SECTION 6: CONDITIONAL PROBABILITY

EXERCISES

Exercise 1: The pdf of continuous random variable X is given in below:

$$f(x)=\begin{cases}\dfrac{c}{\sqrt{x}}, & 0<x<4\\ 0, & otherwise\end{cases}$$

a) Find the constant c.

b) Find the conditional pdf $f(x|X<1)$.

c) Find the conditional cdf $F(x|X<1)$.

d) Find conditional expectation $E(X|X<1)$.

e) Find conditional variance $V(X|X<1)$

Solution:

a) $\int_{R_x} f(x)dx=1\Rightarrow\int_0^4\frac{c}{\sqrt{x}}dx=1\Rightarrow 2c\sqrt{x}\Big|_0^4=4c\Rightarrow 4c=1\Rightarrow c=\frac{1}{4}$

$$f(x)=\begin{cases}\dfrac{1}{4\sqrt{x}}, & 0<x<4\\ 0, & otherwise\end{cases}$$

b) $f(x|X<1)=\dfrac{f(x)}{P(X<1)}=\dfrac{\frac{1}{4\sqrt{x}}}{\int_0^1\frac{1}{4\sqrt{x}}dx}=\dfrac{\frac{1}{4\sqrt{x}}}{\frac{\sqrt{x}}{2}\Big|_0^1}=\dfrac{1}{2\sqrt{x}}$

$$f(x|X<1)=\begin{cases}\dfrac{1}{2\sqrt{x}}, & 0<x<1\\ 0, & otherwise\end{cases}$$

c) $F(x|X<1)=\dfrac{P((X\le x)\cap(X<1))}{P(X<1)}=\dfrac{P(X\le x)}{P(X<1)}=\dfrac{F(x)}{F(1)}$

$$F(x)=\int_0^x\frac{1}{4\sqrt{t}}dt=\frac{\sqrt{t}}{2}\Big|_0^x=\frac{\sqrt{x}}{2}\quad\Rightarrow\quad F(x)=\begin{cases}\dfrac{\sqrt{x}}{2}, & 0<x<4\\ 0, & x<0\\ 1, & x\ge 4\end{cases}$$

$$F\left(x \mid X<1\right)=\frac{F(x)}{F(1)}=\frac{\frac{\sqrt{x}}{2}}{\frac{\sqrt{1}}{2}}=\sqrt{x} \quad \Rightarrow \quad F\left(x \mid X<1\right)=\begin{cases}\sqrt{x}, & 0<x<1 \\ 0, & x<0 \\ 1, & x\geq 1\end{cases}$$

*Second way of finding $f\left(x \mid X<1\right)$ is $f\left(x \mid X<1\right)=\dfrac{d\,F\left(x \mid X<1\right)}{dx}=\dfrac{d\left(\sqrt{x}\right)}{dx}=\dfrac{1}{2\sqrt{x}}$

$$f\left(x \mid X<1\right)=\begin{cases}\dfrac{1}{2\sqrt{x}}, & 0<x<1 \\ 0, & otherwise\end{cases}$$

d) $E\left(X \mid X<1\right)=\int_0^1 x\, f\left(x \mid X<1\right)dx=\int_0^1 x\frac{1}{2\sqrt{x}}\,dx=\int_0^1 \frac{\sqrt{x}}{2}dx=\left.\frac{x^{3/2}}{3}\right|_0^1=\frac{1}{3}$

e) $E\left(X^2 \mid X<1\right)=\int_0^1 x^2\, f\left(x \mid X<1\right)dx=\int_0^1 x^2\frac{1}{2\sqrt{x}}\,dx=\int_0^1 \frac{x\sqrt{x}}{2}dx=\left.\frac{x^{5/2}}{5}\right|_0^1=\frac{1}{5}$

$$V\left(X \mid X<1\right)=E\left(X^2 \mid X<1\right)-\left[E\left(X \mid X<1\right)\right]^2=\frac{1}{5}-\left(\frac{1}{3}\right)^2=\frac{4}{45}$$

Exercise 2: The pmf of discrete random variable X is given in below:

$$p(x)=\begin{cases}\dfrac{x}{15}, & x=1,2,3,4,5 \\ 0. & otherwise\end{cases}$$

- **a)** Find the conditional pmf $p\left(x \mid X\leq 3\right)$.
- **b)** Find conditional expectation $E\left(X \mid X\leq 3\right)$.
- **c)** Find the conditional cdf $F\left(x \mid X\leq 3\right)$
- **d)** Find the conditional probability $P\left(3<X\leq 5 \mid X\leq 4\right)$.
- **e)** Find the conditional cdf. $F\left(x \mid X>2\right)$

