Oscillations in Harmonic Analysis

Carlos O. Huaricapcha

Summer 18, San Francisco State University

Notes written from Dr. Eyvindur Palsson

Problem sets

I'd love to hear your feedback. Feel free to email me at coscohua@mail.sfsu.edu.
See git:icarlitoss for updates.

Contents

1 Introduction

← July 2, 2018

Combinatorics

1.0.1 Erdős problem

1. Erdős distinct distance problem (1946).
   What is the least number of distinct distances determined by N points in a plane.

Example 1.1. We have four points $(0,1),(2,2),(0,0),(1,0)$, and if we start listing the distances between each of them we obtain the following:

$$1, 1, \sqrt{2}, \sqrt{5}, \sqrt{5}, \sqrt{8}.$$

However, we care about the distinct number; hence we get a new list

$$1, \sqrt{2}, \sqrt{5}, \sqrt{8}.$$

Upper bound: Notice that the first list is obtain by getting the distance between two points, hence

$$\binom{N}{2} = \frac{1}{2}N^2 - \frac{1}{2}N \sim N^2.$$

are obtained random.

To analyze what happens with the lower bound, we look at the following example.

Example 1.2. (a) Say N is a perfect square.

(b) Then, we have an square of $\sqrt{N}$ lattice.

(c) Now, let's count

$$(\text{distinct distcnes})^2.$$

(d) We obtain the list:

$$1, 2, \ldots, 2N.$$

(e) Hence we get no more than $\sim N$.

(f) Notice that $a^2 + b^2 = 3$ has no solution (number theory). Hence our list (d) have holes.

(g) Hence,

$$\#\text{ distinct distance} \sim \frac{N}{\sqrt{\log(N)}} \text{ as } N \to \infty.$$

Conjecture. Conjecture (Erdős 1946): The answer for Erdos 1946 should be in the order of $\frac{N}{\sqrt{\log(N)}}$ as $N \to \infty$.

Theorem (Erdős 1946)**.** *At least* $\sim \sqrt{N}$ *as* $N \to \infty$.

Theorem (Guth, Katz 2015)**.** *At least* $\sim \frac{N}{\log(N)}$ *as* $N \to \infty$.

2. Crescent Configurations N points in the plane such that distance d_1 appears 1 times, d_2 appears 2 times, and so on, until d_{N-1} appears $N-1$ times. ($N-1$ distinct distances). It is possible to achieve this if you place equally spaced points on a line. Additionally require general position no more than 2 points on a line, and no more than 3 points on a circle. Call a crescent configuration.

Example 1.3. We can go from 3 pts to 8 points in crescent configuration. It is unknown if we could find a 9 point or higher order configurations.

Conjecture (Erdős)**.** Eventually they do not exist.

Question: Find many (all) crescent configurations for some N.

Example 1.4. Given $N = 4$ in $\mathbb{R}^2$.

Question: For $N = 5$. Many known but not all.

2 Review on Vector Spaces

← July 3, 2018

2.1 Inner Product

An *inner product* is a map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

that satisfies for all $f, g, h \in V$, and $\alpha, \beta \in F$.

1. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0 \Rightarrow f = 0$.
2. $\langle f, g \rangle = \langle g, f \rangle$.
3. $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$.

Example 2.1. In $\mathbb{R}^d$, we have

$$\langle x, y \rangle = x \cdot y = x_1 y_1 + \cdots + x_d y_d.$$

Example 2.2. In $\mathbb{C}^d$, we have

$$\langle x, y \rangle = x \cdot y = x_1 \bar{y_1} + \cdots + x_d \bar{y_d}.$$

Example 2.3. In $\mathcal{C}[a, b]$, we have

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx$$

2.2 Norm

A *norm* is a function

$$|| \cdot || : V \to \mathbb{R}$$

that satisfies for all $f, g \in V$ and $\alpha \in F$.

1. $||f|| \geq 0$ and if $||f|| = 0$ then $f = 0$.

2. $||\alpha f|| = |\alpha| ||f||$.
3. $||f + g|| \leq ||f|| + ||g|| \quad$ (triangle inequality).

In in a vector space V with inner product $\langle \cdot, \cdot \rangle$ get a norm for free.

$$||f|| := \sqrt{\langle f, f \rangle}.$$

Definition (Cauchy-Schwartz Inequality)**.**

$$|\langle f, g \rangle| \leq ||f|| ||g||.$$

<u>Hint for Problem set:</u> Implies triangle inequality.

$$\begin{aligned} ||f + g|| &= \langle f + g, f + g \rangle \\ &= ||f||^2 + \langle f, g \rangle + \langle \bar{f}, g \rangle + ||g||^2 \\ \text{(by CS)} &\leq ||f||^2 + 2||f|| ||g|| + ||g||^2 \\ &= (||f|| + ||g||)^2. \end{aligned}$$

Example 2.4. On $\mathcal{C}[a, b]$ with

$$\langle f, g \rangle = \frac{1}{b - a} \int_a^b f(x) \overline{g(x)} dx$$

get

$$||f||_2 = \left(\frac{1}{b - a} \int_a^b |f(x)|^2 dx \right)^{1/2}$$

2.3 Orthogonality

Sat $f, g \in V$ are orthogonal if

$$\langle f, g \rangle = 0.$$

Say $\{\phi_1, \ldots, \phi_n\}$ are orthogonal if $\langle \phi_j, \phi_k \rangle = 0$ whenever $j \neq k$ and $\phi_j \neq 0$ for all j.

If in addition $||\phi_j|| = 1$ for all j then $\{\phi_1, \ldots, \phi_n\}$ is *orthonormal.*
<u>Note:</u> Remember Gram-Schmidt.

Theorem (Pythagorean Theorem)**.** *If* $\langle f, g \rangle =$ *then* $||f + g||^2 = ||f||^2 + ||g||^2$.

