Let g^+ be a nonzero upper bound for g and g^- be a nonzero lower bound for g, with $|g^+| > |g^-|$. Then, consider a partition $\mathcal{P}_1$ on $[a, x_0 - \delta]$ and $\mathcal{P}_2$ on $[x_0 + \delta, b]$, such that $\frac{\varepsilon}{3} > U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)$ and $\frac{\varepsilon}{3} > U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)$. Then,

$$U(g, \mathcal{P}_1 \cup \mathcal{P}_2) \leq U(g, \mathcal{P}_1) + U(g, \mathcal{P}_2) + g^+(x_0 + \delta - (x_0 + \delta)) = U(g, \mathcal{P}_1) + U(g, \mathcal{P}_2) + g^+(2\delta) \tag{1}$$

and

$$L(g, \mathcal{P}_1 \cup \mathcal{P}_2) \geq L(g, \mathcal{P}_1) + L(g, \mathcal{P}_2) + g^-(2\delta)) \tag{2}$$

So, by multiplying both sides of (2) by and adding them to (1), get

$$\begin{aligned}
U(g, \mathcal{P}_1 \cup \mathcal{P}_2) - L(g, \mathcal{P}_1 \cup \mathcal{P}_2) &\leq U(g, \mathcal{P}_1) + U(g, \mathcal{P}_2) + g^+(2\delta) - L(g, \mathcal{P}_1) - L(g, \mathcal{P}_2) - g^-(2\delta)) \\
&= U(g, \mathcal{P}_1) - L(g, \mathcal{P}_1) + U(g, \mathcal{P}_2) - L(g, \mathcal{P}_2) + g^+(2\delta) - g^-(2\delta)) \\
&= U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1) + U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2) + (g^+ - g^-)(2\delta) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + (g^+ - g^-)(2\delta)
\end{aligned}$$

So, we just need to pick a δ that forces $(g^+ - g^-)(2\delta) < \frac{\varepsilon}{3}$. Obviously, $\delta < \frac{\varepsilon}{2(g^+ - g^-)}$ suffices, so g is integrable on $[a, b]$ by the Riemann Criterion. Now, just need to show that

$$\int_a^b g = \int_a^b f \tag{3}$$

I'll show this by showing that $U(g, \mathcal{P}_1 \cup \mathcal{P}_2) - U(f, \mathcal{P}_1 \cup \mathcal{P}_2)$ can be made as small as we want, through a similar argument to above.

$$\begin{aligned}
U(g, \mathcal{P}_1 \cup \mathcal{P}_2) - U(f, \mathcal{P}_1 \cup \mathcal{P}_2) &= \sum_{i=0}^{n} M_{ig} - \sum_{i=0}^{n} M_{if} && \text{Where } M_{ig} \text{ and } M_{if} \text{ are the } M_i\text{s for } g \text{ and } f \text{ respectively} \\
&= \sum_{i=1}^{n} (M_{gi} - M_{fi})(t_{i-1} - t_i) && f = g, \forall x \notin [x_0 - \delta, x_0 + \delta] \text{ ,all terms 0 except in } \delta\text{-interval} \\
&= (M_{gi} - M_{fi})(2\delta)
\end{aligned}$$

And we know $M_{gi} \leq g^+$ and $M_{fi} \geq f^-$ (where f^- is a lower bound for f), so we can say $M_{gi} - M_{fi} \leq g^+ - f^-$ (and again, we just pick an upper bound for g so that the difference is not 0). Then, we have:

$$(M_{gi} - M_{fi})(2\delta) \leq (g^+ - f^-)(2\delta)$$

So we want to pick a δ forcing $(g^+ - f^-)(2\delta) < \varepsilon$ for a given ε, and $\delta < \frac{\varepsilon}{2(g^+ - f^-)}$ suffices, so that means $\forall \varepsilon \exists \mathcal{P} : U(g, \mathcal{P}_1 \cup \mathcal{P}_2) - U(f, \mathcal{P}_1 \cup \mathcal{P}_2) < \varepsilon \Rightarrow U(g) = U(f)$, and since we know g and f are integrable, we know $U(g) = L(g) = U(f) = L(f) \Rightarrow \int_a^b g = \int_a^b f$, and we're done.