ASTR/OPTI 428/528 Part 1
Wave Propagation in Random Media
Lecture 4: Field Statistics and Propagation I

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A Random Electric Field

Consider a random electric field: in $\Psi(x,y,z)$.

- Start with a plane wave propagating in the $+\hat{z}$ direction.
- Even when light propagates for long distances through strong turbulence, the scattering angles are small.
- We will use the convention that the main propagation is in the z direction and the transverse direction is

$$\mathbf{x} = (x,y).$$

- A plane wave propagating along z is $\mathrm{e}^{i(\omega t - kz)}$, where $k = 2\pi/\lambda$ and $c = \omega/k$.
- We can write a more general scattered field as the main plane wave along z and a much slower-varying complex amplitude

$$\Psi(\mathbf{x},z,t) = \psi(\mathbf{x},z,t)\mathrm{e}^{i(\omega t - kz)}$$

We will study the complex amplitude $\psi(\mathbf{x},z,t)$.

Studying the field in a plane

Let's examine the demodulated field in the $z = 0$ plane.

- The modulation doesn't affect the intensity, so we can pull it out and ignore it.
- We can write the complex amplitude as a phase and an amplitude:

$$\psi(\mathbf{x}, z = 0, t) = a(\mathbf{x}, t)\mathrm{e}^{i\phi(\mathbf{x},t)}$$

How can we study this field?

- Interfere the field with itself at 2 points using an interferometer.
- Look at the incoming light using a telescope.
- Measure the irradiance over the $\mathbf{x}$ plane using a camera.
- Look at the correlation of irradiance over the $\mathbf{x}$ plane.

Fluctuating Interferometer output

Consider an interferometer that adds the field from two points in the transverse plane and computes the resulting intensity:

$$
\begin{aligned}
J_{out}(\mathbf{x}_1,\mathbf{x}_2,t) &= |\psi(\mathbf{x}_1,t)+\psi(\mathbf{x}_2,t)|^2 \\
&= \left(\psi(\mathbf{x}_1,t)+\psi(\mathbf{x}_2,t)\right)\left(\psi(\mathbf{x}_1,t)+\psi(\mathbf{x}_2,t)\right)^* \\
&= |\psi(\mathbf{x}_1,t)|^2+|\psi(\mathbf{x}_2,t)|^2+\psi(\mathbf{x}_1,t)\psi^*(\mathbf{x}_2,t)+\psi(\mathbf{x}_1,t)^*\psi(\mathbf{x}_2,t) \\
&= I(\mathbf{x}_1,t)+I(\mathbf{x}_2,t)+2\Re\left\{\psi(\mathbf{x}_1,t)\psi^*(\mathbf{x}_2,t)\right\}.
\end{aligned}
$$

Average Interferometer Output

This output may change with time and we want to measure the average value.

- Use a time average

$$\langle J_{out}(\mathbf{x}_1,\mathbf{x}_2)\rangle = \frac{1}{T}\int_0^T J_{out}(\mathbf{x}_1,\mathbf{x}_2,t)\mathrm{d}t$$

- Or use an ensemble average. If the problem is *ergodic*, the averages are the same.

$$\langle J_{out}(\mathbf{x}_1,\mathbf{x}_2)\rangle = \langle I(\mathbf{x}_1)\rangle + \langle I(\mathbf{x}_2)\rangle + 2\Re\langle \psi(\mathbf{x}_1)\psi^*(\mathbf{x}_2)\rangle .$$

If the average irradiance does not vary across the observation plane

$$\langle J_{out}(\mathbf{x}_1,\mathbf{x}_2)\rangle = 2\langle I\rangle + 2\Re\langle \psi(\mathbf{x}_1)\psi^*(\mathbf{x}_2)\rangle .$$

Mutual Coherence Function

We can therefore use an interferometer to measure the second moment of the field, also known as the *Mutual Coherence Function* (MCF)

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2) = \langle \psi(\mathbf{x}_1) \psi^*(\mathbf{x}_2) \rangle$$

Looking again at the average output of the interferometer...

$$\langle J_{out}(\mathbf{x}_1, \mathbf{x}_2) \rangle = 2 \langle I \rangle + 2\Re \{ \Gamma(\mathbf{x}_1, \mathbf{x}_2) \} .$$

Introducing a $\pi/2$ phase shift in the interferometer gives the imaginary part, allowing us to measure the full complex MCF.

