Nitigya

HW 9 OPTI-330

Q.1

(a) $\vec{F} = D \cdot \vec{f}$ $\qquad \vec{F_k} = \sum_{n=0}^{N-1} f_n \exp\left(-2\pi i \frac{kn}{N}\right)$

where

$$D_K = \sum_{n=0}^{N-1} \exp\left(-2\pi i \frac{kn}{N}\right)$$

defining $\omega = \exp\left(-\frac{2\pi i}{N}\right)$ so elements of D will be $\omega^{(kn)}$

so for $N=8$ D will be a 8×8 matrix $D = \begin{array}{c} \\ k=0 \\ \vdots \\ k= \end{array} \begin{array}{c} n=0 \ldots \\ \begin{bmatrix} \omega^{kn} & \cdots \\ \vdots & \\ \vdots & \end{bmatrix} \end{array}$

(b) $D =$

$$D_8 = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccccc} n=0 & n=1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} \end{array}$$

$\forall\ \omega = e^{-\frac{2\pi i}{N}}$

(c) By observation it is clear that

$$D_8(k,n) = D_8(n,k) \Rightarrow \text{symmetric matrix}$$

$$D_8 = [D_8]^T \quad \text{transpose} = \text{matrix}$$

also $\exp(-2\pi(i/N))l = \exp(-2\pi(i/N)\, l\,\mathrm{mod}(N))$

$$w^l = w^{l\,\mathrm{mod}N}$$

$l\,\mathrm{mod}\,N$ operator returns the remainder of l/N

$$w^9 = w^1 \qquad w^{10} = w^2 \qquad w^{11} = w^3 \quad \ldots\ldots$$

we can simplify our matrix

d part at the end (matlab code)

Q.2

$f(x) \rightarrow$ function

$f(n) \rightarrow$ my sampled function

(a) Upsample by 10 & then down sample by 20

$f(n)$

$$F_k = \sum_{n=0}^{N-1} f_n \exp\left(-2\pi i \frac{nk}{N}\right)$$

Now for upsampling mathematically

$$f_u[n] = \begin{cases} f[n/10] & n = 10k \quad k \equiv \text{integer} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{F.T of } f_{u,s}[n] = \sum f[n/10] \exp\left(-2\pi i \frac{nk}{N}\right) \quad \frac{n}{10} = n'$$

$$n = 10n'$$

$$= \sum_{n'} f[n'] \exp\left(-2\pi i \frac{\boxed{k10}n'}{N}\right)$$

$$= \sum_{n'} f[n'] \exp\left(-2\pi i \frac{\xi n'}{N}\right)$$

$$= F(10k)$$

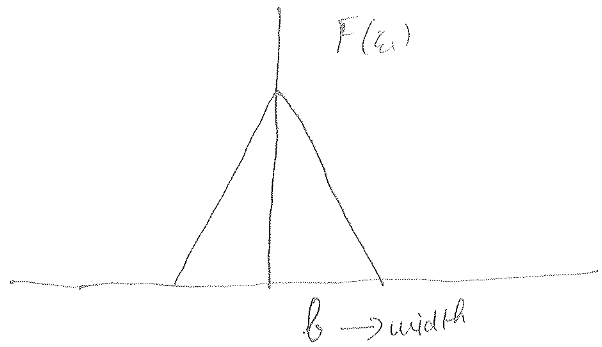
where $k \rightarrow$ the frequency term in discreet space

