Nitigya Kathuria

OPTI -330 - 2014

HW3 - Sol^n

Q.1) Prove

(a) $\delta(x-x_1) \otimes f(x) = f(x-x_1)$

$\delta(x-x_1) \otimes f(x) = f(x) \otimes \delta(x-x_1)$ → Commutative property

$\delta(x-x_1) f(x) = \int \delta(\alpha-x_1) f(x-\alpha)\, d\alpha$

$= \int f(x-\alpha)\, \delta(\alpha-x_1)\, d\alpha$

from the sifting property of delta function

$= f(x-x_1)$

(b) $\delta(x+x_1) \otimes f(x+x_2)$

$= \int \delta(\alpha+x_1)\, f(x-(\alpha-x_2))\, d\alpha$

$= \int f(x-(\alpha-x_2))\, \delta(\alpha-(-x_1)) =$

from sifting property

$= f(x-(-x_1-x_2))$

$= f(x+x_1+x_2)$

(c) $\delta\left(\frac{x-x_1}{b}\right) \circledast f\left(\frac{x-x_2}{d}\right)$

$$= \int \delta\left(\frac{\alpha-x_1}{b}\right) f\left(\frac{x-\alpha-x_2}{d}\right) d\alpha$$

$$\frac{\alpha}{b} = \mu$$

$$\alpha = b\mu$$

$$d\alpha = |b| d\mu$$

$$= \int \delta\left(\frac{b\mu-x_1}{b}\right) f\left(\frac{x-b\mu-x_2}{d}\right) b\,d\mu$$

$$= |b| \int f\left(\frac{x-b\mu-x_2}{d}\right) \delta\left(\mu-\frac{x_1}{b}\right) d\mu$$

By sifting property

$$= |b| \quad f\left(\frac{x-\cancel{b}\,x_1/\cancel{b}-x_2}{d}\right) \quad = |b|\, f\left(\frac{x-x_1-x_2}{d}\right)$$

Q.2(a) Convolution and sketch

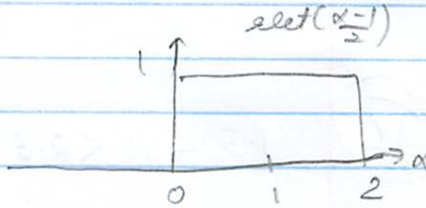
(a) $\left[\text{rect}\left(\frac{x-1}{2}\right) + \delta(x-1)\right] \otimes \text{rect}(x)$

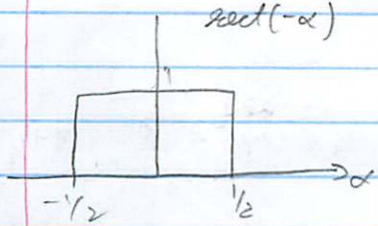
$\text{rect}\left(\frac{x-1}{2}\right) \otimes \text{rect}(x) + \delta(x-1) \otimes \text{rect}(x)$

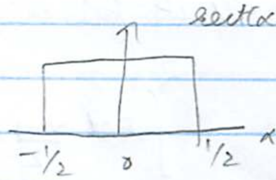
$$\underbrace{\int \text{rect}\left(\frac{\alpha-1}{2}\right) \text{rect}(x-\alpha)\, d\alpha}_{\mu(x)} + \int \delta(\alpha-1)\, \text{rect}(x-\alpha)\, d\alpha$$

$\mu(x) \quad + \quad \text{rect}(x-1)$

$\mu(x) = \int \text{rect}\left(\frac{\alpha-1}{2}\right) \text{rect}(x-\alpha)\, d\alpha$