2.4 Projections

Let $\phi \in B$ with $||\phi|| = 1$. The projection of f in the direction of ϕ is

$$\text{proj}_\phi(f) := \langle f, \phi \rangle \phi.$$

Example 2.5. Project $f = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ on $\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$\text{proj}_\phi(f) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Definition. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. The proejction of f onto W_n is.

$$\text{proj}_{W_n}(f)L = \langle f_1, \phi_1 \rangle \phi_1 + \ldots + \langle f_n, \phi_n \rangle \phi_n.$$

Theorem.

$$\text{proj}_{W_n}(f) = 0 \iff f \in W_n.$$

Theorem. *$f - \text{proj}_{W_n}$ is orthonormal to every vector in W_n.*

3 Fourier Series

← July 5, 2018

Theorem. *Let $f \in V$. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. The element $winW_n$ that minimizes $||f - w||$ is*

$$w = \text{proj}_{W_n}(f). \qquad \leftarrow \textit{Best approximation}$$

Proof. Write:

$$w = \sum_{i=1}^{n} \beta_i \phi_i$$

and set

$$\alpha_i = \langle f, \phi_i \rangle, \quad i = 1, \ldots, n.$$

Then,

$$\begin{aligned}
||f - w||^2 &= \langle f - w, f - w \rangle \\
&= ||f||^2 - \langle f, \sum_{i=1}^{n} \beta_i \phi_i \rangle - \overline{\langle f, \sum_{i=1}^{n} \beta_i \phi_i \rangle} + \langle \sum_{i=1}^{n} \beta_i \phi_i, \sum_{j=1}^{n} \beta_j \phi_j \rangle \\
&= ||f||^2 - \sum_{i=1}^{n} \overline{\beta_i} \langle f, \phi_i \rangle - \sum_{i=1}^{n} \overline{\overline{\beta_i}}\ \overline{\langle f, \phi_i \rangle} + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \overline{\beta_j} \langle \phi_i, \phi_j \rangle \\
&= ||f||^2 - \sum_{i=1}^{n} \overline{\beta_i} \alpha_i - \sum_{i=1}^{n} \beta_i \overline{\alpha_i} + \sum_{i=1}^{n} |\beta_i|^2 \\
&= ||f||^2 + \sum_{i=1}^{n} (\beta_i - \alpha_i) \overline{\beta_i - \alpha_i} - \sum_{i=1}^{n} |\alpha_i|^2 \\
&= ||f||^2 - \sum_{i=1}^{n} |\langle f, \phi_i \rangle|^2 + \sum_{i=1}^{n} |\beta_i - \alpha_i|^2.
\end{aligned}$$

So minimized if $\beta_i = \alpha_i = \langle f, \phi_i \rangle$. □

Remark:

$$\sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i \quad \text{best approximation to } f.$$

Corollary.

$$\sum_{i=1}^{n} |\langle f, \phi_i \rangle|^2 \leq ||f||^2.$$

Bessel's inequality:

$$\sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2 \leq ||f||^2.$$

Riemann-Lebesgue Lemma:

$$\lim_{i \to \infty} \langle f, \phi_i \rangle = 0.$$

<u>Motivation:</u>

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \neq 0 \\ 1 & n = m = 0. \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(nx)\sin(mx)dx = \begin{cases} \frac{1}{2} & \text{if } n = m \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(nx)\cos(mx)dx = 0.$$

Note that

$$1, \sqrt{2}\cos(x), \sqrt{2}\sin(x), \sqrt{2}\cos(2x), \sqrt{2}\sin(2x), \ldots$$

orthonormal sequence on $[0, 2\pi]$.
Best finite approximation up to level n of a function f is

$$a_0 \cdot 1 + a_1\sqrt{2}\cos(x) + b_1\sqrt{2}\sin(x) + \ldots$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot 1 dx$$

$$a_j = \frac{1}{2\pi} \int_0^{2\pi} f(x)\sqrt{2}\cos(jx)dx$$

$$b_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot \sqrt{2}\sin(kx)dx.$$

Trigonometric Series:

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx))$$

if in addition

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\cos(nx)dx$$

and

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\sin(nx)dx$$

then called *Fourier Series*.

Simplification: We could have started with the orthonormal sequence

$$e^{inx}, n \in \mathbb{Z}, \text{ on } [0, 2\pi].$$

Then,

$$\begin{aligned}
\langle e^{inx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{inx} \overline{e^{imx}} dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} dx \\
&= \begin{cases} \frac{1}{2\pi} \left[\frac{1}{i(n-m)} e^{i(n-m)x} \right]_0^{2\pi} = 0 & \text{if } n \neq m \\ \\ \frac{1}{2} \int_0^{2\pi} 1 dx = 1 & \text{if } n = m. \end{cases}
\end{aligned}$$

Give best approximation up to level $n > 0$ of a function f

$$c_{-n} e^{i(-n)x} + \cdots + c_0 \cdot 1 + \cdots + c_n e^{inx}$$

with corresponding series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{e^{inx}} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx.$$

Note:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

$$= \begin{cases} \frac{1}{2}(A_n - iB_n) & \text{if } n > 0 \\ \frac{1}{2} A_0 & \text{if } n = 0 \\ \frac{1}{2}\left(A_{|n|} + iB_{|n|}\right) & \text{if } n < 0. \end{cases} .$$

For $n > 0$, we have

$$c_n e^{inx} + c_{-n} e^{i(-n)x} = A_n \cos(nx) + B_n \sin(nx)$$

All the same terms as before!

Intervals:

e^{inx} works on any interval of length 2π.
$e^{2\pi inx/N}$ works on any interval of length L.

Summary:
Fourier Series: If f is integrable on $[a, b]$ of length of L then the n^{th} *Forier coefficient* of f is defined by

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi inx/L} dx, \quad n \in \mathbb{Z}$$

and its Forier series is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-2\pi inx/L}.$$

Question: Does $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-2\pi inx/L}$ converges to $f(x)$? ← July 6, 2018
Let's first look at the following example.

Example 3.1. Let $f(x) = x$ on $[0, 2n]$. Then,

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}dx = \frac{i}{n}, \quad n \neq 0.$$

and

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} xdx = \pi.$$

Fourier series

$$\sum_{n=-\infty}^{\infty} \frac{i}{n}e^{inx} = \pi - 2\left(\frac{1}{1}\sin(x) + \frac{1}{2}\sin(2x) + \ldots\right)$$

$$\uparrow \frac{i}{n}e^{inx} + \frac{i}{-n}e^{-inx} = -\frac{2}{n}\sin(nx).$$

Yields π if $x = 0$ or $x = 2\pi$ while $f(0) = 0$ and $f(2\pi) = 2\pi$.

Consider: [insert graph 1 vs 2 here]:

Note: Fourier series is periodic on $\mathbb{R}$.