$$= F\left(\frac{k}{1/10}\right)$$

or if we say ξ_s is the sampling fa $(k \equiv \xi_s)$

$$= F\left(\frac{\xi_s}{1/10}\right)$$

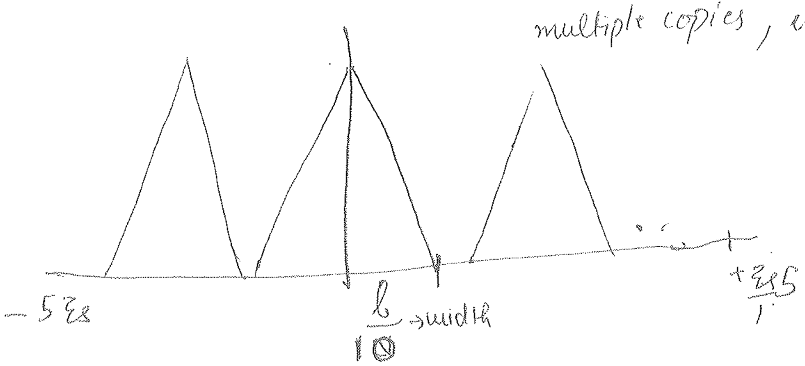
upsampling effect will be to compress the DFT

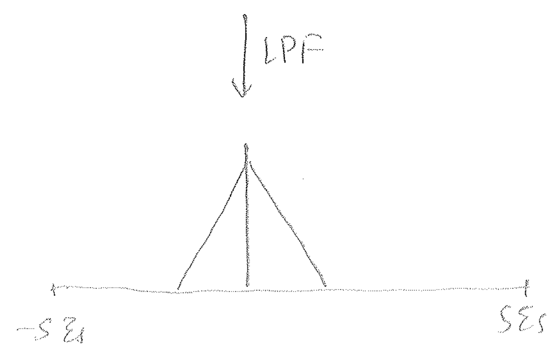


~~F(10ξ)~~

F(10ξs)

multiple copies, we filter the centre one





Now the downsampling operation is preceded by a LPF operation downsampling by 20 so we multiply by the rect function

$$F_{M,D} = F_s(10\xi)\,\text{rect}\left(\frac{\xi}{\xi_s/20}\right)$$

Mathematically the Down sampling operation is $f_D(n) = \begin{cases} f[20n] & n = \frac{K}{20} \\ 0 & \end{cases}$

so in freq domain the ~~freq~~ of signal will be

$$\frac{10\xi_s}{20} = \frac{\xi_s}{2}$$

$F_{M,D} = F_s\left(\frac{\xi_s}{2}\right)\text{rect}\left(\frac{\xi}{\xi_s/20}\right)$ → Here no information is lost in the downsampling

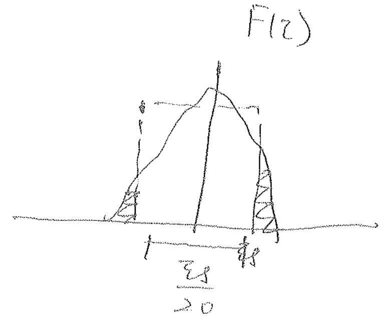
(b) downsampling then upsampling

$f(n) \xrightarrow{FT} = F(\xi_s)$

Downsampling first so have to pass the signal through a LPF to prevent aliasing

LPF operation $\rightarrow$ multiply by $\mathrm{rect}\left(\frac{\xi_s}{\frac{\xi_s}{20}}\right)$

so $F_{D,s} = F(\xi_s)\,\mathrm{rect}\left(\frac{\xi}{\frac{\xi_s}{20}}\right)$



so after downsample operatio

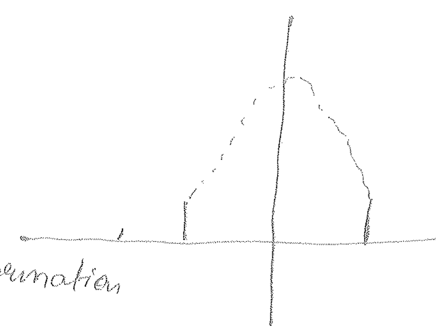
$F_{D,s} = F\left(\frac{\xi_s}{20}\right)$

function is defined only from $-\frac{\xi_s}{40}$ to $+\frac{\xi_s}{40}$

doing this LPF operation would lead to loss of information. which was not the case in part (a)

