Impulse Function & Derivatives

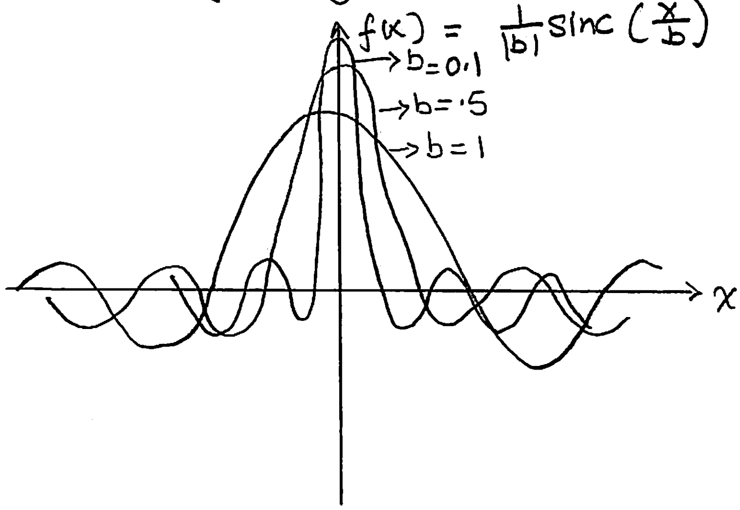
Impulse (or Dirac Delta) Function

In the last class we considered several pulse functions such as rectangle, triangle, sinc, Gaussian, and sinc^2 function which we didn't consider in detail last time. Note that all of these functions have: different widths, different full-width half-max (FWHM), different number of zeros etc. One important feature of all these functions is that all have a total integrated area of $|b|$.

Now, let us consider the sinc(·) function, as plotted below,

$$f(x) = \frac{1}{|b|}\,\text{sinc}\left(\frac{x}{b}\right)$$

By normalizing by $\frac{1}{|b|}$ all these functions have an integrated area of unity.



As we decrease the scaling parameter b, note how the peak of the sinc function increases and the width decreases to maintain an overall area of 1.

Consider, what happens as we allow the scaling parameter to go to 0. We can define the impulse function, also known as the Dirac delta function, as

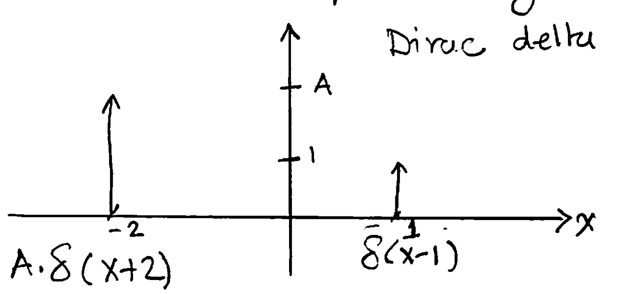
$$\delta(x) = \lim_{b \to 0} \frac{1}{|b|} \operatorname{sinc}\left(\frac{x}{b}\right)$$

or equivalentely

$$\delta(x) = \lim_{b \to 0} \frac{1}{|b|} \operatorname{rect}\left(\frac{x}{b}\right) \text{ or } \leftarrow \delta(x) = \lim_{b \to 0} \frac{1}{|b|} \operatorname{Gaus}\left(\frac{x}{b}\right) \quad \text{[Gaskill's Defn]}$$

Equivalently, we could have used any other function, such as rect, triangle or any other pulse function with unity area to define $\delta(x)$.

This function is what we refer to as a generalized function. As $b \to 0$, the width goes to 0 and the peak goes to ∞. Graphically, we denote the Dirac delta function $\delta(x-x_0)$ as an arrow at location $x = x_0$ to represent it's infinite magnitude and zero width.



The δ-function is best described by its integral properties: Fundamentally, we can state

$$\int_{x_1}^{x_2} f(x)\,\delta(x-x_0)\,dx = \begin{cases} f(x_0) & x_1 < x < x_2 \\ 0 & \text{otherwise} \end{cases}$$

Note that this integral definition does not really depend on the particular form of $\delta(x)$, but only its <u>sifting</u> property. This will become very important when we discuss linear systems.