Solution:

a) $p\left(x \mid X\leq 3\right)=\dfrac{p(x)}{P\left(X\leq 3\right)}=\dfrac{\frac{x}{15}}{\frac{1}{15}\left(1+2+3\right)}=\dfrac{\frac{x}{15}}{\frac{6}{15}}=\dfrac{x}{6} \Rightarrow p(x)=\begin{cases}\dfrac{x}{6}, & x=1,2,3 \\ 0, & otherwise\end{cases}$

b) $E\left(X \middle| X \le 3\right) = \sum_{x=1}^{3} x\, p\left(x \middle| X \le 3\right) = \sum_{x=1}^{3} x \frac{x}{6} = \sum_{x=1}^{3} \frac{x^2}{6} = \frac{14}{6} = \frac{7}{3}$

c) $F\left(x \middle| X \le 3\right) = \frac{P\left(\left(X \le x\right) \cap \left(X \le 3\right)\right)}{P\left(X \le 3\right)} = \frac{P\left(X \le x\right)}{P\left(X \le 3\right)} = \frac{F\left(x\right)}{F\left(3\right)}$

$$F\left(x\right) = \sum_{t=1}^{x} \frac{t}{15} = \frac{1}{15}\frac{x\left(x+1\right)}{2} = \frac{x\left(x+1\right)}{30} \quad \Rightarrow \quad F\left(x\right) = \begin{cases} \frac{x\left(x+1\right)}{30}, & x = 1,2,3,4,5 \\ 0, & x < 1 \\ 1, & x \ge 5 \end{cases}$$

$$F\left(x \middle| X \le 3\right) = \frac{F\left(x\right)}{F\left(3\right)} = \frac{\frac{x\left(x+1\right)}{30}}{\frac{12}{30}} = \frac{x\left(x+1\right)}{12}$$

$$F\left(x \middle| X \le 3\right) = \begin{cases} \frac{x\left(x+1\right)}{12}, & x = 1,2,3 \\ 0, & x < 1 \\ 1, & x \ge 3 \end{cases}$$

d) $P\left(3 < X \le 5 \middle| X \le 4\right) = \frac{P\left(\left(3 < X \le 5\right) \cap \left(X \le 4\right)\right)}{P\left(X \le 4\right)} = \frac{P\left(3 < X \le 4\right)}{P\left(X \le 4\right)} = \frac{P\left(X = 4\right)}{F\left(4\right)} = \frac{\frac{4}{15}}{\frac{4.5}{30}} = \frac{2}{5}$

e)

$$F\left(x \middle| X > 2\right) = \frac{P\left(\left(X \le x\right) \cap \left(X > 2\right)\right)}{P\left(X > 2\right)} = \frac{P\left(2 < X \le x\right)}{1 - P\left(X \le 2\right)} = \frac{F\left(x\right) - F\left(2\right)}{1 - F\left(2\right)} = \frac{\frac{x\left(x+1\right)}{30} - \frac{2.3}{30}}{1 - \frac{2.3}{30}} = \frac{x\left(x+1\right) - 6}{24}$$

$$F\left(x \middle| X > 2\right) = \begin{cases} \frac{x\left(x+1\right) - 6}{24}, & x = 3,4,5 \\ 0, & x < 3 \\ 1, & x \ge 5 \end{cases}$$

Exercise 3: The pdf of continuous random variable X is given in below:

$$f(x)=\begin{cases}\frac{1}{56}(x+3), & 0\le x\le 8\\ 0, & otherwise\end{cases}$$

a) Find the conditional probability $P\left(X\le 5 \middle| 2\le X\le 7\right)$.

b) Find the conditional pdf $f\left(x \middle| X\ge 3\right)$.

c) Find the conditional cdf $F\left(x \middle| X\ge 3\right)$

d) Find the conditional probability $P\left(\frac{5}{2}<X<\frac{9}{2}\middle| X\ge 3\right)$.

