Oscillations in Harmonic Analysis

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Problem sets

Contents

1 Introduction

Combinatorics

1.0.1 Erdős problem

1. Erdős distinct distance problem (1946).
   What is the least number of distinct distances determined by N points in a plane.

Example 1.1. We have four points $(0, 1), (2, 2), (0, 0), (1, 0)$, and if we start listing the distances between each of them we obtain the following:

$$1, 1, \sqrt{2}, \sqrt{5}, \sqrt{5}, \sqrt{8}.$$

However, we care about the distinct number; hence we get a new list

$$1, \sqrt{2}, \sqrt{5}, \sqrt{8}.$$

Upper bound: Notice that the first list is obtain by getting the distance between two points, hence

$$\binom{N}{2} = \frac{1}{2}N^2 - \frac{1}{2}N \sim N^2.$$

are obtained random.

To analyze what happens with the lower bound, we look at the following example.

Example 1.2. (a) Say N is a perfect square.

(b) Then, we have an square of $\sqrt{N}$ lattice.

(c) Now, let's count

$$(\text{distinct distcnes})^2.$$

(d) We obtain the list:

$$1, 2, \ldots, 2N.$$

(e) Hence we get no more than $\sim N$.

(f) Notice that $a^2 + b^2 = 3$ has no solution (number theory). Hence our list (d) have holes.

(g) Hence,

$$\# \text{ distinct distance} \sim \frac{N}{\sqrt{\log(N)}} \text{ as } N \to \infty.$$

Conjecture. Conjecture (Erdős 1946): The answer for Erdos 1946 should be in the order of $\frac{N}{\sqrt{\log(N)}}$ as $N \to \infty$.

Theorem (Erdős 1946). *At least $\sim \sqrt{N}$ as $N \to \infty$.*

Theorem (Guth, Katz 2015). *At least $\sim \frac{N}{\log(N)}$ as $N \to \infty$.*

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2. Crescent Configurations N points in the plane such that distance d_1 appears 1 times, d_2 appears 2 times, and so on, until d_{N-1} appears $N-1$ times. ($N-1$ distinct distances). It is possible to achieve this if you place equally spaced points on a line. Additionally require general position no more than 2 points on a line, and no more than 3 points on a circle. Call a crescent configuration.

Example 1.3. We can go from 3 pts to 8 points in crescent configuration. It is unknown if we could find a 9 point or higher order configurations.

Conjecture (Erdős)**.** Eventually they do not exist.

Question: Find many (all) crescent configurations for some N.

Example 1.4. Given $N = 4$ in $\mathbb{R}^2$.

Question: For $N = 5$. Many known but not all.

2 Review on Vector Spaces

2.1 Inner Product

An *inner product* is a map
$$\langle \cdot, \cdot \rangle : V \times V \to F$$
that satisfies for all $f, g, h \in V$, and $\alpha, \beta \in F$.

1. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0 \Rightarrow f = 0$.

2. $\langle f, g \rangle = \langle g, f \rangle$.

3. $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$.

Example 2.1. In $\mathbb{R}^d$, we have
$$\langle x, y \rangle = x \cdot y = x_1 y_1 + \cdots + x_d y_d.$$

Example 2.2. In $\mathbb{C}^d$, we have
$$\langle x, y \rangle = x \cdot y = x_1 \bar{y}_1 + \cdots + x_d \bar{y}_d.$$

Example 2.3. In $\mathcal{C}[a, b]$, we have
$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx$$

2.2 Norm

A *norm* is a function
$$|| \cdot || : V \to \mathbb{R}$$
that satisfies for all $f, g \in V$ and $\alpha \in F$.

1. $||f|| \geq 0$ and if $||f|| = 0$ then $f = 0$.

2. $||\alpha f|| = |\alpha|\,||f||$.

3. $||f + g|| \leq ||f|| + ||g||$ (triangle inequality).

In in a vector space V with inner product $\langle \cdot, \cdot \rangle$ get a norm for free.

$$||f|| := \sqrt{\langle f, f \rangle}.$$

Definition (Cauchy-Schwartz Inequality).

$$|\langle f, g \rangle| \leq ||f||\,||g||.$$

<u>Hint for Problem set:</u> Implies triangle inequality.

$$\begin{aligned}
||f + g|| &= \langle f + g, f + g \rangle \\
&= ||f||^2 + \langle f, g \rangle + \langle \bar{f}, g \rangle + ||g||^2 \\
(\text{by CS}) &\leq ||f||^2 + 2||f||\,||g|| + ||g||^2 \\
&= (||f|| + ||g||)^2.
\end{aligned}$$

Example 2.4. On $\mathcal{C}[a, b]$ with

$$\langle f, g \rangle = \frac{1}{b - a} \int_a^b f(x)\overline{g(x)}dx$$

get

$$||f||_2 = \left(\frac{1}{b - a} \int_a^b |f(x)|^2 dx \right)^{1/2}$$

2.3   Orthogonality

Sat $f, g \in V$ are orthogonal if

$$\langle f, g \rangle = 0.$$

Say $\{\phi_1, \ldots, \phi_n\}$ are orthogonal if $\langle \phi_j, \phi_k \rangle = 0$ whenever $j \neq k$ and $\phi_j \neq 0$ for all j.

If in addition $||\phi_j|| = 1$ for all j then $\{\phi_1, \ldots, \phi_n\}$ is *orthonormal*.
<u>Note:</u> Remember Gram-Schmidt.

Theorem (Pythagorean Theorem). *If* $\langle f, g \rangle = $ *then* $||f + g||^2 = ||f||^2 + ||g||^2$.

2.4   Projections

Let $\phi \in B$ with $||\phi|| = 1$. The projection of f in the direction of ϕ is

$$\text{proj}_\phi(f) := \langle f, \phi \rangle \phi.$$

Example 2.5. Project $f = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ on $\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$\text{proj}_\phi(f) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Definition. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. The proejction of f onto W_n is.

$$\text{proj}_{W_n}(f)L = \langle f_1, \phi_1 \rangle \phi_1 + \ldots + \langle f_n, \phi_n \rangle \phi_n.$$

Theorem.

$$\text{proj}_{W_n}(f) = 0 \iff f \in W_n.$$

Theorem. $f - \text{proj}_{W_n}$ is orthonormal to every vector in W_n.

