

Oscillations in Harmonic Analysis

Carlos O. Huaricapcha

Summer 18, San Francisco State University

Notes written from Dr. Eyvindur Palsson

[Problem sets](#)

I'd love to hear your feedback. Feel free to email me at coscohua@mail.sfsu.edu.
See [git:icarlitoss](#) for updates.

Contents

1	Introduction	3
1.0.1	Erdős problem	3
2	Review on Vector Spaces	4
2.1	Inner Product	4
2.2	Norm	4
2.3	Orthogonality	5
2.4	Projections	5
3	Fourier Series	6
3.1	Domains	9
3.2	Uniqueness	9
3.3	Partial Sums	9
3.4	Convolution	11
4	Kernels	12
4.1	Direchlet Kernel	12
4.2	Good Kernels on the circle	12
4.3	Fej'er Kernel	13
4.4	Parseval's identity	14
5	Fourier Series II	14
5.1	Pointwise Convergence of Fourier Series	14
5.2	Uniform Convergence of Fourier Series	15
6	Fourier Transforms on \mathbb{R}	16
6.1	The Schwartz space	16
7	Inner Product and Norm on $\mathcal{S}(\mathbb{R})$	19
8	Fourier Transform on \mathbb{R}^d	20
8.1	Rotations	20
8.2	Coarea Formula / Polar coordinates	21
8.3	Schwartz space	21
8.4	Radical Functions	22
8.5	Fourier Transform of σ in \mathbb{R}^3	22

1 Introduction

← July 2, 2018

Combinatorics

1.0.1 Erdős problem

1. Erdős distinct distance problem (1946).

What is the least number of distinct distances determined by N points in a plane.

Example 1.1. We have four points $(0, 1), (2, 2), (0, 0), (1, 0)$, and if we start listing the distances between each of them we obtain the following:

$$1, 1, \sqrt{2}, \sqrt{5}, \sqrt{5}, \sqrt{8}.$$

However, we care about the distinct number; hence we get a new list

$$1, \sqrt{2}, \sqrt{5}, \sqrt{8}.$$

Upper bound: Notice that the first list is obtain by getting the distance between two points, hence

$$\binom{N}{2} = \frac{1}{2}N^2 - \frac{1}{2}N \sim N^2.$$

are obtained random.

To analyze what happens with the lower bound, we look at the following example.

Example 1.2. (a) Say N is a perfect square.

(b) Then, we have an square of \sqrt{N} lattice.

(c) Now, let's count

$$(\text{distinct distcnes})^2.$$

(d) We obtain the list:

$$1, 2, \dots, 2N.$$

(e) Hence we get no more than $\sim N$.

(f) Notice that $a^2 + b^2 = 3$ has no solution (number theory). Hence our list (d) have holes.

(g) Hence,

$$\# \text{ distinct distance} \sim \frac{N}{\sqrt{\log(N)}} \text{ as } N \rightarrow \infty.$$

Conjecture. Conjecture (Erdős 1946): The answer for Erdos 1946 should be in the order of $\frac{N}{\sqrt{\log(N)}}$ as $N \rightarrow \infty$.

Theorem (Erdős 1946). *At least $\sim \sqrt{N}$ as $N \rightarrow \infty$.*

Theorem (Guth, Katz 2015). *At least $\sim \frac{N}{\log(N)}$ as $N \rightarrow \infty$.*

2. Crescent Configurations N points in the plane such that distance d_1 appears 1 times, d_2 appears 2 times, and so on, until d_{N-1} appears $N - 1$ times. ($N - 1$ distinct distances). It is possible to achieve this if you place equally spaced points on a line. Additionally require general position no more than 2 points on a line, and no more than 3 points on a circle. Call a crescent configuration.

Example 1.3. We can go from 3 pts to 8 points in crescent configuration. It is unknown if we could find a 9 point or higher order configurations.

Conjecture (Erdős). Eventually they do not exist.

Question: Find many (all) crescent configurations for some N .

Example 1.4. Given $N = 4$ in \mathbb{R}^2 .

Question: For $N = 5$. Many known but not all.

2 Review on Vector Spaces

← July 3, 2018

2.1 Inner Product

An *inner product* is a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow F$$

that satisfies for all $f, g, h \in V$, and $\alpha, \beta \in F$.

1. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0 \Rightarrow f = 0$.
2. $\langle f, g \rangle = \langle g, f \rangle$.
3. $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$.

Example 2.1. In \mathbb{R}^d , we have

$$\langle x, y \rangle = x \cdot y = x_1 y_1 + \cdots + x_d y_d.$$

Example 2.2. In \mathbb{C}^d , we have

$$\langle x, y \rangle = x \cdot y = x_1 \bar{y}_1 + \cdots + x_d \bar{y}_d.$$

Example 2.3. In $\mathcal{C}[a, b]$, we have

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx$$

2.2 Norm

A *norm* is a function

$$\| \cdot \| : V \rightarrow \mathbb{R}$$

that satisfies for all $f, g \in V$ and $\alpha \in F$.

