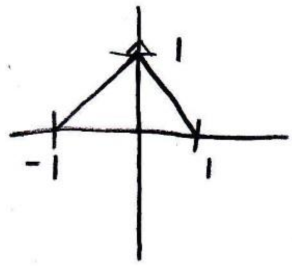
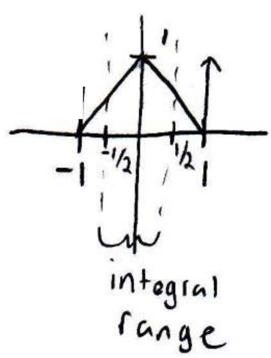
OPti 330 HW 2 Solutions

1. a) $$\int_{-2}^{2} \mathrm{tri}(y)\,\delta(y)\,dy = \mathrm{tri}(0) = \boxed{1}$$

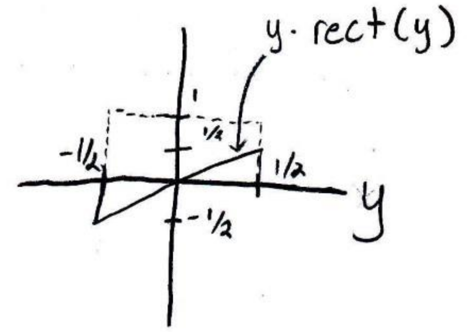


b) $$\int_{-1/2}^{1/2} \mathrm{tri}(y)\,\delta(y-1)\,dy = \boxed{0}$$



1. (cont)

c) $f(x) = \int_{-\infty}^{x} y \cdot \text{rect}(y)\, dy$



regions:

$-\infty \le x < -1/2$

$-1/2 \le x < 1/2$

$1/2 \le x \le \infty$

$-\infty \le x < -1/2$:

$$\int_{-\infty}^{x} 0 \, dy = 0$$

$-1/2 \le x \le 1/2$:

$$\int_{-1/2}^{x} y \, dy = \left.\frac{y^2}{2}\right|_{-1/2}^{x} = \frac{x^2}{2} + \frac{1/4}{2} = \frac{1}{2}(x^2 - 1/4)$$

$1/2 \le x \le \infty$

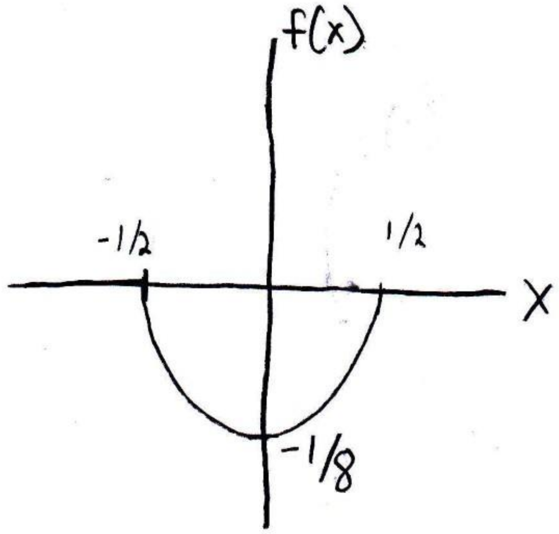
$$\int_{1/2}^{x} 0 \, dy = 0$$

1. (Cont.)

c) (cont.)

$$f(x) = \begin{cases} \int_{-\infty}^{x} 0\,dy & -\infty \le x < -1/2 \\ \int_{-\infty}^{-1/2} 0\,dy + \int_{-1/2}^{x} y\,dy & -1/2 \le x < 1/2 \\ \int_{-\infty}^{-1/2} 0\,dy + \int_{-1/2}^{1/2} y\,dy + \int_{1/2}^{x} 0\,dy & 1/2 \le x \le \infty \end{cases}$$

$$f(x) = \begin{cases} 0 & 1/2 \le |x| \le \infty \\ \frac{1}{2}\left(x^2 - 1/4\right) & 0 \le |x| < 1/2 \end{cases}$$

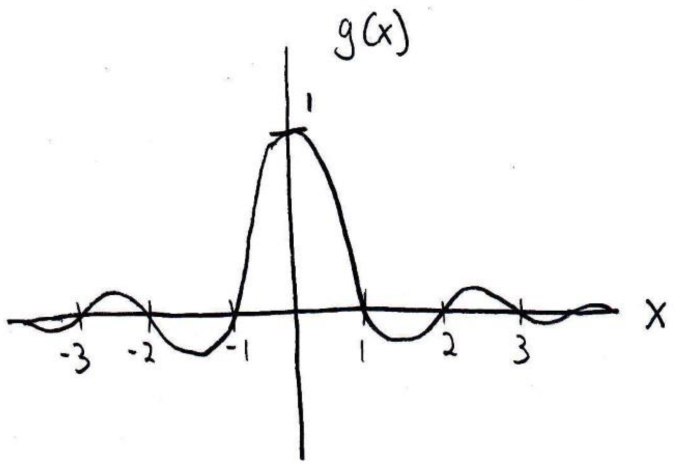


1. (cont.)

