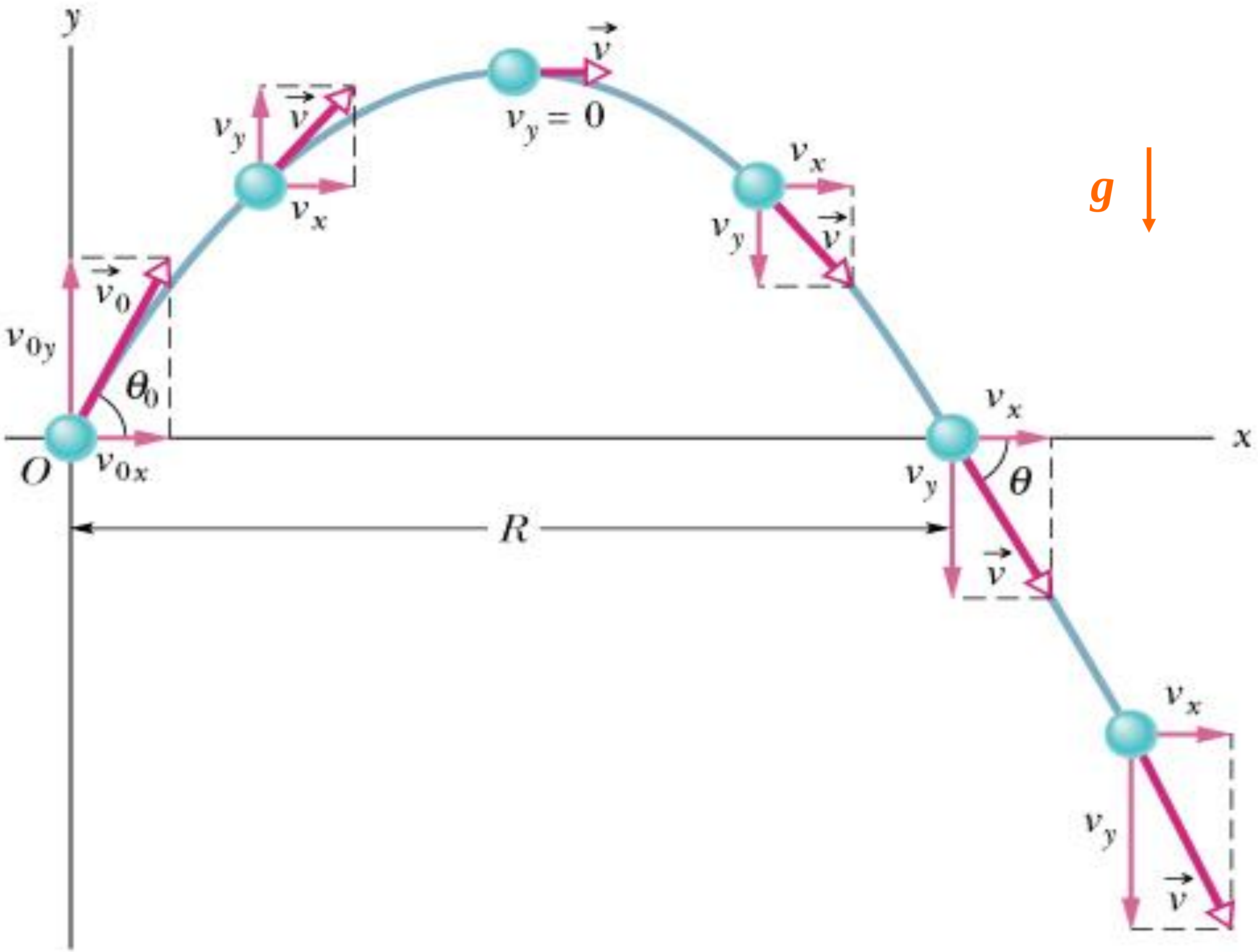
$$\int_{t_o}^{t_1} adt = \left[\text{Area under the } a \text{ versus } t \text{ curve between } t_o \text{ and } t_1\right]$$



