Mon. Not. R. Astron. Soc. **288**, 1049–1059 (1997)

Spin-down of an oblique rotator with a current-starved outer magnetosphere

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Accepted 1997 March 10. Received 1997 March 3; in original form 1996 October 16

ABSTRACT

A variant of the vacuum-dipole model of rotation-powered pulsars is presented that accounts for the observed spin-down properties of all three pulsars with braking indices measured from absolute pulse numbering (the Crab, PSR B0540 – 69 and PSR B1509 – 58). In the model, the neutron star *and inner magnetosphere* are treated phenomenologically as a single unit, a magnetized, perfectly conducting sphere of radius r_{v} rotating rigidly *in vacuo*. The 'vacuum radius' r_{v} corresponds to the innermost point in the magnetosphere where field-aligned flow breaks down and the plasma becomes three-dimensional, that is, the point where cyclotron losses occur slowly enough to allow electrons (or positrons) to move an appreciable distance before decaying to the $l=0$ Landau state. For young, Crab-like pulsars, one typically finds $r_* \ll r_{\mathrm{v}} < r_{\mathrm{L}}$, where r_* is the stellar radius, and r_{L} is the light-cylinder distance. The model therefore differs from standard vacuum-dipole theories, in which the dipole has radius r_* and is treated as point-like.

Three observable pulsar parameters – the rotation frequency ω, its time derivative $\dot{\omega}$, and the angle α between the rotation and magnetic axes – uniquely determine r_{v} and hence the electromagnetic braking torque exerted on the star, calculated from the Deutsch radiation fields of the rotating dipole. With no free parameters, the theory yields braking index values $n=\omega\ddot{\omega}/\dot{\omega}^2$ for the Crab, PSR B0540 – 69 and PSR B1509 – 58 that agree with timing data to 4 per cent. The second deceleration parameter $m=\omega^2\dddot{\omega}/\dot{\omega}^3$ is also predicted for each object, but cannot be verified to useful precision using data available at present. The relationship between our idealized yet successful spin-down model and a genuine pulsar magnetosphere is discussed, and further observational tests of the theory are proposed.

Key words: stars: neutron – pulsars: general – stars: rotation.

1 INTRODUCTION

All rotation-powered pulsars spin down systematically as they age. In the absence of glitches and timing noise, the rotation frequency ω appears to evolve secularly according to a braking law of the form $\dot{\omega} \propto \omega^n$, where n is called the braking index, the overdot denotes differentiation with respect to time, and the constant of proportionality is always negative. Measurements of n have been carried out for four young, fast pulsars to date (the Crab, PSR B0540 – 69, PSR B1509 – 58 and Vela), yielding $n<3$ in each case (Manchester & Peterson 1989; Nagase et al. 1990; Lyne, Pritchard & Graham-Smith 1993; Kaspi et al. 1994; Boyd et al. 1995; Lyne et al. 1996). The results are surprising from a theoretical viewpoint, because most magnetospheric theories require spin-down power to be transported mainly as Poynting flux at the light cylinder, implying $n=3$ if the stellar magnetic field is dipolar and $n>3$ if it contains higher order multipoles. This conclusion holds regardless of the specific form that the Poynting flux takes, whether it be associated with vacuum electromagnetic radiation (Deutsch 1955; Ostriker & Gunn 1969), a force-free magnetosphere (Goldreich & Julian 1969), or a relativistic magnetohydrodynamic wind (Michel 1969; Goldreich & Julian 1970).

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Several explanations of the observed spin-down behaviour have been advocated in the literature, each with its advantages and disadvantages. For example, one can postulate that energy is transported primarily by particles at the light cylinder, and that the ratio of Poynting flux to kinetic-energy flux is different for different objects, leading to a distribution of n values less than 3. However, it is difficult to generate a kinetic-energy-dominated flow from pair cascades near the polar cap (Daugherty & Harding 1982; Arons 1983) or in an outer gap (Cheng, Ho & Ruderman 1986a; Chiang & Romani 1992, 1994). Also, a kinetic-energy-dominated flow is expected to emit X-rays and gamma rays with a luminosity comparable to the spin-down luminosity, yet this is rarely observed (Arons 1992). Blandford & Romani (1988) showed that a Poynting-flux-dominated outflow can have $n<3$ if the spin and magnetic axes counteralign over a characteristic spin-down time-scale, or if the stellar magnetic moment increases (e.g., due to thermoelectric currents in the crust, or resurrection of a buried field; see Blandford, Applegate & Hernquist 1983 and Muslimov & Page 1996). The latter scenario is viable, but the former leaves many old pulsars as nearly orthogonal rotators, with frequency-independent pulse–interpulse morphologies, whereas the actual number of such objects is modest (Arons 1992). Yet another possibility is that the conduction current flowing on polar magnetic field lines is large enough to distort the magnetosphere, so that the spin-down torque is fully developed at some characteristic radius not equal to the light-cylinder distance (Blandford & Romani 1988; Arons 1992). Again, however, the resulting electromagnetic stresses are expected to generate radiation losses above the observed level (Arons 1992).

A drawback of all the above ideas is that they do not lead easily to a quantitative, predictive, and therefore testable theory of spin-down that uniquely determines the braking index and related quantities from observable pulsar parameters. Fundamentally, this is because it is extremely difficult to calculate the magnetospheric structure of either aligned or oblique rotators in a self-consistent manner (Arons 1992; Mestel 1992). In this paper, however, we demonstrate that one can construct a predictive and accurate theory of spin-down without a self-consistent analysis of the global magnetosphere. We present a variant of the vacuum-dipole model of Ostriker & Gunn (1969) which, although idealized and phenomenological, nevertheless reproduces the observed braking indices of the Crab, PSR B0540 − 69 and PSR B1509 − 58 to an accuracy of 4 per cent, given just three observable parameters (ω, $\dot{\omega}$ and the angle α between the spin and magnetic axes). In Section 2, the physical features of the model and its relationship to a genuine pulsar magnetosphere are described, followed by calculations of the braking torque and resulting spin-down behaviour. The calculated spin-down properties are then compared with observations in Section 3, and additional tests of the theory are proposed for the future.

2 PHENOMENOLOGICAL THEORY OF SPIN-DOWN

In this section, we investigate the spin-down properties of a rotation-powered pulsar modelled as a magnetized, conducting sphere rotating *in vacuo*. The physical features of the model are outlined in Section 2.1; in particular, the effective radius of the conducting sphere is defined and calculated in terms of basic pulsar parameters. We then compute the stellar braking torque due to vacuum dipole radiation in Section 2.2, and hence the braking index $n=\omega\ddot{\omega}/\dot{\omega}^2$ and second deceleration parameter $m=\omega^2\dddot{\omega}/\dot{\omega}^3$ (Section 2.3). The analogy between our idealized, phenomenological model and a genuine pulsar magnetosphere is discussed in Sections 2.4 and 2.5.

2.1 Rotating magnet of variable size

We suppose that the magnetosphere of a rotation-powered pulsar is divided into two concentric, spherical regions: (i) an inner magnetosphere of radius r_v (defined below), which acts as a corotating extension of the perfectly conducting stellar interior, with the dipolar stellar field 'frozen in'; and (ii) an outer magnetosphere beyond r_v, which is populated by outflowing plasma, yet acts as an electrodynamic vacuum because the conduction currents therein are reduced by relativistic effects to being much smaller than the displacement currents associated with the rotating fields (see Section 2.5). The radius r_v, which we call the 'vacuum radius', is defined to be the innermost point within the magnetosphere where the plasma can become three-dimensional and field-aligned flow breaks down, that is, the innermost location where cyclotron losses are slow enough for outflowing electrons and positrons to travel an appreciable distance ($\sim r_v$) before decaying to the $l=0$ Laudau state. This choice of r_v is primarily motivated by the empirical success of the resulting theory, which reproduces the observed braking indices of the Crab, PSR B0540 − 69 and PSR B1509 − 58 to 4 per cent with no free parameters, but it is also justified by semi-quantitative physical arguments in Sections 2.4 and 2.5. A truly self-consistent theoretical justification will require a more complete understanding of magnetospheric structure than is presently at hand.