1. $\|f\| \geq 0$ and if $\|f\| = 0$ then $f = 0$.

2. $\|\alpha f\| = |\alpha| \|f\|$.
3. $\|f + g\| \leq \|f\| + \|g\|$ (triangle inequality).

In a vector space V with inner product $\langle \cdot, \cdot \rangle$ get a norm for free.

$$\|f\| := \sqrt{\langle f, f \rangle}.$$

Definition (Cauchy-Schwartz Inequality).

$$|\langle f, g \rangle| \leq \|f\| \|g\|.$$

Hint for Problem set: Implies triangle inequality.

$$\begin{aligned} \|f + g\|^2 &= \langle f + g, f + g \rangle \\ &= \|f\|^2 + \langle f, g \rangle + \langle f, g \rangle + \|g\|^2 \\ &\quad (\text{by CS}) \leq \|f\|^2 + 2\|f\| \|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2. \end{aligned}$$

Example 2.4. On $\mathcal{C}[a, b]$ with

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx$$

get

$$\|f\|_2 = \left(\frac{1}{b-a} \int_a^b |f(x)|^2 dx \right)^{1/2}$$

2.3 Orthogonality

Say $f, g \in V$ are orthogonal if

$$\langle f, g \rangle = 0.$$

Say $\{\phi_1, \dots, \phi_n\}$ are orthogonal if $\langle \phi_j, \phi_k \rangle = 0$ whenever $j \neq k$ and $\phi_j \neq 0$ for all j .

If in addition $\|\phi_j\| = 1$ for all j then $\{\phi_1, \dots, \phi_n\}$ is *orthonormal*.

Note: Remember Gram-Schmidt.

Theorem (Pythagorean Theorem). If $\langle f, g \rangle = 0$ then $\|f + g\|^2 = \|f\|^2 + \|g\|^2$.

2.4 Projections

Let $\phi \in B$ with $\|\phi\| = 1$. The projection of f in the direction of ϕ is

$$\text{proj}_\phi(f) := \langle f, \phi \rangle \phi.$$

Example 2.5. Project $f = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ on $\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$\text{proj}_\phi(f) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Definition. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \dots, \phi_n\}$. The projection of f onto W_n is.

$$\text{proj}_{W_n}(f)L = \langle f_1, \phi_1 \rangle \phi_1 + \dots + \langle f_n, \phi_n \rangle \phi_n.$$

Theorem.

$$\text{proj}_{W_n}(f) = 0 \iff f \in W_n.$$

Theorem. $f - \text{proj}_{W_n}$ is orthonormal to every vector in W_n .

3 Fourier Series

← July 5, 2018

Theorem. Let $f \in V$. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \dots, \phi_n\}$. The element $w \in W_n$ that minimizes $\|f - w\|$ is

$$w = \text{proj}_{W_n}(f). \quad \leftarrow \text{Best approximation}$$

Proof. Write:

$$w = \sum_{i=1}^n \beta_i \phi_i$$

and set

$$\alpha_i = \langle f, \phi_i \rangle, \quad i = 1, \dots, n.$$

Then,

$$\begin{aligned} \|f - w\|^2 &= \langle f - w, f - w \rangle \\ &= \|f\|^2 - \langle f, \sum_{i=1}^n \beta_i \phi_i \rangle - \overline{\langle f, \sum_{i=1}^n \beta_i \phi_i \rangle} + \langle \sum_{i=1}^n \beta_i \phi_i, \sum_{j=1}^n \beta_j \phi_j \rangle \\ &= \|f\|^2 - \sum_{i=1}^n \overline{\beta_i} \langle f, \phi_i \rangle - \sum_{i=1}^n \overline{\beta_i} \overline{\langle f, \phi_i \rangle} + \sum_{i=1}^n \sum_{j=1}^n \beta_i \overline{\beta_j} \langle \phi_i, \phi_j \rangle \\ &= \|f\|^2 - \sum_{i=1}^n \overline{\beta_i} \alpha_i - \sum_{i=1}^n \beta_i \overline{\alpha_i} + \sum_{i=1}^n |\beta_i|^2 \\ &= \|f\|^2 + \sum_{i=1}^n (\beta_i - \alpha_i) \overline{\beta_i - \alpha_i} - \sum_{i=1}^n |\alpha_i|^2 \\ &= \|f\|^2 - \sum_{i=1}^n |\langle f, \phi_i \rangle|^2 + \sum_{i=1}^n |\beta_i - \alpha_i|^2. \end{aligned}$$

So minimized if $\beta_i = \alpha_i = \langle f, \phi_i \rangle$. □

Remark:

$$\sum_{i=1}^n \langle f, \phi_i \rangle \phi_i \quad \text{best approximation to } f.$$

Corollary.

