Complex Numbers Review and Representation of Physical Quantities

Why do we need complex numbers?

For many physical quantities it is much easier to represent and manipulate them in terms of complex numbers instead of real numbers of alone.

The concept of complex numbers comes from continuous functions such as square-root and logarithm that only apply to positive numbers in their traditional definitions. Complex numbers allow us to define such functions over the full range of real numbers.

A crucial concept in complex numbers is the definition of imaginary number

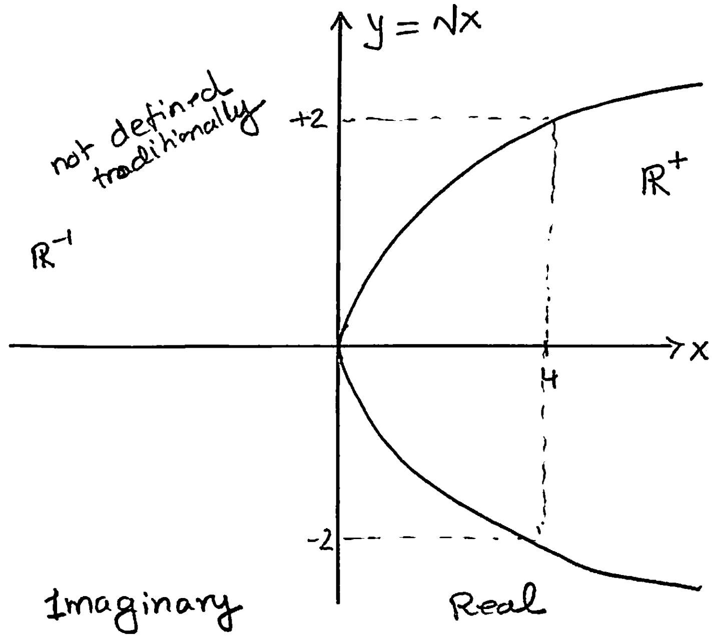
$$j = \sqrt{-1} \qquad -(1)$$

Given this, an arbitrary complex is defined in terms of real & imaginary parts

$$\underbrace{u}_{\text{complex number}} = \underbrace{v}_{\text{real part}} + j\,\underbrace{w}_{\text{imaginary part}} \qquad -(2)$$

where v & w are both real numbers.

Note: j is also sometimes denoted by i.

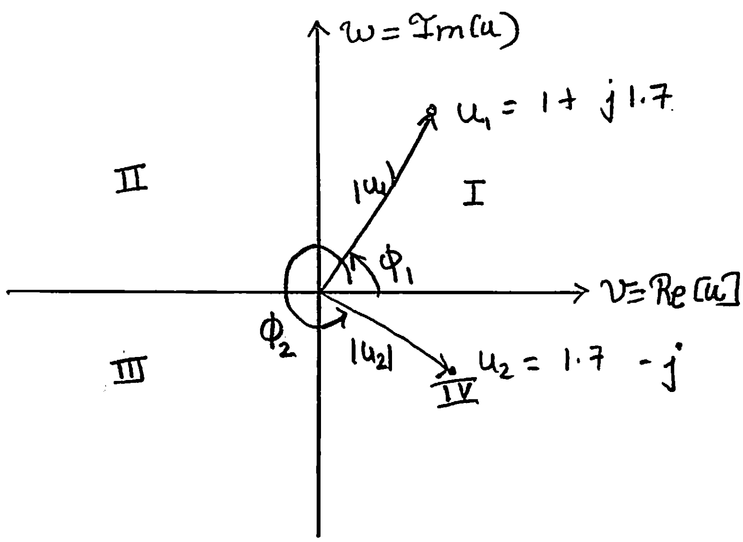


The real & imaginary parts of a complex number are denoted as

$$v = \mathrm{Re}[u] \qquad -(3)$$

$$w = \mathrm{Im}[u] \qquad -(4)$$

Note that you cannot add together a real number and an imaginary number. We treat real and imaginary part of u as orthogonal, in a similar fashion as a two-dimensional vector. We can thus plot a complex number in two dimensions.



