Special Functions

Throughout this course we will be working with scaled and shifted versions of functions. The general notation for a scaled and shifted function will be as follows

$$f(x) \longrightarrow f\left(\frac{x-x_0}{b}\right)$$

where constant x_0 represents the shift and constant b represents the scale.

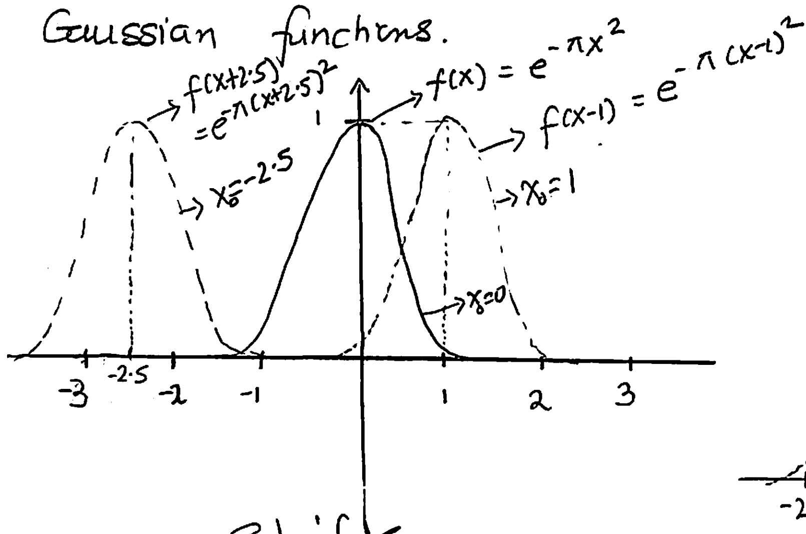
Let us consider an example function, the Gaussian function, to make these concepts more concrete.

$$f(x) = e^{-\pi x^2}$$

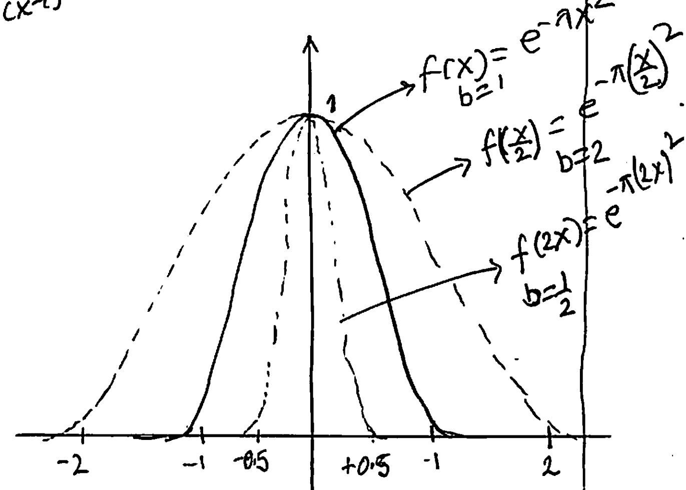
Shift: $$f(x-x_0) = e^{-\pi (x-x_0)^2}$$

Scale: $$f(x/b) = e^{-\pi \left(\frac{x}{b}\right)^2}$$

The figure below show example shifts and scaled Gaussian functions.



Shifts



Scaled

Note that for positive values of x_0, the function shifts to the right, while for negative values of x_0, the shift is towards the left of origin.

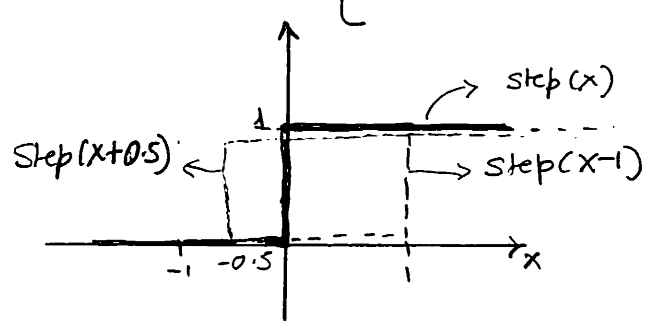
Similarly, for $|b| > 1$ the function expands, and for $|b| < 1$ value it ~~shrinks~~ compress as seen in the previous example.

