SECTION 3: CONDITIONAL PROBABILITY AND INDEPENDENT EVENTS

3.1 Conditional Probability

Conditional probability provides probability of events in a restricted sample space. For example, let us think rolling a die, if we know that the face-up of the die is even, when we ask what the probability of the face-up of the die of 4 is?

Definition 3.1. If A and B are any two events in a sample space S and $P(A)\neq 0$, the conditional probability of B given A is

$$P\left(B\middle|A\right)=\frac{P(A\cap B)}{P(A)}$$

Example 3.1. A die is loaded in such a way that each odd number is twice as likely to occur as each even number. The die is rolled, if it is given that the number of points is greater than 4 or equal 4, what is the probability of the number of points being 6?

The sample space S={1,2,3,4,5,6} hence we assign probability w to each even number and probability $2w$ to each odd number, we find that $2w+w+2w+w+2w+w=9w=1$, $w=1/9$.

A={the points are greater than 4 or equal 4}={4, 5, 6}, here the restricted sample space would be S′={4, 5, 6}

B={the point is 6}={6}

A∩B={6}

The points 4 and 6 are even numbers; their probabilities are 1/9s.
The point 5 is odd number; its probability is 2/9.

$$P\left(B\middle|A\right)=\frac{P(A\cap B)}{P(A)}=\frac{1/9}{4/9}=\frac{1}{4}$$

Example 3.2. A manufacturer of airplane parts knows from past experience that the probability is 0.80 that an order (sipariş) will be ready for shipment (sevkiyat) on time, and it is 0.72 that and order will be ready for shipment on time and will also be delivered on time. What is the probability that such an order will be delivered on time given that it was ready for shipment on time?

R shows the event that an order is ready for shipment on time.
D shows the event that an order is delivered on time.

$P(R)=0.80$, $P(R\cap D)=0.72$ then the reply in the question asked is:

$$P\left(D|R\right) = \frac{P(D\cap R)}{P(R)} = \frac{0.72}{0.80} = 0.90.$$

Theorem 3.1. If *A* and *B* are any two events in a sample space *S* and *P*(*A*)≠0, then

$$P(A\cap B) = P(A)P\left(B|A\right)$$

Example 3.3. Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards if we sample,

a) Without replacement,
b) With replacement.

a) If the first card is not replaced before the second card is drawn the probability getting two aces in succession is $\frac{4}{52}\frac{3}{51} = \frac{1}{221}$
b) If the first card is replaced before the second card is drawn the probability getting two aces in succession is $\frac{4}{52}\frac{4}{52} = \frac{1}{169}$

Theorem 3.2. If *A*, B and *C* are any three events in a sample space *S* and *P*(*A*∩*B*)≠0, then

$$P(A\cap B\cap C) = P(A)P\left(B|A\right)P\left(C|A\cap B\right)$$

Proof: writing $A\cap B\cap C$ as $(A\cap B)\cap \mathrm{C}$ and using the formula of Theorem 3. 1 twice, we get

$$\begin{aligned} P(A\cap B\cap C) &= P[(A\cap B)\cap C] \\ &= P\left(A\cap B\right)P\left(C|A\cap B\right) \\ &= P(A)P\left(B|A\right)P\left(C|A\cap B\right) \end{aligned}$$

3.2 Independent Events

Informally speaking, two events are independent, if the occurrence or non-occurrence of either one does not affect the probability of the occurrence of the other.

Definition 3.2. Two events *A* and *B* are independent if only if

$$P(A\cap B) = P(A)P(B)$$

Example 3.4. A coin is tossed three times and A is the event that a head occurs on each of the first two tosses, B is the event that a tail occurs on the third toss. Find that A and B are independent.

S={ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

A={ HHH, HHT}

B={HHT, HTT, THT, TTT}

A∩B={HHT}

If they are independent, it must be P(A∩B)=P(A)P(B) and so $\frac{1}{8}=\frac{1}{4}\cdot\frac{1}{2}$ and we say that they are independent events.

Theorem 3.3. If *A* and B are independent, then A and B′ are also independent.

Proof: Since $A=(A\cap B)\cup(A\cap B')$ and A∩B and A∩ B′ are mutually exclusive events, and A and B are independent by assuming, we have

$$\begin{aligned} P(A) &= P[(A\cap B)\cup(A\cap B')] \\ &= P(A\cap B)+P(A\cap B') \\ &= P(A)P(B)+P(A\cap B') \end{aligned}$$

It follows that

$$\begin{aligned} P(A\cap B') &= P(A)[1-P(B)] \\ &= P(A)P(B') \end{aligned}$$

and hence that A and B′ are independent.