Average telescope image

We will use Fourier Optics to study the effect of imaging a random field.
The field in the image plane is the Fourier transform of the field seen through the pupil $\Pi(\mathbf{x})$

$$\Psi(k\boldsymbol{\theta},t) = \int \mathrm{d}^2x\, \mathrm{e}^{ik\boldsymbol{\theta}\cdot\mathbf{x}}\Pi(\mathbf{x})\psi(\mathbf{x},t)$$

The instantaneous intensity in the image plane is

$$\Phi(k\boldsymbol{\theta},t) = |\Psi(k\boldsymbol{\theta},t)|^2$$

$$\Phi(k\boldsymbol{\theta},t) = \int \mathrm{d}^2x_1 \int \mathrm{d}^2x_2\, \mathrm{e}^{ik\boldsymbol{\theta}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\Pi(\mathbf{x}_1)\Pi^*(\mathbf{x}_2)\psi(\mathbf{x}_1,t)\psi^*(\mathbf{x}_2,t)$$

The average (long exposure) intensity is

$$\langle\Phi(k\boldsymbol{\theta})\rangle = \int \mathrm{d}^2x_1 \int \mathrm{d}^2x_2\, \mathrm{e}^{ik\boldsymbol{\theta}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\Pi(\mathbf{x}_1)\Pi^*(\mathbf{x}_2)\,\langle\psi(\mathbf{x}_1)\psi^*(\mathbf{x}_2)\rangle$$

Randomly-fluctuating wavefront (random phase)

We will cover this again later. This is just an introduction...
If we can ignore the field's amplitude variations (i.e. ignore scintillation), we can write the field as

$$\psi(\mathbf{x}) = a e^{i\varphi(\mathbf{x})}.$$

Both the interferometer output and the average image depend on the *Mutual Coherence Function* (MCF).

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2) = \langle \psi(\mathbf{x}_1) \psi^*(\mathbf{x}_2) \rangle .$$

Using our fluctuating phase expression, we can write the MCF as

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2) = a^2 \langle \exp\left[i \left(\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2) \right) \right] \rangle .$$

Introducing the Phase Structure Function

If we can justify assuming the *difference* between the phase fluctuations at two points is a zero-mean Gaussian random process, we can use our result

$$\left\langle e^{iq} \right\rangle = e^{-\left\langle q^2 \right\rangle/2}$$

to write

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2) = a^2 \exp\left[-\left\langle (\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2))^2 \right\rangle /2 \right].$$

This mean-square difference of the phase at two points is called the "Phase Structure Function"

$$D_\varphi(\mathbf{x}_1, \mathbf{x}_2) = \left\langle (\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2))^2 \right\rangle.$$

This gives our simple MCF result as

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2) = a^2 \exp\left[-D_\varphi(\mathbf{x}_1, \mathbf{x}_2)/2 \right].$$

Irradiance statistics

Fluctuating Irradiance

$$I(\mathbf{x},t) = |\psi(\mathbf{x},t)|^2$$

Mean Irradiance

$$\langle I(\mathbf{x},t)\rangle = \langle \psi(\mathbf{x},t)\psi^*(\mathbf{x},t)\rangle = \Gamma(\mathbf{x},\mathbf{x})$$

Variance of the irradiance

$$\mathrm{var}\,(I(\mathbf{x},t)) = \left\langle (I - \langle I(\mathbf{x},t)\rangle)^2 \right\rangle = \sigma^2$$

Scintillation Index

$$m^2 = \frac{\sigma^2}{\langle I\rangle^2}$$

Irradiance autocovariance

The fluctuating irradiance can be studied by correlating the fluctuations at two points. This is usually studied using the autocovariance of the irradiance, $\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2)$:

$$\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2) = \langle (I(\mathbf{x}_1) - \langle I(\mathbf{x}_1) \rangle) (I(\mathbf{x}_2) - \langle I(\mathbf{x}_2) \rangle) \rangle$$

$$\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2) = \langle I(\mathbf{x}_1) I(\mathbf{x}_2) \rangle - \langle I(\mathbf{x}_1) \rangle \langle I(\mathbf{x}_2) \rangle$$

If the mean irradiance is independent of position, $\langle I(\mathbf{x}_1) \rangle = \langle I \rangle$, and we can simply write

$$\sigma_I^2(\mathbf{x}_1, \mathbf{x}_2) = C_I(\mathbf{x}_1, \mathbf{x}_2) - \langle I \rangle^2 .$$

Irradiance autocorrelation

This last expression relates the autocovariance to the irradiance autocorrelation

$$C_I(\mathbf{x}_1, \mathbf{x}_2) = \langle I(\mathbf{x}_1) I(\mathbf{x}_2) \rangle .$$

The irradiance is $I = \psi\psi^*$, so we can write

$$\langle I(\mathbf{x}) \rangle = \langle \psi(\mathbf{x}) \psi^*(\mathbf{x}) \rangle = \Gamma(\mathbf{x}, \mathbf{x})$$

and

$$C_I(\mathbf{x}_1, \mathbf{x}_2) = \langle \psi(\mathbf{x}_1) \psi^*(\mathbf{x}_1) \psi(\mathbf{x}_2) \psi^*(\mathbf{x}_2) \rangle .$$

This is a “Fourth Moment” of the random field, and is significantly more complicated than the second moment $\Gamma(\mathbf{x}_1, \mathbf{x}_2)$.

