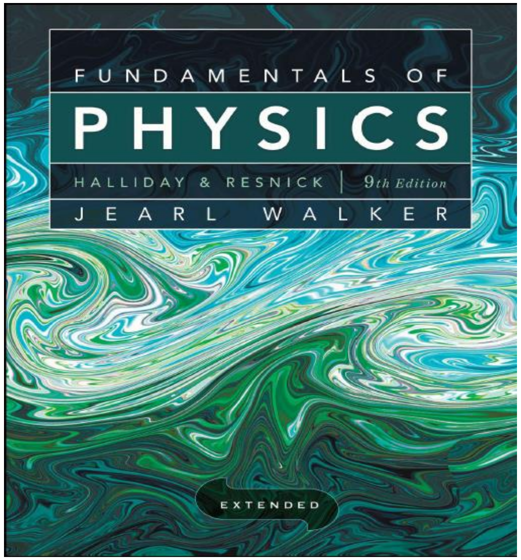
FİZ 137 – 25
CHAPTER 11
Rolling – Torque
Angular Momentum



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FALL SEMESTER

In this chapter we will cover the following topics:

- ***Rolling of circular objects*** and its relationship with ***friction***.

- ***Redefinition of torque*** as a vector to describe rotational problems that are more complicated than the rotation of a rigid body about a fixed axis.

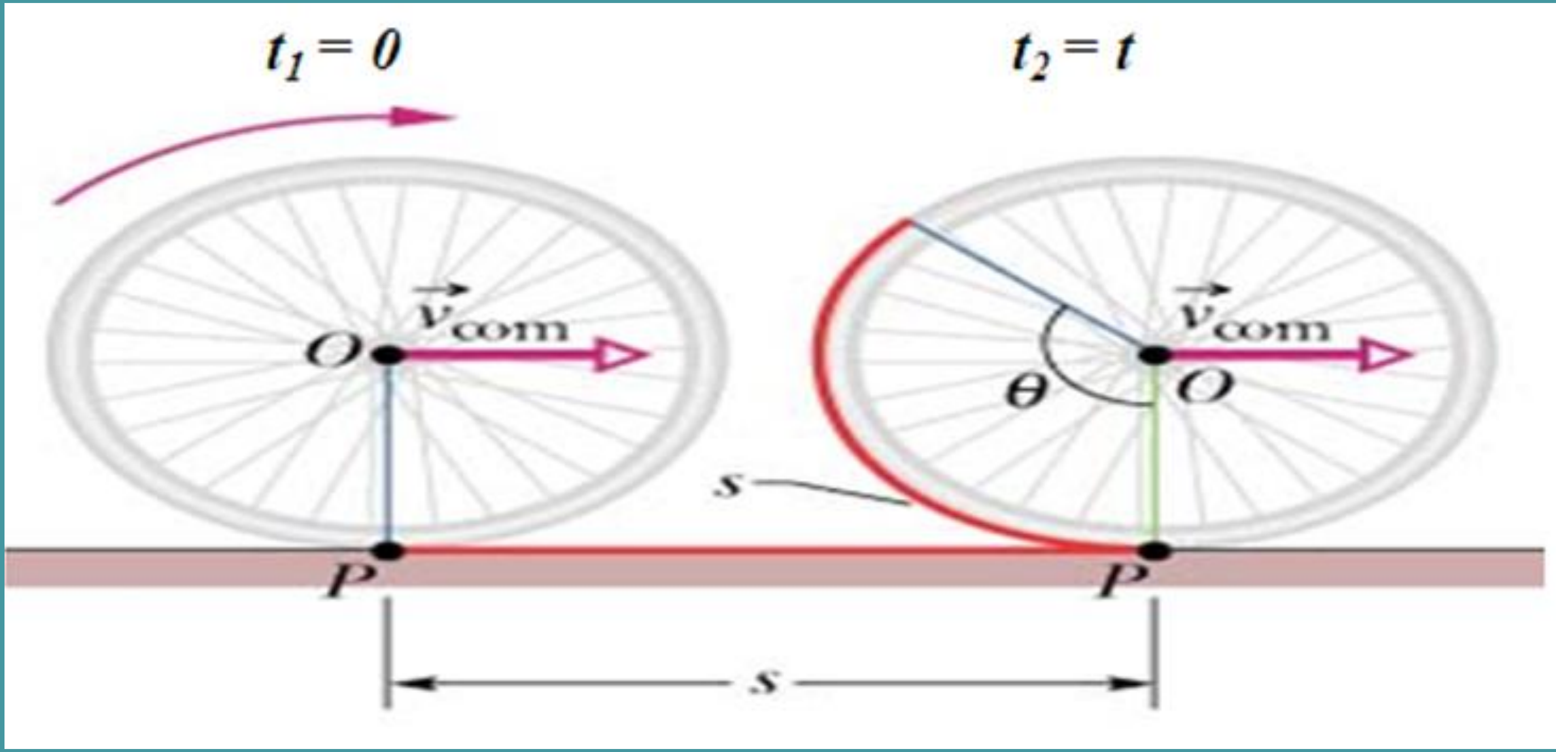
- ***Angular momentum*** of single particles and systems of particles.

- ***Conservation of angular momentum***.

- ***Applications*** of the conservation of angular momentum.

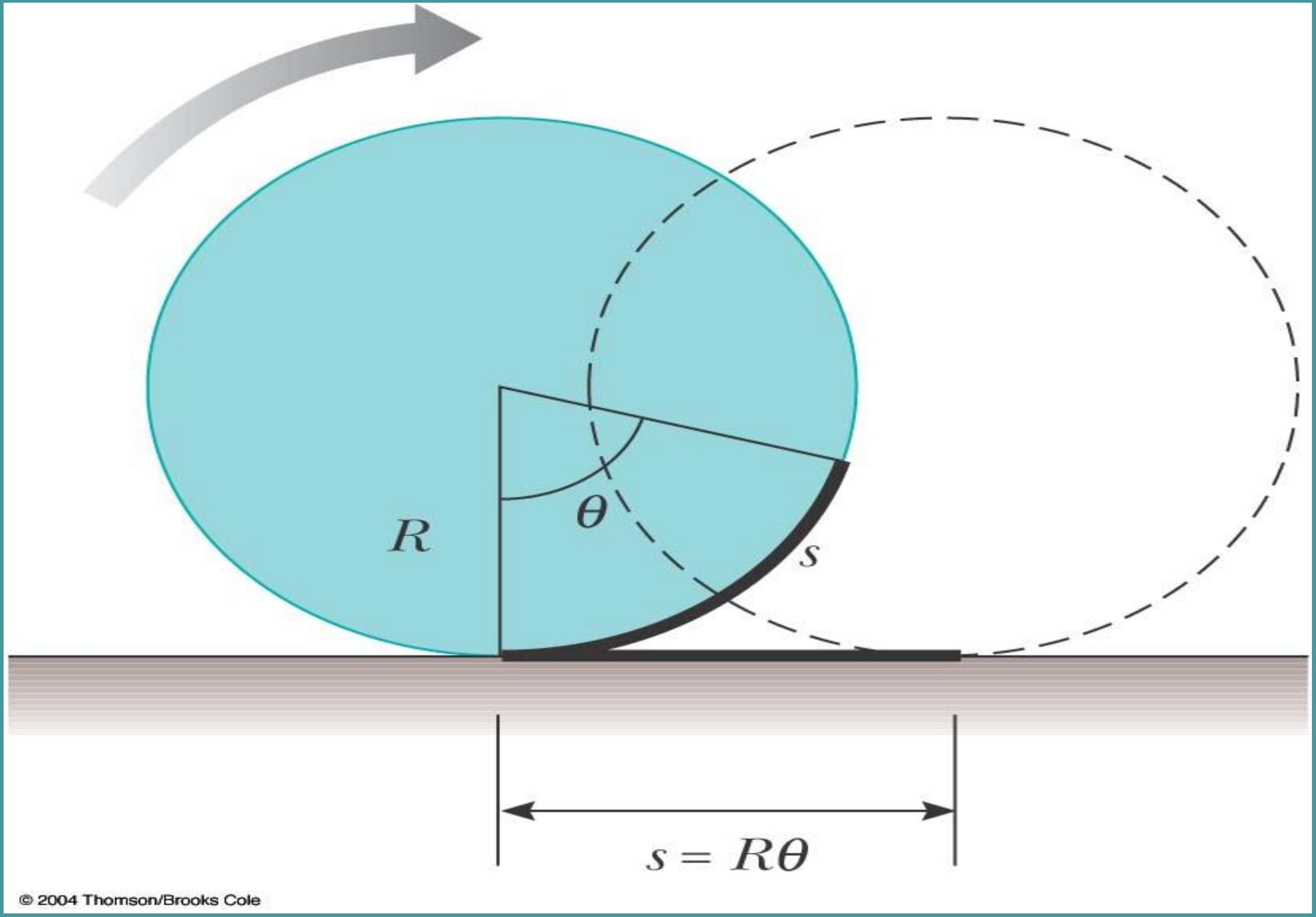
Rolling as Translation and Rotation Combined

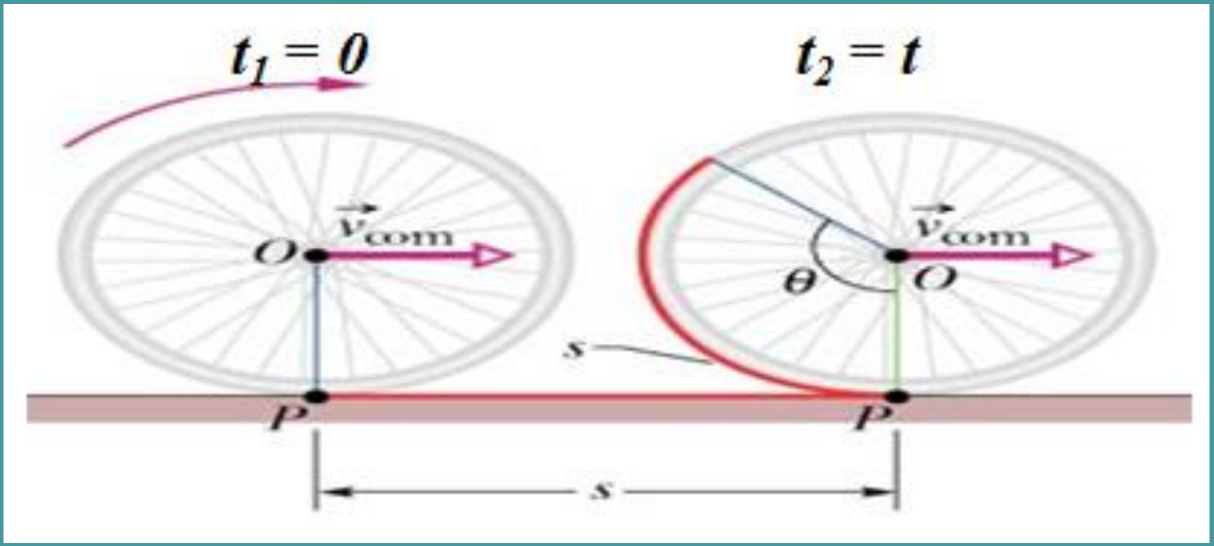
Consider an object with circular cross section that rolls along a surface without slipping.



We can simplify its study by treating it as a combination of translation of the center of mass and rotation of the object about the center of mass

ROLLING ≡ TRANSLATION + ROTATION





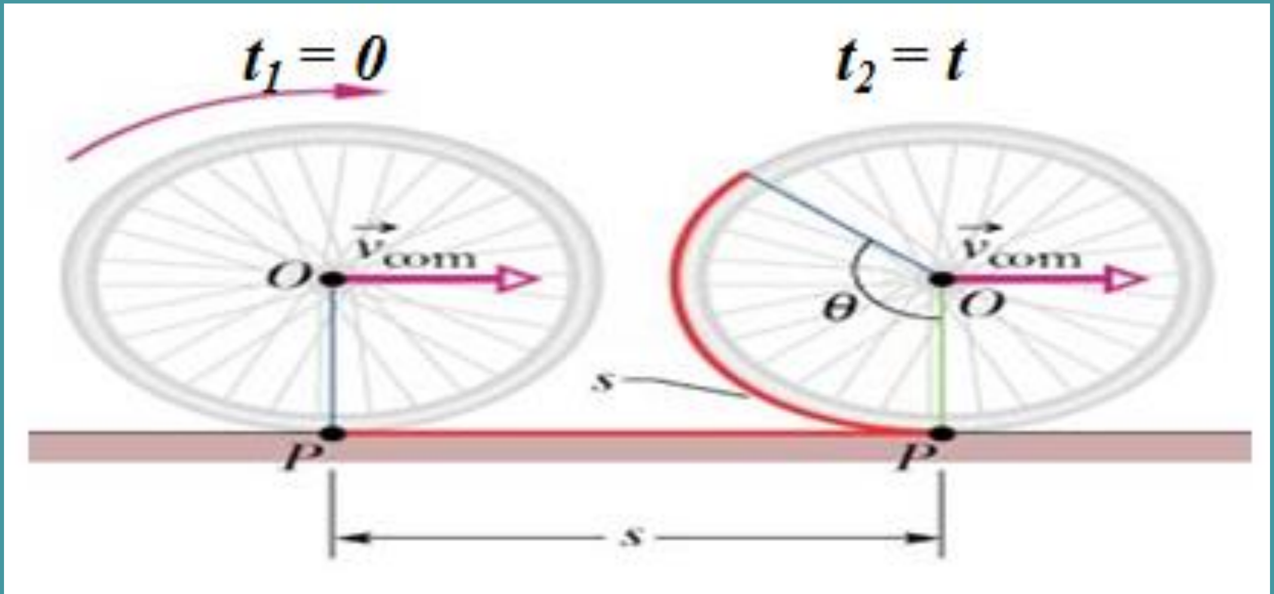
Consider the two snapshots of a rolling bicycle wheel shown in the figure.

