5.2. CONTINUOUS RANDOM VARIABLES

Previously, we discussed the measurement of the current in a thin copper wire. We noted that the results might differ slightly in day-to-day replications because of small variations in variables that are not controlled in our experiment-changes in ambient temperatures (ortam sıcaklıkları), small impurities (kirlilik) in the chemical composition of the wire, current source drifts, and so forth. Another example is the selection of one part from a day's production and very accurately measuring a dimensional length. In practice, there can be small variations in the actual measured lengths due to many causes, such as vibrations, temperature fluctuations, operator differences, calibrations, cutting tool wear (alet takımı aşınması), bearing wear (yatak aşınması), and raw material changes. Even the measurement procedure can produce variations in the final results.

A number of continuous distributions frequently arise in applications. These distributions are described, and example computations of probabilities, means, and variances are provided in the remaining sections of this chapter.

5.2.1. Probability Distributions and Probability Density Functions for Continuous Random Variables

Density functions are commonly used in engineering to describe physical systems. For example, consider the density of a loading on a long, thin beam (kiriş, ışın) as shown in Figure 1. For any point x along the beam, the density can be described by a function (in grams/cm). Intervals with large loadings correspond to large values for the function. The total loading between points a and b is determined as the integral of the density function from a to b. This integral is the area under the density function over this interval, and it can be loosely interpreted as the sum of all the loadings over this interval.

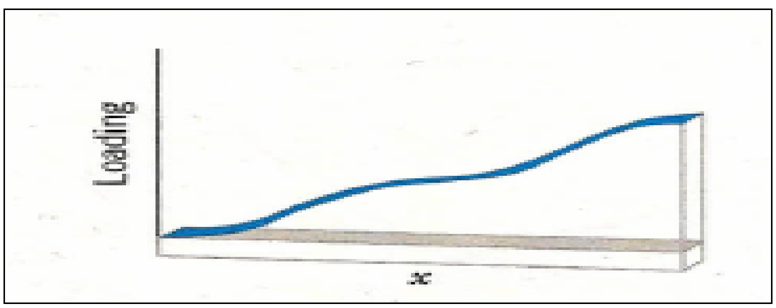

Figure 1. Density function of a loading on a long, thin beam.

Similarly, a **probability density function** $f(x)$ can be used to describe the probability distribution of a **continuous random variable** X. If an interval is likely to contain a value for X, its probability is large and it corresponds to large values for $f(x)$. The probability that X is between a and b is determined as the integral of $f(x)$ from a to b. See Figure 2.

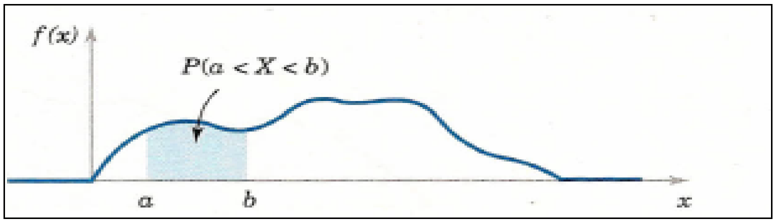

Figure 2. Probability determined from the area under $f(x)$

<u>Definition:</u> For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_a^b f(x)dx = area\ under\ f(x)\ from\ a\ to\ b$

for any a and b

A probability density function provides a simple description of the probabilities associated with a random variable. As long as $f(x)$ is nonnegative and $\int_{-\infty}^{+\infty} f(x)\,dx = 1$, $0 \leq P(a \leq X \leq b) \leq 1$ so that the probabilities are properly restricted. <u>*A probability density function is zero for x values that cannot occur*</u> *and it is assumed to be zero wherever it is not specifically defined.*

If X is a **continuous random variable,** for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$$

Example 1: Let the continuous random variable X denotes the current measured in a thin copper wire in milliamperes. Assume that the range of X is $[0, 20\,mA]$, and assume that the probability density function of X is $f(x) = 0.05\ for\ 0 \leq x \leq 20$. What is the probability that a current measurement is less than 10 milliamperes?

$$P(X < 10) = \int_0^{10} f(x)dx = \int_0^{10} 0.05dx = 0.05x\Big|_0^{10} = 0.05(10-0) = 0.5$$

The probability density function is shown in Figure 3. It is assumed that $f(x) = 0$ wherever it is not specifically defined. The probability requested is indicated by the shaded area in Figure 3.

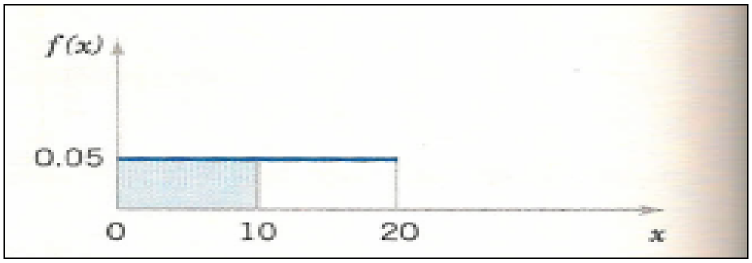

Figure 3. Probability density function for Example 1.

Example 2: Let the continuous random variable X denote the diameter of a hole drilled in a sheet (levha) metal component. The target diameter is 12.5 milliameters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$.