$$\sum_{i=1}^n |\langle f, \phi_i \rangle|^2 \leq \|f\|^2.$$

Bessel's inequality:

$$\sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2 \leq \|f\|^2.$$

Riemann-Lebesgue Lemma:

$$\lim_{i \rightarrow \infty} \langle f, \phi_i \rangle = 0.$$

Motivation:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \cos(nx) \cos(mx) dx &= \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \neq 0 \\ 1 & n = m = 0. \end{cases} \\ \frac{1}{2\pi} \int_0^{2\pi} \sin(nx) \sin(mx) dx &= \begin{cases} \frac{1}{2} & \text{if } n = m \neq 0 \\ 0 & \text{otherwise.} \end{cases} \\ \frac{1}{2\pi} \int_0^{2\pi} \sin(nx) \cos(mx) dx &= 0. \end{aligned}$$

Note that

$$1, \sqrt{2} \cos(x), \sqrt{2} \sin(x), \sqrt{2} \cos(2x), \sqrt{2} \sin(2x), \dots$$

orthonormal sequence on $[0, 2\pi]$.

Best finite approximation up to level n of a function f is

$$a_0 \cdot 1 + a_1 \sqrt{2} \cos(x) + b_1 \sqrt{2} \sin(x) + \dots$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot 1 dx \\ a_j &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \sqrt{2} \cos(jx) dx \\ b_k &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot \sqrt{2} \sin(kx) dx. \end{aligned}$$

Trigonometric Series:

$$\frac{1}{2} A_0 + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx))$$

if in addition

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx$$

and

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

then called *Fourier Series*.

Simplification: We could have started with the orthonormal sequence

$$e^{inx}, n \in \mathbb{Z}, \text{ on } [0, 2\pi].$$

Then,

$$\begin{aligned} \langle e^{inx}, e^{imx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{inx} \overline{e^{imx}} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} dx \\ &= \begin{cases} \frac{1}{2\pi} \left[\frac{1}{i(n-m)} e^{i(n-m)x} \right]_0^{2\pi} = 0 & \text{if } n \neq m \\ \frac{1}{2} \int_0^{2\pi} 1 dx = 1 & \text{if } n = m. \end{cases} \end{aligned}$$

Give best approximation up to level $n > 0$ of a function f

$$c_{-n}e^{i(-n)x} + \dots + c_0 \cdot 1 + \dots + c_n e^{inx}$$

with corresponding series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{e^{inx}} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx.$$

Note:

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx \\ &= \begin{cases} \frac{1}{2}(A_n - iB_n) & \text{if } n > 0 \\ \frac{1}{2}A_0 & \text{if } n = 0 \\ \frac{1}{2}(A_{|n|} + iB_{|n|}) & \text{if } n < 0. \end{cases} \end{aligned}$$

For $n > 0$, we have

$$c_n e^{inx} + c_{-n} e^{i(-n)x} = A_n \cos(nx) + B_n \sin(nx)$$

All the same terms as before!

Intervals:

$$\begin{aligned} e^{inx} &\text{ works on any interval of length } 2\pi. \\ e^{2\pi i n x / N} &\text{ works on any interval of length } L. \end{aligned}$$

Summary:

Fourier Series: If f is integrable on $[a, b]$ of length of L then the n^{th} *Fourier coefficient* of f is defined by

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x / L} dx, \quad n \in \mathbb{Z}$$

and its Fourier series is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-2\pi i n x / L}.$$

Question: Does $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-2\pi i n x / L}$ converges to $f(x)$?
Let's first look at the following example.

← July 6, 2018

Example 3.1. Let $f(x) = x$ on $[0, 2\pi]$. Then,

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} dx = \frac{i}{n}, \quad n \neq 0.$$

and

$$\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} x dx = \pi.$$

Fourier series

$$\sum_{n=-\infty}^{\infty} \frac{i}{n} e^{inx} = \pi - 2 \left(\frac{1}{1} \sin(x) + \frac{1}{2} \sin(2x) + \dots \right)$$

$$\uparrow \frac{i}{n} e^{inx} + \frac{i}{-n} e^{-inx} = -\frac{2}{n} \sin(nx).$$

Yields π if $x = 0$ or $x = 2\pi$ while $f(0) = 0$ and $f(2\pi) = 2\pi$.

Consider: [insert graph 1 vs 2 here]:

Note: Fourier series is periodic on \mathbb{R} .

3.1 Domains

Consider $[a, b]$ on periodic functions on \mathbb{R} in a circle such that $f(x) = F(e^{ix})$ and $x \in [0, 2\pi]$. Hence, functions on the circle are periodic functions $f(x)$ on $[0, 2\pi]$ with $f(0) = f(2\pi)$.

3.2 Uniqueness

Theorem. Suppose f is integrable and bounded on an interval with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x_0) = 0$ whenever f is continuous at x_0 .

3.3 Partial Sums

Let

$$S_N(f)(x) := \sum_{n=-N}^N \hat{f}(n) e^{-2\pi i n x / L} \text{ on } [0, L].$$

Question: In what sense $S_N f \rightarrow f$.