An observer stationary with the ground will see the center of mass O of the wheel move forward with a speed v_{com}.

The point P at which the wheel makes contact with the road also moves with the same speed. During the time interval t between the two snapshots both O and P cover a distance s.

$$v_{com} = \frac{ds}{dt}$$

During t the bicycle rider sees the wheel rotate by an angle θ about O



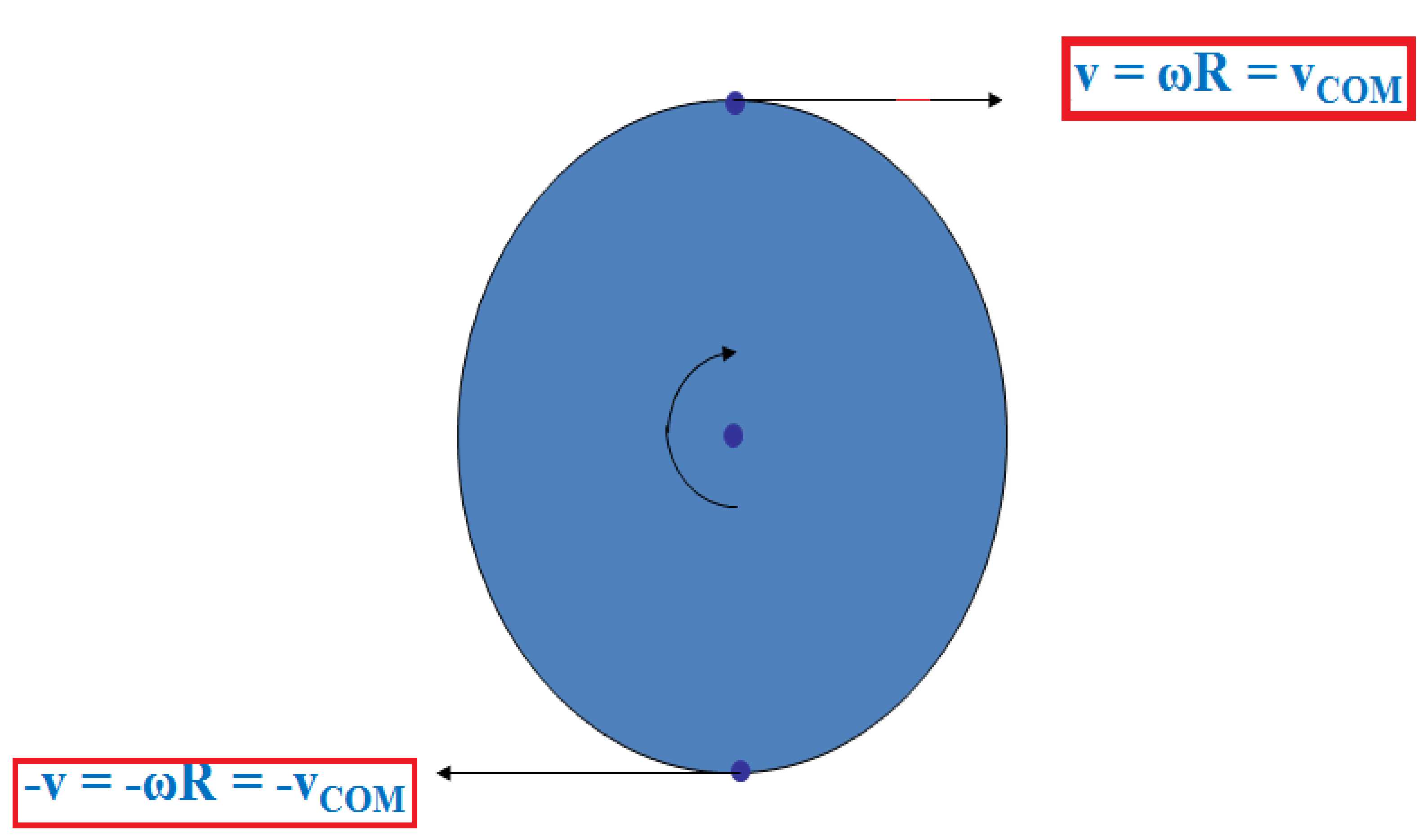
$$s = R\theta \rightarrow \frac{ds}{dt} = R\frac{d\theta}{dt} = R\,\omega$$

If we cambine we get the condition for rolling without slipping.