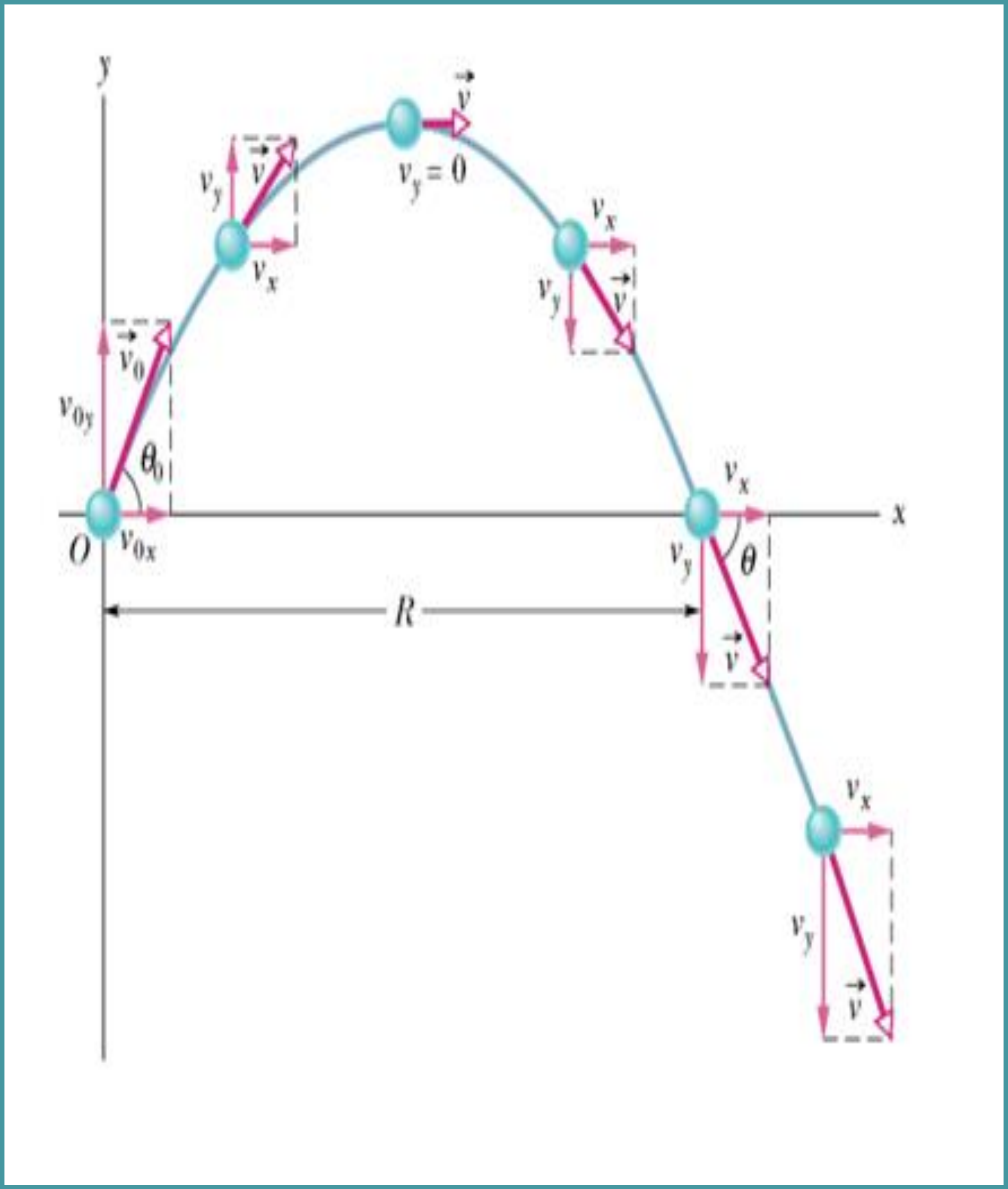
$$a = \frac{dv}{dt} \rightarrow dv = a\,dt \rightarrow \int_{t_o}^{t_1} dv = \int_{t_o}^{t_1} a\,dt \rightarrow v_1 - v_o = \int_{t_o}^{t_1} a\,dt \rightarrow v_1 = v_o + \int_{t_o}^{t_1} a\,dt$$

Projectile Motion

The motion of an object in a vertical plane under the *influence of gravitational force* is known as *"projectile motion"*



The projectile is launched with an initial velocity $\vec{V}_o$



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Here x_o and y_o are the coordinates of the launching point. For many problems the launching point is taken at the origin. In this case

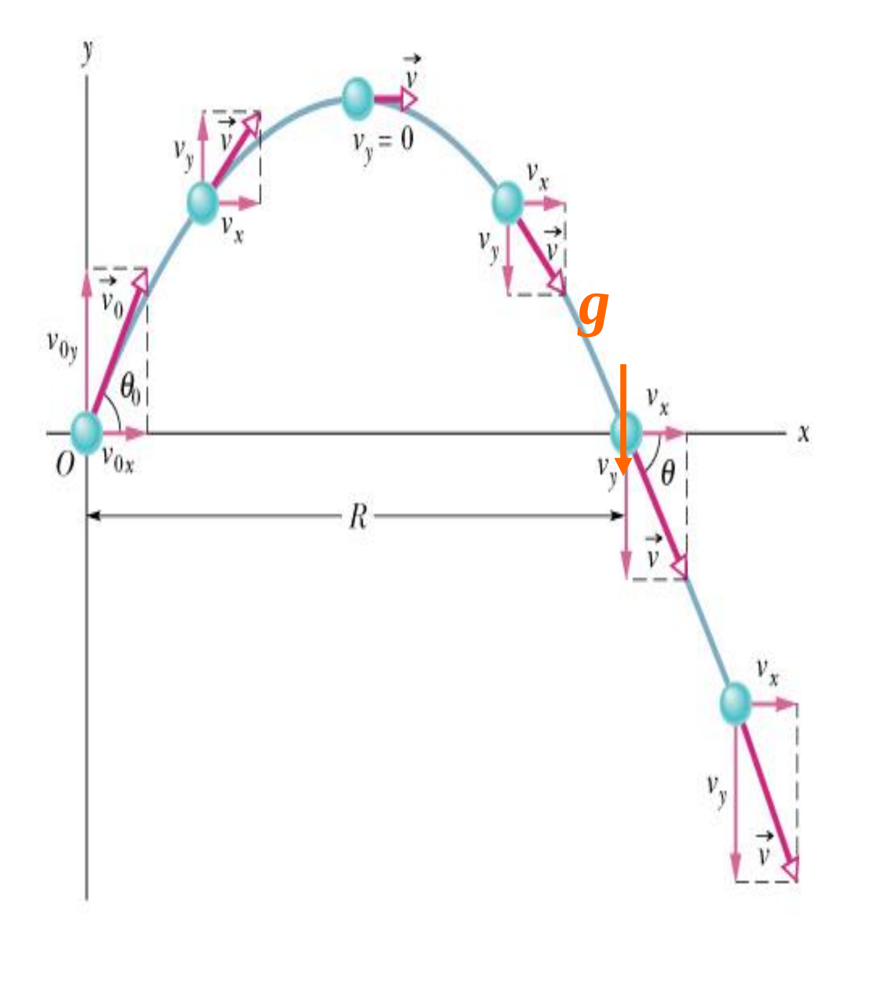
$x_o = 0$ and $y_o = 0$

Note: In this analysis of projectile motion we neglect the effects of air resistance

The horizontal and vertical velocity components are:

$$v_{ox} = v_o \cos\theta_o$$

$$v_{oy} = v_o \sin\theta_o$$



Projectile motion will be analyzed in a horizontal and a vertical motion along the x and y axes.

These two motions are independent of each other.

