ASTR/OPTI 428/528

Lecture 6: Turbulence Theory

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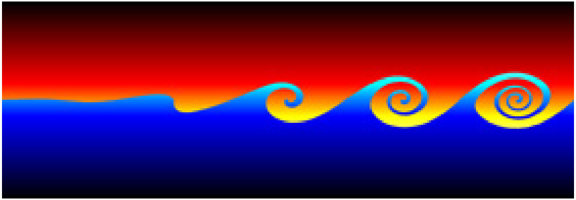
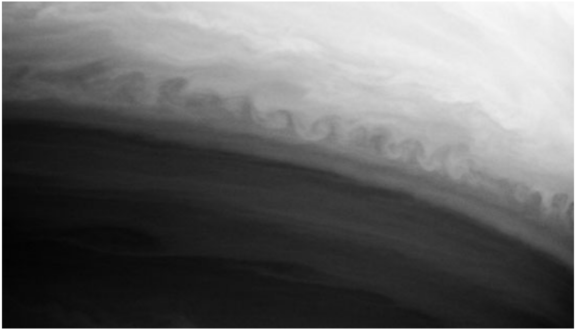
Kelvin–Helmholtz Instability

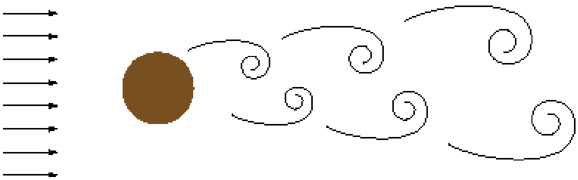


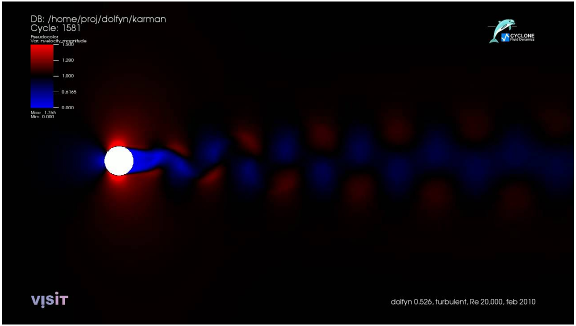
Kelvin–Helmholtz Clouds

Kelvin–Helmholtz on Saturn



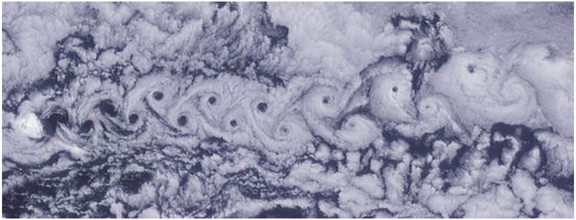
von Kármán Vortex Street