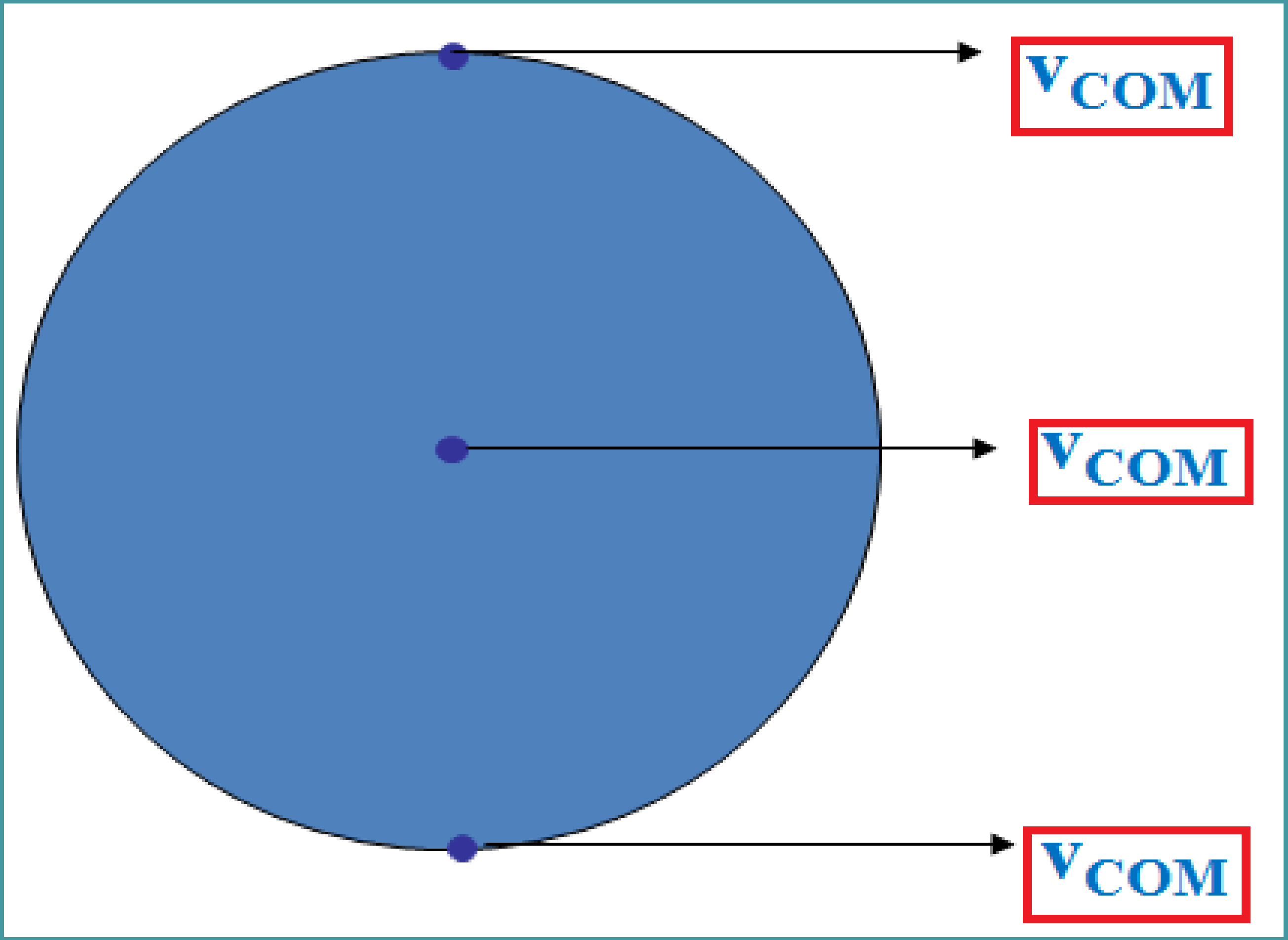
$$v_{com} = \frac{ds}{dt}$$

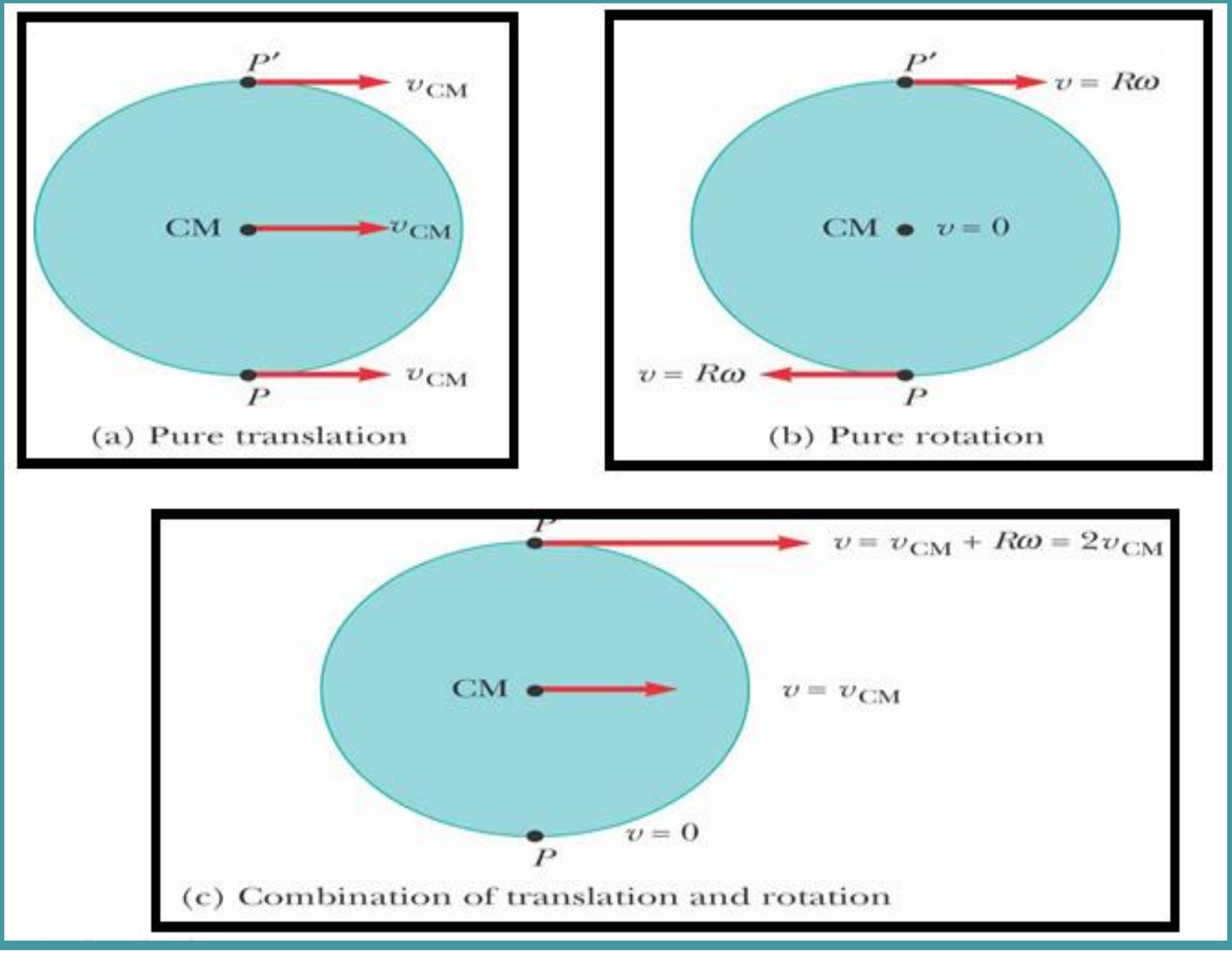
$$\boxed{v_{com} = R\omega}$$

Pure Rotational Motion



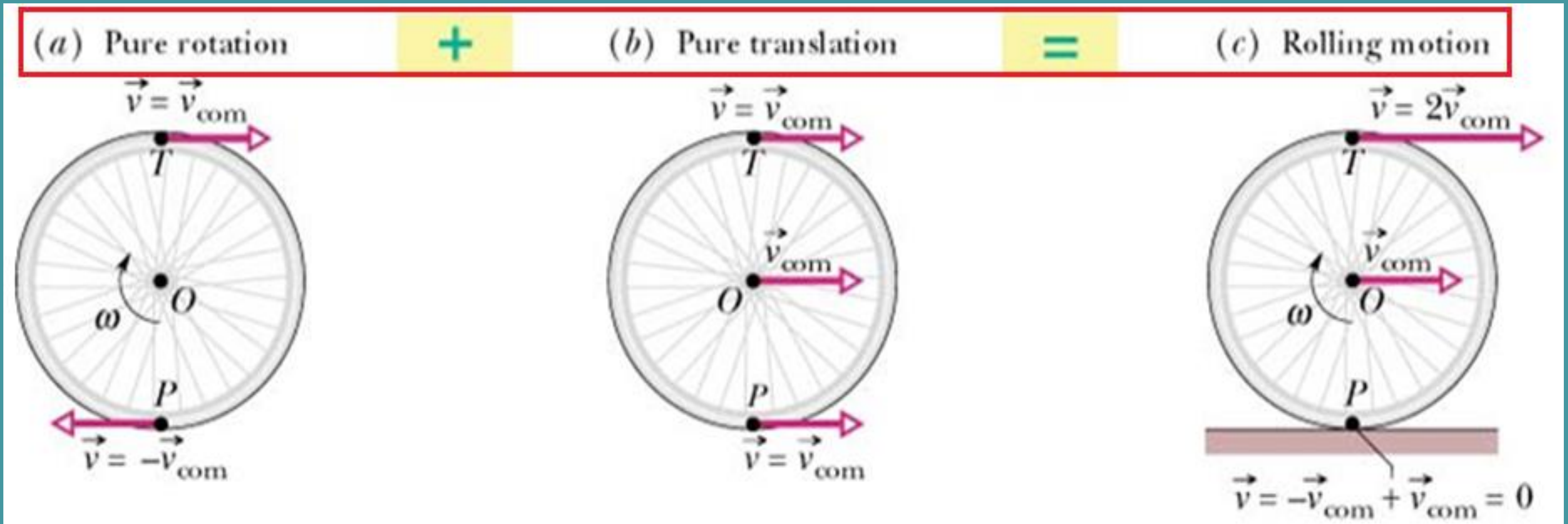
Pure Translational Motion





Rolling can be considered rotating about an axis through the ***com*** while the center of mass moves. At the bottom P is instantaneously at rest.

The wheel also moves slower at the bottom because pure rotation motion and pure translation partially cancel out.

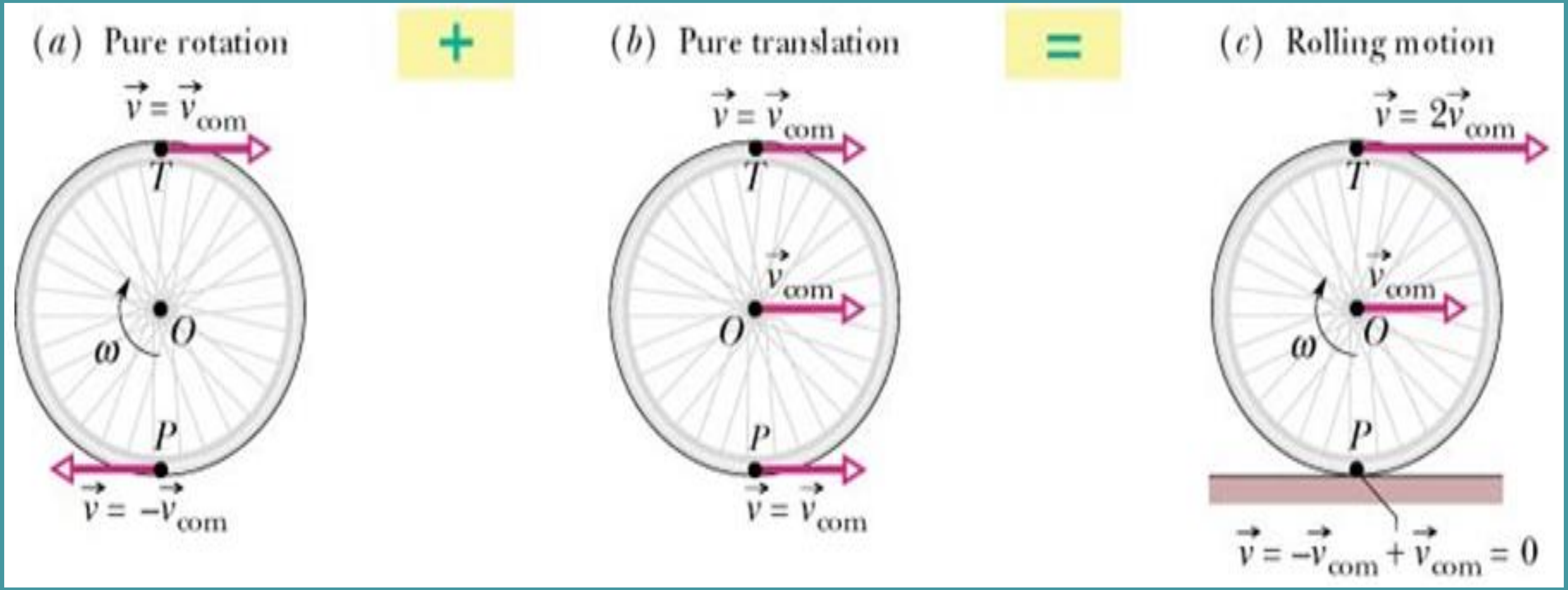


For the translational motion the velocity vector is the same for every point ($\vec{v}_{com}$, see fig.b).

The rotational velocity varies from point to point. Its magnitude is equal to $\boxed{\omega r}$ where r is the distance of the point from O. Its direction is tangent to the circular orbit (see fig.a).

The net velocity is the vector sum of these two terms.

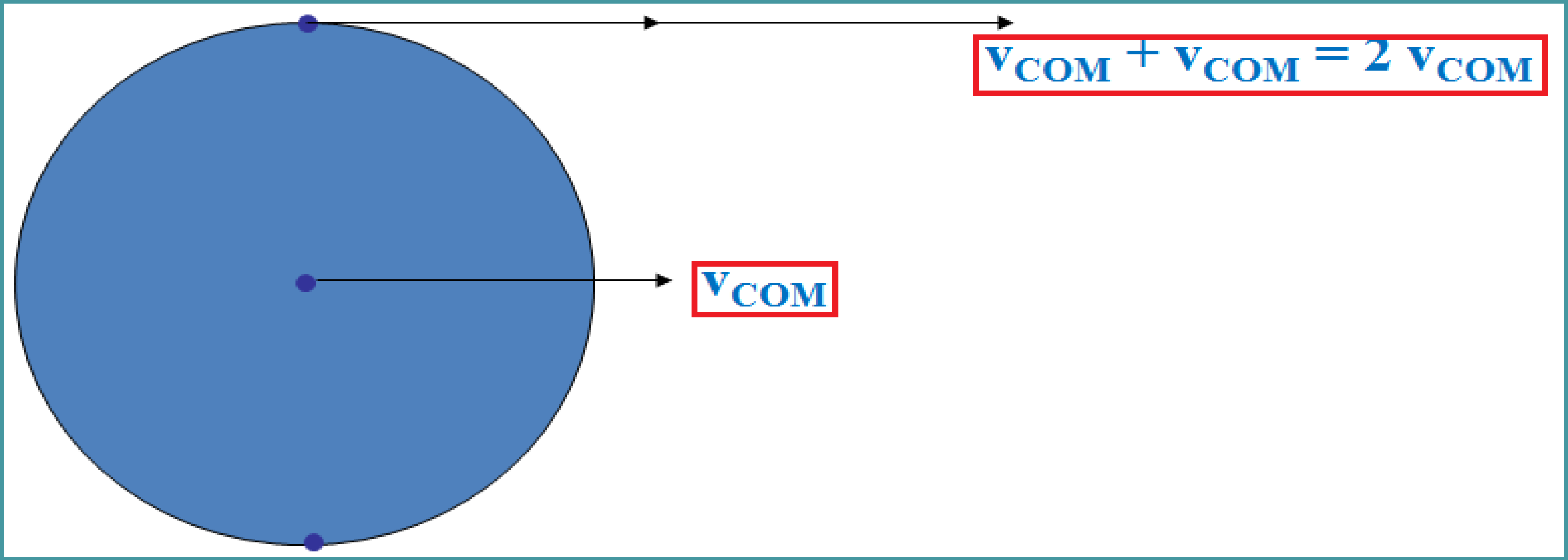
For example the velocity of point P is always zero. The velocity of the center of mass O is $\vec{v}_{com}$ $(r = 0)$. Finally the velocity of the top point T is wqual to $2\vec{v}_{com}$.

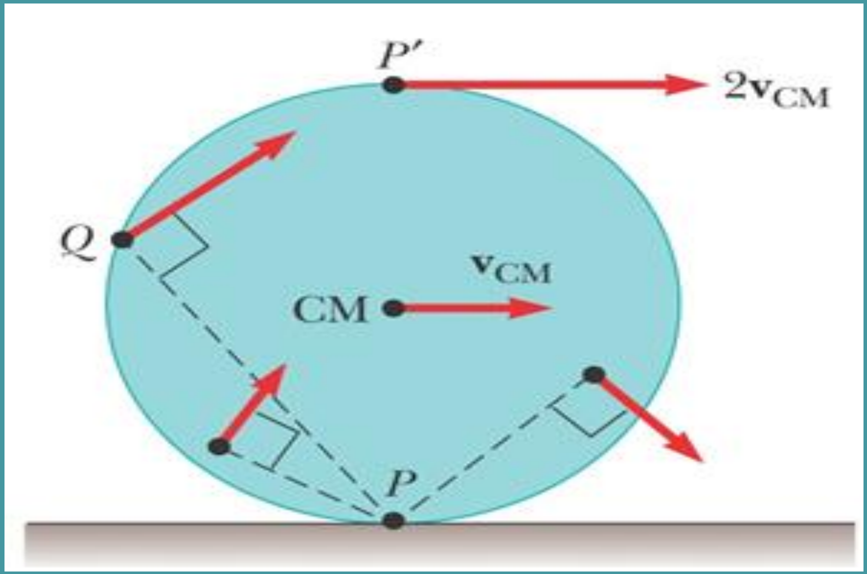


$$\boxed{v_{com} = R\omega}$$

We have seen that rolling is a combination of purely translational motion with speed v_{com} and a purely rotaional motion about the center of mass with angular velocity $\boxed{\omega = \frac{v_{com}}{R}}$. The velocity of each point is the vector sum of the velocities of the two motions.

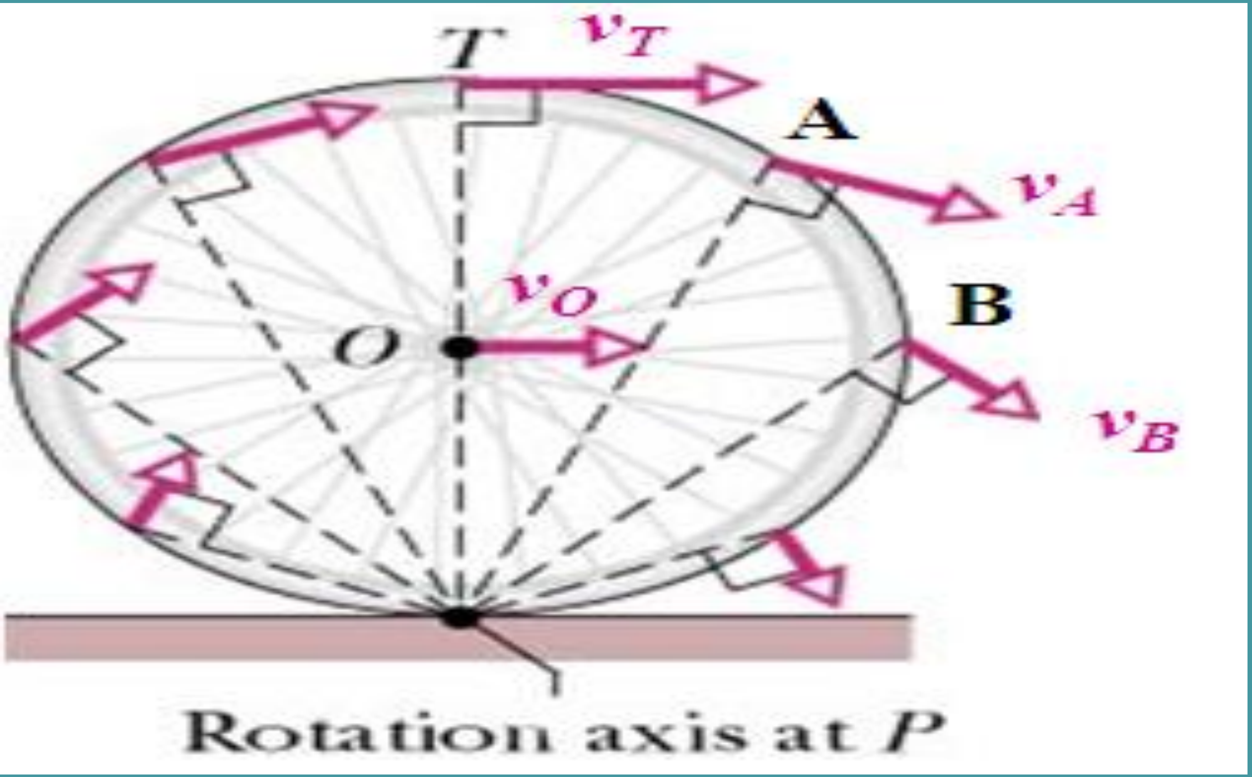
ROLLING



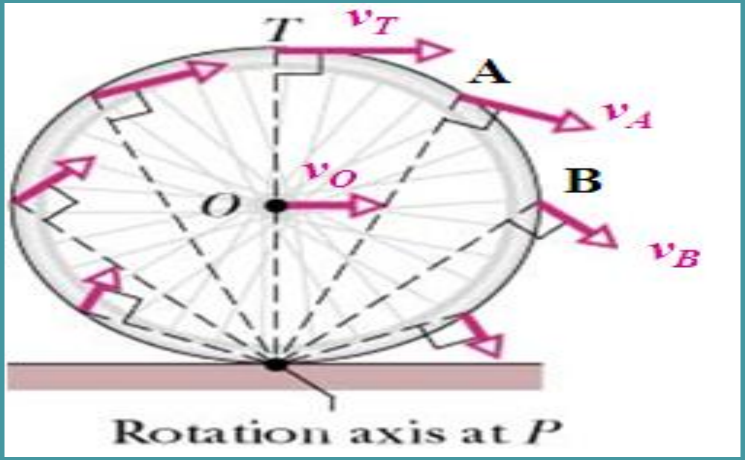


Rolling as Pure Rotation

Another way of looking at rolling is shown in the figure We consider rolling as a pure rotation about an axis of rotation that passes through the contact point P between the wheel and the road. The angular velocity of the rotation is $\boxed{\omega = \frac{v_{com}}{R}}$



In order to define the velocity vector for each point we must know its magnitude as well as its direction. The direction for each point on the wheel points along the tangent to its circular orbit.



