Oscillations in Harmonic Analysis

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Problem sets

I'd love to hear your feedback. Feel free to email me at coscohua@mail.sfsu.edu.
See git:icarlitoss for updates.

Contents

1 Introduction

← July 2, 2018

Combinatorics

1.0.1 Erdős problem

1. Erdős distinct distance problem (1946).
   What is the least number of distinct distances determined by N points in a plane.

Example 1.1. We have four points $(0,1),(2,2),(0,0),(1,0)$, and if we start listing the distances between each of them we obtain the following:

$$1, 1, \sqrt{2}, \sqrt{5}, \sqrt{5}, \sqrt{8}.$$

However, we care about the distinct number; hence we get a new list

$$1, \sqrt{2}, \sqrt{5}, \sqrt{8}.$$

Upper bound: Notice that the first list is obtain by getting the distance between two points, hence

$$\binom{N}{2} = \frac{1}{2}N^2 - \frac{1}{2}N \sim N^2.$$

are obtained random.

To analyze what happens with the lower bound, we look at the following example.

Example 1.2. (a) Say N is a perfect square.

(b) Then, we have an square of $\sqrt{N}$ lattice.

(c) Now, let's count

$$(\text{distinct distcnes})^2.$$

(d) We obtain the list:

$$1, 2, \ldots, 2N.$$

(e) Hence we get no more than $\sim N$.

(f) Notice that $a^2 + b^2 = 3$ has no solution (number theory). Hence our list (d) have holes.

(g) Hence,

$$\#\text{ distinct distance} \sim \frac{N}{\sqrt{\log(N)}} \text{ as } N \to \infty.$$

Conjecture. Conjecture (Erdős 1946): The answer for Erdos 1946 should be in the order of $\frac{N}{\sqrt{\log(N)}}$ as $N \to \infty$.

Theorem (Erdős 1946)**.** *At least* $\sim \sqrt{N}$ *as* $N \to \infty$.

Theorem (Guth, Katz 2015)**.** *At least* $\sim \frac{N}{\log(N)}$ *as* $N \to \infty$.

2. Crescent Configurations N points in the plane such that distance d_1 appears 1 times, d_2 appears 2 times, and so on, until d_{N-1} appears $N-1$ times. ($N-1$ distinct distances). It is possible to achieve this if you place equally spaced points on a line. Additionally require general position no more than 2 points on a line, and no more than 3 points on a circle. Call a crescent configuration.

Example 1.3. We can go from 3 pts to 8 points in crescent configuration. It is unknown if we could find a 9 point or higher order configurations.

Conjecture (Erdős)**.** Eventually they do not exist.

Question: Find many (all) crescent configurations for some N.

Example 1.4. Given $N = 4$ in $\mathbb{R}^2$.

Question: For $N = 5$. Many known but not all.

2 Review on Vector Spaces

← July 3, 2018

2.1 Inner Product

An *inner product* is a map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

that satisfies for all $f, g, h \in V$, and $\alpha, \beta \in F$.

1. $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0 \Rightarrow f = 0$.
2. $\langle f, g \rangle = \langle g, f \rangle$.
3. $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$.

Example 2.1. In $\mathbb{R}^d$, we have

$$\langle x, y \rangle = x \cdot y = x_1 y_1 + \cdots + x_d y_d.$$

Example 2.2. In $\mathbb{C}^d$, we have

$$\langle x, y \rangle = x \cdot y = x_1 \bar{y_1} + \cdots + x_d \bar{y_d}.$$

Example 2.3. In $\mathcal{C}[a, b]$, we have

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx$$

2.2 Norm

A *norm* is a function

$$|| \cdot || : V \to \mathbb{R}$$

that satisfies for all $f, g \in V$ and $\alpha \in F$.

1. $||f|| \geq 0$ and if $||f|| = 0$ then $f = 0$.

2. $||\alpha f|| = |\alpha|||f||$.
3. $||f+g|| \leq ||f|| + ||g|| \quad$ (triangle inequality).

In in a vector space V with inner product $\langle \cdot, \cdot \rangle$ get a norm for free.

$$||f|| := \sqrt{\langle f, f \rangle}.$$

Definition (Cauchy-Schwartz Inequality)**.**

$$|\langle f, g \rangle| \leq ||f||||g||.$$

<u>Hint for Problem set:</u> Implies triangle inequality.

$$\begin{aligned} ||f+g|| &= \langle f+g, f+g \rangle \\ &= ||f||^2 + \langle f, g \rangle + \langle f \bar{,} g \rangle + ||g||^2 \\ \text{(by CS)} &\leq ||f||^2 + 2||f||||g|| + ||g||^2 \\ &= (||f|| + ||g||)^2. \end{aligned}$$

Example 2.4. On $\mathcal{C}[a,b]$ with

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x)\overline{g(x)}dx$$

get

$$||f||_2 = \left(\frac{1}{b-a} \int_a^b |f(x)|^2 dx \right)^{1/2}$$

2.3 Orthogonality

Sat $f, g \in V$ are orthogonal if

$$\langle f, g \rangle = 0.$$

Say $\{\phi_1, \ldots, \phi_n\}$ are orthogonal if $\langle \phi_j, \phi_k \rangle = 0$ whenever $j \neq k$ and $\phi_j \neq 0$ for all j.

If in addition $||\phi_j|| = 1$ for all j then $\{\phi_1, \ldots, \phi_n\}$ is *orthonormal.*
<u>Note:</u> Remember Gram-Schmidt.

Theorem (Pythagorean Theorem)**.** *If* $\langle f, g \rangle =$ *then* $||f+g||^2 = ||f||^2 + ||g||^2$.

