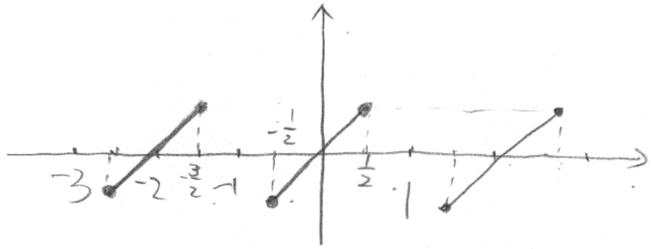
OPTI 330. HW4. Solution

1. $C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(x)\cdot e^{-j2\pi\xi_0 n x}\,dx$.

a. $f(x) = x \cdot rect(x)$, $T=2$, $x \in [-T/2, T/2]$, $\xi_0 = \frac{1}{T} = \frac{1}{2}$



so, $C_n = \frac{1}{2}\int_{-1}^{1} x\,rect(x)\, e^{-j2\pi\xi_0 n x}\,dx = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} x\, e^{-j2\pi\xi_0 n x}\,dx$.

$$= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} x \cdot \frac{1}{-j2\pi\xi_0 n}\, d e^{-j2\pi\xi_0 n x}.$$

$$= \frac{1}{2}\,\frac{1}{-j2\pi\xi_0 n}\left[x\cdot e^{-j2\pi\xi_0 n x}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi\xi_0 n x}\,dx\right]$$

$$= \frac{1}{-4j\pi\xi_0 n}\left[\frac{1}{2}e^{-j\pi\xi_0 n} + \frac{1}{2}e^{j\pi\xi_0 n} - \frac{e^{-j2\pi\xi_0 n x}}{-j2\pi\xi_0 n}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}\right]$$

$$= \frac{1}{-4j\pi\xi_0 n}\left[\frac{1}{2}\,2\cos(\pi\xi_0 n) + \frac{e^{-j2\pi\xi_0 n\frac{1}{2}}\, e^{+j2\pi\xi_0 n\frac{1}{2}}}{j2\pi\xi_0 n}\right]$$

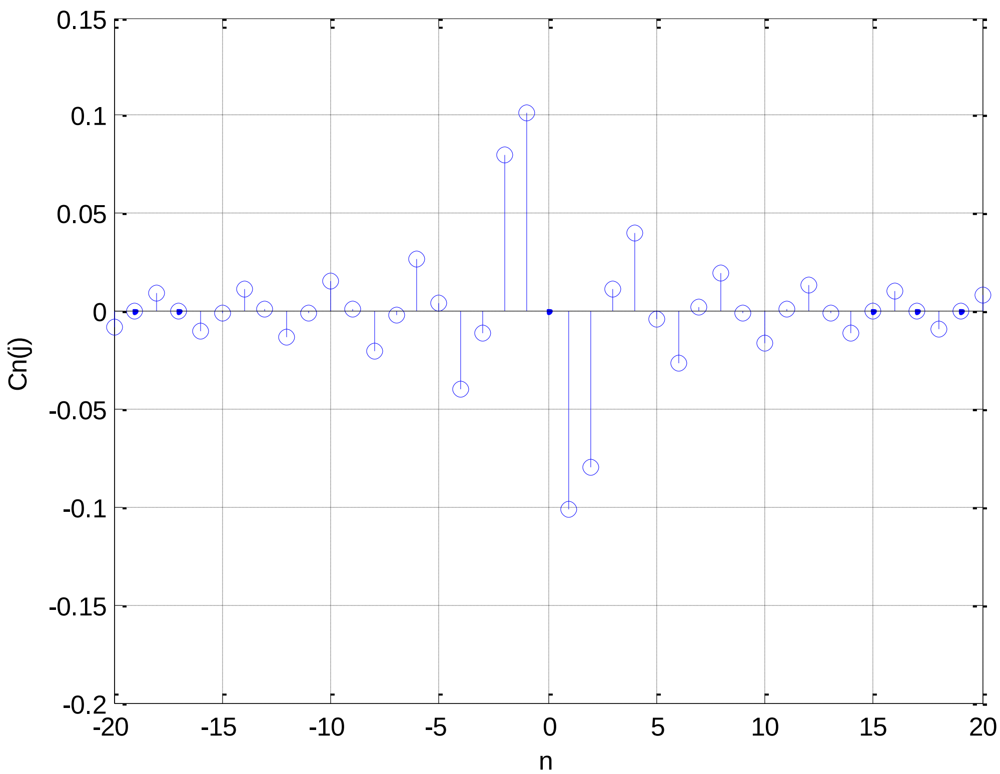
$$= \frac{1}{-4j\pi\xi_0 n}\left[\cos(\pi\xi_0 n) + \frac{-\cancel{2}\cancel{j}\sin(\pi\xi_0 n)}{\cancel{j}\cancel{2}\pi\xi_0 n}\right].$$

