ASTR/OPTI 428/528

Lecture 7: From Kolmogorov to Fried

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Velocity Structure Function

For isotropic turbulence over scales in the **inertial subrange**, the velocity structure function

$$D_v(\mathrm{r}) = \left\langle \|\mathbf{v}(\mathrm{x}+\mathrm{r}) - \mathbf{v}(\mathrm{x})\|^2 \right\rangle$$

has the form

$$D_v(\mathrm{r}) = C_v^2 r^{2/3}.$$

The overall constant C_v^2 is called the *velocity structure constant*.

Index of Refraction Fluctuations

The index of refraction (n) in air is related to:

- Pressure
- Temperature, and
- water vapor content (humidity).

Velocity fluctuations cause variations in the temperature and pressure of the air.

$$\delta v \longrightarrow (\delta T, \delta P)$$

Restoring timescales

The pressure variations are rapidly brought back into equilibrium by pressure waves (i.e. sound waves),

- Temperature variations relax more slowly by
  - conduction (most important)
  - convection
  - radiation

$\Longrightarrow$ Thus the most important link between turbulent velocity and index of refraction is via temperature.

Index of Refraction Structure Function

The end result is that the index of refraction follows the temperature which follows the velocity.

$$\delta n \propto \delta T \propto \delta v$$

This means that the structure function of index of refraction variations

$$D_n(\mathrm{r}) = \left\langle (n(\mathrm{x}+\mathrm{r}) - n(\mathrm{x}))^2 \right\rangle$$

has the power-law form

$$D_n(\mathrm{r}) = C_n^2 r^{2/3}$$

over the inertial subrange of scales.

Phase Fluctuations

The index of refraction fluctuations occur throughout the volume of the propagation medium: $n = n(\mathbf{x}, z, t)$.

It may seem like a trivial point, but this is in 3-dimensional space.

The main observable effect on an electromagnetic wavefront is phase variation entering our instrument.

The local speed of light:

$$c(\mathbf{x}, z, t) = c_0 / n(\mathbf{x}, z, t).$$

The wavefront can be affected by optical path length (OPL) or by geometry (because tilted rays travel farther).

Paraxial Phase Fluctuations

For paraxial rays, the dominant wavefront variation is caused by optical path length variations.

$$\mathsf{OPL} = \int_{z_0}^{z_0+h} n(\mathbf{x}, z)dz.$$

Wavefront arrival time fluctuations are

$$\delta t = \frac{\mathsf{OPL} - \mathsf{OPL}_0}{c_0} = \int_{z_0}^{z_0+h} [n(\mathbf{x}, z) - 1]\, dz/c_0$$

This corresponds to a phase shift of $2\pi\delta t/T = \omega\delta t = kc_0\delta t$. This gives

$$\delta\phi = \int_{z_0}^{z_0+h} \underbrace{[n(\mathbf{x}, z) - 1]}_{\mu(\mathbf{x},z)} kdz \equiv \int_{z_0}^{z_0+h} k\mu(\mathbf{x}, z)dz$$

Stop and Think...

- What about the geometry part of the phase?
- What constraints does this place on scattering parameters in our problems?
- What happens if we are out of the strictly applicable regime? What measurables would change?

Constructing the Phase Structure Function

The phase at some point $\mathbf{x}$

$$\phi(\mathbf{x}) = k \int_{z_0}^{z_0+h} \mathrm{d}z\, \mu(\mathbf{x}, z)$$

$$\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) = k \int_{z_0}^{z_0+h} \mathrm{d}z\, \underbrace{\left(\mu(\mathbf{x}_1, z) - \mu(\mathbf{x}_2, z)\right)}_{\delta\mu(\mathbf{x}_1, \mathbf{x}_2, z)}$$

$$\left(\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\right)^2 = k^2 \int_{z_0}^{z_0+h} \mathrm{d}z_1 \int_{z_0}^{z_0+h} \mathrm{d}z_2\, \delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_1)\delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_2)$$

Phase Structure Function

$$
\begin{aligned}
D_\phi(\mathbf{x}_1, \mathbf{x}_1) &= \left\langle (\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2))^2 \right\rangle \\
&= k^2 \int_{z_0}^{z_0+h} \mathrm{d}z_1 \int_{z_0}^{z_0+h} \mathrm{d}z_2 \, \langle \delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_1) \delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_2) \rangle
\end{aligned}
$$

Markov Approximation

Here we typically make an approximation.

- The turbulence may (or may not) be isotropic, but suppose it is.
- In narrow-angle scattering, the waves typically are within a small angle θ_s of each other.
- This creates an anisotropy in the spatial coherence of the field.
- By considering the integral scaled to the field coherence lengths, we see that δz is effectively scaled by $1/\theta_s$ relative to the transverse scales of δx.

Markov Approximation, cont.