2.4 Projections

Let $\phi \in B$ with $||\phi|| = 1$. The projection of f in the direction of ϕ is

$$\text{proj}_\phi(f) := \langle f, \phi \rangle \phi.$$

Example 2.5. Project $f = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ on $\phi = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

$$\text{proj}_\phi(f) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Definition. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. The proejction of f onto W_n is.

$$\mathrm{proj}_{W_n}(f)L = \langle f_1, \phi_1\rangle\phi_1 + \ldots + \langle f_n, \phi_n\rangle\phi_n.$$

Theorem.

$$\mathrm{proj}_{W_n}(f) = 0 \iff f \in W_n.$$

Theorem. *$f - \mathrm{proj}_{W_n}$ is orthonormal to every vector in W_n.*

3 Fourier Series

← July 5, 2018

Theorem. *Let $f \in V$. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. The element $winW_n$ that minimizes $||f - w||$ is*

$$w = \mathrm{proj}_{W_n}(f). \qquad \leftarrow \textit{Best approximation}$$

Proof. Write:

$$w = \sum_{i=1}^{n} \beta_i\phi_i$$

and set

$$\alpha_i = \langle f, \phi_i\rangle, \quad i = 1, \ldots, n.$$

Then,

$$\begin{aligned}
||f - w||^2 &= \langle f - w, f - w\rangle \\
&= ||f||^2 - \langle f, \sum_{i=1}^{n} \beta_i\phi_i\rangle - \overline{\langle f, \sum_{i=1}^{n} \beta_i\phi_i\rangle} + \langle \sum_{i=1}^{n} \beta_i\phi_i, \sum_{j=1}^{n} \beta_j\phi_j\rangle \\
&= ||f||^2 - \sum_{i=1}^{n} \overline{\beta_i}\langle f, \phi_i\rangle - \sum_{i=1}^{n} \overline{\overline{\beta_i}}\; \overline{\langle f, \phi_i\rangle} + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_i\overline{\beta_j}\langle\phi_i, \phi_j\rangle \\
&= ||f||^2 - \sum_{i=1}^{n} \overline{\beta_i}\alpha_i - \sum_{i=1}^{n} \beta_i\overline{\alpha_i} + \sum_{i=1}^{n} |\beta_i|^2 \\
&= ||f||^2 + \sum_{i=1}^{n} (\beta_i - \alpha_i)\overline{\beta_i - \alpha_i} - \sum_{i=1}^{n} |\alpha_i|^2 \\
&= ||f||^2 - \sum_{i=1}^{n} |\langle f, \phi_i\rangle|^2 + \sum_{i=1}^{n} |\beta_i - \alpha_i|^2.
\end{aligned}$$

So minimized if $\beta_i = \alpha_i = \langle f, \phi_i\rangle$. □

Remark:

$$\sum_{i=1}^{n} \langle f, \phi_i\rangle\phi_i \quad \text{best approximation to } f.$$

Corollary.

$$\sum_{i=1}^{n} |\langle f, \phi_i\rangle|^2 \leq ||f||^2.$$

Bessel's inequality:

$$\sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2 \leq ||f||^2.$$

Riemann-Lebesgue Lemma:

$$\lim_{i \to \infty} \langle f, \phi_i \rangle = 0.$$

<u>Motivation:</u>

$$\frac{1}{2\pi} \int_0^{2\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \neq 0 \\ 1 & n = m = 0. \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(nx)\sin(mx)dx = \begin{cases} \frac{1}{2} & \text{if } n = m \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin(nx)\cos(mx)dx = 0.$$

Note that

$$1, \sqrt{2}\cos(x), \sqrt{2}\sin(x), \sqrt{2}\cos(2x), \sqrt{2}\sin(2x), \ldots$$

orthonormal sequence on $[0, 2\pi]$.
Best finite approximation up to level n of a function f is

$$a_0 \cdot 1 + a_1\sqrt{2}\cos(x) + b_1\sqrt{2}\sin(x) + \ldots$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot 1dx$$

$$a_j = \frac{1}{2\pi} \int_0^{2\pi} f(x)\sqrt{2}\cos(jx)dx$$

$$b_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cdot \sqrt{2}\sin(kx)dx.$$

Trigonometric Series:

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos(nx) + B_n \sin(nx))$$

if in addition

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\cos(nx)dx$$

and

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(x)\sin(nx)dx$$

then called *Fourier Series*.

Simplification: We could have started with the orthonormal sequence

$$e^{inx}, n \in \mathbb{Z}, \text{ on } [0, 2\pi].$$

Then,

$$\begin{aligned}
\langle e^{inx}, e^{inx} \rangle &= \frac{1}{2\pi} \int_0^{2\pi} e^{inx} \overline{e^{imx}} dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} dx \\
&= \begin{cases} \frac{1}{2\pi} \left[\frac{1}{i(n-m)} e^{i(n-m)x} \right]_0^{2\pi} = 0 & \text{if } n \neq m \\ \\ \frac{1}{2} \int_0^{2\pi} 1 dx = 1 & \text{if } n = m. \end{cases}
\end{aligned}$$

Give best approximation up to level $n > 0$ of a function f

$$c_{-n} e^{i(-n)x} + \cdots + c_0 \cdot 1 + \cdots + c_n e^{inx}$$

with corresponding series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{e^{inx}} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx.$$

Note:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx$$

$$= \begin{cases} \frac{1}{2}(A_n - iB_n) & \text{if } n > 0 \\ \frac{1}{2} A_0 & \text{if } n = 0 \\ \frac{1}{2}\left(A_{|n|} + iB_{|n|}\right) & \text{if } n < 0. \end{cases} .$$

For $n > 0$, we have

$$c_n e^{inx} + c_{-n} e^{i(-n)x} = A_n \cos(nx) + B_n \sin(nx)$$

All the same terms as before!