Now, let's look at some properties of δ-function. First, its scaling property.

$$\int_{-\infty}^{\infty} f(x)\,\delta\left(\frac{x-x_0}{b}\right)dx = |b|\int_{-\infty}^{\infty} f(bu)\,\delta\left(u-\frac{x_0}{b}\right)du$$

$$= |b|\,f(x_0)$$

This allows us to say $\delta(x/b) = |b|\,\delta(x)$.

We can tabulate some important properties of δ-function.

1. Definition:

$$\delta(x-x_0) = 0 \qquad x \neq x_0$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

↑ "Sifting" property

2. Scaling properties

$$\delta\left(\frac{x-x_0}{b}\right) = |b|\,\delta(x-x_0)$$

$$\delta(ax-x_0) = \frac{1}{|a|}\,\delta\left(x-\frac{x_0}{a}\right)$$

$$\delta(-x+x_0) = \delta(x-x_0)$$

$$\underbrace{\delta(-x) = \delta(x)}$$

↳ $\delta(x)$ is even function.

3. Products with δ-functions

$$\underbrace{f(x)\,\delta(x-x_0) = f(x_0)\,\delta(x-x_0)}$$

— valid for integral operation on both sides.

$$x\,\delta(x-x_0) = x_0\,\delta(x-x_0)$$

$$\delta(x)\,\delta(x-x_0) = 0 \qquad (x_0 \neq 0)$$

$$\delta(x-x_0)\,\delta(x-x_0) \rightarrow \text{not defined!}$$

4. Integral properties

$$\int_{-\infty}^{\infty} A\,\delta(x-x_0)\,dx = A$$

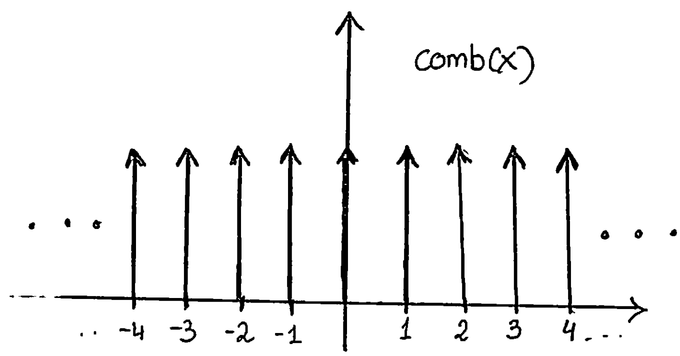
$$\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x-x_0)\,\delta(x-x')\,dx = \delta(x_0-x')$$

Comb Function

In many applications, such as digital imaging systems, it is useful to define an array of evenly spaced delta functions. Such a function is often referred to as the comb function (due to it's shape resembling a comb). Mathematically, the comb function is defined as

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



Note that we can change the interger spacing between delta functions comprising the comb by using the scaling property of δ-function

$$\mathrm{comb}\left(\frac{x}{b}\right) = |b| \sum_{n=-\infty}^{\infty} \delta(x-nb)$$

Here the delta functions are seperated by b, each with an integrated area of $|b|$. Later in this semester we will see that the comb function is integral in developing the sampling theory, using the property:

$$f(x)\left[\frac{1}{|b|}\mathrm{comb}\left(\frac{x}{b}\right)\right] = \sum_{n=-\infty}^{\infty} f(nb)\,\delta(x-nb)$$

Derivatives and Integrals of $\delta(x)$

Let us begin by defining the function

$$u(x) = \int_{-\infty}^{x} \delta(\alpha)d\alpha = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$$

Recall, $u(x)$ is the unit step function, therefore by definition we can say that

$$\delta(x) = \frac{d}{dx}\text{step}(x)$$

Similarly, we can consider the derivative of the δ-function. Let's define

$$\delta'(x) = \frac{d}{dx}\delta(x)$$

We can determine the property of this function by exploring it's integral as follows

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x-x_0)\,dx = \left[f(x)\int \delta'(x-x_0)dx\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\int \delta'(x-x_0)dx$$

$$= f(x)\,\delta(x-x_0)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x-x_0)dx$$

$$= -f'(x_0)$$

This function is often referred to as the <u>doublet function</u>.
Similarly, we can continue this indefinetely

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(n)}(x-x_0)\,dx = (-1)^n f^{(n)}(x_0)$$

where $f^{(n)}$ represents the n^{th} derivative of $f(x)$.