Motion along the x-axis has zero acceleration. Motion along the y-axis has uniform (constant) acceleration $a_y = -g$

Horizontal Motion: $a_x = 0$ The velocity along the x-axis does not change

$$v_x = v_0 \cos\theta_0 \quad \text{(eqs.1)} \qquad x = x_o + (v_o \cos\theta_o)t \quad \text{(eqs.2)}$$

Vertical Motion: $a_y = -g$ Along the y-axis the projectile is in free fal

$$v_y = v_0 \sin\theta_0 - gt \quad \text{(eqs.3)} \qquad y = y_o + (v_0 \sin\theta_0)t - \frac{gt^2}{2} \quad \text{(eqs.4)}$$

If we eliminate t between equations 3 and 4 we get: $v_y^2 - (v_0 \sin\theta_0)^2 = -2g(y - y_o)$

The equation of the path:

$$x = (v_o \cos\theta_o)t \quad \text{(eqs.2)} \qquad\qquad y = (v_0 \sin\theta_0)t - \frac{gt^2}{2} \quad \text{(eqs.4)}$$

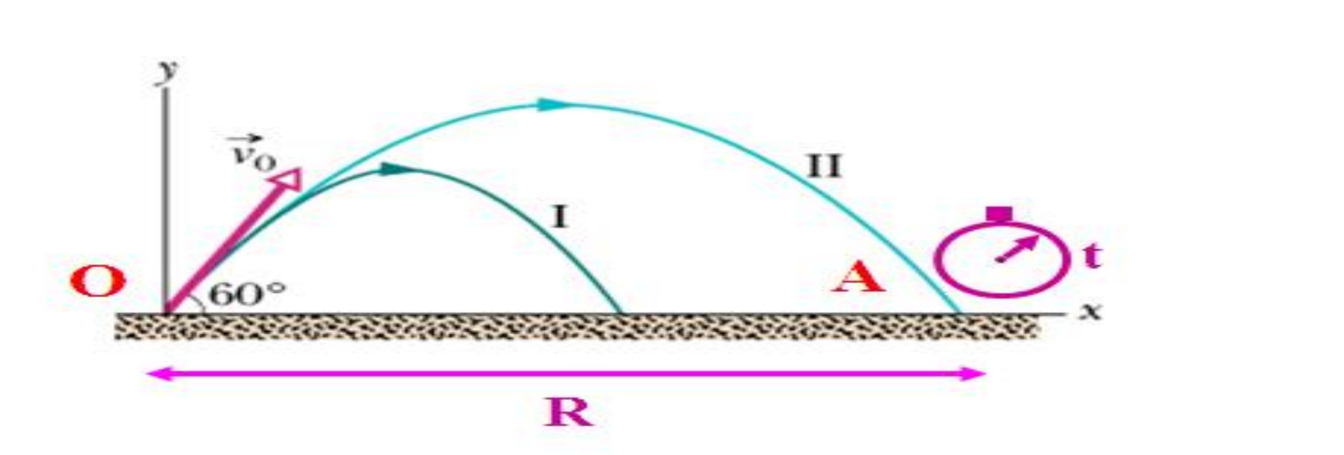
If we eliminate t between equations 2 and 4 we get:

$$y = (\tan\theta_o)x - \frac{g}{2(v_o \cos\theta_o)^2}x^2$$

This equation describes the path of the motion

The path equations has the form: $y = ax + bx^2$ This is the equation of a parabola

Horizontal Range (R)



Horizontal Range: The distance OA is defined as the horizantal range R

At point A we have: $y = 0$ From equation 4 we have:

$(v_0 \sin\theta_0)t - \frac{gt^2}{2} = 0 \rightarrow t\left(v_0 \sin\theta_0 - \frac{gt}{2}\right) = 0$ This equation has two solutions:

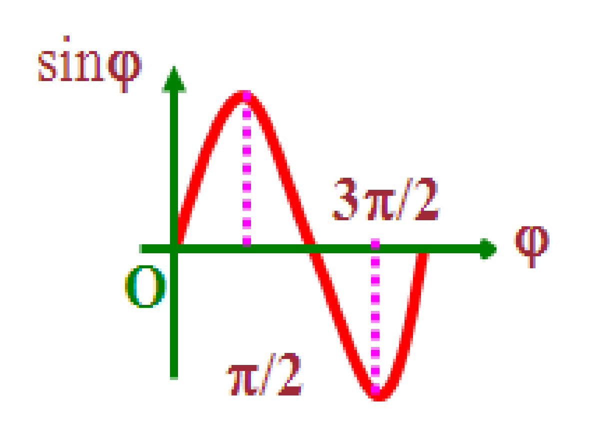
Solution 1. $t = 0$ This solution correspond to point O and is of no interest

Solution 2. $v_0 \sin\theta_0 - \frac{gt}{2} = 0$ This solution correspond to point A

From solution 2 we get: $\boxed{t = \frac{2v_0 \sin\theta_0}{g}}$

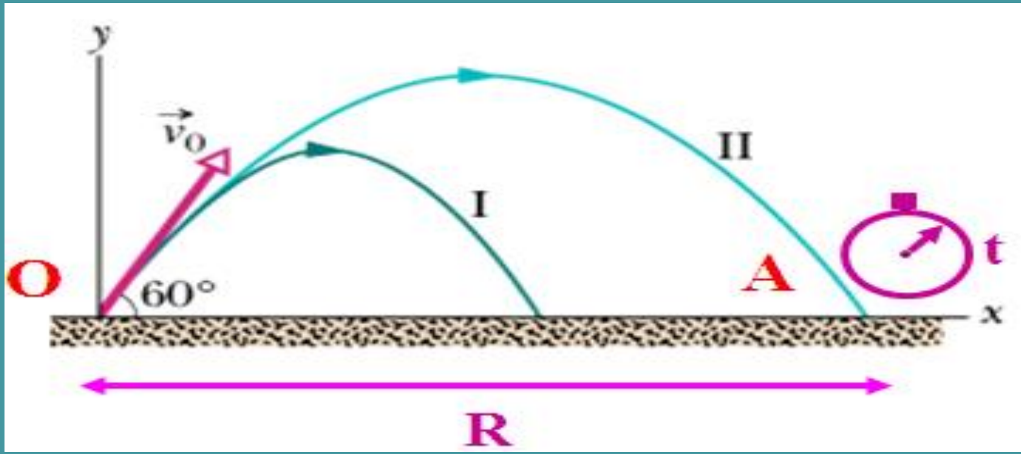
If we substitute t in eqs.2 we get:

$$R = \frac{2v_o^2}{g}\sin\theta_o\cos\theta_o = \boxed{\frac{v_o^2}{g}\sin 2\theta_o = R}$$