For example at point A the velocity vector $\vec{v}_A$ is perpendicular to the dotted line that connects pont A with point B. The speed of each point is given by:

$$\boxed{v = \omega r}$$

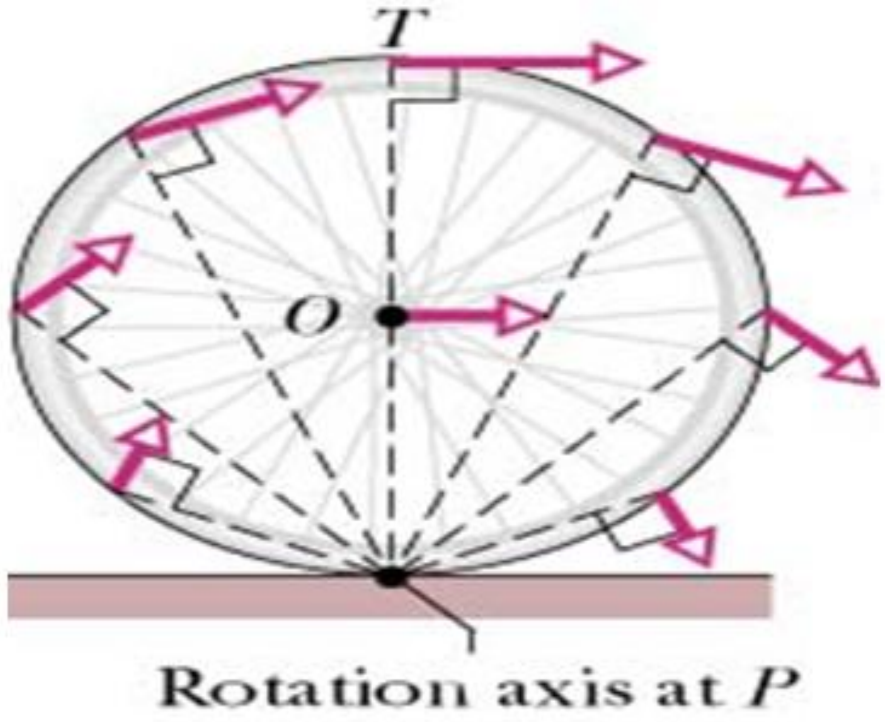
Here r is the distance between a particular point and the contact point P.
For example $\boxed{\text{at point T } r = 2R. \text{ Thus } v_T = 2R\omega = 2v_{com}}$.

$\boxed{\text{For point O } r = R \text{ thus } v_O = \omega R = v_{com}}$ $\boxed{\text{For point P } r = 0 \text{ thus } v_P = 0}$

The Kinetic Energy of Rolling

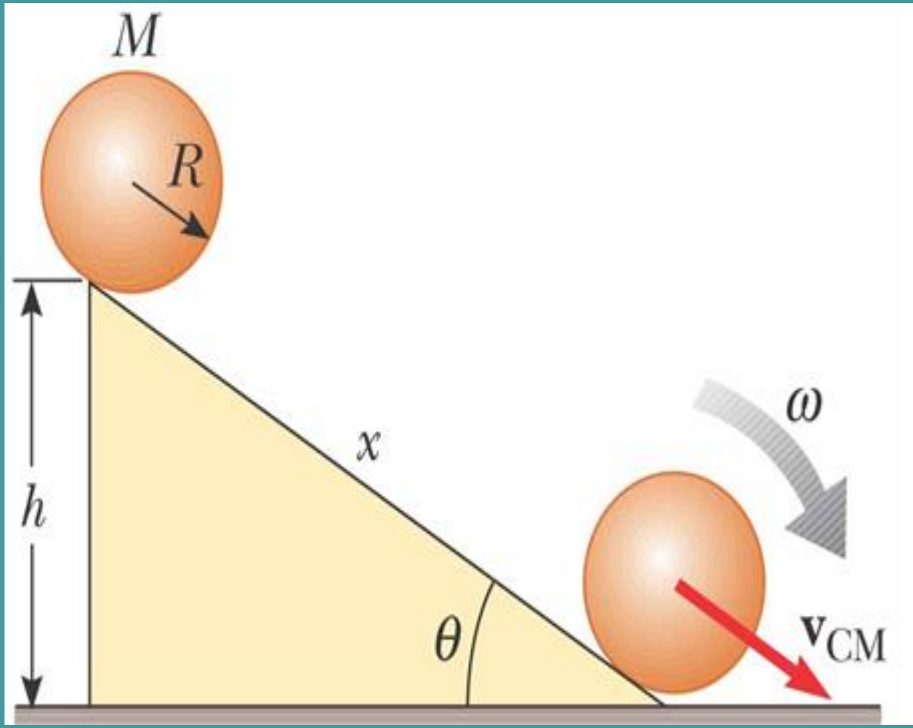
Consider the rolling object shown in the figure

It is easier to calculate the kinetic energy of the rolling body by considering the motion as pure rolling about the contact point P. The rolling object has mass *M* and radius *R*.



$$K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} M v_{com}^2$$

Accelerated Rolling Motion



$$K_i + U_i = K_f + U_f$$

$$Mgh = ½\, I_{KM}\, \omega^2 + ½\, Mv_{KM}{}^2$$

$$v_{km} = \omega R$$

$$K = \frac{1}{2} I_{KM} \left(\frac{v_{KM}}{R} \right)^2 + \frac{1}{2} M v_{KM}^2$$

$$K = \frac{1}{2} \left(\frac{I_{KM}}{R^2} + M \right) v_{KM}^2$$

$$\frac{1}{2} \left(\frac{I_{KM}}{R^2} + M \right) v_{KM}^2 = Mgh$$

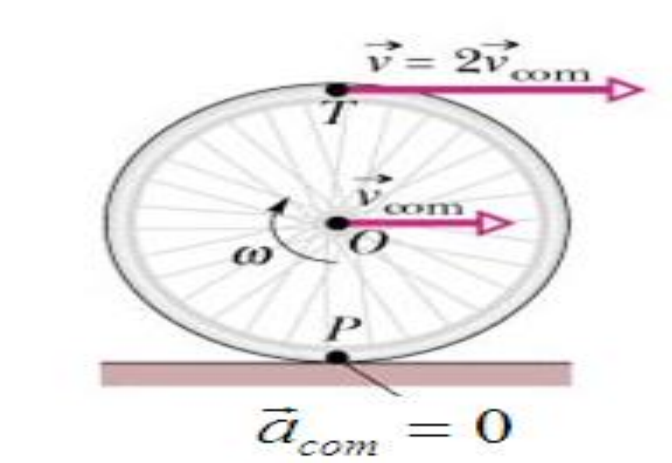
$$v_{KM} = \left(\frac{2gh}{1 + I_{KM} / MR^2} \right)^{1/2}$$

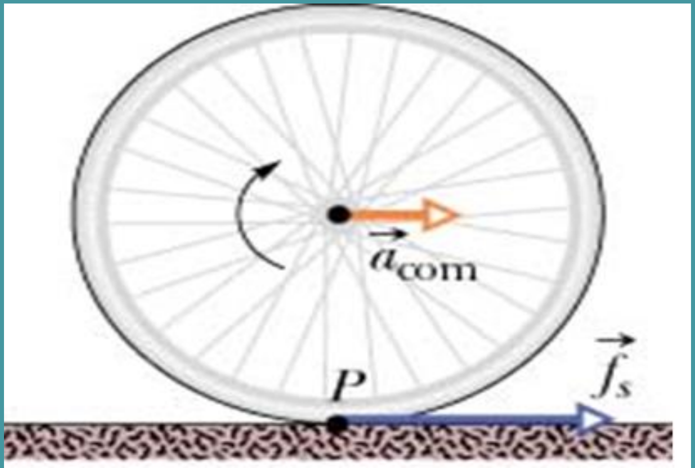
Rolling **is possible *when there is friction* between the surface and the rolling object.**

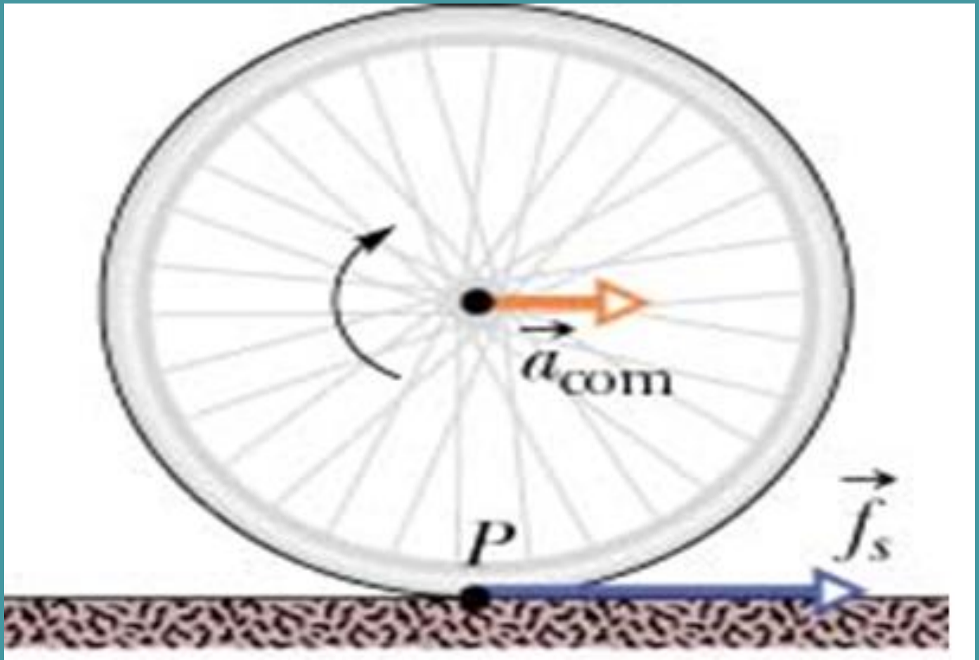
The frictional force **provides the torque to rotate the object.**

Friction and Rolling

When an object rolls with constant speed (see left figure) it has no tendency to slide at the contact point P and thus no frictional force acts there. If a net force acts on the rolling body it results in a non-zero acceleration $\vec{a}_{com}$ for the center of mass (see right figure). If the rolling object accelerates to the right it has the tendency to slide at point P to the left. Thus a static frictional force $\vec{f}_s$ opposes the tendency to slide. The motion is smooth rolling as long as $f_s < f_{s,\max}$







The rolling condition results in a connection beteen the magnitude of the acceleration a_{com} of the center of mass and its angular acceleration α

