II. SCINTILLATION THEORY

2.1. Introduction

The theory of wave propagation in random media has been the subject of a great deal of research. In particular, the case of narrow-angle forward scattering ("Scintillation theory") has been the treated from a wide variety of approaches. The basic propagation equation in scintillation theory is the "parabolic wave equation" [Fock, 1950]. Early investigators attempted to find exact solutions for the fluctuating wave field using perturbation techniques [Barabanenkov, *et al*, 1971]. The resulting formulae were only valid in weak scattering. Eventually, a statistical approach superceded exact solutions as the method of choice. Statistical moments of the wave field were propagated using "moment equations" derived from the parabolic wave equation [de Wolf, 1967; Dolin, 1968; Prokhorov, *et al*, 1975]. These equations were greatly simplified by the approximation that the medium's correlation length in the propagation direction was very small compared with the longitudinal correlation length of the electric field (the "Markov approximation") [Klyatskin, 1969, 1975; Klyatskin and Tatarskii, 1971; Lee and Jokipii, 1975; Zavorotnyi, 1978]. An alternative to the moment equations was provided by writing the fluctuating electric field in terms of a Feynman path integral and constructing the statistical moments [Klyatskin and Tatarskii, 1970; Klyatskin, 1973; Dashen, 1979; Tatarskii and Zavorotnyi, 1980]. The two formulations have been shown to be equivalent and give to identical results [Codona, *et al*, 1986a]. Using either approach, the mean electric field (the "first moment") and the monochromatic mutual coherence function (the "second moment") are easily found [Dashen, 1979; Tatarskii and Zavorotnyi, 1980]. However, the two-point intensity correlation, or the more general case of the fourth moment, has not been solved in closed form. In strong scattering, the fourth moment was approximated by an expansion derived from the moment equation [Fante, 1975] or by operator techniques from the path integral form [Tatarskii and Zavorotnyi, 1980]. The result for the intensity correlation was best understood in the spatial-frequency domain where the series expansion built up an approximation from the low spatial-frequencies [Gochelashvily and Shishov, 1971; Gochelashvily, *et al*, 1974]. By approximately summing the higher-order terms in the expansion series, an approximate form for the high spatial-frequency behavior was found [Gochelashvily and Shishov, 1971]. This "high spatial-frequency approximation" was also found directly from the moment equation [Rumsey, 1975; Gochelashvily and Shishov, 1975] and was related to the nearly Gaussian statistics of the field. The technique of summing the expansion series was quite complicated and did not permit high spatial-frequency expansion terms to be readily found. Although the expansion was also carried out for the full fourth moment, the implications of the fourth moment's symmetries were not fully utilized [Codona, *et al*, 1986b].

This chapter contains a presentation of the basic scintillation theory required for general applications. Since the Feynman path integral permits a clear presentation of the averaging and expansion processes, it will be used to propagate the electric field. We begin with a derivation of the parabolic wave equation with a discussion of the assumptions for the propagating field and the random medium. Next, the Feynman path integral is derived from the parabolic wave equation. Feynman's original form of the path integral is used [Feynman, 1948] instead of the functional operator form employed in much of the Russian literature [Tatarskii and Zavorotnyi, 1980]. Two path integral identities are derived which permit the reduction of the complicated path integral formulas to standard integrals. Following Dashen [1979], the general Green's functions for the second and fourth moments are written in terms of multiple path integrals. The second moment Green's function is found exactly and the fourth moment Green's function is reduced to a double path integral. The fourth moment Green's function is expanded in a "low spatial-frequency" expansion and the identities used to evaluate each term. Finally, the symmetry of the fourth moment is used to find a second series, a "high spatial-frequency expansion," for the fourth moment Green's function.

2.2. The Parabolic Wave Equation.

In all of the problems considered in this dissertation, incoming waves are scattered into a narrow angular spectrum by fluctuations in the medium's index of refraction. These fluctuations are very small relative to the local mean, which may be slowly varying. Depolarization of the electric field through interaction with the medium is not included in the calculations. Also neglected are deterministic effects such as Faraday rotation. However, the effect of medium dispersion *is* included to permit the analysis of problems in both neutral atmosphere and plasma environments. Plasma currents may be neglected as long as the field propagating through the plasma does not perturb the current distribution (i.e. non-linear effects are assumed to be negligible) and the currents themselves are not radiating significant electromagnetic waves. Non-linear effects should be negligible if the radiation has a low spatial power density and has a temporal frequency that is much higher than the plasma frequency. We ignore all path radiance effects since they may be treated separately in the linear regimes considered. The equation describing the propagation of waves under such circumstances is the "Parabolic Wave Equation."

For a medium with no free charges and no currents, Maxwell's equations are written

$$\nabla\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t} \tag{2.2.1}$$

$$\nabla\times\vec{\mathbf{B}} = \frac{1}{c^2}\frac{\partial}{\partial t}(n^2\vec{\mathbf{E}}) \tag{2.2.2}$$

$$\nabla\cdot(n^2\vec{\mathbf{E}}) = 0 \tag{2.2.3}$$

$$\nabla\cdot\vec{\mathbf{B}} = 0 \tag{2.2.4}$$

where $\vec{\mathbf{E}}$ is the electric field, $\vec{\mathbf{B}}$ is the magnetic field, c is the speed of light in the average medium, n is the index of refraction normalized to the mean, and $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is the three-dimensional gradient operator. The index of refraction is considered to be a function of both space and time. The propagation equation is found by taking the curl of eq. (2.2.1), applying the identity

$$\nabla\times(\nabla\times\vec{\mathbf{E}}) = -\nabla^2\vec{\mathbf{E}} + \nabla(\nabla\cdot\vec{\mathbf{E}})$$

and using eq. (2.2.3) leading to

$$\nabla^2\vec{\mathbf{E}} - \nabla(\nabla\cdot\vec{\mathbf{E}}) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(n^2\vec{\mathbf{E}}) \tag{2.2.5}$$

From eq. (2.2.3) we find

$$\nabla\cdot\vec{\mathbf{E}} = -2\vec{\mathbf{E}}\cdot\nabla(\ln n) \tag{2.2.6}$$

Since we are only concerned with small fluctuations in the index of refraction away from unity, we define

$$\mu = n - 1 \tag{2.2.7}$$

To first order in μ, eq. (2.2.6) becomes

$$\nabla\cdot\vec{\mathbf{E}} = -2\vec{\mathbf{E}}\cdot\nabla\mu \tag{2.2.8}$$

Inserting eq. (2.2.8) into eq. (2.2.5) and expanding the resultant term gives

$$\nabla^2\vec{\mathbf{E}} + 2\left[(\vec{\mathbf{E}}\cdot\nabla)\nabla\mu + ((\nabla\mu)\cdot\nabla)\vec{\mathbf{E}} + \vec{\mathbf{E}}\times(\nabla\times(\nabla\mu)) + (\nabla\mu)\times(\nabla\times\vec{\mathbf{E}})\right] \tag{2.2.9}$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(n^2\vec{\mathbf{E}})$$

The fourth term on the left-hand side is identically zero. However, the other terms require closer examination.

The characteristic length scale of the electric field is the wavelength, λ. For a realistic random medium, there is also a characteristic length scale. While such a scale may not be defined for a medium with a pure power-law turbulence spectrum, a characteristic length scale may be defined in a medium which possesses both an inner and an outer scale. A typical definition for the characteristic length scale of a random medium might be the correlation length of μ, the size of the inner scale for the irregularities, or perhaps the normalized radius of curvature of the medium autocorrelation function

$$L_\mu^2 \equiv \left. \frac{<\mu^2>}{-\frac{\partial^2}{\partial z^2} < \mu(\vec{x}, 0)\mu(\vec{x}, z)>} \right|_{z=0}$$

Whatever the definition, L_μ represents a length scale over which the index of refraction fluctuation, μ, undergoes a substantial change in its value. The gradient of $\vec{E}$ has a magnitude of roughly E_{rms}/λ while the gradient of μ has a magnitude of roughly μ_{rms}/L_μ where "rms" denotes the root-mean-square value of the quantity. Estimating the typical magnitudes of the terms on the left-hand side of eq. (2.2.9) gives

$$\nabla^2 \vec{E} \approx E_{rms}/\lambda^2$$

$$(\vec{E} \cdot \nabla)\nabla\mu \approx \mu_{rms} E_{rms}/L_\mu^2$$

$$((\nabla\mu) \cdot \nabla)\vec{E} \approx \mu_{rms} E_{rms}/(L_\mu \lambda)$$

$$(\nabla\mu) \times (\nabla \times \vec{E}) \approx \mu_{rms} E_{rms}/(L_\mu \lambda)$$

As long as $\mu_{rms} \ll 1$ and $L_\mu \gg \lambda$, the Laplacian term will be by far the most important.

Keeping only the first term on the left-hand side of eq. (2.2.9) and expanding the right-hand side, leads to

$$\nabla^2 \vec{E} = \frac{1}{c^2} \left[\vec{E} \frac{\partial^2 n^2}{\partial t^2} + 2 \frac{\partial \vec{E}}{\partial t} \frac{\partial n^2}{\partial t} + n^2 \frac{\partial^2 \vec{E}}{\partial t^2} \right] \tag{2.2.10}$$

The characteristic time scale of the electric field is $2\pi/\omega$, where ω is a temporal frequency typical of the radiation. The characteristic time of the medium is defined in a fashion similar to L_μ and is denoted T_μ. Estimates of the magnitudes of the terms in the brackets are given by

$$\vec{E} \frac{\partial^2 n^2}{\partial t^2} \approx \mu_{rms} E_{rms}/T_\mu^2$$

$$\frac{\partial \vec{E}}{\partial t} \frac{\partial n^2}{\partial t} \approx \mu_{rms} E_{rms}\, \omega/(2\pi T_\mu)$$

$$n^2 \frac{\partial^2 \vec{E}}{\partial t^2} \approx E_{rms}\, \omega^2/(2\pi)^2$$

Since $\mu_{rms} \ll 1$ and the medium is changing on a time scale which is long compared to a wave-period, $T_\mu \gg 2\pi/\omega$, the third term is dominant over the other two. Since $n^2 = 1 + 2\mu + \mu^2 \approx 1 + 2\mu$, a bounding argument on $\mu\partial^2 \vec{E}/\partial t^2$ shows that it too is dominant over the two neglected terms. The wave equation has now been reduced to

$$\nabla^2 \vec{E} = \frac{n^2}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad (2.2.11)$$

Note that at this point the vector nature of the electric field is no longer of importance. Therefore, the electric field will be written as a scalar field E. Equation (2.2.11) describes time-dependent waves traveling in any direction.