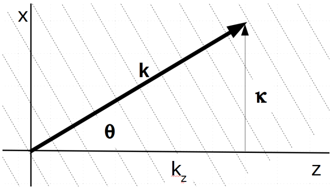
Fourth moments are important in the study of scintillation and things like the correlation of wavefront tilt, focal plane speckle statistics, etc.

Field Statistics

The lesson here is that we can study many aspects of the random field by measuring second and fourth moments of the field.

- Average PSFs and Seeing are described by second moments of the field.
- Scintillation and noise or variance estimates on irradiance-related measurements are often fourth moments of the field.

A General Plane Wave



A plane wave propagating in the direction of the wavevector $\mathbf{k} = (\boldsymbol{\kappa}, k_z)$, $|\mathbf{k}| = k = 2\pi/\lambda$, is written

$$\Psi = \psi e^{i(\omega t - \mathbf{k}\cdot(\mathbf{x},z))}.$$

The amplitude is complex and contains the phase of the plane wave.

Demodulating a Plane Wave

Recall from last time that we want to "demodulate" waves by taking out the "fast" variation in the z direction

$$e^{i(\omega t - kz)}.$$

Note that the k we use here is $k = 2\pi/\lambda$ which is only the correct value of k_z for a plane wave propagating along the z axis. Otherwise it is too small.

$$\begin{aligned}
e^{i(\omega t - \mathbf{k}\cdot(\mathbf{x},z))} &= e^{i(kz - (\boldsymbol{\kappa},k_z)\cdot(\mathbf{x},z))}\, e^{i(\omega t - kz)} \\
&= e^{i(kz - \boldsymbol{\kappa}\cdot\mathbf{x} - k_z z)}\, e^{i(\omega t - kz)} \\
&= \underbrace{e^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}}_{\text{demodulated field}}\; \underbrace{e^{-i(k-k_z)z}}_{\text{propagator}}\; \underbrace{e^{i(\omega t - kz)}}_{\text{reference field}}
\end{aligned}$$

The wavevector components

The overall wavenumber k, and the wavevector components k_z and $\boldsymbol{\kappa}$ are related by Pythagoras

$$k^2 = k_z^2 + \kappa^2,$$

allowing us to write the z component of the wavevector as

$$k_z = \sqrt{k^2 - \kappa^2}.$$

We can now pull out a factor of k

$$k_z = k\sqrt{1 - \kappa^2/k^2}.$$

Terminology: *We will often call the transverse wavevector* $\boldsymbol{\kappa}$ *the "spatial frequency" of the plane wave.*

Small off-axis angles

The propagation angle can be written

$$\sin \boldsymbol{\theta} \equiv (\sin \theta_x, \sin \theta_y) = \frac{\boldsymbol{\kappa}}{k}.$$

If the angle off the z axis is very small (as it is in our scattering problems), we can just use the small-angle approximation

$$\sin \boldsymbol{\theta} = \frac{\boldsymbol{\kappa}}{k} \approx \boldsymbol{\theta} \;\Rightarrow\; \boldsymbol{\kappa} = k\boldsymbol{\theta}.$$

The Paraxial Approximation

For small angles, k_z can be approximated

$$k_z = k\sqrt{1 - \kappa^2/k^2}$$

$$k_z \approx k\left(1 - \frac{\kappa^2}{2k^2}\right)$$

$$k_z \approx k - \frac{\kappa^2}{2k}$$

The Paraxial Plane Wave Propagator

The propagator for our demodulated plane wave was

$$\mathrm{e}^{-i(k-k_z)z} = \mathrm{e}^{-i\kappa^2 z/2k}.$$

That is, a plane wave propagating nearly parallel with the z axis with a spatial frequency $\boldsymbol{\kappa}$, can be propagated by

$$\mathrm{e}^{-i\boldsymbol{\kappa}\cdot\mathbf{x}} \longrightarrow \mathrm{e}^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}\mathrm{e}^{-i\kappa^2 z/2k}.$$

This obviously works the same for a plane wave with any complex amplitude.

A sum of plane waves (Fourier synthesis)

We can write the field[1] as a sum of plane waves at different angles and with different complex amplitudes

$$\psi(\mathbf{x}) = \int \mathrm{d}^2\kappa\, \widetilde{\psi}(\boldsymbol{\kappa}) \mathrm{e}^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}$$

where $\widetilde{\psi}(\boldsymbol{\kappa})$ is the complex amplitude of the plane wave traveling in the direction $\boldsymbol{\theta} = \boldsymbol{\kappa}/k$.