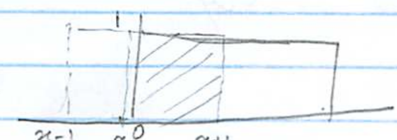


Now for rect$(x-\alpha)$

(I) $x \le -\frac{1}{2}$

no overlap $\quad \mu(x) = 0$

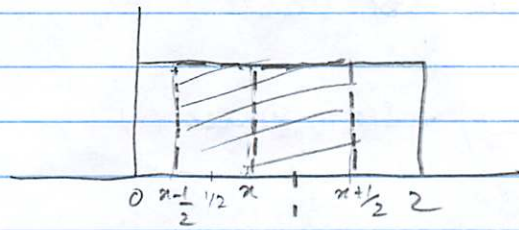
II $-\frac{1}{2} < x < \frac{1}{2}$



limits of integration for overlap region

$$\int_0^{x+\frac{1}{2}} \text{rect}\left(\frac{\alpha-1}{2}\right) \text{rect}(x-\alpha)\, d\alpha = x + \frac{1}{2} - 0 = x + \frac{1}{2}$$

III $\frac{1}{2} < x < \frac{3}{2}$



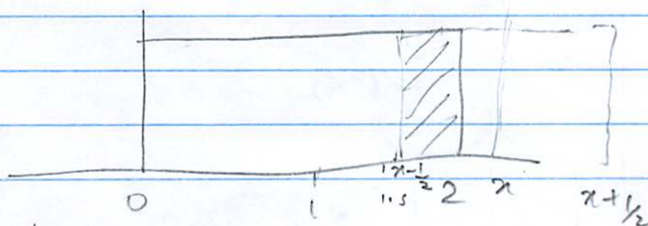
limits of integration for the overlap region

$$= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} rect\left(\frac{\alpha-1}{2}\right) rect(x-\alpha)\, d\alpha$$

$$= x+\frac{1}{2} - \left(x-\frac{1}{2}\right) = \cancel{x}+\frac{1}{2} - \cancel{x}+\frac{1}{2} = \underline{\underline{1}}$$

IV $1.5 < x < 2.5$

limits for integration



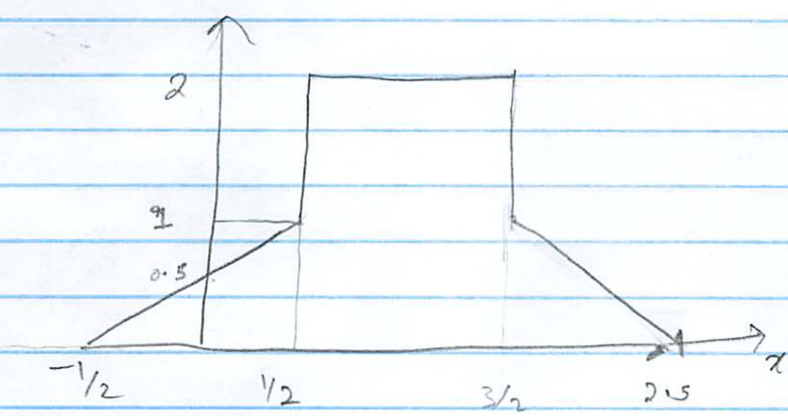
$$\int_{x-\frac{1}{2}}^{2} rect\left(\frac{\alpha-1}{2}\right) rect(x-\alpha)\, d\alpha$$

$$2 - \left(x-\frac{1}{2}\right) = 2 - x + \frac{1}{2} = \underline{\underline{2.5 - x}}$$

V $x > 2.5$ no overlap

So Convolution (contribution of rect$(x-1)$ will be added)

$x \leq -\frac{1}{2}$	0
$-\frac{1}{2} < x < \frac{1}{2}$	$x + 0.5$
$\frac{1}{2} < x < \frac{3}{2}$	$1 + 1 = 2$ (1 → contribution from rect$(x-1)$)
$1.5 < x < 2.5$	$2.5 - x$
$x > 2.5$	0

