SECTION 6: CONDITIONAL PROBABILITY

6.1.Conditional Probability and Cumulative Distribution Functions

Let $f(x)$ be the probability density function (pdf) of a continuous random variable X. Consider an event A. If $P(A) \neq 0$ then the conditional pdf $f(x|A)$ is defined as shown in below:

$$
\begin{aligned}
f(x|A) &= \lim_{\Delta x \to \infty} \frac{P\left[(x \le X \le x + \Delta x)|A\right]}{\Delta x} \\
&= \frac{f(x)}{P(A)}
\end{aligned}
\tag{1}
$$

The cumulative distribution function (cdf) of random variable X,

$$F(x|A) = \frac{P\left[(X \le x) \cap A\right]}{P(A)}. \tag{2}$$

Here, $\left[(X \le x) \cap A\right] = \{s : X(s) \le x \; and \; s \in A\}$

From Eq. (1) and Eq. (2) the following could be written:

$$f(x|A) = \frac{d}{dx} F(x|A) \tag{3}$$

$F(x|A)$ satisfies the following properties:

1) $\lim_{x \to -\infty} F(x|A) = 0 \; and \; \lim_{x \to +\infty} F(x|A) = 1$

Here, $-\infty$ and $+\infty$ represent the lower and upper bounds, respectively, for the domain of the event A.

2) The following equation could be written:

$$
\begin{aligned}
F(x_2|A) - F(x_1|A) &= P\left[(x_1 < X \le x_2)|A\right] \\
&= \frac{P\left[(x_1 < X \le x_2) \cap A\right]}{P(A)}
\end{aligned}
$$

The conditional pdf $f(x|A)$ also shows the all properties of pdf:

1) $f(x|A) = 0, \; x \notin A$

2) $f(x|A) \ge 0, \; x \in A$

3) $\int_{-\infty}^{+\infty} f(x|A) dx = F(+\infty|A) - F(-\infty|A) = 1$

Here, $-\infty$ and $+\infty$ represent the lower and upper bounds, respectively, for the A.

For discrete random variables, conditional probability and conditional cumulative distribution functions are defined in a similar style. Conditional probability function shows the properties of probability function and conditional cumulative distribution function shows the properties of cumulative distribution function.

A Special Example:

Let $f(x)$ be the pdf of a continuous random variable X. Find the conditional cdf $F(x|A)$ and conditional pdf $f(x|A)$.

Solution: Let *A* be $X \le a$, thus,

$$F(x|X \le a) = \frac{P[(X \le x) \cap (X \le a)]}{P(X \le a)} = \frac{P(X \le x)}{P(X \le a)} = \frac{F(x)}{F(a)}$$ is obtained.

Here, $[(X \le x) \cap (X \le a)] = (X \le x)$.

If $x > a$, as $[(X \le x) \cap (X \le a)] = (X \le a)$

then $F(x|X \le a) = \dfrac{P[(X \le x) \cap (X \le a)]}{P(X \le a)} = \dfrac{F(a)}{F(a)} = 1$ could be written.

If the derivative of $F(x|X \le a)$ is taken according to Eq. (3), the conditional pdf is found as in below:

$$\begin{aligned} f(x|X \le a) &= \frac{f(x)}{F(a)} = \frac{f(x)}{P(X \le a)}, \quad && x \le a \\ &= 0, && x > a \end{aligned}$$

Example 1: The random variable *X* of the life lengths of batteries discussed above is associated with a probability density function of the form

$$f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & \text{for } x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the probability that a battery of this type lasts more than 300 hours, given that it already has been in use for more than 200 hours.

Solution 1: We are interested in $P\left(X>3 \middle| X>2\right)=\dfrac{P\left(X>3\right)}{P\left(X>2\right)}$ because the intersection of the events (X > 3) and (X > 2) is the event (X > 3).

$$\frac{P\left(X>3\right)}{P\left(X>2\right)}=\frac{\int_3^\infty \frac{1}{2}e^{-x/2}dx}{\int_2^\infty \frac{1}{2}e^{-x/2}dx}=\frac{e^{-3/2}}{e^{-1}}=e^{-1/2}=0.606$$

Solution 2: Sometimes it is convenient to look at cumulative probabilities of the form

$$\begin{aligned}F\left(x\right)&=P\left(X\le x\right)\\&=\int_0^x \frac{1}{2}e^{-t/2}dt\\&=-e^{-t/2}\Big|_0^x\\&=\begin{cases}1-e^{-x/2}, & x>0\\ 0, & x\le 0\end{cases}\end{aligned}$$

$$P\left(X>3 \middle| X>2\right)=\frac{P\left(X>3\right)}{P\left(X>2\right)}=\frac{1-P\left(X\le 3\right)}{1-P\left(X\le 2\right)}=\frac{1-F\left(3\right)}{1-F\left(2\right)}=\frac{1-\left(1-e^{-3/2}\right)}{1-\left(1-e^{-2/2}\right)}=\frac{e^{-3/2}}{e^{-1}}=e^{-1/2}=0.606$$

Example 2: The pdf of continuous random variable X is given in below:

$$\begin{aligned}f\left(x\right)&=\frac{1}{12}x, 1<x<5\\&=0, \qquad \textit{for other values of } x\end{aligned}$$

- **a)** Find the conditional cdf $F\left(x \middle| X>3\right)$.
- **b)** Find the conditional pdf $f\left(x \middle| X>3\right)$.