3.1 Domains

Consider $[a, b]$ on periodic functions on $\mathbb{R}$ in a circle such that $f(x) = F(e^{ix})$ and $x \in [0, 2\pi]$. Hence, functions on the circle are periodic functions $f(x)$ on $[0, 2\pi]$ with $f(0) = f(2\pi)$.

3.2 Uniqueness

Theorem. *Suppose f is integrable and bounded on an interval with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x_0) = 0$ whenever f is continuous at x_0.*

3.3 Partial Sums

Let

$$S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n)e^{-2\pi inx/L} \text{ on } [0, L].$$

Question: In what sense $S_N f \to f$.

Theorem. *Suppose f is continous on the circle and the Fourier series of f is absolutely convergent*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Then

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

uniformly in x.

Proof. Since

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

converges absolutely and in fact uniformly.
If

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

then g must be continuous.
Since $f-g$ is continuous and $\widehat{f-g}(n) = 0$ for all $n \in \mathbb{Z}$ we conclude by uniqueness theorem that $f = g$.

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\lim_{N\to\infty}\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}\right) e^{-inx}dx \\
&= \lim_{N\to\infty}\sum_{k=-\infty}^{\infty} \hat{f}(k)\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)x}dx.
\end{aligned}$$

Note that:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)x}dx = \begin{cases} 1 & \text{if } k = n \\ 0 & else. \end{cases}$$

Hence, we have that

$$\begin{aligned}
\hat{g}(n) &= \lim_{N\to\infty}\sum_{k=-\infty}^{\infty} \hat{f}(k)\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)x}dx \\
&= \hat{f}(n).
\end{aligned}$$

$\square$

Theorem. *Let $f \in \mathcal{C}^2[0, 2\pi]$ and 2π period. Then*

$$\hat{f}(n) = \mathcal{O}\left(\frac{1}{n^2}\right) \quad \textit{as } |n| \to \infty$$

and thus the Fourier series of f converges absolutely and uniformly to f.

Note: $f(x) = \mathcal{O}(g(x))$ as $x \to a$ means there exists a constant C such that

$$|f(x)| \le C|g(x)| \text{ as } x \to a.$$

Proof. Since f'' is continous on $[0, 2\pi]$, it is bounded. Say

$$|f''(x)| \le B \text{ for all } x \in (0, 2\pi].$$

Then

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \\
&= \frac{1}{in}\widehat{f'}(n) \quad \text{by int. by parts.}
\end{aligned}$$

By iterating $\hat{f}(n) = \frac{1}{i^2n^2}\widehat{f''}(n)$,

$$\begin{aligned}
|\hat{f}(n)| &= \frac{1}{n^2}\left|\frac{1}{2\pi}\int_0^{2\pi} f''(x)e^{-inx}dx\right| \\
&\le \frac{1}{n^2}\left(\frac{1}{2\pi}\int_0^{2\pi} Bdx\right) = \frac{B}{n^2}.
\end{aligned}$$

□

Corollary. $\widehat{f'}(n) = in\hat{f}(n)$.

3.4 Convolution

Given 2π periodic integrable functions f and g on $\mathbb{R}$. Define their *convolution* $f * g$ by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)dy \qquad x \in [-\pi, \pi].$$

Remarks: It has many good properties by most importantly

- $f * g = g * f$.
- $\widehat{f * g} = \hat{f}(n)\hat{g}(n)$.

4 Kernels

← July 9, 2018

4.1 Direchlet Kernel

$$\begin{aligned} S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \\ &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy\right)e^{inx} \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-N}^{N} e^{-in(x-y)}dy\right)e^{inx} \\ &= (f * D_N)(x) \end{aligned}$$

where $D_N(x) = \sum_{n=-N}^{N} e^{inx}$. Facts:

(1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$.

(2) $D_n(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$.

(3) $D_n(x) \le C\min\{N, \frac{1}{|x|}\}$ if $x \in [-\pi, \pi]$, $N \ge 1$.

4.2 Good Kernels on the circle

Definition. A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$ for all $n \ge 1$.

(ii) There exists $M > 0$ such that $\int_{-\pi}^{\pi} |K_n(x)|dx \le M$ for all $n \ge 1$.

(iii) For every $\delta > 0$ $\quad \int_{\delta \le |x| \le \pi} |K_n(x)|dx \to 0$ as $n \to \infty$.

Theorem. *Let $\{K_n\}$ be a family of good kernels and f a bounded and integrable function on the circle. Then*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere then the limit is uniform.

Note: Unfortunately

$$\int_{-\pi}^{\pi} |D_N(x)|dx \ge c\log(N)$$

as N is large. Then D_N is *not* a good kernel.

4.3 Fej'er Kernel

Say a series $\sum_{k=0}^{\infty} c_k$ is Cesáro summable to σ if

$$\lim_{N\to\infty} \sigma_N = \sigma$$

where

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}$$

is the N^{th} Cesáro mean.

Example 4.1. $\sum_{k=0}^{\infty}(-1)^k$ does not converge but Cesáro summable to $\frac{1}{2}$. For Fourier series the N^{th} Cesáro mean is

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\
&= \frac{(f * D_0)(x) + \ldots (f * D_{N-1})(x)}{N} \\
&= \left(f * \left(\frac{D_0 + \cdots + D_{N-1}}{N}\right)\right)(x) \\
&= (f * F_N)(x) \qquad \left(\text{since } F_N = \frac{D_0 + \cdots + D_{N=1}}{N} \to \text{Fej'er kernel}\right).
\end{aligned}$$

<u>Facts:</u>

(1) $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$.

(2) F_N is a good kernel.

Theorem (Weierstrass Approximation Theorem)**.** *For every continuous function $f : \mathbb{R} \to \mathbb{C}$ with period 2π and every $\epsilon > 0$ one can find a trigonometric polynomial P such that*

$$\forall x \qquad |f(x) - P(x)| < \epsilon.$$

Proof.