Equation (2.2.11) may be further simplified by restricting consideration to radiation propagating primarily in a particular direction, say parallel with the z axis. A wave propagating in the z direction through free space with a temporal frequency ω would have the form $2Re\{E e^{i(kz - \omega t)}\}$ where $\omega = ck$, and E is a complex number giving the amplitude and phase of the wave. The time-dependent scattered field in a random medium would have a similar form for each of the constituent temporal frequencies. Since the principle behavior of the electricfield is given by the $e^{i(kz - \omega t)}$ factor, the rest of the function will be relatively slowly varying. This suggests a form for the electric field as

$$\mathrm{E}(\vec{x}, z, t) = \int_{-\infty}^{\infty} e^{i(kz - \omega t)} E(\vec{x}, z, \omega) d\omega \qquad (2.2.12)$$

where $k = \omega/c$ for each value of ω and $E(\vec{x}, z, \omega) = E^*(\vec{x}, z, -\omega)$ since the total electric field must be real. Using inverse Fourier transforms, we find

$$e^{ikz} E(\vec{x}, z, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\, \mathrm{E}(\vec{x}, z, t) dt \qquad (2.2.13)$$

Multiplying eq. (2.2.11) by $e^{i\omega t}/(2\pi)$ and integrating over all time yields

$$\nabla^2(e^{ikz}E) = \frac{1}{2\pi c^2}\int_{-\infty}^{\infty} e^{i\omega t} n^2(\vec{x}, z, t)\frac{\partial^2 E}{\partial t^2}\,dt$$

Integrating the right-hand side twice by parts leads to three terms which may be estimated as above. Since the medium either changes slowly compared to the inverse of all contributing temporal frequencies or the electric field doesn't exist as $t \rightarrow \pm\infty$, the important term is

$$-\frac{\omega^2}{2\pi c^2}\int_{-\infty}^{\infty} e^{i\omega t} n^2(\vec{x}, z, t)E(\vec{x}, z, t)\,dt$$

Moving n^2 outside of the integral is equivalent to ignoring all Doppler shifts due to changes in the medium. Having made this approximation, eq. (2.2.11) reduces to the "Helmholtz Wave Equation" for $e^{ikz}E$

$$\nabla^2(e^{ikz}E) = -n^2k^2Ee^{ikz} \tag{2.2.14}$$

which is true for a given temporal Fourier component of the scalar electric field. The Laplacian may be written

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \vec{x}^2}$$

where $\partial/\partial\vec{x} \equiv (\partial/\partial x, \partial/\partial y)$ is the transverse gradient operator. Making use of this notation and ignoring terms which are quadratic in μ, eq. (2.2.14) reduces to

$$\frac{\partial^2 E}{\partial z^2} + 2ik\frac{\partial E}{\partial z} + \frac{\partial^2 E}{\partial \vec{x}^2} = -2\mu k^2 E \tag{2.2.15}$$

We will now examine the various terms in eq. (2.2.15). The term on the right-hand side contains the only reference to the random medium and must be included. The $\partial^2 E/\partial\vec{x}^2$ term is the source of diffraction and plays a crucial role in the physics. The $2ik\,\partial E/\partial z$ term, when balanced against the medium dependent term, leads to "geometrical optics" and is important in the high frequency limit. The $\partial^2 E/\partial z^2$ term plays an unimportant role in narrow-angle forward scattering and is, in fact, small compared to estimates of $2ik\,\partial E/\partial z$. The reduced electric field , E, hereafter referred to simply as the "electric field," changes only by interaction with the fluctuations in the index of refraction. We will refer to the length scale over which this interaction makes an important change in E as $L_{interaction}$. A reasonable definition for $L_{interaction}$ might be where the longitudinal electric field correlation has reached half of its zero-lag value.

$$< E(z = 0)E^*(z = L_{interaction}) > = \frac{1}{2} \tag{2.2.16}$$

Using a geometrical optics approximation for E

$$E(z) \approx e^{ik\int_0^z \mu(\vec{x}, z')dz'}$$

the definition for $L_{interaction}$ becomes

$$< e^{ik\int_0^{L_{interaction}} \mu(\vec{x}, z)dz} > = \frac{1}{2} \tag{2.2.17}$$

Assuming $\int_0^{L_{interaction}} \mu(\vec{x}, z)dz$ is a zero-mean Gaussian random variable, eq. (2.2.17) becomes

$$e^{-\frac{1}{2}\iint_0^{L_{interaction}} < \mu(\vec{x}, z_1)\mu(\vec{x}, z_2) > dz_1 dz_2} = \frac{1}{2} \tag{2.2.18}$$

Taking the log and using sum and difference coordinates in z, eq. (2.2.18) may be approximated

$$k^2 \int_0^{L_{interaction}} \int_{-\infty}^{\infty} < \mu(\vec{x}, z)\mu(\vec{x}, z + \zeta) > d\zeta dz = 2\ln 2 \tag{2.2.19}$$

where we have assumed that the interaction length scale is long compared to the correlation length of the medium. Using L_μ to denote the equivalent width of the index of refraction autocorrelation function, eq. (2.2.19) leads to an estimate for the interaction length scale as

$$L_{interaction} \approx \frac{\lambda^2}{< \mu^2 > L_\mu}$$

where miscellaneous numerical factors have been ignored. If $L_{interaction}$ is *not* large compared with the correlation length of the medium, the interaction length scale is roughly

$$L_{interaction} \approx \frac{\lambda}{\mu_{rms}}$$

Either way, $L_{interaction}$ is large compared to λ. Returning to eq. (2.2.15) we see that

$$\frac{\partial^2 E}{\partial z^2} \approx E_{rms}/L^2_{interaction}$$

$$2ik\frac{\partial E}{\partial z} \approx E_{rms}/(\lambda L_{interaction})$$

Therefore, as long as the interaction length scale is very large compared to a wavelength, we may ignore the $\partial^2 E/\partial z^2$ term. This leads us to the result

$$\frac{\partial E}{\partial z} = \frac{i}{2k}\frac{\partial^2 E}{\partial \vec{x}^2} + ik\mu(\vec{x}, z)E(\vec{x}, z) \qquad (2.2.20)$$

where μ is, at most, slowly varying in time and space compared to a wave-period or a wavelength. Equation (2.2.20) is referred to as the "Parabolic Wave Equation" and describes paraxial waves propagating principally in the direction of the z axis.

2.3. The Feynman Path Integral.

We are solely concerned with narrow-angle forward scattering and hence, only with conditions under which the parabolic wave equation (PWE) is valid. This "limitation" permits us to make rigorous use of the Huygens-Fresnel integral in constructing an equivalent, functional-integral formulation for paraxial wave propagation: the "Feynman path integral" (FPI). The primary advantage of this parallel formulation is that it presents the mathematics of wave propagation in an extended, random medium in a form which is directly analogous to the thin screen problem. The FPI is mathematically equivalent to the PWE in that either form may be derived from the other [Codona *et al*, 1986a]. The PWE is a *local* description of the dynamics of a paraxial wave, whereas, the path integral is a *global* description of the paraxial response to a point source. Although the two forms are equivalent and give identical answers for the field, the global nature of the FPI permits a more straightforward discussion of the averaging processes. Since the two forms are equivalent, they both have the same limits of validity. For completeness and the consistent introduction of notation, we will now derive the Feynman path integral from the parabolic wave equation.

When the medium is constant, i.e. $\mu = 0$, the PWE becomes

$$\frac{\partial E}{\partial z} = \frac{i}{2k}\frac{\partial^2 E}{\partial \vec{x}^2} \tag{2.3.1}$$

Defining the transverse Fourier transform of E with respect to $\vec{x}$ as

$$\tilde{E}(\vec{\kappa}, z) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{-i\vec{\kappa}\cdot\vec{x}} E(\vec{x}, z)\, d^2x \tag{2.3.2}$$

allows eq. (2.3.1) to be written

$$E(\vec{x}, z+\Delta z) = \int_{-\infty}^{\infty} \exp\left\{-i\vec{\kappa}\cdot\vec{x} - \frac{i\kappa^2\Delta z}{2k}\right\}\tilde{E}(\vec{\kappa}, z)\, d^2\kappa \tag{2.3.3}$$

or

$$E(\vec{x}, z+\Delta z) = \frac{k}{2\pi i\,\Delta z}\int_{-\infty}^{\infty} \exp\left\{\frac{ik(\vec{x}-\vec{x}_0)^2}{2\Delta z}\right\} E(\vec{x}_0, z)\, d^2x_0 \tag{2.3.4}$$

While eqs. (2.3.3) and (2.3.4) are equivalent, they have different names and show different aspects of the physics. Equation (2.3.3) is the "Fresnel transform" and (2.3.4) is the "Huygens-Fresnel integral." The Fresnel transform is used primarily in thin screen problems where it simplifies the calculations. It represents propagation through free space as a transfer function in the spatial frequency domain. The Huygens-Fresnel integral is the "Green's function" approach to free-space propagation. Each point in the starting plane acts like an independently radiating point source whose strength is proportional to $E(\vec{x}_0, z)$. We write the free-space field due to a point source as $G_f(\vec{x}, z; \vec{x}_0)$. Using this notation, eq. (2.3.4) is rewritten as

$$E(\vec{x}, z+\Delta z) = \int_{-\infty}^{\infty} G_f(\vec{x}, \Delta z; \vec{x}_0) E(\vec{x}_0, z) d^2x_0 \tag{2.3.5}$$

A self-consistency requirement of (2.3.5) is that the Green's function should be able to propagate itself. This leads to the "group property" for the free-space Green's function

$$G_f(\vec{x}, z; \vec{x}_0) = \int_{-\infty}^{\infty} G_f(\vec{x}, z-z_1; \vec{x}_1) G_f(\vec{x}_1, z_1; \vec{x}_0) d^2x_1 \tag{2.3.6}$$

One of the most important consequences of the group property is that the strength of the Green's function point sources is determined. If we were to change the strength of the point source that generates G_f, eq. (2.3.6) would no longer be true. The result is that the Green's function point sources must be of a "standard strength" such that the resulting field is

$$G_f(\vec{x}, z; \vec{x}_0) = \frac{k}{2\pi i z} \exp\left\{\frac{ik(\vec{x} - \vec{x}_0)^2}{2z}\right\} \tag{2.3.7}$$

We would like to derive a formula for the field due to a point source of standard strength embedded in a random medium. Our method will be to separate the random medium and the propagation process by using a "sampling" technique, wherein the medium is broken into a series of thin screens separated by free space. This approximation to the random medium is written

$$\mu_s(\vec{x}, z) = \sum_{n=0}^{\infty} \mu(\vec{x}, z_n) \delta(z - n\Delta z) \Delta z \tag{2.3.8}$$

where $z_n = n\Delta z$. As $\Delta z \to 0$, the power spectrum of μ_s becomes equal to the three dimensional power spectrum of μ plus images separated by $1/\Delta z$ in longitudinal spatial frequency. The phase shift introduced by passing through one of these screens is $k\mu(\vec{x}, z_n)\Delta z$. Since the source is of standard strength, the electric field after the first screen at $z = \Delta z$ is

$$E_1(\vec{x}_1, z_1^+) = \frac{k}{2\pi i \Delta z} \exp\left\{\frac{ik(\vec{x}_1 - \vec{x}_0)^2}{2\Delta z} + ik\mu(\vec{x}_1, z_1)\Delta z\right\} \tag{2.3.9}$$