Intervals:

e^{inx} works on any interval of length 2π.
$e^{2\pi inx/N}$ works on any interval of length L.

Summary:
Fourier Series: If f is integrable on $[a, b]$ of length of L then the n^{th} *Forier coefficient* of f is defined by

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) e^{-2\pi inx/L} dx, \quad n \in \mathbb{Z}$$

and its Forier series is given by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-2\pi inx/L}.$$

Question: Does $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-2\pi inx/L}$ converges to $f(x)$? ← July 6, 2018
Let's first look at the following example.

Example 3.1. Let $f(x) = x$ on $[0, 2n]$. Then,

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}dx = \frac{i}{n}, \quad n \neq 0.$$

and

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} xdx = \pi.$$

Fourier series

$$\sum_{n=-\infty}^{\infty} \frac{i}{n}e^{inx} = \pi - 2\left(\frac{1}{1}\sin(x) + \frac{1}{2}\sin(2x) + \dots\right)$$

$$\uparrow \frac{i}{n}e^{inx} + \frac{i}{-n}e^{-inx} = -\frac{2}{n}\sin(nx).$$

Yields π if $x = 0$ or $x = 2\pi$ while $f(0) = 0$ and $f(2\pi) = 2\pi$.

Consider: [insert graph 1 vs 2 here]:

Note: Fourier series is periodic on $\mathbb{R}$.

3.1 Domains

Consider $[a, b]$ on periodic functions on $\mathbb{R}$ in a circle such that $f(x) = F(e^{ix})$ and $x \in [0, 2\pi]$. Hence, functions on the circle are periodic functions $f(x)$ on $[0, 2\pi]$ with $f(0) = f(2\pi)$.

3.2 Uniqueness

Theorem. *Suppose f is integrable and bounded on an interval with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(x_0) = 0$ whenever f is continuous at x_0.*

3.3 Partial Sums

Let

$$S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n)e^{-2\pi inx/L} \text{ on } [0, L].$$

Question: In what sense $S_N f \to f$.

Theorem. *Suppose f is continous on the circle and the Fourier series of f is absolutely convergent*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Then

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

uniformly in x.

Proof. Since

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

converges absolutely and in fact uniformly.
If

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

then g must be continuous.
Since $f - g$ is continuous and $\widehat{f-g}(n) = 0$ for all $n \in \mathbb{Z}$ we conclude by uniqueness theorem that $f = g$.

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\lim_{N\to\infty}\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}\right) e^{-inx}dx \\
&= \lim_{N\to\infty}\sum_{k=-\infty}^{\infty} \hat{f}(k)\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)x}dx.
\end{aligned}$$

Note that:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)x}dx = \begin{cases} 1 & \text{if } k = n \\ 0 & else. \end{cases}$$

Hence, we have that

$$\begin{aligned}
\hat{g}(n) &= \lim_{N\to\infty}\sum_{k=-\infty}^{\infty} \hat{f}(k)\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-n)x}dx \\
&= \hat{f}(n).
\end{aligned}$$

∎

Theorem. *Let $f \in \mathcal{C}^2[0, 2\pi]$ and 2π period. Then*

$$\hat{f}(n) = \mathcal{O}\left(\frac{1}{n^2}\right) \quad \textit{as } |n| \to \infty$$

and thus the Fourier series of f converges absolutely and uniformly to f.

Note: $f(x) = \mathcal{O}(g(x))$ as $x \to a$ means there exists a constant C such that

$$|f(x)| \leq C|g(x)| \text{ as } x \to a.$$

Proof. Since f'' is continous on $[0, 2\pi]$, it is bounded. Say

$$|f''(x)| \leq B \text{ for all } x \in (0, 2\pi].$$

Then

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \\
&= \frac{1}{in}\widehat{f'}(n) \quad \text{by int. by parts.}
\end{aligned}$$

By iterating $\hat{f}(n) = \frac{1}{i^2n^2}\widehat{f''}(n)$,

$$\begin{aligned}
|\hat{f}(n)| &= \frac{1}{n^2}\left|\frac{1}{2\pi}\int_0^{2\pi} f''(x)e^{-inx}dx\right| \\
&\leq \frac{1}{n^2}\left(\frac{1}{2\pi}\int_0^{2\pi} Bdx\right) = \frac{B}{n^2}.
\end{aligned}$$

□

Corollary. $\widehat{f'}(n) = in\hat{f}(n)$.

3.4 Convolution

Given 2π periodic integrable functions f and g on $\mathbb{R}$. Define their *convolution* $f * g$ by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x - y)dy \qquad x \in [-\pi, \pi].$$

Remarks: It has many good properties by most importantly

- $f * g = g * f$.
- $\widehat{f * g} = \hat{f}(n)\hat{g}(n)$.

4 Kernels

← July 9, 2018

4.1 Direchlet Kernel

$$
\begin{aligned}
S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \\
&= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)e^{-iny}dy \right) e^{inx} \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{-in(x-y)}dy \right) e^{inx} \\
&= (f * D_N)(x)
\end{aligned}
$$

where $D_N(x) = \sum_{n=-N}^{N} e^{inx}$. Facts:

(1) $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x)dx = 1$.

(2) $D_n(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}$.

(3) $D_n(x) \leq C \min\{N, \frac{1}{|x|}\}$ if $x \in [-\pi, \pi]$, $N \geq 1$.