This is also a Fourier transform.

This allows us to *decompose* an arbitrary field $\psi(\mathbf{x})$ into a set of plane wave complex amplitudes as

$$\widetilde{\psi}(\boldsymbol{\kappa}) = \frac{1}{(2\pi)^2} \int \mathrm{d}^2x\, \psi(\mathbf{x}) \mathrm{e}^{i\boldsymbol{\kappa}\cdot\mathbf{x}}.$$

[1] We always mean the demodulated field from here on.

Free-Space Paraxial Wave Propagation

How do we propagate a general field?

Starting with the field $\psi(\mathbf{x}, z)$ in the $z = 0$ plane, we

1. Decompose it into plane waves with different angles (equivalent to saying spatial frequencies).

$$\widetilde{\psi}(\boldsymbol{\kappa}, 0) = \frac{1}{(2\pi)^2} \int \mathrm{d}^2x\, \psi(\mathbf{x}, 0) \mathrm{e}^{i\boldsymbol{\kappa}\cdot\mathbf{x}}$$

2. Propagate each plane wave by multiplying each complex amplitude by the appropriate propagator $\mathrm{e}^{-i\kappa^2 z/2k}$.
3. Re-synthesize the field by adding up the propagated plane waves.

$$\psi(\mathbf{x}, z) = \int \mathrm{d}^2\kappa\, \widetilde{\psi}(\boldsymbol{\kappa}, 0) \mathrm{e}^{-i\kappa^2 z/2k} \mathrm{e}^{-i\boldsymbol{\kappa}\cdot\mathbf{x}}$$

Free-Space Propagation in Phase Space

We can write the propagation step as two integrals, one over space and one over spatial frequency (“phase space propagator”)

$$\boxed{\psi(\mathbf{x},z) = \frac{1}{(2\pi)^2}\int \mathrm{d}^2\kappa \int \mathrm{d}^2x'\, \mathrm{e}^{-i\kappa^2 z/2k}\mathrm{e}^{i\boldsymbol{\kappa}\cdot(\mathbf{x}'-\mathbf{x})}\psi(\mathbf{x}',0)}$$

Phase space is usually thought of as a description that talks about position and angle at the same time.
We can do this by replacing $\boldsymbol{\kappa} = k\boldsymbol{\theta}$:

$$\psi(\mathbf{x},z) = \frac{1}{\lambda^2}\int \mathrm{d}^2\theta \int \mathrm{d}^2x'\, \mathrm{e}^{-ikz\theta^2/2}\mathrm{e}^{ik\boldsymbol{\theta}\cdot(\mathbf{x}'-\mathbf{x})}\psi(\mathbf{x}',0)$$

Free-Space Green's Function

Or we can do the $\boldsymbol{\kappa}$ integral first, leaving us with just an integral over the initial plane

$$\psi(\mathbf{x},z) = \int \mathrm{d}^2x' \underbrace{\left[\frac{1}{(2\pi)^2}\int \mathrm{d}^2\kappa\, \mathrm{e}^{-i\kappa^2 z/2k}\mathrm{e}^{i\boldsymbol{\kappa}\cdot(\mathbf{x}'-\mathbf{x})}\right]}_{G(\mathbf{x};\mathbf{x}')}\psi(\mathbf{x}',0)$$

where $G(\mathbf{x};\mathbf{x}')$ is the paraxial free-space Green's Function (i.e. the field due to a point source)

$$G(\mathbf{x};\mathbf{x}') = \frac{1}{i\lambda z}\exp\left\{\frac{-i\pi\left(\mathbf{x}-\mathbf{x}'\right)^2}{2\lambda z}\right\}.$$

This gives the propagation as

$$\boxed{\psi(\mathbf{x},z) = \frac{1}{i\lambda z}\int \mathrm{d}^2x' \exp\left\{\frac{-i\pi\left(\mathbf{x}-\mathbf{x}'\right)^2}{\lambda z}\right\}\psi(\mathbf{x}',0)}$$

The Fresnel Scale

Notice that in the free-space propagation equation, the wavelength and the propagation distance always appear together as λz. The phase of the point source response is within $\pi/2$ of the on-axis phase when

$$\frac{\pi\rho^2}{\lambda z} < \frac{\pi}{2}$$

or

$$\rho = \sqrt{\lambda z/2}.$$

This region over which a propagated point source field can reasonably be thought of as being “in phase” is the Fresnel Zone and has the size of the Fresnel Scale, R_f

$$R_f = \sqrt{\lambda z/2}.$$

We will see that this is a key propagation reference scale. If the wavefront is distorted on scales smaller that the Fresnel scale, the scattering is strong.