The electrodynamic problem of a magnetized, conducting sphere rotating *in vacuo*, losing energy and angular momentum through magnetic dipole radiation, is closely related to the standard vacuum rotator solved by Deutsch (1955) and Ostriker & Gunn (1969), but with one important difference: we take the radius of the rotating dipole to be r_v rather than r_*. Since r_v can be comparable to (though less than) the light-cylinder distance $r_L=c/\omega$, whereas r_* is always much less than r_L, the braking torque in our model differs from Ostriker & Gunn's (1969) point-dipole estimate. Even more importantly, r_v is a function of ω and therefore changes with time, so the braking index n does not equal 3 as for a point dipole.

Before calculating the braking torque for our model, we estimate r_v in terms of fundamental stellar parameters. In a strong magnetic field B, an electron or positron in an excited Landau level $l=l'$ decays to the ground state $l=0$ in a stepwise fashion, with l most often decreasing by one at each step. Once in the ground state, the particle moves strictly along the magnetic field. The lab-frame de-excitation time t_s is approximately the lifetime of the final (and slowest) transition, from $l=1$ to $l=0$. For a particle with Lorentz factor γ, we have (Mészáros 1992)

$$t_s = \frac{3\lambda_e^2}{16\pi^2 r_e c}\left(\frac{B_c}{B}\right)^2 \gamma, \tag{1}$$

where r_e is the classical radius of the electron, λ_e is the Compton wavelength, and B_c is the critical magnetic field

$$B_c = \frac{m_e^2 c^2}{e\hbar} = 4.4 \times 10^{13}\ \mathrm{G}. \tag{2}$$

The de-excitation time is very short at $r = r_*$ ($t_s \approx 10^{-17}\gamma$ s), but it increases rapidly with distance from the stellar surface as B decreases. Assuming a dipole field with magnitude B_* at the stellar poles, and ignoring latitudinal variations, we have $B(r) \approx B_* r_*^3/r^3$, and hence $t_s(r) \propto r^6$. Relativistic particles travel a distance $ct_s(r)$ before decaying to the Landau ground state. Therefore, from our definition of r_v as the radius satisfying $ct_s(r_v) \approx r_v$, we obtain

$$\frac{r_v}{r_*} = \left(\frac{16\pi^2 r_e r_*}{3\lambda_e^2}\right)^{1/5}\left(\frac{B_*}{B_c}\right)^{2/5}\gamma^{-1/5}. \tag{3}$$

The Lorentz factor γ in equation (3) is set by the pair-production physics in the low-altitude acceleration region. Since r_v depends weakly on γ, a detailed model of the acceleration region is not essential; in what follows, we adopt the slot gap posited by Arons (1983) as a plausible working hypothesis. In the slot-gap picture, particles with one sign of charge are freely extracted from the surface of the polar cap as a primary beam with density $n_b \approx n_{GJ}$, where $n_{GJ} = 2\varepsilon_0 \omega B_*/e$ is the Goldreich–Julian density at $r = r_*$. The primary beam is accelerated by a starvation electric field along unfavourably curved magnetic field lines to a Lorentz factor $\gamma_b \approx 10^7$, when enough curvature photons are emitted to initiate a pair cascade. In the cascade, curvature photons decay into electron–positron pairs in the superstrong magnetic field (primarily via the one-photon process), the pairs radiate away their perpendicular momentum almost instantaneously by emitting synchrotron photons, and these photons themselves form further pairs if they exceed an energy threshold. Each primary particle therefore multiplies into a large number κ of secondary pairs, which constitute the bulk of the outflowing plasma. It is the Lorentz factor of the secondaries which features in equation (3).

A number of authors have evaluated κ numerically for curvature-synchrotron cascades of the above type, with the more recent work including Compton effects (Daugherty & Harding 1982, 1996; Arons 1983; Chiang & Romani 1992, 1994; Sturner, Dermer & Michel 1995). Here we fit a power law in B_* and ω to the multiplicity calculations of Daugherty & Harding (1982) (fig. 8 of that paper), obtaining

$$\kappa = 7 \times 10^2 \left(\frac{B_*}{B_c}\right)^{0.9}\left(\frac{\omega}{1\ \mathrm{s}^{-1}}\right)^{1.0}. \tag{4}$$

Equation (4) is accurate to $\sim$25 per cent in the regimes $10 \lesssim \omega \lesssim 200\ \mathrm{s}^{-1}$ and $5 \times 10^{11} \lesssim B_* \lesssim 5 \times 10^{12}$ G, implying an uncertainty of $\sim$5 per cent in r_v. This level of accuracy is tolerable in the context of our phenomenological model, although care must be exercised with pulsars lying outside the above regime. The multiplicity ranges from $\kappa \approx 2 \times 10^4$ for young, Crab-like pulsars to $\kappa \approx 5 \times 10^2$ for ordinary pulsars with $\omega \approx 20\ \mathrm{s}^{-1}$. Daugherty & Harding's (1982) simulations also showed that the total energy density in secondary pairs is comparable to the energy density of the primary beam, implying $\gamma\kappa \approx \gamma_b$ and hence

$$\gamma = 3 \times 10^4 \left(\frac{B_*}{B_c}\right)^{-0.9}\left(\frac{\omega}{1\ \mathrm{s}^{-1}}\right)^{-1.0} \tag{5}$$

for $\gamma_b = 2 \times 10^7$. Taking $r_* = 10$ km, we now combine equations (3) and (5) to obtain

$$x_v = 8 \times 10^2 (\omega r_*/c)^{1.2} (B_*/B_c)^{0.6}, \tag{6}$$

where we define the dimensionless vacuum radius $x_v = \omega r_v/c$. Equation (6) gives x_v in terms of the fundamental pulsar parameters r_*, B_* and ω.

For the magnetospheric model described above to be realistic, the inner corotating magnetosphere must not extend beyond the light cylinder (i.e., $x_v < 1$). Thus the model applies only to pulsars with $\omega^2 B_*/B_c \lesssim 1.4 \times 10^4\ \mathrm{s}^{-2}$, limiting the polar-cap potential drop to $\Delta\Phi_{cap} = B_* r_*^3 \omega^2/c \lesssim 2 \times 10^{17}$ V. *It is intriguing that every known pulsar satisfies this constraint.* Young pulsars typically have $x_v \sim 0.1$–0.8, and ordinary pulsars (with $\omega \approx 20\ \mathrm{s}^{-1}$ and $B_*/B_c \approx 0.02$) have $x_v \sim 10^{-3}$. However, there is no obvious reason why pulsars cannot exist having $x_v \geq 1$. Such objects spin down via $\boldsymbol{J} \times \boldsymbol{B}$ torques instead of vacuum torques, because plasma inertia determines the field structure near the light cylinder rather than the displacement current (Goldreich & Julian 1969; Mestel et al. 1985; Mestel & Shibata 1994). They are otherwise unexceptional, and the fact that not a single one has been found is mysterious.

2.2 Electromagnetic braking torque

In Appendix A, the electromagnetic fields generated by a magnetized, conducting sphere of radius r_0 rotating *in vacuo* are derived exactly following the multipole method of Deutsch (1955). The magnetic field at the surface of the sphere is assumed to be dipolar, with magnitude B_0 at the magnetic poles, and the magnetic axis is inclined at an angle α with respect to the rotation axis. In the standard point-dipole model of a rotation-powered pulsar, one chooses $r_0 = r_* \ll r_L$ (Ostriker & Gunn 1969). Here, in contrast, we take $r_0 = r_v \sim r_L$. We also set $B_0 = B_v = B_* r_*^3/r_v^3$, reflecting the fact that the stellar magnetic field is frozen into the inner corotating magnetosphere. The electromagnetic braking torque on the star is then given by

$$I\dot{\omega} = -\frac{2\pi B_*^2 r_*^6 \omega^3 \sin^2\alpha}{\mu_0 c^3} \times \left[\frac{1}{3(x_v^2+1)} + \frac{x_v^4}{5(x_v^6 - 3x_v^4 + 36)}\right] \tag{7}$$

from equation (A10). In equation (7), the moment of inertia I is, strictly speaking, that of the star and corotating magnetosphere combined, but the mass of plasma in the region $r_* \leq r \leq r_v$ is negligible compared to the stellar mass, yielding $I = I_*$ for all practical purposes. Although equation (7) is formally valid for arbitrary $x_v \geq 0$, in reality a model with $x_v \geq 1$ is not physically meaningful, as discussed above.