then upsampling process by 10

$F_{D,u} = F\left(\frac{10\xi_s}{20}\right)$



the signal with less information will now be compressed

Q(3)

$$u(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\xi,\eta)\, e^{j2\pi(\xi x+\eta y)}\, e^{jz\frac{2\pi}{\lambda}\sqrt{1-\lambda^2(\xi^2+\eta^2)}}\, d\xi\, d\eta$$

now for the expression

$\sqrt{1+x}$ if we expand using the Taylor series formula we get

$\sqrt{1+x} = 1+\frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ if we assume x is small

then $\sqrt{1+x} \approx 1+\frac{1}{2}x$

so for the term

$$\sqrt{1-\lambda^2(\xi^2+\eta^2)} = 1-\frac{1}{2}\lambda^2(\xi^2+\eta^2)$$

putting this in eq of field

$$u(x,y,z) = \iint_{-\infty}^{\infty} U(\xi,\eta)\, e^{j2\pi(\xi x+\eta y)}\, e^{jz\frac{2\pi}{\lambda}\left(1-\frac{1}{2}\lambda^2(\xi^2+\eta^2)\right)}\, d\xi\, d\eta$$

$$= \iint_{-\infty}^{\infty} U(\xi,\eta)\, e^{j2\pi(\xi x+\eta y)}\, e^{jz\frac{2\pi}{\lambda}}\, e^{-jz\pi\lambda(\xi^2+\eta^2)}\, d\xi\, d\eta$$

$$= e^{jzk}\iint_{-\infty}^{\infty} \underbrace{U(\xi,\eta)}_{\text{fun 1}}\, \underbrace{e^{-jz\pi\lambda(\xi^2+\eta^2)}}_{\text{fun 2}}\, \underbrace{e^{j2\pi(\xi x+\eta y)}}_{\text{inverse F-T Kernal}}\, d\xi\, d\eta$$

$\forall\ K = \frac{2\pi}{\lambda}$

This is quite similar to inverse fourier transform integral

Also using the convolution theorem

$$= e^{jkz}\, u(x,y) \circledast F^{-1}(\text{function 2})$$

1

if we use the following fourier transform pair

If $H(\xi,\eta) = e^{jkz_{12}} \, e^{-j\pi\lambda z_{12}(\xi^2+\eta^2)}$

iF.T $\downarrow\uparrow$ FT

then $h(x,y)$

$$\Downarrow \quad \frac{e^{jkz_{12}}}{j\lambda z_{12}} \exp\left\{ jk\frac{\pi}{\lambda z_{12}}(x^2+y^2)\right\}$$

so using this information in (1) we get

$$u(x,y,z) = \frac{e^{jkz}}{j\lambda z} \; u(x,y) \circledast \cdot e^{j\frac{\pi}{\lambda z}(x^2+y^2)}$$

doing convolution

$$= \frac{e^{jkz}}{j\lambda z} \iint_{-\infty}^{\infty} u(x_1,y_1)\, e^{\left(j\frac{\pi}{\lambda z}\left((x-x_1)^2+(y-y_1)^2\right)\right)} dx_1 dy_1$$

$\forall$ x & y are coordinates on the obs. screen

fresnel diffraction integral

$$= \frac{e^{jkz}}{j\lambda z} \, e^{j\frac{\pi}{\lambda z}(x^2+y^2)} \, F\left\{ u(x_1,y_1)\, e^{j\frac{\pi}{\lambda z}(x_1^2+y_1^2)} \right\}\Bigg|_{\xi=\frac{x_2}{\lambda z},\ \eta=\frac{y_2}{\lambda z}}$$

Fresnel diffraction pattern can be computed by taking the Fourier Transform of the aperture field modified by a phase curvature term in the aperture plane.