Consider the θ_s scaling to shrink the effect of μ (i.e. n) correlations effectively to a delta function.

$$\langle \delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_1)\delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_2)\rangle \approx \langle \delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_1)\delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_1)\rangle$$
$$L_\mu(\mathbf{x}_1, \mathbf{x}_2)\delta(z_1 - z_2)$$

Markov Phase Structure Function

Assume that h is bigger than the longitudinal correlation length. Then we can approximate

$$\int_{z_0}^{z_0+h} \mathrm{d}z_2 \, \langle \delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_1)\delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_2)\rangle = \underbrace{\left\langle (\delta\mu(\mathbf{x}_1, \mathbf{x}_2, z_1))^2 \right\rangle}_{D_n(\delta\mathbf{x}=\mathbf{x}_1-\mathbf{x}_2, \delta z=0; z)} L_\mu(\mathbf{x}_1 - \mathbf{x}_2)$$

Effect of δz integration

Notice that for $\delta x \gg L_{\mu 0}$ we can expect $L_\mu(\delta \mathbf{x}) \propto 2.91 \, \|\delta \mathbf{x}\|$ (the number comes from numerical evaluation for Kolmogorov turbulence).

$$D_\phi(\mathbf{x}_1, \mathbf{x}_2) = k^2 L_\mu(\delta \mathbf{x}) \int_{z_0}^{z_0+h} \mathrm{d}z \, \underbrace{D_n(\delta \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2, \delta z = 0; z)}_{C_n^2(z) \|\delta \mathbf{x}\|^{2/3}}$$

$$D_\phi(\delta \mathbf{x}) = 2.91 k^2 \, \|\delta \mathbf{x}\| \int_{z_0}^{z_0+h} \mathrm{d}z C_n^2(z) \, \|\delta \mathbf{x}\|^{2/3}$$

$$D_\phi(\delta \mathbf{x}) = 2.91 k^2 \, \|\delta \mathbf{x}\|^{5/3} \int_{z_0}^{z_0+h} C_n^2(z) \, \mathrm{d}z$$

The simple phase coherence length

In a simple sense, the phase structure function measures the mean-square difference of the phases measured at two points. This gives a natural coherence length ℓ_ϕ where $\left\langle \delta\phi^2 \right\rangle = 1\,\text{radian}^2$ that we might imagine would be our reference.
If we wrote things that way, we would simply write

$$D_\phi(r) = \left(r/\ell_\phi\right)^{5/3}.$$

The Fried Length

We don't do that though.

- David Fried computed the size of a circular telescope pupil which, for projected Kolmogorov turbulence, has an rms phase variation of 1 radian.
- It is the largest size pupil before your PSF core starts to break up into speckles.
- He called this length r_0.

In terms of r_0 the phase structure function becomes

$$D_\phi(r) = 6.88\,(r/r_0)^{5/3}\,.$$

Relationship between r_0 and C_n^2

We have two ways to write D_ϕ over the inertial subrange. They have the same power law and so should be equal since they are just different ways to describe the same thing...

$$D_\phi(r) = 6.88\,(r/r_0)^{5/3}$$

and

$$D_\phi(r) = 2.91 k^2 r^{5/3} \int C_n^2(z)\,\mathrm{d}z$$

Therefore,

$$6.88 r_0^{-5/3} = 2.91 (2\pi)^2 \lambda^{-2} \int C_n^2(z)\,\mathrm{d}z$$

Relationship between r_0 and C_n^2, cont.

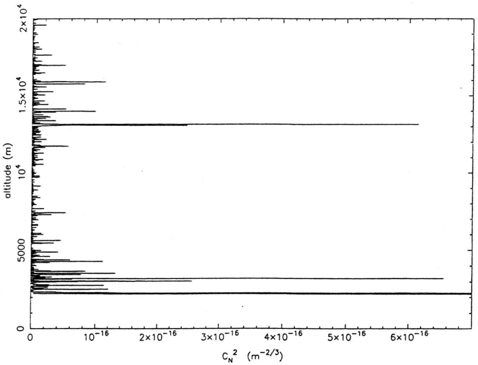
$$6.88 r_0^{-5/3} = (2\pi)^2 \lambda^{-2} \int C_n^2(z)\,\mathrm{d}z$$

$$r_0^{-5/3} = 2.91 \frac{(2\pi)^2}{6.88} \frac{\int C_n^2(z)\,\mathrm{d}z}{\lambda^2}$$

$$r_0 = \left(\frac{6.88}{2.91(2\pi)^2}\right)^{3/5} \frac{\lambda^{6/5}}{[\int C_n^2(z)\,\mathrm{d}z]^{3/5}}$$

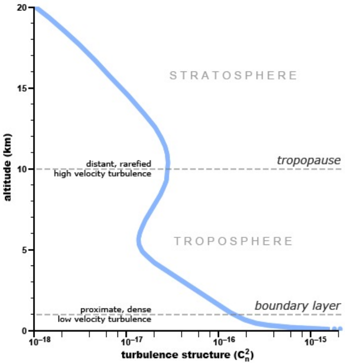
$$r_0 = \frac{0.1847\lambda^{6/5}}{[\int C_n^2(z)\,\mathrm{d}z]^{3/5}}$$

Typical Atmospheric Turbulence Profile



An actual measured turbulence C_n^2 profile.

Hufnagel-Valley C_n^2 Profile



$$C_n^2(h) \approx \underbrace{C_n^2(0)\mathrm{e}^{-h/100}}_{ground\,layer} + \underbrace{2.7\times10^{-16}\mathrm{e}^{-h/1500}}_{low-altitude} + \underbrace{8.148\times10^{-56}U^2h^{10}\mathrm{e}^{-h/1000}}_{tropopause}$$

Looking along a non-Zenith angle

In astronomy, the turbulence profile is often quoted as being only a function of altitude h, $C_n^2(h)$. If we look along a path that is not vertical but with a zenith angle ζ, we spend more time in each layer and the formula for the Fried length becomes

$$r_0 = \frac{0.1847\lambda^{6/5}}{\left[\int C_n^2(h)\,\sec\zeta \mathrm{d}h\right]^{3/5}}$$

or

$$r_0 = \frac{0.1847\lambda^{6/5}\,(\sec\zeta)^{-3/5}}{\left[\int C_n^2(h)\,\mathrm{d}h\right]^{3/5}}.$$

Typical Observing Run

Don't take mean profiles too seriously. YMMV.