Define $\forall n \in \mathbb{N}$

$$f(x) = \begin{cases} n+\frac{1}{2}-\frac{1}{n2^{n+1}} < x < n+\frac{1}{2} & n^2 2^{n+2}x + (2n - n^2 2^{n+2}(n+\frac{1}{2}) \\ n+\frac{1}{2} \le x < n+\frac{1}{2}+\frac{1}{n2^{n+1}} & -n^2 2^{n+2}x + (2n + n^2 2^{n+2}(n+\frac{1}{2}) \\ else & 0 \end{cases}$$

This function looks pretty bad, but it's describing just describing a function everywhere 0 except in between natural numbers, where it forms a triangle(with straight line sides) of height $2n$ and base $\frac{1}{2*2^n}$.

First, this function is continuous. In order to prove this, note that f is obviously continuous on each piece, so just need to check the transition points.

Check for $x = n+\frac{1}{2}-\frac{1}{n2^{n+1}}$. $f(x) = 0$,

$$\begin{aligned} \lim_{x \to n+\frac{1}{2}-\frac{1}{n2^{n+1}}^+} f(x) &= n^2 2^{n+2}(n+\frac{1}{2}-\frac{1}{n2^{n+1}}) + (2n - n^2 2^{n+2}(n+\frac{1}{2})) \\ &= 0 \end{aligned}$$

Similarly, we can check for $x = n+\frac{1}{2}(f(x) = 2n)$:

$$\begin{aligned} \lim_{x \to n+\frac{1}{2}^-} f(x) &= n^2 2^{n+2}(n+\frac{1}{2}) + (2n - n^2 2^{n+2}(n+\frac{1}{2})) \\ &= 2n \end{aligned}$$

Finally, check for $x = n+\frac{1}{2}+\frac{1}{n2^{n+1}}(f(x) = 0)$:

$$\begin{aligned} \lim_{x \to n+\frac{1}{2}+\frac{1}{n2^{n+1}}^-} f(x) &= -n^2 2^{n+2}(n+\frac{1}{2}+\frac{1}{n2^{n+1}}) + (2n + n^2 2^{n+2}(n+\frac{1}{2})) \\ &= 0 \end{aligned}$$

So we now know f is continuous on $[0,\infty)$.

The only pieces of f that could be negative are the lines. However, since they're lines, and the first line starts at 0 and goes to $2n$ and the second line starts at $2n$ and ends at 0 this function is always nonnegative.

To show it has no upper bound, note that at $n+\frac{1}{2}$ f takes the value $2n$. So, for any $M \in \mathbb{R}$, $f(ceil(M)+\frac{1}{2}) = 2((ceil(M)+1) > M$.

Now, we need to show that the integral of f converges to 1. Since f is always nonnegative, we know that $F(x) = \int_0^x f(x)$ is nondecreasing. Since between triangles $\int_{n+\frac{1}{2}+\frac{1}{n2^{n+1}}}^{n+\frac{1}{2}-\frac{1}{n2^{n+1}}} f = 0$, we know that at every $n \in \mathbb{N}$, $F(n)$ is just given by the sum of the area of the previous $n-1$ triangles. In other words, $\forall n \in \mathbb{N}$:

$$\begin{aligned} F(n) &= \sum_{i=1}^{n-1}\left[\int_{i+\frac{1}{2}-\frac{1}{i2^{i+1}}}^{i+\frac{1}{2}} i^2 2^{i+1}x + (2i - i^2 2^{i+1}(i+\frac{1}{2})) + \int_{i+\frac{1}{2}}^{i+\frac{1}{2}+\frac{1}{i2^{i+1}}} -i^2 2^{i+1}x + (2i + i^2 2^{i+1}(i+\frac{1}{2}))\right] \\ &= \sum_{i=1}^{n-1}\left[(i^2 2^{i+1}x^2 + (2i - i^2 2^{i+1}(i+\frac{1}{2}x)|_{i+\frac{1}{2}-\frac{1}{i2^{i+1}}}^{i+\frac{1}{2}} + (-i^2 2^{i+1}x^2 + (2i + i^2 2^{i+1}(i+\frac{1}{2}x)|_{i+\frac{1}{2}}^{i+\frac{1}{2}-\frac{1}{i2^{i+1}}}\right] \\ &= \sum_{i=1}^{n-1}\left[2^{-i-1} + 2^{-i-1}\right] \\ &= \sum_{i=1}^{n-1} 2^{-i} \end{aligned}$$

So, that means for $n \in \mathbb{N}, F(n) = \sum_{i=1}^{n-1} 2^{-i}$ Now we just need to find $lim_{x\to\infty}F(x)$. Since $F(x)$ is nondecreasing, $\forall n \in \mathbb{N} n \le x \le n+1 \Rightarrow F(n) \le F(x) \le F(n+1)$. Also, note that since $F(n) = \sum_{i=1}^{n-1} 2^{-i}, lim_{n\to\infty}F(n) = lim_{n\to\infty}\sum_{i=1}^{n-1} 2^{-i} = 1$. Or,

$$\forall \varepsilon > 0 \forall n \in \mathbb{N} \exists M \in \mathbb{N} : n > M \to |F(n) - 1| < \varepsilon$$

Fix an ε, and let $M' = ceil(M) + 2$. Then, $x > M' \Rightarrow \exists n \in \mathbb{N} : n + 1 \geq x \geq n > m$. That implies that $F(n) \leq F(x) \leq F(n+1)$, and since $n > M$, we have $F(n) - 1 \leq F(x) - 1 \leq F(n+1) - 1 \Rightarrow |F(x) - 1| \leq max\{|F(n) - 1|, |F(n+1) - 1|\} < \varepsilon$. Therefore, $lim_{x \to \infty} F(x) = 1$, as desired.