The Huygens-Fresnel integral allows us to continue the propagation by permitting us to write

$$E_{n+1}(\vec{x}_{n+1}, z_{n+1}^{+}) = \tag{2.3.10}$$

$$\frac{k}{2\pi i\,\Delta z} \int_{-\infty}^{\infty} d^2x_n \exp\left\{\frac{ik(\vec{x}_{n+1} - \vec{x}_n)^2}{2\Delta z} + ik\,\mu(\vec{x}_{n+1}, z_{n+1})\Delta z\right\} E_n(\vec{x}_n, z_n^{+})$$

Repeated application of (2.3.10) yields

$$E_N(\vec{x}_N, z_N) = \left[\frac{k}{2\pi i\,\Delta z}\right]^N \int_{-\infty}^{\infty}\!\!\ldots\!\int d^2x_1 \ldots d^2x_{N-1} \tag{2.3.11}$$

$$\exp\left\{\frac{ik}{2}\sum_{n=1}^{N}\left[\frac{\vec{x}_n - \vec{x}_{n-1}}{\Delta z}\right]^2 \Delta z + ik\sum_{n=1}^{N}\mu(\vec{x}_n, z_n)\Delta z\right\}$$

As the distance between the screens goes to zero, eq. (2.3.11) will approach the Green's function for the electric field. Setting $z_N = R$ and taking $N \to \infty$ in (2.3.11) leads to

$$G(\vec{x}, R; \vec{x}_0) = \lim_{N \to \infty} E_N(\vec{x}, R) \tag{2.3.12}$$

Introducing the notation for the multiple integration operator

$$\int D\{\vec{x}(z)\} \equiv \lim_{N \to \infty}\left[\frac{kN}{2\pi i R}\right]^N \int_{-\infty}^{\infty}\!\!\ldots\!\int d^2x_1 \ldots d^2x_{N-1} \tag{2.3.13}$$

the Green's function is written

$$G(\vec{x}, R; \vec{x}_0) = \int D\{\vec{x}(z)\}\exp\left\{\frac{ik}{2}\int_0^R (\vec{x}'(z))^2 dz + ik\int_0^R \mu(\vec{x}(z), z)dz\right\} \tag{2.3.14}$$

This is the Feynman path integral solution to the parabolic wave equation. The function $\vec{x}(z)$ is referred to as the "path" and is considered the integration variable in the path integral. The path takes on all possible continuous and discontinuous functions joining ($\vec{x}_0$, $z = 0$) and ($\vec{x}_f$, $z = R$) (two-dimensional Hilbert space) in the functional integration domain. Discontinuous paths cause the $\frac{ik}{2}\int_0^R (\vec{x}'(z))^2 dz$ term in the exponential to diverge, leading to phase cancellation and giving no contribution to the path integral. Thus, the *contributing* domain of the path integral is really all continuous functions joining the endpoints.

2.4. Two Important Path Integral Identities.

The path integral formalism introduced in the preceding section permits us to construct statistical moments of the electric field directly, allowing a clear presentation of the averaging processes involved and greatly reducing the complexity of the algebraic manipulations. However, even with these advantages, the path integrals must still be evaluated, a task that can be extremely difficult. Fortunately, we will be able to write our expressions in a form that may be evaluated immediately, given two path integral identities. The purpose of this section is to present these results, making path integrals practical for use in scintillation theory. The important results are eqs. (2.4.1) and (2.4.20). Once the reader is familiar with path integral manipulations, the extended medium calculations become as simple as the familiar thin screen problem.

The basic result is a simple consequence of the construction given in the last section. The electric field Green's function result for free space is the parabolic approximation to a spherical wave. Therefore, writing the free space electric field for a point source as a path integral yields

$$\int D\{\vec{x}(z)\}\exp\left\{\frac{ik}{2}\int_0^R(\vec{x}'(z))^2dz\right\} = \frac{k}{2\pi iR}\exp\left\{\frac{ik(\vec{x}_f-\vec{x}_0)^2}{2R}\right\} \tag{2.4.1}$$

where $\vec{x}_0$ and $\vec{x}_f$ are the transverse coordinates of the point source and the receiver, respectively. This formula is true by construction.

An important double path integral that we will be using in our spectrum expansions is

$$\iint D\{\vec{y}(z)\}D\{\vec{x}(z)\}\exp\left\{-ik\int_0^R \vec{x}(z)\cdot(\vec{y}''(z)-\vec{f}(z))dz\right\}F\{\vec{y}(z)\} \tag{2.4.2}$$

where $\vec{f}(z)$ is some given vector function and $F\{\vec{y}(z)\}$ is some functional of $\vec{y}(z)$. Without loss of generality, $\vec{x}(z)$ can be considered to be zero at $z = 0$ and $z = R$. The problem wherein the path integral arose will set boundary conditions for $\vec{y}(z)$. The form of this double path integral is reminiscent of the Fourier formula for a Dirac delta function. If $\vec{y}''(z)$ differs from $f(z)$, the $\vec{x}(z)$ path integral will oscillate and cancel; giving a zero result. Consider the differential equation

$$\vec{g}''(z) = \vec{f}(z) \tag{2.4.3}$$

where $\vec{g}(z)$ satisfies the boundary conditions on $\vec{y}(z)$. Based on the analogy to a delta function, we might consider that the double path integral is proportional to $F\{\vec{g}(z)\}$. This is, in fact, the case. However, finding the constant of proportionality is considerably more difficult. The result of the following derivation will be a "delta functional" which permits us to write our expansions in terms of standard, Riemann integrals instead of Feynman path integrals.

We begin the derivation by making a change of variables in eq. (2.4.2) to

$$\vec{v}(z) = \vec{y}(z) - \vec{g}(z)$$

where $\vec{g}(z)$ satisfies (2.4.3) and the boundary conditions on $\vec{y}(z)$. This transforms (2.4.2) to

$$\iint D\{\vec{x}(z)\}D\{\vec{v}(z)\}\exp\left\{-ik\int_0^R \vec{x}(z)\cdot\vec{v}''(z)dz\right\}F\{\vec{g}(z)+\vec{v}(z)\} \tag{2.4.4}$$

We can expand $F\{\vec{g}(z)+\vec{v}(z)\}$ in a vector functional Taylor series about $\vec{g}(z)$. A standard functional Taylor series is written [Tatarskii, 1971, page 445]

$$F\{g(z)+v(z)\} = F\{g(z)\} + \int_0^R dz_1\, v(z_1)\frac{\delta}{\delta\xi(z_1)} F\{g(z)+\xi(z)\}\Bigg|_{\xi(z)=0} \tag{2.4.5}$$

$$+\frac{1}{2!}\int_0^R dz_1\, v(z_1)\frac{\delta}{\delta\xi(z_1)}\int_0^R dz_2\, v(z_2)\frac{\delta}{\delta\xi(z_2)} F\{g(z)+\xi(z)\}\Bigg|_{\xi(z)=0} + \ldots$$

The operator $\frac{\delta}{\delta\xi(z)}$ is a functional derivative and is defined

$$\frac{\delta}{\delta\xi(z')} F\{\xi(z)\} = \lim_{\Delta\to 0}\frac{F\{\xi(z)+\delta\xi(z)\} - F\{\xi(z)\}}{\int_\Delta \delta\xi(z)dz} \tag{2.4.6}$$

where $z' \in \Delta$ and $\delta\xi(z)$ is non-zero only for $z \in \Delta$. A two dimensional, functional gradient is defined

$$\frac{\delta F\{\vec{x}(z)\}}{\delta\vec{x}(z')} = \left[\frac{\delta F\{(x(z),y(z))\}}{\delta x(z')}, \frac{\delta F\{(x(z),y(z))\}}{\delta y(z')}\right] \tag{2.4.7}$$

By analogy with eq. (2.4.5), we write a vector functional Taylor series as

$$F\{\vec{g}(z)+\vec{v}(z)\} = F\{\vec{g}(z)\} + \int_0^R dz_1\, \vec{v}(z)\cdot\frac{\delta}{\delta\vec{s}(z_1)} F\{\vec{g}(z)+\vec{s}(z)\}\Bigg|_{\vec{s}(z)=0} \tag{2.4.8}$$

$$+\frac{1}{2!}\int_0^R dz_1\, \vec{v}(z_1)\cdot\frac{\delta}{\delta\vec{s}(z_1)}\int_0^R dz_2\, \vec{v}(z_2)\cdot\frac{\delta}{\delta\vec{s}(z_2)} F\{\vec{g}(z)+\vec{s}(z)\}\Bigg|_{\vec{s}(z)=0} + \ldots$$

An operator notation "short-hand" for the expansion in eq. (2.4.8) is

$$F\{\vec{g}(z)+\vec{v}(z)\} = \exp\left\{\int_0^R dz'\, \vec{v}(z')\cdot\frac{\delta}{\delta\vec{s}(z')}\right\} F\{\vec{g}(z)+\vec{s}(z)\}\Bigg|_{\vec{s}(z)=0} \tag{2.4.9}$$

where the "e to an operator" form is treated as a short-hand notation for the series. The exponential is formally Taylor expanded and *then* the operator series is allowed to operate on whatever is to the right.