The torque (7) reduces to the formula for a point dipole,

$$I\dot{\omega} = -\frac{2\pi B_*^2 r_*^6 \omega^3 \sin^2 \alpha}{3\mu_0 c^3}, \tag{8}$$

in the limit $x_v \to 0$ (Ostriker & Gunn 1969), and is roughly halved relative to the point-dipole value for young, Crab-like pulsars with $x_v \approx 1$. The point-dipole formula (8), with $\alpha = \pi/2$, is widely used in observational work to derive an *apparent* dipole field strength B_d from measurements of the period P and period derivative $\dot{P}$, with $B_d = 3.3 \times 10^{19} (P\dot{P})^{1/2}$ G for a 10-km star with $I = 10^{38}$ kg m^2 (e.g. Lyne & Graham-Smith 1990). Rewriting (7) to eliminate ω and $\dot{\omega}$ in favour of B_d, we obtain

$$\frac{B_d}{\sin \alpha} = B_* \left[\frac{1}{x_v^2 + 1} + \frac{3x_v^4}{5(x_v^6 - 3x_v^4 + 36)} \right]^{1/2}. \tag{9}$$

The left-hand side of equation (9) is a function of observed parameters only. For $x_v < 1$, the second term on the right-hand side is at most 4 per cent of the first and can be safely neglected for most pulsars.

If energy transport is Poynting-flux-dominated at the light cylinder, the braking torque acting on the star is given approximately by the electromagnetic value (7), with a negligible contribution from the mechanical torque exerted by outflowing plasma. On the other hand, if energy transport is kinetic-energy-dominated, the above expression for the electromagnetic torque is still correct (since the outer magnetosphere still behaves as an electrodynamic vacuum), but the electromagnetic torque cannot be identified as the *total* torque. In this paper, we analyse only the former case. It is important to realize that the existence of a current-starved outer magnetosphere behaving like an electrodynamic vacuum does *not* automatically imply that energy transport is Poynting-flux-dominated in the region $r > r_v$. It is possible for the charge and conduction current densities to be negligible even when the mechanical energy of the outflowing plasma exceeds the energy of the electromagnetic fields. We therefore emphasize that ascribing spin-down to an electromagnetic torque involves an assumption that energy transport is Poynting-flux-dominated at r_L – an assumption that appears justified a posteriori by the empirical success of the model.

2.3 Braking index and second deceleration parameter

The evolution of ω as a function of time is usually characterized by two observable quantities, the braking index $n = \omega\ddot{\omega}/\dot{\omega}^2$ and second deceleration parameter $m = \omega^2 \dddot{\omega}/\dot{\omega}^3$ (e.g. Blandford & Romani 1988; Lyne et al. 1993; Kaspi et al. 1994). Both n and m can be computed theoretically from equations (6) and (7) by assuming that r_*, B_*, α and I do not vary on the spin-down time-scale $\omega/\dot{\omega}$; equation (6) implies $\dot{x}_v/x_v = p\dot{\omega}/\omega$ $(p = 1.2)$ under these conditions. Exact but complicated expressions for n and m are derived in Appendix B (equations B4 and B7 respectively), and are used in Section 3 to compare the theory with observations. Here, for illustrative purposes, we present approximate forms of n and m obtained by neglecting the second term in equation (7) and its derivatives:

$$n = 3 - \frac{2px_v^2}{x_v^2 + 1}, \tag{10}$$

$$m = n(2n - 1) - \frac{4p^2 x_v^2}{(x_v^2 + 1)^2}. \tag{11}$$

Equations (10) and (11) are accurate to 5 per cent for $0 \le x_v \le 0.8$ and $0 \le x_v \le 0.6$ respectively, but break down near $x_v = 1$. Besides affording quick, accurate estimates of n and m for many pulsars, they also illustrate two important consequences of our spin-down model. First, the braking index is always between 1.8 and 3.0, in line with observations. Secondly, the property $m = n(2n-1)$ derived by Blandford & Romani (1988) for power-law spin-down is modified, because the torque at any instant is proportional to a non-power-law function of x_v, and x_v itself changes with time as ω evolves through equation (6). Note that $n < 3$ is achieved by changing the effective radius of the rotating, magnetized star, *not* by distorting the magnetosphere and developing the torque at a surface inside the light cylinder, as alternative models suggest (Blandford & Romani 1988; Arons 1992).

We are now in a position to calculate n and m for any pulsar, given the three observable parameters ω, B_d and α (or, equivalently, ω, $\dot{\omega}$ and α). This is accomplished in two steps. First, we solve equations (6) and (9) simultaneously for x_v and B_*. Then, we substitute x_v into equations (B4) and (B7) to obtain n and m; alternatively, the approximations (10) and (11) can be used in their ranges of validity. *We emphasize that the theory involves no free parameters.* Given ω, B_d (or $\dot{\omega}$) and α – all observable in principle, although α is sometimes difficult to measure accurately – the theory predicts n and m from first principles, and is therefore highly falsifiable.

Before applying the theory to timing observations in Section 3, we examine the physical basis of our two-piece (inner–outer) magnetospheric model in the next two sections.

2.4 Corotating inner magnetosphere

Our idealized assumption that the inner magnetosphere $r_* \le r \le r_v$ acts as a corotating extension of the stellar interior is clearly justified in a broad sense: the stellar magnetic field is frozen into the highly conducting magnetospheric plasma throughout most of the volume $r_* \le r \le r_v$. Nevertheless, the assumption is an oversimplification for the following three reasons. (i) Plasma in the low-altitude acceleration layer $r_* \le r \lesssim 2r_*$ departs from corotation wherever a starvation electric field $E_\parallel = \boldsymbol{E \cdot B}/|\boldsymbol{B}| \neq 0$ develops due to inertial effects, e.g., in a slot gap (Arons 1983), polar gap (Usov & Melrose 1995), or dissipation layer (Mestel & Shibata 1994). The departure is slight in relative terms because inertial effects manifest themselves as small perturbations of an essentially force-free magnetosphere ($E_\parallel \ll E_\perp$, where $E_\perp$ is the corotation electric field; see Cheng & Ruderman 1977b). (ii) The corotation assumption is strongly violated near the light cylinder, where inertial stresses are large and a significant toroidal component of magnetic field is added to the dipolar stellar field (Goldreich & Julian 1969). However, the empirical success of our idealized model for objects with r_v as large as $0.8r_L$ (e.g. PSR B0540−69) suggests that inertial stresses do not influence spin-down dramatically unless r_v is more than 80 per cent of r_L. (iii) The inner corotating magnetosphere is not really spherical; it is divided into polar and equatorial sectors threaded by open and closed magnetic field lines respectively (Goldreich & Julian 1969).

Cheng & Ruderman (1977a) calculated the lowest order inertial perturbations of a force-free magnetosphere. They found that, as the primary beam ($n_b = n_{GJ}$) and secondary pair plasma ($\gamma_+ = \gamma_- \ll \gamma_b$ and $n_+ = n_- \gg n_b$ at the base of the cascade) flow outwards along curved magnetic field lines, current continuity forces the electron and positron components to stream relative to each other. Consequently, the secondary pair plasma does not remain neutral in the bulk frame, and the lab-frame current density adjusts to remain Goldreich–Julian everywhere. This property, predicated on field-aligned flow, breaks down in the outer magnetosphere $r \geq r_v$ (see Section 2.5).