If a part with a diameter (çap) larger than 12.60 milliameters is scrapped (ıskartaya çıkartılmış), what proportion of parts is scrapped? The density function and the requested probability are shown in Figure 4. A part is scrapped if $X > 12.60$. Now,

$$P(X > 12.60) = \int_{12.60}^{\infty} f(x)dx = \int_{12.60}^{\infty} 20e^{-20(x-12.5)}dx = -e^{-20(x-12.5)}\Big|_{12.6}^{\infty} = -\left(e^{-\infty} - e^{-20(12.6-12.5)}\right) = e^{-2} = 0.135$$

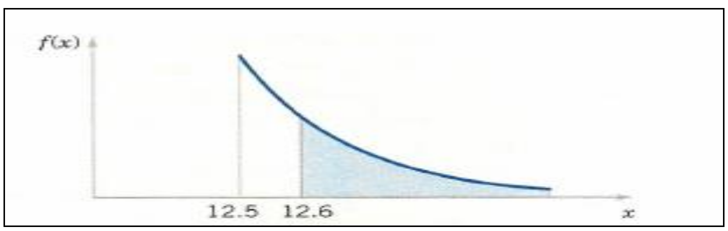

Figure 4. Probability density function for Example 2.

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)dx = -e^{-20(x-12.5)}\Big|_{12.5}^{12.6} = -\left(e^{-2} - e^{0}\right) = 1 - e^{-2} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

5.2.2. Cumulative Distribution Function of a Continuous Random Variable

<u>Definition:</u> The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \quad for \; -\infty < x < +\infty$$

Extending the definition of $f(x)$ to the entire real line enables us to define the cumulative distribution function for all real numbers. The following example illustrates the definition.

Example 3: For the copper current measurement in Example 1, the cumulative distribution function of the random variable X consists of three expressions. If $x<0$, $f(x)=0$. Therefore,

$F(x)=0, x<0$ and $F(x)=\int_0^x f(t)dt=\int_0^x 0.05dt=0.05x,\ for\ 0\le x\le 20$.

Finally, $F(x)=\int_0^x f(t)dt=1,\ for\ x\ge 20$

Therefore, $F(x)=\begin{cases} 0 & x<0 \\ 0.05x & 0\le x\le 20 \\ 1 & x\ge 20 \end{cases}$

The plot of $F(x)$ is shown in Figure 5.

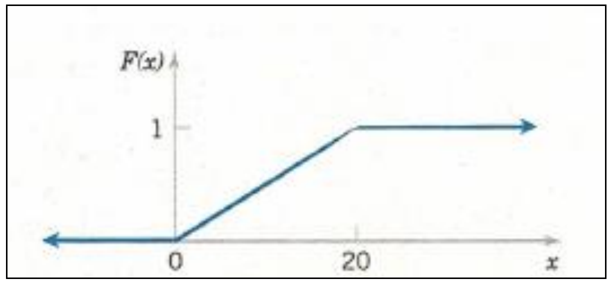

Figure 5. Cumulative distribution function for Example 3.

!!! Notice that in the definition of $F(x)$ ***any < can be changed to ≤ and vice vers***a. That is, $F(x)$ can be defined as either $0.05x$ or 0 at the end-point $x=0$, and $F(x)$ can be defined as either $0.05x$ or 1 at the end-point $x=20$. In other words, $F(x)$ is a continuous function. For a discrete random variable, $F(x)$ is not a continuous function. Sometimes, a continuous random variable is defined as one that has a continuous cumulative distribution function.

Example 4: For the drilling operation in Example 2, $F(x)$ consists of two expressions. $F(x)=0\ for\ x<12.5$.

Therefore, $F(x)=\begin{cases} 0 & x<12.5 \\ 1-e^{-20(x-12.5)} & x\ge 12.5 \end{cases}$

Figure 6 displays a graph of $F(x)$.

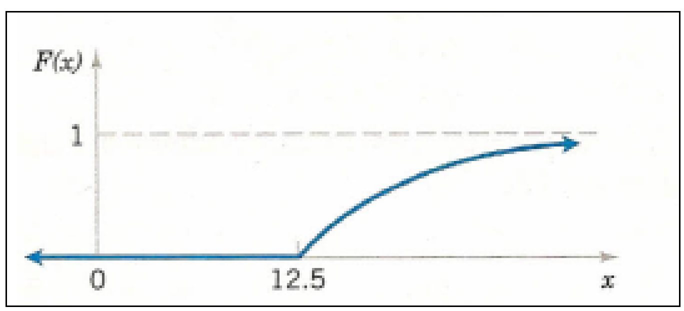

Figure 6. Cumulative distribution function for Example 4.

The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating. Recall that the fundamental theorem of calculus states that $\frac{d}{dx}\int_{-\infty}^{x} f(t)dt=f(x)$ ***Then, given*** $F(x)$**,** $f(x)=\frac{dF(x)}{dx}$ ***as long as the derivative exists.***

Example 5: The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F(x)=\begin{cases} 0 & x<0 \\ 1-e^{-0.01x} & x\geq 0 \end{cases}$$

Determine the probability density function of X. What proportion of reactions is complete within 200 milliseconds? Using the result that the probability density function is the derivative of the $F(x)$, we obtain

$$f(x)=\begin{cases} 0 & x<0 \\ 0.01e^{-0.01x} & x\geq 0 \end{cases}$$

The probability that a reaction completes within 200 milliseconds is

$$P(X<200)=F(200)=1-e^{-2}=0.8647$$

5.2.3. Mean and Variance of a Continuous Random Variable

The mean and variance of a continuous random variable are defined similarly to a discrete random variable. Integration replaces summation in the definitions. *If a probability density function is viewed as a loading on a beam as in Figure 1, the mean is the balance point.*

Definition: Suppose X is a continuous random variable with probability density function $f(x)$.

