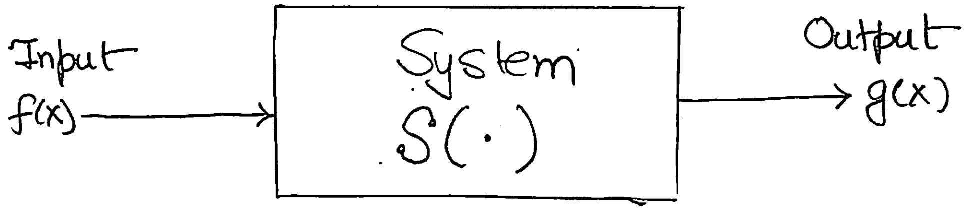
Linear, Shift-Invariant Systems



Block-diagram of a System.

The figure above shows schematic flow diagram that can be used to describe the flow of information, from input to output, in systems in mathematical physics. In this lecture we will develop a mathematical model of how the physical system works. This mathematical model is denoted by the system operator $S(\cdot)$ as shown in the figure above. One of our overarching goals in this course is to understand how to describe such systems - especially optical imaging system as *linear, shift-invariant systems*.

Linearity

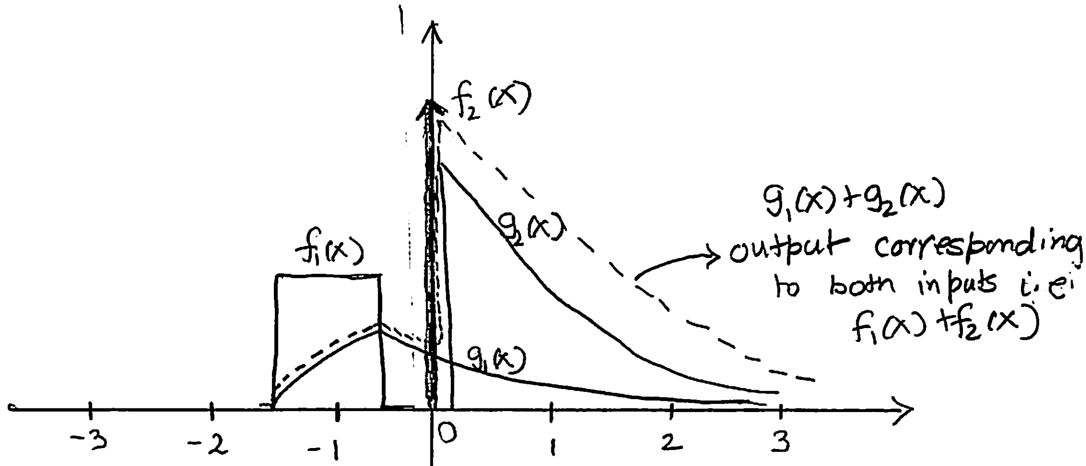
A linear system is a special system who's operator $S(\cdot)$ obeys the following relationship.

For any pair of inputs $f_1(x)$ and $f_2(x)$ with corresponding outputs $g_1(x)$ and $g_2(x)$

$$\boxed{S(\alpha f_1(x) + \beta f_2(x)) = \alpha g_1(x) + \beta g_2(x)} \quad (1)$$