The velocity difference $\Delta v = |v_+ - v_-|$ between the electrons and positrons in the lab frame is given by (Cheng & Ruderman 1977a)

$$\frac{\Delta v}{c} \approx \frac{n_b}{n_\pm} \approx \kappa^{-1}, \tag{12}$$

and the bulk Lorentz factor of the pair plasma is $\gamma \approx \gamma_b/\kappa$ (Section 2.1). Together, these results imply $\gamma_- = (2\Delta v/c)^{-1/2}$ and $\gamma_+ = 2\gamma - \gamma_-$ in the lab frame, and hence $\tilde{\gamma}_\pm = \gamma/2\gamma_\pm$ in the bulk frame. For Crab parameters ($\gamma_b = 2\times10^7$, $\kappa = 2\times10^4$; see Daugherty & Harding 1982), we find $\gamma_- = 100$, $\tilde{\gamma}_- = 5$, $\gamma_+ = 1900$ and $\tilde{\gamma}_+ = 0.3$.

2.5 Current-starved outer magnetosphere

The empirical success (see Section 3) of our modified vacuum-dipole model poses a puzzling physical question: why does the outer magnetosphere $r \geq r_v$ behave like an electrodynamic vacuum, even though it is filled with plasma $\sim 10^4$ times the Goldreich–Julian number density? If the electromagnetic fields are approximate solutions of the source-free, harmonically driven Maxwell equations, the displacement current J_D must clearly be large compared to the conduction current J in the region $r \geq r_v$. Here we argue on physical grounds that this is indeed the case. The argument is in three parts. (i) In the region $r \geq r_v$, where the plasma is not forced to be one-dimensional, particles develop momentum components perpendicular to $\boldsymbol{B}$ via Compton scattering off soft X-ray photons (e.g., from an outer gap). (ii) As a consequence, the relative streaming between electrons and positrons in the bulk frame is washed out, and the plasma reverts to being neutral in that frame, with $E_\parallel \neq 0$ in the lab frame. (iii) Relativistic kinematics limits J in a bulk-neutral pair plasma to $\kappa n_{GJ} ec/\gamma$, even though the number density is κn_{GJ}, implying $J_D \gg J$ for typical pulsar parameters. It is in the latter sense that the outer magnetosphere is said to be ‘current-starved’.

We first consider Compton scattering of pairs with $\gamma \sim 10^3$ (produced in a low-altitude cascade) off soft X-ray photons with $h\nu \approx 1$ keV at $r \approx r_v$ (e.g., secondary synchrotron photons from an outer gap). The optical depth for this process is $\tau_C = n_\gamma \sigma_T r_v$, where n_γ is the number density of 1-keV photons at r_v, and σ_T is the Thomson cross-section. Cheng, Ho & Ruderman (1986b) used the observed spectrum of the Crab's high-energy pulsed emission to estimate $n_\gamma \approx 10^{20}(h\nu/m_ec^2)^{-1.8}$ m^{-3} for $10^{-3} \lesssim h\nu/m_ec^2 \lesssim 1$ (equations [7.1] and [7.2] of their paper). For $h\nu \approx 1$ keV, we find $\tau_C \approx 5r_v/r_* \gg 1$, i.e., the outer magnetosphere is opaque to Compton scattering of low-altitude pairs, at least for young, Crab-like pulsars. Consequently, an initially one-dimensional pair plasma becomes three-dimensional in the region $r \geq r_v$ as pairs scatter out of the $l = 0$ Landau state.

In the bulk frame of the pair plasma, we have $\tilde{\gamma}_- = 5$ and $\tilde{\gamma}_+ = 0.3$ at $r = r_v$ for Crab-like parameters (see Section 2.4). Also in that frame, the soft X-ray photons discussed above are Lorentz-boosted to an energy $h\tilde{\nu} \approx \gamma h\nu \approx 1$ MeV. Since $h\tilde{\nu}$ is of order $\tilde{\gamma}_\pm m_ec^2$, each Compton scattering event typically changes the bulk-frame energy of the scattered electron or positron by $\Delta\tilde{\gamma}_\pm \sim \tilde{\gamma}_\pm$. Multiple scattering therefore washes out the relative streaming between the two species, so that the plasma is neutral in the bulk frame in the region $r > r_v$. A non-zero $E_\parallel$ develops in the lab frame, and the pair plasma is accelerated both parallel and perpendicular to the magnetic field, attaining a Lorentz factor $\gamma_w \approx 10^6$ (Hoshino et al. 1992; Gallant & Arons 1994) before escaping through the light cylinder as a relativistic wind.

If a pure pair plasma is neutral in its bulk frame, it can be shown (Melatos & Melrose 1996) that the lab-frame conduction current $\boldsymbol{J}$ obeys the inequality

$$J/nec < (\Gamma^2 \sin^2\chi + \cos^2\chi)^{-1/2}, \tag{13}$$

where $n = n_+ + n_-$ is the aggregate number density, $\boldsymbol{U}$ is the bulk 3-velocity, $\Gamma = (1 - U^2/c^2)^{-1/2}$ is the bulk Lorentz factor, and χ is the angle between $\boldsymbol{U}$ and $\boldsymbol{J}$. This effect has the following physical origin. Under a Lorentz boost from the bulk frame to the lab frame, the aggregate number density increases by a factor Γ ($n = \tilde{n}\Gamma$), velocity components perpendicular to $\boldsymbol{U}$ decrease by Γ, and velocity components parallel to $\boldsymbol{U}$ are close to c. The components of the conduction current therefore transform as $J_\parallel = \tilde{J}_\parallel\Gamma$ and $J_\perp = \tilde{J}_\perp$, while the conduction-current upper bound in the bulk frame transforms as $\tilde{n}ec = nec/\Gamma$. (Note that $\parallel$ and $\perp$ refer to directions relative to $\boldsymbol{U}$, not $\boldsymbol{B}$.) Together with $\tilde{J} < \tilde{n}ec$, these results imply equation (13) (Melatos & Melrose 1996). For highly relativistic bulk velocities, (13) reduces to $J < nec/\Gamma$ except for a narrow range of χ values satisfying $\sin\chi \lesssim \Gamma^{-1} \ll 1$.

Let us now compare J_D and J in the outer magnetosphere. From (13), we obtain $J(r) = \kappa n_{GJ} ec/\gamma_w \approx \varepsilon_0 \omega B(r) c\kappa/\gamma_w$, because the flow is not field-aligned (i.e., $\chi \gtrsim \gamma_w^{-1}$). The displacement current is given approximately by $J_D(r) \approx \varepsilon_0 \omega E \approx \varepsilon_0 \omega^2 r B(r)$. We therefore find

$$J_D(r)/J(r) = \gamma_w r/\kappa r_L, \tag{14}$$

and hence $J_D \gg J$ in the region $r \geq r_v$ for typical parameters ($\gamma_w \approx 10^6$, $\kappa \approx 2\times10^4$, $r_v/r_L \approx 0.1$); the outer magnetosphere is current-starved and behaves like an electrodynamic vacuum. In the inner magnetosphere $r_* \leq r \leq r_v$, the conduction current is larger by a factor γ_w (charge separation in the bulk frame, $\chi = 0$), giving $J_D/J \approx r/\kappa r_L \ll 1$ (and $J_D/J \approx r_*/r_L \ll 1$ below the pair formation front). Hence the inner magnetosphere does not behave like an electrodynamic vacuum. Note that, in Section 2.1, the transition to a current-starved plasma is postulated to occur abruptly at r_v, whereas in reality it is likely to occur gradually over a distance $\sim r_v$.

Equation (14) implies that the relativistic wind beyond the light cylinder is also current-starved. A pulsar wind obeys the asymptotic scalings $J_D \propto r^{-1}$ and $J \propto r^{-2}$, yielding $J_D(r)/J(r) = rJ_D(r_L)/r_L J(r_L) \gg 1$. The electromagnetic fields in the wind are therefore approximately the far-zone radiation fields of a magnetic dipole rotating *in vacuo* (Usov 1975, 1994; Melatos & Melrose 1996; cf. Coroniti 1990). This property carries implications beyond the effects on spin-down discussed in this paper. Particle acceleration by the radiation fields has been invoked to explain non-thermal radiation from cosmological gamma-ray bursters (Usov 1994), and a self-consistent model of the wind (Melatos & Melrose 1996) correctly predicts the ratio of Poynting flux to kinetic-energy flux required to reproduce the observed wind–nebula interaction of the Crab (Kennel & Coroniti 1984).