The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu=E(X)=\int_{-\infty}^{+\infty} xf(x)dx$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2=V(X)=\int_{-\infty}^{+\infty}(x-\mu)^2 f(x)dx=\int_{-\infty}^{+\infty}x^2 f(x)dx-\mu^2=E(X^2)-\mu^2,\ where\ \mu=E(X)$$

Hence, also it can be computed as $\sigma^2=\mathrm{V}(\mathrm{X})=\mathrm{E}(\mathrm{X}^2)-[\mathrm{E}(\mathrm{X})]^2$

The **standard deviation** of X is $\sigma=\sqrt{\sigma^2}$

The equivalence of two formulas for variance can be derived as one, as was done for discrete random variables.

Example 6: For the copper current measurement in Example 1, the mean of X is
$f(x)=0.05,\quad for\ 0\leq x\leq 20$ otherwise $f(x)=0.$

$$E(X)=\int_0^{20} x0.05dx=0.05\left(\left.\frac{x^2}{2}\right|_0^{20}\right)=\frac{5}{200}\left(20^2-0^2\right)=10$$

The variance of X is;

$$V(X)=\int_0^{20}(x-10)^2 f(x)dx=\int_0^{20}(x-10)^2 0.05dx=0.05(x-10)\Big|_0^{20}=\frac{100}{3}$$

u=x-10 then du=dx and $0.05\int_{-10}^{10}\mathrm{u}^2\mathrm{du}=0.05\left.\frac{\mathrm{u}^3}{3}\right|_{-10}^{10}=\frac{5}{300}\left(10^3-(-10)^3\right)=\frac{5\times2000}{300}=\frac{100}{3}$

or

$$E(X^2)=\int_0^{20}x^2 0.05dx=0.05\left(\left.\frac{x^3}{3}\right|_0^{20}\right)=\frac{400}{3}\ \text{ and }\ \mathrm{V}(\mathrm{X})=\mathrm{E}\left(\mathrm{X}^2\right)-\left[\mathrm{E}(\mathrm{X})\right]^2=\frac{400}{3}-10^2=\frac{100}{3}$$

The expected value of a function $g(X)$ of a continuous random variable is defined similarly to a function of a discrete random variable.

<u>Expected Value of a Function of a Continuous Random Variable:</u>

If X is continuous random variable with probability density function $f(x)$,

$$E\left[g(X)\right]=\int_{-\infty}^{+\infty}g(x)f(x)dx$$

Example 7: In Example 1, X is the current measured in milliamperes. What is the expected value of the squared current? Now, $g(X)=X^2$.

$$E(X^2)=\int_0^{20}x^2 0.05dx=0.05\left(\left.\frac{x^3}{3}\right|_0^{20}\right)=\frac{1}{60}20^3=\frac{400}{3}$$

In the previous example, the expected value of X^2 does not equal $E(X)$ squared. However, in the special case that $g(X)=aX+b$ for any constants a and b, $E\left[g(X)\right]=aE(X)+b$. This can be shown from the properties of integrals.

For constants a and b,

$E(aX+b)=aE(X)+b$ and $V(aX+b)=a^2V(X)$

Example 8: For the drilling operation in Example 2, the mean of X is

$$E(X)=\int_{12.5}^{\infty} xf(x)dx=\int_{12.5}^{\infty} x20e^{-20(x-12.5)}dx$$

Integration by parts can be used to show that $\int udv = uv| - \int vdu$

$$\mathrm{x=u \Rightarrow dx=du \quad dv=e^{-20x}dx \;\Rightarrow\; v=-\frac{e^{-20x}}{20}}$$

$$E(X)=\int_{12.5}^{+\infty} x20e^{-20(x-12.5)}dx=e^{250}\int_{12.5}^{+\infty} 20\underbrace{x}_{u}\underbrace{e^{-20x}dx}_{dv}$$

$$=e^{250}\left(-xe^{-20x}\Big|_{12.5}^{+\infty}+\int_{12.5}^{+\infty} e^{-20x}dx\right)$$

$$=e^{250}\left(12.5e^{-250}-\frac{e^{-20x}}{20}\Bigg|_{12.5}^{+\infty}\right)=12.5+\frac{1}{20}=12.55$$

The variance of X is $V(X)=\int_{12.5}^{\infty}(x-12.55)^2 f(x)dx$; although more difficult, integration by parts can be used two times to show that $V(X)=0.0025$

Variance also could be calculated as in below:

$$E(X^2)=\int_{12.5}^{+\infty} x^2 20e^{-20(x-12.5)}dx=e^{250}\int_{12.5}^{+\infty} 20\underbrace{x^2}_{u}\underbrace{e^{-20x}dx}_{dv}$$

$$=e^{250}\left(-x^2e^{-20x}\Big|_{12.5}^{+\infty}+2\int_{12.5}^{+\infty}\underbrace{x}_{u}\underbrace{e^{-20x}dx}_{dv}\right)$$

$$=e^{250}\left[(12.5^2)e^{-250}+2\left(-\frac{xe^{-20x}}{20}\Bigg|_{12.5}^{+\infty}+\int_{12.5}^{+\infty} e^{-20x}dx\right)\right]$$

$$=e^{250}\left[(12.5^2)e^{-250}+\frac{25e^{-250}}{20}-\frac{2e^{-20x}}{400}\Bigg|_{12.5}^{+\infty}\right]$$

$$=e^{250}\left[(12.5^2)e^{-250}+\frac{25e^{-250}}{20}+\frac{2e^{-250}}{400}\right]=156.25+1.25+0.005=157.505$$

$$V(X)=E(X^2)-\left[E(X)\right]^2=157.505-\underbrace{(12.55^2)}_{156.5025}=0.0025$$