$$= \frac{1}{-4j\pi\xi_0 n}\left[\cos(\pi\xi_0 n) - \frac{\sin(\pi\xi_0 n)}{\pi\xi_0 n}\right]$$

$$= \frac{1}{-4j\pi\xi_0 n}\left[\cos(\pi\xi_0 n) - \mathrm{sinc}(\xi_0 n)\right]$$

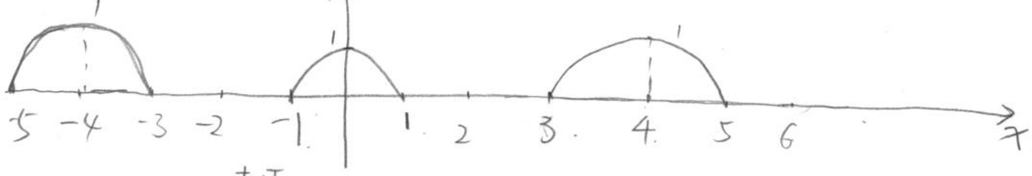
$$= \frac{1}{-j2\pi n}\left[\cos(\tfrac{\pi}{2}n) - \mathrm{sinc}(\tfrac{n}{2})\right].$$

$$= \frac{j}{2\pi n}\left[\cos(\tfrac{\pi}{2}n) - \mathrm{sinc}(\tfrac{n}{2})\right]$$



1(a)

b. $f(x) = (1-x^2) \times \mathrm{rect}(x/2)$, $T=4$, $\xi_0 = \frac{1}{T} = \frac{1}{4}$



$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(x)\, e^{-j2\pi\xi_0 n x}\, dx$$

$$= \frac{1}{4}\int_{-2}^{2} f(x)\, e^{-j2\pi\xi_0 n x}\, dx$$

$$= \frac{1}{4}\int_{-2}^{2} (1-x^2)\times \mathrm{rect}(x/2)\, e^{-j2\pi\xi_0 n x}\, dx$$

$$= \frac{1}{4}\int_{-1}^{1} (1-x^2)\, e^{-j2\pi\xi_0 n x}\, dx$$

$$= \frac{1}{4}\left[\int_{-1}^{1} e^{-j2\pi\xi_0 n x}\, dx - \int_{-1}^{1} x^2 e^{-j2\pi\xi_0 n x}\, dx\right]$$

$$= \frac{1}{4}\left[\frac{e^{-j2\pi\xi_0 n} - e^{j2\pi\xi_0 n}}{-j2\pi\xi_0 n} - \int_{-1}^{1} x^2 \frac{d e^{-j2\pi\xi_0 n x}}{-j2\pi\xi_0 n}\right]$$

$$= \frac{1}{4}\cdot\frac{1}{-j2\pi\xi_0 n}\left[-2j\sin(2\pi\xi_0 n) - \int_{-1}^{1} x^2\, d e^{-j2\pi\xi_0 n x}\right]$$

$$= \frac{1}{-j8\pi\xi_0 n}\left\{-2j\sin(2\pi\xi_0 n) - \left[x^2 e^{-j2\pi\xi_0 n x}\Big|_{-1}^{1} - \int_{-1}^{1} e^{-j2\pi\xi_0 n x}\, 2x\, dx\right]\right\}$$