Example 3.5. A sharpshooter (keskin nişancı) hits a target with probability 0.75. Assuming independence, find the probabilities getting

a) a hit followed by two misses
b) two hits and a miss in any order

a) "HMM" hence $P(HMM)=0.75\times 0.25^2$
b) "HHM", "HMH", "MHH" these are mutually exclusive and hitting or missing a target are independent events from **Theorem 3.3.**
$P(HHM)+P(HMH)+P(MHH)=3\times 0.75^2\times 0.25$

Definition 3.3. Events $A_1, A_2, ..., A_k$ are independent if and only if the probability of intersection of any 2,3,…, or k of these events equals the product of their respective probabilities.

For three events A, B, and C for example, independence requires that

$$\begin{aligned} &P(A\cap B)=P(A)P(B) \\ &P(A\cap C)=P(A)P(C) \\ &P(B\cap C)=P(B)P(C) \\ &\textit{and } P(A\cap B\cap C)=P(A)P(B)P(C) \end{aligned}$$

Note that three or more events can be **pairwise independent** without being independent.

Example 3.6. A coin is loaded so the probabilities of heads and tails are 0.52 and 0.48, respectively. If the coin is tossed three times, what are the probabilities of getting

a) All heads$\Rightarrow$ because of independency $P(HHH)=P(H)\underbrace{P\left(H\left|H\right.\right)}_{P(H)}\underbrace{P\left(H\left|H\cap H\right.\right)}_{P(H)}=0.52^3$

b) Two tails and a head in that order $\Rightarrow$ because of independency $P(TTH)=P(T)\underbrace{P\left(T\left|T\right.\right)}_{P(T)}\underbrace{P\left(H\left|T\cap T\right.\right)}_{P(H)}=0.48^2\times 0.52$

Example 3.7. There are 90 applicants for a job with the news department of television station. Some of them are college graduates and some are not, some of them at least three years' experience and some have not, with the exact breakdown being

	College graduates	Not College graduates	Total
At least three years' experience	18	9	27
Less than three years' experience	36	27	63
Total	54	36	90

If the order in which the applicants are interviewed by the station manager is random, G is the event that the first applicant interviewed is a college graduate, and T is the event the first applicant interviewed has at least three years 'experience determine each of the probabilities given below:

a) P(G), P(T), P(T′)
b) $P\left(T\left|G\right.\right)$
c) $P\left(G'\left|T'\right.\right)$

a) $P(G)=\frac{54}{90}=\frac{3}{5}\quad P(G')=\frac{36}{90}=\frac{2}{5}\quad P(T)=\frac{27}{90}=\frac{3}{10}\quad P(T')=\frac{63}{90}=\frac{7}{10}$

b) $P\left(T\left|G\right.\right)=\frac{P(G\cap T)}{P(G)}=\frac{18/90}{54/90}=\frac{18}{54}=\frac{1}{3}$

c) $P\left(G'\left|T'\right.\right)=\frac{P(T'\cap G')}{P(T')}=\frac{27/90}{63/90}=\frac{27}{63}=\frac{3}{7}$

Example 3.8. Suppose that in Vancouver, B.C. the probability that a rainy day is followed by a rainy day is 0.80 and the probability that a sunny day is followed by a rainy day is 0.60. Find the probabilities that a rainy day is followed by

a) A rainy day, a sunny day, and another rainy day,
b) Two rainy days and then two sunny days

$P\left(R\left|R\right.\right)=0.80\quad P\left(R\left|S\right.\right)=0.60$

a) $P\left(RSR\left|R\right.\right)=P\left(R\left|R\right.\right)P\left(S\left|R\cap R\right.\right)P\left(R\left|R\cap R\cap S\right.\right)=0.80\times 0.20\times 0.60=0.096$

b) $P\left(RRSS\left|R\right.\right)=P\left(R\left|R\right.\right)P\left(R\left|R\cap R\right.\right)P\left(S\left|R\cap R\cap R\right.\right)P\left(S\left|R\cap R\cap R\cap S\right.\right)$

$$= 0.80 \times 0.80 \times 0.20 \times 0.40$$
$$= 0.0512$$