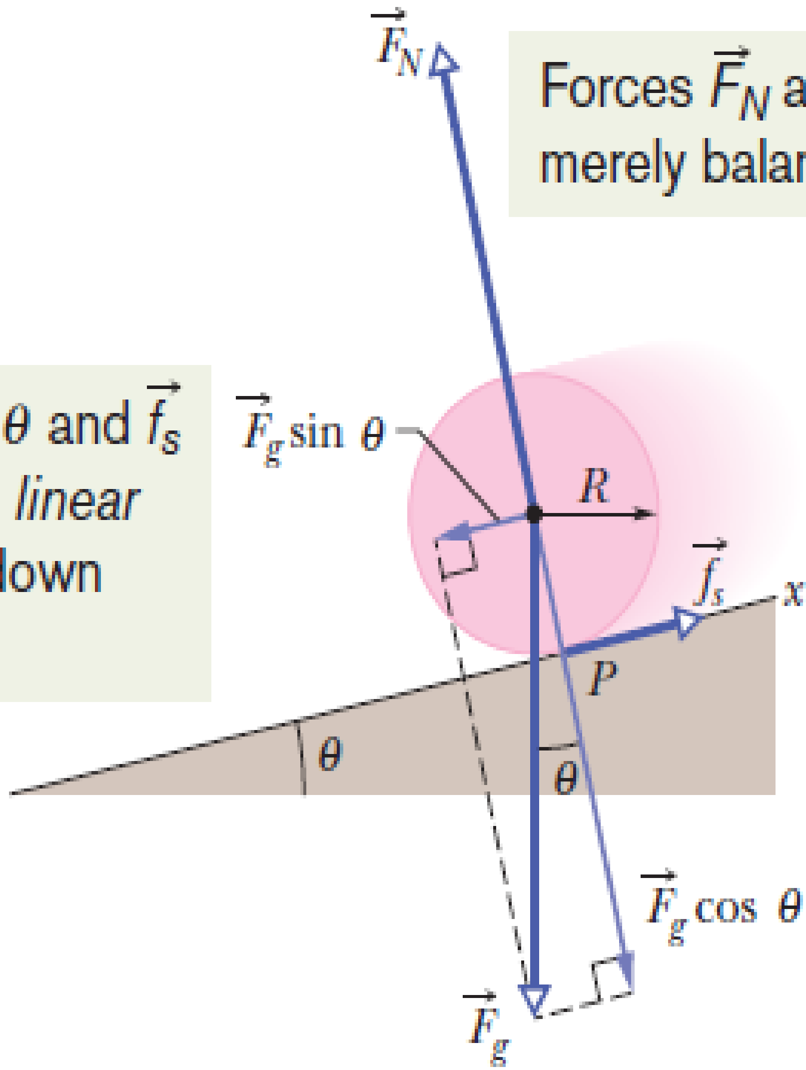
$v_{com} = \omega R$ We take time derivetives of both sides $\rightarrow$ $a_{com} = \frac{dv_{com}}{dt} = R\frac{d\omega}{dt} = R\alpha$

$$\boxed{a_{com} = R\alpha}$$

Rolling Down a Ramp

Forces $\vec{F}_N$ and $\vec{F}_g \cos\theta$ merely balance.

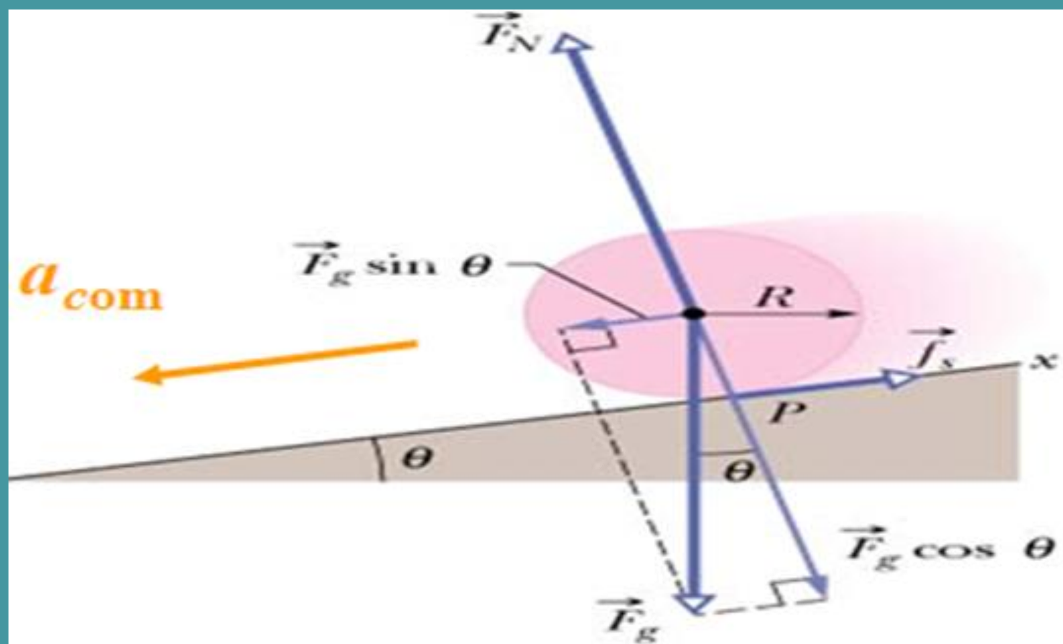
Forces $\vec{F}_g \sin\theta$ and $\vec{f}_s$ determine the *linear* acceleration down the ramp.



The torque due to $\vec{f}_s$ determines the *angular* acceleration around the com.

Rolling Down a Ramp

Consider a round uniform body of mass M and radius R rolling down an inclined plane of angle θ. We will calculate the acceleration a_{com} of the center of mass along the x-axis using Newton's second law for the translational and rotational motion



A **static frictional force** acts at the point of contact *P and is* ***directed up*** the ramp. Because if the body were to slide at *P*, ***it would slide down*** the ramp. The frictional force opposing the sliding must be ***up the ramp.***

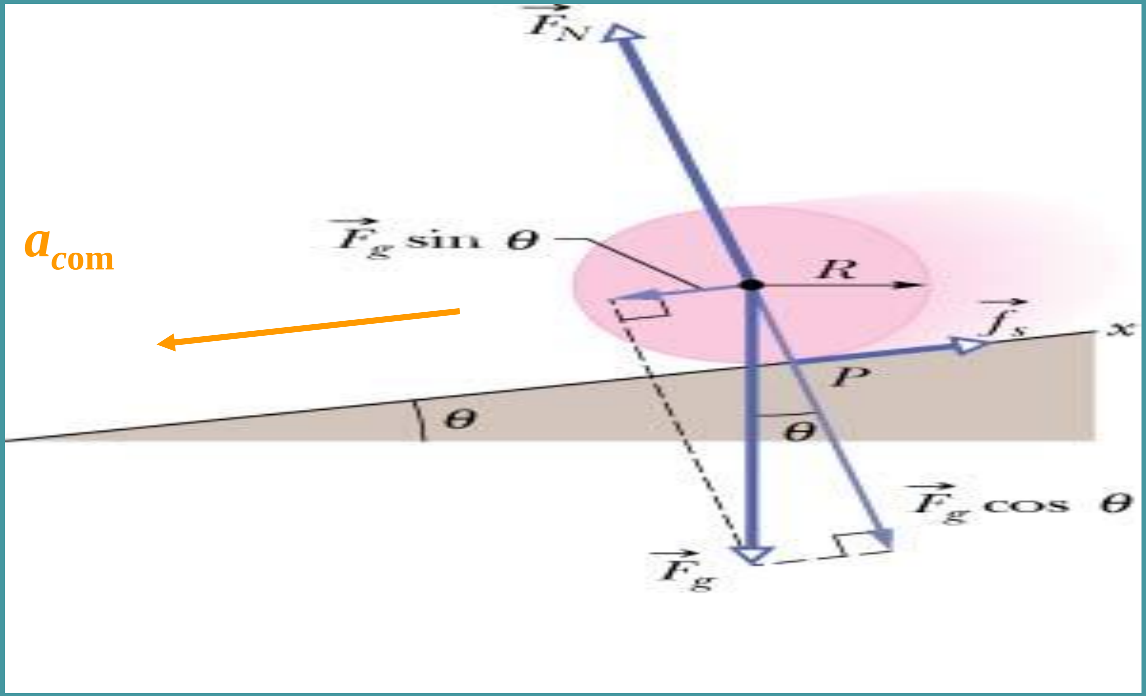
Newton's second law for motion along the x-axis: $\boxed{f_s - Mg\sin\theta = Ma_{com}}$ (eqs.1)

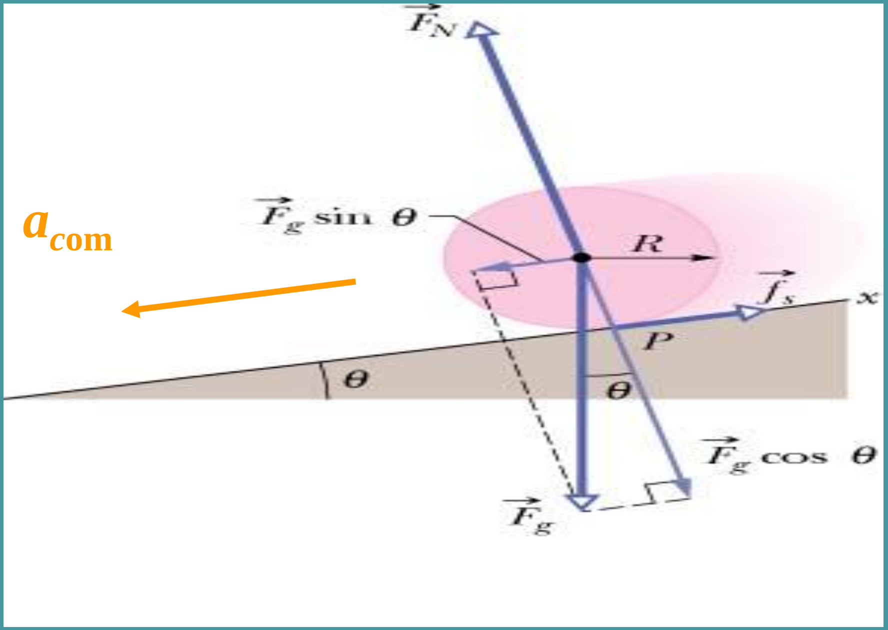
Newton's second law for rotation about the center of mass: $\boxed{\tau = Rf_s = I_{com}\alpha}$

$\alpha = -\dfrac{a_{com}}{R}$ We substitute α in the second equation and get: $Rf_s = -I_{com}\dfrac{a_{com}}{R} \rightarrow$

$f_s = -I_{com}\dfrac{a_{com}}{R^2}$ (eqs.2) We substitute f_s from equation 2 into equation 1 $\rightarrow$

$$\boxed{-I_{com}\frac{a_{com}}{R^2} - Mg\sin\theta = Ma_{com}}$$





$$-I_{com}\frac{a_{com}}{R^2} - Mg\sin\theta = Ma_{com}$$

$$|a_{com}| = \frac{g\sin\theta}{1+\dfrac{I_{com}}{MR^2}}$$

Cylinder

$$I_1 = \frac{MR^2}{2}$$

$$a_1 = \frac{g\sin\theta}{1 + I_1 / MR^2}$$

$$a_1 = \frac{g\sin\theta}{1 + MR^2 / 2MR^2}$$

$$a_1 = \frac{g\sin\theta}{1 + 1/2}$$

$$a_1 = \frac{2g\sin\theta}{3} = \underline{(0.67)g\sin\theta}$$

Hoop

$$I_2 = MR^2$$

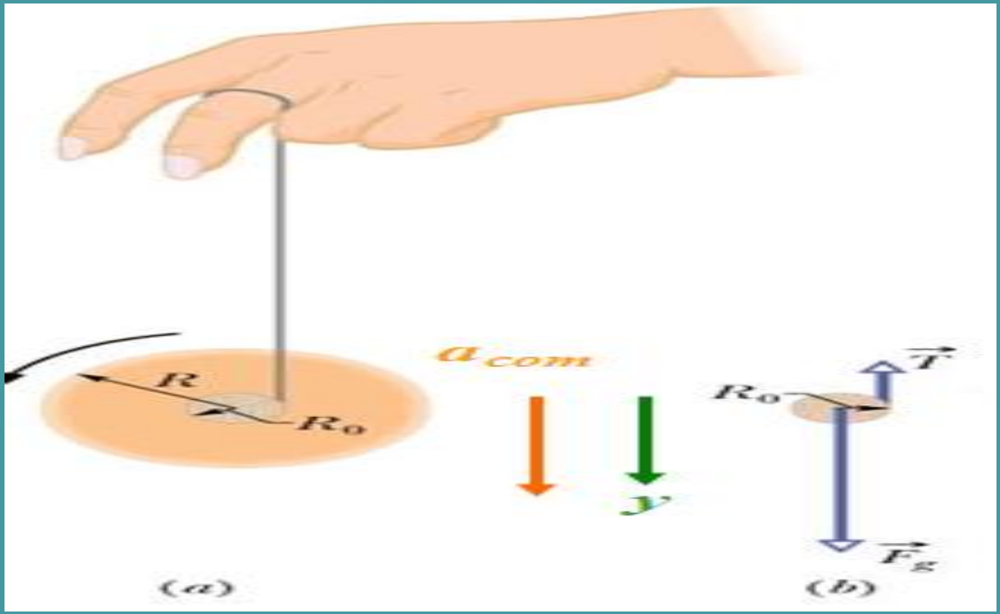
$$a_2 = \frac{g\sin\theta}{1 + I_2 / MR^2}$$

$$a_2 = \frac{g\sin\theta}{1 + MR^2 / MR^2}$$

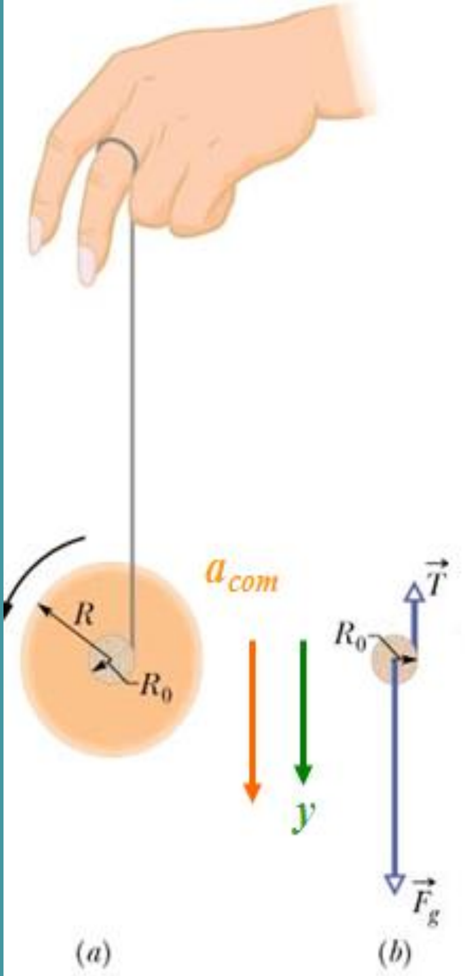
$$a_2 = \frac{g\sin\theta}{1 + 1}$$

$$a_2 = \frac{g\sin\theta}{2} = \underline{(0.5)g\sin\theta}$$

YO - YO



1. Instead of rolling down a ramp at angle θ with the horizontal, the yo-yo rolls down a string at angle $\theta = 90°$ with the horizontal.
2. Instead of rolling on its outer surface at radius R, the yo-yo rolls on an axle of radius R_0 (Fig. 11-9*a*).
3. Instead of being slowed by frictional force $\vec{f}_s$, the yo-yo is slowed by the force $\vec{T}$ on it from the string (Fig. 11-9*b*).