3 Fourier Series

Theorem. *Let $f \in V$. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. The element w in W_n that minimizes $||f - w||$ is*

$$w = \text{proj}_{W_n}(f). \qquad \leftarrow \text{ Best approximation}$$

Proof. Write:

$$w = \sum_{i=1}^{n} \beta_i \phi_i$$

and set

$$\alpha_i = \langle f, \phi_i \rangle, \quad i = 1, \ldots, n.$$

Then,

$$
\begin{aligned}
||f - w||^2 &= \langle f - w, f - w \rangle \\
&= ||f||^2 - \langle f, \sum_{i=1}^{n} \beta_i \phi_i \rangle - \overline{\langle f, \sum_{i=1}^{n} \beta_i \phi_i \rangle} + \langle \sum_{i=1}^{n} \beta_i \phi_i, \sum_{j=1}^{n} \beta_j \phi_j \rangle \\
&= ||f||^2 - \sum_{i=1}^{n} \overline{\beta_i} \langle f, \phi_i \rangle - \sum_{i=1}^{n} \overline{\overline{\beta_i}} \; \overline{\langle f, \phi_i \rangle} + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_i \overline{\beta_j} \langle \phi_i, \phi_j \rangle \\
&= ||f||^2 - \sum_{i=1}^{n} \overline{\beta_i} \alpha_i - \sum_{i=1}^{n} \beta_i \overline{\alpha_i} + \sum_{i=1}^{n} |\beta_i|^2 \\
&= ||f||^2 + \sum_{i=1}^{n} (\beta_i - \alpha_i)\overline{\beta_i - \alpha_i} - \sum_{i=1}^{n} |\alpha_i|^2 \\
&= ||f||^2 - \sum_{i=1}^{n} |\langle f, \phi_i \rangle|^2 + \sum_{i=1}^{n} |\beta_i - \alpha_i|^2.
\end{aligned}
$$

So minimized if $\beta_i = \alpha_i = \langle f, \phi_i \rangle$. $\qquad\square$

<u>Remark:</u>

$$\sum_{i=1}^{n} \langle f, \phi_i \rangle \phi_i \quad \text{best approximation to } f.$$

Corollary.

$$\sum_{i=1}^{n} |\langle f, \phi_i \rangle|^2 \leq ||f||^2.$$

Bessel's inequality:

$$\sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2 \leq ||f||^2.$$

Riemann-Lebesgue Lemma:

$$\lim_{i \to \infty} \langle f, \phi_i \rangle = 0.$$

Motivation:

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \neq 0 \\ 1 & n = m = 0. \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(nx)\sin(mx)dx = \begin{cases} \frac{1}{2} & \text{if } n = m \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(nx)\cos(mx)dx = 0.$$

Note that

$$1, \sqrt{2}\cos(x), \sqrt{2}\sin(x), \sqrt{2}\cos(2x), \sqrt{2}\sin(2x), \ldots$$

orthonormal sequence on $[0, 2\pi]$.
Best finite approximation up to level n of a function f is

$$a_0 \cdot 1 + a_1 \sqrt{2}\cos(x) + b_1 \sqrt{2}\sin(x) + \ldots$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot 1 \, dx$$

$$a_j = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sqrt{2}\cos(jx) dx$$

$$b_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot \sqrt{2}\sin(kx) dx.$$

Trigonometric Series:

$$\frac{1}{2} A_0 + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx))$$

if in addition

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx$$

and

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

then called *Fourier Series*.

Simplification: We could have started with the orthonormal sequence

$$e^{inx}, n \in \mathbb{Z}, \text{ on } [0, 2\pi].$$

Then,

$$\langle e^{inx}, e^{inx} \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{inx}\overline{e^{imx}}dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x}dx$$

$$= \begin{cases} \frac{1}{2\pi}\left[\frac{1}{i(n-m)}e^{i(n-m)x}\right]_0^{2\pi} = 0 & \text{if } n \neq m \\[2mm] \frac{1}{2}\int_0^{2\pi} 1dx = 1 & \text{if } n = m. \end{cases}$$

Give best approximation up to level $n > 0$ of a function f

$$c_{-n}e^{i(-n)x} + \cdots + c_0 \cdot 1 + \cdots + c_n e^{inx}$$

with corresponding series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{e^{inx}}dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx.$$

Note:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x)\cos(nx)dx - i\frac{1}{2\pi} \int_0^{2\pi} f(x)\sin(nx)dx$$

$$= \begin{cases} \frac{1}{2}(A_n - iB_n) & \text{if } n > 0 \\ \frac{1}{2}A_0 & \text{if } n = 0 \\ \frac{1}{2}\left(A_{|n|} + iB_{|n|}\right) & \text{if } n < 0. \end{cases}.$$

For $n > 0$, we have

$$c_n e^{inx} + c_{-n}e^{i(-n)x} = A_n \cos(nx) + B_n \sin(nx)$$

All the same terms as before!

Intervals:

$$e^{inx} \text{ works on any interval of length } 2\pi.$$
$$e^{2\pi inx/N} \text{ works on any interval of length } L.$$

Summary:
Fourier Series: If f is integrable on $[a, b]$ of length of L then the nth *Forier coefficient* of f is defined by

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x)e^{-2\pi inx/L}dx, \quad n \in \mathbb{Z}$$

and its Forier series is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-2\pi inx/L}.$$

Question: Does $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-2\pi inx/L}$ converges to $f(x)$?
Let's first look at the following example.

Example 3.1. Let $f(x) = x$ on $[0, 2n]$. Then,

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} xe^{-inx} dx = \frac{i}{n}, \quad n \neq 0.$$

and

$$\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} x dx = \pi.$$

Fourier series

$$\sum_{n=-\infty}^{\infty} \frac{i}{n} e^{inx} = \pi - 2\left(\frac{1}{1}\sin(x) + \frac{1}{2}\sin(2x) + \dots\right)$$

$$\uparrow \frac{i}{n}e^{inx} + \frac{i}{-n}e^{-inx} = -\frac{2}{n}\sin(nx).$$

Yields π if $x = 0$ or $x = 2\pi$ while $f(0) = 0$ and $f(2\pi) = 2\pi$.

Consider: [insert graph 1 vs 2 here]:

Note: Fourier series is periodic on $\mathbb{R}$.

3.1 Domains

Consider $[a, b]$ on periodic functions on $\mathbb{R}$ in a circle such that $f(x) = F(e^{ix})$ and $x \in [0, 2\pi]$.
Hence, functions on the circle are periodic functions $f(x)$ on $[0, 2\pi]$ with $f(0) = f(2\pi)$.