The exponential operator appearing in eq. (2.4.9) can itself be considered as a functional of $\frac{\delta}{\delta \vec{s}(z')}$ and may also be functionally Taylor expanded as

$$\exp\left\{\int_0^R dz\,\vec{v}(z)\cdot\frac{\delta}{\delta\vec{s}(z)}\right\} = \exp\left\{\int_0^R dz'\frac{\delta}{\delta\vec{s}(z')}\cdot\frac{\delta}{\delta\vec{\xi}(z')}\right\}\exp\left\{\int_0^R \vec{v}(z)\cdot\vec{\xi}(z)dz\right\}\Bigg|_{\vec{\xi}(z)=0} \tag{2.4.10}$$

Expanding $F\{\vec{g}(z)+\vec{v}(z)\}$ in this type of second order functional Taylor series yields

$$F\{\vec{g}(z)+\vec{v}(z)\} = \tag{2.4.11}$$

$$\exp\left\{\int_0^R dz'\frac{\delta}{\delta\vec{s}(z')}\cdot\frac{\delta}{\delta\vec{\xi}(z')}\right\}\exp\left\{\int_0^R \vec{v}(z)\cdot\vec{\xi}(z)dz\right\}F\{\vec{g}(z)+\vec{s}(z)\}\Bigg|_{\vec{\xi}(z)=0 \text{ and } \vec{s}(z)=0}$$

Since the second exponential is simply a functional, it commutes with the functional F.

$$F\{\vec{g}(z)+\vec{v}(z)\} = \tag{2.4.12}$$

$$\exp\left\{\int_0^R dz'\frac{\delta}{\delta\vec{s}(z')}\cdot\frac{\delta}{\delta\vec{\xi}(z')}\right\}F\{\vec{g}(z)+\vec{s}(z)\}\Bigg|_{\vec{s}(z)=0}\exp\left\{\int_0^R \vec{v}(z)\cdot\vec{\xi}(z)dz\right\}\Bigg|_{\vec{\xi}(z)=0}$$

Formally collapsing the $\vec{s}$ expansion, we write

$$F\{\vec{g}(z)+\vec{v}(z)\} = F\{\vec{g}(z)+\frac{\delta}{\delta\vec{\xi}(z')}\}\exp\left\{\int_0^R \vec{v}(z)\cdot\vec{\xi}(z)dz\right\}\Bigg|_{\vec{\xi}(z)=0} \tag{2.4.13}$$

The unusual argument in F on the right hand side is defined only through its functional Taylor expansion. Finally, making the substitution $\vec{\xi}(z) = -i\vec{s}(z)$ we find

$$F\{\vec{g}(z)+\vec{v}(z)\} = F\{\vec{g}(z)+i\frac{\delta}{\delta\vec{s}(z')}\}\exp\left\{-i\int_0^R \vec{v}(z)\cdot\vec{s}(z)dz\right\}\Bigg|_{\vec{s}(z)=0} \tag{2.4.14}$$

Using eq. (2.4.14) to expand F in eq. (2.4.4) leads to

$$\iint D\{\vec{x}(z)\}D\{\vec{v}(z)\}\exp\left\{-ik\int_0^R \vec{x}(z)\cdot\vec{v}''(z)dz\right\} \tag{2.4.15}$$

$$F\{\vec{g}(z)+i\frac{\delta}{\delta\vec{s}(z')}\}\exp\left\{-i\int_0^R \vec{v}(z)\cdot\vec{s}(z)dz\right\}\Bigg|_{\vec{s}(z)=0}$$

F is now an operator that may be pulled outside of the path integrals.

$$F\{\vec{g}(z)+i\frac{\delta}{\delta\vec{s}(z')}\}\iint D\{\vec{x}(z)\}D\{\vec{v}(z)\} \tag{2.4.16}$$

$$\exp\left\{-i\int_0^R \vec{v}(z)\cdot(k\vec{x}''(z)+\vec{s}(z))\,dz\right\}\Bigg|_{\vec{s}(z)=0}$$

Introduce a new variable, $\vec{u}(z)$, defined by

$$\vec{u}''(z)=\vec{x}''(z)+\frac{1}{k}\vec{s}(z)$$

where $\vec{s}(z)$ is our functional Taylor series' dummy expansion path and $\vec{u}(z)$ is to be zero at $z=0$ and $z=R$. The double path integral becomes

$$F\{\vec{g}(z)+i\frac{\delta}{\delta\vec{s}(z')}\}\iint D\{\vec{x}(z)\}D\{\vec{v}(z)\}\exp\left\{ik\int_0^R \vec{v}'(z)\cdot\vec{u}'(z)\,dz\right\} \tag{2.4.17}$$

where the endpoints of both $\vec{u}(z)$ and $\vec{v}(z)$ are zero.

The double path integral may now be evaluated. Taking the magnitude-squared of eq. (2.4.1) with $\vec{x}_0$ and $\vec{x}_f$ both zero yields

$$\iint D\{\vec{x}(z)\}D\{\vec{y}(z)\}\exp\left\{\frac{ik}{2}\int_0^R\left[(\vec{x}'(z))^2-(\vec{y}'(z))^2\right]dz\right\}=\left[\frac{k}{2\pi R}\right]^2 \tag{2.4.18}$$

Changing path variables to

$$\vec{u}(z) = \frac{1}{2}(\vec{x}(z) + \vec{y}(z))$$

$$\vec{v}(z) = \vec{x}(z) - \vec{y}(z)$$

we can rewrite eq. (2.4.18) as

$$\iint D\{\vec{u}(z)\}D\{\vec{v}(z)\}\exp\left\{ik\int_0^R \vec{u}'(z)\cdot\vec{v}'(z)\,dz\right\} = \left[\frac{k}{2\pi R}\right]^2 \tag{2.4.19}$$

This is the same double path integral as in eq. (2.4.17). Note especially that it is independent of $\vec{s}(z)$. Therefore, the functional derivative in F will give zero when F is expanded; allowing us to write

$$\iint D\{\vec{y}(z)\}D\{\vec{x}(z)\}\exp\left\{-ik\int_0^R \vec{x}(z)\cdot(\vec{y}''(z) - \vec{f}(z))dz\right\}F\{\vec{y}(z)\} \tag{2.4.20}$$

$$= \left[\frac{k}{2\pi R}\right]^2 F\{\vec{g}(z)\}$$

where $\vec{g}''(z) = \vec{f}(z)$ and $\vec{g}(z)$ satisfies the boundary conditions for $\vec{y}(z)$. This equation will be referred to as the "delta-functional" result.

The results from this section are the "free-space Green's function," eq. (2.4.1), and the "delta-functional" result, eq. (2.4.20).

2.5. Statistical Moments of the Electric Field.

We are primarily interested in two aspects of the fluctuating electric field: fluctuations in the angular spectrum of plane waves and intensity scintillations. The angular spectrum is the average power impinging on the receiver from various directions. It is well known that the average angular spectrum is the Fourier transform of the averaged complex electric field coherence [Booker and Clemmow, 1950]. After having propagated through a random medium, the intensity is a random function of position in the receiver plane. The averaged spatial power spectrum of the intensity pattern describes the scale size distribution of intensity variance. By the Wiener-Khintchine theorem [Ratcliffe, 1956], this is the Fourier transform of the two-point intensity correlation. The complex field coherence is given by the "second moment" of the electric field and the two-point intensity correlation is a special case of the "fourth moment" of the field. We will develop a general approach for computing the second and fourth moments of the field for an arbitrary field incident on an extended random medium. For convenience and clarity of presentation, the Feynman path integral approach will be used to propagate the fields from an arbitrary input field distribution. This technique, combined with the Markov approximation, has been shown to be equivalent to solving the standard moment equations [Codona, *et al*, 1986a].

The electric field in the $z = R$ plane due to a point source at $(\vec{x}_0, z = 0)$ is given by the Green's function $G(\vec{x}, R; \vec{x}_0)$. In section 2.3 we found the Green's function to be given by the Feynman path integral (FPI)

$$G(\vec{x}, R; \vec{x}_0) = \int D\{\vec{x}(z)\} \exp\left\{ \frac{ik}{2} \int_0^R (\vec{x}'(z))^2 dz + ik \int_0^R \mu(\vec{x}(z), z) dz \right\} \tag{2.5.1}$$

Since the fluctuations in the index of refraction are random, the Green's function is also random. Employing linearity for the superposition of the electric field, we can use the Green's function to construct the fluctuating field due to any arbitrary input field. Writing the electric field in the $z = 0$ plane as $E_0(\vec{x}_0)$, we find the resultant electric field in the $z = R$ plane to be

$$E(\vec{x}, R) = \int_{-\infty}^{\infty} G(\vec{x}, R; \vec{x}_0) E_0(\vec{x}_0) d^2 x_0 \tag{2.5.2}$$

We may use eq. (2.5.2) to construct any statistical moment of the field for any input condition. The second moment is found by multiplying eq. (2.5.2) with $\vec{x} = \vec{x}_1$ by the complex conjugate of eq. (2.5.2) with $\vec{x} = \vec{x}_2$ and averaging over any random processes affecting the final field

$$X(\vec{x}_1, \vec{x}_2, R) \equiv \langle E(\vec{x}_1, R)E^*(\vec{x}_2, R)\rangle \tag{2.5.3}$$

$$= \iint_{-\infty}^{\infty} \langle G(\vec{x}_1, R; \vec{x}_{1,0})G^*(\vec{x}_2, R; \vec{x}_{2,0})E_0(\vec{x}_{1,0})E_0^*(\vec{x}_{2,0})\rangle \; d^2x_{1,0}d^2x_{2,0}$$

Since we have assumed that the electric field does not affect the medium through which it propagates, the input field, E_0, is statistically independent of the fluctuations in μ. Furthermore, the assumption of forward-scattering implies that the fluctuations in μ have no effect on the input field. However, E_0 itself may be the result of random processes. These considerations break the ensemble average in eq. (2.5.3) into two parts: an average over the random medium between $z = 0$ and $z = R$ and an average over any random processes that might affect the input field. (Processes affecting the input field might include propagation through earlier random media or random fluctuations in the source of the waves.) The independence of these averages permits us to write eq. (2.5.3) as

$$X(\vec{x}_1, \vec{x}_2, R) = \iint_{-\infty}^{\infty} G_2(\vec{x}_1, \vec{x}_2, R; \vec{x}_{1,0}, \vec{x}_{2,0})X_0(\vec{x}_{1,0}, \vec{x}_{2,0})d^2x_{1,0}d^2x_{2,0} \tag{2.5.4}$$

where

$$X_0(\vec{x}_{1,0}, \vec{x}_{2,0}) \equiv \langle E_0(\vec{x}_{1,0})E_0^*(\vec{x}_{2,0})\rangle_{\textit{processes affecting } E_0} \tag{2.5.5}$$

and

$$G_2(\vec{x}_1, \vec{x}_2, R; \vec{x}_{1,0}, \vec{x}_{2,0}) \equiv \langle G(\vec{x}_1, R; \vec{x}_{1,0})G^*(\vec{x}_2, R; \vec{x}_{2,0})\rangle_{\textit{medium}} \tag{2.5.6}$$

G_2 is the Green's function that propagates the second moment. A basic difference from the field Green's function is the dimensionality of the input and output spaces. Moreover, the second moment Green's function is *not* random, being the result of averaging over the random medium.

