Determinants

Let $A=[a_{ij}]$ be an $n\times n$ matrix.

If $n=1$, then $|A|=a_{11}$.

If $n=2$, then $|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

We will define the determinant of A by assuming that determinants of $(n-1)\times(n-1)$ matrices are already defined.

Let $A = [a_{ij}]$ be an $n \times n$ matrix.

Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i-th row and j-th column of A.

The determinant $|M_{ij}| = \det(M_{ij})$ is called the minor of a_{ij}.

The cofactor A_{ij} of a_{ij} is defined as

$$A_{ij} = (-1)^{i+j} \, |M_{ij}|$$

Example: Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$

Then $M_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 2 \end{bmatrix}$ and $|M_{12}| = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 8 - 42 = -34$

$M_{23} = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}$ and $|M_{23}| = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 3 + 7 = 10$

$A_{12} = (-1)^{1+2} \, |M_{12}| = (-1)(-34) = 34$

$A_{23} = (-1)^{2+3} \, |M_{23}| = (-1)(10) = -10$

Let $A=[a_{ij}]$ be an $n\times n$ matrix. Then

expansion of $\det(A)$ along with the ith row

$$\det(A)=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}$$

and

expansion of $\det(A)$ along with the jth column

$$\det(A)=a_{1j}A_{1j}+a_{2j}A_{2j}+\cdots+a_{nj}A_{nj}$$

Example

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix} = (-1)^{3+1} \cdot 3 \cdot \begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} + (-1)^{3+2} \cdot 0 \cdot \begin{vmatrix} 1 & -3 & 4 \\ -4 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix}$$

$$+ (-1)^{3+3} \cdot 0 \cdot \begin{vmatrix} 1 & 2 & 4 \\ -4 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} + (-1)^{3+4} \cdot (-3) \cdot \begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

$$= 1 \cdot 3 \cdot 20 + 0 + 0 + (-1) \cdot (-3) \cdot (-4) = 48$$

$$\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 2 \cdot \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} + (-1)^{2+1} \cdot 2 \cdot \begin{vmatrix} -3 & 4 \\ -2 & 3 \end{vmatrix} + 0$$

$$= 2 \cdot (3+6) + (-1) \cdot 2 \cdot (-9+8)$$

$$= 2 \cdot 9 + 2 = 20$$

and

$$\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix} = (-1)^{3+1} \cdot 2 \cdot \begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} + 0 + (-1)^{3+3} \cdot (-2) \cdot \begin{vmatrix} 1 & 2 \\ -4 & 2 \end{vmatrix}$$

$$= 2 \cdot (2+6) + 0 + (-2) \cdot (2+8)$$

$$= 16 + (-20) = -4$$

Properties of Determinant

(1) $|A| = |A^T|$

(2) $B = A_{R_i \leftrightarrow R_j}$ or $B = A_{C_i \leftrightarrow C_j} \Rightarrow |B| = -|A|$

(3) If two rows (columns) of A are equal, then $|A| = 0$

(4) If a row (column) of A is entirely of zeros, then $|A| = 0$.

(5) $B = A_{kR_i + R_j}$ or $B = A_{kC_i + C_j}$ where k is a real number, then $|B| = |A|$.

(6) $B = A_{kR_i}$ or $B = A_{kC_i}$, then $|B| = k|A|$.

(7) The determinant of a triangular matrix is the product of elements on the main diagonal.

(8) If E is an elementary matrix, then

$|EA| = |E|\,|A|$ and $|AE| = |A|\cdot|E|$ for any square matrix A.

(9) A is an $n\times n$ matrix. A is invertible $\Leftrightarrow$ $|A| \neq 0$.

(10) A is an $n\times n$ matrix. Then

$AX = 0$ has a nontrivial solution $\Leftrightarrow$ $|A| = 0$

(11) A and B are $n\times n$ matrices. $|AB| = |A|\,|B|$.