3.2 Uniqueness

Theorem. *Suppose f is integrable and bounded on an interval with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$.*
Then $f(x_0) = 0$ whenever f is continuous at x_0.

3.3 Partial Sums

Let

$$S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n)e^{-2\pi inx/L} \text{ on } [0, L].$$

Question: In what sense $S_N f \to f$.

Theorem. *Suppose f is continous on the circle and the Fourier series of f is absolutely convergent*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Then

$$\lim_{N \to \infty} S_N(f)(x) = f(x)$$

uniformly in x.

Proof. Since

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

converges absolutely and in fact uniformly.
If

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

then g must be continuous.
Since $f - g$ is continuous and $\widehat{f - g}(n) = 0$ for all $n \in \mathbb{Z}$ we conclude by uniqueness theorem that $f = g$.

$$\hat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} g(x)e^{-inx}dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\lim_{N \to \infty} \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx} \right) e^{-inx}dx$$

$$= \lim_{N \to \infty} \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{2\pi} \int_0^{2\pi} e^{i(k-n)x}dx.$$

Note that:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(k-n)x}dx = \begin{cases} 1 & \text{if } k = n \\ 0 & else. \end{cases}$$

Hence, we have that

$$\hat{g}(n) = \lim_{N \to \infty} \sum_{k=-\infty}^{\infty} \hat{f}(k) \frac{1}{2\pi} \int_0^{2\pi} e^{i(k-n)x}dx$$

$$= \hat{f}(n).$$

$\square$

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Theorem. *Let $f \in \mathcal{C}^2[0, 2\pi]$ and 2π period. Then*

$$\hat{f}(n) = \mathcal{O}\left(\frac{1}{n^2}\right) \quad \text{as } |n| \to \infty$$

and thus the Fourier series of f converges absolutely and uniformly to f.

Note: $f(x) = \mathcal{O}(g(x))$ as $x \to a$ means there exists a constant C such that

$$|f(x)| \le C|g(x)| \text{ as } x \to a.$$

Proof. Since f'' is continous on $[0, 2\pi]$, it is bounded. Say

$$|f''(x)| \le B \text{ for all } x \in (0, 2\pi].$$

Then

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

$$= \frac{1}{in} \widehat{f'}(n) \quad \text{by int. by parts.}$$

By iterating $\hat{f}(n) = \frac{1}{i^2 n^2} \widehat{f''}(n)$,

$$|\hat{f}(n)| = \frac{1}{n^2} \left| \frac{1}{2\pi} \int_0^{2\pi} f''(x) e^{-inx} dx \right|$$

$$\le \frac{1}{n^2} \left(\frac{1}{2\pi} \int_0^{2\pi} B dx \right) = \frac{B}{n^2}.$$

$\square$

Corollary. $\widehat{f'}(n) = in\hat{f}(n)$.

3.4 Convolution

Given 2π periodic integrable functions f and g on $\mathbb{R}$. Define their *convolution* $f * g$ by

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) dy \qquad x \in [-\pi, \pi].$$

Remarks: It has many good properties by most importantly

- $f * g = g * f$.
- $\widehat{f * g} = \hat{f}(n)\hat{g}(n)$.

4 Kernels

4.1 Direchlet Kernel

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$$

$$= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)e^{-iny}dy \right) e^{inx}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{-in(x-y)}dy \right) e^{inx}$$

$$= (f * D_N)(x)$$

where $D_N(x) = \sum_{n=-N}^{N} e^{inx}$. <u>Facts:</u>

(1) $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)dx = 1$.

(2) $D_n(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$.

(3) $D_n(x) \leq C \min\{N, \frac{1}{|x|}\}$ if $x \in [-\pi, \pi]$, $N \geq 1$.

4.2 Good Kernels on the circle

Definition. A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if

(i) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)dx = 1$ for all $n \geq 1$.

(ii) There exists $M > 0$ such that $\int_{-\pi}^{\pi} |K_n(x)|dx \leq M$ for all $n \geq 1$.

(iii) For every $\delta > 0$ $\int_{\delta \leq |x| \leq \pi} |K_n(x)|dx \to 0$ as $n \to \infty$.

Theorem. *Let $\{K_n\}$ be a family of good kernels and f a bounded and integrable function on the circle. Then*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere then the limit is uniform.

<u>Note:</u> Unfortunately

$$\int_{-\pi}^{\pi} |D_N(x)|dx \geq c\log(N)$$

as N is large. Then D_N is *not* a good kernel.

4.3 Fej'er Kernel

Say a series $\sum_{k=0}^{\infty} c_k$ is Cesáro summable to σ if

$$\lim_{N \to \infty} \sigma_N = \sigma$$

where

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}$$

is the Nth Cesáro mean.

Example 4.1. $\sum_{k=0}^{\infty} (-1)^k$ does not converge but Cesáro summable to $\frac{1}{2}$. For Fourier series the Nth Cesáro mean is

$$
\begin{aligned}
\sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\
&= \frac{(f * D_0)(x) + \ldots (f * D_{N-1})(x)}{N} \\
&= \left(f * \left(\frac{D_0 + \cdots + D_{N-1}}{N} \right) \right)(x) \\
&= (f * F_N)(x) \qquad \left(\text{since } F_N = \frac{D_0 + \cdots + D_{N=1}}{N} \to \text{Fej'er kernel} \right).
\end{aligned}
$$

Facts:

(1) $F_N(x) = \frac{1}{N} \frac{\sin^2(Nx/2)}{\sin^2(x/2)}$.

(2) F_N is a good kernel.