Example 9: The weekly demand X, in hundreds of gallon, for propane (bir gaz türü) at a certain supply station has a density function given by

$$f(x) = \begin{cases} \dfrac{x}{4}, & 0 \le x \le 2 \\ \dfrac{1}{2}, & 2 < x \le 3 \\ 0, & elsewhere \end{cases}$$

Find the expected weekly demand.

Solution:

$$E(X) = \int_{\mathbb{R}x} xf(x)dx = \int_0^2 \frac{x^2}{4}dx + \int_2^3 \frac{x}{2}dx = \left.\frac{x^3}{12}\right|_0^2 + \left.\frac{x^2}{4}\right|_2^3$$
$$= \left(\frac{8}{12}\right) + \left(\frac{9}{4} - \frac{4}{4}\right) = \frac{23}{12} = 1.92$$

5.2.4. Expectation of a Function of a Continuous Random Variable

Let X be a random variable with density f(x). The expected value of a function g(·) of the random variable, denoted E[g(X)], is defined to be

$$E\left[g(X)\right] = \int_{-\infty}^{+\infty} g(x)\, f(x) dx \qquad \text{if the integral is defined.}$$

5.2.5. Properties of Expectation

5.2.5.1. Constants

$$E[a] = \int_{-\infty}^{+\infty} a\, f(x) dx = a \int_{-\infty}^{+\infty} f(x)\, dx = a$$

5.2.5.2. Constants Multiplied by a Random Variable

$$E[aX] = \int_{-\infty}^{+\infty} a\, x\, f(x) dx = a \int_{-\infty}^{+\infty} x\, f(x)\, dx = a\, E[X]$$

5.2.5.3. Constants Multiplied by a Function of a Random Variable

$$E\left[a\,g(X)\right]=\int_{-\infty}^{+\infty} a\,g(x)f(x)\,dx=a\int_{-\infty}^{+\infty} g(x)\,f(x)\,dx=aE\left[g(X)\right]$$

5.2.5.4. Sums of Expected Values

Let X be a continuous random variable with density function f(x) and let $g_1(X)$, $g_2(X)$, $g_3(X), \ldots, g_k(X)$ be k functions of X. Also let $c_1, c_2, c_3, \ldots, c_k$ be k constants.

$$E\left[c_1 g_1(X)+c_2 g_2(X)+c_3 g_3(X)+\cdots+c_k g_k(X)\right]=E\left[c_1 g_1(X)\right]+E\left[c_2 g_2(X)\right]+E\left[c_3 g_3(X)\right]+\cdots+E\left[c_k g_k(X)\right]$$

<u>EXERCISES</u>

EXERCISES 5

Exercise 5.1: Suppose that $f(x)=e^{-x},\ \ for\ \ x>0$. Determine the following properties:

a) $P(1<X)$ **b)** $P(1<X<2.5)$ **c)** $P(X=3)$ **d)** $P(X<4)$ **e)** $P(3\le X)$

Solution:

a) $P(1<X)=P(X>1)=\int_1^{\infty} f(x)dx=\int_1^{\infty} e^{-x}dx=-e^{-x}\Big|_1^{\infty}=-\left(e^{-\infty}-e^{-1}\right)=\dfrac{1}{e}$

b) $P(1<X<2.5)=\int_1^{2.5} f(x)dx=\int_1^{2.5} e^{-x}dx=-e^{-x}\Big|_1^{2.5}=-\left(e^{-2.5}-e^{-1}\right)=\dfrac{1}{e}-\dfrac{1}{e^{2.5}}$

c) $P(X=3)=0$

d) $P(X<4)=\int_1^{4} f(x)dx=\int_1^{4} e^{-x}dx=-e^{-x}\Big|_1^{4}=-\left(e^{-4}-e^{-1}\right)=\dfrac{1}{e}-\dfrac{1}{e^{4}}$

e) $P(3\le X)=\int_3^{\infty} f(x)dx=\int_3^{\infty} e^{-x}dx=-e^{-x}\Big|_3^{\infty}=-\left(e^{-\infty}-e^{-3}\right)=\dfrac{1}{e^{3}}$

Exercise 5.2: Suppose that $f(x)=x/8\ \ for\ 3<x<5$. Determine the following properties:

a) $P(X<4)$ **b)** $P(X>3.5)$ **c)** $P(4<X<5)$ **d)** $P(X<4.5)$ **e)** $P(X<3.5\ or\ X>4.5)$