By using the FPI form of the field Green's function in eq. (2.5.6), we may find a formula for the second moment Green's function:

$$G_2(\vec{x}_1, \vec{x}_2, R; \vec{x}_{1,0}, \vec{x}_{2,0}) = \tag{2.5.7}$$

$$\iint D\{\vec{x}_1(z)\} D\{\vec{x}_2(z)\} \exp\left\{\frac{ik}{2}\int_0^R \left[(\vec{x}_1{}'(z))^2 - (\vec{x}_2{}'(z))^2\right] dz\right\}$$

$$< \exp\left\{-ik\int_0^R \left[\mu(\vec{x}_1(z), z) - \mu(\vec{x}_2(z), z)\right] dz\right\}>$$

where $\vec{x}_1(z)$ is a dummy integration path joining $\vec{x}_{1,0}$ at $z = 0$ and $\vec{x}_1$ at $z = R$ and, likewise, $\vec{x}_2(z)$ joins $\vec{x}_{2,0}$ and $\vec{x}_2$. The ensemble average brackets have been brought inside the path integrals to the only random portion of the integrand. Since μ is a zero-mean random variable, the integral of μ over z is also zero-mean. The statistics of μ are usually assumed to be Gaussian at each point in the random medium. However, even if this is not the case, the central-limit theorem may be invoked to argue that the integral over z is a zero-mean Gaussian random variable. Using the characteristic function for a zero-mean Gaussian random variable Q, the average of e^{iQ} is given by

$$< e^{iQ} > = e^{-\frac{1}{2}<Q^2>} \tag{2.5.8}$$

Applying eq. (2.5.8) to eq. (2.5.7), we find

$$G_2(\vec{x}_1, \vec{x}_2, R; \vec{x}_{1,0}, \vec{x}_{2,0}) = \tag{2.5.9}$$

$$\iint D\{\vec{x}_1(z)\} D\{\vec{x}_2(z)\} \exp\left\{\frac{ik}{2}\int_0^R \left[(\vec{x}_1{}'(z))^2 - (\vec{x}_2{}'(z))^2\right] dz\right.$$

$$\left. - k^2 \int_0^R\!\!\int_0^R \left[B_\mu(0, z) - B_\mu(\vec{x}_1(z_1) - \vec{x}_2(z_2), z_1, z_2)\right] dz_1 dz_2\right\}$$

where $B_\mu(\Delta\vec{x}, z_1, z_2) = < \mu(\vec{x}, z_1)\mu(\vec{x} + \Delta\vec{x}, z_2)>$ is the correlation function of μ assuming statistical homogeneity in the transverse direction. If the medium is locally homogeneous in z, the double path integral of B_μ over the two z coordinates may be approximated by a single z integral of a reduced correlation function.

$$\int_0^R\int_0^R B_\mu(\Delta\vec{x}, \Delta z = z_1 - z_2, z = \frac{z_1 + z_2}{2}) dz_1 dz_2 \approx \int_0^R A_\mu(\Delta\vec{x}, z) dz \qquad (2.5.10)$$

where

$$A_\mu(\Delta\vec{x}, z) \equiv \int_{-\infty}^{\infty} B_\mu(\Delta\vec{x}, \Delta z, z) d(\Delta z) \qquad (2.5.11)$$

This "Markov approximation" is good when the angular spectrum is very narrow [Zavorotnyi, 1978; Lee and Jokipii, 1975]. Writing the "wave structure function density" as

$$D(\Delta\vec{x}, z) = 2k^2 \left[A_\mu(0, z) - A_\mu(\Delta\vec{x}, z)\right] \qquad (2.5.12)$$

the second moment Green's function reduces to

$$G_2(\vec{x}_1, \vec{x}_2, R; \vec{x}_{1,0}, \vec{x}_{2,0}) = \int\int \mathrm{D}\{\vec{x}_1(z)\}\mathrm{D}\{\vec{x}_2(z)\} \qquad (2.5.13)$$

$$\exp\left\{\frac{ik}{2}\int_0^R \left[(\vec{x}_1{}'(z))^2 - (\vec{x}_2{}'(z))^2\right] dz - \frac{1}{2}\int_0^R D(\vec{x}_1(z) - \vec{x}_2(z), z) dz\right\}$$

A change of variables within path integrals must have a Jacobian of unity. Introducing the variables

$$\vec{u}(z) = \frac{1}{2}(\vec{x}_1(z) + \vec{x}_2(z)) \qquad (2.5.14)$$

$$\vec{v}(z) = \vec{x}_1(z) - \vec{x}_2(z) \qquad (2.5.15)$$

with the corresponding formulae for the end points, the second moment Green's function becomes

$$G_2(\vec{u}, \vec{v}, R; \vec{u}_0, \vec{v}_0) = \tag{2.5.16}$$

$$\iint \mathbf{D}\{\vec{u}(z)\}\mathbf{D}\{\vec{v}(z)\}\exp\left\{ik\int_0^R \vec{u}'(z)\cdot\vec{v}'(z)dz - \frac{1}{2}\int_0^R D(\vec{v}(z), z)dz\right\}$$

The structure function density, $D(\vec{v}(z), z)$, is only a function of the difference path $\vec{v}(z)$ because of the transverse statistical homogeneity of the medium. Equation (2.5.16) is nearly in the form of the identity, eq. (2.4.20). Introducing the "straight-line path" joining the end-points as

$$\vec{f}(\vec{x}, z) = (1 - \frac{z}{R})\vec{x}(0) + \frac{z}{R}\vec{x}(R) \tag{2.5.17}$$

we write the integration paths as the straight-line paths plus the deviations about them

$$\vec{u}(z) = \vec{f}(\vec{u}, z) + \vec{u}_1(z)$$

$$\vec{v}(z) = \vec{f}(\vec{v}, z) + \vec{v}_1(z)$$

These new variables transform the integral in the exponential phase factor in eq. (2.5.16) to

$$ik\int_0^R \vec{u}'(z)\cdot\vec{v}'(z)dz = \frac{ik}{R}(\vec{u} - \vec{u}_0)\cdot(\vec{v} - \vec{v}_0) + ik\int_0^R \vec{u}_1{}'(z)\cdot\vec{v}_1{}'(z)dz \tag{2.5.18}$$

where $\vec{u}_0 \equiv \vec{u}(0)$, $\vec{u} \equiv \vec{u}(R)$, $\vec{v}_0 \equiv \vec{v}(0)$, and $\vec{v} \equiv \vec{v}(R)$. The second moment Green's function becomes

$$G_2(\vec{u}, \vec{v}, R; \vec{u}_0, \vec{v}_0) = \mathbf{e}^{\frac{ik}{R}(\vec{u} - \vec{u}_0)\cdot(\vec{v} - \vec{v}_0)} \iint \mathbf{D}\{\vec{u}_1(z)\}\mathbf{D}\{\vec{v}_1(z)\} \tag{2.5.19}$$

$$\exp\left\{-ik\int_0^R \vec{u}_1(z)\cdot\vec{v}_1{}''(z)dz - \frac{1}{2}\int_0^R D(\vec{f}(\vec{v}, z) + \vec{v}_1(z), z)dz\right\}$$

which is exactly in the form of eq. (2.4.20). Since, by construction, the values of $\vec{v}_1$ at the end points are both zero, the double path integral may be evaluated, reducing the second moment

Green's function to

$$G_2(\vec{u}, \vec{v}, R\,; \vec{u}_0, \vec{v}_0) = G_2^f(\vec{u}, \vec{v}, R\,; \vec{u}_0, \vec{v}_0)\,e^{-\frac{1}{2}\int_0^R D(\vec{r}(\vec{v}, z), z)dz} \qquad (2.5.20)$$

where G_2^f is the free-space Green's function for the second moment

$$G_2^f(\vec{u}, \vec{v}, R\,; \vec{u}_0, \vec{v}_0) = \left[\frac{k}{2\pi R}\right]^2 e^{\frac{ik}{R}(\vec{u} - \vec{u}_0)\cdot(\vec{v} - \vec{v}_0)} \qquad (2.5.21)$$

or, in the original coordinates,

$$G_2^f(\vec{x}_1, \vec{x}_2, R\,; \vec{x}_{1,0}, \vec{x}_{2,0}) = \left[\frac{k}{2\pi R}\right]^2 e^{\frac{ik}{2R}\left[(\vec{x}_1 - \vec{x}_{1,0})^2 - (\vec{x}_2 - \vec{x}_{2,0})^2\right]} \qquad (2.5.22)$$

By using eq. (2.5.20) in eq. (2.5.4), the second moment a distance R through a random medium may be found for any, arbitrary, input field.

The fourth moment may also be constructed using path integrals and a fourth moment Green's function derived. Using the random Green's function for propagating the field, the fourth moment becomes

$$Z(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R\,; \vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0}) \equiv \qquad (2.5.23)$$

$$< E(\vec{x}_1, R)E^*(\vec{x}_2, R)E(\vec{x}_3, R)E^*(\vec{x}_4, R)>$$

$$= \iiiint_{-\infty}^{\infty} < G(\vec{x}_1, R\,; \vec{x}_{1,0})G^*(\vec{x}_2, R\,; \vec{x}_{2,0})G(\vec{x}_3, R\,; \vec{x}_{3,0})G^*(\vec{x}_4, R\,; \vec{x}_{4,0})$$

$$E_0(\vec{x}_{1,0})E_0^*(\vec{x}_{2,0})E_0(\vec{x}_{3,0})E_0^*(\vec{x}_{4,0})> \; d^2x_{1,0}\, d^2x_{2,0}\, d^2x_{3,0}\, d^2x_{4,0}$$

By noting the statistical independence of the input fields and the field Green's functions, we may average over the two factors separately, defining

$$Z_0(\vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0}) \equiv \qquad (2.5.24)$$

$$< E_0(\vec{x}_{1,0})E_0^*(\vec{x}_{2,0})E_0(\vec{x}_{3,0})E_0^*(\vec{x}_{4,0}) >_{\textit{processes affecting } E_0}$$

and

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R; \vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0}) = \tag{2.5.25}$$

$$< G(\vec{x}_1, R; \vec{x}_{1,0})G^*(\vec{x}_2, R; \vec{x}_{2,0})G(\vec{x}_3, R; \vec{x}_{3,0})G^*(\vec{x}_4, R; \vec{x}_{4,0}) >_{medium}$$

where the angle brackets denote ensemble averaging over the relevant random processes. Using these definitions, the fourth moment in eq. (2.5.23) becomes

$$Z(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R) = \iiiint_{-\infty}^{\infty} G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R; \vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0}) \tag{2.5.26}$$

$$Z_0(\vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0})d^2x_{1,0}\, d^2x_{2,0}\, d^2x_{3,0}\, d^2x_{4,0}$$

Once again, we can use the FPI formulation of the electric field Green's to form the fourth moment Green's function

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R; \vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0}) = \tag{2.5.27}$$

$$\iiiint \mathrm{D}\{\vec{x}_1(z)\}\mathrm{D}\{\vec{x}_2(z)\}\mathrm{D}\{\vec{x}_3(z)\}\mathrm{D}\{\vec{x}_4(z)\}$$

$$\exp\left\{\frac{ik}{2}\int_0^R \left[(\vec{x}_1{}'(z))^2 - (\vec{x}_2{}'(z))^2 + (\vec{x}_3{}'(z))^2 - (\vec{x}_4{}'(z))^2\right]dz\right\}$$

$$< \exp\left\{-ik\int_0^R \left[\mu(\vec{x}_1(z), z) - \mu(\vec{x}_2(z), z) + \mu(\vec{x}_3(z), z) - \mu(\vec{x}_4(z), z)\right]dz\right\} >$$

Treating the integrated combination of index of refraction fluctuations as a zero-mean, Gaussian random variable, we use eq. (2.5.8) to write