Theorem (Weierstrass Approximation Theorem). *For every continuous function $f : \mathbb{R} \to \mathbb{C}$ with period 2π and every $\epsilon > 0$ one can find a trigonometric polynomial P such that*

$$\forall x \qquad |f(x) - P(x)| < \epsilon.$$

Proof.

$$
\begin{aligned}
\sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\
&= (f * F_N)(x).
\end{aligned}
$$

is a trigonometric polynomial and $(f * F_N) \to f$ uniformly because f continuous and F_N is a good kernel. $\square$

Mean Square Convergence of Fourier Series
Let f be a continuous function on the circle. Then

$$||f - S_N(f)||_2 \to 0 \text{ as } N \to \infty.$$

Proof. By the Weierstrass Approximation Theorem for a given $\epsilon > 0$ there exists a trigonometric polynomial P, say of degree M, such that

$$|f(x) - P(x)| < \epsilon \quad \text{for all } x \in [0, 2\pi]$$

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and thus

$$||f - P||_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - P(x)|^2 fx \right)^2 < \epsilon.$$

By the best approximation theorem

$$||f - S_N(f)||_2 < \epsilon$$

whenever $N \geq M$. □

4.4 Parseval's identity

From general theory about orthonormal sequences Bessel's inequality:

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq ||f||_2^2.$$

Riemman-Lebesgue Lemma: $\hat{f}(n) \to 0$ as $|n| \to \infty$, and

$$\int_0^{2\pi} f(x) \sin(nx) dx \to 0,$$

$$\int_0^{2\pi} f(x) \cos(nx) dx \to 0.$$

Theorem (Parsval's Identity).

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq ||f||_2^2$$

where we assume f is continuous on the circle.

Proof. From proof of best approximation theorem

$$||f||_2^2 = ||f - S_N(f)||_2^2 + \sum_{|n| \leq N} |\hat{f}(n)|^2.$$

□

5 Fourier Series II

5.1 Pointwise Convergence of Fourier Series

Theorem (Pointwise Convergence of Fourier Series). *July 10, 2018] Let f be a square integrable function on the circle which is differentiable at x_0. Then*

$$S_N(f)(x_0) \to f(x_0) \ as \ N \to \infty.$$

Proof.

$$S_N(f)(x_0) - f(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x_0 - t) D_N(t) dt - f(x_0) \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(t) dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x_0 - t) - f(x_0)) \frac{\sin((N + \frac{1}{2})t)}{\sin(\frac{t}{2})} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \boxed{\frac{f(x_0 - t) - f(x)}{t} \frac{t}{\sin(t/2)} \cos(t/2)} \sin(Nt) dt$$

$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \boxed{\frac{f(x_0 - t) - f(x_0)}{t} \frac{t}{\sin(t/2)} \sin(t/2)} \cos(Nt) dt.$$

As $t \to 0$, the right side becomes

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) \cdot 2 \cdot 1 \sin(Nt) dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x_0) \cdot 2 \cdot 0 \cos(Nt) dt$$

and away from $t \to 0$ then $\square$ are square integrable like. Conclude by Riemman-Lebesgue lemma

$$S_N(f)(x_0) - f(x_0)| \to 0 \text{ as } N \to \infty.$$

$\square$

Theorem. *Let f be a 'square' integrrable funcion on the circle such that $f'(x_0^+)$ and $f'(x_0^-)$ exist. Then*

$$S_n(f)(x) \to \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Proof. Similar. $\square$

Non-convergence
There exists an integrable function which is continuous at a point x_0 such that $S_N(f)(x_0)$ does not converge.

$$|f(x) - f(y)| \le C|x - y|^\infty.$$

Gibbs Phenomenon

$$\sum_{k=1}^{N} \frac{1}{k} \sin(kx) = S_N(f)(x) \to f(x) = \frac{\pi}{2} - x \text{ on } (0, 2\pi).$$

For a fixed big N.   (include graph).

5.2   Uniform Convergence of Fourier Series

Theorem (Uniform Convergence of Fourier Series). *Let f be continuously differentiable on the circle. Then its Fourier series converges uniformly to f.*

Proof. Since f' is continuous it is square integrable and Parseval's identity holds. Further an old corollary says

$$\hat{f'}(n) = in\hat{f}(n).$$

and

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$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{\infty} \frac{1}{n} |\widehat{f'}(n)| \leq \underbrace{\left(\sum_{n=-\infty}^{\infty} \frac{1}{n^2} \right)^{1/2}}_{\infty} \underbrace{\left(\sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2 \right)^{1/2}}_{= ||f'||_2 < \infty.} < \infty$$

$\square$

6 Fourier Transforms on $\mathbb{R}$

f on $[0,1]$ period 1, $n \in \mathbb{Z}$.

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x} dx, \quad f(x) \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi i n x}.$$

f on $\mathbb{R}$ (no period), $\zeta \in \mathbb{R}$. We have

$$\hat{f}(\zeta) = \int_{-}^{\infty} \infty^{\infty} f(x)e^{-2\pi i \zeta x} dx, \quad f(x) \stackrel{?}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi i \zeta x} d\zeta.$$

6.1 The Schwartz space

Definition. The Schwartz space on $\mathbb{R}$ denoted $\mathcal{S}(\mathbb{R})$, consists of all indefinitely differentiable functions f such that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \qquad \text{for every } k, l \geq 0.$$

Note: If $f \in \mathcal{S}(\mathbb{R})$ then also

$$x^k f^{(l)}(x) \in \mathcal{S}(\mathbb{R}) \qquad \text{for all } k, l \geq 0.$$

Note: $\mathcal{S}(\mathbb{R})$ is a vector space.

Example 6.1. e^{-ax^2}, for $a > 0$ is in $\mathcal{S}(\mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$ define Fourier transform

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(\zeta)e^{-2\pi i \zeta x} d\zeta.$$

Useful facts:

(i)
$$f(x+h) \quad \stackrel{\text{Fourier transf.}}{\to} \quad e^{2\pi i h \zeta} \hat{f}(\zeta).$$

(ii)
$$f(x)e^{-2\pi i x h} \quad \stackrel{\text{Fourier transf.}}{\to} \quad \hat{f}(\zeta + h).$$

(iii)
$$f(\delta x) \quad \stackrel{\text{Fourier transf.}}{\to} \quad \delta^{-1} \hat{f}(\delta^{-1}\zeta), \delta > 0.$$

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(iv)
$$f'(x) \quad \overset{\text{Fourier transf.}}{\longrightarrow} \quad 2\pi i \zeta \hat{f}(\zeta).$$

(v)
$$-2\pi i x f(x) \quad \overset{\text{Fourier transf.}}{\longrightarrow} \quad \frac{d}{d\zeta} \hat{f}(\zeta).$$

Theorem. *If $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

Proof.
$$|\zeta^k \left(\frac{d}{d\zeta}\right)^l \hat{f}(\zeta)| = |\left(\frac{1}{2\pi i}\right)^k \left(\frac{d}{dx}\right)^k ((-2\pi i x)^l f(x))(\zeta)|$$
$$< \infty.$$

