

**Q3).**

**Proposition:**

Let  $A, B$  be non-empty sets  $A, B \subseteq \mathbb{R}$ ,  $A \cup B = \mathbb{R}$ , and  $a < b, \forall a \in A, \forall b \in B$ . Then, there exists an  $s \in \mathbb{R}$  such that  $a \leq s \leq b, \forall a \in A, b \in B$ . In other words,  $P(13) \Rightarrow P(13')$ .

**Proof:**

By the definition of upper bound,  $B$  is the set of upper bounds of  $A$ , with the possible exception of a maximum in  $A$ . That means  $A$  has an upper bound. Thus, by the completeness axiom,  $A$  has a least upper bound. In other words,  $\exists s \in \mathbb{R}$  such that  $a \leq s, \forall a \in A$  ( $s > a$  unless  $A$  has a maximum, in which case  $s = \max(A)$ ), and, since it is a least upper bound,  $s \leq b, \forall b \in B$  ( $s < b$  unless  $A$  has a maximum, in which case  $s = \max(A)$ ). So,  $a \leq s \leq b, \forall a \in A, b \in B$ , meaning that  $P(13) \Rightarrow P(13')$ .

Next, need to show that  $P(13') \Rightarrow P(13)$ .

**Proposition:**

If  $s \in \mathbb{R}$  and  $S$  has an upper bound,  $S$  has a least upper bound. In other words,  $P(13') \Rightarrow P(13)$ .

**Proof:**

Let  $B = \{x \in \mathbb{R} : x > s, \forall s \in S\}$  and  $A = \{x \in \mathbb{R} : x \leq s\}$ . Since an element of  $\mathbb{R}$  is either in  $B$  or not in  $B$ ,  $A \cup B = \mathbb{R}$ .  $B$  is the set of upper bounds of  $A$ , save for only the maximum of  $A$ , if it exists. Also,  $S \subseteq A$ , since no element of  $S$  is greater than every element of  $S$ . From the definition of  $A$ ,  $\forall a \in A, s \in S, a \leq s$  (Because no element of  $A$  is bigger than any element of  $S$ ). This means that if  $A$  has a least upper bound, so does  $S$ , and the least upper bound of  $A$  is equal to that of  $S$ . So, if  $A$  has a least upper bound, the proof is complete. Apply  $P(13')$ , so  $\exists k \in \mathbb{R} : a \leq k \leq b, \forall a \in A, b \in B$ . This is the definition of a least upper bound. So,  $A$  has a least upper bound, and therefore, so does  $S$ , completing the proof.

Therefore,  $P(13') \Rightarrow P(13)$  and  $P(13) \Rightarrow P(13')$ , so they are equivalent.