Convolution and Correlation

We just learned about a very important property of $\mathcal{L}SI$ systems that we can write the output as a convolution integral with the input and impulse response function as

$$g(x) = \int_{-\infty}^{\infty} f(\alpha)\, h(x-\alpha)\, d\alpha$$

$$\xrightarrow{f(x)} \underset{h(x)}{\boxed{\text{System } \mathcal{L}\{\}}} \longrightarrow g(x) = f(x) \circledast h(x)$$

Notationally, we denote convolution between two functions as

$$f(x) \circledast h(x) = \int_{-\infty}^{\infty} f(\alpha)\, h(x-\alpha)\, d\alpha$$

At this point, convolution appears to be a rather abstract operation. Now we will examine it graphically to gain some deeper insight.

Graphical Convolution

Let us consider two example functions:

$$f(x) = \mathrm{rect}\left(\frac{x-1}{2}\right) \quad \& \quad h(x) = \mathrm{ramp}(x)\, \mathrm{rect}\left(\frac{x+1.5}{3}\right)$$

Note that convolution can be broken into three steps

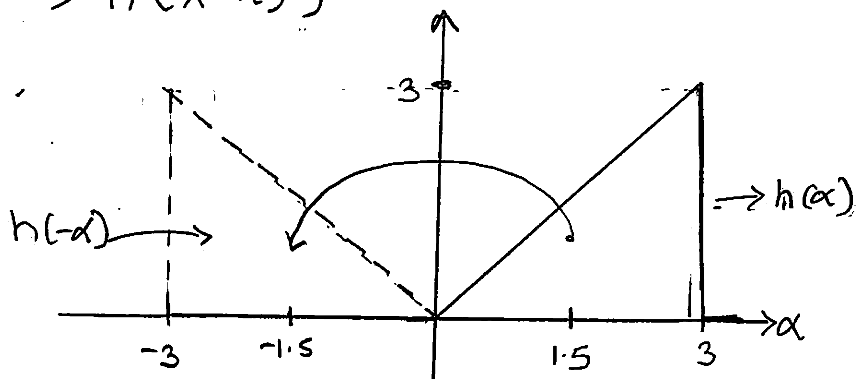
<u>Step 1</u>: Express the two functions in the integral in terms of the two input functions:

$$f(x) \downarrow f(\alpha) \qquad h(x) \downarrow h(x-\alpha)$$

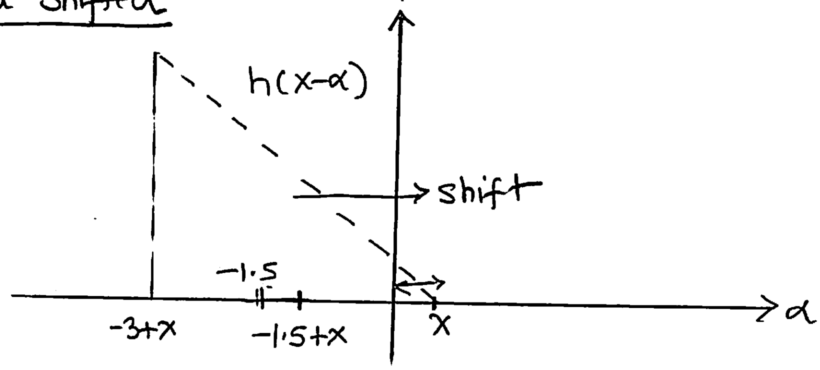
Note that $h(x-\alpha)$ is the original function that has been flipped $(h(x) \to h(-\alpha))$ and shifted $(h(x) \to h(x-\alpha))$