Now, we will consider a number of special functions that will prove to be very useful in representing physical structures, such as an aperture or other physical quantities and/or processes.

Unit Step Function

This function is (usually) used to turn other function 'on' or 'off'. It is defined as:

$$\mathrm{Step}(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ 1 & x>0 \end{cases}$$



The step function can be used to make another function single sided. For example: $\mathrm{Step}(x)\cdot\cos(2\pi x)$ is only non-zero for $x \geq 0$, as shown in the next figure.

Function Transformations

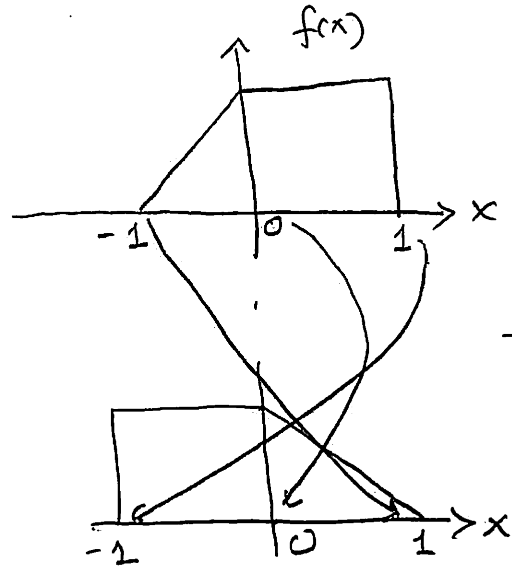
* **Flip**

$$f(x) \longrightarrow f(-x)$$

argument sign change

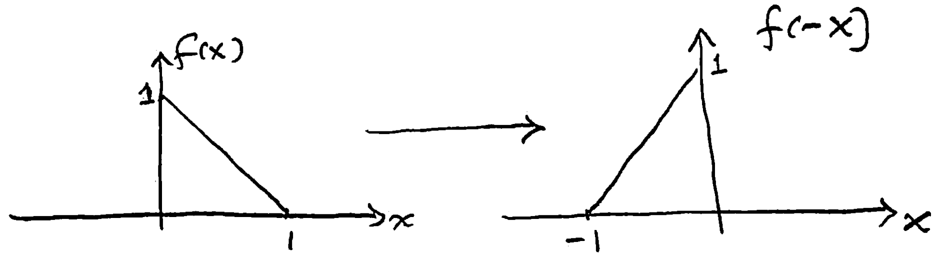
Transform	→	original
$x=-1$	→	$x=1$
$x=0$	→	$x=0$
$x=1$	→	$x=-1$

Ex:

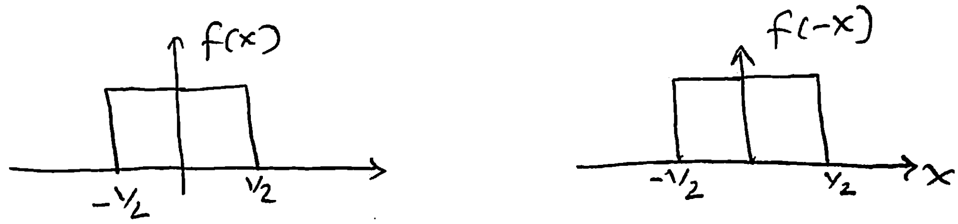


→ reflect around y-axis

Ex:



Ex:



Functions symmetric about origin do not change with $f(x) \rightarrow f(-x)$ transformation.

Note:

$f(x) \longrightarrow f(-x)$ → flip around y-axis

! Not same !