$$= \frac{1}{-j8\pi\xi_0 n}\left\{-2j\sin(2\pi\xi_0 n) - \left(e^{-j2\pi\xi_0 n} - e^{j2\pi\xi_0 n}\right) + 2\int_{-1}^{1} e^{-j2\pi\xi_0 n x}\, x\, dx\right\}$$

$$= \frac{1}{-j8\pi\xi_0 n}\left\{-2j\sin(2\pi\xi_0 n) - (-2j\sin(2\pi\xi_0 n)) + 2\int_{-1}^{1} e^{-j2\pi\xi_0 n x}\, x\, dx\right\}$$

$$= \frac{2}{-j8\pi\xi_0 n}\cdot\int_{-1}^{1} e^{-j2\pi\xi_0 n x}\, x\, dx$$

$$= \frac{1}{-j4\pi\xi_0 n}\frac{1}{j2\pi\xi_0 n}\int_{-1}^{1} x \, d\, e^{-j2\pi\xi_0 n x}$$

$$= \frac{1}{-2\cdot(2\pi\xi_0 n)^2}\left(x\cdot e^{-j2\pi\xi_0 n x}\Big|_{-1}^{1} - \int_{-1}^{1} e^{-j2\pi\xi_0 n x}dx \right)$$

$$= \frac{1}{-2(2\pi\xi_0 n)^2}\cdot\left[e^{-j2\pi\xi_0 n} + e^{j2\pi\xi_0 n} - \frac{e^{-j2\pi\xi_0 n} - e^{j2\pi\xi_0 n}}{-j2\pi\xi_0 n} \right]$$

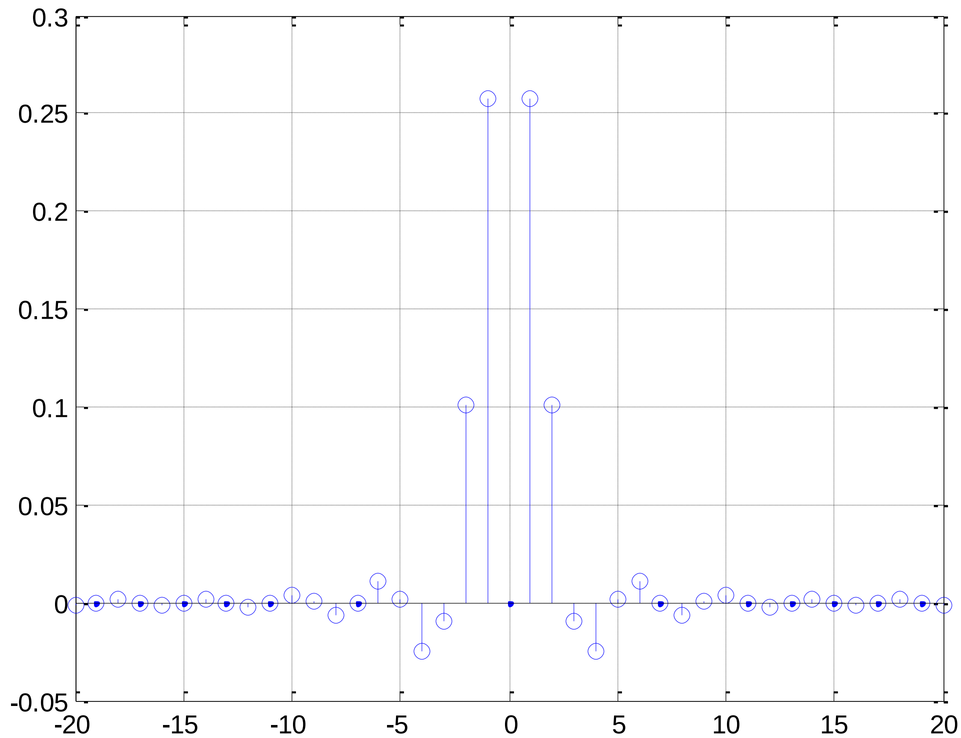
$$= \frac{1}{-2(2\pi\xi_0 n)^2}\cdot\left[2\cos(2\pi\xi_0 n) - \frac{-2j\sin(2\pi\xi_0 n)}{-j2\pi\xi_0 n} \right]$$