$h(x) = \mathrm{ramp}(x)\,\mathrm{rect}\left(\frac{x-1.5}{3}\right)$

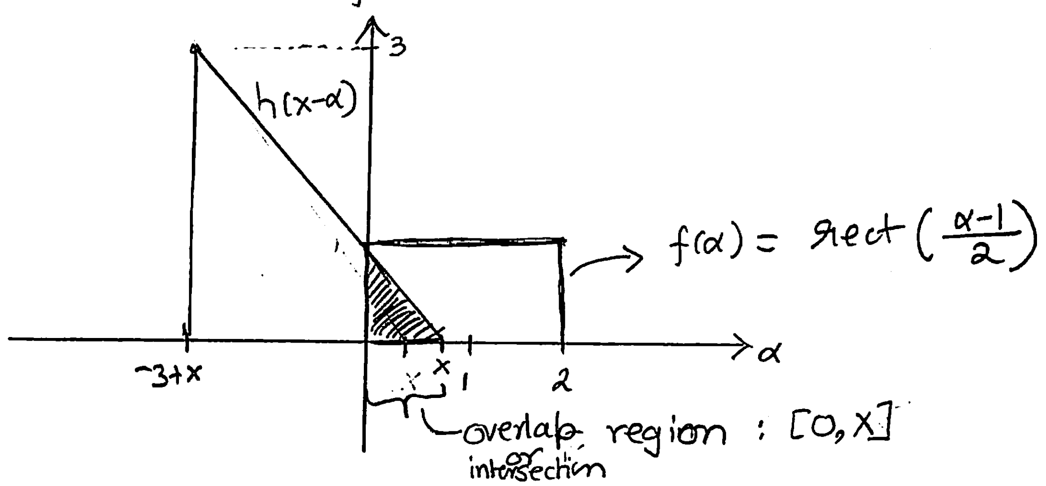
Flip.



Flip and Shifted



Step 2 : For a given output value 'x' find the overlap between $f(\alpha)$ & $h(x-\alpha)$



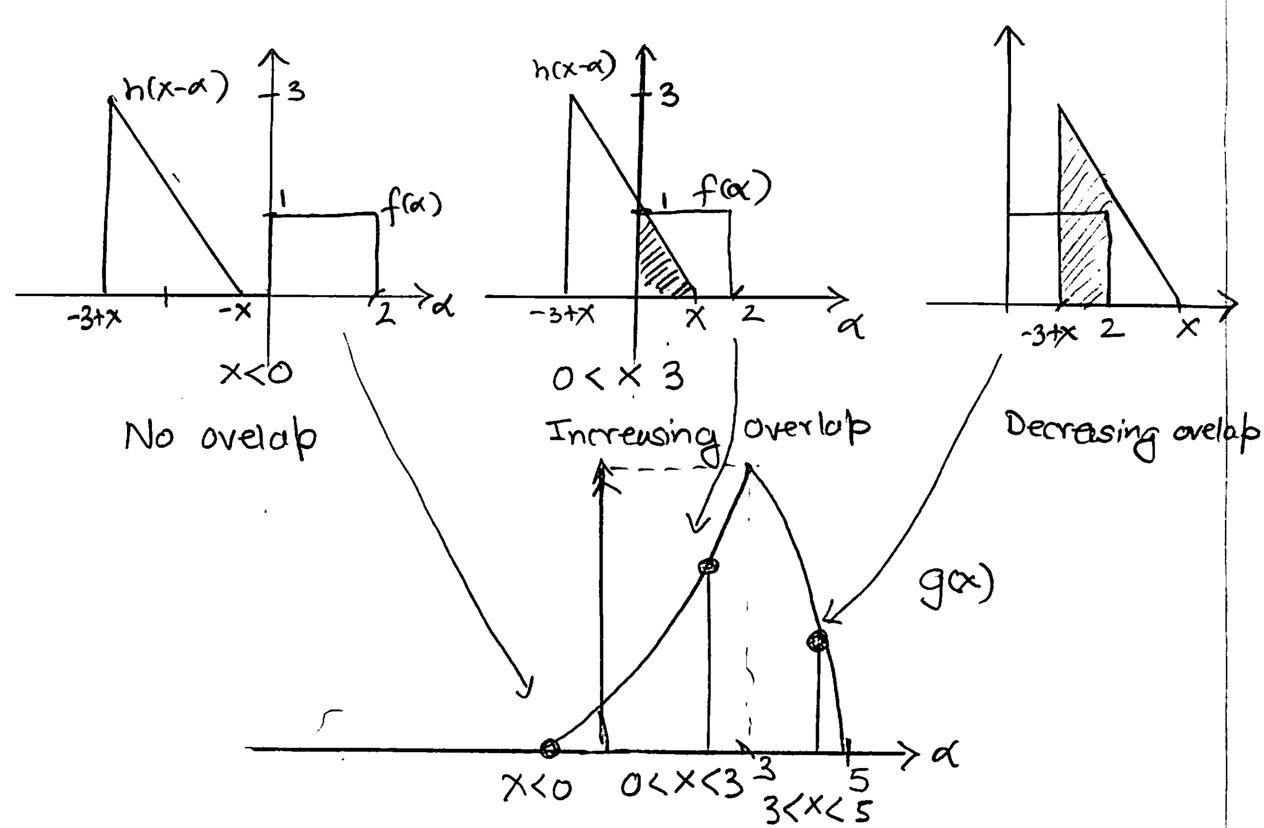
Step 3 : Compute the integral (i.e $f(\alpha) \cdot h(x-\alpha)$) in the overlap region