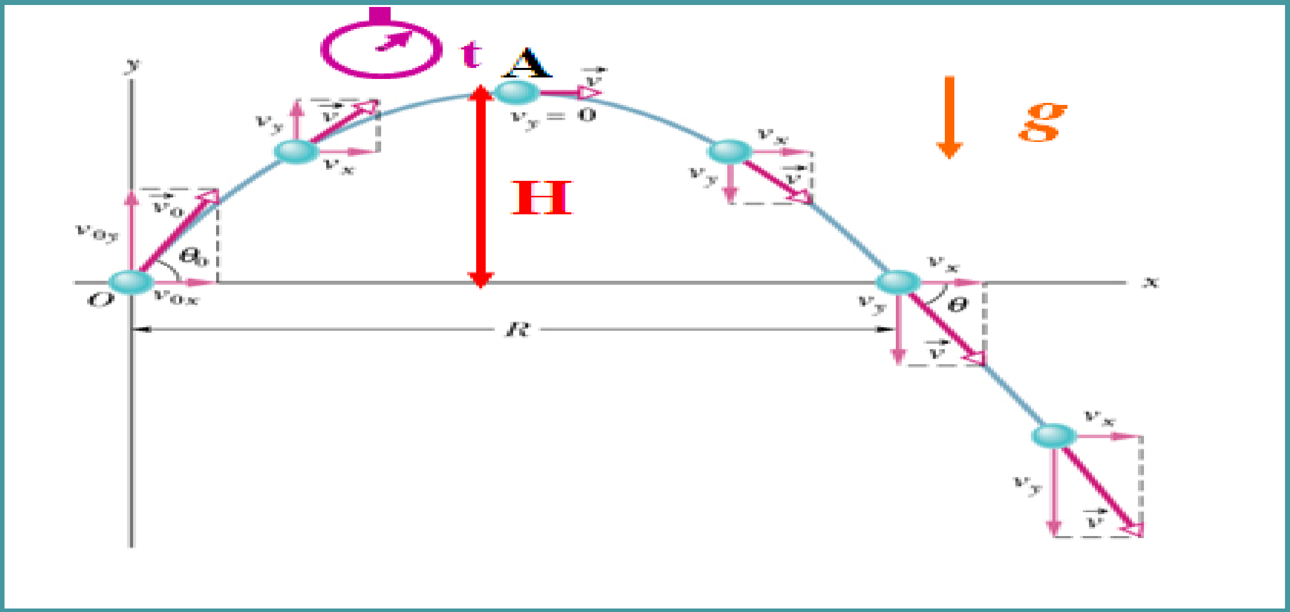


$$2\sin A\cos A = \sin 2A$$

R has its maximum value when $\theta_o = 45°$ $\quad R_{max} = \dfrac{v_o^2}{g}$



Maximum Height (H)



$$H = \frac{v_o^2 \sin^2 \theta_o}{2g}$$

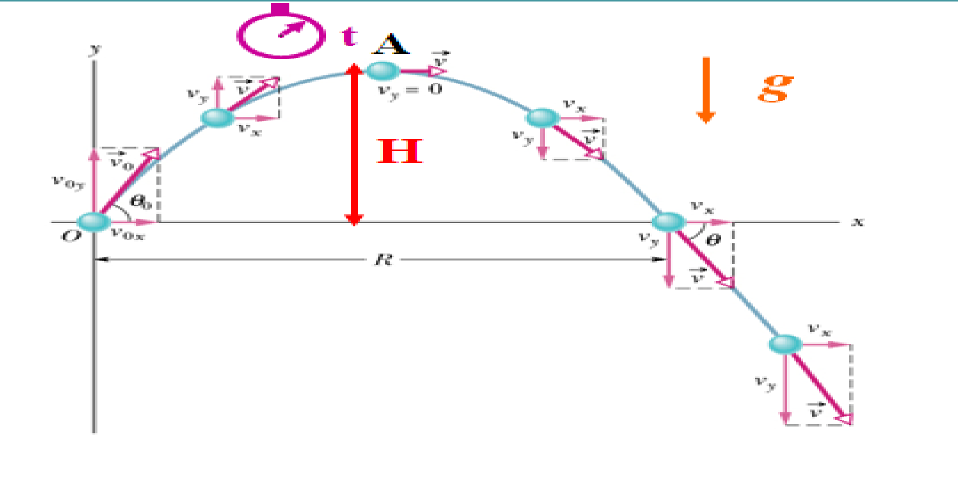
The y-component of the projectile velocity is: $v_y = v_0 \sin\theta_0 - gt$

At point A: $v_y = 0 \quad \rightarrow v_0 \sin\theta_0 - gt \rightarrow t = \dfrac{v_0 \sin\theta_0}{g}$

$$H = y(t) = (v_0 \sin\theta_0)t - \frac{gt^2}{2} = (v_0 \sin\theta_0)\frac{v_0 \sin\theta_0}{g} - \frac{g}{2}\left(\frac{v_0 \sin\theta_0}{g}\right)^2 \rightarrow$$

$$\boxed{H = \frac{v_o^2 \sin^2 \theta_o}{2g}}$$

Maximum Height (H)