$$= \frac{1}{-2(2\pi\xi_0 n)^2}\cdot\left[2\cos(2\pi\xi_0 n) - 2\operatorname{sinc}(2\xi_0 n) \right]$$

$$= \frac{1}{-2(\frac{\pi n}{2})^2}\left[2\cos(\frac{\pi}{2}n) - 2\operatorname{sinc}(\frac{n}{2}) \right]$$

$$= \frac{-2}{\pi^2 n^2}\left[2\cos(\frac{\pi}{2}n) - 2\operatorname{sinc}(\frac{n}{2}) \right]$$

$$= \frac{4}{\pi^2 n^2}\left[\operatorname{sinc}(\frac{n}{2}) - \cos(\frac{\pi}{2}n) \right]$$



1(b)

2. Show that

a. $F(\xi) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\xi x}\, dx$.

$F^*(-\xi) = \int_{-\infty}^{\infty} f^*(x) \cdot e^{j2\pi(-\xi)x}\, dx$

because $f(x)$ is real that is $f^*(x) = f(x)$ so.

$F^*(-\xi) = \int_{-\infty}^{\infty} f^*(x) \cdot e^{-j2\pi\xi x}\, dx = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\xi x}\, dx = F(\xi)$

b.. $g(x) = f(x) \cdot \sin(2\pi\xi_0 x)$

$\mathcal{F}\{g(x)\} = \mathcal{F}\{f(x)\} \otimes \mathcal{F}\{\sin(2\pi\xi_0 x)\}$

$= F(\xi) \otimes \frac{1}{2j}[\delta(\xi-\xi_0) - \delta(\xi+\xi_0)]$

$= \frac{1}{2j} F(\xi) \otimes \delta(\xi-\xi_0) - \frac{1}{2j} F(\xi) \otimes \delta(\xi+\xi_0)$