$$g(x) = \int_0^x \mathrm{rect}\left(\frac{\alpha-1}{2}\right) \mathrm{ramp}(x-\alpha)\, \mathrm{rect}\left(\frac{x-\alpha+1.5}{2}\right) d\alpha$$

$$0 < x < 2$$

Go to step 2 and repeat for other values of x.

For this particular choice of functions, here are calculations for three example values of x.



Refer to the animation (on D2L) for more examples.

Now that you have some insight into the mechanism of convolution, let us apprach it from an analytic perspective.

Direct Evaluation

Once we identify the overlap region we still need to compute the integral. Let us use the two functions we just used for graphical convolution for this exercise.

$$f(x) = \mathrm{rect}\left(\frac{x-1}{2}\right), \quad h(x) = \mathrm{ramp}(x)\,\mathrm{rect}\left(\frac{x-1.5}{3}\right)$$

From the graphical convolution exercise we know that $f(\alpha)$ is only non-zero between $0<\alpha<2$ and $h(x-\alpha)$ is non-zero between $(-3+x)<\alpha<x$. This means the product of these two functions is non-zero in the intersection of these two ranges.

1. Note that when $x<0$, there is <u>no overlap</u>

$$\Rightarrow \quad g(x)=0 \qquad x<0$$

2. When $0\leq x\leq 2$, we know the lower limit of the intersection range is $\underline{0}$ and the upper limit is at $\underline{\alpha = x}$. Thus we can write the convolution integral as

$$g(x) = \int_0^x 1\cdot(x-\alpha)\,d\alpha \qquad 0\leq x\leq 2$$

$f(\alpha)=1$ in $\alpha\in[0,x]$ $\qquad h(x-\alpha)$ in $\alpha\in[0,x]$

$$= x\alpha - \frac{\alpha^2}{2}\Big]_0^x = x^2-\frac{x^2}{2} = \frac{x^2}{2}$$

3. When x is larger than $\underline{2}$ & less than $\underline{3}$ the intersection region is $[0,2]$

$$g(x) = \int_0^2 1.(x-\alpha)\,d\alpha = x\alpha-\frac{\alpha^2}{2}\Big]_{-3+x}^{2}$$