We can express a complex number in terms of a magnitude and a phase/angle

$$\text{Magnitude} \longrightarrow |u| = \sqrt{v^2+w^2} \quad -(5)$$

$$\text{Angle} \longrightarrow \angle u = \phi = \tan^{-1}\left(\frac{w}{v}\right) \quad -(6)$$

Also,

$$v = |u| \cos\phi \quad -(7)$$

$$w = |u| \sin\phi \quad -(8)$$

By using Euler's identity : $e^{j\theta} = \cos\theta + j\sin\theta$ $-(9)$

we can write complex number in polar form

$$u = v + jw = |u|\, e^{j\phi} \quad -(10)$$

Ex:

$$u_1 = 1 + j1.7 = 2e^{j\pi/3}$$

$$u_2 = 1.7 - j = 2e^{j11\pi/6} = 2e^{-j\pi}$$

So we note that inverse tangent function in Eq. (6) is modulo π. Typically angles are specified in radians

An important quantity related to any complex number is its complex conjugate defined as.

$$u^* = v - jw = |u|\, e^{-j\phi} \quad -(11)$$

From this definition of complex conjugate many properties follow that will be discussed next

Complex Algebra

Given two arbitrary complex numbers

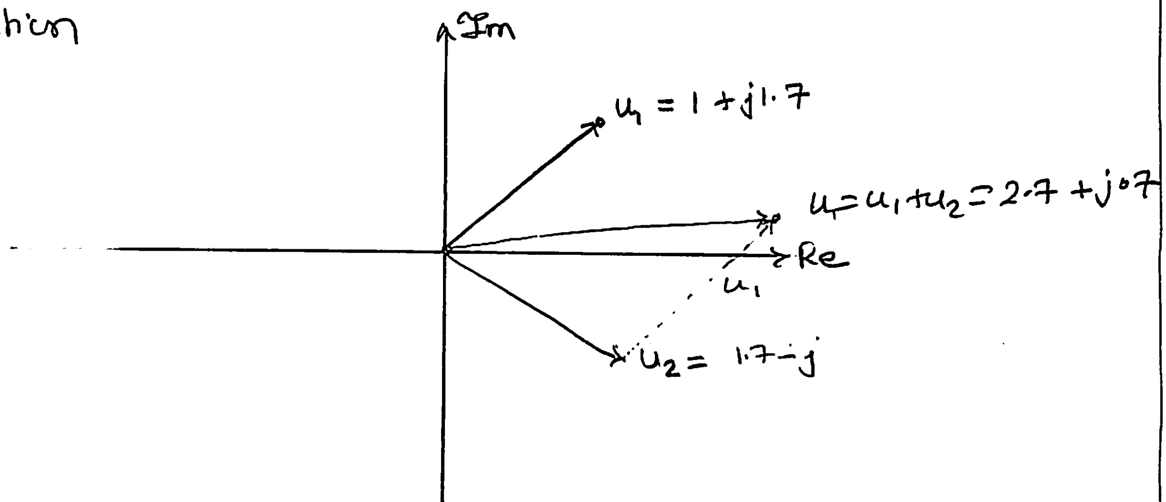
$$u_1 = v_1 + j\omega_1$$

$$u_2 = v_2 + j\omega_2$$

The complex addition is defined as

$$u = u_1 + u_2 = (v_1 + v_2) + j(\omega_1 + \omega_2) \quad -(12)$$

we simply add the real parts and imaginary parts respectively. Geometrically, we can utilize our vector notation of complex number to perform the addition operation



Multiplication can be can be done in either cartesian or polar forms by following normal rules of multiplication

$$u = u_1 \cdot u_2 = (v_1 + j\omega_1)\cdot(v_2 + j\omega_2) = (v_1 v_2 - \omega_1\omega_2) + j(v_1\omega_2 + \omega_1 v_2) \quad -(13)$$

$$= |u_1||u_2| e^{j(\phi_1+\phi_2)} \quad -(14)$$

Note: $j^2 = -1$

Similarly division can be done in either cartesian or polar form, but as we observe polar form is much simpler

$$u = \frac{u_1}{u_2} = \frac{|u_1| e^{j\phi_1}}{|u_2| e^{+j\phi_2}} = \frac{|u_1|}{|u_2|} e^{-j(\phi_1 - \phi_2)} \quad - (15)$$

$$u = \frac{u_1}{u_2} = \frac{v_1 v_2 + w_1 w_2}{v_2^2 + w_2^2} + j\,\frac{v_1 w_2 - v_2 w_1}{v_2^2 + w_2^2} \quad - (16)$$

Some useful properties of complex arthimetic:

$$|u_1 u_2| = |u_1||u_2| \quad - (17)$$

$$\left|\frac{u_1}{u_2}\right| = \frac{|u_1|}{|u_2|} \quad - (18)$$

$$\angle(u_1 \cdot u_2) = \angle u_1 + \angle u_2 \quad - (19)$$

$$(u_1 \cdot u_2)^* = u_1^* u_2^* \quad - (20)$$

$$u_1 \cdot u_1^* = |u_1|^2 e^{j(\phi - \phi)} = |u_1|^2 \quad - (21)$$

$$\frac{1}{u_1} = \frac{1}{|u_1|} e^{-j\phi} = \frac{u_1^*}{|u_1|^2} \quad - (22)$$

Power and roots of complex number are most easily analyzed in polar form using standard rules for these operations. For powers:

$$(u)^n = |u|^n e^{jn\phi} \quad - (23)$$

Root of a complex number is a bit tricker

$$(u)^{1/n} = |u|^{1/n} e^{j\frac{\phi + 2\pi k}{n}} \quad - (24)$$

$$k = 0, 1, \ldots, n-1$$

There are multiple root due to the periodicity of ϕ!