all this valid in assumption that $\lambda^2(\xi^2+\eta^2) \ll 1$

Q·4

Fraunhoffer diffraction

$$\mu(x_2,y_2) = \frac{e^{jkz_{12}}}{j\lambda z_{12}} e^{\frac{j\pi}{\lambda z_{12}}(x_2^2+y_2^2)} \mathcal{F}(\mu(x_1,y_1)) \Big|_{\xi = \frac{x_2}{\lambda z_{12}} \quad \eta = \frac{y_2}{\lambda z_{12}}}$$

$x_1, y_1 \rightarrow$ coordinates in ~~[illegible]~~ aperture

$x_2, y_2 \rightarrow$ coord. in observation screen

$\xi, \eta \rightarrow$ spatial frequency defined as

$$\xi = \frac{x_2}{\lambda z_{12}} \qquad \eta = \frac{y_2}{\lambda z_{12}}$$

for fraunhoffer integral

$$z_{12} > \frac{\pi L_{12}^2}{\lambda} = \frac{\pi(10\times10^{-3})^2}{500\times10^{-9}} = \frac{\pi}{5\times10^{-5}} = 6.2832\times10^4 \text{ m}$$

Intensity in obs plane

$$I(x_2,y_2) = \left(\frac{1}{\lambda z_{12}}\right)^2 \left| F(\mu(x_1,y_1)) \right|^2$$

$$\frac{1}{\lambda z_{12}} = \frac{1}{\frac{100}{500\times10^{-9}\cancel{\pi}} \times \frac{\pi}{5\times10^{\cancel{-5}}}} = \frac{100}{\pi}$$

also

$$F(\mu(x_1,y_1)) = F\left[\text{rect}\left(\frac{x_1}{L_1}, \frac{y_1}{L_1}\right) - \text{rect}\left(\frac{x-x_0}{L_2}, \frac{y-y_0}{L_2}\right)\right]$$

$$= L_1^2 \,\text{sinc}(L_1\xi, L_1\eta) - L_2^2 \,\text{sinc}(L_2\xi, L_2\eta)\, e^{-2\pi i(\xi x_0 + \eta y_0)}$$

$\rightarrow$ replacing ξ & η

$$= L_1^2 \,\text{sinc}\left(\frac{L_1 x_2}{\lambda z_{12}}, \frac{L_1 y_2}{\lambda z_{12}}\right) - L_2^2 \,\text{sinc}\left(\frac{L_2 x_2}{\lambda z_{12}}, \frac{L_2 y_2}{\lambda z_{12}}\right) e^{-2\pi i\left(\frac{x_0 x_2}{\lambda z_{12}} + \frac{y_0 y_2}{\lambda z_{12}}\right)}$$

(a)

$$I(x_2, y_2) = \left(\frac{100}{\pi}\right)^2 \left[(0.1)^4 \operatorname{sinc}^2\left(\frac{10x_2}{\pi}, \frac{10y_2}{\pi}\right) + 0.02^4 \operatorname{sinc}^2\left(\frac{2x_2}{\pi}, \frac{2y_2}{\pi}\right) \right.$$

$$\left. - 2\, 0.1^2\, 0.02^2 \operatorname{sinc}\left(\frac{10x_2}{\pi}, \frac{10y_2}{\pi}\right) \operatorname{Sinc}\left(\frac{2x_2}{\pi}, \frac{2y_2}{\pi}\right) \right]$$

where $\frac{L_1 x_2}{\lambda z_{12}} = \frac{10 \times 10^{-2}}{500 \times 10^{-9} \times \pi}$ 5×10^{-5} $= \frac{10x_2}{\pi}$ $\qquad \frac{L_2 x_2}{\lambda z_{12}} = \frac{2x_2}{\pi}$

(b)

$$\left(\frac{100}{\pi}\right)^2 \left[(0.1)^4 \operatorname{sinc}^2\left(\frac{10x_2}{\pi}, \frac{10y_2}{\pi}\right) + (0.04)^4 \operatorname{Sinc}^2\left(\frac{4x_2}{\pi}, \frac{4y_2}{\pi}\right) \right.$$

$$\left. - 2\, 0.1^2 \times 0.04^2 \operatorname{sinc}\left(\frac{10x_2}{\pi}, \frac{10y_2}{\pi}\right) \operatorname{Sinc}\left(\frac{4x_2}{\pi}, \frac{4y_2}{\pi}\right) \right]$$

