In Orange County, 51% of the adults are males. (It doesn't take too much advanced mathematics to deduce that the other 49% are females.) One adult is randomly selected for a survey involving credit card usage.

- a) Find the prior probability that the selected person is a male.
- b) It is later learned that the selected survey subject was smoking a cigar. Also, 9.5% of males smoke cigars, whereas 1.7% of females smoke cigars (based on data from the Substance Abuse and Mental Health Services Administration).

Use this additional information to find the probability that the selected subject is a male.

Solution

Let's use the following notation:

M = male $\overline{M} = female (or not male)$ C = cigar smoker $\overline{C} = not a cigar smoker.$

- a. Before using the information given in part b, we know only that 51% of the adults in Orange County are males, so the probability of randomly selecting an adult and getting a male is given by P(M) = 0.51.
- b. Based on the additional given information, we have the following:

$$P(\overline{M}) = 0.51$$
 because 51% of the adults are males $P(\overline{M}) = 0.49$ because 49% of the adults are females (not males) $P(C|M) = 0.095$ because 9.5% of the males smoke cigars (That is, the probability of getting someone who smokes cigars, given that the person is a male, is 0.095.) $P(C|\overline{M}) = 0.017$. because 1.7% of the females smoke cigars (That is, the probability of getting someone who smokes cigars, given that the person is a female, is 0.017.)

Let's now apply Bayes' theorem by using the preceding formula with M in place of A, and C in place of B. We get the following result:

$$P(M \mid C) = \frac{P(M) \cdot P(C \mid M)}{[P(M) \cdot P(C \mid M)] + [P(\overline{M}) \cdot P(C \mid \overline{M})]}$$

$$= \frac{0.51 \cdot 0.095}{[0.51 \cdot 0.095] + [0.49 \cdot 0.017]}$$

$$= 0.85329341$$

$$= 0.853 \text{ (rounded)}$$

Proof that, if a binary tree of height h has t terminal vertices, then

$$\lg t \leq h$$

Solution

Here, log function is base 2. We want to show that $t \leq 2^h$.

Using induction, base case with h=0 is clear.

Assuming that the statement is true when height is h-1, we consider a tree with height h, call it T. Remove all terminal vertices and call this new tree T'. The height of T' is h-1 and by inductive hypothesis, there are at most 2^{h-1} terminal vertices. Since, the removed terminal vertices in T are the children of terminal vertices in T', there can be at most 2^h of them.

3)

Assume that the function f, g and h take on only positive values. Prove that, if $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $f(n) = \Theta(h(n))$.

Solution

There are constants c₁ and c₂ such that

$$c_1h(n) \le f(n) \le c_2h(n)$$

for large enough n. The constants in both inequalities are obtained by the given assumptions. (The details are omitted here.)

4) Show that a simple graph is a tree if and only if it is connected, but the deletion of any of its edges makes the graph disconnected.

Solution

- → Direction is clear, since in any tree there is a unique path between any pair of vertices and if we remove the edge uv, u and v are disconnected.
- <-- Show by contrapositive: "If not a tree, then either the graph is not connected or there is a cycle by definition." Since the graph is connected, assume there is a cycle C. Then the deletion of any edge uv on that cycle would leave the graph still connected. Because if any two vertices x and y were connected using a path P that contains the edge uv, then after deleting the edge uv, x and y are still connected by using a path in the union of P and the remaining u,v-path on C. So, the deletion of any edge does not make the graph disconnected.
- 5) Use mathematical induction to show that given n+1 non-negative integers none exceeding 2n, there is at least one integer that divides another integer in this set.

Solution

Clearly true for n=1.

Assume true for all i less than or equal to n. We will prove for n+2 numbers not exceeding

2(n+1).

Case 1: If all numbers are at most 2n, then we are done by Ind. Hypo.

Case 2: If all numbers but one number are at most 2n, again done by Ind. Hypo.

Case 3: If at least two numbers are greater than 2n, then there are exactly two numbers greater than 2n. This means 2n+1 and 2n+2 in this set.

If n+1 is also in this set, then we are done, because n+1 divides 2n+2.

Otherwise, we remove 2n+2 from the set and add n+1 to the set. Call this new set A. By ind. Hypo, there are two integers in A that divide each other. If one of these integers is n+1, then the other number divides n+1 and this number divides also 2n+2, done.