3 APPLICATION TO OBSERVATIONS

The phenomenological theory of spin-down presented in Section 2 is applied below to pulsars with braking indices measured by two different methods: absolute pulse numbering (Section 3.1) and interpolation across glitches by modelling short-term transients (Section 3.2). Additional, independent tests of the theory are also proposed, based on measurements of the second deceleration parameter (Section 3.3) and future timing of new objects (Section 3.4).

3.1 Braking indices of the Crab, PSR B0540 – 69 and PSR B1509 – 58

Table 1 lists theoretical and observed values of n for the Crab, PSR B0540 – 69 and PSR B1509 – 58. These are all young pulsars relatively free of glitches and timing noise, with large values of $\dot{\omega}$. They accumulate rotation phase quickly and stably, enabling n to be measured from absolute pulse numbering based on either radio timing (e.g., Crab and PSR B1509 – 58; see Lyne et al. 1993 and Kaspi et al. 1994) or optical and X-ray timing (e.g., PSR B0540 – 69; see Manchester & Peterson 1989, Nagase et al. 1990 and Boyd et al. 1995). Uncertainties in the measured values are ± 1 in the least significant digit.

Measurements of ω, B_d and α are also listed in Table 1; ω and B_d are known to high accuracy, whereas α is typically uncertain to at least $\pm 10°$. Note that α can be determined by three independent methods in principle for any single object: (i) fitting the polarization swing with a rotating vector model (Radhakrishnan & Cooke 1969; Lyne & Manchester 1988; Rankin 1993a,b); (ii) fitting the high-energy (optical to gamma-ray) pulse profile with an outer-gap emission model (Romani & Yadigaroglu 1995); and (iii) relating the core width W_{core} and/or conal separation W_{cone} of the radio profile to P and α (Rankin 1990, 1993a,b). Unfortunately, method (i) works well only for pulsars with strong polarized emission across a substantial fraction of the pulse, and it has not yet been applied successfully to any of the objects in Table 1. Furthermore, method (iii) yields little information on the Crab and Vela pulsars, because Rankin (1990) *assumed* $\alpha = 90°$ for these two objects in order to calibrate the width-versus-period relations. Finally, it is often the case that only one method out of three is feasible for a particular object, or that sometimes, even when two or three are feasible, they yield inconsistent results. For example, PSR B0540 – 69 is barely detectable at radio wavelengths, so polarization measurements are out of the question, but it does exhibit a broad triple pulse with $W_{cone} = 180°$ and hence $\alpha = 14°$ (consistent with the observed core width). On the other hand, fitting the high-energy emission of PSR B1509 – 58 to an outer gap model yields $\alpha = 60°$, yet $W_{core} = 36°$ implies the inconsistent value $\alpha = 10°$. Vela is similar in this regard.

The theoretical n values in Table 1 are calculated from ω, B_d and α following the recipe in Section 2.3: we solve equations (6) and (9) for x_v and B_*, then substitute x_v into equation (B4). With the exception of Vela, discussed below, the calculated n is within 4 per cent of the observed value – good agreement for a theory with no free parameters. The discrepancy can be traced back to (i) our crude treatment of

Table 1. Theoretical and observed braking indices n and second deceleration parameters m for four young pulsars, tabulated as a function of the observable parameters ω, B_d and α. The calculated values of the vacuum radius x_v and surface field B_* for each object are also listed.

Object	ω (s^{-1})	B_d/B_c	α (°)	n (th.)	n (obs.)	m (th.)	m (obs.)	x_v	B_*/B_c
Crab	188	0.086	80[a]	2.60	2.51	10.1	10.2 ± 0.1	0.45	0.096
			90[c]	2.61		10.3		0.44	0.094
0540–69	125	0.12	14[b]	2.08	2.01	5.73	—	0.86	0.65
1509–58	41.8	0.36	60[a]	2.92	2.84	14.0	14.5 ± 3.6	0.18	0.42
			10[b]	2.54		9.51		0.49	2.3
Vela	70.4	0.079	65[a]	2.96	1.4 ± 0.2	14.5	—	0.13	0.092
			90[c]	2.97		14.5		0.12	0.080

[a]High-energy pulse profile (Romani & Yadigaroglu 1995).
[b]Core width and cone separation versus period (Rankin 1990, 1993a,b).
[c]Width-versus-period calibrator with $\alpha = 90°$ assumed (Rankin 1990, 1993a,b).

the pair-cascade multiplicity κ based on Daugherty & Harding's (1982) simulations, and (ii) our imperfect knowledge of α from observations. As an illustration, if Daugherty & Harding (1982) systematically underestimated κ by 50 per cent, so that the proportionality constant in equation (6) is 9×10^2 rather than 8×10^2, we obtain $n = 2.51$, 2.01 and 2.91 for the Crab, PSR B0540 − 69 and PSR B1509 − 58 respectively.

A recent new measurement of n for PSR B0540 − 69, based in part on timing data from the High-Speed Photometer on the *Hubble Space Telescope*, yielded $n = 2.28 \pm 0.02$ (Boyd et al. 1995). This value differs significantly from the one appearing in Table 1, which was obtained independently by two different groups using two different techniques: ground-based optical timing (Manchester & Peterson 1989) and X-ray timing (Nagase et al. 1990). The origin of the discrepancy has not yet been identified. We note that Boyd et al. (1995) combined X-ray and optical data from different epochs when fitting for the pulsar frequency as a function of time.

3.2 Braking index of Vela

It is impossible to determine the braking index of Vela from absolute pulse numbering, because it glitches too frequently (nine major glitches in 25 yr). Instead, Lyne et al. (1996) used the following method to estimate n for this object. They postulated the existence of a 'point of stability' 100 d after each glitch (when the exponentially decaying transients following a glitch have largely subsided, but timing noise has not accumulated to a significant level), and computed a mean $\ddot{\omega}$ from values of $\dot{\omega}$ at the nine historically available points of stability. Note that the relatively large quoted uncertainty ($n = 1.4 \pm 0.2$) corresponds to the formal least-squares error when fitting a time-independent $\ddot{\omega}$ to the Vela data; it is *not* an estimate of the systematic error arising from identifying points of stability, which is potentially greater. Lyne et al. (1996) claimed their method to be robust in the sense that points of stability between 50 and 200 d after a glitch all yield approximately the same n.

In Table 1, the measured and predicted values of n for Vela are compared. They are discrepant, unlike the situation for the other three objects. Moreover, we are powerless to adjust the theory to produce agreement, because there are no free parameters to adjust. As pointed out above, the Vela measurement is not obtained by the same technique as for the other three pulsars, and it is more susceptible to systematic errors. Nevertheless, if $n = 1.4 \pm 0.2$ is taken at face value, our model fails to describe the spin-down of the Vela pulsar correctly.

3.3 Second deceleration parameter

Simultaneous measurements of n and m exist for two objects, the Crab and PSR B1509 − 58. Kaspi et al. (1994) determined $\dddot{\omega}$ for PSR B1509 − 58 from absolute pulse numbering to an accuracy of ~ 25 per cent, finding $m = 14.5 \pm 3.6$. PSR B1509 − 58 has a high $\dot{\omega}$ and is largely free of glitches and timing noise, so this accuracy will improve over time as more rotation phase accumulates. Lyne et al. (1993) determined $\dddot{\omega}$ for the Crab *indirectly* by fitting the cubic timing residuals according to the simple spin-down law $\dot{\omega} \propto \omega^n$ and then checking for consistency a posteriori. They found $m = 10.2 \pm 0.1$. Glitches and timing noise during their data span precluded a measurement from absolute pulse numbering; indeed, a direct estimate of m may never be possible for the Crab. Lyne et al. (1993) quote a formal accuracy of ~ 1 per cent for their fitting procedure, but the uncertainties introduced by glitches and timing noise are likely to be much greater (Kaspi et al. 1994).

