Then, want to find $\lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-x+1}$.

$$\lim_{x\to\infty} \frac{x^3+4x-7}{7x^2-x+1}$$

$$= \lim_{x\to\infty} \frac{1+\frac{4}{x^2}-\frac{7}{x^3}}{\frac{7}{x}-\frac{1}{x^2}+\frac{1}{x^3}}$$

[redacted] Need to show

$$x>2 \Rightarrow 1+\frac{4}{x^2}-\frac{7}{x^3}>0 \quad \text{and} \quad \lim_{x\to\infty} 1+\frac{4}{x^2}-\frac{7}{x^3}=1$$

and

$$x>0 \Rightarrow \frac{7}{x}-\frac{1}{x^2}+\frac{1}{x^3}>0 \quad \text{and} \quad \lim_{x\to\infty} \frac{7}{x}-\frac{1}{x^2}+\frac{1}{x^3}=0.$$

First, show

$x>2 \Rightarrow 1+\frac{4}{x^2}-\frac{7}{x^3}>0$

Assume by contradiction

$x>2 \Rightarrow 1+\frac{4}{x^2}-\frac{7}{x^3}\leq 0.$

then,

$$1+\frac{4}{x^2}-\frac{7}{x^3}\leq 0$$

$$x^3+4x-7\leq 0$$

But $x>2$, so

$$2^3+4(2)-7 < x^3-4x-7\leq 0$$

so

$$8+1<x^3-4x-7\leq 0$$

contradiction.

Next,

$$\lim_{x\to\infty} 1+\frac{4}{x^2}-\frac{7}{x^3}$$

$$= \lim_{x\to 0^+} 1+4x^2-7x^3$$

$$= 1+\lim_{x\to 0^+} 4x^2-\lim_{x\to 0^+} 7x^3$$

$$=1.$$

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