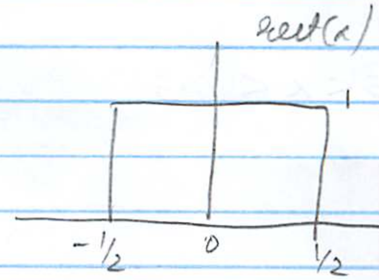


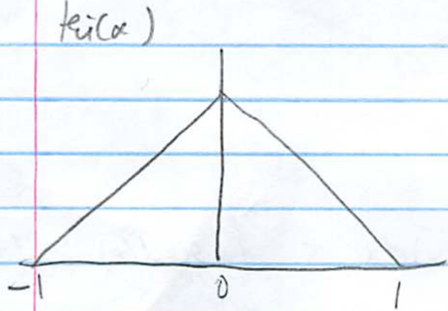
Q.2

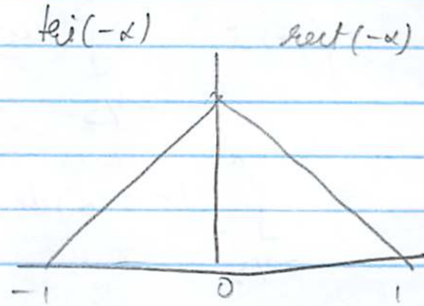
(b) Convol

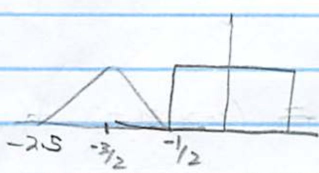
$\text{rect}(x) * \text{tri}(x)$

$$f(x) = \int \text{rect}(\alpha)\, \text{tri}(x-\alpha)\, d\alpha$$

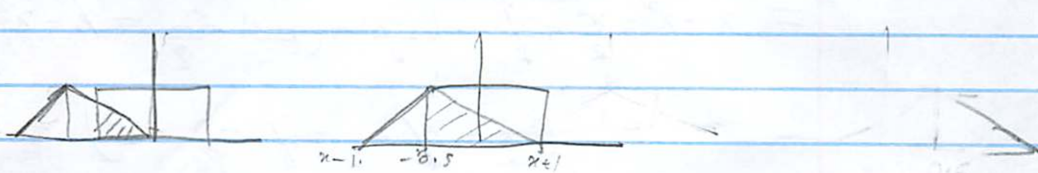








(I) $x < -1.5$ no overlap $f(x) = 0$



(II) $-1.5 < x < -0.5$

$$\int_{-0.5}^{x+1} \text{rect}(\alpha)\, \text{tri}(x-\alpha)\, d\alpha$$

as the limit is till the side of the tri with negative slope

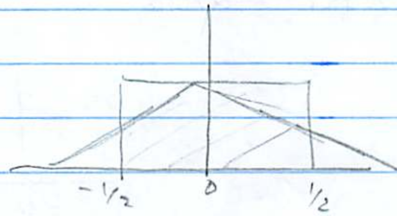
$$= \int_{-0.5}^{x+1} (1+x-\alpha)\, d\alpha = (1+x)\alpha - \frac{\alpha^2}{2}\Big|_{-0.5}^{x+1} = (1+x)(x+1) - \frac{(x+1)^2}{2} - \left(\frac{-1}{2}(1+x) - \frac{0.25}{2}\right)$$

$$= (x+1)^2 - \frac{(x+1)^2}{2} + \frac{(x+1)}{2} + \frac{0.25}{2}$$

$$= \frac{(x+1)^2 + (x+1) + 0.25}{2} = (x^2 + 1 + 2x + x + 1 + 0.25)/2$$

$$= \frac{(x^2 + 3x + 2.25)}{2}$$

III $-0.5 < x < 0.5$



Integration limit.

$$f(x) = \int_{-1/2}^{x} rect(\alpha)\, tri(x-\alpha)\, d\alpha + \int_{x}^{1/2} rect(\alpha)\, tri(x-\alpha)\, d\alpha$$

$$= \int_{-1/2}^{x} (1 - x + \alpha)\, d\alpha + \int_{x}^{1/2} (1 + x - \alpha)\, d\alpha$$

$$= (1-x)\alpha + \frac{\alpha^2}{2}\Big|_{-1/2}^{x} + (1+x)\alpha - \frac{\alpha^2}{2}\Big|_{x}^{1/2} = (1-x)x + \frac{x^2}{2} + \frac{(1-x)}{2} - \frac{1}{8}$$