$= \frac{1}{2j} F(\xi-\xi_0) - \frac{1}{2j} F(\xi+\xi_0)$

$= \frac{1}{2j}[F(\xi-\xi_0) - F(\xi_0+\xi_0)]$

3. Compute and Sketch

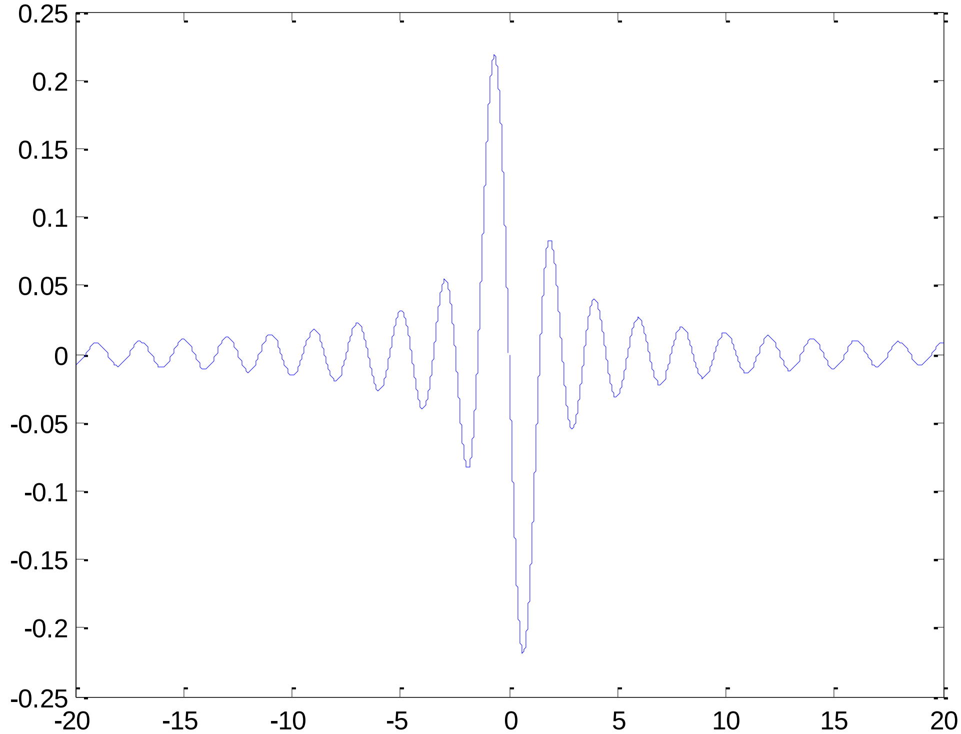
a. $f(x) = x \cdot \mathrm{rect}(x)$

$\mathcal{F}\{f(x)\} = \mathcal{F}\{x \cdot \mathrm{rect}(x)\}$ because $(-j2\pi x)^k f(x) \xrightarrow{\mathcal{F}} F^{(k)}(\xi)$

so $\mathcal{F}\{-j2\pi x \cdot \mathrm{rect}(x)\} = \mathrm{sinc}'(\xi) = \left(\frac{\sin\pi\xi}{\pi\xi}\right)' = \frac{\pi\cos\pi\xi \cdot \xi - \sin\pi\xi}{\pi \cdot \xi^2}$

$= \frac{\pi\xi\cos\pi\xi - \sin\pi\xi}{\pi\xi^2}$

so $\mathcal{F}\{f(x)\} = \frac{j}{2\pi} \cdot \frac{\pi\xi\cos\pi\xi - \sin\pi\xi}{\pi\xi^2} = \frac{j}{2\pi\xi}[\cos\pi\xi - \mathrm{sinc}(\xi)]$



3(a)

(b). $f(x) = (1-x^2) \cdot \mathrm{rect}(x/2)$.

$= \mathrm{rect}(x/2) - x^2 \mathrm{rect}(x/2)$

$\mathrm{rect}(x/2) \xrightarrow{\mathcal{F}} 2\,\mathrm{sinc}(2\xi)$

$(-j2\pi x)^2 \mathrm{rect}(x/2) \xrightarrow{\mathcal{F}} [2\mathrm{sinc}(2\xi)]^{(2)}$

$2\,\mathrm{sinc}^{(2)}(2\xi) = \left[2 \cdot \frac{\sin(\pi 2\xi)}{\pi 2\xi}\right]^{(2)} = \frac{1}{\pi}\left[\frac{\sin(2\pi\xi)}{\xi}\right]^{(2)}$

$= \frac{1}{\pi} \cdot \left[\frac{2\pi\xi\cos(2\pi\xi) - \sin(2\pi\xi)}{\xi^2}\right]'$

$= \frac{1}{\pi} \cdot \left\{\frac{\xi^2[2\pi\cos(2\pi\xi) - 4\pi^2\xi\sin(2\pi\xi) - 2\pi\cos(2\pi\xi)] - 2\xi[2\pi\xi\cos(2\pi\xi) - \sin(2\pi\xi)]}{\xi^4}\right\}$