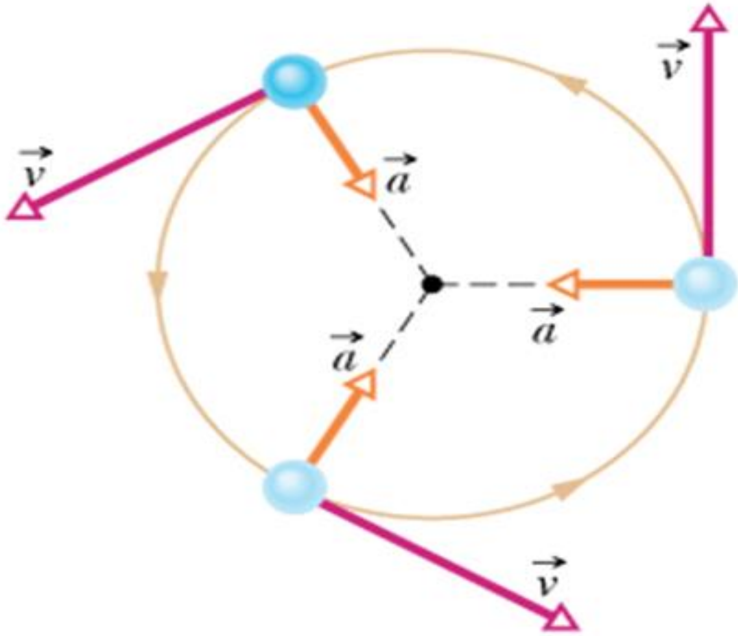
We can calculate the maximum height using the third equation of kinematics for motion along the y-axis: $v_y^2 - v_{yo}^2 = 2a(y - y_o)$

In our problem: $y_o = 0$, $y = H$, $v_{yo} = v_o \sin\theta_o$, $v_y = 0$, and $a = -g \rightarrow$

$$-v_{yo}^2 = -2gH \rightarrow H = \frac{v_{yo}^2}{2g} = \boxed{\frac{v_o^2 \sin^2\theta_o}{2g} = H}$$

Uniform Circular Motion

A particles is in uniform circular motion it moves on a circular path of radius *r* with constant speed *v*.



• Even though the speed is constant, <u>the velocity is not constant</u> as the direction of the velocity vector changes from point to point along the path.

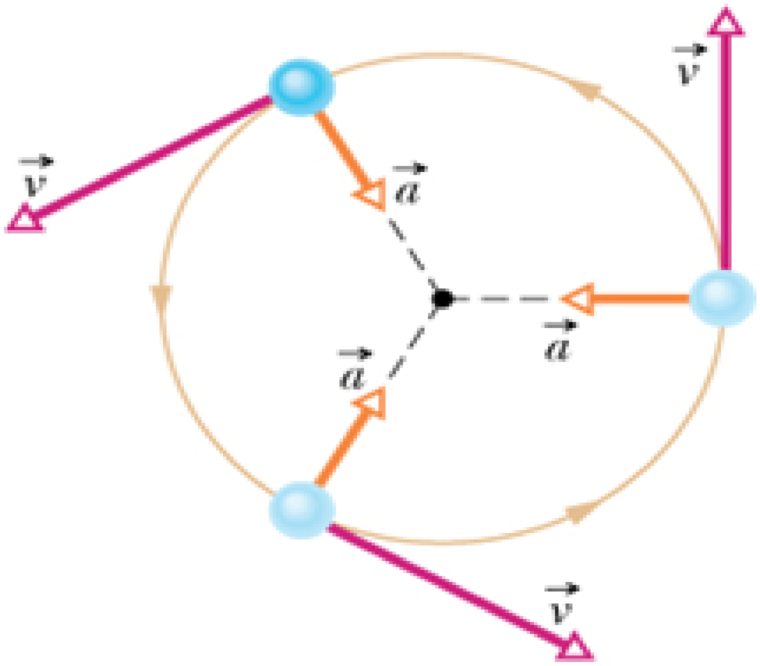
• The fact that the velocity changes means that the ***<u>acceleration is not zero</u>***.

The acceleration in uniform circular motion has the following characteristics:

1. Its vector points towards the center C of the circular path, thus the name **“<u>centripetal</u>”.**

2. Its magnitude *a* is given by the equation:

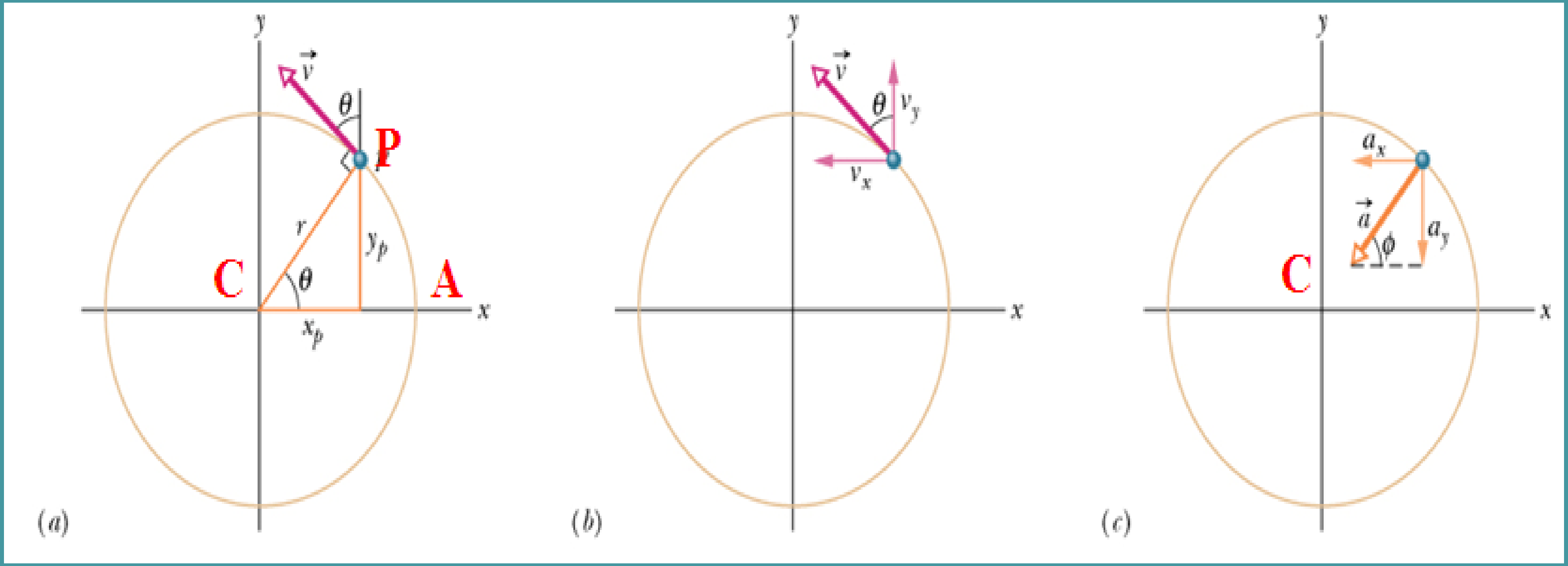
$$\boxed{a = \frac{v^2}{r}}$$



The time *T* it takes to complete a full revolution is known as the "<u>period</u>". It is given by the equation:

$$\boxed{T = \frac{2\pi r}{v}}$$

Proof



$$v_x = -v\sin\theta$$

$$v_y = v\cos\theta$$

Proof

$$\vec{v} = v_x\hat{i} + v_y\hat{j} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j} \qquad \sin\theta = \frac{y_P}{r} \qquad \cos\theta = \frac{x_P}{r}$$

Here x_P and y_P are the coordinates of the rotating particle

$$\vec{v} = \left(-v\frac{y_P}{r}\right)\hat{i} + \left(v\frac{x_P}{r}\right)\hat{j} \quad \text{Acceleration } \vec{a} = \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\frac{dy_P}{dt}\right)\hat{i} + \left(\frac{v}{r}\frac{dx_P}{dt}\right)\hat{j}$$

We note that: $\dfrac{dy_P}{dt} = v_y = v\cos\theta$ and $\dfrac{dx_P}{dt} = v_x = -v\sin\theta$

$$\vec{a} = \left(-\frac{v^2}{r}\cos\theta\right)\hat{i} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{j} \qquad a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{(\cos\theta)^2 + (\sin\theta)^2} = \frac{v^2}{r}$$

$$\boxed{(\cos\theta)^2 + (\sin\theta)^2 = 1}$$

$$\boxed{a = \frac{v^2}{r}}$$

(centripetal accelaration)

$$\tan\phi = \frac{a_y}{a_x} = \frac{-(v^2/r)\sin\theta}{-(v^2/r)\cos\theta} = \tan\theta \rightarrow \phi = \theta \rightarrow \vec{a} \text{ points towards C}$$

Relative Motion

In One Dimension

Relative Motion in One Dimension

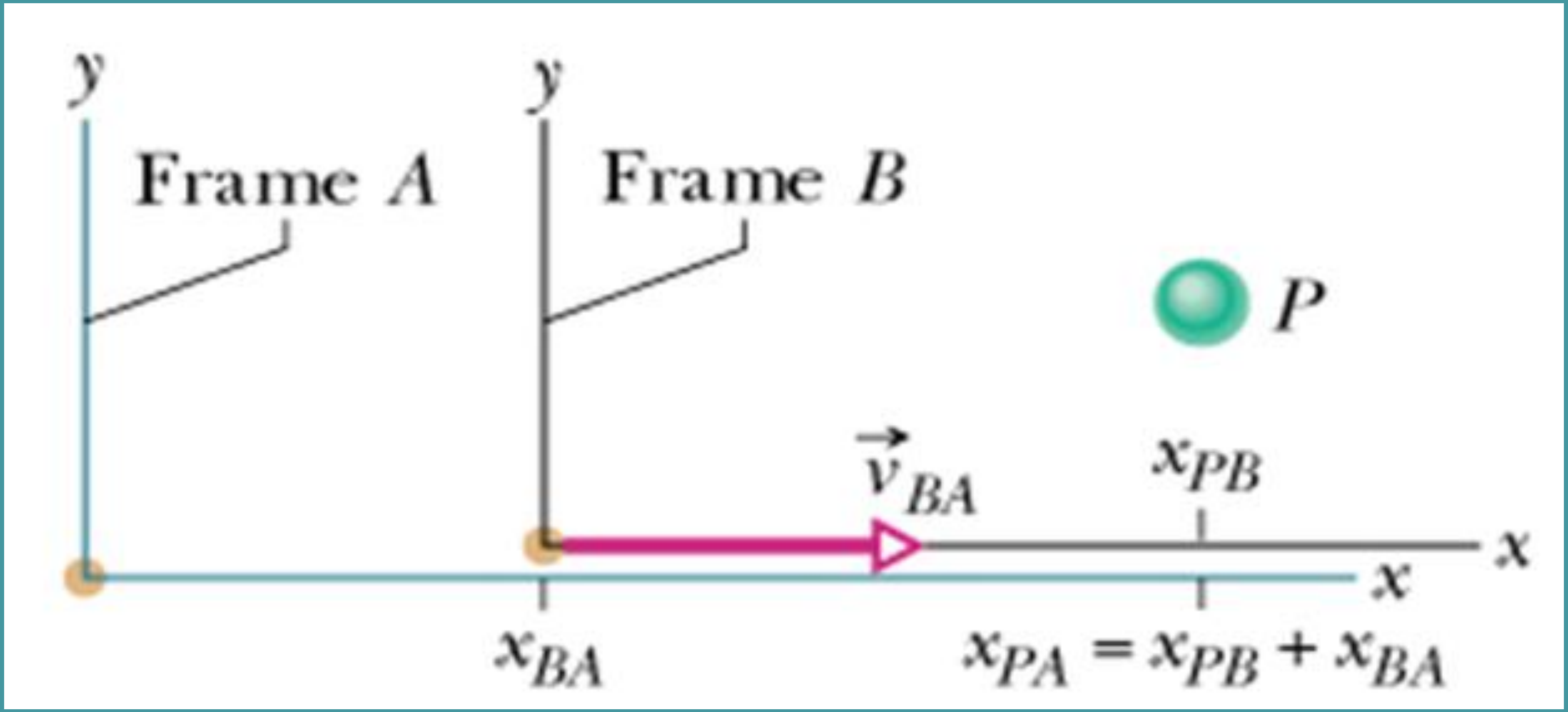
The velocity of a particle **P** determined by two different observers **A** and **B** varies from observer to observer.