(c)

$$\xi x_0 = \frac{x_2 x_0}{\lambda z_{12}} = \cancel{\frac{x_2 x_0}{\pi}} \quad \cancel{\quad} \quad \frac{x_2}{\pi}$$

$\operatorname{Sinc}_1 = \operatorname{sinc}\left(\frac{10x_2}{\pi}, \frac{10y_2}{\pi}\right)$

$\operatorname{Sinc}_2 = \operatorname{sinc}\left(\frac{2x_2}{\pi}, \frac{2y_2}{\pi}\right)$

$$I(x_2, y_2) = \left(\frac{100}{\pi}\right)^2 \left[\left| 0.1^2 \operatorname{sinc}(\;) + 0.02^2 \operatorname{Sinc}(\;) e^{-2\pi i \left(\frac{x_2}{\pi}, \frac{y_2}{\pi}\right)} \right|^2 \right]$$

$$= \left(\frac{100}{\pi}\right)^2 \left[0.1^4 \operatorname{sinc}_1^2(\;) + 0.02^4 \operatorname{sinc}_2^2(\;) - 0.1^2 \times 0.02^2 \operatorname{sinc}_1(\;) \operatorname{sinc}_2(\;) \left(e^{-2\pi i \left(\frac{x_2}{\pi} + \frac{y_2}{\pi}\right)} + e^{+2\pi i \left(\frac{x_2}{\pi}, \frac{y_2}{\pi}\right)} \right) \right]$$

$$= \left(\frac{100}{\pi}\right)^2 \left[0.1^4 \operatorname{sinc}_1^2(\;) + 0.02^4 \operatorname{sinc}_2^2 - 0.1^2 \times 0.02^2 \operatorname{sinc}_1(\;) \operatorname{sinc}_2(\;) \times \underline{2\cos(2x_2 + 2y_2)} \right]$$

Phase term of cos

$$\begin{bmatrix} 2x_2 x_0 \\ 2y_2 y_0 \end{bmatrix}$$

In this case $x_0 = y_0 = 1$

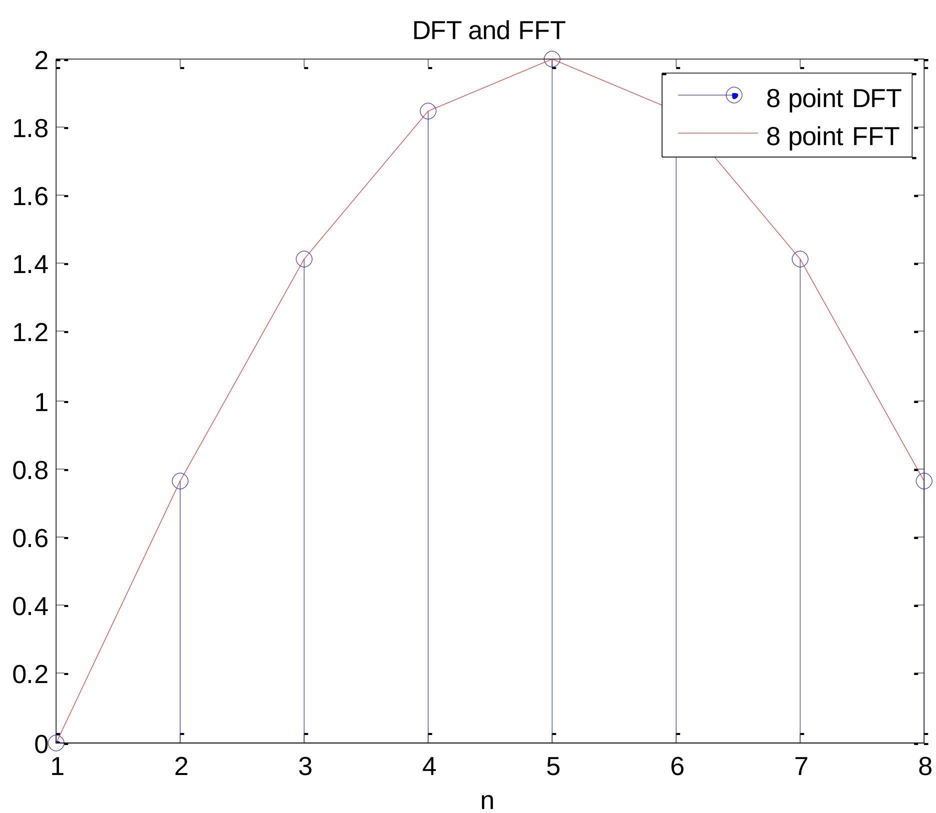
(d)