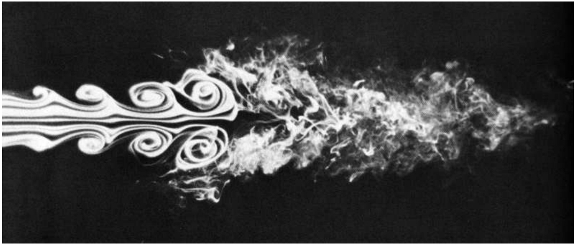


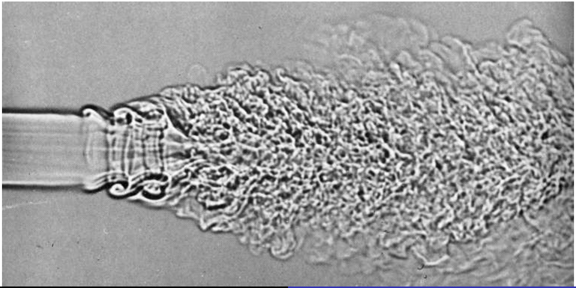
von Kármán Clouds

More von Kármán Clouds

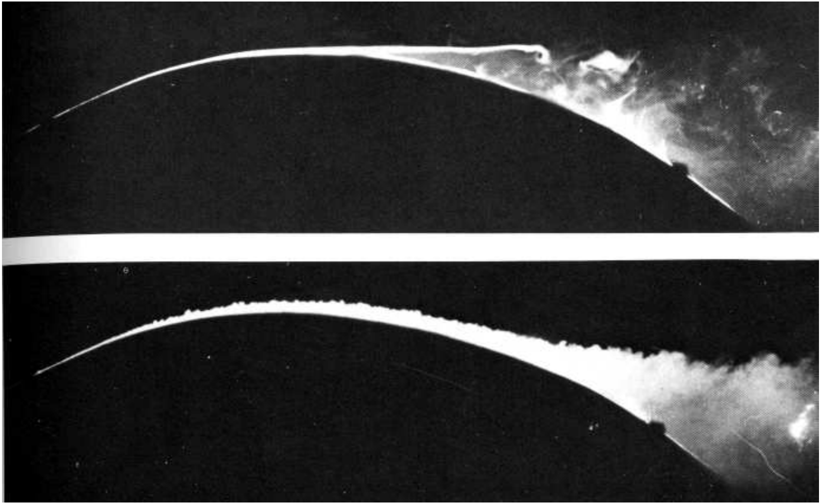


Vortex Instability Turbulence

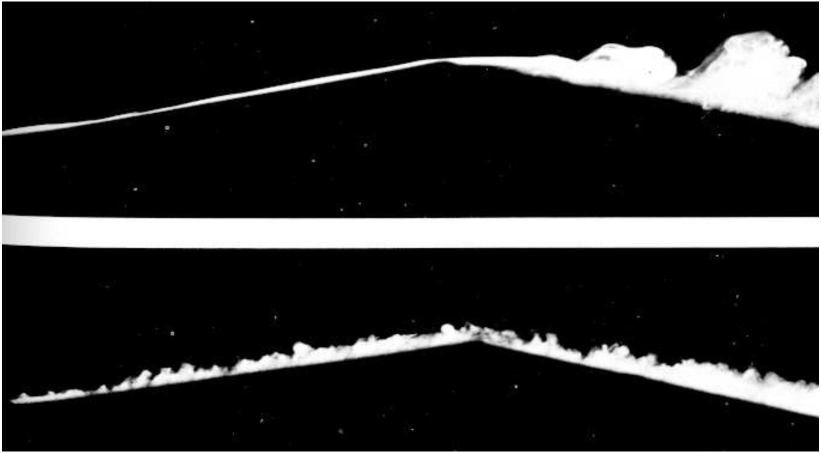




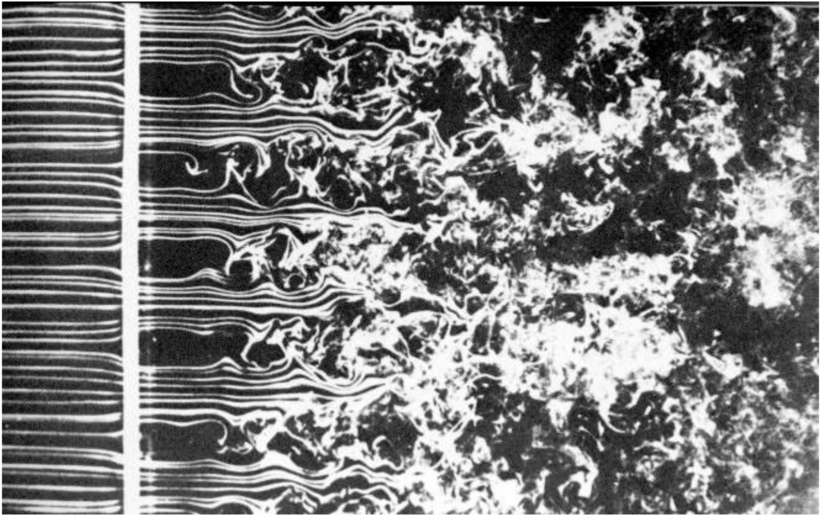
Boundary Layer Turbulence



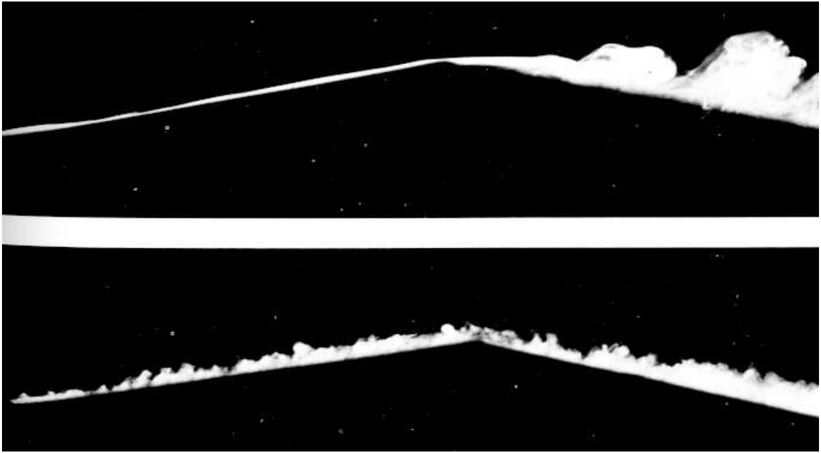
More Boundary Layer Turbulence



Grid Turbulence



More Boundary Layer Turbulence



What is Turbulence?