for all constants α and β

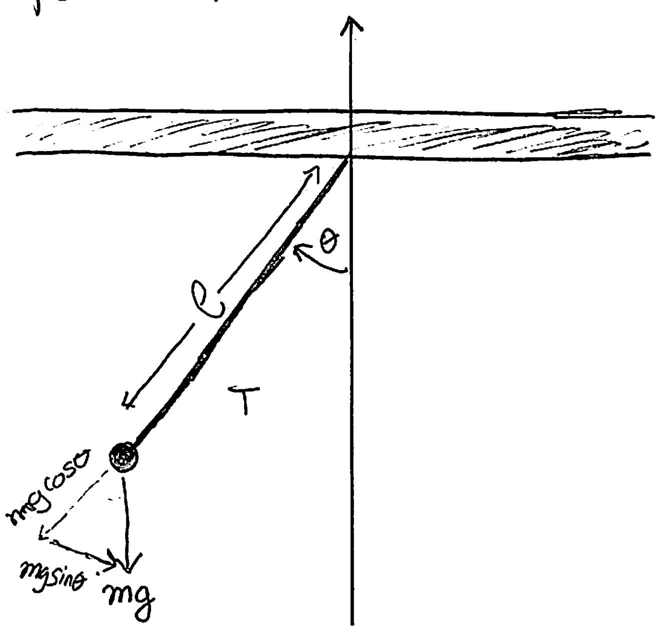
(1) is known as principle of superposition. This is a very important principle, in that it allows us to take an arbitrary input, decompose it into smaller simple pieces, pass each of these pieces through our system, the add up the respective output to yield the total output. Note, as α and β are arbitrary constants, linearity implies that the output is completely independent of the input (upto a scale factor). The figure below shows a simple example of linearity based on a R-C electrical circuit.



Output of RC-circuit with inputs $f_1(x)$ and $f_2(x)$ and both inputs present simultaneously.

Note that this is an approximation, but a very good approximation for a large range of real-world problems.

Now, let us consider a classic problem from mechanics. The figure below shows a free-body diagram for a simple pendulum



Free body diagram of a pendulum

The equation of motion for this system is

$$\underbrace{|\text{Torque}|}_{\text{force} \times \text{length}} \rightarrow \vec{T} = \overbrace{I}^{\text{moment of inertia}} \frac{d^2\theta}{dt^2} \rightarrow \text{angular acceleration}$$

$$-mg\sin\theta \cdot \ell = m\ell^2 \frac{d^2\theta}{dt^2}$$

$(m \neq 0)$

$$\underbrace{\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0}$$

Note that this is a non-linear differential equation in θ due to the presence of the non-linear function $\sin\theta$. However, for small angles $\sin\theta \approx \theta$ (via Taylor series expansion)

Plugging this approximation of $\sin\theta$, we get

$$\underbrace{\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0}_{\text{Linear differential equation (for small angle)}}$$

we know the solution to this differential equation is a sinusoidal oscillation.

Shift-Invariant System

A second important property for the systems we are considering is shift-invariance. This simply means, for a shift-invariant system, shifting the input produces exactly the same output shifted by the same amount.

Mathematically,

$$\mathcal{S}(\underset{\substack{\uparrow\\ \text{input}}}{f(x)}) = \underset{\substack{\uparrow\\ \text{output}}}{g(x)}$$

Shifted input : $f(x-x_0)$ ↳ Shifted by x_0 to right

For a shift-invariant system

$$\mathcal{S}(f(x-x_0)) = \underbrace{g(x-x_0)}_{\text{shifted output}}$$

Let's consider a real optical system that includes aberrations. If we have an on-axis point-source the image formed by the system will have some

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finite spot size determined by spherical aberration and diffraction. As we move off-axis, additional aberrations come into play, changing the point response of the system. The image of the same object will depend upon where it is in the field. Therefore, such a system is not shift-invariant.

However, for many well designed optical systems, the effect of these off-axis aberrations is small enough that we can consider them approximately shift-invariant over some range of field positions or numerical aperture. For example, if the effect of aberrations is smaller than the detector size in a CCD sensor, then the system can usually be considered as shift-invariant. In the strict sense, the point spread function and MTF of the imaging system will be a function of field (position/angle), and we will discuss this in greater detail in a few lectures.

Impulse Response of a Linear Shift-Invariant (LSI) System

For a given LSI system: $S(f(x)) = g(x)$

When the excitation/input to the system is an impulse function (or Dirac-delta function), the output is defined as the impulse response of the system, denoted by $h(x)$.