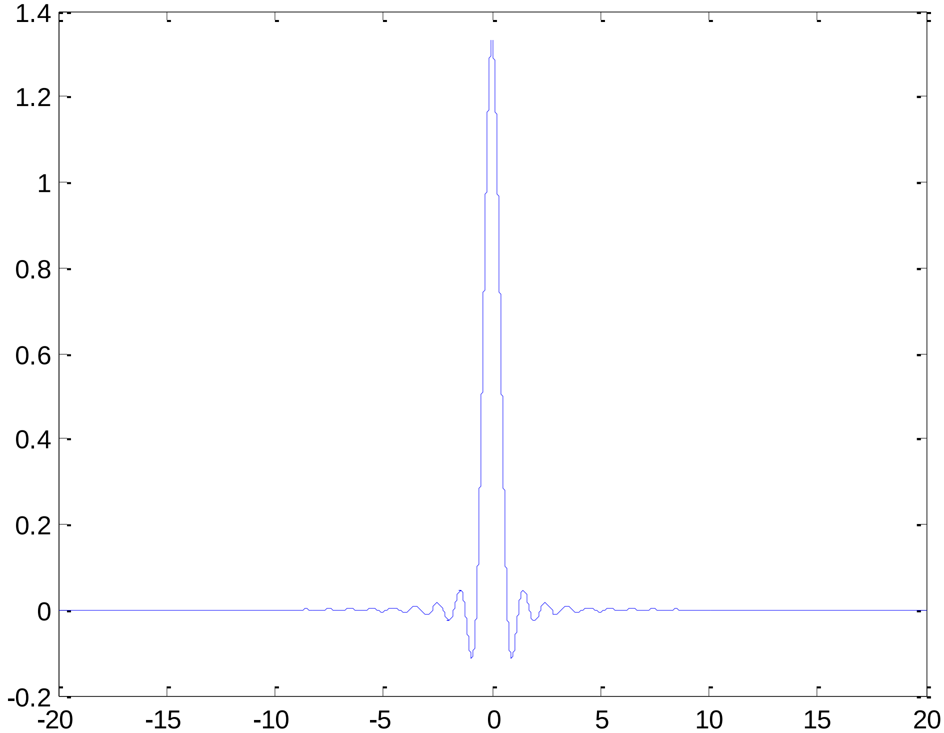
$= \frac{1}{\pi}\, \frac{-\xi^3 4\pi^2 \sin(2\pi\xi) - 4\pi\xi^2\cos(2\pi\xi) + 2\xi\sin(2\pi\xi)}{\xi^4}$

so $x^2 \mathrm{rect}(x/2) \xrightarrow{\mathcal{F}} \frac{-1}{4\pi^3}\left[\frac{2\sin(2\pi\xi)}{\xi^3} - \frac{4\pi^2\sin(2\pi\xi)}{\xi} - \frac{4\pi\cos(2\pi\xi)}{\xi^2}\right]$

so. $x^2 \mathrm{rect}(x/2) \xrightarrow{\mathcal{F}} \frac{-1}{4\pi^2\xi^2}\left[\frac{2\sin(2\pi\xi)}{2\pi\xi} \cdot 2 - 4 \cdot \cos(2\pi\xi)\right] + 2\,\mathrm{sinc}(2\xi)$

$F(\xi) = 2\,\mathrm{sinc}(2\xi) - 2\,\mathrm{sinc}(2\xi) + \frac{1}{4\pi^2\xi^2}[4\,\mathrm{sinc}(2\xi) - 4\cos(2\pi\xi)]$

$= \frac{1}{\pi^2\xi^2}[\mathrm{sinc}(2\xi) - \cos(2\pi\xi)]$.



3(b)

4. Compute and sketch the Fourier Transform

a. $f(x) = \mathrm{rect}(x - x_0)$

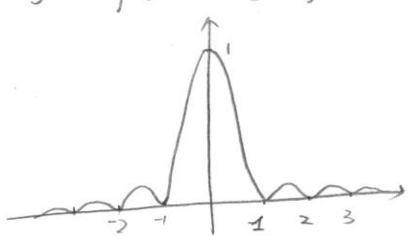
$$F(\xi) = e^{-j2\pi\xi x_0}\,\mathrm{sinc}(\xi)$$

b. $f(x) = \mathrm{sinc}(bx - x_0) = \mathrm{sinc}(b(x - \frac{x_0}{b}))$

$$F(\xi) = e^{-j2\pi\frac{x_0}{b}\xi}\,\frac{1}{|b|}\,\mathrm{rect}\left(\frac{\xi}{b}\right)$$

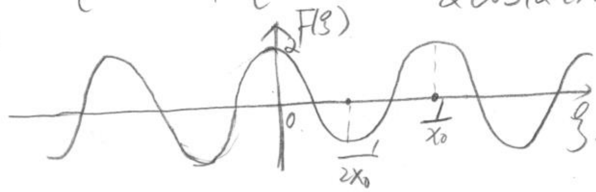
c. $f(x) = \mathrm{rect}(x) \otimes \mathrm{rect}(x)$

$$F(\xi) = \mathcal{F}\{f(x)\} = \mathcal{F}\{\mathrm{rect}(x)\}\cdot\mathcal{F}\{\mathrm{rect}(x)\} = \mathrm{sinc}(\xi)\cdot\mathrm{sinc}(\xi) = \frac{\sin^2(\pi\xi)}{\pi^2\xi^2}$$



d. $f(x) = \delta(x - x_0) + \delta(x + x_0)$

$$F(\xi) = \mathcal{F}\{\delta(x - x_0)\} + \mathcal{F}\{\delta(x + x_0)\}$$
$$= e^{-j2\pi x_0\xi} + e^{j2\pi x_0\xi} = 2\cos(2\pi x_0\xi)$$