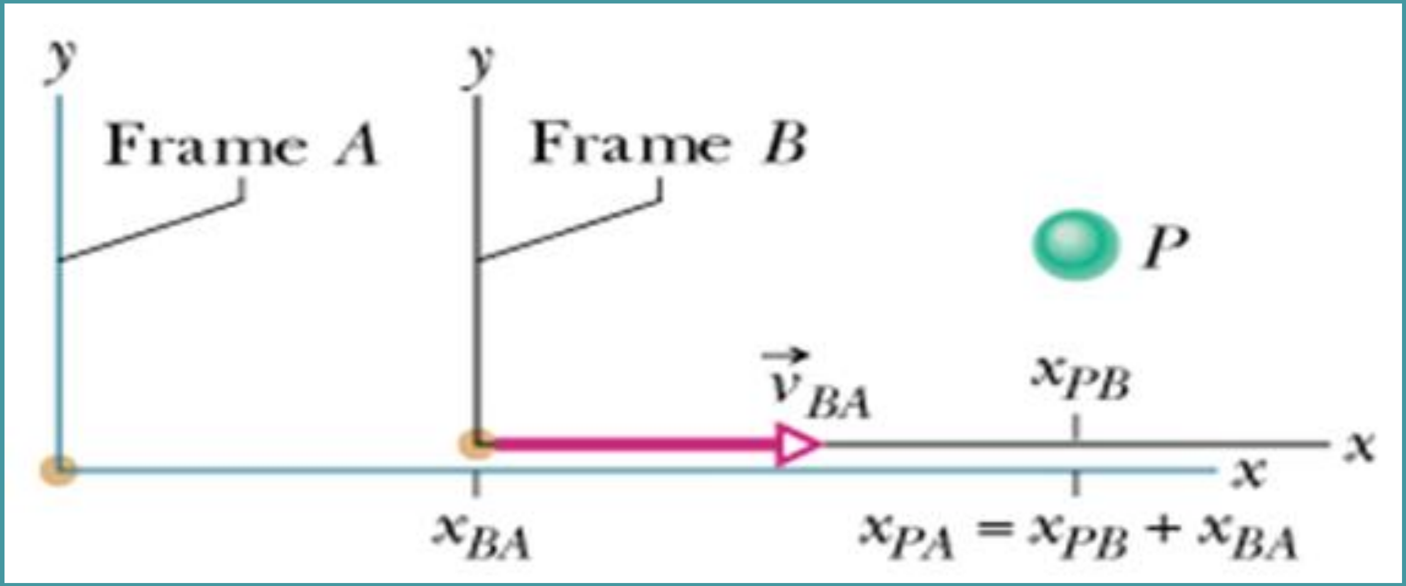
«Transformation equation» of velocities gives the exact relationship between the velocities each observer perceives.

*We assume that observer **B** moves with a known **constant** velocity $\boldsymbol{v_{BA}}$ with respect to observer **A.*** Observer **A** and **B** determine the coordinates of particle **P** to be $\boldsymbol{x_{PA}}$ and $\boldsymbol{x_{PB}}$, respectively.

We assume that observer ***B*** *moves with a known* ***constant*** *velocity* $\mathbf{v}_{BA}$ *with respect to observer* ***A.***

We assume the transformation of velocities between two reference systems which move with respect to each other with constant velocity.





$x_{PA} = x_{PB} + x_{BA}$ Here x_{BA} is the coordinate of B with respect to A

We take derivatives of the above equation: $\dfrac{d}{dt}(x_{PA}) = \dfrac{d}{dt}(x_{PB}) + \dfrac{d}{dt}(x_{BA}) \rightarrow$

$$\boxed{v_{PA} = v_{PB} + v_{BA}}$$

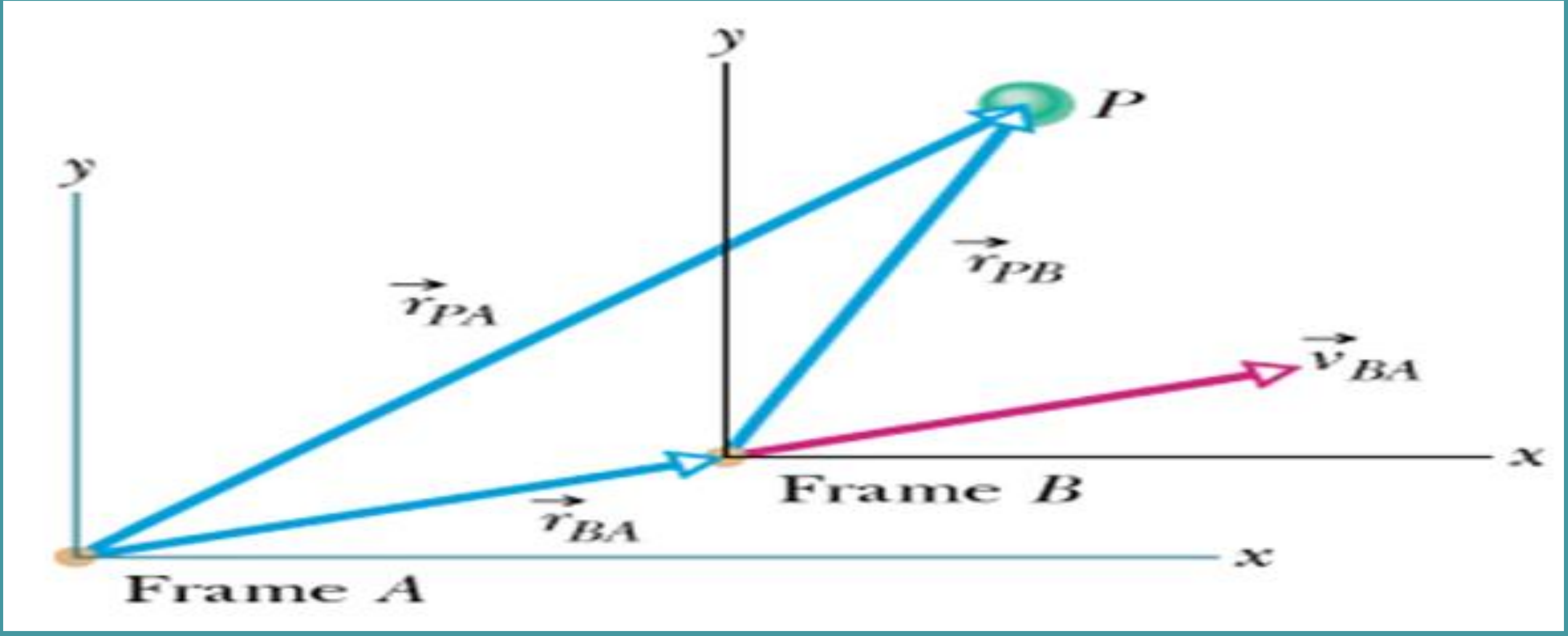
If we take derivatives of the last equation and take into account that $\dfrac{dv_{BA}}{dt} = 0 \rightarrow$ $\boxed{a_{PA} = a_{PB}}$