Solution:

a) $F(x)=\int_0^x \frac{1}{56}(t+3)\,dt=\frac{1}{56}\left(\frac{t^2}{2}+3t\right)\Big|_0^x=\frac{1}{56}\left(\frac{x^2}{2}+3x\right)$

$$F(x)=\begin{cases}\frac{1}{112}\left(x^2+6x\right), & 0\le x\le 8\\ 0, & x<0\\ 1, & x\ge 8\end{cases}$$

$$P\left(X\le 5 \middle| 2\le X\le 7\right)=\frac{P\left((X\le 5)\cap(2\le X\le 7)\right)}{P(2\le X\le 7)}=\frac{P(2\le X\le 5)}{P(2\le X\le 7)}=\frac{F(5)-F(2)}{F(7)-F(2)}$$

$$=\frac{\frac{1}{112}\left[(25+30)-(4+12)\right]}{\frac{1}{112}\left[(49+42)-(4+12)\right]}=\frac{13}{25}$$

b) $f\left(x \middle| X\ge 3\right)=\frac{f(x)}{P(X\ge 3)}=\frac{\frac{1}{56}(x+3)}{1-P(X\le 3)}=\frac{\frac{1}{56}(x+3)}{1-\frac{1}{112}(9+18)}=\frac{2(x+3)}{85}$

$$f\left(x \middle| X\ge 3\right)=\begin{cases}\frac{2(x+3)}{85}, & 3\le x\le 8\\ 0, & otherwise\end{cases}$$

c)

$$F\left(x \middle| X \ge 3\right) = \frac{P\left(\left(X \le x\right) \cap \left(X \ge 3\right)\right)}{P\left(X \ge 3\right)} = \frac{P\left(3 \le X \le x\right)}{1 - P\left(X < 3\right)} = \frac{F\left(x\right) - F\left(3\right)}{1 - F\left(3\right)} = \frac{\frac{1}{112}\left(x^2 + 6x\right) - \frac{27}{112}}{1 - \frac{1}{112}\left(9 + 18\right)}$$

$$= \frac{x^2 + 6x - 27}{85}$$

$$F\left(x \middle| X \ge 3\right) = \begin{cases} \dfrac{x^2 + 6x - 27}{85}, & 3 \le x \le 8 \\ 0, & x < 3 \\ 1, & x \ge 8 \end{cases}$$

d)

$$P\left(\frac{5}{2} < X < \frac{9}{2} \middle| X \ge 3\right) = P\left(X \le \frac{9}{2} \middle| X \ge 3\right) = F\left(\frac{9}{2} \middle| X \ge 3\right)$$

$$= \frac{1}{85}\left(\frac{81}{4} + \frac{54}{2} - 27\right) = \frac{81}{340} = 0.2382$$

Exercise 4: The pmf of discrete random variable X is given in below:

$$p\left(x\right) = \begin{cases} \mathrm{k}\left(x + 3\right), & x = 0, 1, \cdots, 8 \\ 0. & otherwise \end{cases}$$

- **a)** Find the conditional pmf $p\left(x \middle| X \le 5\right)$ and the conditional cdf $F\left(x \middle| X \le 5\right)$
- **b)** Find the conditional pmf $p\left(x \middle| X > 5\right)$ and the conditional cdf $F\left(x \middle| X > 5\right)$
- **c)** Find the conditional cdf $F\left(x \middle| 2 < X \le 5\right)$
- **d)** Find conditional expectation $E\left(2X - 1 \middle| 2 \le X \le 4\right)$.

Solution:

$$\sum_{x=0}^{8} k\left(x + 3\right) = 1 \Rightarrow k\sum_{x=0}^{8}\left(x + 3\right) = k\sum_{x=1}^{9}\left(x - 1 + 3\right) = k\sum_{x=1}^{9}\left(x + 2\right)$$

$$= k\left[\frac{9.10}{2} + 2.9\right] = 63k$$

$$63k = 1 \Rightarrow k = \frac{1}{63}$$

$$F(x)=\sum_{t=0}^{x}\frac{1}{63}(t+3)=\frac{1}{63}\sum_{t=1}^{x+1}(t-1+3)=\frac{1}{63}\sum_{t=1}^{x+1}(t+2)$$

$$=\frac{1}{63}\frac{(x+1)(x+2)}{2}+\frac{4(x+1)}{2}=\frac{(x+1)\left[(x+2)+4\right]}{126}$$

$$F(x)=\begin{cases}\dfrac{(x+1)(x+6)}{126}, & x=0,1,\cdots,8\\ 0, & x<0\\ 1, & x\geq 8\end{cases}$$