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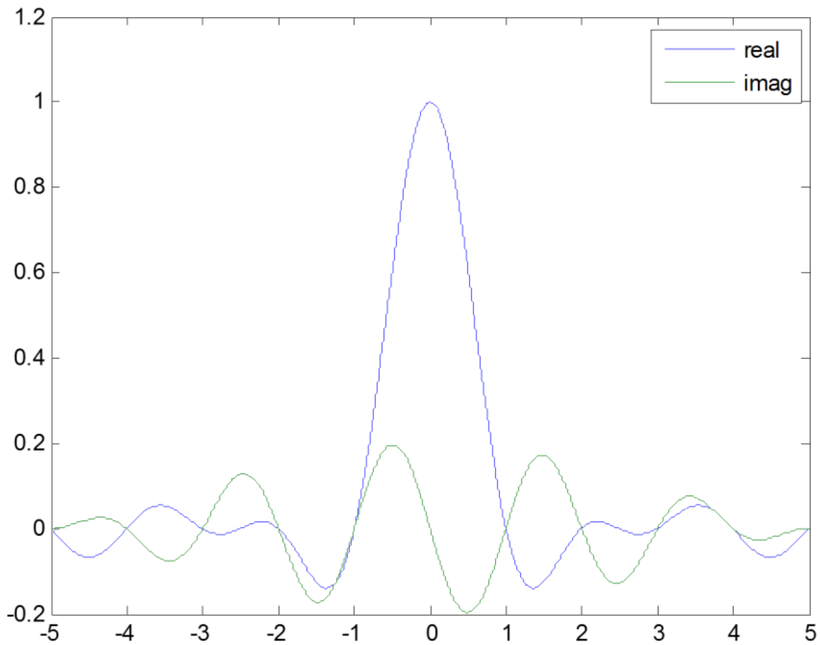
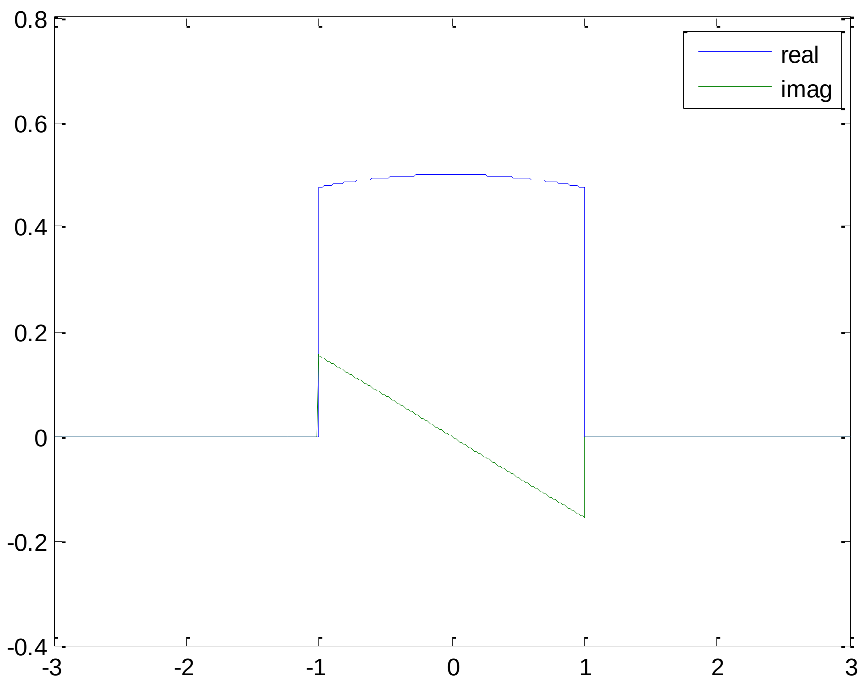

real
imag
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-5
-4
-3
-2
-1
0
1
2
3
4
5