$\square$

Fact:
$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Theorem. *If $f(x) = e^{-\pi x^2}$ then $\hat{f}(\zeta) = e^{-\pi \zeta^2}$.*

Proof. Define $F(\zeta) = \hat{f}(\zeta) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \zeta} dx$. Then $F(0) = 1$ by fact above, and
$$F'(\zeta) = \int_{-\infty}^{\infty} \underbrace{e^{-\pi x^2}(-2\pi x)}_{f'(x)} i e^{-2\pi i x \zeta} dx$$
$$= i \widehat{f'}(\zeta)$$
$$= i(2\pi i \zeta)\hat{f}(\zeta)$$
$$= -2\pi \zeta F(\zeta).$$

Set $G(\zeta) = F(\zeta)e^{\pi \zeta^2}$. Then $G(0) = F(0) = 1$ and
$$G'(\zeta) = \underbrace{F'(\zeta)}_{-2\pi \zeta F(\zeta)} e^{\pi \zeta^2} + F(\zeta)2\pi \zeta e^{\pi \zeta^2} = 0.$$

So, $G(0) = 1$ and thus $F(\zeta) = e^{-\pi \zeta^2}$. $\square$

Define for $\delta > 0$.
$$K_\delta(x) = \delta^{1/2} e^{-\pi x^2/\delta} \text{ then } \hat{K}_\delta(\zeta) = e^{\pi \delta \zeta^2}.$$

(insert picture of graph as δ is small).

Definition. If $f, g \in \mathcal{S}(\mathbb{R})$ their *convolution* is defined by
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)dt.$$

Note: Similar properties as on circle for $f, g \in \mathcal{S}(\mathbb{R})$.

(i) $f * g \in \mathcal{S}(\mathbb{R})$.

(ii) $f * g = g * f$.

(iii) $\widehat{f * g} = \hat{f}\hat{g}$.

Theorem. *The collection $\{K_\delta\}_{\delta > 0}$ is a family of good kernels as $\delta \to 0^+$.*

Theorem. *If $f \in \mathbb{R}$ then $(f * K_\delta)(x) \to f(x)$ uniformly in x as $\delta \to 0^+$.*

Note: Works for any good kernel.

Proposition. *If $f, g \in \mathcal{S}(\mathbb{R})$ then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy.$$

Proof.

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(x)e^{-2\pi ixy}dydx$$

$$= \int_{-\infty}^{\infty} g(y) \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dxdy$$

$$= \int_{-\infty}^{\infty} g(y)\hat{f}(y)dy.$$

$\square$

Theorem (Fourier Inversion)**.** *if $f \in \mathcal{S}(\mathbb{R})$ then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi ix\zeta}d\zeta$.*

Proof. First note that

$$f(0) = \lim_{\delta \to 0^+} (f * K_\delta)(0)$$

$$= \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} f(x)K_\delta(0 - x)dx$$

$$= \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} f(x) \underbrace{\delta^{-1/2}e^{-\pi x^2/\delta}}_{K_\delta(x)}.$$

Also we have that,

$$e^{-\pi x^2} = \widehat{e^{-\pi \delta^2}}(x)$$

$$\Rightarrow \delta^{-1/2}e^{-\pi x^2/\delta} = \underbrace{\delta^{-1/2}\widehat{e^{-\pi \delta^2}}(\delta^{-1/2}x)}_{\widehat{e^{-\pi \delta^2}}(x)}$$

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Then, continuing our first observation

$$f(0) = \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} f(x) \underbrace{\delta^{-1/2} e^{-\pi x^2/\delta}}_{K_\delta(x)}$$

$$= \lim_{\delta \to 0^+} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{-\pi \delta \zeta^2} d\zeta$$

$$\overset{\text{Prop.}}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta) d\zeta$$

$$= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i \delta \zeta} d\zeta.$$

In genreal, let

$$F(y) = f(x + y).$$

Then

$$f(x) = F(0)$$

$$= \int_{-\infty}^{\infty} \hat{F}(\zeta) d\zeta$$

$$= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i x \zeta} d\zeta.$$

$\square$

Corollary. *The Fourier transform is bijective mapping on* $\mathcal{S}(\mathbb{R})$

$$\mathcal{S}(\mathbb{R}) \overset{\mathcal{F}(f)(\zeta) = \int_{\infty}^{\infty} f(x) e^{-2\pi i x \zeta} dx}{\curvearrowright} \mathcal{S}(\mathbb{R}).$$

and

$$\mathcal{S}(\mathbb{R}) \overset{\mathcal{F}^{-1}(g)(\zeta) = \int_{\infty}^{\infty} g(\zeta) e^{-2\pi i x l} dl}{\curvearrowleft} \mathcal{S}(\mathbb{R}).$$

7   Inner Product and Norm on $\mathcal{S}(\mathbb{R})$

Can use the usual

$$\langle f, g \rangle = \int_{\infty}^{\infty} f(x) \overline{g(x)} dx$$

and

$$||f||_2 = \left(\int_{\infty}^{\infty} |f(x)|^2 dx \right)^{1/2}.$$

Theorem (Plancharel). *If* $f \in \mathcal{S}(\mathbb{R})$ *then* $||f^2||_2 = ||f||_2$.

Proof.

$$\widehat{f^*}(\zeta) = \int_{\infty}^{\infty} \overline{f(-x)} e^{-2\pi i x \zeta} dx$$

$$= \overline{\int_{\infty}^{\infty} \overline{f(-x)} e^{-2\pi i (-x) \zeta} dx}$$

$$= \overline{\hat{f}(\zeta)}.$$

Also,

$$
\begin{aligned}
\int_{\infty}^{\infty} |f(x)|^2 dx &= \int_{\infty}^{\infty} f(x)\overline{f(x)}dx \\
&= \int_{\infty}^{\infty} \overline{f(-(-x))}dx \\
&= \int_{\infty}^{\infty} f(x)f^{\#}(0-x)dx \\
&= (f * f^{\#})(0) \\
&\overset{\text{inversion}}{=} \int_{\infty}^{\infty} \widehat{f * f^{\#}}(\zeta)d\zeta \\
&= \int_{\infty}^{\infty} \hat{f}(\zeta)\widehat{f^{\#}}(\zeta)d\zeta \\
&= \int_{\infty}^{\infty} \hat{f}(\zeta)\overline{\hat{f}(\zeta)}(\zeta) \\
&= ||\hat{f}||_2^2.
\end{aligned}
$$

<u>Note:</u> $\mathcal{F}$ is unitary on $\mathcal{S}(\mathbb{R})$ with $|| \cdot ||_2$.
<u>Note:</u>

- Parseval:

$$
\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \right)^{1/2} = ||f||_2.
$$

- Plancharel:

$$
||\hat{f}||_2 = ||f||_2.
$$

$\square$

8 Fourier Transform on $\mathbb{R}^d$

8.1 Rotations

A rotation in $\mathbb{R}^d$ is a linear transformation that fulfills one of the following equivalent conditions

(a) $Rx \cdot Ry = x \cdot y$ for all $x, y \in \mathbb{R}^d$.