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\
&= (f * F_N)(x).
\end{aligned}$$

is a trigonometric polynomial and $(f * F_N) \to f$ uniformly because f continuous and F_N is a good kernel. ☐

Mean Square Convergence of Fourier Series
Let f be a continuous function on the circle. Then

$$||f - S_N(f)||_2 \to 0 \text{ as } N \to \infty.$$

Proof. By the Weierstrass Approximation Theorem for a given $\epsilon > 0$ there exists a trigonometric polynomial P, say of degree M, such that

$$|f(x) - P(x)| < \epsilon \quad \text{ for all } x \in [0, 2\pi]$$

and thus

$$||f - P||_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - P(x)|^2 fx \right)^2 < \epsilon.$$

By the best approximation theorem

$$||f - S_N(f)||_2 < \epsilon$$

whenever $N \geq M$. □

4.4 Parseval's identity

From general theory about orthonormal sequences Bessel's inequality:

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq ||f||_2^2.$$

Riemman-Lebesgue Lemma: $\hat{f}(n) \to 0$ as $|n| \to \infty$, and

$$\int_0^{2\pi} f(x) \sin(nx) dx \to 0,$$

$$\int_0^{2\pi} f(x) \cos(nx) dx \to 0.$$

Theorem (Parsval's Identity)**.**

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq ||f||_2^2$$

where we assume f is continuous on the circle.

Proof. From proof of best approximation theorem

$$||f||_2^2 = ||f - S_N(f)||_2^2 + \sum_{|n| \leq N} |\hat{f}(n)|^2.$$

□

5 Fourier Series II

5.1 Pointwise Convergence of Fourier Series

Theorem (Pointwise Convergence of Fourier Series)**.** *July 10, 2018] Let f be a square integrable function on the circle which is differentiable at x_0. Then* ← [

$$S_N(f)(x_0) \to f(x_0) \text{ as } N \to \infty.$$

Proof.

$$
\begin{aligned}
S_N(f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0-t)D_N(t)dt - f(x_0)\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x_0-t)-f(x_0))\frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{\frac{f(x_0-t)-f(x)}{t}\frac{t}{\sin(t/2)}\cos(t/2)}\sin(Nt)dt \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{\frac{f(x_0-t)-f(x_0)}{t}\frac{t}{\sin(t/2)}\sin(t/2)}\cos(Nt)dt.
\end{aligned}
$$

As $t \to 0$, the right side becomes

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x_)\cdot 2\cdot 1\sin(Nt)dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x_0)\cdot 2\cdot 0\cos(Nt)dt
$$

and away from $t \to 0$ then $\square$ are square integrable like. Conclude by Riemman-Lebesgue lemma

$$
S_N(f)(x_0) - f(x_0)| \to 0 \text{ as } N \to \infty.
$$

$\square$

Theorem. *Let f be a 'square' integrrable funcion on the circle such that $f'(x_0^+)$ and $f'(x_0^-)$ exist. Then*

$$
S_n(f)(x_) \to \frac{f(x_0^+)+f(x_0^-)}{2}.
$$

Proof. Similar. $\square$

Non-convergence

There exists an integrable function which is continuous at a point x_0 such that $S_N(f)(x_0)$ does not converge.

$$
|f(x)-f(y)| \le C|x-y|^{\infty}.
$$

Gibbs Phenomenon

$$
\sum_{k=1}^{N}\frac{1}{k}\sin(kx) = S_N(f)(x) \to f(x) = \frac{\pi}{2} - x \text{ on } (0,2\pi).
$$

For a fixed big N. (include graph).

5.2 Uniform Convergence of Fourier Series

Theorem (Uniform Convergence of Fourier Series)**.** *Let f be continuously differentiable on the circle. Then its Fourier series converges uniformly to f.*

Proof. Since f' is continuous it is square integrable and Parseval's identity holds. Further an old corollary says

$$
\hat{f}'(n) = in\hat{f}(n).
$$

and

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{\infty} \frac{1}{n} |\widehat{f'}(n)| \leq \underbrace{\left(\sum_{n=-\infty}^{\infty} \frac{1}{n^2}\right)^{1/2}}_{\infty} \underbrace{\left(\sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2\right)^{1/2}}_{=||f'||_2 < \infty.} < \infty$$

□

6 Fourier Transforms on $\mathbb{R}$

f on $[0,1]$ period 1, $n \in \mathbb{Z}$.

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx, \quad f(x) \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

f on $\mathbb{R}$ (no period), $\zeta \in \mathbb{R}$. We have

$$\hat{f}(\zeta) = \int_{-} \infty^{\infty} f(x) e^{-2\pi i \zeta x} dx, \quad f(x) \stackrel{?}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i \zeta x} d\zeta.$$

6.1 The Schwartz space

Definition. The Schwartz space on $\mathbb{R}$ denoted $\mathcal{S}(\mathbb{R})$, consists of all indefinitely differentiable functions f such that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \qquad \text{for every } k, l \geq 0.$$

Note: If $f \in \mathcal{S}(\mathbb{R})$ then also

$$x^k f^{(l)}(x) \in \mathcal{S}(\mathbb{R}) \qquad \text{for all } k, l \geq 0.$$

Note: $\mathcal{S}(\mathbb{R})$ is a vector space.

Example 6.1. e^{-ax^2}, for $a > 0$ is in $\mathcal{S}(\mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$ define Fourier transform

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(\zeta) e^{-2\pi i \zeta x} d\zeta.$$

Useful facts:

(i)
$$f(x+h) \stackrel{\text{Fourier transf.}}{\rightarrow} e^{2\pi i h \zeta} \hat{f}(\zeta).$$

(ii)
$$f(x) e^{-2\pi i x h} \stackrel{\text{Fourier transf.}}{\rightarrow} \hat{f}(\zeta + h).$$

(iii)
$$f(\delta x) \stackrel{\text{Fourier transf.}}{\rightarrow} \delta^{-1} \hat{f}(\delta^{-1} \zeta), \delta > 0.$$

(iv)

$$f'(x) \overset{\text{Fourier transf.}}{\to} 2\pi i\zeta \hat{f}(\zeta).$$

(v)

$$-2\pi i x f(x) \overset{\text{Fourier transf.}}{\to} \frac{d}{d\zeta}\hat{f}(\zeta).$$

← July 11, 2018

Theorem. *If $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

Proof.

$$\begin{aligned} |\zeta^k \left(\frac{d}{d\zeta}\right)^l \hat{f}(\zeta)| &= |\left(\frac{1}{2\pi i}\right)^k \left(\frac{d}{dx}\right)^k ((-2\pi i x)^l f(x))(\zeta)| \\ &< \infty. \end{aligned}$$

□

Fact:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Theorem. *If $f(x) = e^{-\pi x^2}$ then $\hat{f}(\zeta) = e^{-\pi \zeta^2}$.*