Theorem. Suppose f is continuous on the circle and the Fourier series of f is absolutely convergent

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Then

$$\lim_{N \rightarrow \infty} S_N(f)(x) = f(x)$$

uniformly in x .

Proof. Since

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

converges absolutely and in fact uniformly.

If

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

then g must be continuous.

Since $f - g$ is continuous and $\widehat{f - g}(n) = 0$ for all $n \in \mathbb{Z}$ we conclude by uniqueness theorem that $f = g$.

$$\begin{aligned} \hat{g}(n) &= \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\lim_{N \rightarrow \infty} \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} \right) e^{-inx} dx \\ &= \lim_{N \rightarrow \infty} \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{2\pi} \int_0^{2\pi} e^{i(k-n)x} dx. \end{aligned}$$

Note that:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(k-n)x} dx = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{else.} \end{cases}$$

Hence, we have that

$$\begin{aligned} \hat{g}(n) &= \lim_{N \rightarrow \infty} \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{2\pi} \int_0^{2\pi} e^{i(k-n)x} dx \\ &= \hat{f}(n). \end{aligned}$$

□

Theorem. Let $f \in \mathcal{C}^2[0, 2\pi]$ and 2π period. Then

$$\hat{f}(n) = \mathcal{O}\left(\frac{1}{n^2}\right) \text{ as } |n| \rightarrow \infty$$

and thus the Fourier series of f converges absolutely and uniformly to f .

Note: $f(x) = \mathcal{O}(g(x))$ as $x \rightarrow a$ means there exists a constant C such that

$$|f(x)| \leq C|g(x)| \text{ as } x \rightarrow a.$$

Proof. Since f'' is continuous on $[0, 2\pi]$, it is bounded. Say

$$|f''(x)| \leq B \text{ for all } x \in (0, 2\pi].$$

Then

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\ &= \frac{1}{in} \hat{f}'(n) \text{ by int. by parts.} \end{aligned}$$

By iterating $\hat{f}(n) = \frac{1}{i^2 n^2} \widehat{f''}(n)$,

$$\begin{aligned} |\hat{f}(n)| &= \frac{1}{n^2} \left| \frac{1}{2\pi} \int_0^{2\pi} f''(x) e^{-inx} dx \right| \\ &\leq \frac{1}{n^2} \left(\frac{1}{2\pi} \int_0^{2\pi} B dx \right) = \frac{B}{n^2}. \end{aligned}$$

□

Corollary. $\widehat{f'}(n) = in\hat{f}(n)$.

3.4 Convolution

Given 2π periodic integrable functions f and g on \mathbb{R} . Define their *convolution* $f * g$ by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) dy \quad x \in [-\pi, \pi].$$

Remarks: It has many good properties by most importantly

- $f * g = g * f$.
- $\widehat{f * g} = \hat{f}(n) \hat{g}(n)$.

4 Kernels

← July 9, 2018

4.1 Direchlet Kernel

$$\begin{aligned}
 S_N(f)(x) &= \sum_{n=-N}^N \hat{f}(n) e^{inx} \\
 &= \sum_{n=-N}^N \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \right) e^{inx} \\
 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^N e^{-in(x-y)} dy \right) e^{inx} \\
 &= (f * D_N)(x)
 \end{aligned}$$

where $D_N(x) = \sum_{n=-N}^N e^{inx}$. Facts:

- (1) $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) dx = 1$.
- (2) $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$.
- (3) $D_N(x) \leq C \min\{N, \frac{1}{|x|}\}$ if $x \in [-\pi, \pi]$, $N \geq 1$.

4.2 Good Kernels on the circle

Definition. A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if

- (i) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1$ for all $n \geq 1$.
- (ii) There exists $M > 0$ such that $\int_{-\pi}^{\pi} |K_n(x)| dx \leq M$ for all $n \geq 1$.
- (iii) For every $\delta > 0$ $\int_{\delta \leq |x| \leq \pi} |K_n(x)| dx \rightarrow 0$ as $n \rightarrow \infty$.

Theorem. Let $\{K_n\}$ be a family of good kernels and f a bounded and integrable function on the circle. Then

$$\lim_{n \rightarrow \infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x . If f is continuous everywhere then the limit is uniform.

Note: Unfortunately

$$\int_{-\pi}^{\pi} |D_N(x)| dx \geq c \log(N)$$

as N is large. Then D_N is *not* a good kernel.

4.3 Fej'er Kernel

Say a series $\sum_{k=0}^{\infty} c_k$ is Cesáro summable to σ if

$$\lim_{N \rightarrow \infty} \sigma_N = \sigma$$

where

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}$$

is the N^{th} Cesáro mean.

Example 4.1. $\sum_{k=0}^{\infty} (-1)^k$ does not converge but Cesáro summable to $\frac{1}{2}$. For Fourier series the N^{th} Cesáro mean is

$$\begin{aligned} \sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\ &= \frac{(f * D_0)(x) + \cdots + (f * D_{N-1})(x)}{N} \\ &= \left(f * \left(\frac{D_0 + \cdots + D_{N-1}}{N} \right) \right)(x) \\ &= (f * F_N)(x) \quad \left(\text{since } F_N = \frac{D_0 + \cdots + D_{N-1}}{N} \rightarrow \text{Fej'er kernel} \right). \end{aligned}$$

Facts:

$$(1) \quad F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

(2) F_N is a good kernel.

