Q3a

Define $P_1(x) = \frac{1}{x^2}$, and $P_n(x) = P_{n-1}(x)\frac{1}{x^2} + p'_{n-1}(x)$. Note that $P_n(x)$ will always be a polynomial. Now, define

$$f(x) = \begin{cases} x > 0 & e^{-\frac{1}{x}} \\ x \leq 0 & 0 \end{cases}$$

So, notice that f is continuous, since $lim_{x \to 0^+} f(x) = lim_{x \to 0^-} f(x) = 0$. Now, need to show it is infinitely differentiable. It's obvious that f is infinitely differentiable on $(-\infty, 0), (0, \infty)$, so we just need to show that $lim_{x \to 0^+} f^{(n)}(x) = lim_{x \to 0^-} f^{(n)}(x)$. Obviously $lim_{x \to 0^-} f^{(n)}(x) = 0$. First, I claim that for $x > 0$, $f^{(n)}(x) = P_n(x)f(x)$. Proof (by induction)
Base case: $n = 1$:

$$f'(x) = \frac{1}{x^2}e^{\frac{-1}{x}}$$

Now, assume $f^{(n-1)}(x) = P_{n-1}(x)f(x)$. Then, need to show for $f^{(n)}$.

$$\begin{aligned} f^{(n)} &= \frac{d}{dx}f^{(n-1)}(x) \\ &= \frac{d}{dx}P_{n-1}(x)f(x) \\ &= P'_{n-1}(x)f(x) + \frac{1}{x^2}P_{n-1}(x)f(x) \\ &= f(x)(P'_{n-1}(x) + \frac{1}{x^2}P_{n-1}(x)) \\ &= f(x)P_n(x) \end{aligned}$$

So, $lim_{x \to 0^+} = f^{(n)}(x) = lim_{x \to 0^+} P_n(x)e^{\frac{-1}{x}} = lim_{x \to 0^+} \frac{P_n(x)}{e^{\frac{1}{x}}} = lim_{x \to \infty} \frac{P_n(\frac{1}{x})}{e^x} = 0$.

So, f is everywhere differentiable.

Q3b

$$f(x) = \begin{cases} |x| \geq 1 & 0 \\ |x| \leq 1 & e^{\frac{x^2}{x^2-1}} \end{cases}$$

Again, define a rational function:

$$\begin{aligned} P_1(x) &= \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} \\ P_n(x) &= P'_{n-1}(x) + P_{n-1}(x)\frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} \end{aligned}$$

We claim, for $-1 < x < 1$, $f^{(n)}(x) = P_n(x)f(x)$.
Base case $n = 1$:

$$f'(x) = f(x)\frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} = f(x)P_1(x)$$

Assume $f^{(n-1)}(x) = P_{n-1}(x)f(x)$. Then,

$$\begin{aligned}
f^{(n)}(x) &= \frac{d}{dx} f^{(n-1)}(x) \\
&= \frac{d}{dx} P_{n-1}(x)f(x) \\
&= P'_{n-1}(x)f(x) + \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} P_{n-1}(x)f(x) \\
&= f(x)(P'_{n-1}(x) + \frac{2x(x^2-1) - x^2(2x)}{(x^2-1)^2} P_{n-1}(x)) \\
&= f(n)P_n(x)
\end{aligned}$$

Notice that both f and P_n are continuous for all $-1 < x < 1$, so we know f is infinitely differentiable for all x except maybe -1 and 1. So, just need to check those limits.

$$\begin{aligned}
\lim_{x \to -1^+} f^{(n)}(x) &= \lim_{x \to -1^+} e^{\frac{x^2}{x^2-1}} P_n(x) \\
&= \lim_{u \to 0^-} e^{\frac{u+1}{u}} P_n(\sqrt{u+1}) \\
&= \lim_{u \to -\infty} e^{\frac{\frac{1}{u}+1}{\frac{1}{u}}} P_n(\sqrt{\frac{1}{u}+1}) \\
&= e \lim_{u \to -\infty} e^{u} P_n(\sqrt{\frac{1}{u}+1}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to 1^-} f^{(n)}(x) &= \lim_{x \to 1^-} e^{\frac{x^2}{x^2-1}} P_n(x) \\
&= \lim_{u \to 0^-} e^{\frac{u+1}{u}} P_n(\sqrt{u+1}) \\
&= \lim_{u \to -\infty} e^{\frac{\frac{1}{u}+1}{\frac{1}{u}}} P_n(\sqrt{\frac{1}{u}+1}) \\
&= e \lim_{u \to -\infty} e^{u} P_n(\sqrt{\frac{1}{u}+1}) \\
&= 0
\end{aligned}$$