Solution:

a) $P(X<4)=\int_3^4 \frac{x}{8}dx=\left.\frac{x^2}{16}\right|_3^4=\frac{16}{16}-\frac{9}{16}=\frac{7}{16}=0.4375$

b) $P(X>3.5)=\int_{3.5}^5 \frac{x}{8}dx=\left.\frac{x^2}{16}\right|_{3.5}^5=\frac{25}{16}-\frac{12.25}{16}=\frac{12.75}{16}=0.7969$

c) $P(4<X<5)=\int_4^5 \frac{x}{8}dx=\left.\frac{x^2}{16}\right|_4^5=\frac{25}{16}-\frac{16}{16}=\frac{9}{16}=0.5625$

d) $P(X<4.5)=\int_3^{4.5} \frac{x}{8}dx=\left.\frac{x^2}{16}\right|_3^{4.5}=\frac{20.25}{16}-\frac{9}{16}=\frac{11.25}{16}=0.7031$

e)

$$P(X<3.5\ or\ X>4.5)=\int_3^{3.5}\frac{x}{8}dx+\int_{4.5}^5\frac{x}{8}dx=\left.\frac{x^2}{16}\right|_3^{3.5}+\left.\frac{x^2}{16}\right|_{4.5}^5$$
$$=\left(\frac{12.25}{16}-\frac{9}{16}\right)+\left(\frac{25}{16}-\frac{20.25}{16}\right)=\frac{8}{16}=\frac{1}{2}=0.5$$

Exercise 5.3: Suppose that $f(x)=1.5x^2\ \ for\ -1<x<1.$ Determine the following probabilities:

a) $P(0<X)$ **b)** $P(0.5<X)$ **c)** $P(-0.5\le X\le 0.5)$ **d)** $P(X<-2)$ **e)** $P(X<0\ or\ X>-0.5)$
f) Determine x such that $P(X>x)=0.05$

Solution:

a) $P(0<X)=\int_0^1 1.5x^2dx=1.5\left.\frac{x^3}{3}\right|_0^1=\left.\frac{x^3}{2}\right|_0^1=\frac{1}{2}=0.5$

b) $P(0.5<X)=\int_{0.5}^1 1.5x^2dx=1.5\left.\frac{x^3}{3}\right|_{0.5}^1=\left.\frac{x^3}{2}\right|_{0.5}^1=\frac{1}{2}-\frac{0.5^3}{2}=\frac{0.875}{2}=0.4375$

c) $P(-0.5\le X\le 0.5)=\int_{-0.5}^{0.5} 1.5x^2dx=1.5\left.\frac{x^3}{3}\right|_{-0.5}^{0.5}=\left.\frac{x^3}{2}\right|_{-0.5}^{0.5}=\frac{0.125}{2}-\left(\frac{-0.125}{2}\right)=0.125$

d) $P(X<-2)=0$

e) $P\left(X < 0 \; or \; X > -0.5\right) = P(-1 < X < 1) = 1$

f) $P\left(X > x\right) = 1 - P\left(X \le x\right) = 0.05 \quad \Rightarrow \quad P\left(X \le x\right) = 0.95$

$$P\left(X \le x\right) = \int_{-1}^{x} 1.5t^2 dt = 1.5\frac{t^3}{3}\bigg|_{-1}^{x} = \frac{t^3}{2}\bigg|_{-1}^{x} = \frac{x^3}{2} - \frac{-1}{2} = \frac{x^3+1}{2} \quad \Rightarrow \quad \frac{x^3+1}{2} = 0.95 \quad \Rightarrow \quad x^3 = 0.9 \Rightarrow x = 0.9655$$

EXERCISES 6

Exercise 6.1: Suppose the cumulative distribution function of the random variable X is

$$F\left(x\right) = \begin{cases} 0 & x < 0 \\ 0.2x & 0 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

Determine the following:

a) $P\left(X < 2.8\right)$ **b)** $P\left(X > 1.5\right)$ **c)** $P\left(X < -2\right)$ **d)** $P\left(X > 6\right)$

Solution:

a) $P\left(X < 2.8\right) = F\left(2.8\right) = 0.2 \times 2.8 = 0.56$

b) $P\left(X > 1.5\right) = 1 - P\left(X \le 1.5\right) = 1 - F\left(1.5\right) = 1 - \left(0.2 \times 1.5\right) = 0.7$

c) $P\left(X < -2\right) = 0$

d) $P\left(X > 6\right) = 1 - P\left(X \le 6\right) = 1 - 1 = 0$

Exercise 6.2: Determine the cumulative distribution function for $f\left(x\right) = e^{-x} \; for \; x > 0$.

Solution:

$$F\left(x\right) = P\left(X \le x\right) = \int_{0}^{x} e^{-t}\, dt = -e^{-t}\Big|_{0}^{x} = -\left(e^{-x} - e^{0}\right) = 1 - e^{-x}$$

$$F\left(x\right) = \begin{cases} 1 - e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$$

Exercise 6.3: Determine the cumulative distribution function for $f(x) = e^{-(x-4)}$ *for* $x > 4$.

Solution:

$$F(x) = P(X \le x) = \int_4^x e^{-(t-4)}\,dt = -e^{-(t-4)}\Big|_4^x = -\left(e^{-(x-4)} - e^{-(4-4)}\right) = 1 - e^{-(x-4)}$$

$$F(x) = \begin{cases} 1 - e^{-(x-4)}, & x > 4 \\ 0, & x < 4 \end{cases}$$

Exercise 6.4: Determine the cumulative distribution function for $f(x) = 1.25$ for $74.6 < x < 75.4$. Use the cumulative distribution function to determine the probability that a length exceeds 75 millimeters.