Theorem (Weierstrass Approximation Theorem). *For every continuous function $f : \mathbb{R} \rightarrow \mathbb{C}$ with period 2π and every $\epsilon > 0$ one can find a trigonometric polynomial P such that*

$$\forall x \quad |f(x) - P(x)| < \epsilon.$$

Proof.

$$\begin{aligned} \sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\ &= (f * F_N)(x). \end{aligned}$$

is a trigonometric polynomial and $(f * F_N) \rightarrow f$ uniformly because f continuous and F_N is a good kernel. \square

Mean Square Convergence of Fourier Series

Let f be a continuous function on the circle. Then

$$\|f - S_N(f)\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Proof. By the Weierstrass Approximation Theorem for a given $\epsilon > 0$ there exists a trigonometric polynomial P , say of degree M , such that

$$|f(x) - P(x)| < \epsilon \quad \text{for all } x \in [0, 2\pi]$$

and thus

$$\|f - P\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - P(x)|^2 dx \right)^{1/2} < \epsilon.$$

By the best approximation theorem

$$\|f - S_N(f)\|_2 < \epsilon$$

whenever $N \geq M$. □

4.4 Parseval's identity

From general theory about orthonormal sequences Bessel's inequality:

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \|f\|_2^2.$$

Riemman-Lebesgue Lemma: $\hat{f}(n) \rightarrow 0$ as $|n| \rightarrow \infty$, and

$$\int_0^{2\pi} f(x) \sin(nx) dx \rightarrow 0,$$

$$\int_0^{2\pi} f(x) \cos(nx) dx \rightarrow 0.$$

Theorem (Parseval's Identity).

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \|f\|_2^2$$

where we assume f is continuous on the circle.

Proof. From proof of best approximation theorem

$$\|f\|_2^2 = \|f - S_N(f)\|_2^2 + \sum_{|n| \leq N} |\hat{f}(n)|^2.$$

□

5 Fourier Series II

5.1 Pointwise Convergence of Fourier Series

Theorem (Pointwise Convergence of Fourier Series). *July 10, 2018] Let f be a square integrable function on the circle which is differentiable at x_0 . Then* ← [

$$S_N(f)(x_0) \rightarrow f(x_0) \text{ as } N \rightarrow \infty.$$

Proof.

$$\begin{aligned}
S_N(f)(x_0) - f(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0 - t) D_N(t) dt - f(x_0) \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x_0 - t) - f(x_0)) \frac{\sin((N + \frac{1}{2})t)}{\sin(\frac{t}{2})} dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{f(x_0 - t) - f(x_0)}{t} \frac{t}{\sin(t/2)} \cos(t/2) \right] \sin(Nt) dt \\
&\quad + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{f(x_0 - t) - f(x_0)}{t} \frac{t}{\sin(t/2)} \sin(t/2) \right] \cos(Nt) dt.
\end{aligned}$$

As $t \rightarrow 0$, the right side becomes

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) \cdot 2 \cdot 1 \sin(Nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x_0) \cdot 2 \cdot 0 \cos(Nt) dt$$

and away from $t \rightarrow 0$ then \square are square integrable like. Conclude by Riemman-Lebesgue lemma

$$|S_N(f)(x_0) - f(x_0)| \rightarrow 0 \text{ as } N \rightarrow \infty.$$

\square

Theorem. Let f be a 'square' integrable function on the circle such that $f'(x_0^+)$ and $f'(x_0^-)$ exist. Then

$$S_n(f)(x) \rightarrow \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Proof. Similar. \square

Non-convergence

There exists an integrable function which is continuous at a point x_0 such that $S_N(f)(x_0)$ does not converge.

$$|f(x) - f(y)| \leq C|x - y|^\infty.$$

Gibbs Phenomenon

$$\sum_{k=1}^N \frac{1}{k} \sin(kx) = S_N(f)(x) \rightarrow f(x) = \frac{\pi}{2} - x \text{ on } (0, 2\pi).$$

For a fixed big N . (include graph).

5.2 Uniform Convergence of Fourier Series

Theorem (Uniform Convergence of Fourier Series). Let f be continuously differentiable on the circle. Then its Fourier series converges uniformly to f .

Proof. Since f' is continuous it is square integrable and Parseval's identity holds. Further an old corollary says

$$\hat{f}'(n) = in\hat{f}(n).$$

and

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{\infty} \frac{1}{n} |\hat{f}'(n)| \leq \underbrace{\left(\sum_{n=-\infty}^{\infty} \frac{1}{n^2} \right)^{1/2}}_{\infty} \underbrace{\left(\sum_{n=-\infty}^{\infty} |\hat{f}'(n)|^2 \right)^{1/2}}_{= \|f'\|_2} < \infty.$$

□

6 Fourier Transforms on \mathbb{R}

f on $[0, 1]$ period 1, $n \in \mathbb{Z}$.