Proof. Define $F(\zeta) = \hat{f}(\zeta) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\zeta} dx$. Then $F(0) = 1$ by fact above, and

$$\begin{aligned} F'(\zeta) &= \int_{-\infty}^{\infty} \underbrace{e^{-\pi x^2}(-2\pi x)}_{f'(x)} i e^{-2\pi i x\zeta} dx \\ &= i\widehat{f'}(\zeta) \\ &= i(2\pi i\zeta)\hat{f}(\zeta) \\ &= -2\pi\zeta F(\zeta). \end{aligned}$$

Set $G(\zeta) = F(\zeta)e^{\pi\zeta^2}$. Then $G(0) = F(0) = 1$ and

$$G'(\zeta) = \underbrace{F'(\zeta)}_{-2\pi\zeta F(\zeta)} e^{\pi\zeta^2} + F(\zeta) 2\pi\zeta e^{\pi\zeta^2} = 0.$$

So, $G(0) = 1$ and thus $F(\zeta) = e^{-\pi\zeta^2}$. □

Define for $\delta > 0$.

$$K_\delta(x) = \delta^{1/2} e^{-\pi x^2/\delta} \text{ then } \hat{K}_\delta(\zeta) = e^{\pi\delta\zeta^2}.$$

(insert picture of graph as δ is small).

Definition. If $f, g \in \mathcal{S}(\mathbb{R})$ their *convolution* is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Note: Similar properties as on circle for $f, g \in \mathcal{S}(\mathbb{R})$.

(i) $f * g \in \mathcal{S}(\mathbb{R})$.

(ii) $f * g = g * f$.

(iii) $\widehat{f * g} = \hat{f}\hat{g}$.

Theorem. *The collection* $\{K_\delta\}_{\delta > 0}$ *is a family of good kernels as* $\delta \to 0^+$.

Theorem. *If* $f \in \mathbb{R}$ *then* $(f * K_\delta)(x) \to f(x)$ *uniformly in* x *as* $\delta \to 0^+$.

<u>Note:</u> Works for any good kernel.

Proposition. *If* $f, g \in \mathcal{S}(\mathbb{R})$ *then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy.$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(x)e^{-2\pi ixy}dydx \\
&= \int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dxdy \\
&= \int_{-\infty}^{\infty} g(y)\hat{f}(y)dy.
\end{aligned}$$

□

Theorem (Fourier Inversion)**.** *if* $f \in \mathcal{S}(\mathbb{R})$ *then* $f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi ix\zeta}d\zeta$.

Proof. First note that

$$\begin{aligned}
f(0) &= \lim_{\delta \to 0^+} (f * K_\delta)(0) \\
&= \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} f(x)K_\delta(0 - x)dx \\
&= \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} f(x)\ \underbrace{\delta^{-1/2}e^{-\pi x^2/\delta}}_{K_\delta(x)}.
\end{aligned}$$

Also we have that,

$$\begin{aligned}
e^{-\pi x^2} &= \widehat{e^{-\pi\delta^2}}(x) \\
\Rightarrow \delta^{-1/2}e^{-\pi x^2/\delta} &= \underbrace{\delta^{-1/2}\widehat{e^{-\pi\delta^2}}(\delta^{-1/2}x)}_{\widehat{e^{-\pi\delta^2}}(x)}
\end{aligned}$$

Then, continuing our first observation

$$
\begin{aligned}
f(0) &= \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} f(x) \underbrace{\delta^{-1/2} e^{-\pi x^2/\delta}}_{K_\delta(x)} \\
&= \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{-\pi \delta \zeta^2} d\zeta \\
&\overset{\text{Prop.}}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta) d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i \delta \zeta} d\zeta.
\end{aligned}
$$

In genreal, let

$$F(y) = f(x+y).$$

Then

$$
\begin{aligned}
f(x) &= F(0) \\
&= \int_{-\infty}^{\infty} \hat{F}(\zeta) d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i x \zeta} d\zeta.
\end{aligned}
$$

□

Corollary. *The Fourier transform is bijective mapping on* $\mathcal{S}(\mathbb{R})$

$$\mathcal{S}(\mathbb{R}) \overset{\mathcal{F}(f)(\zeta)=\int_{\infty}^{\infty} f(x)e^{-2\pi i x\zeta}dx}{\curvearrowright} \mathcal{S}(\mathbb{R}).$$

and

$$\mathcal{S}(\mathbb{R}) \overset{\mathcal{F}^{-1}(g)(\zeta)=\int_{\infty}^{\infty} g(\zeta)e^{-2\pi i x l}dl}{\curvearrowleft} \mathcal{S}(\mathbb{R}).$$

7 Inner Product and Norm on $\mathcal{S}(\mathbb{R})$

Can use the usual

$$\langle f, g \rangle = \int_{\infty}^{\infty} f(x)\overline{g(x)}dx$$

and

$$||f||_2 = \left(\int_{\infty}^{\infty} |f(x)|^2 dx \right)^{1/2}.$$

Theorem (Plancharel)**.** *If* $f \in \mathcal{S}(\mathbb{R})$ *then* $||f^2||_2 = ||f||_2$.

Proof.

$$
\begin{aligned}
\widehat{f^*}(\zeta) &= \int_{\infty}^{\infty} \overline{f(-x)} e^{-2\pi i x \zeta} dx \\
&= \overline{\int_{\infty}^{\infty} \overline{f(-x)} e^{-2\pi i (-x) \zeta} dx} \\
&= \overline{\hat{f}(\zeta)}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int_{\infty}^{\infty} |f(x)|^2 dx &= \int_{\infty}^{\infty} f(x)\overline{f(x)} dx \\
&= \int_{\infty}^{\infty} \overline{f(-(-x))} dx \\
&= \int_{\infty}^{\infty} f(x) f^{\#}(0-x) dx \\
&= (f * f^{\#})(0) \\
&\overset{\text{inversion}}{=} \int_{\infty}^{\infty} \widehat{f * f^{\#}}(\zeta) d\zeta \\
&= \int_{\infty}^{\infty} \hat{f}(\zeta)\widehat{f^{\#}}(\zeta) d\zeta \\
&= \int_{\infty}^{\infty} \hat{f}(\zeta)\overline{\hat{f}(\zeta)}(\zeta) \\
&= ||\hat{f}||_2^2.
\end{aligned}
$$

<u>Note:</u> $\mathcal{F}$ is unitary on $\mathcal{S}(\mathbb{R})$ with $|| \cdot ||_2$.
<u>Note:</u>

- Parseval:

$$
\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \right)^{1/2} = ||f||_2.
$$

- Plancharel:

$$
||\hat{f}||_2 = ||f||_2.
$$

□

8 Fourier Transform on $\mathbb{R}^d$

← <u>July 12, 2018</u>

8.1 Rotations

A rotation in $\mathbb{R}^d$ is a linear transformation that fulfills one of the following equivalent conditions

(a) $Rx \cdot Ry = x \cdot y$ for all $x, y \in \mathbb{R}^d$.