4.2 Good Kernels on the circle

Definition. A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if

(i) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)dx = 1$ for all $n \geq 1$.

(ii) There exists $M > 0$ such that $\int_{-\pi}^{\pi} |K_n(x)|dx \leq M$ for all $n \geq 1$.

(iii) For every $\delta > 0$ $\quad \int_{\delta \leq |x| \leq \pi} |K_n(x)|dx \to 0$ as $n \to \infty$.

Theorem. *Let $\{K_n\}$ be a family of good kernels and f a bounded and integrable function on the circle. Then*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere then the limit is uniform.

Note: Unfortunately

$$\int_{-\pi}^{\pi} |D_N(x)|dx \geq c\log(N)$$

as N is large. Then D_N is *not* a good kernel.

4.3 Fej'er Kernel

Say a series $\sum_{k=0}^{\infty} c_k$ is Cesáro summable to σ if

$$\lim_{N\to\infty} \sigma_N = \sigma$$

where

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}$$

is the N^{th} Cesáro mean.

Example 4.1. $\sum_{k=0}^{\infty}(-1)^k$ does not converge but Cesáro summable to $\frac{1}{2}$. For Fourier series the N^{th} Cesáro mean is

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\
&= \frac{(f * D_0)(x) + \ldots (f * D_{N-1})(x)}{N} \\
&= \left(f * \left(\frac{D_0 + \cdots + D_{N-1}}{N}\right)\right)(x) \\
&= (f * F_N)(x) \qquad \left(\text{since } F_N = \frac{D_0 + \cdots + D_{N=1}}{N} \to \text{Fej'er kernel}\right).
\end{aligned}$$

<u>Facts:</u>

(1) $F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}$.

(2) F_N is a good kernel.

Theorem (Weierstrass Approximation Theorem)**.** *For every continuous function $f : \mathbb{R} \to \mathbb{C}$ with period 2π and every $\epsilon > 0$ one can find a trigonometric polynomial P such that*

$$\forall x \qquad |f(x) - P(x)| < \epsilon.$$

Proof.

$$\begin{aligned}
\sigma_N(f)(x) &= \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N} \\
&= (f * F_N)(x).
\end{aligned}$$

is a trigonometric polynomial and $(f * F_N) \to f$ uniformly because f continuous and F_N is a good kernel. □

Mean Square Convergence of Fourier Series

Let f be a continuous function on the circle. Then

$$||f - S_N(f)||_2 \to 0 \text{ as } N \to \infty.$$

Proof. By the Weierstrass Approximation Theorem for a given $\epsilon > 0$ there exists a trigonometric polynomial P, say of degree M, such that

$$|f(x) - P(x)| < \epsilon \quad \text{ for all } x \in [0, 2\pi]$$

and thus

$$||f - P||_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x) - P(x)|^2 fx \right)^2 < \epsilon.$$

By the best approximation theorem

$$||f - S_N(f)||_2 < \epsilon$$

whenever $N \geq M$. $\square$

4.4 Parseval's identity

From general theory about orthonormal sequences Bessel's inequality:

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq ||f||_2^2.$$

Riemman-Lebesgue Lemma: $\hat{f}(n) \to 0$ as $|n| \to \infty$, and

$$\int_0^{2\pi} f(x) \sin(nx) dx \to 0,$$

$$\int_0^{2\pi} f(x) \cos(nx) dx \to 0.$$

Theorem (Parsval's Identity)**.**

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq ||f||_2^2$$

where we assume f is continuous on the circle.

Proof. From proof of best approximation theorem

$$||f||_2^2 = ||f - S_N(f)||_2^2 + \sum_{|n| \leq N} |\hat{f}(n)|^2.$$

$\square$

5 Fourier Series II

5.1 Pointwise Convergence of Fourier Series

← [

Theorem (Pointwise Convergence of Fourier Series)**.** *July 10, 2018] Let f be a square integrable function on the circle which is differentiable at x_0. Then*

$$S_N(f)(x_0) \to f(x_0) \text{ as } N \to \infty.$$

Proof.

$$\begin{aligned}
S_N(f)(x_0) - f(x_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0 - t)D_N(t)dt - f(x_0)\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x_0 - t) - f(x_0))\frac{\sin((N+\frac{1}{2})t)}{\sin(\frac{t}{2})}dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{\frac{f(x_0 - t) - f(x)}{t}\frac{t}{\sin(t/2)}\cos(t/2)}\sin(Nt)dt \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{\frac{f(x_0 - t) - f(x_0)}{t}\frac{t}{\sin(t/2)}\sin(t/2)}\cos(Nt)dt.
\end{aligned}$$

As $t \to 0$, the right side becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x_)\cdot 2\cdot 1\sin(Nt)dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x_0)\cdot 2\cdot 0\cos(Nt)dt$$

and away from $t \to 0$ then $\Box$ are square integrable like. Conclude by Riemman-Lebesgue lemma

$$S_N(f)(x_0) - f(x_0)| \to 0 \text{ as } N \to \infty.$$

$\Box$

Theorem. *Let f be a 'square' integrrable funcion on the circle such that $f'(x_0^+)$ and $f'(x_0^-)$ exist. Then*

$$S_n(f)(x_) \to \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Proof. Similar. $\Box$

Non-convergence

There exists an integrable function which is continuous at a point x_0 such that $S_N(f)(x_0)$ does not converge.

$$|f(x) - f(y)| \leq C|x - y|^{\infty}.$$

Gibbs Phenomenon

$$\sum_{k=1}^{N}\frac{1}{k}\sin(kx) = S_N(f)(x) \to f(x) = \frac{\pi}{2} - x \text{ on } (0, 2\pi).$$

For a fixed big N. (include graph).