$f(x) \longrightarrow -f(x)$ → flip around x-axis

* Scaling

$$f(x) \longrightarrow f(bx)$$

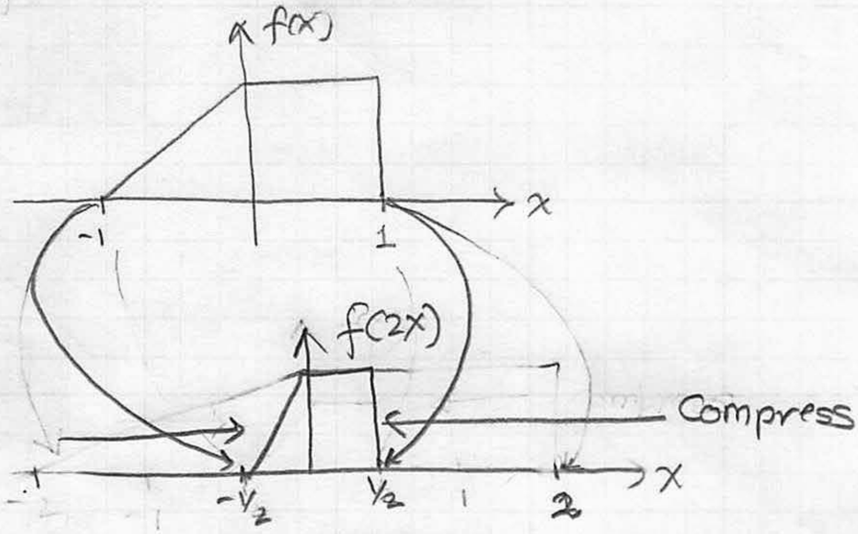
$|b| > 1 \longrightarrow$ compress

$|b| \leq 1 \longrightarrow$ expand

Ex:

$$f(x) \longrightarrow f(2x)$$

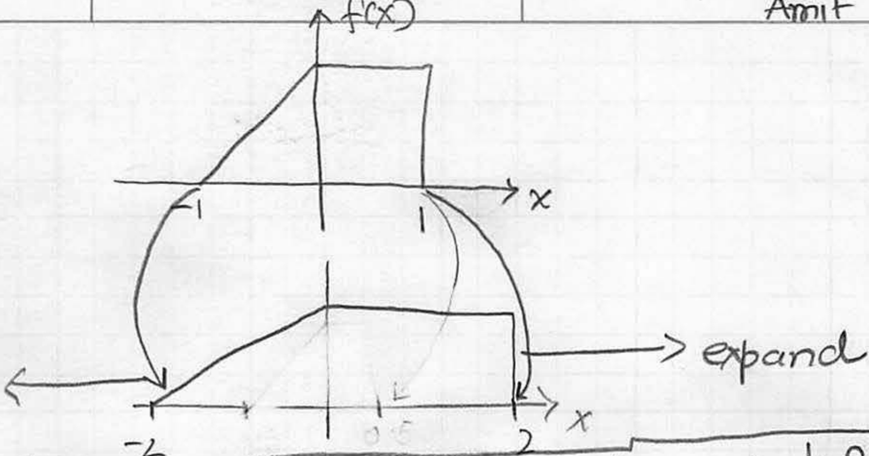
Transform		Original
$x = -0.5$	$\longrightarrow$	$x = -1$
$x = 0$	$\longrightarrow$	$x = 0$
$x = +0.5$	$\longrightarrow$	$x = 1$



Ex:

$$f(x) \rightarrow f(.5x) = f(x/2)$$

Transform		Original
$x = -2$	$\longrightarrow$	$x = -1$
$x = 0$	$\longrightarrow$	$x = 0$
$x = +2$	$\longrightarrow$	$x = +1$



$$\boxed{f(x) \leftrightarrow f(bx) \neq f(x) \leftrightarrow b f(x)}$$

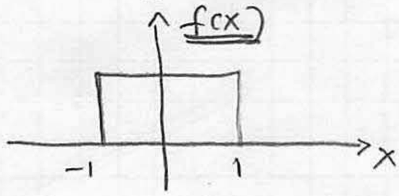
* Shifting

$$f(x) \longrightarrow f(x - x_0)$$

$x_0 > 0 \longrightarrow$ shift to right

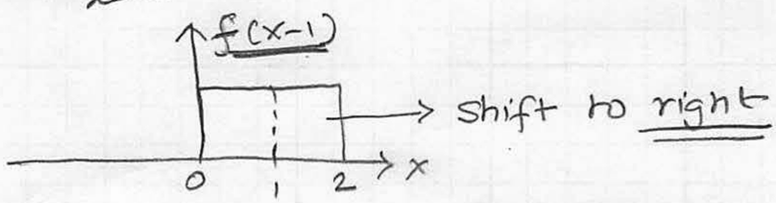
$x_0 < 0 \longrightarrow$ shift to left

Ex:



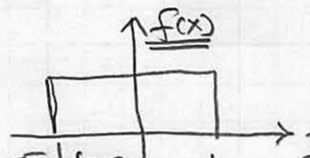
$$f(x) \longrightarrow f(x-1)$$

Transform		Original
$x = 0$	$\longrightarrow$	$x = -1$
$x = 1$	$\longrightarrow$	$x = 0$
$x = 2$	$\longrightarrow$	$x = -1$

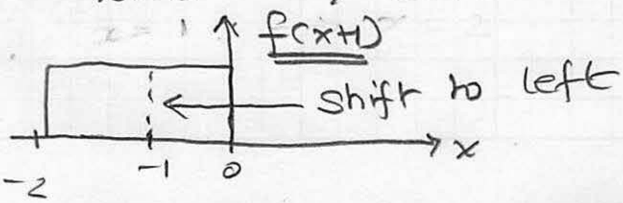


Ex:

$$f(x) \rightarrow f(x+1)$$



Transform		Original
$x = -2$	$\longrightarrow$	$x = -1$
$x = -1$	$\longrightarrow$	$x = 0$
$x = 0$	$\longrightarrow$	$x = 1$



* Shifting and Scaling

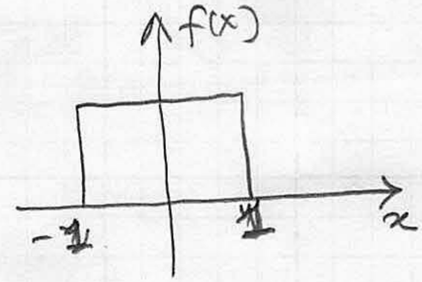
$$f(x) \longrightarrow f\left(b \cdot (x - x_0)\right)$$

↳ shift

then

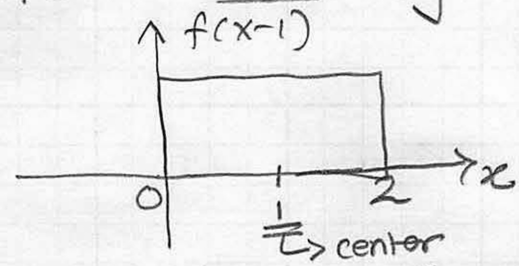
↳ scale (around x_0)

Ex:

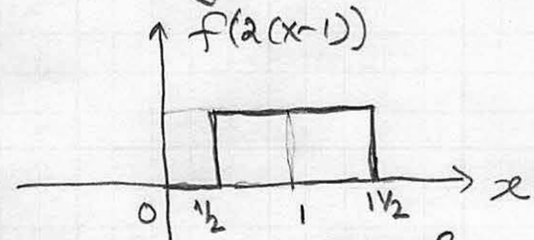


$$f(x) \longrightarrow f\left(2(x-1)\right)$$

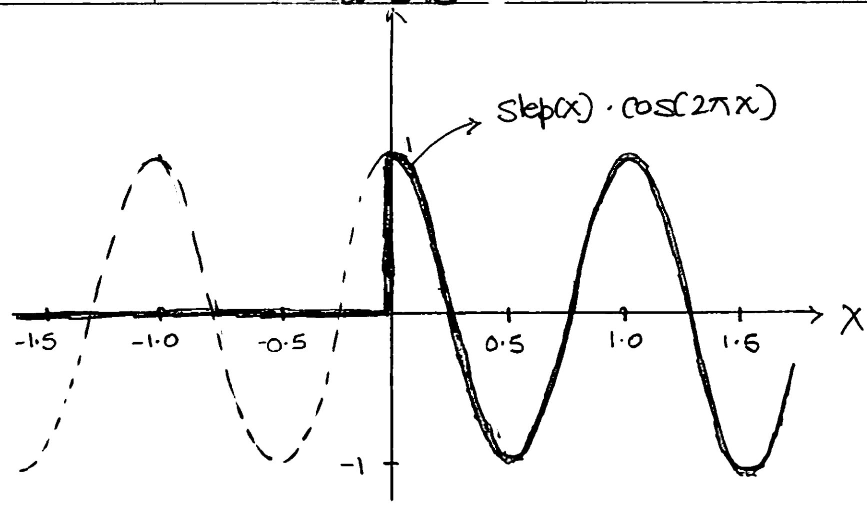
Shift to right by 1



Scale by 2 (compress)