Solution:

- **a)** The cdf of X, $F\left(x\right)=\int_1^x \frac{1}{12}tdt=\frac{1}{24}\left(x^2-1\right)$.

For $x>3$, $F\left(x \middle| X>3\right)=\dfrac{P\left[\left(X\le x\right)\cap\left(X>3\right)\right]}{P\left(X>3\right)}=\dfrac{P\left(3<X\le x\right)}{P\left(X>3\right)}=\dfrac{F\left(x\right)-F\left(3\right)}{1-F\left(3\right)}=\dfrac{x^2-9}{16}$ is obtained.

For $x\le 3$, $F\left(x \middle| X>3\right)=\dfrac{P\left[\left(X\le x\right)\cap\left(X>3\right)\right]}{P\left(X>3\right)}=0$ is obtained.

b) The conditional pdf, $f\left(x \middle| X>3\right)=\dfrac{d}{dx}F\left(x \middle| X>3\right)=\dfrac{1}{8}x,\ \ 3<x<5$

$$=0,\quad \textit{for other values of } x$$

6.1.1. The Law of Total Probability

Let $A_1, A_2, \cdots, A_n$ events satisfy the conditions given in below:

a) $A_i \cap A_j = \varnothing, i \neq j$
b) $A_1 \cup A_2 \cup \cdots \cup A_n = S$
It could be showed that the equality of $F(x)=P(A_1)F\left(x \middle| A_1\right)+\cdots+P(A_n)F\left(x \middle| A_n\right)$exists.

6.2.Conditional Expected Value

Consider the event A. We will show the conditional expected value of random variable X for any event A as $E\left(X \middle| A\right)$.

If X is a continuous random variable,

$$E\left(X \middle| A\right)=\int_A x\ f\left(x \middle| A\right)dx \tag{4}$$

And discrete random variable,

$$E\left(X \middle| A\right)=\sum_A x\ p\left(x \middle| A\right) \tag{5}$$

Example 3: The pdf of continuous random variable X is given in below:

$$f(x)=\frac{1}{12}x, 1<x<5$$
$$=0,\qquad \textit{for other values of } x$$

Find conditional expectation $E\left(X \middle| X \geq 2\right)$.

Solution:

The conditional pdf

$$f\left(x \middle| X \geq 2\right)=\frac{f(x)}{P(X \geq 2)}=\frac{\frac{1}{12}x}{\int_2^5 \frac{1}{12}xdx}=\frac{2}{21}x,\quad 2 \leq x \leq 5$$

$$=0,\quad \textit{for other values of } x$$

The conditional expected value, $E\left(X \mid X \geq 2\right)=\int_{2}^{5} x \frac{2}{21} x dx=\frac{26}{7}$.

If $A_1, A_2, \cdots, A_n$ are disjoint events and sums of them gives the sample space S, then, considering Eq. (1) and Eq. (4),
$E(X)=P(A_1)E\left(X \mid A_1\right)+\cdots+P(A_n)E\left(X \mid A_n\right)$ could be written.

6.3.Conditional Variance

The conditional variance of random variable X (continuous or discrete) $V\left(X \mid A\right)$ could be found by the help of conditional probability functions.

If X is a continuous random variable,

$$E\left(X^2 \mid A\right)=\int_A x^2 f\left(x \mid A\right) dx \tag{6}$$

$$V\left(X \mid A\right)=E\left(X^2 \mid A\right)-\left[E\left(X \mid A\right)\right]^2 \tag{7}$$

If X is a discrete random variable,

$$E\left(X^2 \mid A\right)=\sum_A x^2 p\left(x \mid A\right) \tag{8}$$

$$V\left(X \mid A\right)=E\left(X^2 \mid A\right)-\left[E\left(X \mid A\right)\right]^2 \tag{9}$$

Example 4: Find conditional variance $V\left(X \mid X \geq 2\right)$ for Example 2.

$E\left(X^2 \mid X \geq 2\right)=\int_{2}^{5} x^2 . \frac{2}{21} x dx=\frac{203}{14}$ and $V\left(X \mid X \geq 2\right)=\frac{203}{14}-\left(\frac{26}{7}\right)^2=\frac{69}{98}$