5.2 Uniform Convergence of Fourier Series

Theorem (Uniform Convergence of Fourier Series)**.** *Let f be continuously differentiable on the circle. Then its Fourier series converges uniformly to f.*

Proof. Since f' is continuous it is square integrable and Parseval's identity holds. Further an old corollary says

$$\hat{f}'(n) = in\hat{f}(n).$$

and

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{n=-\infty}^{\infty} \frac{1}{n} |\widehat{f'}(n)| \le \underbrace{\left(\sum_{n=-\infty}^{\infty} \frac{1}{n^2}\right)^{1/2}}_{\infty} \underbrace{\left(\sum_{n=-\infty}^{\infty} |\widehat{f'}(n)|^2\right)^{1/2}}_{= ||f'||_2 < \infty.} < \infty$$

□

6 Fourier Transforms on $\mathbb{R}$

f on $[0,1]$ period 1, $n \in \mathbb{Z}$.

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx, \quad f(x) \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

f on $\mathbb{R}$ (no period), $\zeta \in \mathbb{R}$. We have

$$\hat{f}(\zeta) = \int_{-} \infty^{\infty} f(x) e^{-2\pi i \zeta x} dx, \quad f(x) \stackrel{?}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i \zeta x} d\zeta.$$

6.1 The Schwartz space

Definition. The Schwartz space on $\mathbb{R}$ denoted $\mathcal{S}(\mathbb{R})$, consists of all indefinitely differentiable functions f such that

$$\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty \qquad \text{for every } k, l \ge 0.$$

Note: If $f \in \mathcal{S}(\mathbb{R})$ then also

$$x^k f^{(l)}(x) \in \mathcal{S}(\mathbb{R}) \qquad \text{for all } k, l \ge 0.$$

Note: $\mathcal{S}(\mathbb{R})$ is a vector space.

Example 6.1. e^{-ax^2}, for $a > 0$ is in $\mathcal{S}(\mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$ define Fourier transform

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} f(\zeta) e^{-2\pi i \zeta x} d\zeta.$$

Useful facts:

(i)
$$f(x+h) \stackrel{\text{Fourier transf.}}{\rightarrow} e^{2\pi i h \zeta} \hat{f}(\zeta).$$

(ii)
$$f(x) e^{-2\pi i x h} \stackrel{\text{Fourier transf.}}{\rightarrow} \hat{f}(\zeta + h).$$

(iii)
$$f(\delta x) \stackrel{\text{Fourier transf.}}{\rightarrow} \delta^{-1} \hat{f}(\delta^{-1} \zeta), \delta > 0.$$

(iv)

$$f'(x) \overset{\text{Fourier transf.}}{\rightarrow} 2\pi i\zeta \hat{f}(\zeta).$$

(v)

$$-2\pi i x f(x) \overset{\text{Fourier transf.}}{\rightarrow} \frac{d}{d\zeta}\hat{f}(\zeta).$$

← July 11, 2018

Theorem. *If $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

Proof.

$$\begin{aligned}|\zeta^k \left(\frac{d}{d\zeta}\right)^l \hat{f}(\zeta)| &= |\left(\frac{1}{2\pi i}\right)^k \left(\frac{d}{dx}\right)^k ((-2\pi i x)^l f(x))(\zeta)| \\ &< \infty.\end{aligned}$$

□

Fact:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

Theorem. *If $f(x) = e^{-\pi x^2}$ then $\hat{f}(\zeta) = e^{-\pi\zeta^2}$.*

Proof. Define $F(\zeta) = \hat{f}(\zeta) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\zeta} dx$. Then $F(0) = 1$ by fact above, and

$$\begin{aligned}F'(\zeta) &= \int_{-\infty}^{\infty} \underbrace{e^{-\pi x^2}(-2\pi x)}_{f'(x)} i e^{-2\pi i x\zeta} dx \\ &= i\widehat{f'}(\zeta) \\ &= i(2\pi i\zeta)\hat{f}(\zeta) \\ &= -2\pi\zeta F(\zeta).\end{aligned}$$

Set $G(\zeta) = F(\zeta)e^{\pi\zeta^2}$. Then $G(0) = F(0) = 1$ and

$$G'(\zeta) = \underbrace{F'(\zeta)}_{-2\pi\zeta F(\zeta)} e^{\pi\zeta^2} + F(\zeta)2\pi\zeta e^{\pi\zeta^2} = 0.$$

So, $G(0) = 1$ and thus $F(\zeta) = e^{-\pi\zeta^2}$. □

Define for $\delta > 0$.

$$K_\delta(x) = \delta^{1/2} e^{-\pi x^2/\delta} \text{ then } \hat{K}_\delta(\zeta) = e^{\pi\delta\zeta^2}.$$

(insert picture of graph as δ is small).

Definition. If $f, g \in \mathcal{S}(\mathbb{R})$ their *convolution* is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Note: Similar properties as on circle for $f, g \in \mathcal{S}(\mathbb{R})$.

(i) $f * g \in \mathcal{S}(\mathbb{R})$.

(ii) $f * g = g * f$.

(iii) $\widehat{f * g} = \hat{f}\hat{g}$.

Theorem. *The collection* $\{K_\delta\}_{\delta>0}$ *is a family of good kernels as* $\delta \to 0^+$.

Theorem. *If* $f \in \mathbb{R}$ *then* $(f * K_\delta)(x) \to f(x)$ *uniformly in* x *as* $\delta \to 0^+$.

<u>Note:</u> Works for any good kernel.