Transform		Original
$x = 0$	$\longrightarrow$	$x = -2$
$x = 0.5$	$\longrightarrow$	$x = -1$
$x = 1$	$\longrightarrow$	$x = 0$
$x = 1.5$	$\longrightarrow$	$x = 1$



Note that we can make a function left-sided by using step(−x).

Note: The step function is discontinuous:

$$\lim_{x\to 0^+} \text{step}(x) \neq \lim_{x\to 0^-} \text{step}(x) \neq \text{step}(0) = \frac{1}{2}$$

Such a function is physically not realizable, however, it is still a very useful mathematical tool in practice.

Signum Function

The step function can be defined in terms of the Signum function, denoted by sgn(x),

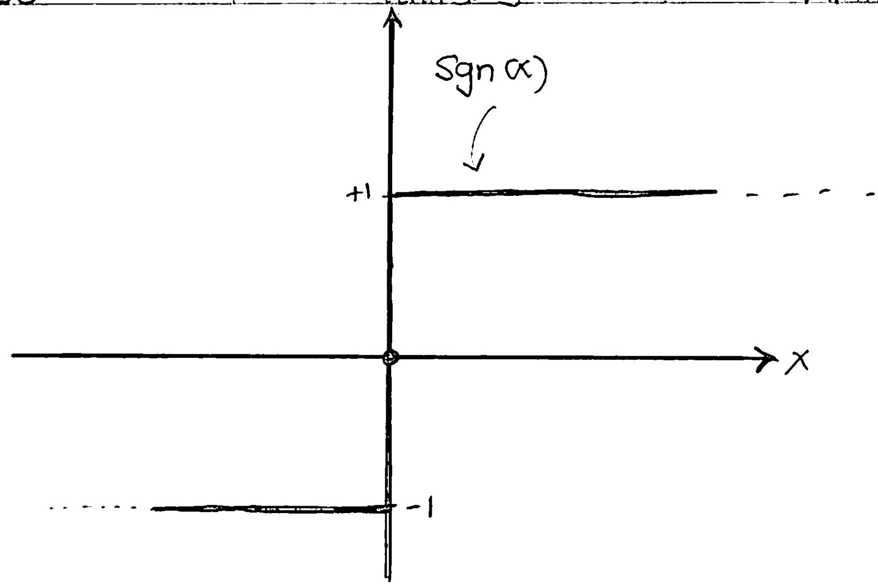
$$\text{Step}(x) = \frac{1}{2}(1 + \text{sgn}(x))$$

This implies the following defn of sgn(x):

$$\text{sgn}(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$$

Note that the signum function, sgn(x), is also discontinuous.

$$\lim_{x\to 0^+} \text{sgn}(x) \neq \lim_{x\to 0^-} \text{sgn}(x)$$



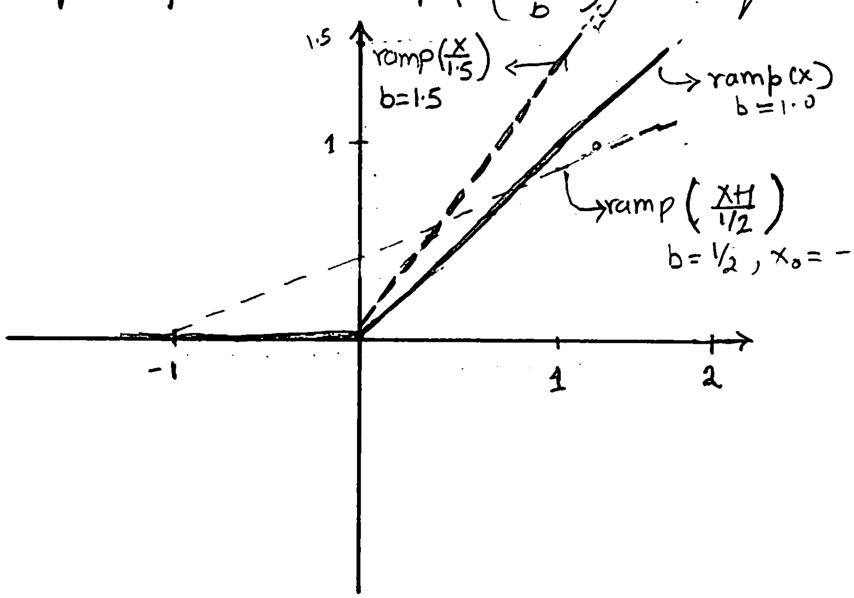
Basically, sgn(x) yields the sign of the argument. This function will be important when we define the Fourier Transform of step function later in the semester.

