We will begin by reviewing Maxwell's equations and their solution in source-free media in terms of time-harmonic fields.

Time-harmonic Maxwell's Equations

You should recall from OPTI 310 the fundamental form of Maxwell's equations as follows:

$$\left.\begin{aligned}
\nabla \times E &= -\frac{\partial B}{\partial t} && -(1)\\
\nabla \times H &= J + \frac{\partial D}{\partial t} && -(2)\\
\nabla \cdot D &= \rho && -(3)\\
\nabla \cdot B &= 0 && -(4)
\end{aligned}\right\}$$ spatio-temporal partial-differential equations

$E \equiv$ Electric field amplitude $\quad$ $H \equiv$ Magnetic field amplitude

$D \equiv$ Electric flux density $\quad$ $B \equiv$ Magnetic flux density

$J \equiv$ Current density $\quad$ $\rho \equiv$ Electric charge density

In general,

$$\left.\begin{aligned} D &= \bar{\bar{\epsilon}} E \\ B &= \bar{\bar{\mu}} H \end{aligned}\right\}$$ Constitutive relation

Note: $\bar{\bar{\epsilon}}$ & $\bar{\bar{\mu}}$ are tensors (3x3) in general

Here we consider source-free media where

$$J = 0 \ \& \ \rho = 0 . \qquad -(5)$$

This assumption is reasonable for many applications, especially in optics.

Furthermore, we also assume that media is linear and isotropic

$$\text{Isotropic} \longrightarrow \left.\begin{aligned} D &= \epsilon E \\ B &= \mu H \end{aligned}\right\} - (6)$$

where ϵ & μ are scalar, electric & magnetic permitivities

Now let us consider time-harmonic field which can be expressed as:

$$E(r,t) = Re\{\tilde{E}(r)^* e^{j\omega t}\} \quad -(7)$$

$r \equiv$ is position (x, y, z)
$t \equiv$ time
$\tilde{E} \equiv$ Complex vector phasor
$\omega \equiv$ angular frequency : $2\pi f$

Plugging (5)-(7) in (1)-(4) we get,

$$\left.\begin{aligned} \nabla \times E &= -j\omega\mu H & -(6) \\ \nabla \times H &= +j\omega\epsilon E & -(7) \\ \nabla \cdot E &= 0 & -(8) \\ \nabla \cdot H &= 0 & -(9) \end{aligned}\right\} \text{Time-harmonic source-free Maxwell's Equations.}$$

This is a much simpler and it is relatively easier to solve problems using time-harmonic representation. However, naturally the question arises: what happens when the fields are not time-harmonic?

In that case we can still solve the probem frequency-by-frequency and put everything together using the following superposition of time-harmonic fields. This applicable to Maxwell's equations as they are linear.

$$E(r,t) = \mathrm{Re}\left[\int_0^{\infty} \tilde{E}(r,\omega)^{*} e^{j\omega t} d\omega\right] \quad -(10)$$

Note that Eq.(10) is one form of Fourier transform which will be very important to us this semester.

Scalar Diffraction Theory

Here we will do a quick review of scalar diffraction theory that you learned in OPTI 310 last semester.

Assume that the fields are monochromatic and linearly polarized.

$$E(x,y,z;t) = a(x,y,z)\cdot\cos[2\pi f_0 t - \phi(x,y,z)] \quad -(11)$$

As we mentioned earlier it is easier mathematically to consider complex fields. The complex form of the spatial component can be expressed as