Check

$$\left((\omega)^{1/n}\right)^n = |u| e^{j\phi} e^{j2\pi k}$$

but $e^{j2\pi k} = \cos(2\pi k) + j\sin(2\pi k) = 1$

We note this property from the definition of the imaginary number itself:

$$\sqrt{-1} = \sqrt{e^{j\pi}} = \underset{\downarrow \atop j}{e^{j\pi/2}}, \underset{\downarrow \atop -j}{e^{j3\pi/2}} \quad -(15)$$

Representation of Physical Quantities

Now, we can use what we have learned about complex numbers and algebra to represent physical functions.

$$v(x) = \mathrm{Re}[A e^{j\phi} e^{j2\pi \xi_0 x}] = \mathrm{Re}[\tilde{u}^*(\xi_0) e^{j2\pi\xi_0 x}]$$

$\tilde{u}(\xi_0)$ is the complex phasor of the signal $v(x)$ and it is a function of frequency ξ_0. This is known as the frequency domain representation.

We may ask ourselves why go through the trouble of doing this? Let us consider an example to illustrate the power of complex representations.

Consider

Tedious method!

$$v(x) = A_1 \cos(2\pi\xi_0 x) + A_2 \cos(2\pi\xi_0 x + \phi) \quad -(16)$$

Using the identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$v(x) = A_1 \cos(2\pi\xi_0 x) + A_2[\cos(2\pi\xi_0 x)\cos\phi - \sin(2\pi\xi_0 x)\sin(\phi)]$$

$$= (A_1 + A_2\cos\phi)\cos(2\pi\xi_0 x) - A_2\sin(\phi)\sin(2\pi\xi_0 x)$$

$$= \sqrt{(A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2} \times \left[\left(\frac{A_1 + A_2\cos\phi}{\sqrt{(A_1+A_2\cos\phi)^2 + (A_2\sin\phi)^2}} \right) \cos(2\pi\xi_0 x) + \left(\frac{-A_2\sin\phi}{\sqrt{(A_1+A_2\cos\phi)^2+(A_2\sin\phi)^2}} \right) \sin(2\pi\xi_0 x) \right]$$

$$= B_1 \cos(2\pi\xi_0 x) + B_2 \sin(2\pi\xi_0 x) \quad -(18)$$

Let us define

$$\Phi = \tan^{-1}\left[\frac{-A_2\sin\phi}{A_1 + A_2\cos\phi}\right] \quad -(19)$$

Note that $|B_1| < 1$ & $|B_2| < 1$ and $B_1^2 + B_2^2 = 1$

So,

$$B_1 = \cos\Phi \quad \& \quad B_2 = \sin\Phi \quad -(20)$$

(19) + (20) $\longrightarrow$ (18) we get

$$v(x) = \sqrt{(A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2}\,[\cos\Phi\cos(2\pi\xi_0 x) + \sin\Phi\sin(2\pi\xi_0 x)]$$

or

$$\boxed{v(x) = \sqrt{(A_1 + A_2\cos\phi)^2 + (A_2\sin\phi)^2}\,\cos(2\pi\xi_0 x - \Phi)} \quad -(21)$$

This seems like a very tedious method.

Shorter method

Let's try again, now using complex phasors:

$$u_1 = A_1\cos(2\pi\xi_0 x + 0)$$

$$\downarrow$$

phasor form $\quad \tilde{u}_1 = A_1 e^{j0} = A_1 \quad -(22)$

$$u_2 = A_2 \cos(2\pi \xi_0 x + \phi)$$
$$\downarrow$$
$$\tilde{u}_2 = A_2 e^{-j\phi} \qquad - (23)$$

$$\tilde{u} = \tilde{u}_1 + \tilde{u}_2 = (A_1 + A_2 \cos\phi) - j A_2 \sin\phi \quad - (24)$$

More concisely we can write

$$v(x) = \mathrm{Re}[\tilde{u}^* e^{j2\pi \xi_0 x}]$$
or
$$\boxed{v(x) = A \cos(2\pi \xi_0 x - \Phi)} \qquad - (25)$$