$$+ \frac{(1+x)}{2} - \frac{1}{8} - (1+x)x + \frac{x^2}{2}$$

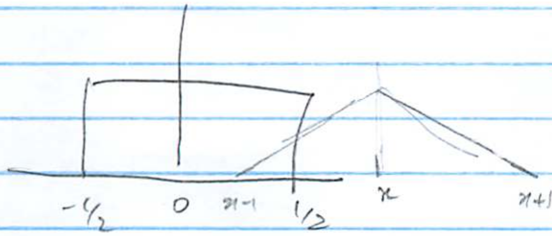
$$= \cancel{x} - \cancel{x^2} + \frac{\cancel{x^2}}{\cancel{2}} + \frac{\cancel{x^2}}{\cancel{2}} + \left(\frac{1-x}{2}\right) + \left(\frac{1+x}{2}\right) - \frac{2}{8} - \cancel{x} - x^2$$

$$= \left(\frac{1}{2} - \frac{\cancel{x}}{\cancel{2}} + \frac{1}{2} + \frac{\cancel{x}}{\cancel{2}} - \frac{2}{8_4} - x^2\right) = 1 - \frac{1}{4} - x^2 = \frac{3}{4} - x^2$$

$\frac{1}{2} < x < 1.5$

(IV)

$\frac{1}{2} < x < \frac{3}{2}$



$$\int_{x-1}^{1/2} \mathrm{rect}(\alpha)\,\mathrm{tri}(x-\alpha)\,d\alpha = \int_{x-1}^{1/2} (1+\alpha-x)\,d\alpha$$

$$= (1-x)\alpha + \frac{\alpha^2}{2} \Big|_{x-1}^{1/2}$$

$$= (1-x)\frac{1}{2} + \frac{1}{8} - (1-x)(x-1) - \frac{(x-1)^2}{2}$$

$$= \frac{(1-x)}{2} + \frac{1}{8} + (x-1)^2 - \frac{(x-1)^2}{2}$$

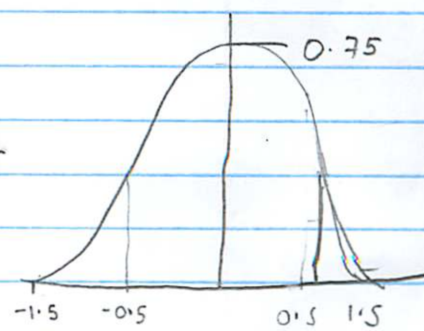
$$= \frac{(1-x)}{2} + \frac{1}{8} + \frac{(x-1)^2}{2}$$

$$= \frac{(1-x+x^2+1-2x)}{2} + \frac{1}{8} = \frac{(x^2-3x+1)}{2} + \frac{1}{8}$$

$$= \cancel{x}$$

(V) $x > \frac{3}{2}$ $\to$ no overlap. $f(x) = 0$

$$f(x) = \begin{cases} 0 & x < -\frac{3}{2} \\ \frac{(x^2+3x+2.25)}{2} & -1.5 < x < -0.5 \\ \frac{3}{4} - x^2 & -0.5 < x < 0.5 \\ \frac{(x^2-3x+2)}{2} + \frac{1}{8} & 0.5 < x < 1.5 \\ 0 & x > 1.5 \end{cases}$$



Q.2
(c)

$$tri(x) * (\delta(x-2) + \delta(x+2))$$

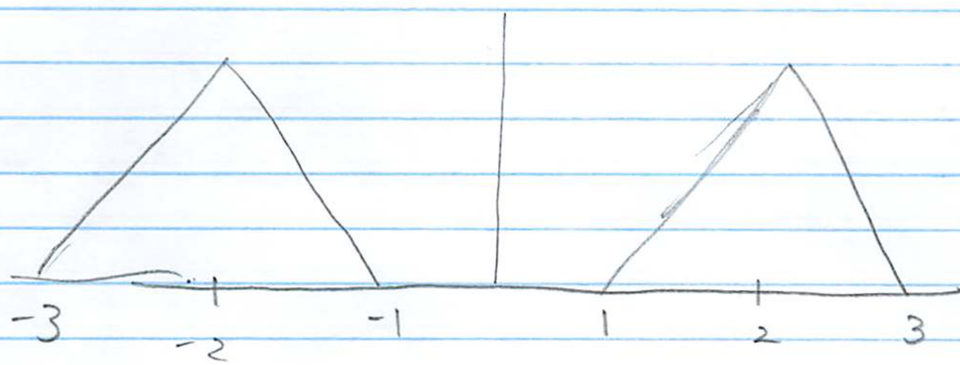
$$= tri(x) * \delta(x-2) \quad + \quad tri(x) * \delta(x+2)$$

$$= \int tri(\alpha)\, \delta(x-(\alpha+2))\, d\alpha \quad + \quad \int \delta(\alpha+2)\, tri(x-\alpha)\, d\alpha$$

$$= \int tri(\alpha)\, \delta(-\alpha + (x-2)) \quad + \int tri(x-\alpha)\, \delta(\alpha - (-2))\, d\alpha$$