$$= \left[2x-\frac{4}{2}\right]$$

$$= 2(x-1) \qquad 2\leq x\leq 3$$

$$=$$

4. When x is larger than $\underline{3}$ and less than $\underline{5}$ the intersection region is $[(-3+x), 2]$

$$g(x) = \int_{-3+x}^{2} 1.\ (x-\alpha)\ d\alpha$$

$$= x\alpha - \frac{\alpha^2}{2} \Big]_{-3+x}^{2}$$

$$= \frac{5}{2} - \frac{x^2}{2} + 2x \qquad 3 \leq x \leq 5$$

5. When x is larger than $\underline{5}$ there is no intersection

$$\Rightarrow \quad g(x) = 0 \qquad x > 5$$

Now, we can write the convolution in terms of these 5 regions

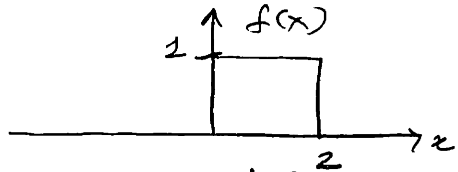
$$g(x) = f(x) \circledast h(x) = \begin{cases} 0 & x \leq 0 \\ x^2/2 & 0 \leq x \leq 2 \\ 2(x-1) & 2 \leq x \leq 3 \\ \frac{5}{2} - \frac{x^2}{2} + 2x & 3 \leq x \leq 5 \\ 0 & x > 5 \end{cases}$$

see page 5.a

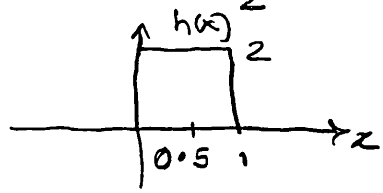
Correlation

Another important operation that appears in many physical optics problems is the correlation operation. Correlation is a measure of similarity between two functions & is defined as, → contd on page 6

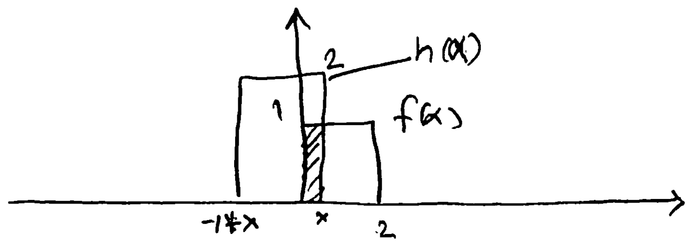
Ex: $g(x) = f(x) \circledast h(x)$



$f(x) = rect\left(\frac{x-1}{2}\right)$



$h(x) = 2\, rect(x-0.5)$



Region 1:

$x < 0$ no overlap $\Rightarrow$ $g(x) = 0$

Region 2: Partial overlap

$0 < x < 1$ $\displaystyle\int_0^x 2\,d\alpha = 2\alpha\Big]_0^x = 2x$

Region 3:

$1 < x < 2$ full overlap $\displaystyle\int_{-1+x}^{x} 2\,d\alpha = 2\alpha\Big]_{-1+x}^{x}$