Mathematically,

$$\boxed{\begin{aligned} \mathcal{L}(\delta(x)) &= h(x) \\ \mathcal{L}(\delta(x-x_0)) &= h(x-x_0) \end{aligned}}$$

We will denote LSI systems with the special operator $\mathcal{L}(\;)$ throughout the remainder of the course.

Let us see what are the implications of this result. We know from the property of the delta function that we can write an arbitrary function $f(x)$ as superposition of delta functions:

$$f(x) = \int_{-\infty}^{\infty} f(\alpha)\,\delta(x-\alpha)\,d\alpha$$

In this equation we have expressed $f(x)$ as a continuous summation (i.e. integral) of shifted delta functions, $\delta(x-\alpha)$, each weighted by the constant $f(\alpha)$. Now, we can operate on $f(x)$ in this equation. We get

$$\begin{aligned} \mathcal{L}(f(x)) &= \mathcal{L}\left(\int_{-\infty}^{\infty} f(\alpha)\,\delta(x-\alpha)\,d\alpha\right) \\ &= \int_{-\infty}^{\infty} f(\alpha)\,\mathcal{L}(\delta(x-\alpha))\,d\alpha \\ &= \int_{-\infty}^{\infty} f(x)\;h(x-\alpha)\,d\alpha \end{aligned}$$

Thus

$$\boxed{\mathcal{L}(f(x)) = \int_{-\infty}^{\infty} f(x)\cdot h(x-\alpha)\,d\alpha}$$

This equation is a special integral known as a ~~Convolutional integral~~. We will talk more about convolution next time. This result tells us that if we know the response of a LSI system to a δ-function, we can compute the response to an arbitrary input! All we have to do is split the input into a summation of small point sources that are shifted and weighted, and then add up the corresponding shifted and weighted impulse responses to get the output.

Eigenfunctions of LSI systems

An eigenfunction of a system satisfies the special property that the output of the system is a scaled version of the input. Formally,

$$S\{\psi(x;\xi_0)\} = \mathcal{H}(\xi_0)\,\psi(x;\xi_0)$$

The $\psi(x;\xi_0)$ function is an eigenfunction and the complex constant is the corresponding eigenvalue.

Now, let us consider a system defined by the operator $\mathcal{L}\{\}$. If the only thing we know about $\mathcal{L}\{\}$ is that it is a LSI operator, what can we say about it's eigenfunctions?

Let's examine what happens if the input to our system is a complex exponential of frequency ξ.

$$\psi(x:\xi) = e^{j2\pi\xi x}$$

$$\mathcal{L}\{\psi(x,\xi)\} = g(x;\xi)$$

By convolution integral:

$$g(x;\xi) = \int_{-\infty}^{\infty} h(x-\alpha)\,\psi(\alpha)\,d\alpha$$

$$= \int_{-\infty}^{\infty} h(x-\alpha)\,e^{j2\pi\xi\alpha}\,dx$$

let $x-\alpha = \beta$

$dx = d\beta$

$$= \int_{-\infty}^{\infty} h(\beta)\,e^{j2\pi\xi(x+\beta)}\,d\beta$$

$$= e^{j2\pi\xi x}\int_{-\infty}^{\infty} h(\beta)\,e^{-j2\pi\xi\beta}\,d\beta$$

$$g(x;\xi) = \underbrace{\left[\int_{-\infty}^{\infty} h(\beta)\, e^{-j2\pi\xi\beta}\right]}_{H(\xi)} \cdot e^{j2\pi\xi x}$$

$$g(x;\xi) = H(\xi)\, e^{j2\pi\xi x}$$

$\Rightarrow$ $e^{j2\pi\xi x}$: is an eigenfunction of any $\mathcal{L}SI$ system

$H(\xi)$: corresponding eigenvalue

This an extremely powerful result, given that without any specific knowledge about the system or it's operator $\mathcal{L}\{\}$ except it is $\mathcal{L}SI$, this result always holds!