Ramp Function

The ramp function is defined as the integral of the step function as.

$$\text{ramp}(x) = \int_{-\infty}^{x} \text{step}(x)\, dx = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$$

Note the slope of the ramp$\left(\left(\frac{x-x_0}{b}\right)\right)$ is equal to $1/b$.



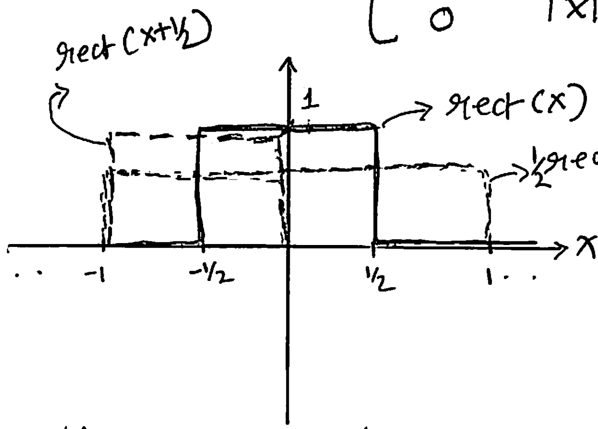
Rectangular Function

The rectangular function is used often to approximate a number of physical processes in physical optics such as slits and rectangular apertures. The rectangular function can be defined in terms of a unit step function as

$$\text{rect}(x) = \text{step}(x+\tfrac{1}{2}) - \text{step}(x-\tfrac{1}{2})$$

or equivalently

$$\text{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = 1/2 \\ 0 & |x| > 2 \end{cases}$$



Let us consider the area under the rectangular function. We define

$$\text{Area} \equiv A = \int_{-\infty}^{\infty} \text{rect}\left(\frac{x-x_0}{b}\right) dx = \int_{-|b|/2+x_0}^{+|b|/2+x_0} 1 \cdot dx$$

$$= \frac{|b|}{2} + x_0 - \left(-\frac{|b|}{2} + x_0\right)$$

$$= |b|$$

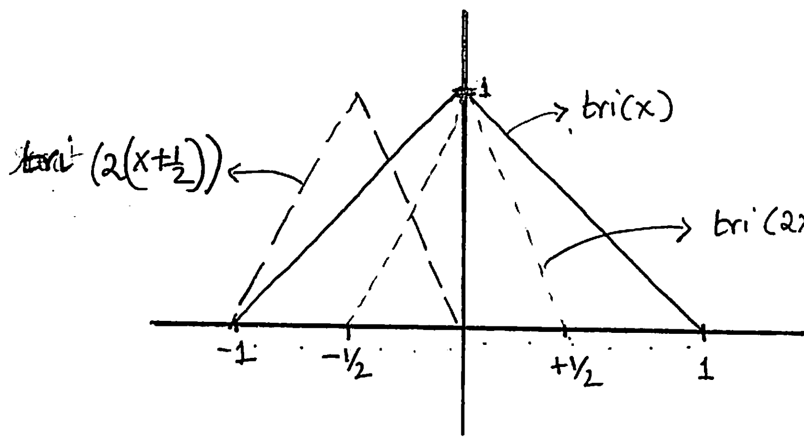
So to insure that a rectangular function has unit area in general we need to scale it by A.

$$\frac{1}{|b|}\text{rect}\left(\frac{x-x_0}{b}\right) \longrightarrow \text{Unit area.}$$