Blandford & Romani (1988) pointed out that a power-law braking torque of the form $\dot{\omega} \propto \omega^n$ implies the relation $m = n(2n - 1)$. The spin-down mechanism proposed in this paper predicts deviations from this relation, as given by (11) and (B7). The theoretical m values for the Crab and PSR B1509 − 58 are consistent with those observed, as Table 1 indicates, but this gives us little real information about the accuracy of the theory, because the measurements of $\dddot{\omega}$ are very uncertain. It will become possible to test the theory with increasing precision as timing data for PSR B1509 − 58 is collected over a longer time span. To illustrate the precision required, note that the difference between the theoretical values of $n(2n - 1)$ and m for this object is ≈ 0.2. One therefore needs to measure $\dddot{\omega}$ to better than 2 per cent.

3.4 Further tests of the theory

In the previous section, we compare theory and observation directly for objects where an estimate of α exists. When α is not known, the theory can be tested indirectly in the following ways.

(1) Equations (B4) and (B7) imply $2.0 < n \leq 3.0$ and $5.6 < m \leq 15$ for all pulsars regardless of ω, B_d and α, provided we have $p = 1.2$ and $0 \leq r_v < r_L$. If any pulsar is found lying well outside these bounds, it is either not covered by the theory (i.e., it has $r_v \geq r_L$ from equation 6), or else the theory itself is incomplete. Vela is ostensibly such an object, but its measured n may be significantly corrupted by systematic errors, as discussed above.

(2) Equation (B7) implies $m < n(2n - 1)$ for all pulsars, regardless of ω, B_d and α.

(3) *The theory predicts that n approaches 3.0 as a pulsar ages.* To see why, suppose a pulsar is born with $x_v < 1$ (and possibly $x_v \sim 1$), so that it has $n < 3$ and the theory applies. As the pulsar ages, $x_v \propto \omega^p$ decreases ($p > 0$), and the corotating inner magnetosphere resembles a point dipole ever more closely as time elapses. It is, of course, very difficult to measure n directly for old pulsars, because one must observe for many years before enough rotation phase accumulates. However, it may be possible to do so indirectly, for example, by fitting an $n = 3$ spin-down law to the data and asking whether the residuals are statistically consistent with a purely stochastic process. Alternatively, one can argue that the trajectories of young pulsars on the P–$\dot{P}$ diagram would eventually place them in the upper right corner of the diagram ($P > 0.5$ s, $B_d > 10^{13}$ G), where no objects have been found, unless n approaches 3 at late times. We will analyse the movement of pulsars on the P–$\dot{P}$ diagram in a future paper.

One item of indirect evidence for $n \rightarrow 3$ comes from Vela. It is thought to be the oldest of the four pulsars in Table 1, based on the age of its supernova remnant, and indeed its *theoretical* braking index is the closest to 3. Of course, its observed braking index is the smallest of the four. It will remain difficult to say anything firm about Vela until it is understood to what extent systematic errors affect its measured n.

4 CONCLUSION

In this paper, a modified version of the vacuum-dipole model of rotation-powered pulsars is proposed, in which the neutron star *and its inner magnetosphere* are regarded as a single unit, a magnetized, conducting sphere rotating *in vacuo*. The radius of the sphere is chosen to be the innermost radius r_v where cyclotron de-excitation occurs slowly enough for particles to move an appreciable distance before decaying to their Landau ground states (Section 2.1). Beyond r_v, the magnetosphere behaves like an electrodynamic vacuum because relativistic effects limit the conduction current to being much smaller than the displacement current (i.e., the outer magnetosphere is current-starved; see Section 2.5).

One can calculate the braking torque in the above model from the Deutsch radiation fields of a magnetized, conducting sphere rotating *in vacuo* (Section 2.2). From the torque, one then obtains the braking index n and second deceleration parameter m in terms of just three observable parameters: ω, B_d and α. The recipe for calculating n and m is to solve equations (6) and (9) simultaneously for x_v and B_*, then substitute x_v into equations (B4) and (B7), or the more convenient approximate formulae (10) and (11). The theory contains no adjustable parameters.

Predicted braking indices for the Crab, PSR B0540 − 69 and PSR B1509 − 58 are $n = 2.61$, 2.08 and 2.92 respectively, compared to the measured values $n = 2.51$, 2.01 and 2.84 obtained from absolute pulse numbering. For Vela, n is predicted to be 2.97, against a measured value of 1.4 based on a technique that interpolates across glitches by modelling short-term transients; it is not clear to what extent this measurement is corrupted by systematic errors. Note that $n < 3$ is achieved by changing the effective radius of the rotating, magnetized star, not by distorting the magnetosphere and developing the torque at a surface inside the light cylinder as alternative models suggest (Blandford & Romani 1988; Arons 1992). Table 1 shows x_v ranging from 0.1 to 0.9 for the four objects studied, and B_*/B_c ranging from 0.1 to 0.7, up to five times higher than the apparent dipole field deduced from P and $\dot{P}$. Predictions of m are also consistent with presently available data for the Crab and PSR 1509 − 58, but a precise test of the theory must await better $\dddot{\omega}$ measurements. At the moment, PSR B1509 − 58 is the best candidate for testing these predictions, but constraints from other objects will improve over time, and there is the hope that more high-$\dot{\omega}$, low-timing-noise pulsars will be discovered in the future. The theory also predicts that n approaches 3 for all pulsars as they age.

We infer, from the empirical success of the theory, that its underlying physical assumptions are approximately correct: (i) the outer magnetosphere behaves like an electrodynamic vacuum despite being filled with plasma at $\sim 10^4$ times the Goldreich–Julian density; (ii) spin-down power is transported mainly as Poynting flux at the light cylinder, so that the electromagnetic torque is a good approximation to the total torque; and (iii) the inner magnetosphere is a corotating extension of the star. Assumption (i) is also supported by the empirical success of outer-gap models (which assume vacuum radiation fields near r_L) in fitting observed high-energy pulse profiles from young pulsars (Chiang & Romani 1992, 1994; Romani & Yadigaroglu 1995), as well as by wave-like pulsar-wind models that correctly predict the ratio of Poynting flux to kinetic-energy flux at the wind termination shock (Melatos & Melrose 1996).

Several improvements can be made to the theory presented in Section 2. A better understanding of the transition from an inner, corotating magnetosphere to an outer, current-starved region is needed; here the transition is assumed to occur abruptly at a spherical surface, clearly an idealization. The physics of low-altitude curvature-synchrotron pair cascades, which partly governs the location of r_v through equation (4), is also treated crudely, and this is probably why the model systematically overestimates n by ~ 4 per cent. On the observational front, better estimates of the angle α between the rotation and magnetic axes are required, as are more precise n and m measurements for even more objects. Finally, the spin-down mechanism proposed here has important implications for the surface magnetic fields of neutron stars and their evolutionary tracks on the P–$\dot{P}$ diagram. These issues will be taken up in a forthcoming paper.

ACKNOWLEDGMENTS

I am grateful to Jon Arons and Vicky Kaspi for several fruitful discussions, and to Don Melrose for commenting on the manuscript. This work was supported by NASA Grant NAG5-2756 and NSF Grant AST-93-15455.