(b) $|Rx| = |x|$ for all $x \in \mathbb{R}^d$.

(c) $R^t = R^{-1}$.

(d) $\det(R) = \pm 1$.

Useful:

$$
\int_{R^d} f(Rx)dx = \int_{R^d} f(x)dx.
$$

8.2 Coarea Formula / Polar coordinates

Write $\gamma = (\cos(\theta), \sin(\theta))$. For a function on g on S' define

$$\int_{S'} g(\gamma)d\sigma(\gamma) = \int_0^{2\pi} g(\cos(\theta), \sin(\theta))d\theta.$$

Then can write

$$\int_{\mathbb{R}^2} f(x)dx = \int_0^{2\pi} \int_0^\infty f(r\cos(\theta), r\sin(\theta))rdrd\theta$$
$$= \int_{S'} f(r\gamma)rdrd\sigma(\gamma).$$

For g on S^2 define

$$\int_{S^2} g(\gamma)d\sigma(\gamma) = \int_0^{2\pi} \int_0^\pi g(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))\sin(\phi)d\phi d\theta.$$

Then.. finish here. In general can write

8.3 Schwartz space

$$\mathcal{S}(\mathbb{R}^d) = \{f \text{ indefinitely differentiable and } \sup_{x \in \mathbb{R}^d} |x^\alpha \left(\frac{\delta}{\delta x}\right)^\beta f(x)| < \infty\}.$$

Example 8.1. $e^{-a|x|^2}$ for $a > 0$ is in $\mathcal{S}(\mathbb{R}^d)$.

Fourier Transform

$$\hat{f}(\zeta) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\zeta}dx, \quad \zeta \in \mathbb{R}^d.$$

Useful facts:

(i)
$$f(x+h) \quad \overset{\text{Fourier transf.}}{\to} \quad \hat{f}(\zeta)e^{2\pi i h\zeta}.$$

(ii)
$$f(x)e^{-2\pi i xh} \quad \overset{\text{Fourier transf.}}{\to} \quad \hat{f}(\zeta + h).$$

(iii)
$$f(\delta x) \quad \overset{\text{Fourier transf.}}{\to} \quad \delta^{-d}\hat{f}(\delta^{-1}\zeta), \delta > 0.$$

(iv)
$$f'(x) \quad \overset{\text{Fourier transf.}}{\to} \quad 2\pi i\zeta \hat{f}(\zeta).$$

(v)
$$(-2\pi i x)^\alpha f(x) \quad \overset{\text{Fourier transf.}}{\to} \quad \left(\frac{d}{d\zeta}\right)^\alpha \hat{f}(\zeta).$$

(vi)

$$f(Rx) \quad \overset{\text{Fourier transf.}}{\longrightarrow} \quad \hat{f}(R\zeta), \quad R \text{ rotation.}$$

$$\int_{\mathbb{R}^d} f(Rx)e^{-2\pi i x \zeta}dx = \int_{\mathbb{R}^d} f(x)e^{-2\pi R^{-1}x\zeta}dx$$

$$= \int_{\mathbb{R}^d} f(x)e^{-2\pi i x R\zeta}dx$$

$$= \hat{f}(R\zeta).$$

Theorem. *If $f \in \mathcal{S}(\mathbb{R}^d)$ then $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$.*

8.4   Radical Functions

$f(x)$ is *radial* if there is a function f_0 defines on $\mathbb{R}_+$ such that

$$f(x) = f_0(|x|).$$

Note: f radial $\iff$ $f(Rx) = f(x)$ for all rotations R.

Fact: The Fourier transform of a radial function is radial. Because,

$$f(Rx) = f(x) \forall R$$

$$\Rightarrow \hat{f}(R\zeta) = \hat{f}(\zeta) \forall R.$$

Example 8.2. $e^{-a|x|^2}$ is radial.

$$\widehat{e^{-\pi|x|^2}}(\zeta) = e^{-\pi|\zeta|^2}.$$

Theorem (Inverse & Plancharz). *Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\zeta)e^{2\pi i x \zeta}d\zeta$$

and

$$||\hat{f}||_2 = ||f||_2.$$

8.5   Fourier Transform of σ in $\mathbb{R}^3$.

$$\hat{\sigma}(\zeta) := \int_{S^2} e^{-2\pi i \zeta \gamma}d\sigma(\zeta)$$

Theorem.

$$\hat{\sigma}(\zeta) = \frac{2\sin(2\pi|\zeta|)}{|\zeta|}.$$

Proof. First note LHS is radial in ζ.

$$\int_{S^2} e^{-2\pi i R\zeta\gamma}d\sigma(\gamma) = \int_{S^2} e^{-2\pi i \zeta \cdot R^{-1}}\gamma d\sigma(\gamma)$$

$$= \int_{S^2} e^{-2\pi i \zeta \cdot \gamma}d\sigma(\gamma). \qquad \boxed{\gamma^{\text{new}} = R^{-1}\gamma^{\text{old}}}$$

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So if $|\zeta| = \rho$ then enough to prove the identity with $\zeta = (0,0,\rho)$.
Use spherical coordinates

$$
\begin{aligned}
\int_{S^2} e^{-2\pi i (0,0,\rho) \cdot \gamma} d\sigma(\gamma) &= \int_0^{2\pi} \int_0^{\pi} e^{-2\pi i \rho \cos(\phi)} \sin(\phi) d\phi d\theta \\
&= 2\pi \int_{-1}^1 e^{2\pi i \rho u} du \qquad (u = \cos(\phi)) \\
&= 2\pi \frac{1}{2\pi i \rho} \left[e^{2\pi i \rho u} \right]_{-1}^1 \\
&= \frac{2 \sin(2\pi \rho)}{\rho} \\
&= \frac{2 \sin(2\pi |\zeta|)}{|\zeta|}.
\end{aligned}
$$

$\square$

8.6 Stationary Phase

For $\phi \in C^\infty$ and $a \in C_c^\infty$ define

$$
I(\lambda) = \int_{\mathbb{R}^d} e^{-\pi i \lambda \phi(x)} a(x) dx.
$$

Question: How does $I(\lambda)$ behave as $\lambda \to \infty$?
Idea: Much cancellation if λ large.
Warning: If $\phi(x) = c$ then

$$
I(\lambda) = e^{-pii\lambda c} \int_{\mathbb{R}^d} a(x) dx
$$

so no decay. We then need conditions on ϕ.