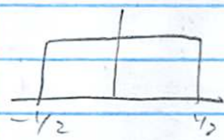
$$= \int tri(\alpha)\, \delta(-(\alpha-(x-2)) \quad + \quad tri(x-(-2)))$$

$$= tri(x-2) \quad + \quad tri(x+2)$$

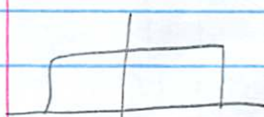


Q. 3 autocorrelation of rect(x)

$$\gamma_{ff}(x) = \int f(x)\, f^*(\alpha - x)\, d\alpha$$

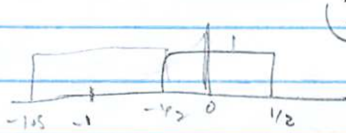


$$\int rect(\alpha)\, rect(\alpha - x)\, d\alpha$$



(I) $x < -1$ no overlap

(II) $-1 < x < 0$ $\displaystyle\int_{-1/2}^{x+1/2} rect(\alpha)\, rect(\alpha - x)\, d\alpha$



$$= x + \frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$= x + 1$$

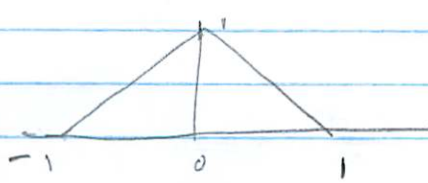
III $0 < x < 1$

Integration limit $\left[\left(x - \frac{1}{2}\right), \frac{1}{2}\right)$

$$\int_{x-\frac{1}{2}}^{1/2} rect(\alpha)\, rect(\alpha - x)\, d\alpha$$

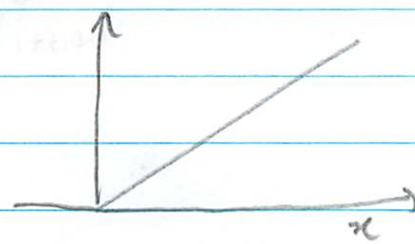
$$= \frac{1}{2} - x + \frac{1}{2} = 1 - x$$

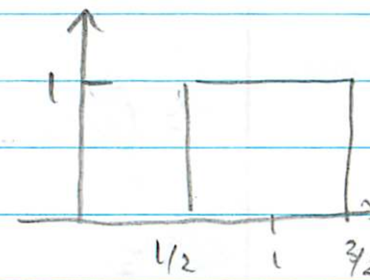
IV $x > 1$ no overlap

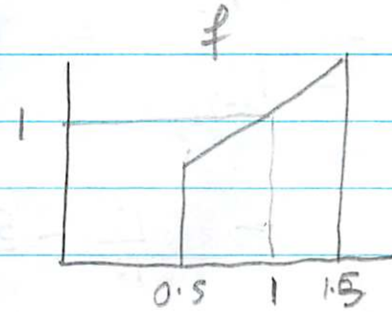


Q.3

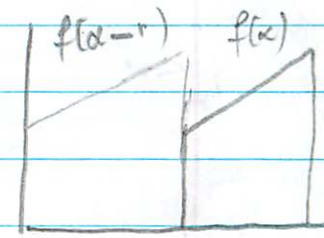
(b) $[\text{ramp}(x)\,\text{rect}(x-1)] = f(x)$

f(x) 