Proposition. *If* $f, g \in \mathcal{S}(\mathbb{R})$ *then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)dy.$$

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\hat{g}(x)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(x)e^{-2\pi ixy}dydx \\ &= \int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dxdy \\ &= \int_{-\infty}^{\infty} g(y)\hat{f}(y)dy.\end{aligned}$$

∎

Theorem (Fourier Inversion)**.** *if* $f \in \mathcal{S}(\mathbb{R})$ *then* $f(x) = \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi ix\zeta}d\zeta$.

Proof. First note that

$$\begin{aligned}f(0) &= \lim_{\delta\to 0^+}(f * K_\delta)(0) \\ &= \lim_{\delta\to 0^+}\int_{-\infty}^{\infty} f(x)K_\delta(0-x)dx \\ &= \lim_{\delta\to 0^+}\int_{-\infty}^{\infty} f(x)\underbrace{\delta^{-1/2}e^{-\pi x^2/\delta}}_{K_\delta(x)}.\end{aligned}$$

Also we have that,

$$\begin{aligned}e^{-\pi x^2} &= \widehat{e^{-\pi\delta^2}}(x) \\ \Rightarrow \delta^{-1/2}e^{-\pi x^2/\delta} &= \underbrace{\delta^{-1/2}\widehat{e^{-\pi\delta^2}}(\delta^{-1/2}x)}_{\widehat{e^{-\pi\delta^2}}(x)}\end{aligned}$$

Then, continuing our first observation

$$
\begin{aligned}
f(0) &= \lim_{\delta\to 0^+}\int_{-\infty}^{\infty} f(x)\ \underbrace{\delta^{-1/2}e^{-\pi x^2/\delta}}_{K_\delta(x)} \\
&= \lim_{\delta\to 0^+}\int_{-\infty}^{\infty} \hat{f}(\zeta)e^{-\pi\delta\zeta^2}d\zeta \\
&\overset{\text{Prop.}}{=} \int_{-\infty}^{\infty} \hat{f}(\zeta)d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi i\delta\zeta}d\zeta.
\end{aligned}
$$

In genreal, let

$$F(y) = f(x+y).$$

Then

$$
\begin{aligned}
f(x) &= F(0) \\
&= \int_{-\infty}^{\infty} \hat{F}(\zeta)d\zeta \\
&= \int_{-\infty}^{\infty} \hat{f}(\zeta)e^{2\pi ix\zeta}d\zeta.
\end{aligned}
$$

$\square$

Corollary. *The Fourier transform is bijective mapping on* $\mathcal{S}(\mathbb{R})$

$$\mathcal{S}(\mathbb{R}) \overset{\mathcal{F}(f)(\zeta)=\int_{\infty}^{\infty} f(x)e^{-2\pi ix\zeta}dx}{\curvearrowright} \mathcal{S}(\mathbb{R}).$$

and

$$\mathcal{S}(\mathbb{R}) \overset{\mathcal{F}^{-1}(g)(\zeta)=\int_{\infty}^{\infty} g(\zeta)e^{-2\pi ixl}dl}{\curvearrowleft} \mathcal{S}(\mathbb{R}).$$

7 Inner Product and Norm on $\mathcal{S}(\mathbb{R})$

Can use the usual

$$\langle f, g\rangle = \int_{\infty}^{\infty} f(x)\overline{g(x)}dx$$

and

$$||f||_2 = \left(\int_{\infty}^{\infty} |f(x)|^2 dx\right)^{1/2}.$$

Theorem (Plancharel)**.** *If* $f \in \mathcal{S}(\mathbb{R})$ *then* $||f^2||_2 = ||f||_2$.

Proof.

$$
\begin{aligned}
\widehat{f^*}(\zeta) &= \int_{\infty}^{\infty} \overline{f(-x)}e^{-2\pi ix\zeta}dx \\
&= \overline{\int_{\infty}^{\infty} \overline{f(-x)}e^{-2\pi i(-x)\zeta}dx} \\
&= \overline{\hat{f}(\zeta)}.
\end{aligned}
$$

Also,

$$\begin{aligned}
\int_{\infty}^{\infty} |f(x)|^2 dx &= \int_{\infty}^{\infty} f(x)\overline{f(x)}dx \\
&= \int_{\infty}^{\infty} \overline{f(-(-x))}dx \\
&= \int_{\infty}^{\infty} f(x)f^{\#}(0-x)dx \\
&= (f * f^{\#})(0) \\
&\overset{\text{inversion}}{=} \int_{\infty}^{\infty} \widehat{f * f^{\#}}(\zeta)d\zeta \\
&= \int_{\infty}^{\infty} \hat{f}(\zeta)\widehat{f^{\#}}(\zeta)d\zeta \\
&= \int_{\infty}^{\infty} \hat{f}(\zeta)\overline{\hat{f}(\zeta)}(\zeta) \\
&= ||\hat{f}||_2^2.
\end{aligned}$$

Note: $\mathcal{F}$ is unitary on $\mathcal{S}(\mathbb{R})$ with $||\cdot||_2$.
Note:

- Parseval:
$$\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\right)^{1/2} = ||f||_2.$$

- Plancharel:
$$||\hat{f}||_2 = ||f||_2.$$

□

8 Fourier Transform on $\mathbb{R}^d$

← July 12, 2018

8.1 Rotations

A rotation in $\mathbb{R}^d$ is a linear transformation that fulfills one of the following equivalent conditions

(a) $Rx \cdot Ry = x \cdot y$ for all $x, y \in \mathbb{R}^d$.

(b) $|Rx| = |x|$ for all $x \in \mathbb{R}^d$.

(c) $R^t = R^{-1}$.

