Q1a) Claim: If $\lim_{x\to\infty} f(x)=1$ and $\forall x>2\ f(x)>0$, and

$\lim_{x\to\infty} g(x)=0$ and $g(x)>0\ \forall x>0$,

then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Proof: given $\forall \varepsilon>0\ \exists \delta_f \in \mathbb{R} : x>\delta_f \Rightarrow |f(x)-1|<\varepsilon$,

$\forall \varepsilon>0\ \exists \delta_g \in \mathbb{R} : x>\delta_g \Rightarrow |g(x)|<\varepsilon$.

Note: Can take $\delta_f>2$ and $\delta_g>0$ without loss of generality.

Want to show $\forall c \in \mathbb{R}\ \exists \delta \in \mathbb{R} : x>\delta \Rightarrow \frac{f(x)}{g(x)}>c$.

Can take $c>1$ and $\delta>0$ without loss of generality.

Since $\delta_f>2$, $f(x)>0$. So,

$|f(x)-1|<\varepsilon \Rightarrow -\varepsilon< f(x)-1<\varepsilon \Rightarrow 0<f(x)<\varepsilon+1$.

Take δ_f such that $x>\delta_f \Rightarrow f(x)<c$.

Similarly, since $\delta_g>0$, $g(x)>0$. So,

$|g(x)|<\varepsilon \Rightarrow 0<g(x)<\varepsilon$.

Take δ_g such that $x>\delta_g \Rightarrow g(x) < \frac{f(x)}{c}$.

Take $\delta = \max\{\delta_g, \delta_f\}$.

Then, $x>\delta \Rightarrow x>\delta_g$ and $x>\delta_f$. So,

$0<f(x)<c$ and $0<g(x)<\frac{f(x)}{c}<1$

So, $\frac{f(x)}{g(x)} > \frac{f(x)}{\frac{f(x)}{c}} = c$

So, $f(x)>c$.

QED.