Wave Propagation in Random Media

Lecture 3: Studying a random field

Dr. Johanan L. Codona

Steward Observatory, N408B
University of Arizona
jlcodona@gmail.com

Jan 22, 2015

Random Variables

- A random variable is an instance drawn from of a set of values.

$$q \in Q$$

  - To say a particular number is “random” *doesn't* make sense.
    - e.g. π is not a random number.
  - To say a particular number was chosen at random from a set *does* make sense.
    - e.g. The digits of π are random in some sense.

Probability Distributions

- Define valid set: e.g $x \in \mathbb{R}$.
- Probability $P(x)$, $x \in \mathbb{Q}$

$$\int_{\mathbb{Q}} P(x) \mathrm{d}x = 1$$

- Mean or Expectation of $f(x)$

$$\langle f(x) \rangle = \int f(x) P(x) \mathrm{d}x$$

Statistical Moments

$$\langle x^n \rangle = \int x^n P(x) \mathrm{d}x = M_n$$

- Mean:

$$\langle x \rangle = \int x P(x) \mathrm{d}x$$

- Variance:

$$\mathrm{var}(x) = \left\langle (x - \langle x \rangle)^2 \right\rangle = \int (x - \langle x \rangle)^2 P(x) \mathrm{d}x$$

$$\mathrm{var}(x) \equiv \left\langle x^2 \right\rangle - \langle x \rangle^2 = M_2 - M_1^2$$

- Standard Deviation:

$$\sigma = \sqrt{\mathrm{var}(x)} = \left\langle (x - \langle x \rangle)^2 \right\rangle^{1/2}$$

We will often call the standard deviation the "rms" (root-mean-square) or the "sigma" of some quantity.

Characteristic function

$$\left\langle e^{i\kappa x}\right\rangle = \int e^{i\kappa x} P(x)\mathrm{d}x = \widetilde{P}(\kappa)$$

We can derive moments from this by taking derivatives:

$$\left.\frac{\mathrm{d}^n}{\mathrm{d}\kappa^n}\left\langle e^{i\kappa x}\right\rangle\right|_{\kappa=0} = i^n \int x^n P(x)\mathrm{d}x = i^n \left\langle x^n\right\rangle$$

$$\left\langle x^n\right\rangle = i^{-n} \left.\frac{\mathrm{d}^n\widetilde{P}(\kappa)}{\mathrm{d}\kappa^n}\right|_{\kappa=0}$$

We can also express the probability distribution in terms of the moments using Taylor Series:

$$\widetilde{P}(\kappa) = \left\langle e^{i\kappa x}\right\rangle = \left\langle \sum_{n=0}^{\infty} \frac{(i\kappa x)^n}{n!}\right\rangle = \sum_{n=0}^{\infty} \frac{(i\kappa)^n}{n!}\left\langle x^n\right\rangle$$

Sums of Independent Random Variables

Say x and y are random variables with possibly different probability distributions $P_x(x)$ and $P_y(y)$.

- If a is a constant, what is the probability distribution P_z of $z = x + a$?

$$P_z(z) = P_x(z-a)$$

- What is the probability distribution of $z = x + y$?

$$P_z(z) = \int P_x(z-y)P_y(y)\mathrm{d}y = P_x * P_y.$$

$\Longrightarrow$ Therefore the characteristic functions multiply.

$$\widetilde{P}_z(\kappa) = \widetilde{P}_x(\kappa)\widetilde{P}_y(\kappa).$$

Gaussian (Normal) random variables

If you average N independent random variables together with the same $P(x)$

$$z = \frac{1}{N}\sum_{n=1}^{N} x_n$$

The characteristic function of z will stretch out around $\kappa = 0$ and being raised to the power N causes almost any starting probability distribution to end up looking like a Gaussian (since the Fourier transform of a Gaussian is a Gaussian).
This is also true for both real ($x \in \mathbb{R}$) and complex ($x \in \mathbb{C}$) random variables.
This is called the Central Limit Theorem.
$\Longrightarrow$ Adding enough unrelated things together tends toward a Gaussian distribution.

An important result

We will often assume things like phase have a Gaussian probability distribution.
Suppose we have a random phasor

$$\psi = \mathrm{e}^{i\phi}$$

where ϕ is a zero-mean Gaussian random variable.
$\therefore P_\phi(\phi)$ is a Gaussian and the average of ψ is

$$\left\langle \mathrm{e}^{i\phi} \right\rangle = \widetilde{P}_\phi(\kappa)\Big|_{\kappa=1} = \exp\left\{ -\frac{\left\langle \phi^2 \right\rangle}{2} \right\}.$$

You should do the algebra for this and prove it to yourself. We will use this result a lot.