(d) $\det(R) = \pm 1$.

Useful:

$$\int_{R^d} f(Rx)dx = \int_{R^d} f(x)dx.$$

8.2 Coarea Formula / Polar coordinates

Write $\gamma = (\cos(\theta), \sin(\theta))$. For a function on g on S' define

$$\int_{S'} g(\gamma) d\sigma(\gamma) = \int_0^{2\pi} g(\cos(\theta), \sin(\theta)) d\theta.$$

Then can write

$$\begin{aligned}\int_{\mathbb{R}^2} f(x)dx &= \int_0^{2\pi} \int_0^{\infty} f(r\cos(\theta), r\sin(\theta)) r dr d\theta \\ &= \int_{S'} f(r\gamma) r dr d\sigma(\gamma).\end{aligned}$$

For g on S^2 define

$$\int_{S^2} g(\gamma) d\sigma(\gamma) = \int_0^{2\pi} \int_0^{\pi} g(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) \sin(\phi) d\phi d\theta.$$

Then.. finish here. In general can write

8.3 Schwartz space

$$\mathcal{S}(\mathbb{R}^d) = \{f \text{ indefinitely differentiable and } \sup_{x\in\mathbb{R}^d} |x^\alpha \left(\frac{\delta}{\delta x}\right)^\beta f(x)| < \infty\}.$$

Example 8.1. $e^{-a|x|^2}$ for $a > 0$ is in $\mathcal{S}(\mathbb{R}^d)$.

<u>Fourier Transform</u>

$$\hat{f}(\zeta) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \zeta} dx, \quad \zeta \in \mathbb{R}^d.$$

<u>Useful facts:</u>

(i)
$$f(x+h) \overset{\text{Fourier transf.}}{\to} \hat{f}(\zeta) e^{2\pi i h \zeta}.$$

(ii)
$$f(x) e^{-2\pi i x h} \overset{\text{Fourier transf.}}{\to} \hat{f}(\zeta + h).$$

(iii)
$$f(\delta x) \overset{\text{Fourier transf.}}{\to} \delta^{-d} \hat{f}(\delta^{-1}\zeta), \delta > 0.$$

(iv)
$$f'(x) \overset{\text{Fourier transf.}}{\to} 2\pi i \zeta \hat{f}(\zeta).$$

(v)
$$(-2\pi i x)^\alpha f(x) \overset{\text{Fourier transf.}}{\to} \left(\frac{d}{d\zeta}\right)^\alpha \hat{f}(\zeta).$$

(vi)

$$f(Rx) \overset{\text{Fourier transf.}}{\rightarrow} \hat{f}(\mathbb{R}\zeta), \quad R \text{ rotation.}$$

$$\begin{aligned}\int_{\mathbb{R}^d} f(Rx)e^{-2\pi ix\zeta}dx &= \int_{\mathbb{R}^d} f(x)e^{-2\pi R^{-1}x\zeta}dx \\ &= \int_{\mathbb{R}^d} f(x)e^{-2\pi ixR\zeta}dx \\ &= \hat{f}(R\zeta).\end{aligned}$$

Theorem. *If $f \in \mathcal{S}(\mathbb{R}^d)$ then $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$.*

8.4 Radical Functions

$f(x)$ is *radial* if there is a function f_0 defines on $\mathbb{R}_+$ such that

$$f(x) = f_0(|x|).$$

Note: f radial $\iff$ $f(Rx) = f(x)$ for all rotations R.

Fact: The Fourier transform of a radial function is radial. Because,

$$\begin{gathered}f(Rx) = f(x)\forall R \\ \Rightarrow \hat{f}(R\zeta) = \hat{f}(\zeta)\forall R.\end{gathered}$$

Example 8.2. $e^{-a|x|^2}$ is radial.

$$\widehat{e^{-\pi|x|^2}}(\zeta) = e^{-\pi|\zeta|^2}.$$

Theorem (Inverse & Plancharz)**.** *Suppose $f \in \mathcal{S}(\mathbb{R}^d)$. Then*

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\zeta)e^{2\pi ix\zeta}d\zeta$$

and

$$||\hat{f}||_2 = ||f||_2.$$

8.5 Fourier Transform of σ in $\mathbb{R}^3$.

$$\hat{\sigma}(\zeta) := \int_{S^2} e^{-2\pi i\zeta\gamma}d\sigma(\zeta)$$

Theorem.

$$\hat{\sigma}(\zeta) = \frac{2\sin(2\pi|\zeta|}{|\zeta|}.$$

Proof. First note LHS is radial in ζ.

$$\begin{aligned}\int_{S^2} e^{-2\pi iR\zeta\gamma}d\sigma(\gamma) &= \int_{S^2} e^{-2\pi i\zeta\cdot R^{-1}}\gamma d\sigma(\gamma) \\ &= \int_{S^2} e^{-2\pi i\zeta\cdot\gamma}d\sigma(\gamma). \qquad \boxed{\gamma^{\text{new}} = R^{-1}\gamma^{\text{old}}}\end{aligned}$$

So if $|\zeta| = \rho$ then enough to prove the identity with $\zeta = (0, 0, \rho)$.
Use spherical coordinates

$$\begin{aligned}\int_{S^2} e^{-2\pi i(0,0,\rho)\cdot\gamma} d\sigma(\gamma) &= \int_0^{2\pi}\int_0^{\pi} e^{-2\pi i\rho\cos(\phi)} \sin(\phi)d\phi d\theta \\ &= 2\pi\int_{-1}^{1} e^{2\pi i\rho u} du \qquad\qquad (u = \cos(\phi)) \\ &= 2\pi\frac{1}{2\pi i\rho}\left[e^{2\pi i\rho u}\right]_{-1}^{1} \\ &= \frac{2\sin(2\pi\rho)}{\rho} \\ &= \frac{2\sin(2\pi|\zeta|)}{|\zeta|}.\end{aligned}$$

