Properties of Convolution

1. Commutative Property -

Starting from the definition

$$f(x) \circledast h(x) = \int_{-\infty}^{\infty} f(\alpha)\, h(x-\alpha)\, d\alpha$$

$$(x-\alpha) = \beta \qquad d\alpha = -d\beta$$

$$= \int_{-\infty}^{\infty} f(x-\beta)\, h(\beta)\, d\beta$$

$$= h(x) \circledast f(x) \qquad \blacksquare$$

Thus, we see that the order of function in convolution does not matter. We can use this property to select which function we want to "flip and shift". One order may be easier to evaluate than the other!

2. Distributive Property -

As the convolution operation is linear, it should not be surprising that

$$f(x) \circledast [a\, h_1(x) + b\, h_2(x)] = a\, f(x) \circledast h_1(x) + b\, f(x) \circledast h_2(x)$$

3. Shift Invariance

Let us consider what happens when one function is shifted in convolution

$$f(x) \circledast h(x) = g(x)$$

$$f(x-x_0) \circledast h(x) = ?$$

$$f(x-x_0) \circledast h(x) = \int_{-\infty}^{\infty} f(\alpha - x_0)\, h(x-\alpha)\, d\alpha$$

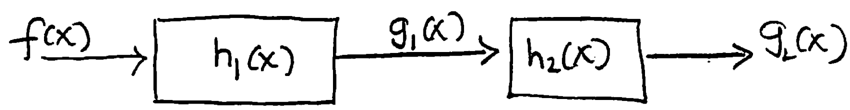
$$= \int_{-\infty}^{\infty} f(\beta)\, h(x - x_0 - \beta)\, d\beta \qquad \begin{aligned} \alpha - x_0 &= \beta \\ d\alpha &= d\beta \end{aligned}$$

$$= g(x - x_0) \quad \blacksquare$$

Thus, we see that the convolution operator is shift-invariant!

4. Associative property

Consider what happens when the output of one system forms the input to another system as shown below.



$$g_2(x) = g_1(x) \circledast h_2(x) = [f(x) \circledast h_1(x)] \circledast h_2(x)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha)\, h_1(\beta - \alpha)\, d\alpha \;\; h_2(x-\beta)\, d\beta$$

$$= \int_{-\infty}^{\infty} f(\alpha) \left[\int_{-\infty}^{\infty} h_1(\beta-\alpha)\, h_2(x-\beta)\, d\beta \right] d\alpha$$

$$= \int_{-\infty}^{\infty} f(\alpha) \underbrace{[h_2(x) \circledast h_1(x-\alpha)]}_{g(x-\alpha)} d\alpha$$

$$= \int_{-\infty}^{\infty} f(\alpha)\, g(x-\alpha)\, d\alpha = f(x) \circledast g(x-\alpha)$$

$$\equiv f(x) \circledast [h_1(x) \circledast h_2(x)] \quad \blacksquare$$

If we define $g(x) = h_1(x) \circledast h_2(x)$, this property tells us that we can group input, output and impulse response conveniently, regardless of the order of systems!

Convolution with δ-functions and Comb functions

1. δ-function

This is an important property

$$f(x) \circledast \delta(x) = \int_{-\infty}^{\infty} f(\alpha)\, \delta(x-\alpha)\, d\alpha$$

using sifting property

$$= f(x) \blacksquare$$

Similarly, convolving with derivative of δ-function as

$$f(x) \circledast \delta^{(k)}(x) = \int_{-\infty}^{\infty} f(\alpha)\, \delta^{(k)}(x-\alpha)\, d\alpha$$

$$= f^{(k)}(x) \blacksquare$$

2. Comb-function

Now, we extend convolution to the comb function.

$$f(x) \circledast \mathrm{comb}(x) = f(x) \circledast \sum_{n=-\infty}^{\infty} \delta(x-n)$$

using distributive property

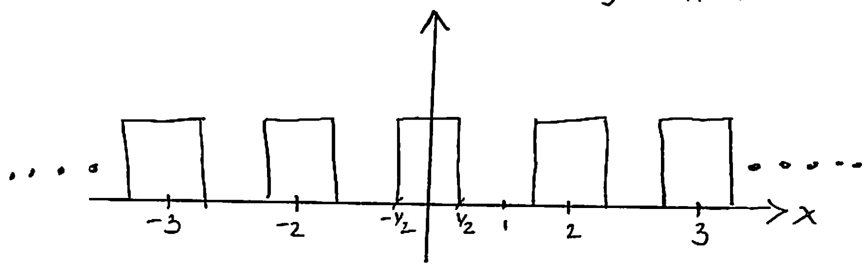
$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha)\, \delta(x-\alpha-n)\, d\alpha$$

$$= \sum_{n=-\infty}^{\infty} f(x-n) \blacksquare$$

Thus, we see that convolution produces "copies" of the input function $f(x)$, however, each copy is offset by integer values of the comb function.

For example, let's consider the convolution:

$$\mathrm{rect}(x) \circledast \mathrm{comb}\left(\frac{x}{2}\right) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(x-2n).$$



From this figure we note an interesting property. Given an arbitrary function $f(x)$ that is zero for $x<0$ & $x>T$, we take the convolution

$$f_p(x) = f(x) \circledast \frac{1}{T}\mathrm{comb}\left(\frac{x}{T}\right)$$

the result $f_p(x)$ is a <u>periodic function</u> with period T, with a single period described by $f(x)$.