ASTR/OPTI 428/528

Lecture 11: PSFs, Speckles, and Strehl Ratio

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Fourier Optics

From Lecture 5...

- Fourier optics is an extremely simple optical model for an imaging system
- The only specifiable part in the imaging system is the pupil mask: $\Pi(\mathbf{x}) \in \mathbb{C}$.
- Fourier optics relates a general pupil plane field (ψ) to the resulting image plane field (Ψ).

Fourier Optics

- Even simpler... use spatial frequency $\boldsymbol{\kappa}$ and just talk about the *angular spectrum* of plane waves

$$\Psi(\boldsymbol{\kappa}) = \int \mathrm{e}^{i\boldsymbol{\kappa}\cdot\mathbf{x}}\psi(\mathbf{x})\,\mathrm{d}^2x.$$

The image plane is just a scaled version of the spatial frequency.

- The pupil field is the incident field times the complex pupil mask: $\psi(\mathbf{x}) = \Pi(\mathbf{x})\psi_0(\mathbf{x})$
- With an incident plane wave from angle $\boldsymbol{\theta} = \boldsymbol{\kappa}/k$, amplitude α and phase φ:

$$\psi(\mathbf{x}) = \Pi(\mathbf{x})\alpha\mathrm{e}^{ik\boldsymbol{\theta}\cdot\mathbf{x}+i\varphi}$$

- In general:

$$\psi(\mathbf{x}) = \Pi(\mathbf{x})\alpha(\mathbf{x})\mathrm{e}^{i\varphi(\mathbf{x})}$$

Image Intensity

- A long exposure averages over the fluctuating irradiance and gives us

$$I_{image}(\boldsymbol{\kappa}) \propto \Phi(\boldsymbol{\kappa}) \equiv \left\langle |\Psi(\boldsymbol{\kappa})|^2 \right\rangle = \langle \Psi(\boldsymbol{\kappa}) \Psi^*(\boldsymbol{\kappa}) \rangle$$

- The inverse Fourier transform is easily written in terms of the pupil field

$$\langle \mathcal{O}(\boldsymbol{\xi}) \rangle \equiv \mathcal{F}^{-1} \{\Phi(\boldsymbol{\kappa})\} = \int \left\langle \psi(\mathbf{x}' + \boldsymbol{\xi}/2) \psi^*(\mathbf{x}' - \boldsymbol{\xi}/2) \right\rangle \mathrm{d}^2 x'$$

This is also the average OTF $\mathcal{O}(\boldsymbol{\xi})$.

Simplifying the average OTF

Writing in terms of our fluctuating amplitude and phase...

$$\mathcal{O}(\boldsymbol{\xi}) = \int \left\langle \alpha(\mathbf{x}' + \boldsymbol{\xi}/2)\alpha(\mathbf{x}' - \boldsymbol{\xi}/2)\mathrm{e}^{i(\varphi(\mathbf{x}'+\boldsymbol{\xi}/2)-\varphi(\mathbf{x}'-\boldsymbol{\xi}/2))} \right\rangle$$

$$\Pi(\mathbf{x}' + \boldsymbol{\xi}/2)\Pi^*(\mathbf{x}' - \boldsymbol{\xi}/2)\mathrm{d}^2x'$$

Ignore Scintillation

Now we make a big assumption that we can ignore scintillation when talking about the PSF. This is somewhat harder to believe when written in the pupil plane, but you can check the validity of the assumption by using a simulation.

It will not always be true, so be careful!

Ignoring scintillation means $\alpha(\mathbf{x})$ is a constant.

$$\mathcal{O}(\boldsymbol{\xi}) = \alpha^2 \int \left\langle \mathrm{e}^{i(\varphi(\mathbf{x}'+\boldsymbol{\xi}/2)-\varphi(\mathbf{x}'-\boldsymbol{\xi}/2))} \right\rangle \Pi(\mathbf{x}'+\boldsymbol{\xi}/2)\Pi^*(\mathbf{x}'-\boldsymbol{\xi}/2)\mathrm{d}^2x'.$$

Then, if $\delta\varphi = \varphi_1 - \varphi_2$ is a zero-mean Gaussian random process,

$$\mathcal{O}(\boldsymbol{\xi}) = \alpha^2 \int \mathrm{e}^{-D_\phi(\mathbf{x}'+\boldsymbol{\xi}/2,\mathbf{x}'-\boldsymbol{\xi}/2)/2} \Pi(\mathbf{x}'+\boldsymbol{\xi}/2)\Pi^*(\mathbf{x}'-\boldsymbol{\xi}/2)\mathrm{d}^2x'.$$

where

$$D_\phi(\mathbf{x}_1, \mathbf{x}_2) = \left\langle (\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2))^2 \right\rangle.$$

Stationarity and Isotropy

If the phase fluctuations are statistically stationary

$$D_\phi(\mathbf{x}_1, \mathbf{x}_2) \to D_\phi(\mathbf{x}_1 - \mathbf{x}_2).$$

This means that

$$D_\phi(\mathbf{x}' + \boldsymbol{\xi}/2, \mathbf{x}' - \boldsymbol{\xi}/2) \to D_\phi(\boldsymbol{\xi}).$$

If, in addition, the phase fluctuations are statistically isotropic

$$D_\phi(\mathbf{x}_1, \mathbf{x}_2) \to D_\phi(\|\mathbf{x}_1 - \mathbf{x}_2\|)$$

and

$$D_\phi(\mathbf{x}' + \boldsymbol{\xi}/2, \mathbf{x}' - \boldsymbol{\xi}/2) \to D_\phi(\xi).$$

WARNING! These may not be good assumptions!