$\square$

8.6 Stationary Phase

← July 13, 2018

For $\phi \in C^\infty$ and $a \in C_c^\infty$ define

$$I(\lambda) = \int_{\mathbb{R}^d} e^{-\pi i\lambda\phi(x)}a(x)dx.$$

Question: How does $I(\lambda)$ behave as $\lambda \to \infty$?
Idea: Much cancellation if λ large.
Warning: If $\phi(x) = c$ then

$$I(\lambda) = e^{-pii\lambda c}\int_{\mathbb{R}^d} a(x)dx$$

so no decay. We then need conditions on ϕ.

Definition. A map f is called a *diffeomorphism* if it is a bijection and it is differentiable and its inverse is differentiable as well. Say it is smooth if $f \in C^\infty,\ f^{-1} \in C^\infty$.

Note: Suppose $\phi_1 = \phi \circ G$ where G is smooth diffeomorphism. Then

$$\begin{aligned}\int e^{-\pi i\lambda\phi_2(x)}a(x)dx &= \int e^-\pi i\lambda_1(G^{-1}x)a(x)dx \\ &= \int e^-\pi i\lambda_1(y)\underbrace{a(Gy)|J_g(y)|}_{\text{smooth}}dy \qquad\qquad (y = G^{-1}x).\end{aligned}$$

So bounds independent of $a(x)$ will be smooth diffeomorphism invariant.

Two Cases: Neighborhoods of points p where

$$\underbrace{\nabla\phi(p) = 0}_{\text{stationary}} \quad \text{and} \quad \underbrace{\nabla\phi(p) \neq 0}_{\text{on-stationary}}.$$

Lemma (Straightening Lemma)**.** *Suppose $\omega \in \mathbb{R}^n$ is open, $f : \omega \to \mathbb{R}$ is C^∞, $p \in \omega$ and $\nabla f(p) \neq 0$. Then there are neighborhoods U and V of 0 and p respectively and a smooth diffeomorphism $G : U \to V$ with*

$$G(0) = p$$

and

$$(f \circ G)(x) = f(p) + x_n.$$

Theorem (Inverse Function Theorem)**.** *Let $A \subseteq \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}^n$ be of class C^1. Let $x_0 \in A$ and suppose $|Df(x_0)| \neq 0$. Then there is a neighborhood U of x_0 in A and an open neighborhood W of $f(x_0)$ such that $f(U) = W$ and f has a C^1 inverse $f^{-1} : W \to U$. More over $y \in W, x = f^{-1}(y)$ we have*

$$Df^{-1}(y) = [Df(x)]^{-1}.$$

If f is of call C^p, then so is f^{-1}.

<u>Note:</u> In $\mathbb{R}$: $(f^{-1})^{-1}(f(a)) = \frac{1}{f'(a)}$.

Theorem (Implicit Function Theorem)**.** *Let $A \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be an open set and let $f; A \to \mathbb{R}^m$ be a function of class C^p. Suppose $(x_0, y_0) \in A$ and $F(x_0, y_0) = 0$. Consider*

$$\nabla = \begin{vmatrix} \frac{\delta F_1}{\delta y_1} & \cdots & \frac{\delta F_1}{\delta y_m} \\ \vdots & \ddots & \vdots \\ \frac{\delta F_m}{\delta y_1} & \cdots & \frac{\delta F_m}{\delta y_m} \end{vmatrix}$$

evaluated at (x_0, y_0) and suppose $\nabla \neq 0$. Then there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of x_0 and a neighborhood V of y_0 in $\mathbb{R}^m$ and a unique function $f : U \to V$ such that $F(x, f(x)) = 0$ for all $x \in U$. Furthermore f is of class C^p.

Proof. Define $G : A \to \mathbb{R}^n \times \mathbb{R}^m$ by $G(x, y) = (x, F(x, y))$. Then G is of class C^p and

$$|DG(x_0, y_0)| = \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{\delta F_1}{\delta x_1} & \cdots & \cdots & \frac{\delta F_1}{\delta x_n} & \frac{\delta F_1}{\delta y_1} & \cdots & \frac{\delta F_1}{\delta y_m} \\ \vdots & \ddots & & & & \ddots & \\ \frac{\delta F_m}{\delta x_1} & \cdots & & \frac{\delta F_m}{\delta x_n} & \frac{\delta F_m}{\delta y_1} & \cdots & \frac{\delta F_m}{\delta y_m} \end{vmatrix} \neq 0$$

so by Inverse Function Theorem there is an open set W containing $(x_0, 0$ and S open set containing (x_0, y_0) such that $G(S) = W$ and G has C^p inverse $G^{-1} : W \to S'$. There exist open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ with $x_0 \in U$ and $y_0 \in V$ such that $U \times V \subseteq S$.
Let $G(U \times V) = Y$. Now G^{-1} is of the form

$$G^{-1}(x, w) = (x, H(x, w)).$$

Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ given by $\pi(x, y) = y$. Then

$$\begin{aligned} F(x, H(x, w)) &= \pi \circ G(x, H(x, w)) \\ &= \pi \circ G \circ G^{-1}(x, w) \\ &= W. \end{aligned}$$

Define $f : U \to V$ by $f(x) = H(x, 0)$. Then

$$F(x, f(x)) = 0$$

. Since H is of class C^p then so must f and since H unique then so must f. $\square$