$= 2x - [-2+2x]$

$= +2 + 2x - 2x$

$= 2$

Region 4: Partial overlap

$2 < x < 3$ partial overlap

$$\int_{-1+x}^{2} 2\,d\alpha = 2\alpha\Big]_{-1+x}^{2}$$

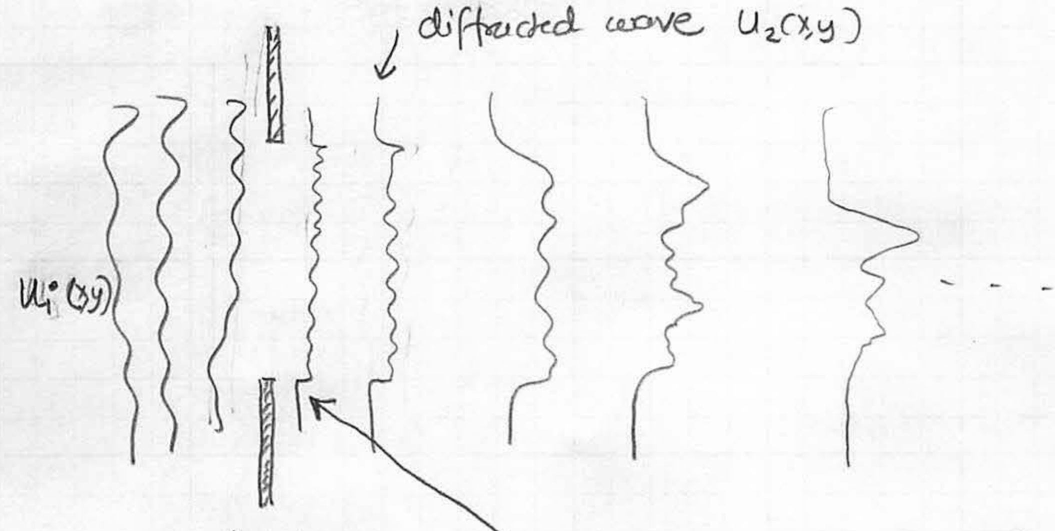
$= 4 - [-2+2x]$

$= 4 + 2 - 2x = 6 - 2x$

Region 5 : No overlap $x > 3 \Rightarrow g(x) = 0$

$$g(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 < x < 1 \\ 2 & 1 < x < 2 \\ 6-2x & 2 < x < 3 \\ 0 & x > 3 \end{cases}$$

Application of Convolution: Fresnel Diffraction



$$u_1(x_1,y_1) = P(x_1,y_1) \cdot U_i(x_1,y_1)$$

Starting from Kirchoff's approximation to Rayleigh-Sommerfeld's diffraction integral:

$$u_2(x_2,y_2) = \iint_{-\infty}^{\infty} u_1(x_1,y_1)\, \frac{z}{j\lambda r} \underbrace{\frac{e^{jkr}}{r}}_{\text{quadratic approximation}}\, dx_1 dx_2$$

Fresnel Diffraction

$$\boxed{u_2(x_2,y_2) = \frac{e^{jkz}}{j\lambda z} \iint_{-\infty}^{\infty} u_1(x_1,y_1)\, e^{j\frac{\pi}{\lambda z}\left\{(x_2-x_1)^2+(y_2-y_1)^2\right\}}\, dx_1 dx_2}$$

In 1D

$$\boxed{u_2(x_2) = \frac{e^{jkz}}{j\lambda z} \iint_{-\infty}^{\infty} u_1(x_1)\, e^{j\frac{\pi}{\lambda z}\left\{(x_2-x_1)^2\right\}}\, dx_1}$$

1D-Convolution

$$\boxed{u_2(x) = u_1(x) \circledast e^{j\frac{\pi}{\lambda z}x^2}}$$

$$\boxed{u_2(x,y) = u_1(x,y) \circledast e^{j\frac{\pi}{\lambda z}x^2+y^2}}$$ 2D convolution

$$\gamma_{fg}(x) = f(x) \star g(x) = \int_{-\infty}^{\infty} f(\alpha)\, g^*(\alpha - x)\, d\alpha$$

We observe that correlation closely resembles the convolution operation, but the argument of one function is flipped. Given the complex conjugate definition (*) we have

$$\gamma_{gf}(x) = \int_{-\infty}^{\infty} g(\alpha)\, f^*(\alpha - x)\, d\alpha$$

$$= \int_{-\infty}^{\infty} f^*(\beta)\, g(\beta + x)\, d\beta$$

$$= \gamma_{fg}^*(-x)$$

$$\gamma_{gf}(x) = \gamma_{fg}^*(-x) \quad \blacksquare$$

Also note that,

$$f(x) \circledast g^*(-x) = \gamma_{fg}(x). \quad \blacksquare$$

When $f(x) = g(x)$, $\gamma_{fg}(x) = \gamma_{ff}(x)$ is known as the auto-correlation function

$$\gamma_{ff}(x) = \int_{-\infty}^{\infty} f(\alpha)\, f^*(\alpha - x)\, d\alpha$$

This is a measure of self-similarity. Due to the symmetry in definition we get

$$\gamma_{ff}(x) = \gamma_{ff}^*(-x) \quad \blacksquare$$

meaning the function is Hermitian. Correlation has another property that $|\gamma_{ff}(x)| \leq |\gamma_{ff}(0)|$ $\blacksquare$