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx, \quad f(x) \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

f on \mathbb{R} (no period), $\zeta \in \mathbb{R}$. We have

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \zeta x} dx, \quad f(x) \stackrel{?}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i \zeta x} d\zeta.$$

6.1 The Schwartz space

Definition. The Schwartz space on \mathbb{R} denoted $\mathcal{S}(\mathbb{R})$, consists of all indefinitely differentiable functions f such that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \quad \text{for every } k, l \geq 0.$$

Note: If $f \in \mathcal{S}(\mathbb{R})$ then also

$$x^k f^{(l)}(x) \in \mathcal{S}(\mathbb{R}) \quad \text{for all } k, l \geq 0.$$

Note: $\mathcal{S}(\mathbb{R})$ is a vector space.

Example 6.1. e^{-ax^2} , for $a > 0$ is in $\mathcal{S}(\mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$ define Fourier transform

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \zeta x} dx.$$

Useful facts:

(i)

$$f(x+h) \xrightarrow{\text{Fourier transf.}} e^{2\pi i h \zeta} \hat{f}(\zeta).$$

(ii)

$$f(x) e^{-2\pi i x h} \xrightarrow{\text{Fourier transf.}} \hat{f}(\zeta + h).$$

(iii)

$$f(\delta x) \xrightarrow{\text{Fourier transf.}} \delta^{-1} \hat{f}(\delta^{-1} \zeta), \delta > 0.$$

(iv)

$$f'(x) \xrightarrow{\text{Fourier transf.}} 2\pi i \zeta \hat{f}(\zeta).$$

(v)

$$-2\pi i x f(x) \xrightarrow{\text{Fourier transf.}} \frac{d}{d\zeta} \hat{f}(\zeta).$$

← July 11, 2018

Theorem. If $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.

Proof.

$$\begin{aligned} |\zeta^k \left(\frac{d}{d\zeta} \right)^l \hat{f}(\zeta)| &= \left| \left(\frac{1}{2\pi i} \right)^k \left(\frac{d}{dx} \right)^k ((-2\pi i x)^l f(x))(\zeta) \right| \\ &< \infty. \end{aligned}$$

□

Fact:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Theorem. If $f(x) = e^{-\pi x^2}$ then $\hat{f}(\zeta) = e^{-\pi \zeta^2}$.

Proof. Define $F(\zeta) = \hat{f}(\zeta) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \zeta} dx$. Then $F(0) = 1$ by fact above, and

$$\begin{aligned} F'(\zeta) &= \int_{-\infty}^{\infty} \underbrace{e^{-\pi x^2} (-2\pi i x)}_{f'(x)} e^{-2\pi i x \zeta} dx \\ &= i \hat{f}'(\zeta) \\ &= i(2\pi i \zeta) \hat{f}(\zeta) \\ &= -2\pi \zeta F(\zeta). \end{aligned}$$

Set $G(\zeta) = F(\zeta) e^{\pi \zeta^2}$. Then $G(0) = F(0) = 1$ and

$$G'(\zeta) = \underbrace{F'(\zeta)}_{-2\pi \zeta F(\zeta)} e^{\pi \zeta^2} + F(\zeta) 2\pi \zeta e^{\pi \zeta^2} = 0.$$

So, $G(0) = 1$ and thus $F(\zeta) = e^{-\pi \zeta^2}$.

□

Define for $\delta > 0$.

$$K_\delta(x) = \delta^{1/2} e^{-\pi x^2 / \delta} \text{ then } \hat{K}_\delta(\zeta) = e^{\pi \delta \zeta^2}.$$

(insert picture of graph as δ is small).

Definition. If $f, g \in \mathcal{S}(\mathbb{R})$ their *convolution* is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t) g(t) dt.$$

Note: Similar properties as on circle for $f, g \in \mathcal{S}(\mathbb{R})$.

(i) $f * g \in \mathcal{S}(\mathbb{R})$.

(ii) $f * g = g * f$.

(iii) $\widehat{f * g} = \hat{f}\hat{g}$.

Theorem. The collection $\{K_\delta\}_{\delta>0}$ is a family of good kernels as $\delta \rightarrow 0^+$.

Theorem. If $f \in \mathcal{S}$ then $(f * K_\delta)(x) \rightarrow f(x)$ uniformly in x as $\delta \rightarrow 0^+$.

Note: Works for any good kernel.

Proposition. If $f, g \in \mathcal{S}(\mathbb{R})$ then

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy.$$

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\hat{g}(x)dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y)e^{-2\pi ixy}dydx \\ &= \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx dy \\ &= \int_{-\infty}^{\infty} g(y)\hat{f}(y)dy. \end{aligned}$$

□

Theorem (Fourier Inversion). if $f \in \mathcal{S}(\mathbb{R})$ then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi i x \zeta}d\zeta$.

