MAT 254 - Midterm Exam - 18 May 2020

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Question 1:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ 2x_1 + 5x_2 + 3x_3 &= 4 \\ x_1 + 2x_2 + a^2 x_3 &= -a \end{aligned} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 2 & 5 & 3 & 4 \\ 1 & 2 & a^2 & -a \end{array}\right] \Longrightarrow$$

$$\begin{array}{c} -R_1 + R_3 \to R_3 \\ -2R_1 + R_2 \to R_2 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & -1 & 5 & 0 \\ 0 & -1 & a^2+1 & -a-2 \end{array}\right] \xrightarrow[-R_2 + R_3 \to R_3]{3R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 14 & 2 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & a^2-4 & -a-2 \end{array}\right] \Longrightarrow$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 14 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & a^2-4 & -a-2 \end{array}\right]$$

if $a = 2$ then

$$\left[\begin{array}{ccc|c} 1 & 0 & 14 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

last row will be a bad row. Thus the system has no solutions.

if $a = -2$ then

$$\left[\begin{array}{ccc|c} 1 & 0 & 14 & 2 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There will be 2 leading terms and 3 unknowns. Thus the system has infinitely many solutions.

if a is other than -2 and 2, then there will be 3 leading terms and 3 unknowns. Thus the system has unique a solution.

a) no solution $\to$ if $a = 2$

b) infinitely many solutions $\to$ if $a = -2$

c) a unique solution $\to$ if $a \neq 2$ and $a \neq -2$

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Question 2:

$$A = \begin{bmatrix} 4 & 2 & 0 & 1 & 6 \\ 2 & 3 & 1 & -1 & 2 \\ 0 & 2 & 0 & 2 & -1 \\ 0 & 0 & 0 & 5 & 0 \\ 1 & 1 & 0 & 3 & -2 \end{bmatrix}$$

Most of zeros on 3rd column and 4th row. I selected 4th row.

a) $|A| = ?$

$$|A| = (-1)^{4+1}.0.[\ ] + (-1)^{4+2}.0.[\ ] + (-1)^{4+3}.0.[\ ] +$$

$$(-1)^{4+4}.5.\begin{bmatrix} 4 & 2 & 0 & 6 \\ 2 & 3 & 1 & 2 \\ 0 & 2 & 0 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix} + (-1)^{4+5}.0.[\ ]$$

$$|A| = 5.\left((-1)^{1+3}.0.[\ ] + (-1)^{2+3}.1.\begin{bmatrix} 4 & 2 & 6 \\ 0 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix} + (-1)^{3+3}.0.[\ ] + (-1)^{4+3}.0.[\ ]\right)$$

$$|A| = 5.\left(-1.\left((-1)^{1+1}.4.\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} + (-1)^{2+1}.0.[\ ] + (-1)^{3+1}.1.\begin{bmatrix} 2 & 6 \\ 2 & -1 \end{bmatrix}\right)\right)$$

$$|A| = 5.(-1.(4.-3 + 0 + -10)) = 5.-1.-22 = \underline{\underline{110}}$$

b) $|(-1)A| = -|A| = -110$

c) $A \xrightarrow{R_1 \leftrightarrow R_3} B \xrightarrow{-500R_3 + R_1} C \xrightarrow{\frac{1}{3}R_2} D$

$|A| = 110$

$|B| = -110$

$|C| = -110$

$|D| = \frac{-110}{3}$

$B = A_{R_1 \leftrightarrow R_3} \Rightarrow |B| = -|A|$

$C = B_{kR_3 + R_1} \Rightarrow |C| = |B|$

$D = C_{k.R_2} \Rightarrow |D| = k.|C|$

d) $|B^T D^{-1}| = |B^T|.|D^{-1}| \Rightarrow |B|.\frac{1}{|D|} = -110.\frac{3}{-110} = 3$

$|B^T| = |B|$

$|D^{-1}| = \frac{1}{|D|}$

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Question 3: $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$ $A^{-1} = ?$ $[A|I] \to [I|A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3 \to R_3]{R_2+R_1 \to R_1}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \xrightarrow[-R_3/2]{-R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \xrightarrow{-R_3+R_2 \to R_2}$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 3/2 & -3/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 3/2 & -3/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

Question 4:

a) True, this is definition of vector space.

b) Let $A = I$ and $B = I$, then $\operatorname{rank}(A) = 3$ and $\operatorname{rank}(B) = 3$.

$\operatorname{rank}(AB) = \operatorname{rank}(I.I) = \operatorname{rank}(I) = 3$, So, False.

c) True, if $cb = 0$, then c or b equal to 0 and matrix A may be as follows:

$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ or $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ Let's check this matrices.

1- $0.\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓ $\qquad 0.\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

2- $\begin{bmatrix} ax_1 & bx_1 \\ 0 & dx_1 \end{bmatrix} + \begin{bmatrix} ax_2 & bx_2 \\ 0 & dx_2 \end{bmatrix} = \begin{bmatrix} a(x_1+x_2) & b(x_1+x_2) \\ 0 & d(x_1+x_2) \end{bmatrix}$ ✓ $\qquad \begin{bmatrix} ax_1 & 0 \\ cx_1 & dx_1 \end{bmatrix} + \begin{bmatrix} ax_2 & 0 \\ cx_2 & dx_2 \end{bmatrix} = \begin{bmatrix} a(x_1+x_2) & 0 \\ c(x_1+x_2) & d(x_1+x_2) \end{bmatrix}$ ✓

3- $k.\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} ak & bk \\ 0 & dk \end{bmatrix}$ ✓ $\qquad k.\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} = \begin{bmatrix} ak & 0 \\ ck & dk \end{bmatrix}$ ✓

So, W is a subspace of M_{2x2}