Random Processes

A random process is an ordered set of random variables.

$$X = \{x_1, x_2, x_3, \ldots, x_N\} \equiv \{x_n\}$$

- The individual variables may or may not be independent.
  - In general, the variables are correlated.
- This can be studied by looking at statistical moments relating the various elements of X.
  - mean: $\langle X \rangle = \{\langle x_n \rangle\}$
  - Second moment: $\langle x_n x_m \rangle$
  - Third moment: $\langle x_n x_m x_l \rangle$
  - Fourth moment: $\langle x_a x_b x_c x_d \rangle$
  - etc.

Continuous Random Processes

- A random process can consist of a continuous set of random variables in multiple dimensions.

$$\psi(x,y,z,t)$$

- The individual values (or value at a location) of a random process can be any type of number, including real or complex.
- For complex random variables, the various moments can include conjugates on different factors.
  - These are both second moments for $(\psi_a, \psi_b) \in \mathbb{C}$: $\langle \psi_a \psi_b \rangle$ and $\left\langle \psi_a \psi_b^* \right\rangle$.
    They look similar, but behave very differently.

Balanced Moments

- In our complex field problems, the phase will often wander over many radians.
  - This will cause any moments that do not have equal numbers of conjugated and un-conjugated factors to have small values.
  - This will always happen for odd moments (i.e. become small when large phase wander is present).

A Random Electric Field

Consider a random electric field in (x, y, z).

- Start with a plane wave propagating in the $+\hat{z}$ direction.
- Even when light propagates for long distances through strong turbulence, the scattering angles are small.
- We will use the convention that the main propagation is in the z direction and the transverse direction is

$$\mathbf{x} = (x, y).$$

- A plane wave propagating along z is $\mathrm{e}^{i(\omega t - kz)}$, where $k = 2\pi/\lambda$ and $c = \omega/k$.
- We can write a more general scattered field as the main plane wave along z and a much slower-varying complex amplitude

$$\Psi(\mathbf{x}, z, t) = \psi(\mathbf{x}, z, t)\mathrm{e}^{i(\omega t - kz)}$$

We will study the complex amplitude $\psi(\mathbf{x}, z, t)$.

Studying the field in a plane

Let's examine the demodulated field in the $z = 0$ plane.

- The modulation doesn't affect the intensity, so we can pull it out and ignore it.
- We can write the complex amplitude as a phase and an amplitude:

$$\psi(\mathbf{x}, z = 0, t) = \alpha(\mathbf{x}, t)\mathrm{e}^{i\phi(\mathbf{x}, t)}$$

How can we study this field?

- Interfere the field with itself at 2 points using an interferometer.
- Look at the incoming light using a telescope.
- Measure the irradiance over the $\mathbf{x}$ plane using a camera.
- Look at the correlation of irradiance over the $\mathbf{x}$ plane.

Fluctuating Interferometer output

Consider an interferometer that adds the field from two points in the transverse plane and computes the resulting intensity:

$$
\begin{aligned}
J_{out}(\mathbf{x}_1,\mathbf{x}_2,t) &= |\psi(\mathbf{x}_1,t)+\psi(\mathbf{x}_2,t)|^2 \\
&= (\psi(\mathbf{x}_1,t)+\psi(\mathbf{x}_2,t))\,(\psi(\mathbf{x}_1,t)+\psi(\mathbf{x}_2,t))^* \\
&= |\psi(\mathbf{x}_1,t)|^2+|\psi(\mathbf{x}_2,t)|^2+\psi(\mathbf{x}_1,t)\psi^*(\mathbf{x}_2,t)+\psi(\mathbf{x}_1,t)^*\psi(\mathbf{x}_2,t) \\
&= I(\mathbf{x}_1,t)+I(\mathbf{x}_2,t)+2\Re\{\psi(\mathbf{x}_1,t)\psi^*(\mathbf{x}_2,t)\}\,.
\end{aligned}
$$

Average Interferometer Output

This output may change with time and we want to measure the average value.

- Use a time average

$$\langle J_{out}(\mathbf{x}_1,\mathbf{x}_2)\rangle = \frac{1}{T}\int_0^T J_{out}(\mathbf{x}_1,\mathbf{x}_2,t)\mathrm{d}t$$

- Or use an ensemble average. If the problem is *ergodic*, the time and ensemble averages are equal.

$$\langle J_{out}(\mathbf{x}_1,\mathbf{x}_2)\rangle = \langle I(\mathbf{x}_1)\rangle + \langle I(\mathbf{x}_2)\rangle + 2\Re\langle \psi(\mathbf{x}_1)\psi^*(\mathbf{x}_2)\rangle .$$

If the average irradiance does not vary across the observation plane

$$\langle J_{out}(\mathbf{x}_1,\mathbf{x}_2)\rangle = 2\langle I\rangle + 2\Re\langle \psi(\mathbf{x}_1)\psi^*(\mathbf{x}_2)\rangle .$$

Mutual Coherence Function

We can therefore use an interferometer to measure the second moment of the field, also known as the *Mutual Coherence Function* (MCF)

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2) = \langle \psi(\mathbf{x}_1) \psi^*(\mathbf{x}_2) \rangle$$

Looking again at the average output of the interferometer...

$$\langle J_{out}(\mathbf{x}_1, \mathbf{x}_2) \rangle = 2 \langle I \rangle + 2\Re \{ \Gamma(\mathbf{x}_1, \mathbf{x}_2) \} .$$

Introducing a $\pi/2$ phase shift in the interferometer gives the imaginary part, allowing us to measure the full complex MCF.