Proof. First note that

$$\begin{aligned} f(0) &= \lim_{\delta \rightarrow 0^+} (f * K_\delta)(0) \\ &= \lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} f(x)K_\delta(0-x)dx \\ &= \lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} f(x) \underbrace{\delta^{-1/2}e^{-\pi x^2/\delta}}_{K_\delta(x)}dx. \end{aligned}$$

Also we have that,

$$\begin{aligned} e^{-\pi x^2} &= \widehat{e^{-\pi \delta^2}}(x) \\ \Rightarrow \delta^{-1/2}e^{-\pi x^2/\delta} &= \underbrace{\delta^{-1/2}\widehat{e^{-\pi \delta^2}}(\delta^{-1/2}x)}_{\widehat{e^{-\pi \delta^2}}(x)} \end{aligned}$$

Then, continuing our first observation

$$\begin{aligned}
f(0) &= \lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} f(x) \underbrace{\delta^{-1/2} e^{-\pi x^2 / \delta}}_{K_\delta(x)} dx \\
&= \lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{-\pi \delta \zeta^2} d\zeta \\
&\stackrel{\text{Prop.}}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta) d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i \delta \zeta} d\zeta.
\end{aligned}$$

In general, let

$$F(y) = f(x + y).$$

Then

$$\begin{aligned}
f(x) &= F(0) \\
&= \int_{-\infty}^{\infty} \hat{F}(\zeta) d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i x \zeta} d\zeta.
\end{aligned}$$

□

Corollary. *The Fourier transform is bijective mapping on $\mathcal{S}(\mathbb{R})$*

$$\mathcal{S}(\mathbb{R}) \xrightarrow{\mathcal{F}(f)(\zeta) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \zeta} dx} \mathcal{S}(\mathbb{R}).$$

and

$$\mathcal{S}(\mathbb{R}) \xrightarrow{\mathcal{F}^{-1}(g)(\zeta) = \int_{-\infty}^{\infty} g(x) e^{-2\pi i x \zeta} dx} \mathcal{S}(\mathbb{R}).$$

7 Inner Product and Norm on $\mathcal{S}(\mathbb{R})$

Can use the usual

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

and

$$\|f\|_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{1/2}.$$

Theorem (Plancharel). *If $f \in \mathcal{S}(\mathbb{R})$ then $\|f^2\|_2 = \|f\|_2$.*

Proof.

$$\begin{aligned}
\widehat{f^2}(\zeta) &= \int_{-\infty}^{\infty} \overline{f(-x)} e^{-2\pi i x \zeta} dx \\
&= \overline{\int_{-\infty}^{\infty} f(-x) e^{-2\pi i (-x) \zeta} dx} \\
&= \overline{\hat{f}(\zeta)}.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^2 dx &= \int_{-\infty}^{\infty} f(x) \overline{f(x)} dx \\
&= \int_{-\infty}^{\infty} \overline{f(-(-x))} dx \\
&= \int_{-\infty}^{\infty} f(x) f^{\#}(0-x) dx \\
&= (f * f^{\#})(0) \\
&\stackrel{\text{inversion}}{=} \int_{-\infty}^{\infty} \widehat{f * f^{\#}}(\zeta) d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) \widehat{f^{\#}}(\zeta) d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) \overline{\hat{f}(\zeta)} d\zeta \\
&= \|\hat{f}\|_2^2.
\end{aligned}$$

Note: \mathcal{F} is unitary on $\mathcal{S}(\mathbb{R})$ with $\|\cdot\|_2$.

Note:

- Parseval:

$$\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \right)^{1/2} = \|f\|_2.$$

- Plancharel:

$$\|\hat{f}\|_2 = \|f\|_2.$$

□

8 Fourier Transform on \mathbb{R}^d

← July 12, 2018

8.1 Rotations

A rotation in \mathbb{R}^d is a linear transformation that fulfills one of the following equivalent conditions

- (a) $Rx \cdot Ry = x \cdot y$ for all $x, y \in \mathbb{R}^d$.
- (b) $|Rx| = |x|$ for all $x \in \mathbb{R}^d$.
- (c) $R^t = R^{-1}$.
- (d) $\det(R) = \pm 1$.

Useful:

$$\int_{\mathbb{R}^d} f(Rx) dx = \int_{\mathbb{R}^d} f(x) dx.$$

8.2 Coarea Formula / Polar coordinates

Write $\gamma = (\cos(\theta), \sin(\theta))$. For a function on g on S^1 define

$$\int_{S^1} g(\gamma) d\sigma(\gamma) = \int_0^{2\pi} g(\cos(\theta), \sin(\theta)) d\theta.$$

Then can write

$$\begin{aligned} \int_{\mathbb{R}^2} f(x) dx &= \int_0^{2\pi} \int_0^\infty f(r \cos(\theta), r \sin(\theta)) r dr d\theta \\ &= \int_{S^1} \int_0^\infty f(r\gamma) r dr d\sigma(\gamma). \end{aligned}$$

For g on S^2 define

$$\int_{S^2} g(\gamma) d\sigma(\gamma) = \int_0^{2\pi} \int_0^\pi g(\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)) \sin(\phi) d\phi d\theta.$$

Then.. finish here. In general can write

8.3 Schwartz space

$$\mathcal{S}(\mathbb{R}^d) = \{f \text{ indefinitely differentiable and } \sup_{x \in \mathbb{R}^d} |x^\alpha \left(\frac{\delta}{\delta x}\right)^\beta f(x)| < \infty\}.$$

Example 8.1. $e^{-a|x|^2}$ for $a > 0$ is in $\mathcal{S}(\mathbb{R}^d)$.