(b) $|Rx| = |x|$ for all $x \in \mathbb{R}^d$.

(c) $R^t = R^{-1}$.

(d) $\det(R) = \pm 1$.

Useful:

$$
\int_{R^d} f(Rx) dx = \int_{R^d} f(x) dx.
$$

8.2 Coarea Formula / Polar coordinates

Write $\gamma = (\cos(\theta), \sin(\theta))$. For a function on g on S' define

$$\int_{S'} g(\gamma)d\sigma(\gamma) = \int_0^{2\pi} g(\cos(\theta), \sin(\theta))d\theta.$$

Then can write

$$\begin{aligned}\int_{\mathbb{R}^2} f(x)dx &= \int_0^{2\pi}\int_0^{\infty} f(r\cos(\theta), r\sin(\theta))rdrd\theta \\ &= \int_{S'} f(r\gamma)rdrd\sigma(\gamma).\end{aligned}$$

For g on S^2 define

$$\int_{S^2} g(\gamma)d\sigma(\gamma) = \int_0^{2\pi}\int_0^{\pi} g(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))\sin(\phi)d\phi d\theta.$$

Then.. finish here. In general can write

8.3 Schwartz space

$$\mathcal{S}(\mathbb{R}^d) = \{f \text{ indefinitely differentiable and } \sup_{x\in\mathbb{R}^d} |x^\alpha \left(\frac{\delta}{\delta x}\right)^\beta f(x)| < \infty\}.$$

Example 8.1. $e^{-a|x|^2}$ for $a > 0$ is in $\mathcal{S}(\mathbb{R}^d)$.

<u>Fourier Transform</u>

$$\hat{f}(\zeta) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\zeta}dx, \quad \zeta \in \mathbb{R}^d.$$

<u>Useful facts:</u>

(i)

$$f(x+h) \overset{\text{Fourier transf.}}{\rightarrow} \hat{f}(\zeta)e^{2\pi i h\zeta}.$$

(ii)

$$f(x)e^{-2\pi i x h} \overset{\text{Fourier transf.}}{\rightarrow} \hat{f}(\zeta+h).$$

(iii)

$$f(\delta x) \overset{\text{Fourier transf.}}{\rightarrow} \delta^{-d}\hat{f}(\delta^{-1}\zeta), \delta > 0.$$

(iv)

$$f'(x) \overset{\text{Fourier transf.}}{\rightarrow} 2\pi i\zeta\hat{f}(\zeta).$$

(v)

$$(-2\pi i x)^\alpha f(x) \overset{\text{Fourier transf.}}{\rightarrow} \left(\frac{d}{d\zeta}\right)^\alpha \hat{f}(\zeta).$$

(vi)

$$f(Rx) \overset{\text{Fourier transf.}}{\rightarrow} \hat{f}(\mathbb{R}\zeta), \quad R \text{ rotation.}$$

$$\begin{aligned}\int_{\mathbb{R}^d} f(Rx)e^{-2\pi ix\zeta}dx &= \int_{\mathbb{R}^d} f(x)e^{-2\pi R^{-1}x\zeta}dx \\ &= \int_{\mathbb{R}^d} f(x)e^{-2\pi ixR\zeta}dx \\ &= \hat{f}(R\zeta).\end{aligned}$$

Theorem. *If $f \in \mathcal{S}(\mathbb{R}^d)$ then $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$.*

8.4 Radical Functions

$f(x)$ is *radial* if there is a function f_0 defines on $\mathbb{R}_+$ such that

$$f(x) = f_0(|x|).$$

Note: f radial $\iff$ $f(Rx) = f(x)$ for all rotations R.

Fact: The Fourier transform of a radial function is radial. Because,

$$\begin{aligned}f(Rx) &= f(x)\forall R \\ \Rightarrow \hat{f}(R\zeta) &= \hat{f}(\zeta)\forall R.\end{aligned}$$

Example 8.2. $e^{-a|x|^2}$ is radial.

$$\widehat{e^{-\pi|x|^2}}(\zeta) = e^{-\pi|\zeta|^2}.$$

Theorem (Inverse & Plancharz)**.** *Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\zeta)e^{2\pi ix\zeta}d\zeta$$

and

$$||\hat{f}||_2 = ||f||_2.$$

8.5 Fourier Transform of σ in $\mathbb{R}^3$.

$$\hat{\sigma}(\zeta) := \int_{S^2} e^{-2\pi i\zeta\gamma}d\sigma(\zeta)$$

Theorem.

$$\hat{\sigma}(\zeta) = \frac{2\sin(2\pi|\zeta|)}{|\zeta|}.$$

Proof. First note LHS is radial in ζ.

$$\begin{aligned}\int_{S^2} e^{-2\pi iR\zeta\gamma}d\sigma(\gamma) &= \int_{S^2} e^{-2\pi i\zeta\cdot R^{-1}}\gamma d\sigma(\gamma) \\ &= \int_{S^2} e^{-2\pi i\zeta\cdot\gamma}d\sigma(\gamma). \qquad \boxed{\gamma^{\text{new}} = R^{-1}\gamma^{\text{old}}}\end{aligned}$$

So if $|\zeta| = \rho$ then enough to prove the identity with $\zeta = (0,0,\rho)$.
Use spherical coordinates

$$\begin{aligned}
\int_{S^2} e^{-2\pi i(0,0,\rho)\cdot\gamma} d\sigma(\gamma) &= \int_0^{2\pi}\int_0^{\pi} e^{-2\pi i\rho\cos(\phi)}\sin(\phi)d\phi d\theta \\
&= 2\pi\int_{-1}^{1} e^{2\pi i\rho u}du \qquad\qquad (u=\cos(\phi)) \\
&= 2\pi\frac{1}{2\pi i\rho}\left[e^{2\pi i\rho u}\right]_{-1}^{1} \\
&= \frac{2\sin(2\pi\rho)}{\rho} \\
&= \frac{2\sin(2\pi|\zeta|)}{|\zeta|}.
\end{aligned}$$

□

8.6 Stationary Phase

← July 13, 2018

For $\phi \in C^\infty$ and $a \in C_c^\infty$ define

$$I(\lambda) = \int_{\mathbb{R}^d} e^{-\pi i\lambda\phi(x)}a(x)dx.$$

Question: How does $I(\lambda)$ behave as $\lambda \to \infty$?
Idea: Much cancellation if λ large.
Warning: If $\phi(x) = c$ then

$$I(\lambda) = e^{-pii\lambda c}\int_{\mathbb{R}^d} a(x)dx$$

so no decay. We then need conditions on ϕ.