Back to our averaged PSF...

Ignoring scintillation and assuming stationarity

$$\mathcal{O}(\boldsymbol{\xi}) = \alpha^2 \mathrm{e}^{-D_\phi(\boldsymbol{\xi})/2} \int \Pi(\mathbf{x}' + \boldsymbol{\xi}/2)\Pi^*(\mathbf{x}' - \boldsymbol{\xi}/2)\mathrm{d}^2x'.$$

The integral is the unscattered OTF, $\mathcal{O}_0(\boldsymbol{\xi})$

$$\mathcal{O}(\boldsymbol{\xi}) = \alpha^2 \mathrm{e}^{-D_\phi(\boldsymbol{\xi})/2} \mathcal{O}_0(\boldsymbol{\xi}).$$

This simple result (including all of its implicit assumptions), is the basis for virtually all of the Adaptive Optics literature.

Canonical Phase Structure Function Model

The 5/3 Kolmogorov structure function is usually improved with an inner scale and an outer scale.
Below the inner scale the structure function is a square-law $\propto \xi^2$ and beyond the outer scale it goes to

$$D_\phi(s) \to 2\left\langle \varphi^2 \right\rangle$$

Extremely Seeing-Limited Images

The standard imaging result for a long exposure is

$$\mathcal{O}(\boldsymbol{\xi}) = I_0 \mathrm{e}^{-D_\phi(\boldsymbol{\xi})/2} \mathcal{O}_0(\boldsymbol{\xi}).$$

If the phase aberrations are severe, $D/r_0 \gg 1$, then the average OTF can be approximated

$$\mathcal{O}(\boldsymbol{\xi}) = I_0 \mathrm{e}^{-D_\phi(\boldsymbol{\xi})/2} \mathcal{O}_0(0)$$

where

$$\mathcal{O}_0(0) = \int \left|\Pi(\mathbf{x}')\right|^2 \mathrm{d}^2 x'$$

which is usually equal to the area of the pupil A.

$$\mathcal{O}(\boldsymbol{\xi}) = I_0 A \mathrm{e}^{-D_\phi(\boldsymbol{\xi})/2}.$$

Diffraction-Limited Images

If the phase aberrations are relatively weak, say $\sigma_\varphi \lesssim 1$, $\mathrm{e}^{-D_\phi/2}$ does not totally limit the resolution of the imaging system. We can make a different approximation. A good way to think about it is $\mathrm{e}^{-D_\phi(s)/2} = 1$ near $s = 0$ and drops to a minimum of $\mathrm{e}^{-D_\phi(s)/2} \to \mathrm{e}^{-\sigma_\varphi^2}$ when $s \gg L_0$. This means that we can rewrite $\mathrm{e}^{-D_\phi(s)/2}$ as (not an approximation!)

$$\mathrm{e}^{-D_\phi(s)/2} = \left\{ \mathrm{e}^{-D_\phi(s)/2} - \mathrm{e}^{-\sigma_\varphi^2} \right\} + \mathrm{e}^{-\sigma_\varphi^2}.$$

Diffraction-Limited Images

$$e^{-D_\phi(s)/2} = \left\{ e^{-D_\phi(s)/2} - e^{-\sigma_\varphi^2} \right\} + e^{-\sigma_\varphi^2}.$$

If we call the value $S = e^{-\sigma_\varphi^2}$, then the first part (in braces) runs from 0 to $1-S$ and is localized near $s=0$, similar to the seeing-limited case. The second term is just a constant.

$$e^{-D_\phi(s)/2} = (1-S)h(s) + S$$

where

$$h(s) = \frac{e^{-D_\phi(s)/2} - e^{-\sigma_\varphi^2}}{1 - e^{-\sigma_\varphi^2}}$$

The Maréchal Approximation

Therefore, in weaker scattering,

$$\mathcal{O}(\boldsymbol{\xi}) = I_0\mathcal{O}_0(\boldsymbol{\xi})\mathrm{e}^{-D_\phi(\boldsymbol{\xi})/2} = I_0\mathcal{O}_0(\boldsymbol{\xi})\left[(1-S)h(s)+S\right]$$

$$\mathcal{O}(\boldsymbol{\xi}) = (1-S)I_0\mathcal{O}_0(\boldsymbol{\xi})h(s) + SI_0\mathcal{O}_0(\boldsymbol{\xi}).$$

Often, $h(s)$ drops to zero much faster than the ideal MTF, allowing us to make the approximation

$$\mathcal{O}(\boldsymbol{\xi}) = (1-S)I_0\mathcal{O}_0(0)h(s) + SI_0\mathcal{O}_0(\boldsymbol{\xi}).$$

The Maréchal Approximation

Fourier transforming back into the image space, we find

$$\Phi(k\boldsymbol{\theta}) = (1-S)\Phi_{speckles}(k\boldsymbol{\theta}) + S\Phi_0(k\boldsymbol{\theta}).$$

Note that the PSF peak falls off as S (until it gets lost in the speckle halo).
The approximation that the average PSF peak normalized to the unscattered peak (the Strehl ratio) is $S = \mathrm{e}^{-\sigma_\varphi^2}$ is called the Maréchal approximation.