The Yo-Yo

Consider a yo-yo of mass M, radius R, and axle radius R_o rolling down a string. We will calculate the acceleration a_{com} of the center of its mass along the y-axis using Newton's second law for the tralational and rotational motion as we did in the previous problem

Newton's second law for motion along the y-axis:

$\boxed{Mg - T = Ma_{com}}$ (eqs.1)

Newton's second law for rotation about the center of mass:

$\boxed{\tau = R_o T = I_{com}\alpha}$ Angular acceleration $\boxed{\alpha = \frac{a_{com}}{R_o}}$.

We substitute α in the second equation and get:

$T = I_{com}\frac{a_{com}}{R_o^2}$ (eqs.2) We substitute T from equation 2 into equation 1 $\rightarrow$

$$Mg - I_{com}\frac{a_{com}}{R_o^2} = Ma_{com} \rightarrow \quad a_{com} = \frac{g}{1 + \frac{I_{com}}{MR_o^2}}$$

Torque Revisited

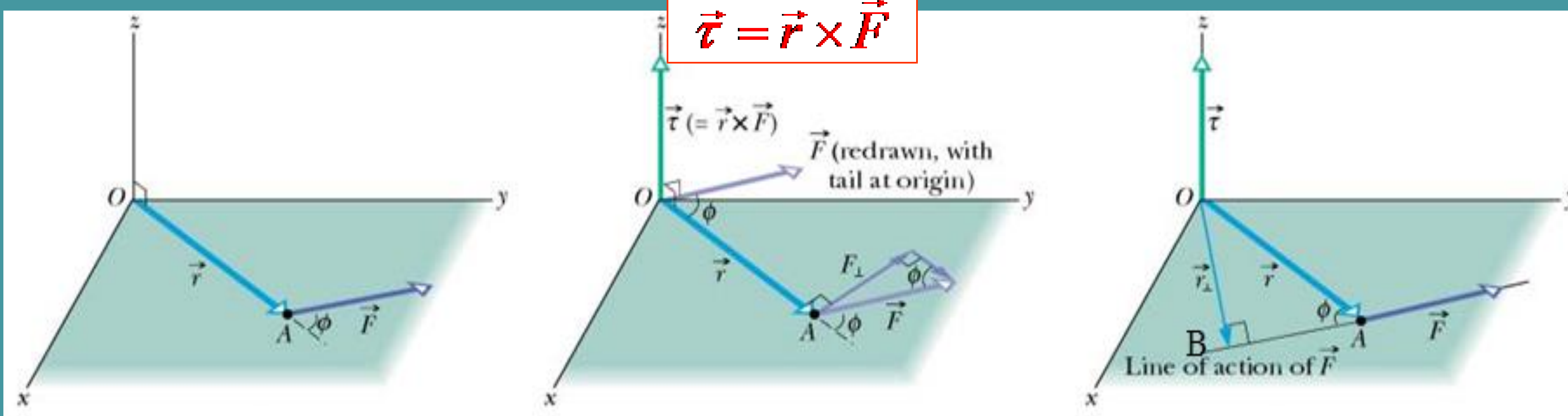
In chapter 10 we defined the torque τ of a rigid body rotating about a fixed axis with each particle in the body moving on a circular path. We now expand the definition of torque so that it can describe the motion of a particle than moves along any path relative to a fixed point. If $\vec{r}$ is the position vector of a particle on which a force $\vec{F}$ is acting, the torque $\vec{\tau}$ is defined as: $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$

In the example shown in the figure both $\vec{r}$ and $\vec{F}$ lie in the *xy*-plane. Using the right hand rule we can see that the direction of $\vec{\tau}$ is along the *z*-axis.

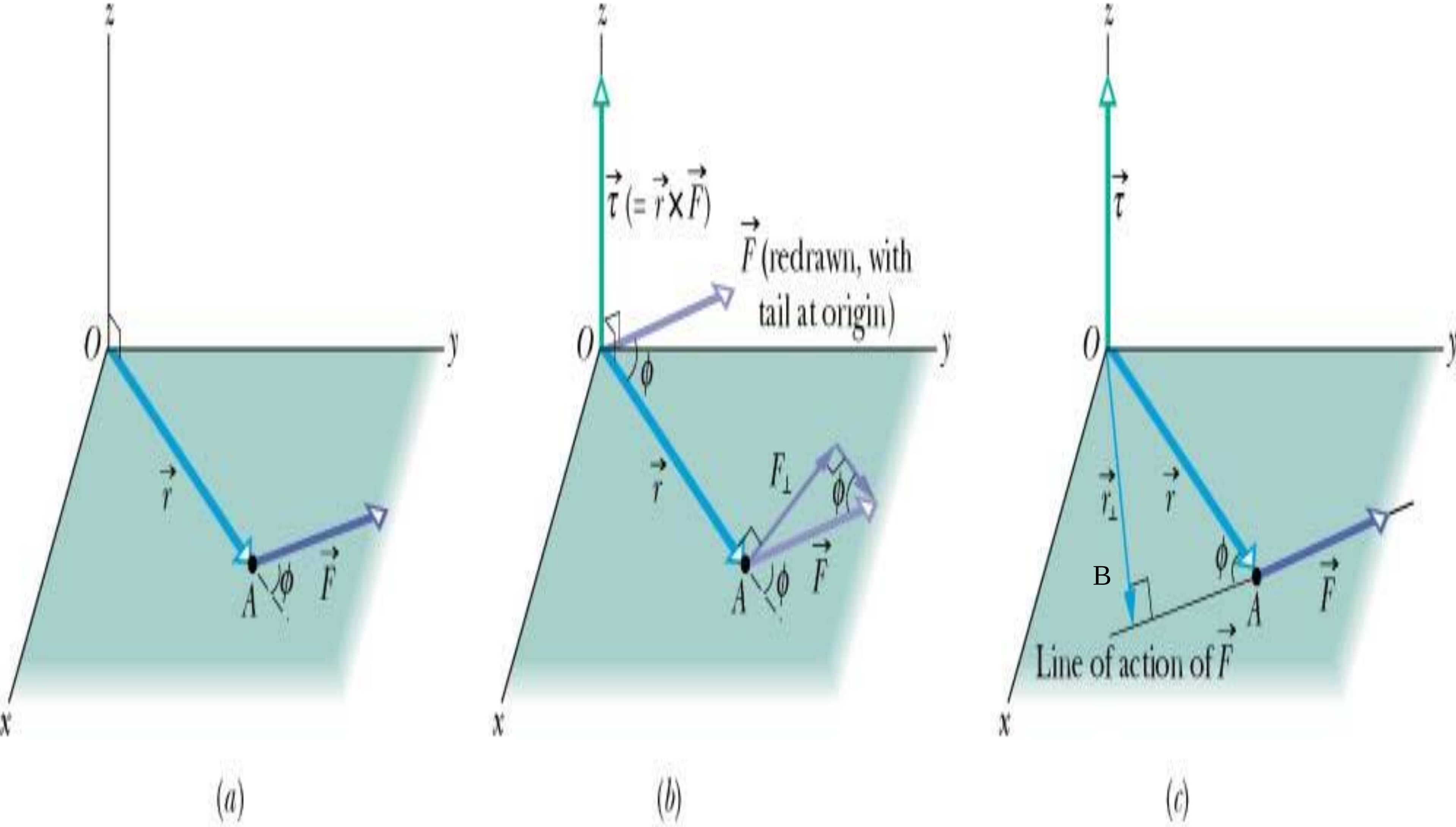
The magnitude of the torque vector $\tau = rF\sin\phi$, where ϕ is the angle between $\vec{r}$ and $\vec{F}$. From triangle OAB we have: $r\sin\phi = r_{\perp} \rightarrow$

$\boxed{\tau = r_{\perp}F}$, in agreement with the definition of chapter 10.

$$\vec{\tau} = \vec{r} \times \vec{F}$$



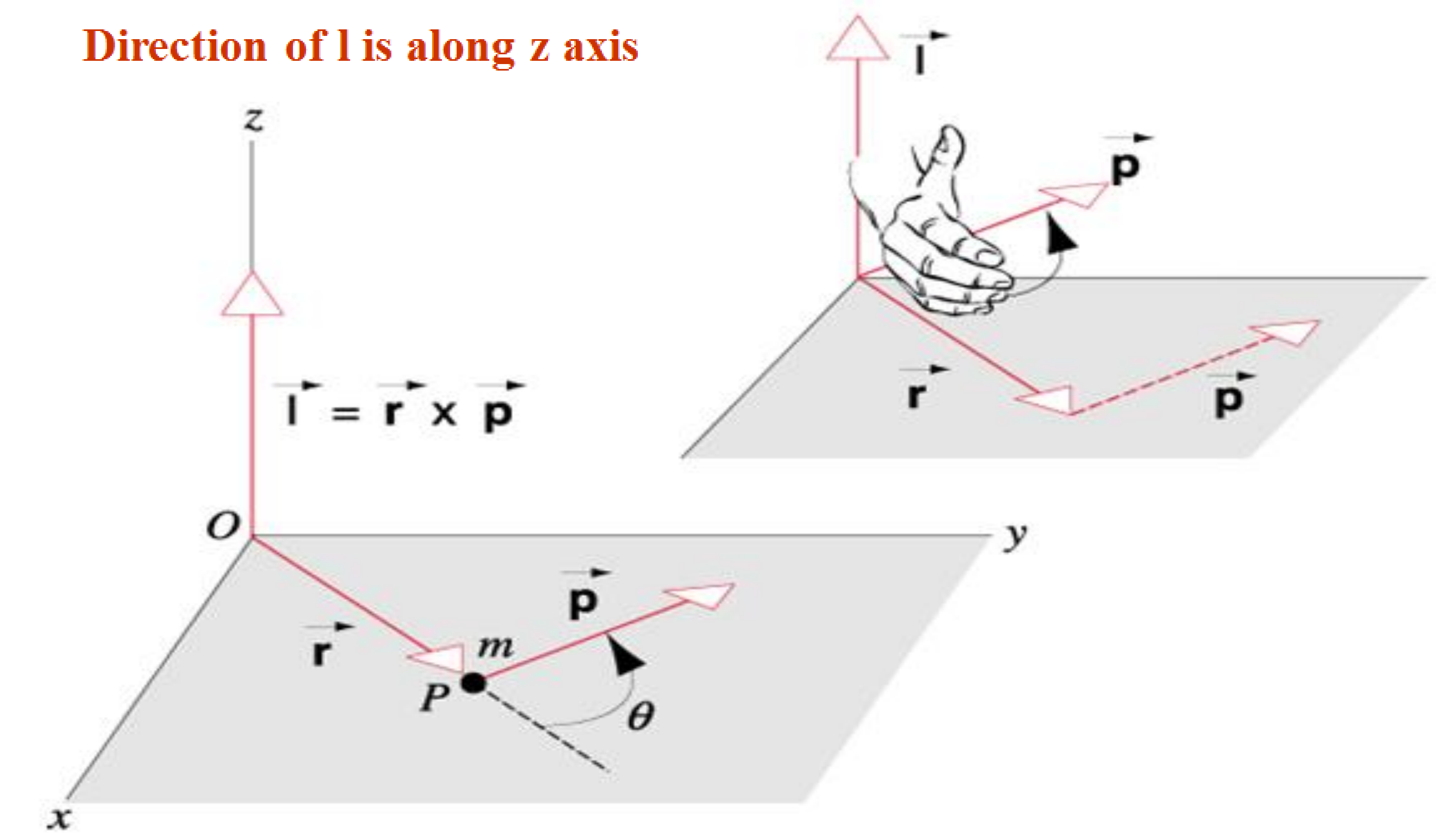
(a) (b) (c)