$$I(x_2, y_2) = \left(\frac{100}{\pi}\right)^2 \Big[0.1^4 \operatorname{sinc}^2\left(\frac{10x_2}{\pi}, \frac{10y_2}{\pi}\right) + 0.02^2 \operatorname{sinc}^2\left(\frac{2x_2}{\pi}, \frac{2y_2}{\pi}\right)$$

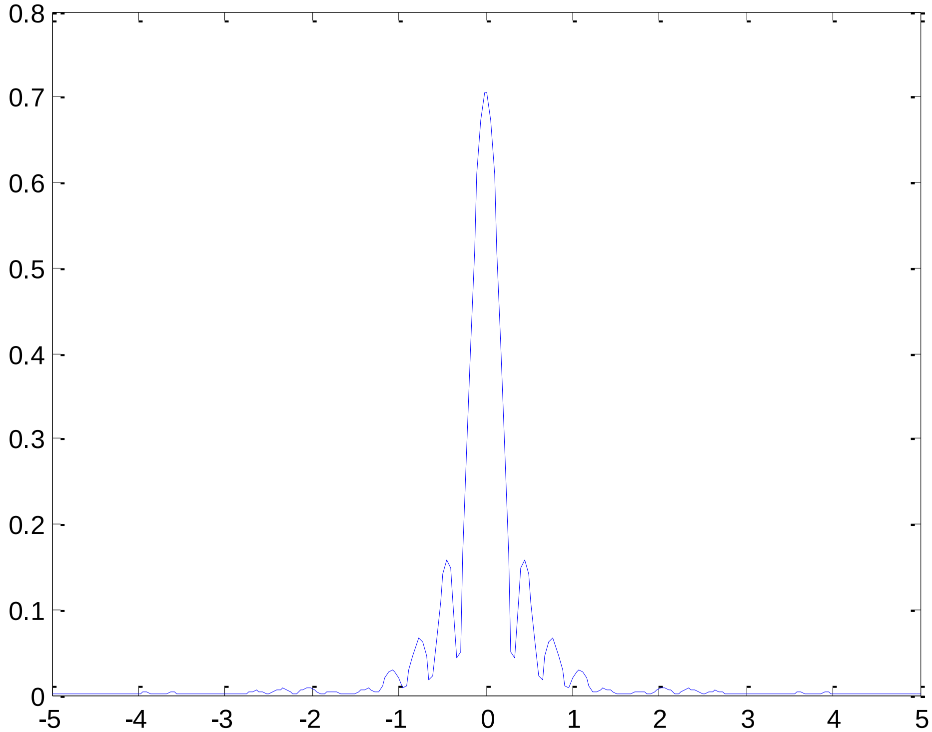
$$- (0.1)^2 (0.02)^2 \operatorname{sinc}\left(\frac{10x_2}{\pi}, \frac{10y_2}{\pi}\right) \operatorname{sinc}\left(\frac{2x_2}{\pi}, \frac{2y_2}{\pi}\right) \times 2\cos(2(2x_2 + 2y_2)) \Big]$$