REFERENCES

Arons J., 1983, ApJ, 266, 215
Arons J., 1992, in Gil J. A., Hankins T. H., Rankin J. M., eds, The Magnetospheric Structure and Emission Mechanisms of Radio Pulsars. Pedagogical Univ. Press, Zielona Góra, p. 56
Blandford R. D., Romani R. W., 1988, MNRAS, 234, 57P
Blandford R. D., Applegate J. H., Hernquist L., 1983, MNRAS, 204, 1025
Boyd P. T. et al., 1995, ApJ, 448, 365
Cheng A. F., Ruderman M. A., 1977a, ApJ, 212, 800
Cheng A. F., Ruderman M. A., 1977b, ApJ, 214, 598
Cheng K. S., Ho C., Ruderman M. A., 1986a, ApJ, 300, 500
Cheng K. S., Ho C., Ruderman M. A., 1986b, ApJ, 300, 522
Chiang J., Romani R. W., 1992, ApJ, 400, 629
Chiang J., Romani R. W., 1994, ApJ, 436, 754
Coroniti F. V., 1990, ApJ, 349, 538
Daugherty J. K., Harding A. K., 1982, ApJ, 252, 337
Daugherty J. K., Harding A. K., 1996, ApJ, 458, 278
Deutsch A. J., 1955, Ann. d'Astrophys., 18, 1
Gallant Y. A., Arons J., 1994, ApJ, 435, 230
Goldreich P., Julian W. H., 1969, ApJ, 157, 869
Goldreich P., Julian W. H., 1970, ApJ, 160, 971
Hoshino M., Arons J., Gallant Y. A., Langdon A. B., 1992, ApJ, 390, 454
Kaspi V. M., Manchester R. N., Siegman B., Johnston S., Lyne A. G., 1994, ApJ, 422, L86
Kennel C. F., Coroniti F. V., 1984, ApJ, 283, 694
Lyne A. G., Graham-Smith F., 1990, Pulsar Astronomy. Cambridge Univ. Press, Cambridge

Lyne A. G., Manchester R. N., 1988, MNRAS, 234, 477
Lyne A. G., Pritchard R. S., Graham-Smith F., 1993, MNRAS, 265, 1003
Lyne A. G., Pritchard R. S., Graham-Smith F., Camilo F., 1996, Nat, 381, 497
Manchester R. N., Peterson B. A., 1989, ApJ, 342, L23
Melatos A., Melrose D. B., 1996, MNRAS, 279, 1168
Mestel L., 1992, in Blandford R. D., Hewish A., Lyne A. G., Mestel L., eds, Pulsars as Physics Laboratories. Royal Society, London, p. 93
Mestel L., Shibata S., 1994, MNRAS, 271, 621
Mestel L., Robertson J. A., Wang Y.-M., Westfold K. C., 1985, MNRAS, 217, 443
Mészáros P., 1992, High-Energy Radiation from Magnetized Neutron Stars. Univ. Chicago Press, Chicago, p. 124
Michel F. C., 1969, ApJ, 158, 727
Muslimov A., Page D., 1996, ApJ, 458, 347
Nagase F., Deeter J., Lewis W., Dotani T., Makino F., Mitsuda K., 1990, ApJ, 351, L13
Ostriker J. P., Gunn J. E., 1969, ApJ, 157, 1395
Radhakrishnan V., Cooke D. J., 1969, Astrophys. Lett., 3, 225
Rankin J. M., 1990, ApJ, 352, 247
Rankin J. M., 1993a, ApJ, 405, 285
Rankin J. M., 1993b, ApJS, 85, 145
Romani R. W., Yadigaroglu I.-A., 1995, ApJ, 438, 314
Sturner S. J., Dermer C. D., Michel F. C., 1995, ApJ, 445, 736
Usov V. V., 1975, Ap&SS, 32, 375
Usov V. V., 1994, MNRAS, 267, 1035
Usov V. V., Melrose D. B., 1995, Aust. J. Phys., 48, 571

APPENDIX A: DERIVATION OF THE BRAKING TORQUE

The electromagnetic fields generated by a magnetized, conducting sphere rotating *in vacuo* were written down by Deutsch (1955). Let the radius of the sphere be r_0, let its angular frequency be ω, and let α denote the angle between its rotation and magnetic axes. Define spherical polar coordinates (r, θ, ϕ) centred on the sphere, with the rotation axis lying in the z direction. If the frozen-in magnetic field is dipolar at $r=r_0$, with magnitude B_0 at the poles, then the electromagnetic field components outside the sphere are given by the *real parts* of

$$B_r = B_0\left\{\left(\frac{x_0}{x}\right)^3 \cos\alpha\cos\theta + \frac{x_0}{h_1(x_0)}\frac{h_1(x)}{x}\sin\alpha\sin\theta\, \mathrm{e}^{\mathrm{i}\eta}\right\}, \tag{A1}$$

$$B_\theta = \frac{B_0}{2}\left\{\left(\frac{x_0}{x}\right)^3 \cos\alpha\sin\theta + \left[\frac{x_0^2 h_2(x)}{x_0 h_2'(x_0)+h_2(x_0)} + \frac{x_0}{h_1(x_0)}\frac{x h_1'(x)+h_1(x)}{x}\right]\sin\alpha\cos\theta\,\mathrm{e}^{\mathrm{i}\eta}\right\}, \tag{A2}$$

$$B_\phi = \frac{\mathrm{i}B_0}{2}\left[\frac{x_0^2 h_2(x)}{x_0 h_2'(x_0)+h_2(x_0)}\cos 2\theta + \frac{x_0}{h_1(x_0)}\frac{x h_1'(x)+h_1(x)}{x}\right]\sin\alpha\,\mathrm{e}^{\mathrm{i}\eta}, \tag{A3}$$

$$E_r = \frac{\omega r_0 B_0}{2}\left[-\frac{1}{2}\left(\frac{x_0}{x}\right)^4\cos\alpha(3\cos 2\theta+1) + \frac{3x_0}{x_0 h_2'(x_0)+h_2(x_0)}\frac{h_2(x)}{x}\sin\alpha\sin 2\theta\,\mathrm{e}^{\mathrm{i}\eta}\right], \tag{A4}$$

$$E_\theta = \frac{\omega r_0 B_0}{2}\left\{-\left(\frac{x_0}{x}\right)^4\cos\alpha\sin 2\theta + \left[\frac{x_0}{x_0 h_2'(x_0)+h_2(x_0)}\frac{x h_2'(x)+h_2(x)}{x}\cos 2\theta - \frac{h_1(x)}{h_1(x_0)}\right]\sin\alpha\,\mathrm{e}^{\mathrm{i}\eta}\right\}, \tag{A5}$$

$$E_\phi = \frac{\mathrm{i}\omega r_0 B_0}{2}\left[\frac{x_0}{x_0 h_2'(x_0)+h_2(x_0)}\frac{x h_2'(x)+h_2(x)}{x} - \frac{h_1(x)}{h_1(x_0)}\right]\sin\alpha\cos\theta\,\mathrm{e}^{\mathrm{i}\eta}, \tag{A6}$$

where h_1 and h_2 are first- and second-order Hankel functions of the first kind, and we write $x=\omega r/c$, $x_0=\omega r_0/c$ and $\eta=\phi-\omega t$. The expressions for B_ϕ and E_θ given by Deutsch (1955) contain minor typographical errors, which are corrected above.

The electromagnetic braking torque $I\dot{\omega}$ exerted on the rotating sphere is equal to the integral over its surface S (or any other closed surface) of the z component of angular momentum flux through S:

$$I\dot{\omega} = -\varepsilon_0\int_S\left[(\boldsymbol{x}\times\boldsymbol{E})_z\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{S} + c^2(\boldsymbol{x}\times\boldsymbol{B})_z\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} - \tfrac{1}{2}(E^2+c^2B^2)(\boldsymbol{x}\times\mathrm{d}\boldsymbol{S})_z\right] \tag{A7}$$

$$= -\varepsilon_0 r_0^3\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin^2\theta\,(E_r E_\phi + c^2 B_r B_\phi). \tag{A8}$$

In (A8), the field components E_r, E_ϕ, B_r and B_ϕ are all evaluated at $x=x_0$. Upon carrying out the θ and ϕ integrals, we arrive at the result

$$I\dot{\omega}=\frac{2\pi B_0^2 r_0^3 \sin^2\alpha}{\mu_0}\left\{\frac{1}{3}\,\mathrm{Im}\left[\frac{x_0 h'_1(x_0)+h_1(x_0)}{h_1(x_0)}\right]-\frac{1}{5}\,\mathrm{Im}\left[\frac{x_0^2 h_2(x_0)}{x_0 h'_2(x_0)+h_2(x_0)}\right]\right\} \tag{A9}$$

$$=-\frac{2\pi B_0^2 r_0^6\omega^3 \sin^2\alpha}{\mu_0 c^3}\left[\frac{1}{3(x_0^2+1)}+\frac{x_0^4}{5(x_0^6-3x_0^4+36)}\right], \tag{A10}$$

where the definitions of the Hankel functions as products of polynomials and exponentials are used to obtain (A10) from (A9).