Definition. A map f is called a *diffeomorphism* if it is a bijection and it is differentiable and its inverse is differentiable as well. Say it is smooth if $f \in C^\infty,\ f^{-1} \in C^\infty$.

Note: Suppose $\phi_1 = \phi \circ G$ where G is smooth diffeomorphism. Then

$$\begin{aligned}
\int e^{-\pi i\lambda\phi_2(x)}a(x)dx &= \int e^-\pi i\lambda_1(G^{-1}x)a(x)dx \\
&= \int e^-\pi i\lambda_1(y)\underbrace{a(Gy)|J_g(y)|}_{\text{smooth}}dy \qquad\qquad (y = G^{-1}x).
\end{aligned}$$

So bounds independent of $a(x)$ will be smooth diffeomorphism invariant.

Two Cases: Neighborhoods of points p where

$$\underbrace{\nabla\phi(p) = 0}_{\text{stationary}} \quad \text{and} \quad \underbrace{\nabla\phi(p) \neq 0}_{\text{on-stationary}}.$$

Lemma (Straightening Lemma). *Suppose $\omega \in \mathbb{R}^n$ is open, $f : \omega \to \mathbb{R}$ is C^∞, $p \in \omega$ and $\nabla f(p) \neq 0$. Then there are neighborhoods U and V of 0 and p respectively and a smooth diffeomorphism $G : U \to V$ with*

$$G(0) = p$$

and

$$(f \circ G)(x) = f(p) + x_n.$$

Theorem (Inverse Function Theorem). *Let $A \subseteq \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}^n$ be of class C^1. Let $x_0 \in A$ and suppose $|Df(x_0)| \neq 0$. Then there is a neighborhood U of x_0 in A and an open neighborhood W of $f(x_0)$ such that $f(U) = W$ and f has a C^1 inverse $f^{-1} : W \to U$. More over $y \in W, x = f^{-1}(y)$ we have*

$$Df^{-1}(y) = [Df(x)]^{-1}.$$

If f is of call C^p, then so is f^{-1}.

Note: In $\mathbb{R}$: $(f^{-1})^{-1}(f(a)) = \frac{1}{f'(a)}$.

Theorem (Implicit Function Theorem). *Let $A \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set and let $f; A \to \mathbb{R}^m$ be a function of class C^p. Suppose $(x_0, y_0) \in A$ and $F(x_0, y_0) = 0$. Consider*

$$\nabla = \begin{vmatrix} \frac{\delta F_1}{\delta y_1} & \dots & \frac{\delta F_1}{\delta y_m} \\ \vdots & \ddots & \vdots \\ \frac{\delta F_m}{\delta y_1} & \dots & \frac{\delta F_m}{\delta y_m} \end{vmatrix}$$

evaluated at (x_0, y_0) and suppose $\nabla \neq 0$. Then there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of x_0 and a neighborhood V of y_0 in $\mathbb{R}^m$ and a unique function $f : U \to V$ such that $F(x, f(x)) = 0$ for all $x \in U$. Furthermore f is of class C^p.

Proof. Define $G : A \to \mathbb{R}^n \times \mathbb{R}^m$ by $G(x, y) = (x, F(x, y))$. Then G is of class C^p and

$$|DG(x_0, y_0)| = \begin{vmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & & \ddots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ \frac{\delta F_1}{\delta x_1} & \dots & \dots & \frac{\delta F_1}{\delta x_n} & \frac{\delta F_1}{\delta y_1} & \dots & \frac{\delta F_1}{\delta y_m} \\ \vdots & \ddots & & & & \ddots & \\ \frac{\delta F_m}{\delta x_1} & \dots & & \frac{\delta F_m}{\delta x_n} & \frac{\delta F_m}{\delta y_1} & \dots & \frac{\delta F_m}{\delta y_m} \end{vmatrix} \neq 0$$

so by Inverse Function Theorem there is an open set W containing $(x_0, 0)$ and S open set containing (x_0, y_0) such that $G(S) = W$ and G has C^p inverse $G^{-1} : W \to S'$. There exist open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ with $x_0 \in U$ and $y_0 \in V$ such that $U \times V \subseteq S$.
Let $G(U \times V) = Y$. Now G^{-1} is of the form

$$G^{-1}(x, w) = (x, H(x, w)).$$

Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ given by $\pi(x, y) = y$. Then

$$\begin{aligned} F(x, H(x, w)) &= \pi \circ G(x, H(x, w)) \\ &= \pi \circ G \circ G^{-1}(x, w) \\ &= W. \end{aligned}$$

Define $f : U \to V$ by $f(x) = H(x, 0)$. Then

$$F(x, f(x)) = 0$$

. Since H is of class C^p then so must f and since H unique then so must f. □

Lemma. *Let $A \subseteq \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}$ be a unction of class $C^p, p \geq 1$. Let $p \in A$ and suppose $f(p) = 0$ and $\nabla f(p) \neq 0$. Then there is an open set U, an open set V containing p and a function $h : U \to V$ of class C^p with inverse $h^{-1} : V \to U$ of class C^p such that*

$$f(h(x_1, \ldots, x_n)) = x_n.$$

Proof. Since $\nabla(p) \neq 0$ there exists i such that $\frac{\delta f}{\delta x_i} \neq 0$. Define $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_n, x_{i+1}, \ldots, x_{n-1}, x_i).$$

The map g is linear so $f \circ g$ is of class C^p and

$$\frac{\delta(f \circ g)}{\delta x_n}(g^{-1}(p)) = \frac{\delta f}{\delta x_i}(p) \neq 0.$$