$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

Angular Momentum

For a single particle→ $\vec{\ell} = \vec{r} \times \vec{p}$



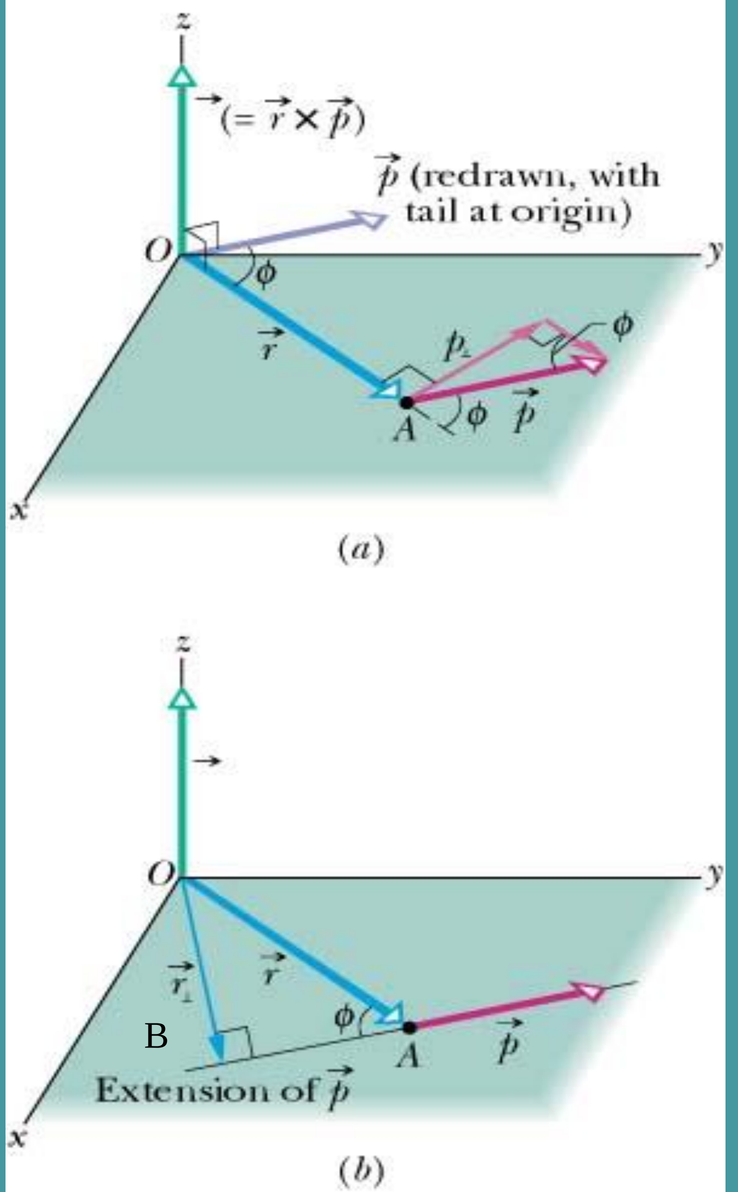
Angular Momentum

The counterpart of linear momentum $\vec{p} = m\vec{v}$ in rotational motion is a new vector known as angular momentum.

The new vector is defined as follows: $\boxed{\vec{\ell} = \vec{r} \times \vec{p}}$

In the example shown in the figure both $\vec{r}$ and $\vec{p}$ lie in the xy-plane. Using the right hand rule we can see that the direction of $\vec{\ell}$ is along the z-axis.

The magnitude of angular momentum $\ell = rmv\sin\phi$, where ϕ is the angle between $\vec{r}$ and $\vec{p}$. From triangle OAB we have: $r\sin\phi = r_\perp \rightarrow \boxed{\ell = r_\perp mv}$



Note: Angular momentum depends on the choice of the origin O. If the origin is shifted in general we get a different value of $\vec{\ell}$

SI unit for angular momentum: $\mathrm{kg.m^2/s}$ Sometimes the equivalent J.s is used

$$\boxed{\vec{\ell} = \vec{r} \times \vec{p} = m\left(\vec{r} \times \vec{v}\right)}$$

$$\boxed{\ell = r_{\perp}\mathrm{mv}}$$

Newton's Second Law in Angular Form

Newton's second law for linear motion has the form: $\boxed{\vec{F}_{net} = \frac{d\vec{p}}{dt}}$ Below we will derive the angular form of Newton's second law for a particle.

$$\boxed{\vec{\ell} = m\left(\vec{r}\times\vec{v}\right)} \rightarrow \boxed{\frac{d\vec{\ell}}{dt} = m\frac{d}{dt}\left(\vec{r}\times\vec{v}\right)} = \boxed{m\left(\vec{r}\times\frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt}\times\vec{v}\right)} = \boxed{m\left(\vec{r}\times\vec{a} + \vec{v}\times\vec{v}\right)}$$

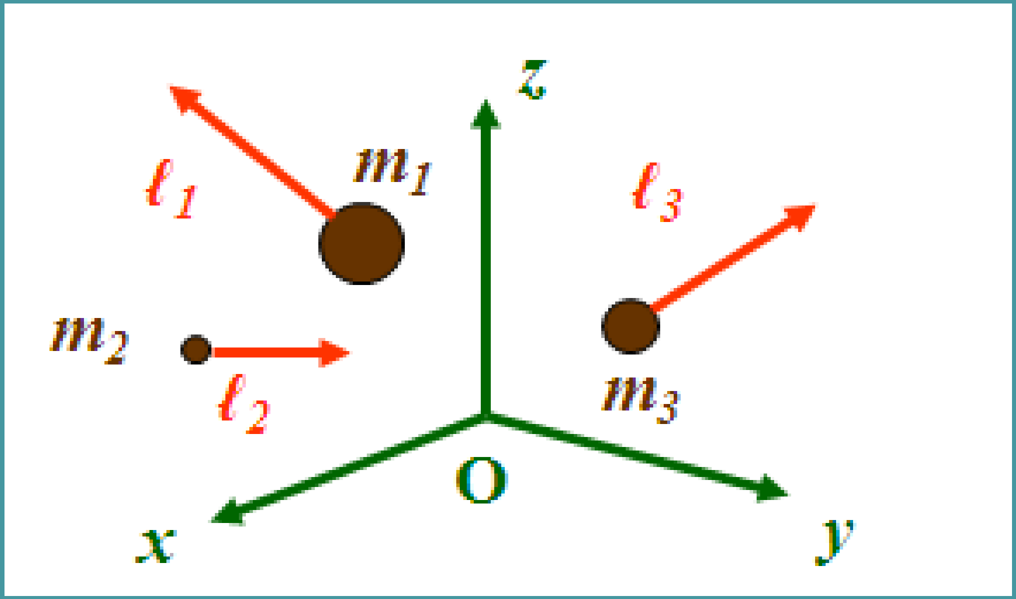
$$\boxed{\vec{v}\times\vec{v} = 0} \rightarrow \boxed{\frac{d\vec{\ell}}{dt} = m\left(\vec{r}\times\vec{a}\right)} = \boxed{\left(\vec{r}\times m\vec{a}\right)} = \boxed{\left(\vec{r}\times\vec{F}_{net}\right)} = \boxed{\vec{\tau}_{net}}$$

Thus: $\boxed{\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}}$ Compare with: $\boxed{\vec{F}_{net} = \frac{d\vec{p}}{dt}}$

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}}$$

The Angular Momentum of a System of Particles

We will now explore Newton's second law in angular form for a system of n particles that have angular momentum $\vec{\ell}_1, \vec{\ell}_2, \vec{\ell}_3, \ldots, \vec{\ell}_n$



The angular momentum $\vec{L}$ of the system is $\boxed{\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \ldots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i}$

The time derivative of the angular momentum is $\dfrac{d\vec{L}}{dt} = \sum_{i=1}^{n} \dfrac{d\vec{\ell}_i}{dt}$

The time derivative for the angular momentum of the i-th particle $\dfrac{d\vec{\ell}_i}{dt} = \vec{\tau}_{net,i}$

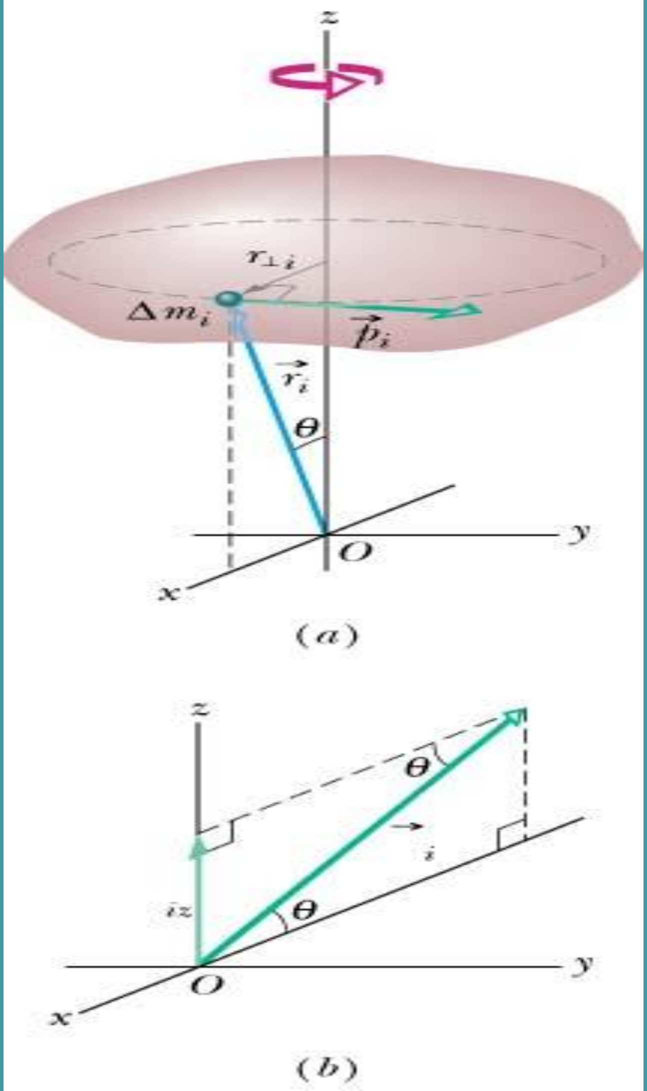
Where $\vec{\tau}_{net,i}$ is the net torque on the particle. This torque has contributions from external as well as internal forces between the particles of the system. Thus

$\boxed{\dfrac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{\tau}_{net,i} = \vec{\tau}_{net}}$ Here $\vec{\tau}_{net}$ is the net torque due to all the external forces.

By virtue of Newton's third law the vector sum of all internal torques is zero.

Thus Newton's second law for a system in angular form takes the form:

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{net}}$$



Angular Momentum of a Rigid Body Rotating about a Fixed Axis

We take the z-axis to be the fixed rotation axis. We will determine the z-component of the net angular momentum. The body is divided n elements of mass Δm_i that have a position vector $\vec{r}_i$

The angular momentum $\vec{\ell}_i$ of the i-the element is: $\boxed{\vec{\ell}_i = \vec{r}_i \times \vec{p}_i}$

Its magnitude $\boxed{\ell_i = r_i p_i \left(\sin 90°\right) = r_i \Delta m_i v_i}$ The z-compoment ℓ_{iz} of ℓ_i is: $\boxed{\ell_{iz} = \ell_i \sin\theta = \left(r_i \sin\theta\right)\left(\Delta m_i v_i\right) = r_{\perp i} \Delta m_i v_i}$

The z-component of the angular momentum L_z is the sum:

$$\boxed{L_z = \sum_{i=1}^{n} \ell_{iz}} \boxed{= \sum_{i=1}^{n} r_{\perp i} \Delta m_i v_i} \boxed{= \sum_{i=1}^{n} r_{\perp i} \Delta m_i \left(\omega r_{i\perp}\right)} \boxed{= \omega\left(\sum_{i=1}^{n} \Delta m_i r_{i\perp}^2\right)}$$

The sum $\sum_{i=1}^{n} \Delta m_i r_{i\perp}^2$ is the rotational inertia I of the rigid body

Thus: $\boxed{L_z = I\omega}$

$$\boxed{L_z = I\omega}$$

Conservation of Angular momentum

For any system of particles (including a rigid body) Newton's second law in angular form is: $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}_{net}}$

If the net external torque $\boxed{\vec{\tau}_{net} = 0}$ then we have: $\boxed{\frac{d\vec{L}}{dt} = 0} \rightarrow$