4(a)


real
imag
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-3
-2
-1
0
1
2
3

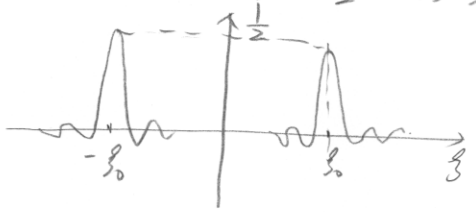
4(b)

e. $f(x) = \text{rect}(x) \cdot \cos(2\pi\xi_0 x)$

$F(\xi) = \mathcal{F}\{\text{rect}(x)\} \otimes \mathcal{F}\{\cos(2\pi\xi_0 x)\}$

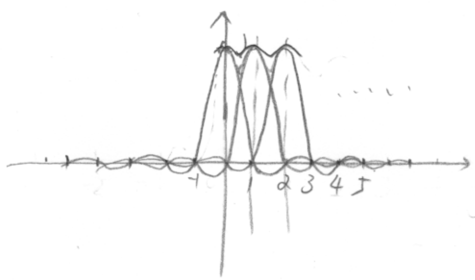
$= \text{sinc}(\xi) \otimes \frac{1}{2}[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)]$

$= \frac{1}{2}\text{sinc}(\xi-\xi_0) + \frac{1}{2}\text{sinc}(\xi+\xi_0)$



f. $f(x) = \text{rect}(x) \cdot \text{comb}(x)$

$F(\xi) = \text{sinc}(\xi) \otimes \text{comb}(\xi)$



5. $f(x) = \left| e(x) + \sin(2\pi\xi_0 x) \right|^2$

$= [e(x) + \sin(2\pi\xi_0 x)] \cdot [e(x) + \sin(2\pi\xi_0 x)]^*$

$= [e(x) + \sin(2\pi\xi_0 x)] \cdot [e^*(x) + \sin(2\pi\xi_0 x)]$

$= e(x) \cdot e^*(x) + e^*(x) \cdot \sin(2\pi\xi_0 x) + e(x) \cdot \sin(2\pi\xi_0 x) + \sin^2(2\pi\xi_0 x)$

assume $\mathcal{F}\{e(x)\} = E(\xi)$

so $\mathcal{F}\{e^*(x)\} = E^*(-\xi)$

$$\mathcal{F}\{e(x)\cdot e^*(x)\} = E(\xi)\otimes E^*(-\xi)$$

$$\mathcal{F}\{e^*(x)\cdot \sin(2\pi\xi_0 x)\} = \frac{1}{2j}\left[E^*(-\xi+\xi_0) - E^*(-\xi-\xi_0)\right]$$

$$\mathcal{F}\{e(x)\sin(2\pi\xi_0 x)\} = \frac{1}{2j}\left[E(\xi-\xi_0) - E(\xi+\xi_0)\right]$$

$$\mathcal{F}\{\sin^2(2\pi\xi_0 x)\} = \mathcal{F}\left\{\frac{1}{2}\left[2\sin^2(2\pi\xi_0 x) - 1\right] + \frac{1}{2}\right\}$$

$$= -\frac{1}{2}\mathcal{F}\{1-2\sin^2(2\pi\xi_0 x)\} + \frac{1}{2}\delta(\xi)$$

$$= -\frac{1}{2}\mathcal{F}\{\cos(4\pi\xi_0 x)\} + \frac{1}{2}\delta(\xi)$$

$$= -\frac{1}{2}\left[\frac{1}{2}\delta(\xi-2\xi_0) + \frac{1}{2}\delta(\xi+2\xi_0)\right] + \frac{1}{2}\delta(\xi)$$

$$= \frac{1}{2}\delta(\xi) - \frac{1}{4}\delta(\xi-2\xi_0) - \frac{1}{4}\delta(\xi+2\xi_0)$$

assume $e(x)$ is real

so $e(x) = e^*(x)$, $E(\xi) = E^*(-\xi)$