Definition. A map f is called a *diffeomorphism* if it is a bijection and it is differentiable and its inverse is differentiable as well. Say it is smooth if $f \in C^\infty$, $f^{-1} \in C^\infty$.

Note: Suppose $\phi_1 = \phi \circ G$ where G is smooth diffeomorphism. Then

$$
\begin{aligned}
\int e^{-\pi i \lambda \phi_2(x)} a(x) dx &= \int e^{-\pi i \lambda_1 (G^{-1} x)} a(x) dx \\
&= \int e^{-\pi i \lambda_1 (y)} \underbrace{a(Gy)|J_g(y)|}_{\text{smooth}} dy \qquad (y = G^{-1} x).
\end{aligned}
$$

So bounds independent of $a(x)$ will be smooth diffeomorphism invariant.

Two Cases: Neighborhoods of points p where

$$
\underbrace{\nabla \phi(p) = 0}_{\text{stationary}} \quad \text{and} \quad \underbrace{\nabla \phi(p) \neq 0}_{\text{on-stationary}}.
$$

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Lemma (Straightening Lemma). *Suppose $\omega \in \mathbb{R}^n$ is open, $f : \omega \to \mathbb{R}$ is C^∞, $p \in \omega$ and $\nabla f(p) \neq 0$. Then there are neighborhoods U and V of 0 and p respectively and a smooth diffeomorphism $G : U \to V$ with*

$$G(0) = p$$

and

$$(f \circ G)(x) = f(p) + x_n.$$

Theorem (Inverse Function Theorem). *Let $A \subseteq \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}^n$ be of class C^1. Let $x_0 \in A$ and suppose $|Df(x_0)| \neq 0$. Then there is a neighborhood U of x_0 in A and an open neighborhood W of $f(x_0)$ such that $f(U) = W$ and f has a C^1 inverse $f^{-1} : W \to U$. More over $y \in W, x = f^{-1}(y)$ we have*

$$Df^{-1}(y) = [Df(x)]^{-1}.$$

If f is of call C^p, then so is f^{-1}.

<u>Note:</u> In $\mathbb{R} : \ (f^{-1})^{-1}(f(a)) = \frac{1}{f'(a)}$.

Theorem (Implicit Function Theorem). *Let $A \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set and let $f; A \to \mathbb{R}^m$ be a function of class C^p. Suppose $(x_0, y_0) \in A$ and $F(x_0, y_0) = 0$. Consider*

$$\nabla = \begin{vmatrix} \frac{\delta F_1}{\delta y_1} & \cdots & \frac{\delta F_1}{\delta y_m} \\ \vdots & \ddots & \vdots \\ \frac{\delta F_m}{\delta y_1} & \cdots & \frac{\delta F_m}{\delta y_m} \end{vmatrix}$$

evaluated at (x_0, y_0) and suppose $\nabla \neq 0$. Then there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of x_0 and a neighborhood V of y_0 in $\mathbb{R}^m$ and a unique function $f : U \to V$ such that $F(x, f(x)) = 0$ for all $x \in U$. Furthermore f is of class C^p.

Proof. Define $G : A \to \mathbb{R}^n \times \mathbb{R}^m$ by $G(x, y) = (x, F(x, y))$. Then G is of class C^p and

$$|DG(x_0, y_0)| = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{\delta F_1}{\delta x_1} & \cdots & \cdots & \frac{\delta F_1}{\delta x_n} & \frac{\delta F_1}{\delta y_1} & \cdots & \frac{\delta F_1}{\delta y_m} \\ \vdots & \ddots & & & & \ddots & \\ \frac{\delta F_m}{\delta x_1} & \cdots & & \frac{\delta F_m}{\delta x_n} & \frac{\delta F_m}{\delta y_1} & \cdots & \frac{\delta F_m}{\delta y_m} \end{vmatrix} \neq 0$$

so by Inverse Function Theorem there is an open set W containing $(x_0, 0)$ and S open set containing (x_0, y_0) such that $G(S) = W$ and G has C^p inverse $G^{-1} : W \to S'$. There exist open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ with $x_0 \in U$ and $y_0 \in V$ such that $U \times V \subseteq S$.
Let $G(U \times V) = Y$. Now G^{-1} is of the form

$$G^{-1}(x, w) = (x, H(x, w)).$$

Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ given by $\pi(x, y) = y$. Then

$$F(x, H(x, w)) = \pi \circ G(x, H(x, w))$$
$$= \pi \circ G \circ G^{-1}(x, w)$$
$$= W.$$

Define $f : U \to V$ by $f(x) = H(x, 0)$. Then

$$F(x, f(x)) = 0$$

. Since H is of class C^p then so must f and since H unique then so must f. $\qquad\square$

Lemma. *Let $A \subseteq \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}$ be a unction of class $C^p, p \geq 1$. Let $p \in A$ and suppose $f(p) = 0$ and $\nabla f(p) \neq 0$. Then there is an open set U, an open set V containing p and a function $h : U \to V$ of class C^p with inverse $h^{-1} : V \to U$ of class C^p such that*

$$f(h(x_1, \ldots, x_n)) = x_n.$$

Proof. Since $\nabla(p) \neq 0$ there exists i such that $\frac{\delta f}{\delta x_i} \neq 0$. Define $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_n, x_{i+1}, \ldots, x_{n-1}, x_i).$$

The map g is linear so $f \circ g$ is of class C^p and

$$\frac{\delta(f \circ g)}{\delta x_n}(g^{-1}(p)) = \frac{\delta f}{\delta x_i}(p) \neq 0.$$