(12) If A is invertible, then $|A^{-1}| = \dfrac{1}{|A|}$.

Example: Given that $\begin{vmatrix} 1 & a & 2 \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4$. Compute $\begin{vmatrix} x^2 & ax & 2x \\ -x & 1 & b \\ ax & 2 & 3b \end{vmatrix}$

$$\begin{vmatrix} x^2 & ax & 2x \\ -x & 1 & b \\ ax & 2 & 3b \end{vmatrix} = x \begin{vmatrix} x & a & 2 \\ -x & 1 & b \\ ax & 2 & 3b \end{vmatrix} = x.x \begin{vmatrix} 1 & a & 2 \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4x^2$$

$$\begin{bmatrix} a+1 & a+2 & 2+3b \\ -1 & 1 & b \\ a & 2 & 3b \end{bmatrix} = \begin{bmatrix} 1 & a & 2 \\ -1 & 1 & b \\ a & 2 & 3b \end{bmatrix}_{R_3+R_1 \to R_1} \quad \text{so} \quad \begin{vmatrix} a+1 & a+2 & 2+3b \\ -1 & 1 & b \\ a & 2 & 3b \end{vmatrix} = 4$$

Let $A=[a_{ij}]$ be an $n\times n$ matrix. Then

$$\operatorname{adj}(A)=\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

is the $n\times n$ matrix whose (i,j)-th entry is the cofactor A_{ji} of a_{ji}.

This matrix is called the adjoint of A.

For any square matrix A we have

$$A \operatorname{adj}(A) = (\operatorname{adj} A) A = \det(A) I$$

If $\det(A) \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Example: Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$. Compute adj (A).

$A_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18$ $\quad A_{12} = (-1)^{1+2} \begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} = 17$ $\quad A_{13} = (-1)^{1+3} \begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} = -6$

$A_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6$ $\quad A_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} = -10$ $\quad A_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} = -2$

$A_{31} = (-1)^{3+1} \begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} = -10$ $\quad A_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = -1$ $\quad A_{33} = (-1)^{3+3} \begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} = 28$

$$\text{adj}(A) = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

Example:

Consider $A=\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$. $\operatorname{adj}(A)=\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$

$\det(A)=-94$. Thus

$$A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)=\begin{bmatrix} \frac{18}{94} & \frac{6}{94} & \frac{10}{94} \\ -\frac{17}{94} & \frac{10}{94} & \frac{1}{94} \\ \frac{6}{94} & \frac{2}{94} & -\frac{28}{94} \end{bmatrix}$$

An Application: Cramer's Rule

Let
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

$A = [a_{ij}]_{n \times n}$

$AX = B$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If $\det(A) \neq 0$, then the system has the unique solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_i is the matrix obtained from A by replacing the i-th column of A by B.

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Example:

$$-2x_1 + 3x_2 - x_3 = 1$$
$$x_1 + 2x_2 - x_3 = 4$$
$$-2x_1 - x_2 + x_3 = -3$$

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$$

$$x_1 = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{|A|} = \frac{-4}{-2} = 2, \qquad x_2 = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{|A|} = \frac{-6}{-2} = 3$$

$$x_3 = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{|A|} = \frac{-8}{-2} = 4.$$

Since $|A| \neq 0$, the system has the unique solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

The following statements are equivalent for an $n \times n$ matrix A:

(1) A is invertible.

(2) $AX = 0$ has only the trivial solution

(3) A is row (column) equivalent to I_n.

(4) $AX = B$ has the unique solution for every B.

(5) A is a product of elementary matrices.

(6) $\det(A) \neq 0$.

The trace of an $n\times n$ matrix $A=[a_{ij}]$ is defined by $tr(A)=a_{11}+a_{22}+\cdots+a_{nn}$.

- $tr(A+B)=tr(A)+tr(B)$
- $tr(cA)=c\,tr(A)$
- $tr(AB)=tr(BA)$

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May the Math be with you!!