Fourier Transform

$$\hat{f}(\zeta) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \zeta} dx, \quad \zeta \in \mathbb{R}^d.$$

Useful facts:

(i)

$$f(x+h) \xrightarrow{\text{Fourier transf.}} \hat{f}(\zeta) e^{2\pi i h \zeta}.$$

(ii)

$$f(x) e^{-2\pi i x h} \xrightarrow{\text{Fourier transf.}} \hat{f}(\zeta + h).$$

(iii)

$$f(\delta x) \xrightarrow{\text{Fourier transf.}} \delta^{-d} \hat{f}(\delta^{-1} \zeta), \delta > 0.$$

(iv)

$$f'(x) \xrightarrow{\text{Fourier transf.}} 2\pi i \zeta \hat{f}(\zeta).$$

(v)

$$(-2\pi i x)^\alpha f(x) \xrightarrow{\text{Fourier transf.}} \left(\frac{d}{d\zeta}\right)^\alpha \hat{f}(\zeta).$$

(vi)

$$f(Rx) \xrightarrow{\text{Fourier transf.}} \hat{f}(\mathbb{R}\zeta), \quad R \text{ rotation.}$$

$$\begin{aligned} \int_{\mathbb{R}^d} f(Rx) e^{-2\pi i x \zeta} dx &= \int_{\mathbb{R}^d} f(x) e^{-2\pi i R^{-1} x \zeta} dx \\ &= \int_{\mathbb{R}^d} f(x) e^{-2\pi i x R \zeta} dx \\ &= \hat{f}(R\zeta). \end{aligned}$$

Theorem. If $f \in \mathcal{S}(\mathbb{R}^d)$ then $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$.

8.4 Radial Functions

$f(x)$ is *radial* if there is a function f_0 defines on \mathbb{R}_+ such that

$$f(x) = f_0(|x|).$$

Note: f radial $\iff f(Rx) = f(x)$ for all rotations R .

Fact: The Fourier transform of a radial function is radial. Because,

$$\begin{aligned} f(Rx) &= f(x) \forall R \\ \Rightarrow \hat{f}(R\zeta) &= \hat{f}(\zeta) \forall R. \end{aligned}$$

Example 8.2. $e^{-a|x|^2}$ is radial.

$$\widehat{e^{-\pi|x|^2}}(\zeta) = e^{-\pi|\zeta|^2}.$$

Theorem (Inverse & Plancharz). Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Then

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\zeta) e^{2\pi i x \zeta} d\zeta$$

and

$$\|\hat{f}\|_2 = \|f\|_2.$$

8.5 Fourier Transform of σ in \mathbb{R}^3 .

$$\hat{\sigma}(\zeta) := \int_{S^2} e^{-2\pi i \zeta \gamma} d\sigma(\gamma)$$

Theorem.

$$\hat{\sigma}(\zeta) = \frac{2 \sin(2\pi|\zeta|)}{|\zeta|}.$$

Proof. First note LHS is radial in ζ .

$$\begin{aligned} \int_{S^2} e^{-2\pi i R \zeta \gamma} d\sigma(\gamma) &= \int_{S^2} e^{-2\pi i \zeta \cdot R^{-1} \gamma} d\sigma(\gamma) \\ &= \int_{S^2} e^{-2\pi i \zeta \cdot \gamma} d\sigma(\gamma). \end{aligned} \quad \boxed{\gamma^{\text{new}} = R^{-1} \gamma^{\text{old}}}$$

So if $|\zeta| = \rho$ then enough to prove the identity with $\zeta = (0, 0, \rho)$.
 Use spherical coordinates

$$\begin{aligned}
 \int_{S^2} e^{-2\pi i(0,0,\rho) \cdot \gamma} d\sigma(\gamma) &= \int_0^{2\pi} \int_0^\pi e^{-2\pi i \rho \cos(\phi)} \sin(\phi) d\phi d\theta \\
 &= 2\pi \int_{-1}^1 e^{2\pi i \rho u} du \quad (u = \cos(\phi)) \\
 &= 2\pi \frac{1}{2\pi i \rho} [e^{2\pi i \rho u}]_{-1}^1 \\
 &= \frac{2 \sin(2\pi \rho)}{\rho} \\
 &= \frac{2 \sin(2\pi |\zeta|)}{|\zeta|}.
 \end{aligned}$$

□