Solution:

$$F(x) = P(X \le x) = \int_{74.6}^x 1.25\,dt = 1.25t\Big|_{74.6}^x = 1.25(x - 74.6) = 1.25x - 93.25$$

$$F(x) = \begin{cases} 0, & x < 74.6 \\ 1.25x - 93.25, & 74.6 < x < 75.4 \\ 1, & x \ge 75.4 \end{cases}$$

$$P(X > 75) = 1 - P(X \le 75) = 1 - F(75) = 1 - (1.25 \times 75 - 93.25) = 1 - 0.5 = 0.5$$

EXERCISES 7

Exercise 7.1: Suppose $\mathrm{f}(\mathrm{x}) = 0.125\mathrm{x}, \quad \text{for } 0 < \mathrm{x} < 4$. Determine the mean and variance of X.

Solution:

$$E(X) = \int_0^4 x f(x)\,dx = \int_0^4 0.125x^2\,dx = 0.125\frac{x^3}{3}\Big|_0^4 = 2.6667$$

$$E(X^2) = \int_0^4 x^2 f(x)\,dx = \int_0^4 0.125x^3\,dx = 0.125\frac{x^4}{4}\Big|_0^4 = 8$$

$$V(X) = E(X^2) - [E(X)]^2 = 8 - (2.6667)^2 = 0.8888$$

Exercise 7.2: Suppose that $f(x)=\frac{x}{8}$ for $3<x<5$. Determine the mean and variance for X.

Solution:

$$E(X)=\int_3^5 xf(x)dx=\int_3^5 \frac{x^2}{8}dx=\left.\frac{x^3}{24}\right|_3^5=\frac{125}{24}-\frac{27}{24}=\frac{98}{24}=4.083$$

$$E(X^2)=\int_3^5 x^2 f(x)dx=\int_3^5 \frac{x^3}{8}dx=\left.\frac{x^4}{32}\right|_3^5=\frac{625}{32}-\frac{81}{32}=\frac{544}{32}=17$$

$$V(X)=E(X^2)-[E(X)]^2=17-(4.083)^2=0.3291$$

Exercise 7.3: The thickness of a conductive (iletken) coating (kaplama) in micrometers has a density function of $600x^{-2}$ for $100\ \mu m<x<120\ \mu m$.

a) Determine the mean and variance of the coating thickness.

b) If the coating costs \$0.50 per micrometer of thickness on each part, what is the average cost of the coating per part?

Solution:

a) $E(X)=\int_{100}^{120} xf(x)dx=\int_{100}^{120} 600x^{-1}\,dx=600\int_{100}^{120} x^{-1}\,dx=600\ln x\Big|_{100}^{120}=600(\ln 120-\ln 100)=109.39\ \mu m$

$$E(X^2)=\int_{100}^{120} x^2 f(x)dx=\int_{100}^{120} 600\,dx=600x\Big|_{100}^{120}=600(120-100)=12000$$

$$V(X)=E(X^2)-[E(X)]^2=12000-(109.39)^2=33.82\ \mu m^2$$

b) The average cost of the coating per part is 0.5×109.39=54.7 dollars.

Exercise 7.4: For a given teller (banka memuru) in a bank, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x)=\begin{cases}3x^2, & 0\le x\le 1\\ 0, & otherwise\end{cases}$$

Find the mean and variance of X.

Solution:

$$E(X)=\int_{-\infty}^{+\infty} xf(x)dx=\int_0^1 x\left(3x^2\right)dx=\int_0^1 3x^3dx=3\left[\frac{x^4}{4}\right]_0^1=\frac{3}{4}=0.75$$

Thus, on average, the teller spends 75% of his time each week directly serving customers.
To compute V(X), we first find $E\left(X^2\right)$:

$$E\left(X^2\right)=\int_{-\infty}^{+\infty} x^2 f(x)dx=\int_0^1 x^2\left(3x^2\right)dx=\int_0^1 3x^4dx=3\left[\frac{x^5}{5}\right]_0^1=\frac{3}{5}=0.60$$

$$V(X)=E\left(X^2\right)-\left[E(X)\right]^2=0.60-(0.75)^2=0.6-0.5625=0.0375$$

Exercise 7.5: Integration by parts is required. The probability density function for the diameter of a drilled hole in millimeters is $10e^{-10(x-5)}$ for $x>5$ mm. Although the target diameter is 5 millimeters, vibrations, tool wear, and other nuisances produce diameters larger than 5 millimeters.

a) Determine the mean and variance of the diameter of the holes.

b) Determine the probability that a diameter exceeds 5.1 millimeters.

Solution:

a) $E(X)=\int_5^{\infty} xf(x)dx=\int_5^{\infty}10xe^{-10(x-5)}\,dx$

$$\int udv=uv\Big|-\int vdu \quad \Rightarrow u=x \text{ and } \quad dv=10e^{-10(x-5)} \Rightarrow du=dx \quad \text{and} \quad v=-e^{-10(x-5)}$$

$$E(X)=\int_5^{\infty} xf(x)dx=\int_5^{\infty}10xe^{-10(x-5)}\,dx=-xe^{-10(x-5)}\Big|_5^{\infty}+\int_5^{\infty}e^{-10(x-5)}dx$$

$$=-xe^{-10(x-5)}\Big|_5^{\infty}-\frac{e^{-10(x-5)}}{10}\Bigg|_5^{\infty}=5+\frac{1}{10}=5.1 \quad mm$$