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R; \vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0}) = \tag{2.5.28}$$

$$\iiiint D\{\vec{x}_1(z)\}D\{\vec{x}_2(z)\}D\{\vec{x}_3(z)\}D\{\vec{x}_4(z)\}$$

$$\exp\left\{\frac{ik}{2}\int_0^R\left[(\vec{x}_1'(z))^2 - (\vec{x}_2'(z))^2 + (\vec{x}_3'(z))^2 - (\vec{x}_4'(z))^2\right]dz\right\}$$

$$\exp\left\{-\frac{k^2}{2}\int_0^R\int_0^R\left[4B_\mu(0, z_1, z_2) - 2B_\mu(\vec{x}_1(z_1) - \vec{x}_2(z_2), z_1, z_2) + 2B_\mu(\vec{x}_1(z_1) - \vec{x}_3(z_2), z_1, z_2)\right.\right.$$

$$- 2B_\mu(\vec{x}_1(z_1) - \vec{x}_4(z_2), z_1, z_2) - 2B_\mu(\vec{x}_2(z_1) - \vec{x}_3(z_2), z_1, z_2)$$

$$\left.\left.+ 2B_\mu(\vec{x}_2(z_1) - \vec{x}_4(z_2), z_1, z_2) - 2B_\mu(\vec{x}_3(z_1) - \vec{x}_4(z_2), z_1, z_2)\right]dz_1 dz_2\right\}$$

Utilizing the Markov approximation, eq. (2.5.10), and the definition of the wave structure function density, eq. (2.5.12), the fourth moment Green's function becomes

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R; \vec{x}_{1,0}, \vec{x}_{2,0}, \vec{x}_{3,0}, \vec{x}_{4,0}) = \tag{2.5.29}$$

$$\iiiint D\{\vec{x}_1(z)\}D\{\vec{x}_2(z)\}D\{\vec{x}_3(z)\}D\{\vec{x}_4(z)\}$$

$$\exp\left\{\frac{ik}{2}\int_0^R\left[(\vec{x}_1'(z))^2 - (\vec{x}_2'(z))^2 + (\vec{x}_3'(z))^2 - (\vec{x}_4'(z))^2\right]dz\right\}$$

$$\exp\left\{-\frac{1}{2}\int_0^R\left[-D(\vec{x}_1(z) - \vec{x}_2(z), z) + D(\vec{x}_1(z) - \vec{x}_3(z), z) - D(\vec{x}_1(z) - \vec{x}_4(z), z)\right.\right.$$

$$\left.\left.- D(\vec{x}_2(z) - \vec{x}_3(z), z) + D(\vec{x}_2(z) - \vec{x}_4(z), z) - D(\vec{x}_3(z) - \vec{x}_4(z), z)\right]dz\right\}$$

Changing path variables according to the unitary transformation [Rumsey, 1975]

$$\begin{bmatrix} \vec{\alpha}(z) \\ \vec{\beta}(z) \\ \vec{\gamma}(z) \\ \vec{\delta}(z) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \vec{x}_1(z) \\ \vec{x}_2(z) \\ \vec{x}_3(z) \\ \vec{x}_4(z) \end{bmatrix} \tag{2.5.30}$$

gives the fourth moment Green's function

$$G_4(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}, R\,;\, \vec{\alpha}_0, \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \tag{2.5.31}$$

$$\iiiint D\{\vec{\alpha}(z)\}D\{\vec{\beta}(z)\}D\{\vec{\gamma}(z)\}D\{\vec{\delta}(z)\}\exp\left\{ik\int_0^R \left[\vec{\alpha}'(z)\cdot\vec{\delta}'(z) + \vec{\beta}'(z)\cdot\vec{\gamma}'(z)\right]dz\right\}$$

$$\exp\left\{-\frac{1}{2}\int_0^R \Big[D(\vec{\beta}(z) + \vec{\delta}(z), z) + D(\vec{\beta}(z) - \vec{\delta}(z), z) + D(\vec{\gamma}(z) + \vec{\delta}(z), z)\right.$$

$$\left. + D(\vec{\gamma}(z) - \vec{\delta}(z), z) - D(\vec{\beta}(z) + \vec{\gamma}(z), z) - D(\vec{\beta}(z) - \vec{\gamma}(z), z)\Big]dz\right\}$$

Since the medium-dependent factor is independent of $\vec{\alpha}(z)$, the $\vec{\alpha}(z)$ and $\vec{\delta}(z)$ path integrals may be evaluated using the identity, eq. (2.4.20), reducing G_4 to

$$G_4(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}, R\,;\, \vec{\alpha}_0, \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \left[\frac{k}{2\pi R}\right]^2 e^{\frac{ik}{R}(\vec{\alpha} - \vec{\alpha}_0)\cdot(\vec{\delta} - \vec{\delta}_0)} \iint D\{\vec{\beta}(z)\}D\{\vec{\gamma}(z)\} \tag{2.5.32}$$

$$\exp\left\{ik\int_0^R \vec{\beta}'(z)\cdot\vec{\gamma}'(z)dz - \frac{1}{2}\int_0^R \Big[D(\vec{\beta}(z) + \vec{f}(\vec{\delta}(z), z), z) + D(\vec{\beta}(z) - \vec{f}(\vec{\delta}(z), z), z)\right.$$

$$+ D(\vec{\gamma}(z) + \vec{f}(\vec{\delta}(z), z), z) + D(\vec{\gamma}(z) - \vec{f}(\vec{\delta}(z), z), z)$$

$$\left. - D(\vec{\beta}(z) + \vec{\gamma}(z), z) - D(\vec{\beta}(z) - \vec{\gamma}(z), z)\Big]dz\right\}$$

Unfortunately, eq. (2.5.32) may not be simplified any further for a general structure function.

However, we may still make progress by employing an expansion technique for the remaining path integrals.

The important results from this section are: the second moment Green's function, eq. (2.5.20), and the double path integral for the fourth moment Green's function, eq. (2.5.32).

2.6. Series Expansions for the Fourth Moment Green's Function.

In the last section we derived the Green's function for the fourth moment Green's function of the electric field. However, unlike the second moment Green's function, the fourth moment Green's function (FMGF) could not be simplified beyond the double path integral form given in eq. (2.5.32). In the present section, we will derive a series expansion for the FMGF and , by employing the symmetries of the full fourth moment, we will find a second series expansion for the FMGF. The two series will generate a "low" and "high spatial-frequency" expansion for the intensity spectrum (the Fourier transform of the two-point intensity correlation).

The fourth moment, as defined in eq. (2.5.23), has the symmetry

$$Z(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, R) = Z(\vec{x}_1, \vec{x}_4, \vec{x}_3, \vec{x}_2, R) \tag{2.6.1}$$

among others. Referring to the transformation given in eq. (2.5.30), this symmetry is equivalent to swapping $\vec{\beta}$ and $\vec{\gamma}$. That is

$$Z(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}, R) = Z(\vec{\alpha}, \vec{\gamma}, \vec{\beta}, \vec{\delta}, R) \tag{2.6.2}$$

This symmetry also holds for the FMGF, provided the input coordinates are treated similarly.

$$G_4(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\alpha}_0, \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = G_4(\vec{\alpha}, \vec{\gamma}, \vec{\beta}, \vec{\delta}, R; \vec{\alpha}_0, \vec{\gamma}_0, \vec{\beta}_0, \vec{\delta}_0) \tag{2.6.3}$$

This symmetry will play a critical role in the development of a symmetric expansion for the FMGF.

We will find it convenient to express the FMGF as the product of two factors:

$$G_4 = G_4^f M \tag{2.6.4}$$

where G_4^f is the FMGF in free space and M gives the effect of the medium. The free-space FMGF is easily found to be

$$G_4(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\alpha}_0, \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \left[\frac{k}{2\pi R}\right]^4 e^{\frac{ik}{R}((\vec{\alpha}-\vec{\alpha}_0)\cdot(\vec{\delta}-\vec{\delta}_0) + (\vec{\beta}-\vec{\beta}_0)\cdot(\vec{\gamma}-\vec{\gamma}_0))} \tag{2.6.5}$$

From eqs. (2.5.32), (2.6.4), and (2.6.5), we find M to be given by

$$M(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \tag{2.6.6}$$

$$\left[\frac{2\pi R}{k}\right]^2 e^{-\frac{ik}{R}(\vec{\beta}-\vec{\beta}_0)\cdot(\vec{\gamma}-\vec{\gamma}_0)} \iint \mathrm{D}\{\vec{\beta}(z)\}\mathrm{D}\{\vec{\gamma}(z)\}$$

$$\exp\left\{ ik\int_0^R \vec{\beta}'(z)\cdot\vec{\gamma}'(z)dz - \frac{1}{2}\int_0^R \Big[D(\vec{\beta}(z) + \vec{f}(\vec{\delta}, z), z) \right.$$

$$+ D(\vec{\beta}(z) - \vec{f}(\vec{\delta}, z), z) + D(\vec{\gamma}(z) + \vec{f}(\vec{\delta}, z), z)$$

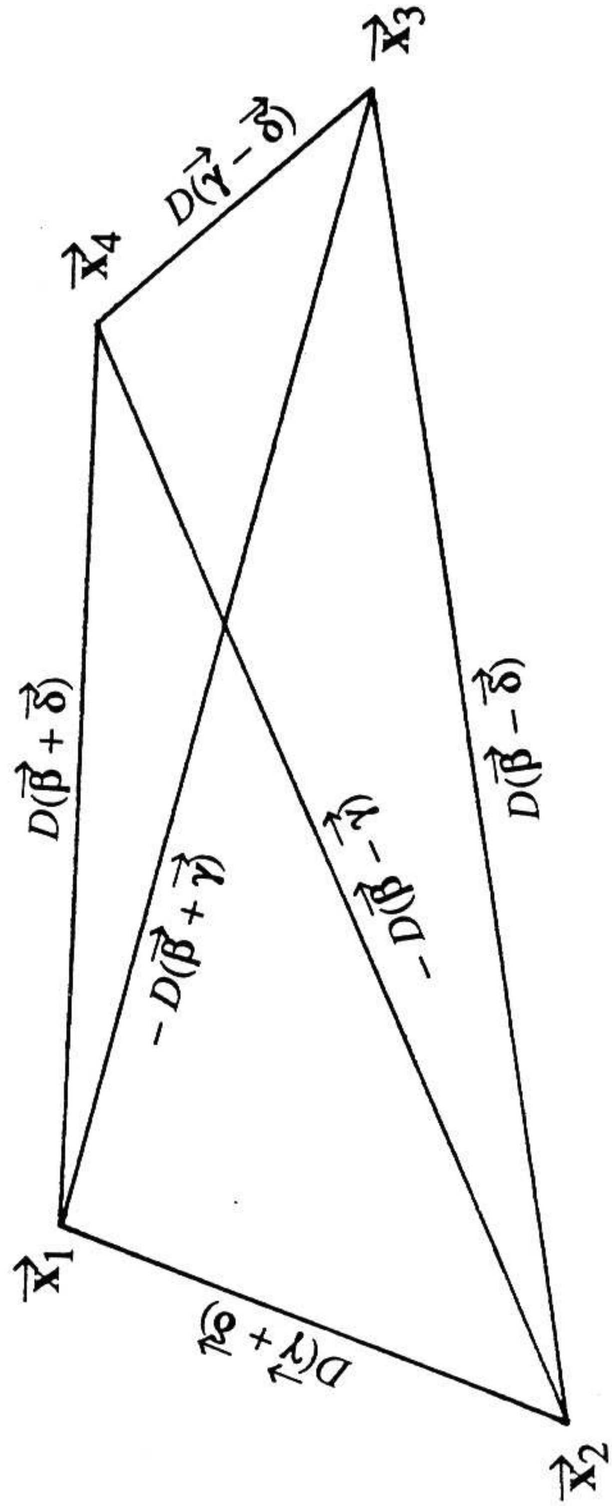
$$\left. + D(\vec{\gamma}(z) - \vec{f}(\vec{\delta}, z), z) - D(\vec{\beta}(z) + \vec{\gamma}(z), z) - D(\vec{\beta}(z) - \vec{\gamma}(z), z) \Big] \right\}$$

where we have dropped $\vec{\alpha}$ and $\vec{\alpha}_0$ since they do not appear in the formula.