a) $p(x|X\leq 5)=\dfrac{p(x)}{P(X\leq 5)}=\dfrac{\frac{1}{63}(x+3)}{F(5)}=\dfrac{\frac{1}{63}(x+3)}{\frac{66}{126}}=\dfrac{x+3}{33}$

$$p(x|X\leq 5)=\begin{cases}\dfrac{x+3}{33}, & x=0,1,2,3,4,5\\ 0, & otherwise\end{cases}$$

$$F(x|X\leq 5)=\frac{P((X\leq x)\cap(X\leq 5))}{P(X\leq 5)}=\frac{\frac{(x+1)(x+6)}{126}}{\frac{66}{126}}=\frac{(x+1)(x+6)}{66}$$

$$F(x|X\leq 5)=\begin{cases}\dfrac{(x+1)(x+6)}{66}, & x=0,1,2,3,4,5\\ 0, & x<0\\ 1, & x\geq 5\end{cases}$$

b) $p(x|X>5)=\dfrac{p(x)}{P(X>5)}=\dfrac{p(x)}{1-P(X\leq 5)}=\dfrac{\frac{1}{63}(x+3)}{1-F(5)}=\dfrac{\frac{1}{63}(x+3)}{1-\frac{66}{126}}=\dfrac{x+3}{30}$

$$p(x|X>5)=\begin{cases}\dfrac{x+3}{30}, & x=6,7,8\\ 0, & otherwise\end{cases}$$

$$F\left(x \middle| X>5\right)=\frac{P\left(\left(X\le x\right)\cap\left(X>5\right)\right)}{P\left(X>5\right)}=\frac{P\left(5<X\le x\right)}{1-P\left(X\le 5\right)}=\frac{F\left(x\right)-F\left(5\right)}{1-F\left(5\right)}$$

$$=\frac{\frac{\left(x+1\right)\left(x+6\right)}{126}-\frac{66}{126}}{\frac{60}{126}}=\frac{\left(x+1\right)\left(x+6\right)-66}{60}$$

$$F\left(x \middle| X>5\right)=\begin{cases}\frac{\left(x+1\right)\left(x+6\right)-66}{60}, & x=6,7,8\\ 0, & x<6\\ 1, & x\ge 8\end{cases}$$

$$F\left(x \middle| 2<X\le 5\right)=\frac{P\left(\left(X\le x\right)\cap\left(2<X\le 5\right)\right)}{P\left(2<X\le 5\right)}=\frac{P\left(2<X\le x\right)}{P\left(2<X\le 5\right)}=\frac{F\left(x\right)-F\left(2\right)}{F\left(5\right)-F\left(2\right)}$$

c)

$$=\frac{\frac{\left(x+1\right)\left(x+6\right)}{126}-\frac{3.8}{126}}{\frac{6.11}{126}-\frac{3.8}{126}}=\frac{\left(x+1\right)\left(x+6\right)-24}{42}$$

$$F\left(x \middle| 2<X\le 5\right)=\begin{cases}\frac{\left(x+1\right)\left(x+6\right)-24}{42}, & x=3,4,5\\ 0, & x<3\\ 1, & x\ge 5\end{cases}$$

d) $$p\left(x \middle| 2\le X\le 4\right)=\frac{p\left(x\right)}{P\left(2\le X\le 4\right)}=\frac{p\left(x\right)}{F\left(4\right)-F\left(1\right)}=\frac{\frac{\left(x+3\right)}{63}}{\frac{50-14}{126}}=\frac{\left(x+3\right)}{18}$$

$$p\left(x \middle| 2\le X\le 4\right)=\begin{cases}\frac{\left(x+3\right)}{18}, & x=2,3,4\\ 0, & otherwise\end{cases}$$

$$E\left(2X-1 \middle| 2\le X\le 4\right)=\sum_{x=2}^{4}\left(2x-1\right)p\left(x \middle| 2\le X\le 4\right)=\sum_{x=2}^{4}\left(2x-1\right)\frac{\left(x+3\right)}{18}$$

$$=\frac{94}{18}=\frac{47}{9}$$

Exercise 5: The pdf of continous random variable X is given in below:

$$p(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1\le x<2 \\ 0, & otherwise \end{cases}$$

Find the conditional pdf $f\left(x|X\ge 1\right)$ and the conditional cdf $F\left(x|X\ge 1\right)$

