

On any interval  $[a, b]$  in  $[0, 1]$ , there is an irrational number, since the irrationals are dense. So, on any interval, the infimum of  $f$  is 0, so all lower sums are 0. In that case, to satisfy the Riemann Criterion, it suffices to show that for all  $\varepsilon > 0$  we can find a partition such that  $U(f, \mathcal{P}) < \varepsilon$ . We know that the max value of  $f$  on  $[0, 1] = 1$ , since no fraction in lowest terms will have a denominator less than 1, and 1 is a rational number  $\frac{1}{1}$ . The same argument can be applied to the first interval,  $[0, t_1]$ , since  $0 = \frac{0}{1}$ . It follows, then, that  $M_1 = M_n = 1$ . Similarly, on  $(0, 1)$ , the max value of  $f$  is  $\frac{1}{2}$ .

So, take a partition  $\mathcal{P} = 0, \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1$ . The entries are evenly spaces, with  $\frac{1}{k}$  between them, and so  $n = k$ . Given this, we can write the upper sum.

$$\begin{aligned}
 U(f, \mathcal{P}) &= \sum_{i=1}^n M_i (T_{i-1} - T_i) \\
 &= \sum_{i=1}^n M_i \frac{1}{k} \\
 &= \frac{1}{k} \sum_{i=1}^k M_i \\
 &= \frac{1}{k} \left( \left[ \sum_{i=1}^{k-1} M_i \right] + 1 \right) \\
 &= \frac{1}{k} \left( \left[ \sum_{i=2}^{k-1} M_i \right] + 2 \right) \\
 &= \frac{1}{k} \sum_{i=2}^{k-1} M_i + \frac{2}{k} \\
 &\leq \frac{k-2}{2k} + \frac{4}{2k} \\
 &= \frac{k+2}{2k}
 \end{aligned}$$

So if we can just find a value of  $k$  that makes  $\frac{k+2}{2k} < \varepsilon$  we're done. And that's pretty easy.

$$\begin{aligned}
 \frac{k+2}{2k} &< \varepsilon \\
 k+2 &< 2k\varepsilon \\
 2 &< k(2\varepsilon - 1) \\
 \frac{2}{2\varepsilon - 1} &< k
 \end{aligned}$$

So given an  $\varepsilon$ , simply pick  $k > \frac{2}{2\varepsilon - 1}$  and you're done.