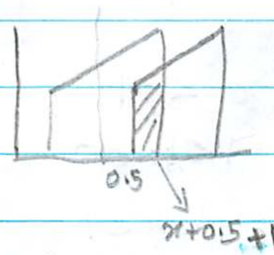
f(x) 

$$r_{ff} = \int_{-\infty}^{\infty} f(\alpha)\, f^*(\alpha - x)\, d\alpha$$



$x < -1$ no overlap $r_{ff} = 0$

$-1 < x < 0$



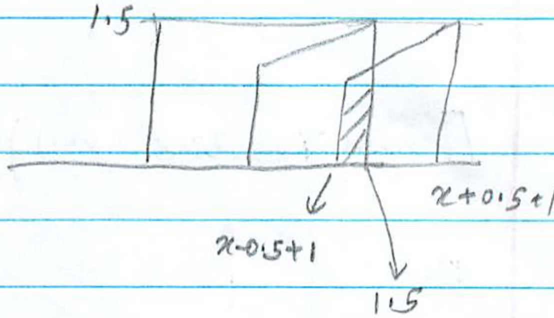
Integration limit

$0.5 \to x + 0.5 + 1$

$$\int_{0.5}^{x+0.5+1} \alpha(\alpha - x)\,d\alpha = \int_{0.5}^{x+0.5} \alpha^2 - x\alpha = \frac{\alpha^3}{3} - \frac{x\alpha^2}{2}\Big|_{0.5}^{x+0.5}$$

$$\frac{(x+1.5)^3}{3} - \frac{x(x+1.5)^2}{2} - \left[\frac{(0.5)^3}{3} - \frac{x(0.5)^2}{2}\right] = \frac{13}{12} + \frac{5x}{4} - \frac{x^3}{6}$$

II $0 < x < 1$

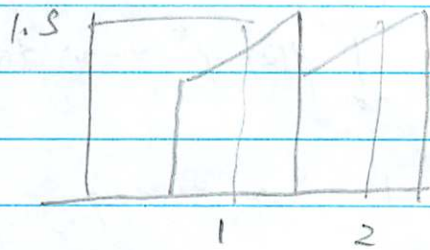


$$= \int_{x+\frac{1}{2}}^{1.5} (\alpha^2 - \alpha x)\, d\alpha$$

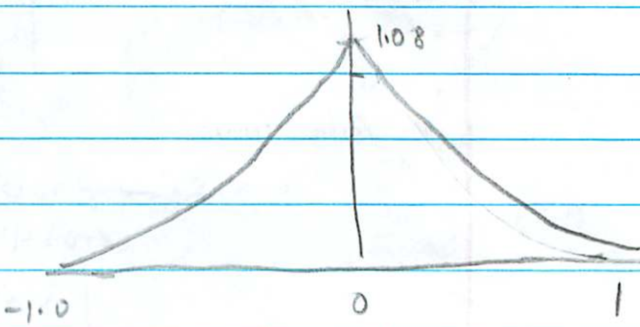
$$= \frac{\alpha^3}{3} - \frac{\alpha^2 x}{2} \Big|_{x+1/2}^{3/2} = \frac{(1.5)^3}{3} - \frac{(1.5)^2 x}{2} - \left[\frac{(x+0.5)^3}{3} - \frac{(x+0.5)^2 x}{2} \right]$$

$$= \frac{13}{12} - \frac{5x}{4} + \frac{x^3}{6}$$

$1 < x$ no overlap



10 γ_{ff}
$$\begin{cases} 0 & x < -1 \\ \frac{13}{12} + \frac{5x}{4} - \frac{x^3}{6} & -1 < x < 0 \\ \frac{13}{2} - \frac{5x}{4} + \frac{x^3}{6} & 0 < x < 1 \\ 0 & 1 < x \end{cases}$$



Q. 4

(a) Show the following

$$f(x) \star g(x) = f(x) \circledast g^*(-x)$$

$$f(x) \star g(x) = \int_{-\infty}^{\infty} f(\alpha)\, g^*(\alpha - x)\, d\alpha$$

$$= \int_{-\infty}^{\infty} f(\alpha)\, g^*\left(\frac{x-\alpha}{-1}\right) d\alpha \quad (1)$$