Turbulence is highly unstable, chaotic, nonlinear, state of fluid flow characterized by mixing. It naturally occurs when the dynamical forces of the flow far exceed the damping forces of viscosity (i.e. high Reynolds Number Re).

Reynolds Number

Reynolds Number is defined as the ratio of inertial over viscous forces.

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{\rho v L}{\mu} = \frac{vL}{\nu}$$

- v is the mean velocity of the object re the fluid.
- L is a characteristic linear dimension.
- μ is the dynamic viscosity of the fluid (Pa·s or N·s/m² or kg/(m·s)).
- ν is the kinematic viscosity ($\nu = \mu/\rho$) (m²/s).
- ρ is the density of the fluid (kg/m³).

Typical evolution leading to turbulent flow

- High Re flow in some sort of a shearing geometry (e.g. wind shear at a boundary).
- The flow interface becomes unstable, leading to vortex shedding.
- The vortices (eddies) grow as they are pulled along with the flow.
- They get big enough to interfere with neighboring eddies.
- Chaotic interactions between eddies dominate the flow.

Self Similarity

Self Similarity

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Inner and Outer Scales

Strictly speaking, self-similar fractal processes do not tend to have well-defined means (like a mean velocity). This is because there is always a larger-scale eddy that messes up the mean. This isn't a problem in the real world because there always exists an "outer scale" beyond which the situation is different. In the present case the outer scale is the scale where energy is pumped into the system—say an unstable layer's vortex shedding scale, or a boundary layer's thickness. However, in the range where the fractal behavior dominates, it is possible to talk about stationary increments where the averages of the differences between the values (of say velocity) are well defined, while averages of the quantities themselves are not. An example of this type of quantity is the "structure function."

Kolmogorov-Obuchow Turbulence

A common model for turbulent flow is is called "fully developed" or "inertial subrange" turbulence. The argument is based on the notion that in high Reynolds Number flows, viscosity should be unimportant. It actually is still important, but only as the mediator of shear.

As the saying goes...

Bigger whirls have little whirls,
Which feed on their velocity;
Little whirls have smaller whirls,
and so on to viscosity.

Kolmogorov-Obuchow Turbulence, cont.

This cascade of energy from large scales to smaller scales leads to a statistically self-similar model that depends only on two parameters:

- the kinematic viscosity ν ,
- and the energy dissipation (or input) rate per unit mass, ϵ.

A Famous Dimensional Analysis

The statistical model describing this turbulence is due to a dimensional analysis argument by *Kolmogorov* and *Obuchow*. The units for the energy dissipation per unit mass are

$$[\epsilon] = Js^{-1}kg^{-1} = m^2 s^3$$

and the kinematic viscosity

$$[\nu] = m^2 s^{-1}.$$

Dimensional Analysis...

These parameters can be combined to form a characteristic time, velocity, and length scale as follows:

$$T_{char} = \sqrt{\nu/\epsilon}$$

$$V_{char} = (\epsilon\nu)^{1/4}$$

$$L_{char} = \nu^{3/4}\epsilon^{-1/4}$$

The form for the velocity structure function can be found by arguing that it should be be self-similar (i.e. a power law) and that it should be explicitly independent of the viscosity.

Velocity structure function

That is, the *velocity structure function* should look something like:

$$D_v(r) = CV_{char}^2 (r/L_{char})^\alpha.$$

Note that the separation variable in this function is in 3-D: $\delta \mathbf{r} = (\delta x, \delta y, \delta z)$.

Dimensional Analysis...

Inserting the values of the characteristic scales in terms of ν and ϵ we find

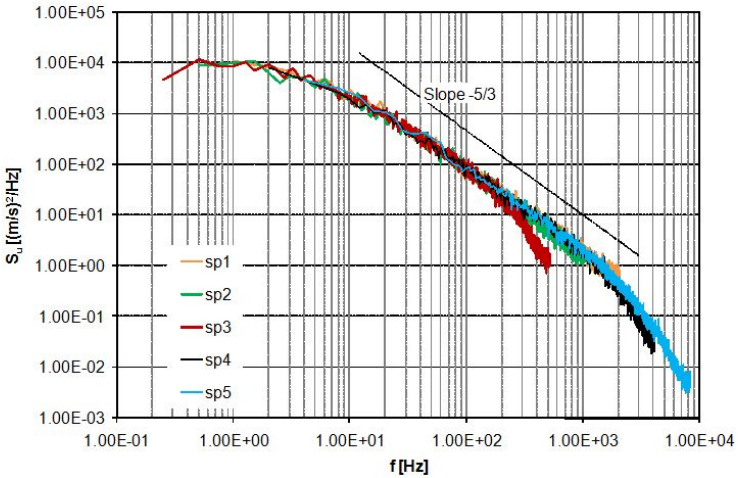
$$D_v(r) = C\epsilon^{1/2+\alpha/4}\nu^{1/2-3\alpha/4}r^{\alpha}.$$

Since this formula shouldn't explicitly depend on viscosity, we must adjust α to make it go away. This only happens when $\alpha = 2/3$, leaving us with the simplified form

$$D_v(r) = C_v^2 r^{2/3}.$$

The overall constant C_v^2 is called the *velocity structure constant*.

