MAT 254 Final Exam 23/06/2020

İbrahim Burak Tanrıkulu 21827852

Question 1:

a) $\text{Ker}\, T = \{(a,b,c,d) : T(a,b,c,d) = 0\} \longrightarrow T(a,b,c,d) = a+c = 0$

if $a+c=0$, then $b+c=0$. Let $a=\alpha$ and $b=\beta$. Then $c = -\alpha$, $d = -\beta$

$\text{Ker}\, T = \{(\alpha, \beta, -\alpha, -\beta) : \alpha, \beta \in \mathbb{R}\}$ $\dim(\text{Ker}\, T) = 2$

b) $\beta = \{(1), (-1)\}$

$\dim(\text{Im}\, T) = 0$

c) $\text{Ker}\, L \neq \{0_v\}$. So L is not one-to-one and T is not isomorphisim

Question 2:

$A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ $S = \{\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,0,1)}\}$

$[L(v_1)]_\beta = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $[L(v_2)]_\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $[L(v_3)]_\beta = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$L(v_1) = L(1,0,0) = c_1 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$L(v_2) = L(0,1,0) = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$L(v_3 = L(0,0,1) = c_1 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Ibrahim Burak Tanrıkulu 21827852

Question 3:

$$S = \{\underset{v_1}{(1,0,-1)}, \underset{v_2}{(-1,1,0)}, \underset{v_3}{(0,1,1)}\}$$

$$w_1 = v_1 = (1,0,-1)$$

$$w_2 = v_2 - \frac{(v_2|w_1)}{(w_1|w_1)} \cdot w_1 = (-1,1,0) + \frac{1}{2} \cdot (1,0,-1) = \left(\frac{-1}{2}, 1, \frac{-1}{2}\right)$$

$$w_3 = v_3 - \frac{(v_3|w_1)}{(w_1|w_1)} \cdot w_1 - \frac{(v_3|w_2)}{(w_2|w_2)} \cdot w_2 = (0,1,1) + \frac{1}{2} \cdot (1,0,-1) - \frac{1}{3} \cdot \left(\frac{-1}{2}, 1, \frac{-1}{2}\right)$$

$$= (0,1,1) + \left(\frac{1}{2}, 0, \frac{1}{2}\right) + \left(\frac{-1}{6}, \frac{1}{3}, \frac{-1}{6}\right)$$

$$= \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$(v_2|w_1) = (-1.1) + 0 + 0 = -1$

$(w_1|w_1) = (1.1) + 0 + (-1.-1) = 2$

$(v_3|w_1) = 0 + 0 + (1.-1) = -1$

$(v_3|w_2) = 0 + (1.1) + (1.\frac{-1}{2}) = \frac{1}{2}$

$(w_2|w_2) = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}$

Orthogonal set $S' = \left\{(1,0,-1), \left(\frac{-1}{2}, 1, \frac{-1}{2}\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)\right\}$

$\|w_1\| = \sqrt{2}$ $\quad \|w_2\| = \sqrt{\frac{3}{2}}$ $\quad \|w_3\| = \frac{2\sqrt{3}}{3}$

Orthonormal set $S'' = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{2\sqrt{\frac{3}{2}}}, \sqrt{\frac{2}{3}}, \frac{-1}{2\sqrt{\frac{3}{2}}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right\}$

Question 4:

$$A = \begin{bmatrix} -4 & 0 & 3 \\ 0 & -1 & 0 \\ -6 & 0 & 5 \end{bmatrix} \qquad \lambda.I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -4 & 0 & 3 \\ 0 & -1 & 0 \\ -6 & 0 & 5 \end{bmatrix} = \begin{bmatrix} \lambda+4 & 0 & -3 \\ 0 & \lambda+1 & 0 \\ 6 & 0 & \lambda-5 \end{bmatrix}$$

$$\det(\lambda I - A) = (-1)^{2+2} \cdot (\lambda+1) \cdot \begin{vmatrix} \lambda+4 & -3 \\ -6 & \lambda-5 \end{vmatrix} = (\lambda+1) \cdot ((\lambda+4) \cdot (\lambda-5) + 18)$$

$$= (\lambda+1) \cdot (\lambda^2 - \lambda - 2) = (\lambda+1)^2 \cdot (\lambda-2) = 0$$

from $(\lambda+1)^2 . (\lambda-2)$

$\lambda_1 = \lambda_2 = -1 \quad \lambda_3 = 2$

for $\lambda = -1 \rightarrow \begin{bmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 6 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\begin{bmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 6 & 0 & -6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_2 = \alpha \\ x_3 = \beta \\ x_1 - x_3 = 0 \quad x_1 = \beta \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for $\lambda = 2 \longrightarrow \begin{bmatrix} 6 & 0 & -3 \\ 0 & 3 & 0 \\ 6 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\begin{bmatrix} 6 & 0 & -3 \\ 0 & 3 & 0 \\ 6 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 6 & 0 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_3 = \alpha \\ 2x_1 - x_3 = 0 \\ x_2 = 0 \\ x_1 = \frac{\alpha}{2} \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$