Solution:

$$0<x<1 \quad \Rightarrow \quad F(x)=\int_0^x t\,dt=\left.\frac{t^2}{2}\right|_0^x=\frac{x^2}{2}$$

$$1\le x<2 \quad \Rightarrow \quad F(x)=\int_0^1 x\,dx+\int_1^x (2-t)\,dt=\left.\frac{x^2}{2}\right|_0^1+\left.\left(2t-\frac{t^2}{2}\right)\right|_1^x=\frac{1}{2}+\left[\left(\frac{4x-x^2}{2}\right)-\left(\frac{4-1}{2}\right)\right]=\frac{-x^2+4x-2}{2}$$

$$F(x)=\begin{cases} \dfrac{x^2}{2}, & 0<x<1 \\ \dfrac{-x^2+4x-2}{2}, & 1\le x<2 \\ 0, & x<0 \\ 1, & x\ge 2 \end{cases}$$

$$f\left(x|X\ge 1\right)=\frac{f(x)}{1/2}=\frac{2-x}{1/2}=2(2-x)$$

$$f\left(x|X\ge 1\right)=\begin{cases} 2(2-x), & 1\le x<2 \\ 0, & otherwise \end{cases}$$

$$F\left(x|X\ge 1\right)=\frac{P\left(X\le x\cap X\ge 1\right)}{P\left(X\ge 1\right)}=\frac{P\left(1\le X\le x\right)}{1-P\left(X<1\right)}=\frac{F(x)-F(1)}{1-F(1)}=\frac{\left(-x^2+4x-2\right)/2-1/2}{1-1/2}=-x^2+4x-3$$

$$F\left(x|X\ge 1\right)=\begin{cases} -x^2+4x-3, & 1\le x<2 \\ 0, & x<1 \\ 1, & x\ge 2 \end{cases}$$

or $\quad F\left(x|X\ge 1\right)=\displaystyle\int_1^x f\left(t|t\ge 1\right)dt=\int_1^x 2(2-t)\,dt=2\left.\left(2t-\frac{t^2}{2}\right)\right|_1^x=-x^2+4x-3,\quad 1\le x<2$

Exercise 6: The pmf of discrete random variable X is given in below:

$$p(x)=\begin{cases}\dfrac{x}{a}, & x=1,2\\ \dfrac{x}{25}, & x=3,4,5\\ 0, & otherwise\end{cases}$$

- **a)** Find the constant a.
- **b)** Find the conditional pmf $p(x|X\le 4)$ and the conditional cdf $F(x|X\le 4)$.
- **c)** Find the conditional probability $P(1<X\le 2|X\le 4)$.
- **d)** Find conditional expectation $E(X|X\le 4)$ and conditional variance $V(X|X\le 4)$.

Solution:

a)

$$\sum_{x\in Rx} p(x)=1 \;\Rightarrow\; \sum_{x=1,2}\frac{x}{a}+\sum_{x=3,4,5}\frac{x}{25}=1 \;\Rightarrow\; \frac{(1+2)}{a}+\frac{(3+4+5)}{25}=1\Rightarrow\frac{3}{a}=\frac{13}{25} \;\Rightarrow\; a=\frac{75}{13}$$

$$p(x)=\begin{cases}\dfrac{13x}{75}, & x=1,2\\ \dfrac{x}{25}, & x=3,4,5\\ 0, & otherwise\end{cases}$$

$$p(x|X\le 4)=\frac{p(x)}{P(X\le 4)}=\frac{13x/75}{13(1+2)/75+(3+4)/25}=\frac{13x/75}{20/25}=\frac{13x}{60},\quad x=1,2$$

b)

$$p(x|X\le 4)=\frac{p(x)}{P(X\le 4)}=\frac{x/25}{20/25}=\frac{x}{20},\quad x=3,4$$

$$p(x|X\le 4)=\begin{cases}\dfrac{13x}{60}, & x=1,2\\ \dfrac{x}{20}, & x=3,4\\ 0, & otherwise\end{cases}$$