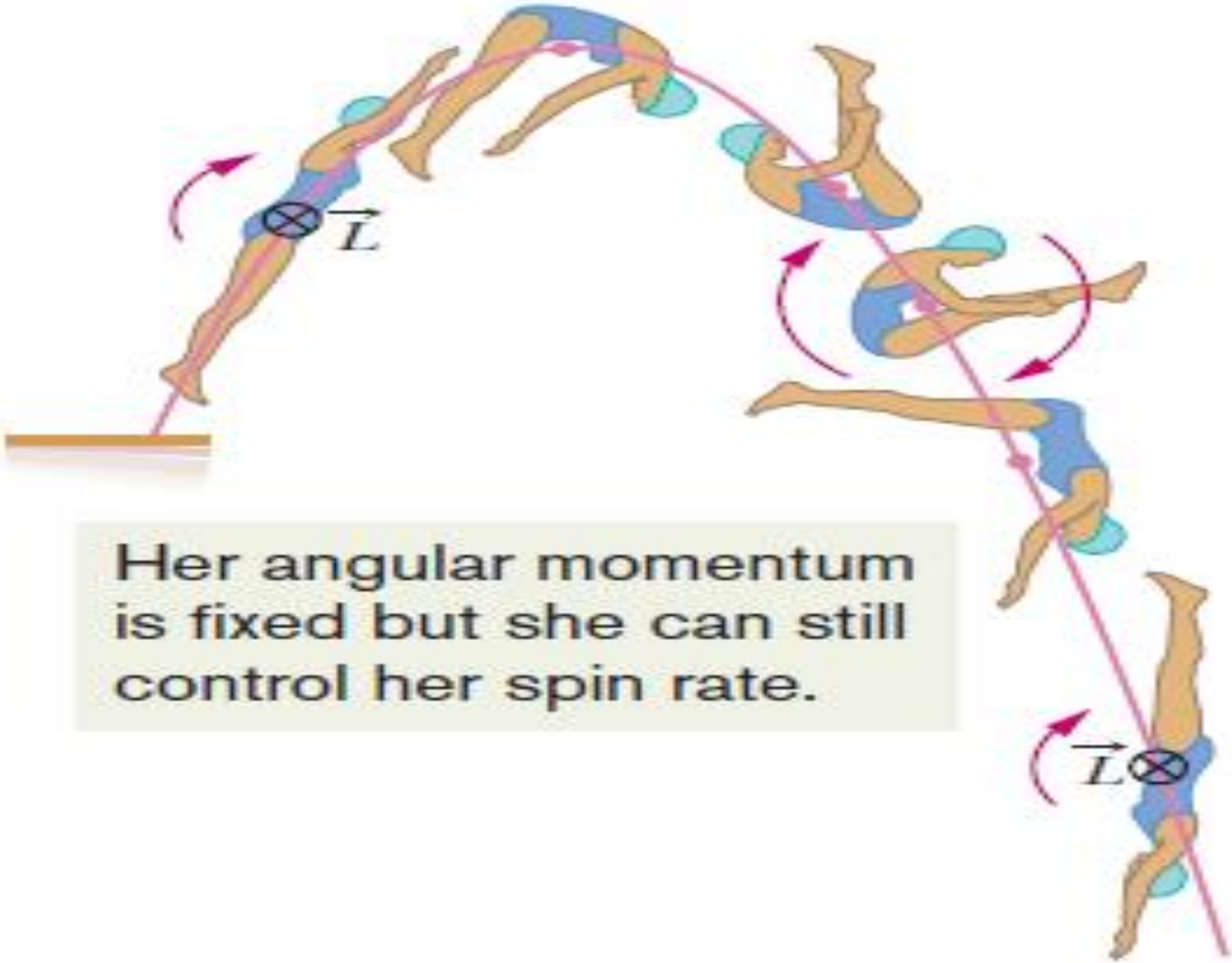
$\boxed{\vec{L} = \text{ a constant}}$ This result is known as the law of the conservation of angular momentum. In words:

$$\boxed{\begin{pmatrix}\text{Net angular momentum} \\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{Net angular momentum} \\ \text{at some later time } t_f\end{pmatrix}}$$

In equation form:

Note: If the component of the external torque along a certain axis is equal to zero, then the componet of the angular momentum of the system along this axis cannot change

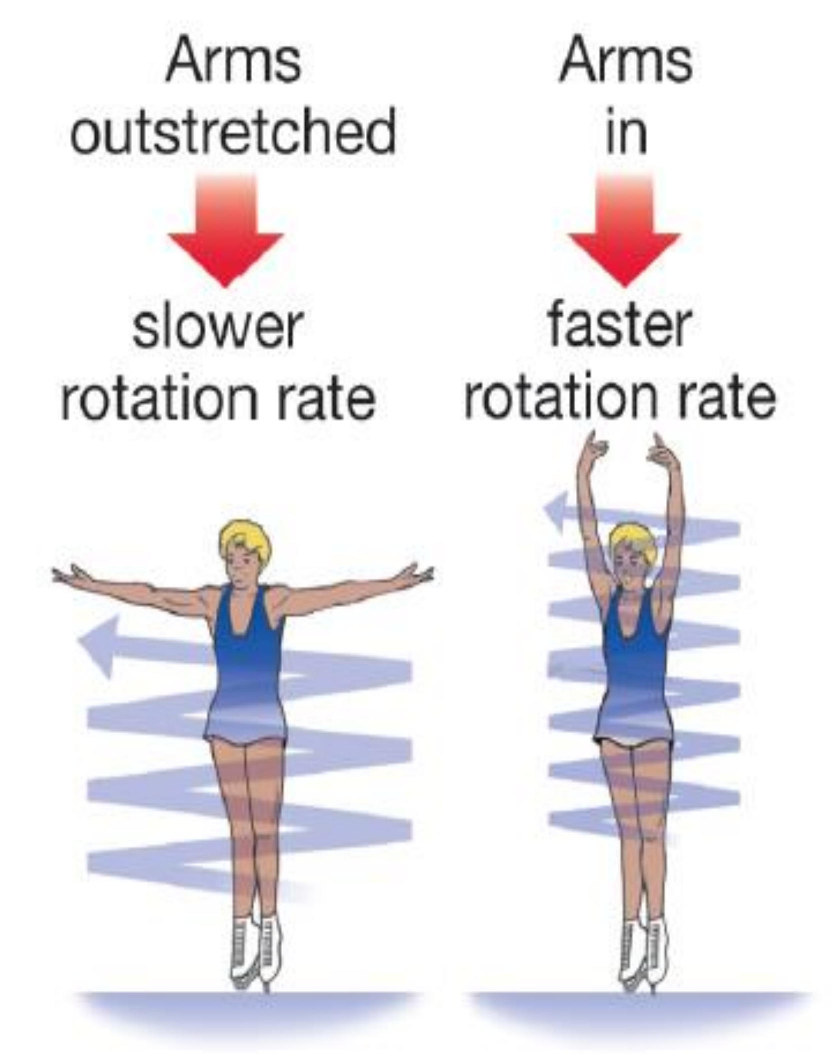
$$\boxed{\vec{L}_i = \vec{L}_f}$$



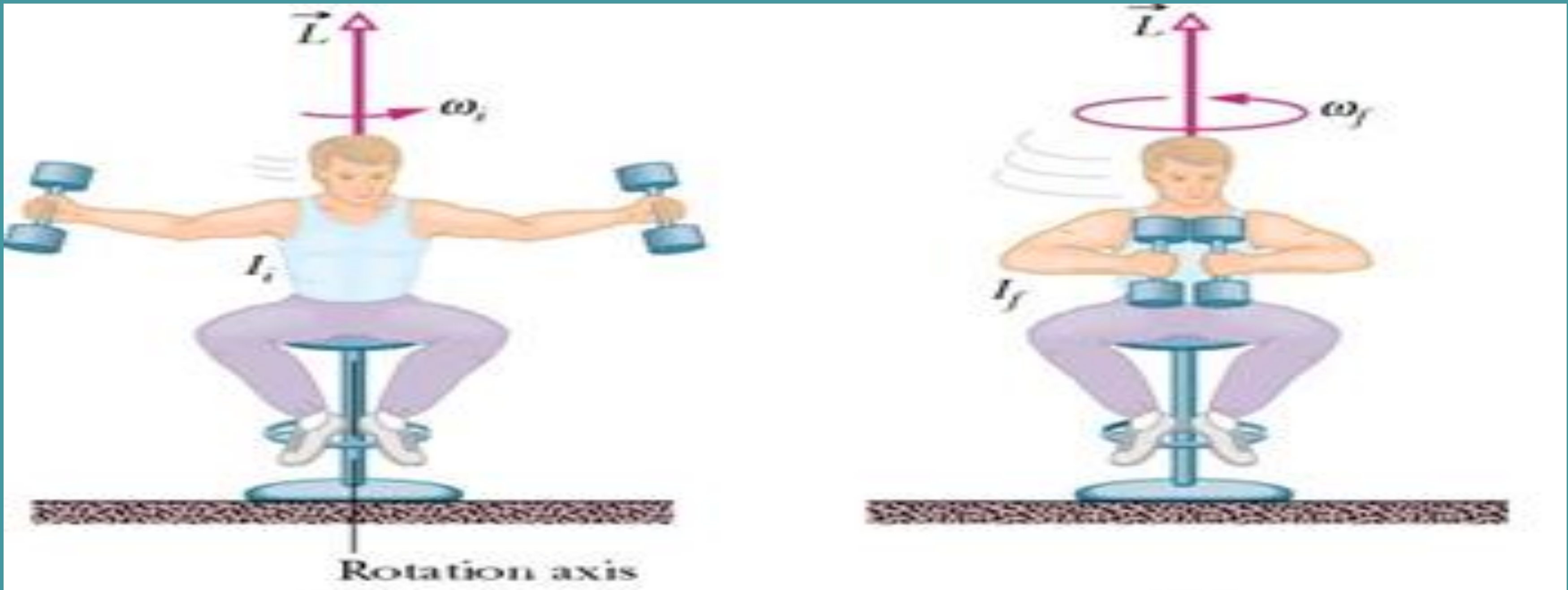
$\vec{L}$
Her angular momentum is fixed but she can still control her spin rate.
$\vec{L}$

- Angular momentum is important because it obeys a conservation law, as does linear momentum.

- The total angular momentum of a closed system stays the same.



Example: The figure shows a student seated on a stool that can rotate freely about a vertical axis. The student who has been set into rotation at an initial angular speed ω_i , holds two dumbbells in his outstretched hands. His angular momentum vector $\vec{L}$ lies along the rotation axis, pointing upward.



(a) (b)

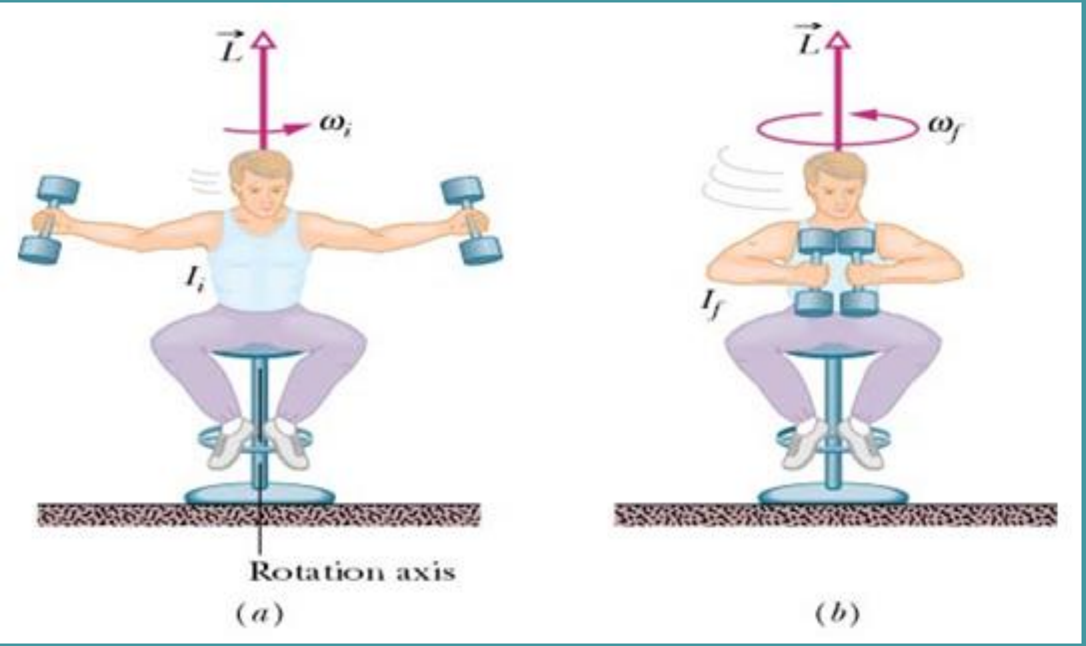
The student then pulls in his hands as shown in fig.b. This action reduces the rotational inertia from an initial value I_i to a smaller final value I_f.

No net external torque acts on the student-stool system. Thus the angular momentum of the system remains unchanged.

Angular momentum at t_i: $L_i = I_i \omega_i$ Angular momentum at t_f: $L_f = I_f \omega_f$

$$\boxed{L_i = L_f} \rightarrow \boxed{I_i \omega_i = I_f \omega_f} \rightarrow \omega_f = \frac{I_i}{I_f} \omega_i \qquad \text{Since } \boxed{I_f < I_i} \rightarrow \frac{I_i}{I_f} > 1 \rightarrow \boxed{\omega_f > \omega_i}$$

The rotation rate of the student in fig.b is faster



Analogies between translational and rotational Motion

Translational Motion		Rotational Motion
x	$\leftrightarrow$	θ
v	$\leftrightarrow$	ω
a	$\leftrightarrow$	α
p	$\leftrightarrow$	ℓ
$K = \frac{mv^2}{2}$	$\leftrightarrow$	$K = \frac{I\omega^2}{2}$
m	$\leftrightarrow$	I
$F = ma$	$\leftrightarrow$	$\tau = I\alpha$
F	$\leftrightarrow$	τ
$P = Fv$	$\leftrightarrow$	$P = \tau\omega$
$\vec{F}_{net} = \frac{d\vec{p}}{dt}$	$\leftrightarrow$	$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}$
$p = mv$	$\leftrightarrow$	$L = I\omega$