Triangle Function

The triangle function creates a pulse, which unlike the rect function, is continuous for all values of argument. The triangle function is def^n as:

$$\mathrm{tri}(x) = \begin{cases} 0 & |x|>1 \\ x+1 & 1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \end{cases}$$



The area under the triangle function is

$$A = \int_{-\infty}^{\infty} \mathrm{tri}\left(\frac{x-x_0}{b}\right) = |b|$$

Therefore, a unit area triangle function is def^n as

$$\frac{1}{|b|}\,\mathrm{tri}\left(\frac{x-x_0}{b}\right) \longrightarrow \text{Unit Area.}$$

Sinc Function

The sinc function is a very important function and is related to the rect function via the Fourier transform. It is defined as (Gaskill's notation)

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

It is interesting to consider the value of sinc function at zero argument values. At first look, it seems that division by zero would cause the function to go to infinity. But a closer inspection reveals

$$\lim_{x\to 0} \text{sinc}(x) = \lim_{x\to 0} \frac{\sin \pi x}{\pi x} \to \left(\frac{0}{0}\right)$$

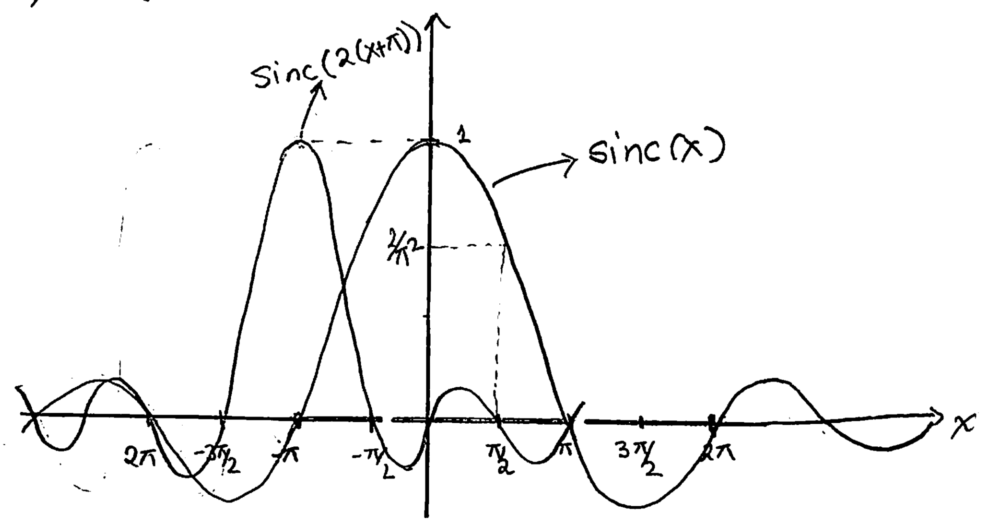
We can apply the L'Hôpital's rule here

$$\lim_{x\to 0} \text{sinc}(x) = \lim_{x\to 0} \frac{\partial \sin\pi x/\partial x}{\partial \pi x/\partial x}$$

$$= \lim_{x\to 0} \frac{\pi \cos \pi x}{\pi}$$

$$= \lim_{x\to 0} \cos \pi x = 1$$

Therefore, we find that: $\text{sinc}(0) = 1$



Note: Many authors use the definition $\text{sinc}(x) = \frac{\sin x}{x}$. Here we will use the notation of Gaskill due to following normalization property

$$\text{Area } A = \int_{-\infty}^{\infty} \text{sinc}\left(\frac{x-x_0}{b}\right) dx = |b|$$

The sinc function has many important properties that are worth remembering.

$$\text{sinc}(x) = 0 \quad \text{at} \quad x = \pm 1, \pm 2, \pm 3, \pm 4 \ldots$$

and ofcourse, the maximum value of sinc(x) occurs at $x=0$.

Gaussian Function

The Gaussian function is defined as (Gaskill):

$$\text{Gaus}(x) = e^{-\pi x^2}$$

As noted in case of sinc function, this particular definition with a factor of π in the argument is preferred due to the following normalization property.

$$A = \int_{-\infty}^{\infty} \text{Gaus}\left(\frac{x-x_0}{b}\right) dx = |b|$$