Thus $(f \circ g) : g^{-1}(A) \subseteq \mathbb{R}^{m-1} \times \mathbb{R}$ is a map as in the Implicit Function Theorem.
Define $G : g^{-1}(A) \to \mathbb{R}^{-1} \times \mathbb{R}$ by $G(x, y) = (x, (f \circ g)(x, y))$. As in proof of I.F.T. there are open set $W \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ with $g^{-1}(p) \in W$ and $(\tilde{p}_1, \ldots, \tilde{p}_{n-1}, 0) \in U$ where $g^{-1}(p) = (\tilde{p}_1, \ldots, \tilde{p}_n)$. such that $G : W \to U$ has an inverse $G^{-1} : U \to W$ of class C^p.

$$(f \circ g) \circ G^{-1}(x_1, \ldots, x_n) = (\pi \circ G) \circ G^{-1}(x1, \ldots, x_n) = x_n.$$

Define $V = g(W)$ and $h : U \to$ by $h = g \circ G^{-1}$. Then h is a C^p function with a C^p inverse $f(h(x_1, \ldots, x_n)) = x_n$. $\qquad\square$

Proposition (Non-stationary phase). *Suppose $\Omega \in \mathbb{R}^n$ is open, $\phi : \omega \to \mathbb{R}$ is C^∞, $p \in \Omega$ and $\nabla\phi(p) \neq 0$. Suppose $a \in C_c^\infty$ has its support in a sufficiently small neighborhood of p. Then*

$$\forall N \exists C_N : \ |I(\lambda)| \leq C_N \lambda^{-N}.$$

Proof. The straightening lemma and smooth diffeomorphism invariance reduce this to the case

$$\phi(x) = x_n + C.$$

Letting $e_n = (0, \ldots, 0, 1)$ get

$$I(\lambda) = \int e^{-\pi i \lambda (x_n + X)} a(x) dx$$
$$= \left(\int a(x) e^{-2\pi i (\frac{\lambda}{2} e_n) \cdot x} dx \right) e^{-\pi i \lambda C}$$
$$= e^{-\pi i \lambda C} \underbrace{\hat{a}(\frac{\lambda}{2} e_n)}_{\in \mathcal{S}(\mathbb{R}^n)}$$

because $a \in C_c^\infty \subseteq \mathcal{S}$ so as much decay as you want. $\qquad\square$

Lemma (Morse Lemma). *Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}^n$ is $C^\infty, p \in \Omega, \nabla f(p) = 0$ and suppose the Hessian matrix*

$$H_f(p) = \left[\frac{\delta^2 f}{\delta x_i \delta x_j}(p) \right]$$

is invertible. Then for a unique R, $0 \le R \le n$, there are neighborhoods U and V of 0 and p respectively and a C^∞ diffeomorphism $G : U \to V$ with $G(0) = p$ and

$$(f \circ G)(x) = f(p) + \maltese_{j=1}^k x_j^2 - \sum_{j=k-1}^n x_j^2.$$

Proof. Without loss of generality, assume $p = 0$ and $f(p) = 0$. Write

$$f(y_1, \ldots, y_n) = \int_0^1 \frac{df}{dt}(ty_1, \ldots, ty_n) dt$$

$$= \int_0^1 \sum_{i=1}^n y_i \frac{\delta f}{\delta y_i}(ty_1, \ldots, ty_n) dt.$$

Thus if we set

$$g_i(y_1, \ldots, y_n) = \int_0^1 \frac{\delta f}{\delta y_i}(ty_1, \ldots, ty_n) dt$$

then

$$f(y_1, \ldots, y_n) = \sum_{i=1}^n y_i g_i(y_1, \ldots, y_n)$$

since critical point at 0 we have

$$g_i(0) = \frac{\delta f}{\delta y_i}(0) = 0$$

so can repeat the process and write

$$g_i(y_1, \ldots, y_n) = \sum_{j=1}^n y_j h_{ij}(y_1, \ldots, y_n)$$

and thus

$$f(y_1, \ldots, y_n) = \sum_{i,j=1}^n y_i y_j h_{ij}(y_1, \ldots, y_n).$$

Can assume $h_{ij} = h_{ji}$ by replacing h_{ij} by $\tilde{h}_{ji} = \frac{h_{ij}+h_{ji}}{2}$ if necessary.

Note that $\frac{\delta^2 f}{\delta x_i \delta x_j}(0) = 2h_{ij}(0)$ so $[h_{ij}(0)]$ is invertible. We can apply a linear coordinate change in $y_1, \ldots, y_n$ to diagonalize

$$\sum_{i,j=1}^n y_i y_j h_{ij}(0)$$

and since $[h_{ij}(0)]$ is invertible the diagonal terms are non-zero so can assume $h_{11}(0) \neq 0$. In fact h_{11} is nonzero in a neighborhood of 0 since it is continuous.

Define new coordinate $(x_1, y2, \ldots, y_n)$ and

$$x_1 = \sqrt{|h_{11}|}\left(y_1 + \sum_{i=2}^{n} y_i \frac{h_{ii}}{h_{11}}\right).$$

Note:

$$\frac{\delta(x_1, y_2, \ldots, y_n)(0)}{\delta(y_1, \ldots, y_n)}(0) = \begin{vmatrix} \sqrt{|h_{11}|} & ? & ? & \ldots & ? \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{vmatrix}$$

is invertible so by Implicit Value Theorem exists diffeomorphism

$$(y_1, \ldots, y_n) \mapsto (x_1, y_2, \ldots, y_n)$$

in a small neighborhood of 0>

$$x_1^2 = |h_{11}|\left(y_1 + \sum_{i=2}^{n} y_i \frac{h_{ii}}{h_{11}}\right)^2$$

$$= h_{11}y_1^2 + 2\sum_{i=2}^{n} y_1 y_i h_{1i} + \frac{(\sum_{i=2}^{n} y_i h_{1i})^2}{h_{11}}$$

if $h_{11} > 0$ and same with minus if $h_{11} < 0$.

$$f(y_1, \ldots, y_n) = y_1^2 h_{11} + 2\sum_{i=2}^{n} y_1 y_i h_{1i} + \sum_{i,j>1} y_i y_j h_{ij}(\ldots)$$

becomes in new coordinates

$$\pm x_1^2 + \sum_{i,j>1} y_i y_j \widetilde{h_{ij}}$$

for new symmetric $\tilde{h}_{ij}$.

...continue to finish proof... $\qquad\qquad\square$