Equation (A10) reduces to the standard expression for the torque exerted on a rotating point dipole in the limit $r_0\to 0$ (Ostriker & Gunn 1969). The point-dipole approximation is an excellent one if r_0 is the stellar radius, since one has $r_* \ll r_{\rm L}$ for all but the fastest pulsars. However, in the model proposed in this paper, r_0 is the effective radius of the (inner) corotating magnetosphere and satisfies $r_0\sim r_{\rm L}$, in which case the torque is reduced to roughly half the point-dipole value. Although (A10) is formally valid for all $x_0\geq 0$, in reality a model with $x_0\geq 1$ is not physically meaningful, because the conducting sphere in the region $1\leq x\leq x_0$ would be rotating at a velocity greater than c. [In the formal radiation calculation leading to (A10), we assume that the sphere is massless, thereby artificially circumventing the problem.] Note that, for $x_0<1$, the second term in (A10) is less than 4 per cent of the first term and can be neglected for many purposes.

An alternative way to derive (A10) is to integrate the far-field Poynting flux through a spherical surface S' at infinity. This yields the spin-down luminosity

$$I\omega\dot{\omega}=-\varepsilon_0 c^2\int_{S'}(\boldsymbol{E}\times\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{S} \tag{A11}$$

$$=-\frac{\varepsilon_0 c^4 x^2}{\omega^2}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\,\sin\theta\,(E_\theta B_\phi-E_\phi B_\theta). \tag{A12}$$

In (A12), the field components E_θ, E_ϕ, B_θ and B_ϕ are evaluated in the limit $x\to\infty$ using the asymptotic forms of the Hankel functions:

$$B_\theta=\frac{\mathrm{i}B_0}{2x}\left[\frac{x_0^2}{x_0 h'_2(x_0)+h_2(x_0)}-\frac{x_0}{h_1(x_0)}\right]\sin\alpha\cos\theta\,\mathrm{e}^{\mathrm{i}(x+\eta)}+O(x^{-2}), \tag{A13}$$

$$B_\phi=-\frac{B_0}{2x}\left[\frac{x_0^2}{x_0 h'_2(x_0)+h_2(x_0)}\cos 2\theta-\frac{x_0}{h_1(x_0)}\right]\sin\alpha\,\mathrm{e}^{\mathrm{i}(x+\eta)}+O(x^{-2}), \tag{A14}$$

$$(E_\theta, E_\phi)=c(B_\phi, -B_\theta). \tag{A15}$$

Upon substituting (A13), (A14) and (A15) into (A12) and integrating over θ and ϕ, we arrive at the result

$$I\omega\dot{\omega}=-\frac{2\pi c^3 B_0^2\sin^2\alpha}{\mu_0\omega^2}\left[\frac{1}{3}\left|\frac{x_0}{h_1(x_0)}\right|^2+\frac{1}{5}\left|\frac{x_0^2}{x_0 h'_2(x_0)+h_2(x_0)}\right|^2\right] \tag{A16}$$

$$=-\frac{2\pi B_0^2 r_0^6\omega^4\sin^2\alpha}{\mu_0 c^3}\left[\frac{1}{3(x_0^2+1)}+\frac{x_0^4}{5(x_0^6-3x_0^4+36)}\right]. \tag{A17}$$

The spin-down luminosity therefore equals the braking torque multiplied by ω, as expected for an isolated rigid rotator (Ostriker & Gunn 1969).

APPENDIX B: DERIVATION OF n AND m

In this appendix, we derive expressions for the braking index n and second deceleration parameter m in terms of the 'vacuum radius' $x_{\rm v}$ defined in Section 2.1. This requires us to calculate the second- and third-order frequency derivatives $\ddot{\omega}$ and $\dddot{\omega}$.

The electromagnetic braking torque calculated in Appendix A implies the proportionality

$$\dot{\omega}\propto\omega^3\left[\frac{1}{3(x_{\rm v}^2+1)}+\frac{x_{\rm v}^4}{5(x^6-3x_{\rm v}^4+36)}\right], \tag{B1}$$

where ω and $x_{\rm v}$ are functions of time but the constant of proportionality is not, because it is composed of stellar parameters (B_*, I_*, r_* and α) which do not vary appreciably over the spin-down time-scale (at least in the model considered here). The relation (B1) can be written purely in terms of ω by noting that we have $x_{\rm v} \propto \omega^p$ ($p \approx 1.2$; see Section 2.1) for any individual pulsar, and hence

$$\frac{\dot{x}_{\rm v}}{x_{\rm v}} = \frac{p\dot{\omega}}{\omega}. \tag{B2}$$

In what follows, we drop the subscript 'v' from $x_{\rm v}$ for the sake of readability.

We first compute $\ddot{\omega}$ and hence $n = \omega\ddot{\omega}/\dot{\omega}^2$ by differentiating (B1) with respect to time:

$$n = \frac{3\dot{\omega}}{\omega^3} \bigg/ \left[\frac{1}{3(x^2+1)} + \frac{x^4}{5(x^6-3x^4+36)}\right] - \frac{2x\dot{x}}{\omega^2}\left[\frac{1}{3(x^2+1)^2} + \frac{x^2(x^6-72)}{5(x^6-3x^4+36)^2}\right] \bigg/ \left[\frac{1}{3(x^2+1)} + \frac{x^4}{5(x^6-3x^4+36)}\right]^2 \tag{B3}$$

$$= 3 - 2px^2\left[\frac{1}{3(x^2+1)^2} + \frac{x^2(x^6-72)}{5(x^6-3x^4+36)^2}\right] \bigg/ \left[\frac{1}{3(x^2+1)} + \frac{x^4}{5(x^6-3x^4+36)}\right]. \tag{B4}$$

Equation (B2) is used to eliminate $\dot{x}$ from (B3), then (B1) is used to rewrite $\dot{\omega}$ in terms of x, yielding (B4). We now differentiate both sides of (B4) with respect to time to obtain an expression for $m = \omega^2\dddot{\omega}/\dot{\omega}^3$. From (B2) and

$$\frac{\omega\dot{n}}{\dot{\omega}} = \frac{\omega}{\dot{\omega}}\left(\frac{\ddot{\omega}}{\dot{\omega}} + \frac{\omega\dddot{\omega}}{\dot{\omega}^2} - \frac{2\omega\ddot{\omega}^2}{\dot{\omega}^3}\right) \tag{B5}$$

$$= n(1-2n) + m, \tag{B6}$$

we find

$$m = n(2n-1) - 4p^2x^4\left[\frac{1}{3(x^2+1)^2} + \frac{x^2(x^6-72)}{5(x^6-3x^4+36)^2}\right]^2 \bigg/ \left[\frac{1}{3(x^2+1)} + \frac{x^4}{5(x^6-3x^4+36)}\right]^2$$

$$+ 4p^2x^2\left[\frac{x^2-1}{3(x^2+1)^3} + \frac{x^2(x^{12}+3x^{10}-468x^6+432x^4+5184)}{5(x^6-3x^4+36)^3}\right] \bigg/ \left[\frac{1}{3(x^2+1)} + \frac{x^4}{5(x^6-3x^4+36)}\right]. \tag{B7}$$

Although n and m are complicated functions of x and p in general, they can be simplified by noting that the second term in (B1) is small compared to the first for $x < 1$ (see Appendix A), as are (by accident) its derivatives. This suggests the following approximate formulae:

$$n = 3 - \frac{2px^2}{x^2+1}, \tag{B8}$$

$$m = n(2n-1) - \frac{4p^2x^2}{(x^2+1)^2}. \tag{B9}$$

We have compared (B8) and (B9) numerically with the exact expressions (B4) and (B7), and have verified the approximations are adequate over most of the range of validity of the model. An accuracy of better than 5 per cent is achieved for $0 \leq x \leq 0.8$ using (B8), and for $0 \leq x \leq 0.6$ using (B9). Where the approximations break down, near $x = 1$, the full expressions (B4) and (B7) must be used.