By the convolution ~~theorem~~

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha)\, g(x-\alpha)\, d\alpha \quad (2)$$

from (1) & (2)

$$f(x) \star g(x) = \int_{-\infty}^{\infty} f(x)\, g^*\left(\frac{x-\alpha}{-1}\right)$$

$$= f(x) * g^*\left(\frac{x}{-1}\right)$$

$$= f(x) * g^*(-x)$$

(b) $\gamma_{ff}(x) = f(x) \star f(x)$

Show that (i) $\gamma_{ff}(x) = \gamma_{ff}(-x)$

(ii) $\gamma_{ff}(x) = \gamma^{*}_{ff}(x)$

auttocorrelat function

$$\gamma_{ff}(x) = \int f(\alpha) f^{*}(\alpha - x) d\alpha$$

Now given $f(x)$ is even and real

so $f^{*}(x) = f(x)$

and $f(x) = f(-x)$

$$\gamma_{ff}(-x) = f(-x) \star f(-x)$$

$$= \int f(-\alpha) f^{*}(\alpha - (\alpha - x)) d\alpha$$

$$= \int f(\alpha) f^{*}(\alpha - x) d\alpha$$

$$= \gamma_{ff}(x)$$

(ii) $\gamma^{*}_{ff}(x) = \left[f(x) \star f^{*}(x) \right]^{*}$

$$= \left[\int f(\alpha) f^{*}(\alpha - x) d\alpha \right]^{*}$$

as $f(x)$ is real

$f(x) = f^{*}(x)$ ①

$$= \left[\int f^{*}(\alpha) \left(f^{*}(\alpha - x) \right)^{*} d\alpha \right]$$

$$= \left[\int f^{*}(\alpha) f(\alpha - x) d\alpha \right]$$ from ①

$$\gamma^{*}_{ff}(x) = \left(\int f(\alpha) f(\alpha - x) d\alpha \right) = \gamma_{ff}(x)$$

(c) If $\gamma_{ff}(x) = f(x) \star f(x)$ then $|\gamma_{ff}(x)| \leq \gamma_{ff}(0)$

To prove this we make use of Schwarz's Inequality

$g(\alpha)$ & $h(\alpha)$ → complex valued functions

$$\left| \int_a^b g(\alpha)\, h(\alpha)\, d\alpha \right| \leq \left[\int_a^b |g(\alpha)|^2 d\alpha \right]^{1/2} \left[\int_a^b |h(\alpha)|^2 d\alpha \right]^{1/2}$$

Setting $g(\alpha) = f(\alpha)$

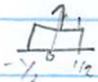
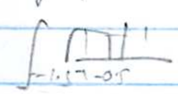
$h(\alpha) = f^*(\alpha - x)$ we get

$$\left| \int_a^b f(\alpha)\, f^*(\alpha - x)\, d\alpha \right| \leq \left[\int_a^b |f(\alpha)|^2 d\alpha \right] \left[\int_a^b |f^*(\alpha - x)|^2 d\alpha \right]^{1/2}$$

setting the limits to $(-\infty, \infty)$

$$|\gamma_{ff}(x)| \leq \left[\int_{-\infty}^{\infty} |f(\alpha)|^2 d\alpha \right]^{1/2} \left[\int_{-\infty}^{\infty} |f^*(\alpha - x)|^2 \right]^{1/2} \quad (1)$$

Now as limits extend from $-\infty$ to ∞, a shifted function would have the same result (i.e the same area under graph

i.e integral for eg $\int$  $=$ $\int$ 

so $\left[\int_{-\infty}^{\infty} |f(\alpha)|^2 d\alpha \right]^{1/2} = \left[\int_{-\infty}^{\infty} |f^*(\alpha - x)|^2 d\alpha \right]^{1/2}$ Also as $f(x)$ is real $f^*(x) = f(x)$

So (1) becomes $|\gamma_{ff}(x)| \leq \int_{-\infty}^{\infty} |f(\alpha)|^2 d\alpha$ | $|f^*|^2 = |f|^2$, $|f|^2 = ff^*$

$\Rightarrow |\gamma_{ff}(x)| \leq \int_{-\infty}^{\infty} f(\alpha)\, f^*(\alpha - 0)\, d\alpha \Rightarrow |\gamma_{ff}(x)| \leq \gamma_{ff}(0)$