Velocity Power Spectra



Index of Refraction Fluctuations

The index of refraction (n) in air is related to:

- Pressure
    - Temperature, and
    - water vapor content (humidity).
- Velocity fluctuations cause variations in the temperature and pressure of the air.

$$\delta v \longrightarrow (\delta T, \delta P)$$

Restoring timescales

- The pressure variations are rapidly brought back into equilibrium by pressure waves (i.e. sound waves),
  - Temperature variations relax more slowly by
    - conduction (most important)
    - convection
    - radiation

$\Longrightarrow$ Thus the most important link between turbulent velocity and index of refraction is via temperature.

Index of Refraction Structure Function

The end result is that the index of refraction follows the temperature which follows the velocity.

$$\delta n \propto \delta T \propto \delta v$$

This means that the structure function of index of refraction variations is of the form:

$$D_n(r) = C_n^2 r^{2/3}.$$

Note that the separation between points, r, is in 3-D.

Phase Fluctuations

The index of refraction fluctuations occur throughout the volume of the propagation medium: $n = n(\mathbf{x}, z, t)$.

It may seem like a trivial point, but this is in 3-dimensional space.

The main observable effect on an electromagnetic wavefront is phase variation entering our instrument.

The local speed of light:

$$c(\mathbf{x}, z, t) = c_0 / n(\mathbf{x}, z, t).$$

The wavefront can be affected by optical path length (OPL) or by geometry (because tilted rays travel farther).

Paraxial Phase Fluctuations

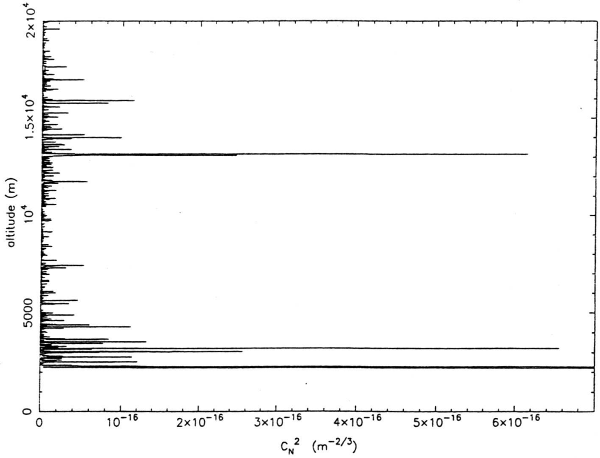
For paraxial rays, the dominant wavefront variation is caused by the OPL variations.
Wavefront arrival time fluctuations are

$$\delta t = \frac{\mathrm{OPL} - \mathrm{OPL}_0}{c_0} = \int_{z_0}^{z_0+h} [n(\mathbf{x},z) - 1]\, dz/c_0$$

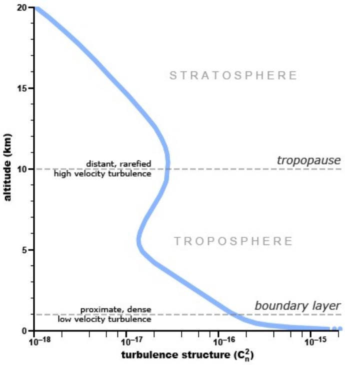
This corresponds to a phase shift of $2\pi\delta t/T = \omega\delta t = kc_0\delta t$. This gives

$$\delta\phi = \int_{z_0}^{z_0+h} \underbrace{[n(\mathbf{x},z) - 1]}_{\mu(\mathbf{x},z)} kdz \equiv \int_{z_0}^{z_0+h} k\mu(\mathbf{x},z)dz$$

Atmospheric Turbulence Profile



Hufnagel-Valley C_n^2 Profile



$$C_n^2(h) \approx \underbrace{C_n^2(0)\mathrm{e}^{-h/100}}_{ground\,layer} + \underbrace{2.7\times 10^{-16}\mathrm{e}^{-h/1500}}_{low-altitude} + \underbrace{8.148\times 10^{-56}U^2h^{10}\mathrm{e}^{-h/1000}}_{tropopause}$$