d) $$g(x) = \int_{-\infty}^{\infty} \text{Sinc}(y) \cdot \delta(x-y)\, dy$$

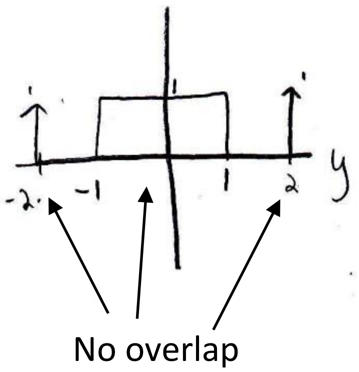
$$= \int_{-\infty}^{\infty} \text{Sinc}(y) \cdot \delta(y-x)\, dy$$

$$= \text{Sinc}(x)$$



1.e (cont.)

e) $$\int_{-\infty}^{x} \underbrace{\mathrm{rect}(y/2)\,[\delta(y-2)+\delta(y+2)]}_{0}\,dy$$



$$= \boxed{0}$$

2. a) $\delta\left(\frac{x}{\beta} - x_0\right) = \delta\left(\frac{x - \beta x_0}{\beta}\right) \overset{\text{Scaling Property}}{=} |\beta|\, \delta(x - x_0\beta)$

b) $\int_{-\infty}^{\infty} f(x)\, \delta^n(x - x_0)\, dx = (-1)^n f^{(n)}(x_0)$

Proof by induction

<u>for n = 1:</u>

$\int_{-\infty}^{\infty} f(x)\, \delta'(x - x_0)\, dx$

$= \left[f(x) \int \delta'(x-x_0)\, dx\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \int \delta'(x - x_0)\, dx$

$= \cancel{f(x)\delta(x-x_0)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\, \delta(x - x_0)\, dx$

$= -f'(x_0) = (-1)^1 f'(x_0)$ ✓

<u>for n = 2:</u>

$\int_{-\infty}^{\infty} f(x)\, \delta''(x - x_0)$

$= \left[f(x) \int \delta''(x - x_0) dx\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \int \delta''(x - x_0)\, dx$

2. b) cont.

$$= \cancel{f(x)\,\delta'(x-x_0)\Big|_{-\infty}^{\infty}} - \underbrace{\int_{-\infty}^{\infty} f'(x)\,\delta'(x-x_0)\,dx}_{\text{use result from } n=1}$$

$$= -\left(-f''(x_0)\right)$$

$$= f''(x_0) = (-1)^2 f^{(2)}(x_0)$$

for $n=i$ $\quad i = 3, 4, \ldots$

$$\int_{-\infty}^{\infty} f(x)\,\delta^{(i)}(x-x_0)$$

$$= \left[f(x) \int \delta^{(i)}(x-x_0)\,dx\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \int \delta^{(i)}(x-x_0)\,dx$$

$$= \cancel{f(x)\,\delta^{i-1}(x-x_0)\,dx\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f'(x)\,\delta^{(i-1)}(x-x_0)\,dx$$

$$= -\int_{-\infty}^{\infty} f'(x)\,\delta^{(i-1)}(x-x_0)\,dx$$

$$\cdots$$

$$= +\int_{-\infty}^{\infty} f''(x)\,\delta^{(i-2)}(x-x_0)\,dx = (-1)^i f^{(i)}(x_0)$$

2. b) cont.

for $n = i+1$:

$$\int_{-\infty}^{\infty} f(x)\, \delta^{(i+1)}(x-x_0)\, dx$$

$$= \left[f(x) \int_{-\infty}^{\infty} \delta^{(i+1)}(x-x_0)\, dx \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \int \delta^{(i+1)}(x-x_0)\, dx$$

$$= \cancel{f(x)\, \delta^{(i)}(x-x_0)\Big|_{-\infty}^{\infty}} - \underbrace{\int_{-\infty}^{\infty} f'(x)\, \delta^{(i)}(x-x_0)\, dx}_{\text{from } n=i \text{ case} \ldots}$$

$$= -\left(f^{(1+i)}(x_0) \right)$$

$$\vdots$$

$$\int_{-\infty}^{\infty} f(x)\, \delta^{(n)}(x-x_0)\, dx = (-1)^n f^{(n)}(x_0)$$

3. a) $\int_{-\infty}^{\infty} \delta(x) e^{j2\pi \xi x} dx$

$= e^{j2\pi \xi (0)} = 1$

integral property

b) $\int_{-\infty}^{\infty} \delta\left(\frac{x-x_0}{b}\right) e^{j2\pi \xi x} dx$

$= |b| \int_{-\infty}^{\infty} \delta(x-x_0) e^{j2\pi \xi x} dx$

Scaling Property

$= |b| e^{j2\pi \xi x_0}$

integral Property