Thus $(f \circ g) : g^{-1}(A) \subseteq \mathbb{R}^{m-1} \times \mathbb{R}$ is a map as in the Implicit Function Theorem.
Define $G : g^{-1}(A) \to \mathbb{R}^{-1} \times \mathbb{R}$ by $G(x, y) = (x, (f \circ g)(x, y))$. As in proof of I.F.T. there are open set $W \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ with $g^{-1}(p) \in W$ and $(\tilde{p}_1, \ldots, \tilde{p}_{n-1}, 0) \in U$ where $g^{-1}(p) = (\tilde{p}_1, \ldots, \tilde{p}_n)$. such that $G : W \to U$ has an inverse $G^{-1} : U \to W$ of class C^p.

$$(f \circ g) \circ G^{-1}(x_1, \ldots, x_n) = (\pi \circ G) \circ G^{-1}(x1, \ldots, x_n) = x_n.$$

Define $V = g(W)$ and $h : U \to$ by $h = g \circ G^{-1}$. Then h is a C^p function with a C^p inverse $f(h(x_1, \ldots, x_n)) = x_n$. □

Proposition (Non-stationary phase)**.** *Suppose $\Omega \in \mathbb{R}^n$ is open, $\phi : \omega \to \mathbb{R}$ is C^∞, $p \in \Omega$ and $\nabla\phi(p) \neq 0$. Suppose $a \in C_c^\infty$ has its support in a sufficiently small neighborhood of p. Then*

$$\forall N \exists C_N : \ |I(\lambda)| \leq C_N \lambda^{-N}.$$

Proof. The straightening lemma and smooth diffeomorphism invariance reduce this to the case

$$\phi(x) = x_n + C.$$

Letting $e_n = (0, \ldots, 0, 1)$ get

$$\begin{aligned} I(\lambda) &= \int e^{-\pi i \lambda(x_n + X)} a(x) dx \\ &= \left(\int a(x) e^{-2\pi i (\frac{\lambda}{2} e_n) \cdot x} dx \right) e^{-\pi i \lambda C} \\ &= e^{-\pi i \lambda C} \underbrace{\hat{a}(\frac{\lambda}{2} e_n)}_{\in \mathcal{S}(\mathbb{R}^n)} \end{aligned}$$

because $a \in C_c^\infty \subseteq \mathcal{S}$ so as much decay as you want. □

Lemma (Morse Lemma)**.** *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}^n$ is $C^\infty, p \in \Omega, \nabla f(p) = 0$ and suppose the Hessian matrix*

$$H_f(p) = \left[\frac{\delta^2 f}{\delta x_i \delta x_j}(p)\right]$$

is invertible. Then for a unique R, $0 \leq R \leq n$, there are neighborhoods U and V of 0 and p respectively and a C^∞ diffeomorphism $G : U \to V$ with $G(0) = p$ and

$$(f \circ G)(x) = f(p) + \text{☼}_{j=1}^{k} x_j^2 - \sum_{j=k-1}^{n} x_j^2.$$

Proof. Without loss of generality, assume $p = 0$ and $f(p) = 0$. Write

$$\begin{aligned} f(y_1, \ldots, y_n) &= \int_0^1 \frac{df}{dt}(ty_1, \ldots, ty_n)dt \\ &= \int_0^1 \sum_{i=1}^{n} y_i \frac{\delta f}{\delta y_i}(ty_1, \ldots, ty_n)dt. \end{aligned}$$

Thus if we set

$$g_i(y_1, \ldots, y_n) = \int_0^1 \frac{\delta f}{\delta y_i}(ty_1, \ldots, ty_n)dt$$

then

$$f(y_1, \ldots, y_n) = \sum_{i=1}^{n} y_i g_i(y_1, \ldots, y_n)$$

since critical point at 0 we have

$$g_i(0) = \frac{\delta f}{\delta y_i}(0) = 0$$

so can repeat the process and write

$$g_i(y_1, \ldots, y_n) = \sum_{j=1}^{n} y_j h_{ij}(y_1, \ldots, y_n)$$

and thus

$$f(y_1, \ldots, y_n) = \sum_{i,j=1}^{n} y_i y_j h_{ij}(y_1, \ldots, y_n).$$

Can assume $h_{ij} = h_{ji}$ by replacing h_{ij} by $\tilde{h_{ji}} = \frac{h_{ij}+h_{ji}}{2}$ if necessary.
Note that $\frac{\delta^2 f}{\delta x_i \delta x_j}(0) = 2h_{ij}(0)$ so $[h_{ij}(0)]$ is invertible. We can apply a linear coordinate change in $y_1, \ldots, y_n$ to diagonalize

$$\sum_{i,j=1}^{n} y_i y_j h_{ij}(0)$$

and since $[h_{ij}(0)]$ is invertible the diagonal terms are non-zero so can assume $h_{11}(0) \neq 0$. In fact h_{11} is nonzero in a neighborhood of 0 since it is continuous.
Define new coordinate $(x_1, y2, \ldots, y_n)$ and

$$x_1 = \sqrt{|h_{11}|}\left(y_1 + \sum_{i=2}^{n} y_i \frac{h_{ii}}{h_{11}}\right).$$

Note:

$$\frac{\delta(x_1, y_2, \ldots, y_n)(0)}{\delta(y_1, \ldots, y_n)}(0) =$$

is invertible so by Implicit Value Theorem exists diffeomorphism

$$(y_1, \ldots, y_n) \mapsto (x_1, y_2, \ldots, y_n)$$

in a small neighborhood of 0>

$$\begin{aligned} x_1^2 &= |h_{11}|\left(y_1 + \sum_{i=2}^{n} y_i \frac{h_{ii}}{h_{11}}\right)^2 \\ &= h_{11}y_1^2 + 2\sum_{i=2}^{n} y_1 y_i h_{1i} + \frac{(\sum_{i=2}^{n} y_i h_{1i})^2}{h_{11}} \end{aligned}$$

if $h_{11} > 0$ and same with minus if $h_{11} < 0$.

$$f(y_1, \ldots, y_n) = y_1^2 h_{11} + 2\sum_{i=2}^{n} y_1 y_i h_{1i} + \sum_{i,j>1} y_i y_j h_{ij}(\ldots)$$

becomes in new coordinates

$$\pm x_1^2 + \sum_{i,j>1} y_i y_j \widetilde{h_{ij}}$$

for new symmetric $\tilde{h_{ij}}$.
...continue to finish proof... $\square$