Since we are primarily interested in the two-point intensity correlation, we are concerned with the situation where, say, $\vec{\alpha} = \vec{\gamma} = \vec{\delta} = 0$ and the correlation baseline is given by $\vec{\beta}$. This corresponds to the case wherein $\vec{x}_1 = \vec{x}_2$ and $\vec{x}_3 = \vec{x}_4$. Choosing to form the intensity correlation by pairing the opposite sets of points is equivalent to using the symmetry discussed above and will result in the same function. We will study the full FMGF for situations close to the two-point intensity correlation, i.e. $\beta \gg \gamma$ and $\beta \gg \delta$. The geometry for a given z along the propagation path is shown in figure 2.6.1. The wave structure function densities in the integrand are shown with their associated baselines. For convenience, we refer to the sum of

Figure 2.6.1

Geometry of the various structure function contributions at a particular value of the longitudinal coordinate, z. When the quadralateral is very narrow with pairs of points close together, the length of the diagonals is nearly the length of the two longer sides. The near cancellation of the structure functions along these segments is the basis of the expansion series for the fourth moment Green's function.


$\vec{x}_3$
$D(\vec{\gamma} - \vec{\delta})$
$\vec{x}_4$
$D(\vec{\beta} + \vec{\delta})$
$D(\vec{\beta} - \vec{\delta})$
$-D(\vec{\beta} - \vec{\gamma})$
$-D(\vec{\beta} + \vec{\gamma})$
$\vec{x}_1$
$D(\vec{\gamma} + \vec{\delta})$
$\vec{x}_2$

the six wave structure function densities as $V(\vec{\beta}, \vec{\gamma}, \vec{\delta}, z)$

$$V(\vec{\beta}(z), \vec{\gamma}(z), \vec{\delta}(z), z) = \tag{2.6.7}$$

$$D(\vec{\gamma}(z) + \vec{\delta}(z), z) + D(\vec{\gamma}(z) - \vec{\delta}(z), z) + D(\vec{\beta}(z) + \vec{\delta}(z), z)$$

$$+ D(\vec{\beta}(z) - \vec{\delta}(z), z) - D(\vec{\beta}(z) + \vec{\gamma}(z), z) - D(\vec{\beta}(z) - \vec{\gamma}(z), z)$$

Since the output coordinates are separated in the configuration mentioned above, we might expect that the path coordinates in the region of path space which is most important to the result, will mimic the separation and give $\beta(z) \gg \gamma(z)$ and $\beta(z) \gg \delta(z)$. If this is the case, the sum of the structure functions along the two diagonals is nearly equal to the sum of the structure functions along the two longer sides. Consequently, the combination of the four structure functions will be small. Treating these four structure function densities separately, the full combination is written

$$V(\vec{\beta}(z), \vec{\gamma}(z), \vec{\delta}(z), z) = V_0(\vec{\gamma}(z), \vec{\delta}(z), z) + V_1(\vec{\beta}(z), \vec{\gamma}(z), \vec{\delta}(z), z) \tag{2.6.8}$$

where

$$V_0(\vec{\gamma}(z), \vec{\delta}(z), z) = D(\vec{\gamma}(z) + \vec{\delta}(z), z) + D(\vec{\gamma}(z) - \vec{\delta}(z), z) \tag{2.6.9}$$

and

$$V_1(\vec{\beta}(z), \vec{\gamma}(z), \vec{\delta}(z), z) = D(\vec{\beta}(z) + \vec{\delta}(z), z) + D(\vec{\beta}(z) - \vec{\delta}(z), z) \tag{2.6.10}$$

$$- D(\vec{\beta}(z) + \vec{\gamma}(z), z) - D(\vec{\beta}(z) - \vec{\gamma}(z), z)$$

While the cancellation of structure functions in V_1 suggest that it will be small, V_0 may have any magnitude. The crucial point about V_0 is that it is independent of $\vec{\beta}(z)$.

The FMGF may now be evaluated in a series, using V_1 as an expansion parameter. Changing path variables in eq. (2.6.6) to the deviations about the straight-line paths as

$$\vec{\beta}_1(z) = \vec{\beta}(z) - \vec{f}(\vec{\beta}, z) \tag{2.6.11}$$

$$\vec{\gamma}_1(z) = \vec{\gamma}(z) - \vec{f}(\vec{\gamma}, z) \tag{2.6.12}$$

the medium factor becomes

$$M(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \left[\frac{2\pi R}{k}\right]^2 \iint D\{\vec{\beta}(z)\} D\{\vec{\gamma}(z)\} \tag{2.6.13}$$

$$\exp\left\{ ik\int_0^R \vec{\beta}_1{}'(z)\cdot\vec{\gamma}_1{}'(z)dz - \frac{1}{2}\int_0^R \left[V_0(\vec{\gamma}_1(z) + \vec{f}(\vec{\gamma}, z), \vec{f}(\vec{\delta}, z), z) \right.\right.$$

$$\left.\left. + V_1(\vec{\beta}_1(z) + \vec{f}(\vec{\beta}, z), \vec{\gamma}_1(z) + \vec{f}(\vec{\gamma}, z), \vec{f}(\vec{\delta}, z), z) \right] dz \right\}$$

The cancellation in V_1 suggests that we perform a Taylor expansion in the V_1 dependent portion of the integrand.

$$\exp\left\{ -\frac{1}{2}\int_0^R V_1 dz \right\} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\frac{1}{2}\int_0^R V_1 dz \right]^n \tag{2.6.14}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^R dz_1 \ldots \int_0^R dz_n \prod_{m=1}^{n} \left\{ -\frac{1}{2} V_1(\vec{\beta}(z_m), \vec{\gamma}(z_m), \vec{\delta}(z_m), z_m) \right\}$$

The expansion of $e^{-\frac{1}{2}\int_0^R V_1 dz}$ will simplify M if we write V_1 in terms of the turbulence spectrum, defined

$$\Phi_\mu(\vec{q}, z) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} B_\mu(\vec{s}, \zeta, z) e^{-i\vec{q}\cdot(\vec{s}, \zeta)} d^2 s \tag{2.6.15}$$

where $\vec{q}$ is a three-dimensional wave vector (or "spatial frequency") and $B_\mu(\vec{s}, \zeta, z) \equiv \langle \mu(\vec{x} - \frac{1}{2}\vec{s}, z - \frac{1}{2}\zeta)\mu(\vec{x} + \frac{1}{2}\vec{s}, z + \frac{1}{2}\zeta) \rangle$ and the statistics are assumed to be homogeneous in $\vec{x}$ and locally homogeneous in z. Using the definition for A_μ as given in eq. (2.5.11), we find

$$A_\mu(\vec{s}, z) = 2\pi \int_{-\infty}^{\infty} \Phi_\mu(\vec{\kappa}, z) e^{i\vec{\kappa}\cdot\vec{s}} d^2\kappa \tag{2.6.16}$$

where $\vec{\kappa}$ is a two-dimensional spatial-frequency. The definition for the wave structure function density, eq. (2.5.12), leads to

$$D(\vec{s}, z) = 4\pi k^2 \int_{-\infty}^{\infty} \left[1 - e^{i\vec{\kappa}\cdot\vec{s}}\right] \Phi_\mu(\vec{\kappa}, z) d^2\kappa \tag{2.6.17}$$

Using this result, we may write V_1 as

$$V_1(\vec{\beta}(z), \vec{\gamma}(z), \vec{\delta}(z), z) = \tag{2.6.18}$$

$$8\pi k^2 \int_{-\infty}^{\infty} e^{i\vec{\kappa}\cdot\vec{\beta}(z)} \left[\cos(\vec{\kappa}\cdot\vec{\gamma}(z)) - \cos(\vec{\kappa}\cdot\vec{\delta}(z))\right] \Phi_\mu(\vec{\kappa}, z) d^2\kappa$$

The n-th term in the expansion, eq. (2.6.14), contains the factor

$$\prod_{m=1}^{n} \left\{ -\frac{1}{2} V_1(\vec{\beta}(z_m), \vec{\gamma}(z_m), \vec{\delta}(z_m), z_m) \right\}$$

written in terms of the turbulence spectrum, this factor becomes

$$(4\pi k^2)^n \int_{-\infty}^{\infty} \!\!\dots\! \int d^2\kappa_1 \dots d^2\kappa_n \, e^{i(\vec{\kappa}_1\cdot\vec{\beta}(z_1) + \dots + \vec{\kappa}_n\cdot\vec{\beta}(z_n))} \tag{2.6.19}$$

$$\prod_{m=1}^{n} \left\{ \Phi_\mu(\vec{\kappa}_m, z_m) \left[\cos(\vec{\kappa}_m\cdot\vec{\delta}(z_m)) - \cos(\vec{\kappa}_m\cdot\vec{\gamma}(z_m))\right] \right\}$$

This leads to the expansion of the medium factor

$$M = \sum_{n=0}^{\infty} M_n$$

where

$$M_n(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \tag{2.6.20}$$

$$\frac{(4\pi k^2)^n}{n!}\left[\frac{2\pi R}{k}\right]^{2}\int_0^R dz_1 \ldots \int_0^R dz_n \int_{-\infty}^{\infty} \ldots \int d^2\kappa_1 \ldots d^2\kappa_n$$

$$\iint D\{\vec{\beta}(z)\}D\{\vec{\gamma}(z)\}\exp\left\{-\frac{1}{2}\int_0^R V_0(\vec{\gamma}_1(z)+\vec{f}(\vec{\gamma},z),\vec{f}(\vec{\delta},z),z)dz\right\}$$

$$\exp\left\{-ik\int_0^R \vec{\gamma}_1{}''(z)\cdot\vec{\beta}_1(z)dz + i(\vec{\kappa}_1\cdot\vec{\beta}_1(z_1)+\ldots+\vec{\kappa}_n\cdot\vec{\beta}_1(z_n))\right.$$

$$\left. + i(\vec{\kappa}_1\cdot\vec{f}(\vec{\beta},z_1)+\ldots+\vec{\kappa}_n\cdot\vec{f}(\vec{\beta},z_n))\right\}$$

$$\prod_{m=1}^{n}\left\{\Phi_\mu(\vec{\kappa}_m,z_m)\left[\cos(\vec{\kappa}_m\cdot\vec{\delta}(z_m)) - \cos(\vec{\kappa}_m\cdot\vec{f}(\vec{\gamma},z_m)+\vec{\kappa}_m\cdot\vec{\gamma}_1(z_m))\right]\right\}$$

which is now in a form that may be evaluated.