$$E(x,y,z) = a(x,y,z)\, e^{j\phi(x,y,z)} \quad -(12)$$

(12) → (11) we get

$$E(x,y,z;t) = \mathrm{Re}\{E^{*}(x,y,z)\, e^{j\omega_0 t}\} \quad -(13)$$

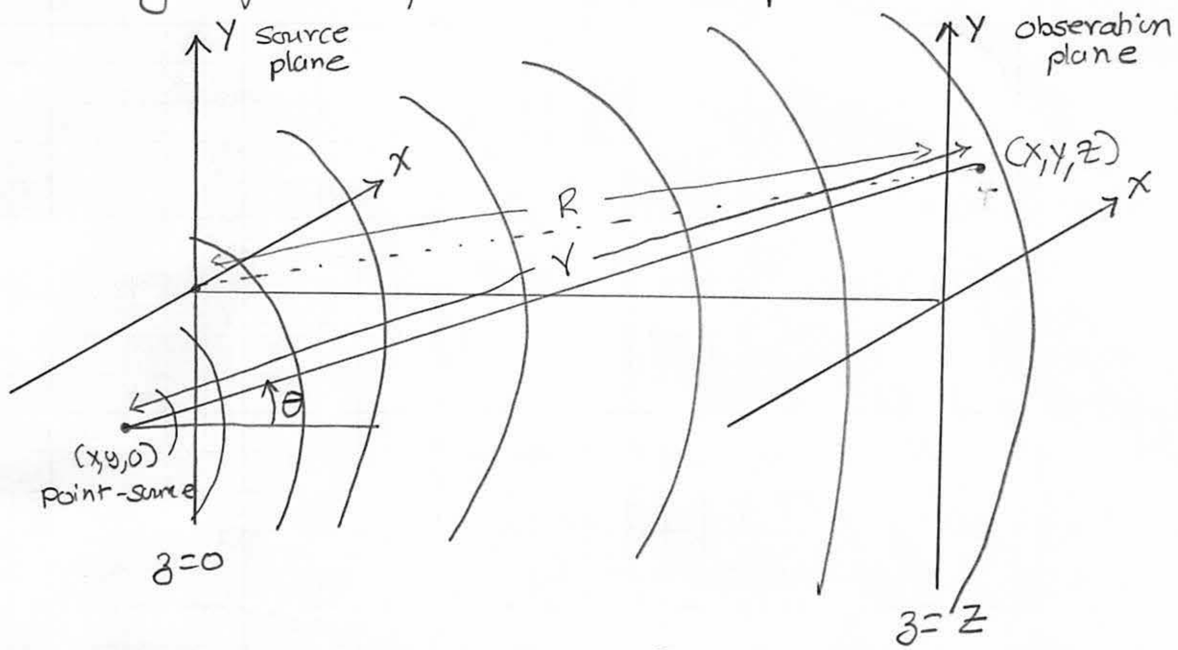
$$\omega_0 = 2\pi f_0$$

Note that while we do our diffraction calculations with optical fields $E(r)$, most optical detectors only respond to the intensity, power or some radiometric quantity. The intensity is given by

$$I(x,y,z) = |E(x,y,z)|^2 \quad - (14)$$

Spherical Waves

In scalar diffraction it is vital that we have a good understanding of the point-source & spherical waves.



Consider the geometry in this figure. Scalar diffraction allows us to write the field in the observation plane as

$$E(x,y,z) = \underbrace{D}_{\text{complex constant}} \overbrace{\cos\theta}^{\text{obliquity factor}} \frac{e^{jkr}}{r} \quad - (15)$$

The obliquity factor accounts for the fact the source is really electromagnetic and therefore radiates like a dipole.

$$\cos\theta = \frac{z}{r} \approx \frac{z}{R} \qquad - (16)$$

where

$$R = \sqrt{X^2+Y^2+z^2}$$ distance from center of aperture

$$r = \sqrt{(X-x)^2 + (Y-y)^2 + z^2}$$

$$\approx R\sqrt{1 - 2\frac{(xX+yY)}{R^2}} \qquad \frac{x^2+y^2}{R^2} \approx 0 \quad (\text{i.e } x^2+y^2 \ll R^2)$$

$$\approx R \cdot \left[1 - \frac{(xX+yY)}{R^2}\right] \qquad \sqrt{1+a} \approx 1 + a/2 \quad (\text{when } a \ll 1)$$

$\hookrightarrow$ (17)

(16) & (17) in (15) we get

$$E(X,Y,z) \approx D \cdot \frac{z}{R} \frac{e^{jk \cdot R\left[1 - \frac{(xX+yY)}{R^2}\right]}}{R}$$

$$= D\left(\frac{z}{R}\right) \frac{e^{jkR}}{R} \cdot e^{-j\frac{k}{R}(xX+yY)}$$

assuming $z \approx R$

$$E(X,Y,z) \approx D \cdot \frac{e^{jkR}}{R} e^{-j\frac{k}{R}(xX+yY)} \qquad - (18)$$

Note that when the source is on axis i.e $x=0, y=0$ we have

$$E(X,Y,z) = D\frac{e^{jkR}}{R} \qquad - (19)$$

In Eq-(19) the surfaces of constant phase are spheres hence the name <u>spherical waves</u>. From onwards as we will be mostly concerned with fields in <u>transverse planes</u> we will express $E(X,Y,z) = E(X,Y)$, where $z = Z$ location is implied.

Huygens-Fresnel Principle

The Huygens-Fresnel principle lets us calculate the fields in (X,Y) plane ($z=Z$) in terms of fields in the plane of (x,y) at ($z=0$). This is a very important principle to diffraction theory we usually take for granted.

More specifically, we can consider each point in (x,y) @ $z=0$ as a point-source with a given amplitude and phase defined by the weight $E(x,y)$. Each of these sources radiates a spherical wave and the field in the observation plane can be simply computed by adding up all the contributions from each point-source. Mathematically,

$$E(X,Y) = \iint_{-\infty}^{\infty} E(x,y) \frac{e^{jkr}}{r} \, dx\, dy \quad - (20)$$

Fraunhofer Diffraction

In the Fraunhofer diffraction regime using (15) & (18) in (20) we can write

$$E(X,Y,Z) \approx \frac{e^{jkR}}{R} \iint_{\infty} E(x,y)\, e^{-jk\left(\frac{xX}{R}+\frac{yY}{R}\right)} dx\,dy$$

making substitutions $k_x = k\frac{X}{R}$ & $k_y = k\frac{Y}{R}$. Note the dimensions of k_x & k_y are $[m^{-1}]$ and can be thought of having the units of <u>spatial frequency</u>.

$$\boxed{E(k_x, k_y) \approx \iint_{-\infty}^{\infty} E(x,y)\, e^{-j(k_x \cdot x + k_y \cdot y)}\, dx\, dy} \quad -(21)$$

Eq. (21) is a very important result. Basically, it tells us that fields in Fraunhofer zone that are diffracted by an aperture are given by the Fourier transform of the field distribution in the aperture. Each point in the observation plane (X, Y) corresponds to a different spatial frequency (k_x, k_y) of the field distribution $E(x,y)$. Compactly we can express this as

$$E(k_x, k_y) = \mathcal{F}\{E(x,y)\}$$

↳ Fourier operator.

Note: Here we have used the linearity of Maxwell's equations and now we see that Fourier transform plays a central role in understanding diffraction.