Average telescope image

We will use Fourier Optics to study the effect of imaging a random field.
The field in the image plane is the Fourier transform of the field seen through the pupil $\Pi(\mathbf{x})$

$$\Psi(k\boldsymbol{\theta},t) = \int \mathrm{d}^2 x \mathrm{e}^{ik\boldsymbol{\theta}\cdot\mathbf{x}} \Pi(\mathbf{x}) \psi(\mathbf{x},t)$$

The instantaneous intensity in the image plane is

$$\Phi(k\boldsymbol{\theta},t) = |\Psi(k\boldsymbol{\theta},t)|^2$$

$$\Phi(k\boldsymbol{\theta},t) = \int \mathrm{d}^2 x_1 \int \mathrm{d}^2 x_2 \, \mathrm{e}^{ik\boldsymbol{\theta}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \Pi(\mathbf{x}_1)\Pi^*(\mathbf{x}_2)\psi(\mathbf{x}_1,t)\psi^*(\mathbf{x}_2,t)$$

The average (long exposure) intensity is

$$\langle\Phi(k\boldsymbol{\theta})\rangle = \int \mathrm{d}^2 x_1 \int \mathrm{d}^2 x_2 \, \mathrm{e}^{ik\boldsymbol{\theta}\cdot(\mathbf{x}_1-\mathbf{x}_2)} \Pi(\mathbf{x}_1)\Pi^*(\mathbf{x}_2) \langle\psi(\mathbf{x}_1)\psi^*(\mathbf{x}_2)\rangle$$

Irradiance statistics

Fluctuating Irradiance

$$I(\mathbf{x},t) = |\psi(\mathbf{x},t)|^2$$

Mean Irradiance

$$\langle I(\mathbf{x},t)\rangle = \langle \psi(\mathbf{x},t)\psi^*(\mathbf{x},t)\rangle = \Gamma(\mathbf{x},\mathbf{x})$$

Variance of the irradiance

$$\mathrm{var}\,(I(\mathbf{x},t)) = \left\langle (I - \langle I(\mathbf{x},t)\rangle)^2 \right\rangle = \sigma^2$$

Scintillation Index

$$m^2 = \frac{\sigma^2}{\langle I\rangle^2}$$

Irradiance autocovariance

The fluctuating irradiance can be studied by correlating the fluctuations at two points. This is usually studied using the autocovariance of the irradiance, $\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2)$:

$$\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2) = \langle (I(\mathbf{x}_1) - \langle I(\mathbf{x}_1) \rangle) (I(\mathbf{x}_2) - \langle I(\mathbf{x}_2) \rangle) \rangle$$

$$\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2) = \langle I(\mathbf{x}_1) I(\mathbf{x}_2) \rangle - \langle I(\mathbf{x}_1) \rangle \langle I(\mathbf{x}_2) \rangle$$

If the mean irradiance is independent of position, $\langle I(\mathbf{x}_1) \rangle = \langle I \rangle$, and we can simply write

$$\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2) = C_I(\mathbf{x}_1, \mathbf{x}_2) - \langle I \rangle^2 .$$

Irradiance autocorrelation

This last expression relates the autocovariance to the irradiance autocorrelation

$$C_I(\mathbf{x}_1, \mathbf{x}_2) = \langle I(\mathbf{x}_1) I(\mathbf{x}_2) \rangle .$$

The irradiance is $I = \psi\psi^*$, so we can write

$$\langle I(\mathbf{x}) \rangle = \langle \psi(\mathbf{x}) \psi^*(\mathbf{x}) \rangle = \Gamma(\mathbf{x}, \mathbf{x})$$

and

$$C_I(\mathbf{x}_1, \mathbf{x}_2) = \langle \psi(\mathbf{x}_1) \psi^*(\mathbf{x}_1) \psi(\mathbf{x}_2) \psi^*(\mathbf{x}_2) \rangle .$$

This is a "Fourth Moment" of the random field, and is significantly more complicated than the second moment $\Gamma(\mathbf{x}_1, \mathbf{x}_2)$.

Fourth moments are important in the study of scintillation and things like the correlation of wavefront tilt, focal plane speckle statistics, etc.

Field Statistics

The lesson here is that we can study many aspects of the random field by measuring second and fourth moments of the field.

- Average PSFs and Seeing are described by second moments of the field.
- Scintillation and noise or variance estimates on irradiance-related measurements are often fourth moments of the field.