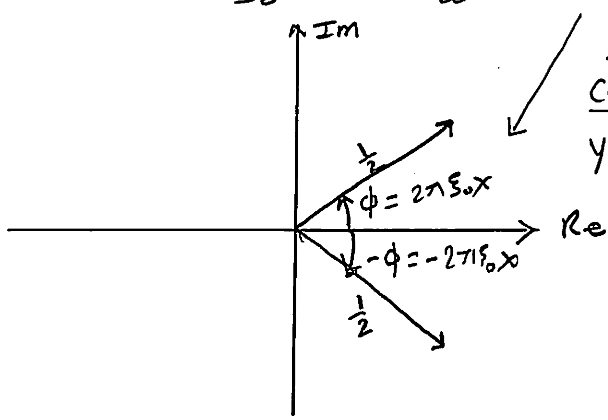
where $A = \sqrt{(A_1 + A_2\cos\phi)^2 + (A_2 \sin\phi)^2}$

$$\Phi = \tan^{-1}\left[\frac{-A_2 \sin\phi}{A_1 + A_2 \cos\phi}\right]$$

Comparing (21) & (25), we see they are identical! However, deriving (25) was <u>much simpler</u> thanks to complex phasor notation!

Another simple example of phasor notation uses Euler's identity to express a physical sinusoid function in complex phasor notation

$$\cos(2\pi \xi_0 x) = \frac{1}{2}[e^{j2\pi\xi_0 x} + e^{-j2\pi \xi_0 x}]$$



super-position of two <u>counter-rotating phasors</u> yields a real sinusoid.

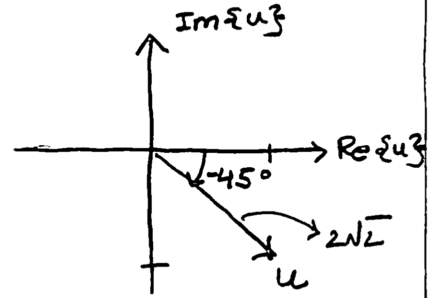
Complex algebra Examples

Ex: Convert to polar form $u = 2 - j2$

$|u| = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2}$

$\angle u = \operatorname{atan}\left(\frac{-2}{2}\right) = -45^\circ$ or $-\pi/4$

$\underline{u = 2\sqrt{2}\, e^{-j\pi/4}}$



Ex: $u = 1 + j$ $\quad v = 1 - j$

1) $u/v = ?$

Convert to polar form

$u = \sqrt{2}\, e^{j\pi/4}$ $\quad v = \sqrt{2}\, e^{-j\pi/4}$

$$\frac{u}{v} = \frac{\cancel{\sqrt{2}}\, e^{j\pi/4}}{\cancel{\sqrt{2}}\, e^{-j\pi/4}} = e^{j\pi/4} \cdot e^{j\pi/4} = e^{j\pi/2} = j$$

$\boxed{\frac{u}{v} = j}$

2) $U \cdot V = ?$ → also simpler in polar form

$u \cdot v = \sqrt{2}\, e^{j\pi/4} \cdot \sqrt{2}\, e^{-j\pi/4} = 2\, e^{j(\pi/4 - \pi/4)}$

$= 2e^{j0}$

$\underline{\underline{U \cdot V = 2}}$

3) $u = 1 + j$ $\quad 1/u = ?$

$\frac{1}{\sqrt{2}\, e^{j\pi/4}} = \frac{1}{\sqrt{2}}\, e^{-j\pi/4}$

$= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

Ex: $u = 1 + j$ $\quad u^{1/4} = ?$

Convert to polar form $\quad |u| = \sqrt{1+1} = \sqrt{2}$

$\angle u = \operatorname{atan}(1/1) = \pi/4$

$u = \sqrt{2}\, e^{j\pi/4}$

$$u^{1/4} = (\sqrt{2})^{1/4} \, e^{j \frac{\pi/4 + k\cdot 2\pi}{4}} \qquad k = 0, 1, 2, 3$$

$$u^{1/4} = (2)^{1/8} e^{j \frac{\pi}{16}}, \; (2)^{1/8} e^{j \frac{9\pi}{16}}$$

$$(2)^{1/8} e^{j \frac{17}{16}\pi}, \; (2)^{1/8} e^{j \frac{25\pi}{16}}$$

<u>check</u>:

$$(u^{1/4})^4 = (2^{1/8})^4 \cdot e^{j \frac{\pi}{4}} = \sqrt{2}\, e^{j\pi/4} \checkmark$$

$$(2^{1/8})^4 \cdot e^{j \frac{9\pi}{4}} = \sqrt{2}\, e^{j\, 2\pi + \pi/4} = \sqrt{2} e^{j\pi/4} \checkmark$$

$$(2^{1/8})^4 \cdot e^{j \frac{17\pi}{4}} = \sqrt{2}\, e^{j\, 4\pi + \pi/4} = \sqrt{2} e^{j\pi/4} \checkmark$$

$$(2^{1/8})^4 \cdot e^{j \frac{25\pi}{4}} = \sqrt{2}\, e^{j\, 6\pi + \pi/4} = \sqrt{2} e^{j\pi/4} \checkmark$$