1. Way:

$$F\left(x \middle| X \le 4\right) = \begin{cases} \sum_{t=1}^{x} \frac{13t}{60} = \frac{13}{60}\frac{x\left(x+1\right)}{2} = \frac{13}{120}x\left(x+1\right), & x = 1,2 \\ \sum_{x=1}^{2} \frac{13x}{60} + \sum_{t=3}^{x} \frac{t}{20} = \frac{13}{60}\left(1+2\right) + \frac{1}{20}\left[\frac{x\left(x+1\right)}{2} - 3\right] = \frac{x^2+x+20}{40}, & x = 3,4 \end{cases}$$

$$F\left(x \middle| X \le 4\right) = \begin{cases} \frac{13}{120}\left(x^2+x\right), & x = 1,2 \\ \frac{1}{40}\left(x^2+x+20\right), & x = 3,4 \\ 0, & x < 1 \\ 1, & x \ge 4 \end{cases}$$

2. Way:

$$F\left(x \middle| X \le 4\right) = P\left(X \le x \middle| X \le 4\right) = \frac{P\left(X \le x \cap X \le 4\right)}{P\left(X \le 4\right)} = \frac{P\left(X \le x\right)}{P\left(X \le 4\right)} = \frac{F\left(x\right)}{F\left(4\right)}$$

$$F\left(x\right) = \sum_{t=1}^{x} \frac{13}{75}t = \frac{13}{75}\frac{x\left(x+1\right)}{2} = \frac{13}{150}\left(x^2+x\right), \quad \mathrm{x} = 1,2$$

$$\mathrm{F}\left(x\right) = \sum_{x=1}^{2} \frac{13x}{75} + \sum_{t=3}^{x} \frac{t}{25} = \frac{13}{75}\left(1+2\right) + \frac{1}{25}\left[\frac{x\left(x+1\right)}{2} - 3\right] = \frac{x^2+x+20}{50}, \quad x = 3,4$$

$$F\left(x\right) = \begin{cases} \frac{13}{150}\left(x^2+x\right), & x = 1,2 \\ \frac{1}{50}\left(x^2+x+20\right), & x = 3,4,5 \\ 0, & x < 1 \\ 1, & x \ge 5 \end{cases}$$

$$F\left(x \middle| X \le 4\right) = \frac{F\left(x\right)}{F\left(4\right)} = \frac{13\left(x^2+x\right)/150}{20/25} = \frac{13}{120}\left(x^2+x\right), \quad x = 1,2$$

$$F\left(x \middle| X \le 4\right) = \frac{F\left(x\right)}{F\left(4\right)} = \frac{\left(x^2+x+20\right)/50}{20/25} = \frac{1}{40}\left(x^2+x+20\right), \quad x = 3,4$$

$$F\left(x \middle| X \le 4\right) = \begin{cases} \dfrac{13}{120}\left(x^2 + x\right), & x = 1, 2 \\ \dfrac{1}{40}\left(x^2 + x + 20\right), & x = 3, 4 \\ 0, & x < 1 \\ 1, & x \ge 4 \end{cases}$$

c) $P\left(1 < X \le 2 \middle| X \le 4\right) = P\left(X = 2 \middle| X \le 4\right) = \dfrac{P\left(X = 2 \cap X \le 4\right)}{P\left(X \le 4\right)} = \dfrac{P\left(X = 2\right)}{X \le 4} = \dfrac{\frac{13}{75}2}{\frac{20}{25}} = \dfrac{13}{30}$.

d)

$$\begin{aligned} E\left(X \middle| X \le 4\right) &= \sum_{x=1}^{2} x\frac{13x}{60} + \sum_{x=3}^{4} x\frac{x}{20} = \sum_{x=1}^{2} \frac{13x^2}{60} + \sum_{x=3}^{4} \frac{x^2}{20} \\ &= \frac{13}{60}\left(1^2 + 2^2\right) + \frac{1}{20}\left(3^2 + 4^2\right) \\ &= 7/3 \end{aligned}$$

$$\begin{aligned} E\left(X^2 \middle| X \le 4\right) &= \sum_{x=1}^{2} x^2\frac{13x}{60} + \sum_{x=3}^{4} x^2\frac{x}{20} = \sum_{x=1}^{2} \frac{13x^3}{60} + \sum_{x=3}^{4} \frac{x^3}{20} \\ &= \frac{13}{60}\left(1^3 + 2^3\right) + \frac{1}{20}\left(3^3 + 4^3\right) \\ &= 13/2 \end{aligned}$$

$V\left(X \middle| X \le 4\right) = E\left(X^2 \middle| X \le 4\right) - \left[E\left(X \middle| X \le 4\right)\right]^2 = 13/2 - \left(7/3\right)^2 = 19/18$.