$$E\left(X^2\right)=\int_5^{\infty} x^2 f(x)dx=\int_5^{\infty}10x^2e^{-10(x-5)}\,dx$$

$$\int udv=uv\Big|-\int vdu \quad \Rightarrow u=x^2 \text{ and } \quad dv=10e^{-10(x-5)} \Rightarrow du=2xdx \quad \text{and} \quad v=-e^{-10(x-5)}$$

$$E\left(X^2\right)=\int_5^{\infty} x^2 f(x)dx=\int_5^{\infty}10x^2e^{-10(x-5)}\,dx=-x^2e^{-10(x-5)}\Big|_5^{\infty}+\frac{1}{5}\int_5^{\infty}10xe^{-10(x-5)}dx$$

$$=25+\frac{1}{5}\times 5.1=26.02$$

$$V(X)=E\left(X^2\right)-\left[E(X)\right]^2=26.02-(5.1)^2=0.01\,mm^2$$

b) $P(X>5.1)=\int_{5.1}^{\infty}10e^{-10(x-5)}dx=-e^{-10(x-5)}\Big|_{5.1}^{\infty}=e^{-1}$

Exercise 7.6: Suppose $\mathrm{f}(\mathrm{x})=\begin{cases}\mathrm{ax}, & 0\le \mathrm{x}\le 1\\ \mathrm{a}, & 1\le \mathrm{x}\le 2\\ 0, & \text{otherwise}\end{cases}$

a) Find the constant a.
b) Determine the mean and variance for X.
c) Determine the cumulative distribution function for X.
d) Determine the following probabilities

$P\left(X\le\frac{3}{2}\right)\quad P\left(X>\frac{1}{2}\right)\quad P\left(0<X<\frac{1}{2}\right)\quad P\left(\frac{1}{3}<X<\frac{1}{2}\right)\quad P(X\le -2)\quad P(X>4)$

Solution:

a) $\int_{Rx} f(x)dx=1\Rightarrow\quad \int_0^2 f(x)dx=\int_0^1 axdx+\int_1^2 adx=a\frac{x^2}{2}\Big|_0^1+ax\Big|_1^2=\frac{3a}{2}=1\Rightarrow a=\frac{2}{3}$

$$\mathrm{f}(\mathrm{x})=\begin{cases}\frac{2}{3}\mathrm{x}, & 0\le \mathrm{x}\le 1\\ \frac{2}{3}, & 1\le \mathrm{x}\le 2\\ 0, & \text{otherwise}\end{cases}$$

b)

$$E(X)=\int_{Rx} xf(x)dx=\int_0^2 xf(x)dx=\int_0^1 x\frac{2}{3}xdx+\int_1^2 x\frac{2}{3}dx=\frac{2}{3}\int_0^1 x^2dx+\frac{2}{3}\int_1^2 xdx$$

$$=\frac{2}{3}\frac{x^3}{3}\Big|_0^1+\frac{2}{3}\frac{x^2}{2}\Big|_1^2=\frac{11}{9}$$

$$E\left(X^2\right)=\int_{Rx} x^2 f(x)dx=\int_0^2 x^2 f(x)dx=\int_0^1 x^2\frac{2}{3}xdx+\int_1^2 x^2\frac{2}{3}dx=\frac{2}{3}\int_0^1 x^3dx+\frac{2}{3}\int_1^2 x^2dx$$

$$=\frac{2}{3}\frac{x^4}{4}\Big|_0^1+\frac{2}{3}\frac{x^3}{3}\Big|_1^2=\frac{1}{6}+\frac{14}{9}=\frac{31}{18}$$

$$V(X)=E\left(X^2\right)-\left[E(X)\right]^2=\frac{31}{18}-\left(\frac{11}{9}\right)^2=\frac{37}{162}$$

c) $$F(x)=\begin{cases} 0, & x<0 \\ \int_0^x \frac{2}{3}tdt=\frac{2}{3}\frac{t^2}{2}\Big|_0^x=\frac{x^2}{3}, & 0\le x\le 1 \\ \int_0^1 \frac{2}{3}xdx+\int_1^x \frac{2}{3}dt=\frac{2}{3}\frac{x^2}{2}\Big|_0^1+\frac{2}{3}t\Big|_1^x=\frac{2x-1}{3}, & 1\le x\le 2 \\ 1, & x\ge 2 \end{cases}$$

d)

$$P\left(X\le\frac{3}{2}\right)=\int_0^{3/2} f(x)dx=\int_0^1\frac{2}{3}xdx+\int_1^{3/2}\frac{2}{3}dx=\frac{2}{3}\frac{x^2}{2}\Big|_0^1+\frac{2}{3}x\Big|_1^{3/2}=\frac{2}{3}$$

or

$$P\left(X\le\frac{3}{2}\right)=F\left(\frac{3}{2}\right)=\frac{2.3/2-1}{3}=\frac{2}{3}$$

$$P\left(X>\frac{1}{2}\right)=\int_{1/2}^{2} f(x)dx=\int_{1/2}^{1}\frac{2}{3}xd+\int_1^2\frac{2}{3}dx=\frac{11}{12}$$

or

$$P\left(X>\frac{1}{2}\right)=1-P\left(X\le\frac{1}{2}\right)=1-F\left(\frac{1}{2}\right)=1-\frac{(1/2)^2}{3}=\frac{11}{12}$$

