I want to start by apologizing because I know there's definitely a better way to prove this so I'm just sorry you need to mark this monstrosity. This proof doesn't explicitly provide a partition on which the upper sum is less than epsilon, but it does show that such a partition must exist.

On any interval [a,b] in [0,1], there is an irrational number, since the irrationals are dense. So, on any interval, the infimum of f is 0, so all lower sums are 0. In that case, to satisfy the Riemann Criterion, it suffices to show that for all $\varepsilon > 0$ we can find a partition such that $U(f, \mathcal{P}) < \varepsilon$. We know that the max value of f on $[0,\, 1] = 1$, since no fraction in lowest terms will have a denominator less than 1, and 1 is a rational number $\frac{1}{1}$. The same argument can be applied to the first interval, $[0,\, t_1]$, since $0 = \frac{0}{1}$. It follows, then, that $M_1 = M_n = 1$. Similarly, on (0, 1), the max value of f is $\frac{1}{2}$.

So, when given a rational number $\frac{a}{b}$, f returns $\frac{1}{b}$. Using this fact, we can place an upper bound on the number of times a number can be contributed to the sum. For example, $f(x) = \frac{1}{3}$ can only occur a *maximum* of two times, as there are only 2 rational numbers with a denominator of 3 in [0,1]. In general, a denominator of $\frac{1}{n}$ can contribute to the sum a maximum of $n-1$ times (and if n isn't prime it will be less than that. Each value of x can be in a maximum of two intervals (consider $x \in \mathcal{P}$, you get $[t_{i-1}, x], [x, t_{1+1}]$). Therefore, M_i will equal $\frac{1}{n}$ a maximum of $2(n-1)$ times.

Define $M_i^p = sup(\{f(x) : x \in [t_{i-1}, t_i]\} \cap \{\frac{1}{q} : q \geq p\})$. Let $\mathcal{P}$ be a partition of even points spaced $\frac{1}{k}$ apart, so $t_{i-1} - t_i = \frac{1}{k}$ and $n = k$. Using these definitions, we can say:

$$
\begin{aligned}
U(f, \mathcal{P}) &= \sum_{i=1}^{k} M_i(t_{i-1} - t_i) \\
&= \sum_{i=1}^{k} M_i \frac{1}{k} \\
&= \frac{1}{k}\left(\left[\sum_{i=2}^{k-1} M_i\right] + 1 + 1\right) && \text{Since } M_0 = M_k = 1 \\
&= \frac{1}{k}\left[\sum_{i=2}^{k-1} M_i\right] + \frac{2}{k} \\
&\leq \frac{1}{k}\left[\sum_{i=2}^{k-1} M_i^3\right] + \frac{2(2-1)}{2k} + \frac{2}{k}
\end{aligned}
$$

Since $M_i = \frac{1}{2}$ a maximum of $2(2-1)$ times, so we can let M_i become M_i^3 because all of the $M_i = \frac{1}{2}$ are accounted for.

$$
\leq \frac{1}{k}\left[\sum_{i=2}^{k-1} M_i^4\right] + \frac{2(3-1)}{3k} + \frac{2}{2k} + \frac{2}{k}
$$

etc...

$$
\begin{aligned}
&\leq \frac{1}{k}\left[\sum_{i=2}^{k-1} M_i^m\right] + \left[\sum_{j=2}^{m-1} \frac{2(j-1)}{jk}\right] + \frac{2}{k} \\
&\leq \frac{1}{k}\left[\sum_{i=2}^{k-1} \frac{1}{m}\right] + \left[\sum_{j=2}^{m-1} \frac{2(j-1)}{jk}\right] + \frac{2}{k} \\
&= \frac{k-2}{km} + \left[\sum_{j=2}^{m-1} \frac{2(j-1)}{jk}\right] + \frac{2}{k}
\end{aligned}
$$

Call this $t(k,m)$. Notice that $\lim_{k\to\infty} t(k,m) = \frac{1}{m}$. Letting k become arbitrarily large corresponds to letting the distance in the partition $\mathcal{P}$ become arbitrarily close to 0. Of course, it never actually becomes 0, but we know from the definition of limits at ∞ that $\forall \epsilon_1 \exists H : \forall k > M, t(k,m) - \frac{1}{m} < \epsilon_1$. So, can pick a k so that $t(k,m) < \frac{1}{m} + \epsilon_1$. Note that in this case $\epsilon_1 \neq \varepsilon$ that was given above. But, since for all m, $U(f,\mathcal{P}) \leq t(k,m)$, we know that $U(f,\mathcal{P}) < \frac{1}{m} + \epsilon_1$ for some partition $\mathcal{P}$ defined by k. And since this is true for all m, we can just pick an m sufficiently large so that $\frac{1}{m} < \frac{\varepsilon}{2}$ and a k so that $t(k,m) - \frac{1}{m} < \frac{\varepsilon}{2}$, and then we have $U(f,\mathcal{P}) \leq t(k,m) < \varepsilon$ on the partition defined by our choice of k.