1d)

```
close all; clear all;
n = linspace(-2,2,9);
n = n(1:end-1); %8ppoint sequence

f=rect(n);  %i have made a function rect that takes in the input variable

 for m=0:length(f)-1;
     for l=0:length(f)-1;
         D(m+1,l+1)=exp(-1i*2*pi*m*l/length(f));% Construct the D matrix
     end;
 end;
% displaythe result with D*f
 FD=abs(D*f');%f'so that we agree with the matrix multiplication
 stem(fftshift(FD))

F=abs(fftshift(fft(fftshift(f,8))));%

hold on
plot(F,'r')
legend('8 point DFT','8 point FFT') ;xlabel('n');title('DFT and FFT')
```



a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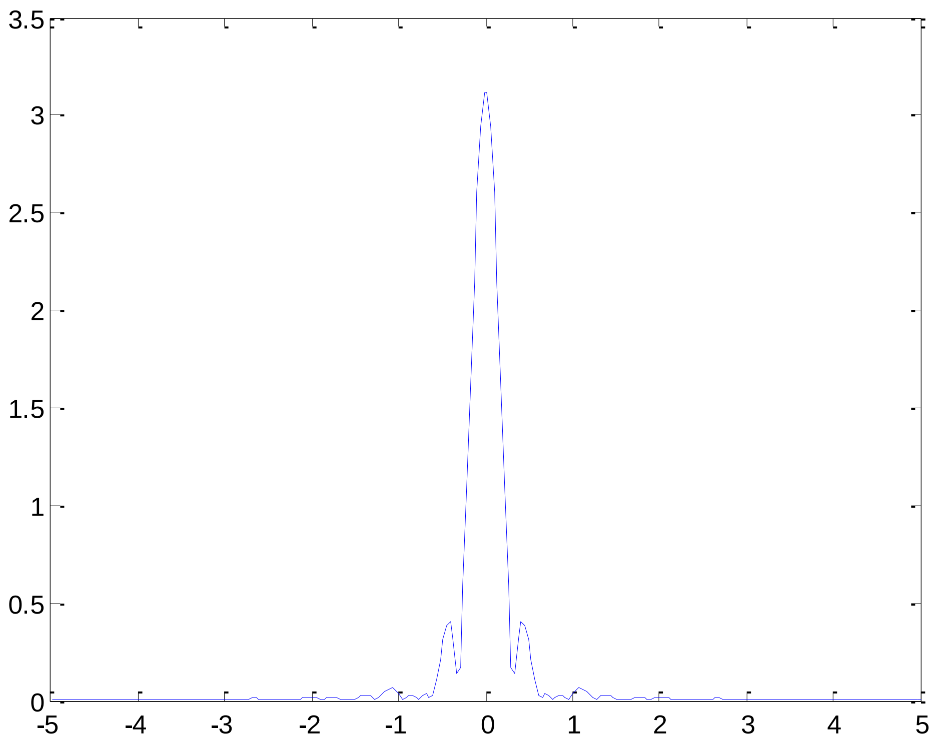
b