The $n = 0$ term is given by

$$M_0(\vec{\beta},\vec{\gamma},\vec{\delta},R;\vec{\beta}_0,\vec{\gamma}_0,\vec{\delta}_0) = \tag{2.6.21}$$

$$\left[\frac{2\pi R}{k}\right]^2\iint D\{\vec{\beta}(z)\}D\{\vec{\gamma}(z)\}\exp\left\{-ik\int_0^R \vec{\beta}_1(z)\cdot\vec{\gamma}_1{}''(z)dz\right\}$$

$$\exp\left\{-\frac{1}{2}\int_0^R V_0(\vec{\gamma}_1(z)+\vec{f}(\vec{\gamma},z),\vec{f}(\vec{\delta},z),z)dz\right\}$$

Since V_0 is independent of $\vec{\beta}_1(z)$, the double path integral is in the form of the identity, eq. (2.4.20). Noting that $\vec{\gamma}_1(z)$ is zero at $z = 0$ and $z = R$ by construction, the effect of the identity is to set $\vec{\gamma}_1(z) = 0$ in V_0, leading to the term

$$M_0(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R\,;\, \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \tag{2.6.22}$$

$$\exp\left\{ -\frac{1}{2}\int_0^R \left[D\left(\vec{f}(\vec{\gamma}, z) + \vec{f}(\vec{\delta}, z), z\right) + D\left(\vec{f}(\vec{\gamma}, z) - \vec{f}(\vec{\delta}, z), z\right) \right] dz \right\}$$

where we have written out V_0 in terms of the wave structure function densities.

For $n > 0$, the calculation is similar, but with a few more complications. Writing the sum $\sum_{m=1}^{n} \vec{\kappa}_m \cdot \vec{\beta}_1(z_m)$ as [Dashen, 1979]

$$\vec{\kappa}_1 \cdot \vec{\beta}_1(z_1) + \ldots + \vec{\kappa}_n \cdot \vec{\beta}_1(z_n) = \int_0^R \vec{\beta}_1(z) \cdot \left[\sum_{m=1}^{n} \vec{\kappa}_m \delta(z - z_m) \right] dz \tag{2.6.23}$$

allows us to write M_n as

$$M_n(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R\,;\, \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \frac{(4\pi k^2)^n}{n\,!} \left[\frac{2\pi R}{k} \right]^{2R} \int_0 dz_1 \ldots \int_0^R dz_n \int_{-\infty}^{\infty} \ldots \int d^2\kappa_1 \ldots d^2\kappa_n \tag{2.6.24}$$

$$\exp\left\{ i\left(\vec{\kappa}_1 \cdot \vec{f}(\vec{\beta}, z_1) + \ldots + \vec{\kappa}_n \cdot \vec{f}(\vec{\beta}, z_n)\right) \right\}$$

$$\iint D\{\vec{\beta}(z)\} D\{\vec{\gamma}(z)\} \exp\left\{ -\frac{1}{2}\int_0^R V_0\left(\vec{\gamma}_1(z) + \vec{f}(\vec{\gamma}, z), \vec{f}(\vec{\delta}, z), z\right) dz \right\}$$

$$\exp\left\{ -ik \int_0^R \vec{\beta}_1(z) \cdot \left[\vec{\gamma}_1{}''(z) - \sum_{m=1}^{n} \frac{\vec{\kappa}_m}{k} \delta(z - z_m) \right] dz \right\}$$

$$\prod_{m=1}^{n} \left\{ \Phi_\mu(\vec{\kappa}_m, z_m) \left[\cos(\vec{\kappa}_m \cdot \vec{f}(\vec{\delta}, z_m)) - \cos(\vec{\kappa}_m \cdot \vec{f}(\vec{\gamma}, z_m) + \vec{\kappa}_m \cdot \vec{\gamma}_1(z_m)) \right] \right\}$$

Since $\vec{\beta}_1(z)$ does not appear in V_0 or in the product factor, the double path integral may also be evaluated using the identity, eq. (2.4.20). The identity requires that $\vec{\gamma}_1(z)$ satisfy

$$\vec{\gamma}_1{}''(z) = \sum_{m=1}^{n} \frac{\vec{\kappa}_m}{k} \delta(z - z_m) \tag{2.6.25}$$

with $\vec{\gamma}_1(0) = \vec{\gamma}_1(R) = 0$. The solution of this ordinary differential equation may be found by linear superposition. Define $g(z; z_m)$ as the solution of the equation

$$\frac{\partial^2 g(z; z_m)}{\partial z^2} = \delta(z - z_m) \tag{2.6.26}$$

subject to the boundary conditions

$$g(0; z_m) = g(R; z_m) = 0$$

and the condition that g be continuous at $z = z_m$, the solution is easily found to be

$$g(z; z_m) = \begin{cases} z(\frac{z_m}{R} - 1) & z < z_m \\ z_m(\frac{z}{R} - 1) & z > z_m \end{cases} \tag{2.6.27}$$

Multiplying eq. (2.6.26) by $\vec{\kappa}_m/k$ and summing on m gives eq. (2.6.25). Therefore, the solution to eq. (2.6.25) is

$$\vec{\gamma}_1(z) = \sum_{m=1}^{n} \frac{\vec{\kappa}_m}{k} g(z; z_m) \tag{2.6.28}$$

Thus, application of the identity to eq. (2.6.24) yields the n-th term in the expansion as

$$M_n(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \frac{(4\pi k^2)^n}{n!} \int_0^R dz_1 \ldots \int_0^R dz_n \int_{-\infty}^{\infty} \ldots \int d^2\kappa_1 \ldots d^2\kappa_n \tag{2.6.29}$$

$$\exp\left\{ i \sum_{m=1}^{n} \vec{\kappa}_m \cdot \vec{f}(\vec{\beta}, z_m) - \frac{1}{2} \int_0^R \left[D\left(\sum_{m=1}^{n} \frac{\vec{\kappa}_m}{k} g(z; z_m) + \vec{f}(\vec{\gamma}, z) + \vec{f}(\vec{\delta}, z), z \right) \right. \right.$$

$$\left. \left. + D\left(\sum_{m=1}^{n} \frac{\vec{\kappa}_m}{k} g(z; z_m) + \vec{f}(\vec{\gamma}, z) - \vec{f}(\vec{\delta}, z), z \right) \right] dz \right\}$$

$$\prod_{m=1}^{n}\left\{\Phi_\mu(\vec{\kappa}_m, z_m)\left[\cos(\vec{\kappa}_m\cdot\vec{f}(\vec{\delta}, z_m)) - \cos(\vec{\kappa}_m\cdot\sum_{j=1}^{n}\frac{\vec{\kappa}_j}{k}g(z_m; z_j) + \vec{\kappa}_m\cdot\vec{f}(\vec{\gamma}, z_m))\right]\right\}$$

Notice that the symmetry in eq. (2.6.2) *does not* carry over to the terms in the series. However, the symmetry of G_4 implies that we may write a second series by writing the symmetric form of the first. This corresponds with examining the fourth moment when $\gamma \gg \beta$ and $\gamma \gg \delta$. The wave structure function densities will then group differently, leading to the expansion for M

$$M = \sum_{n=0}^{\infty} M_{n,symm} \tag{2.6.30}$$

where $M_{n,symm}$ is the symmetric form of M_n defined

$$M_{n,symm}(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = M_n(\vec{\gamma}, \vec{\beta}, \vec{\delta}, R; \vec{\gamma}_0, \vec{\beta}_0, \vec{\delta}_0) \tag{2.6.31}$$

The terms in the symmetric expansion of M are

$$M_{0,symm}(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \tag{2.6.32}$$

$$\exp\left\{-\frac{1}{2}\int_0^R\left[D(\vec{f}(\vec{\beta}, z) + \vec{f}(\vec{\delta}, z), z) + D(\vec{f}(\vec{\beta}, z) - \vec{f}(\vec{\delta}, z), z)\right]dz\right\}$$

and, for $n > 0$,

$$M_{n,symm}(\vec{\beta}, \vec{\gamma}, \vec{\delta}, R; \vec{\beta}_0, \vec{\gamma}_0, \vec{\delta}_0) = \frac{(4\pi k^2)^n}{n!}\int_0^R dz_1\ldots\int_0^R dz_n \int_{-\infty}^{\infty}\ldots\int d^2\kappa_1\ldots d^2\kappa_n \tag{2.6.33}$$

$$\exp\left\{i\sum_{m=1}^{n}\vec{\kappa}_m\cdot\vec{f}(\vec{\gamma}, z_m) - \frac{1}{2}\int_0^R\left[D(\sum_{m=1}^{n}\frac{\vec{\kappa}_m}{k}g(z; z_m) + \vec{f}(\vec{\beta}, z) + \vec{f}(\vec{\delta}, z), z)\right.\right.$$

$$\left.\left. + D(\sum_{m=1}^{n}\frac{\vec{\kappa}_m}{k}g(z; z_m) + \vec{f}(\vec{\beta}, z) - \vec{f}(\vec{\delta}, z), z)\right]dz\right\}$$

$$\prod_{m=1}^{n}\left\{\Phi_\mu(\vec{\kappa}_m, z_m)\left[\cos(\vec{\kappa}_m \cdot \vec{f}(\vec{\delta}, z_m)) - \cos(\vec{\kappa}_m \cdot \sum_{j=1}^{n}\frac{\vec{\kappa}_j}{k} g(z_m; z_j) + \vec{\kappa}_m \cdot \vec{f}(\vec{\beta}, z_m))\right]\right\}$$

Either form of the complete expansion for M generates the entire function. However, if we consider the special case of the two-point intensity correlation, or, better, its Fourier transform (the "intensity spectrum"), the original series for M converges faster at low spatial-frequencies while the symmetric series will converge faster at high spatial-frequencies. Truncated forms of the two series may be used in concert to form an approximation to the intensity spectrum at all spatial-frequencies. This technique is reasonable only when the scattering is strong and the scintillation spatial scales are well separated (i.e. the size of the scattering disk is much larger than the coherence scale). In weak scattering, the low spatial-frequency expansion gives an adequate description for the entire intensity spectrum.

Important results from this section are the decomposition of the fourth moment Green's function into a free-space factor, G_4^f, and a medium-dependent factor, M, eq. (2.6.4), the expansion of M into a series, eqs. (2.6.22) and (2.6.29), and the symmetric form, eqs. (2.6.32) and (2.6.33).