Note: Even though observers A and B measure different velocities for P, they measure the same acceleration

Relative Motion in Two Dimensions

Relative Motion in Two Dimensions

Here we assume that observer **B** moves with a known **constant** velocity $\boldsymbol{v_{BA}}$ with respect to observer **A** in the **xy-plane.**

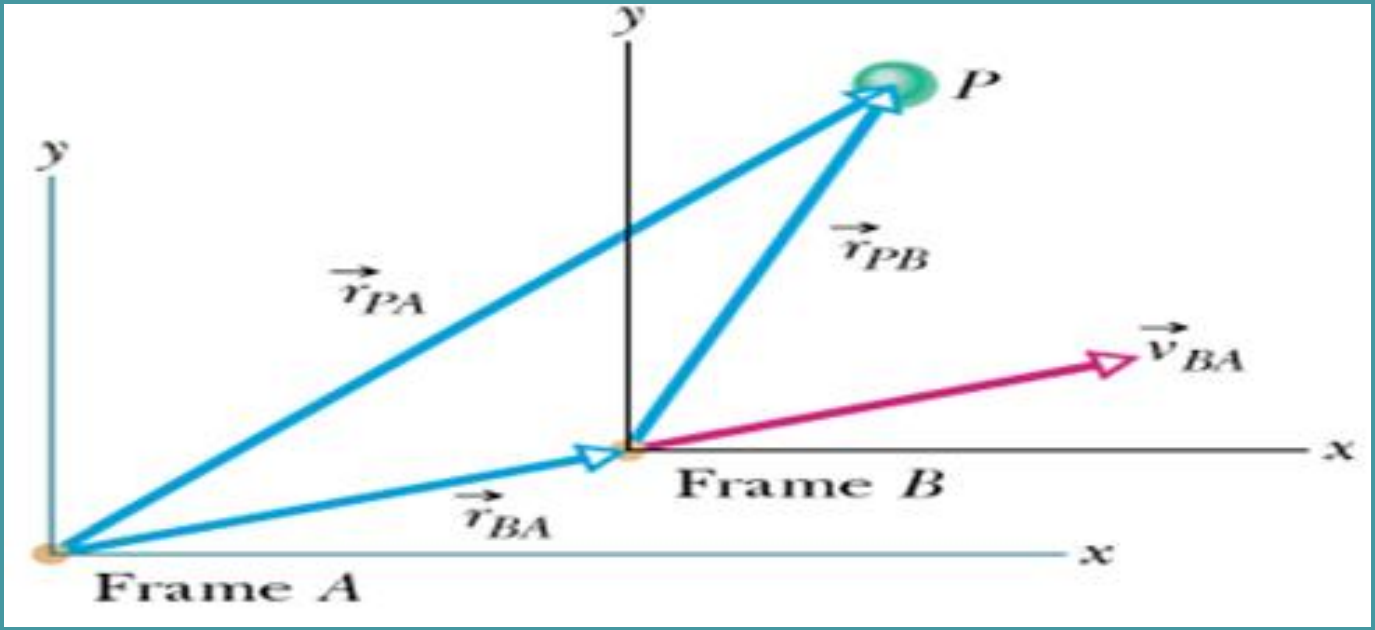


Observers A and B determine the position vector of particle P to be $\vec{r}_{PA}$ and $\vec{r}_{PB}$, respectively.

$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$ We take the time derivative of both sides of the equation

$$\frac{d}{dt}\vec{r}_{PA} = \frac{d}{dt}\vec{r}_{PB} + \frac{d}{dt}\vec{r}_{BA} \rightarrow \vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$

$$\boxed{\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}}$$



If we take the time derivative of both sides of the last equation we have:

$$\frac{d}{dt}\vec{v}_{PA} = \frac{d}{dt}\vec{v}_{PB} + \frac{d}{dt}\vec{v}_{BA}$$ If we take into account that $$\frac{d\vec{v}_{BA}}{dt} = 0 \rightarrow \quad \vec{a}_{PA} = \vec{a}_{PB}$$

Note: As in the one dimensional case, even though observers A and B measure different velocities for P, they measure the same acceleration

In Fig. 4-18, suppose that Barbara's velocity relative to Alex is a constant $v_{BA} = 52$ km/h and car P is moving in the negative direction of the x axis.

(a) If Alex measures a constant $v_{PA} = -78$ km/h for car P, what velocity v_{PB} will Barbara measure?

$$\boxed{\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}}$$

We can attach a frame of reference A to Alex and a frame of reference B to Barbara. Because the frames move at constant velocity relative to each other along one axis, we can use Eq. 4-41 ($v_{PA} = v_{PB} + v_{BA}$) to relate v_{PB} to v_{PA} and v_{BA}.

Calculation: We find

$$-78 \text{ km/h} = v_{PB} + 52 \text{ km/h}.$$

Thus,

$$v_{PB} = -130 \text{ km/h}. \qquad \text{(Answer)}$$