3.5
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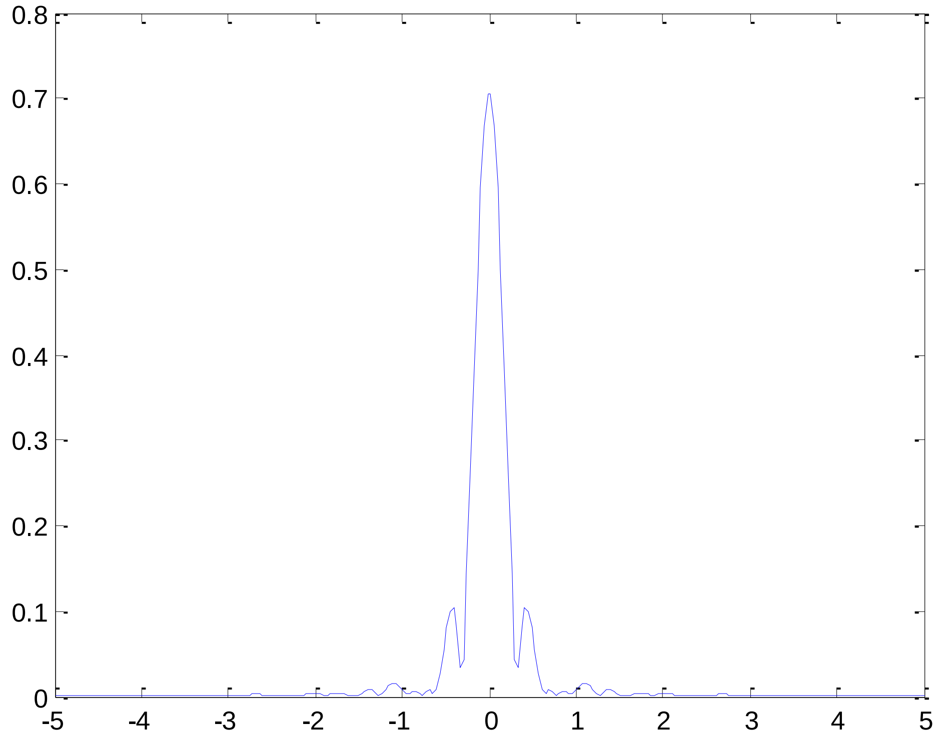
c



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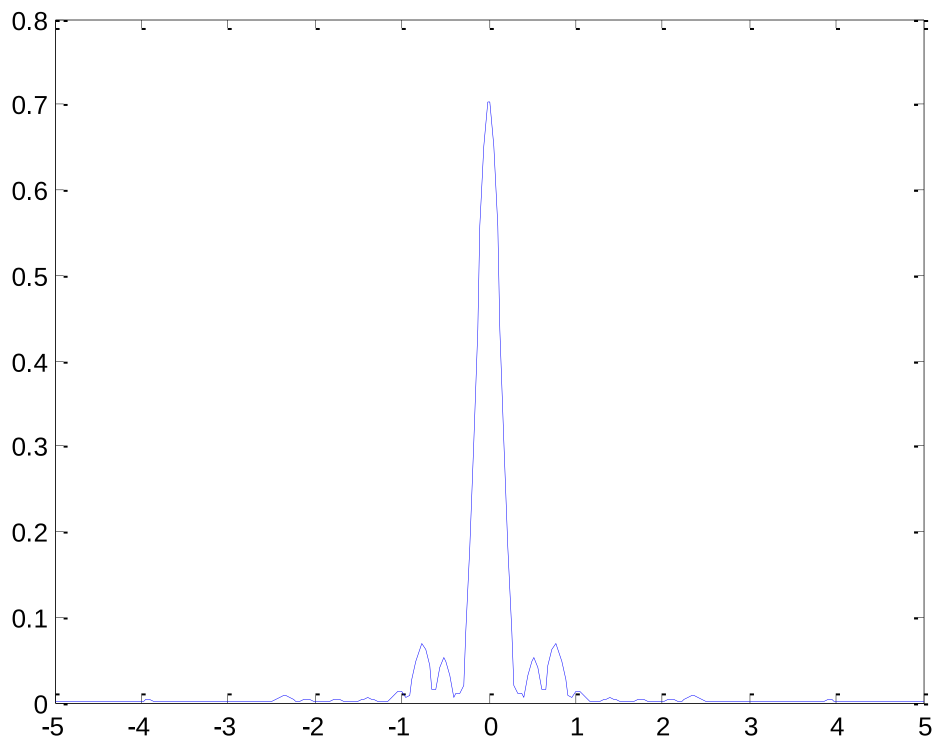
d



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