$$P\left(0<X<\frac{1}{2}\right)=\int_0^{1/2} f(x)dx=\int_0^{1/2}\frac{2}{3}xdx=\frac{2}{3}\frac{x^2}{2}\Big|_0^{1/2}=\frac{1}{12}$$

or

$$P\left(0<X<\frac{1}{2}\right)=F(1/2)-F(0)=\frac{1}{12}-0=\frac{1}{12}$$

$$P\left(\frac{1}{3} < X < \frac{1}{2}\right) = \int_{1/3}^{1/2} \frac{2}{3} x dx = \frac{2}{3}\frac{x^2}{2}\Bigg|_{1/3}^{1/2} = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{108}$$

or

$$P\left(\frac{1}{3} < X < \frac{1}{2}\right) = F(1/2) - F(1/3) = \frac{(1/2)^2}{3} - \frac{(1/3)^2}{3} = \frac{5}{108}$$

$$P(X \le -2) = F(-2) = 0$$

$$P(X > 4) = 1 - P(X \le 4) = 1 - 1 = 0$$

Exercise 7.7: Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0, & x < -1 \\ k(x+1), & -1 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$$

a) Find the constant k.
b) Determine the probability density function for X.
c) Determine the following probabilities:

$$P\left(\frac{-1}{2} \le X \le \frac{1}{2}\right) \quad P(X < 0) \quad P(X > 2) \quad P(X < 2)$$

Solution:

a) k=? $F(1) = 1 \Rightarrow F(1) = k(1+1) = 2k = 1 \Rightarrow k = 1/2 \quad \Rightarrow \quad F(x) = \begin{cases} 0, & x < -1 \\ \dfrac{x+1}{2}, & -1 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$

b) $f(x) = ? \quad f(x) = \dfrac{d}{dx} F(x) = \begin{cases} \dfrac{1}{2}, & -1 \le x \le 1 \\ 0, & otherwise \end{cases}$

$$P\left(\frac{-1}{2} \le X \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \frac{(1/2)+1}{2} - \frac{(-1/2)+1}{2} = \frac{1}{2}$$

c) $P(X < 0) = F(0) = \frac{0+1}{2} = \frac{1}{2}$ $\quad P(X > 2) = 1 - P(X \le 2) = 1 - P(X \le 1) = 1 - F(1) = 1 - 1 = 0$

$$P(X < 2) = P(X \le 1) = 1$$

Exercise 7.8: The probability density function of the net weight in pounds of a packaged chemical herbicide (bitki öldürücü kimyasal) is $f(x) = 2.0$ for $49.75 < x < 50.25$ pounds.

a) Determine the probability that a package weighs more than 50 pounds.

b) How much chemical is contained in 90 % of all packages?

Solution:

a) $P(X > 50) = \int_{50}^{50.25} 2.0\,dx = 2x\Big|_{50}^{50.25} = 0.5$

b) $P(X \le x) = 0.10 \Rightarrow \int_{49.75}^{x} 2\,dt = 2t\Big|_{49.75}^{x} = 2(x - 49.75) \Rightarrow 2x - 99.5 = 0.10 \Rightarrow x = 49.8$

Exercise 7.9: The random variable X of the life lengths of batteries discussed above is associated with a probability density function of the form

$$f(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & for\ x > 0 \\ 0, & elsewhere \end{cases}$$

Find the probability that the life of a particular battery of this type is less than 200 or greater than 400 hours.

Solution: Let A denote the event that X is less than 2, and let B denote the event that X is greater than 4. Then, because A and B are mutually exclusive.

$$P(A \cup B) = P(A) + P(B) = \int_0^2 \frac{1}{2}e^{-x/2}dx + \int_4^\infty \frac{1}{2}e^{-x/2}dx = -e^{-x/2}\Big|_0^2 - e^{-x/2}\Big|_4^\infty = (1 - e^{-1}) + (e^{-2})$$
$$= 1 - 0.368 + 0.135 = 0.767$$

Exercise 7.10: The gap width (boşluk genişliği) is an important property of a magnetic recording head. In coded units, if the width is a continuous random variable over the range from $0 < x < 2$ with $f(x) = 0.5x$, determine the cumulative distribution function of the gap width.

Solution:

$$F(x) = P(X \le x) = \int_0^x 0.5t\,dt = 0.5\left.\frac{t^2}{2}\right|_0^x = 0.25x^2$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.25x^2, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

Exercise 7.11: The distribution function of the random variable *X*, the time (in months) from the diagnosis age until death for one population of AIDS patients, is as follows:

$$F(x) = 1 - e^{-0.03x^{1.2}},\ x > 0$$

a) Find the probability that a randomly selected person from this population survives at least 12 months.

b) Find the probability density function of *X*.

Solution:

a) The probability of surviving at least 12 months is

$$P(X > 12) = 1 - P(X \le 12) = 1 - F(12) = 1 - \left(1 - e^{-0.03(12)^{1.2}}\right) = e^{-0.03(12)^{1.2}} = 0.55$$

55% of this population will survive more than a year from the time of the diagnosis of AIDS.

b) $f(x) = \frac{d}{dx}\left(F(x)\right) = \begin{cases} 0, & x < 0 \\ 0.036x^{0.2}e^{-0.03x^{1.2}} & x \ge 0 \end{cases}$

Notice that both the probability density function and the distribution function are defined for all real values of *X*.