b).

Since the maximum of f is $(e, \frac{1}{e})$, and f decreasing after that, we have

$$\begin{aligned} f(\pi) &< f(e) \\ \frac{log(\pi)}{\pi} &< \frac{log(e)}{e} \\ elog(\pi) &< \pi log(e) \\ exp(elog(\pi)) &< exp(\pi log(e)) \\ \pi^e &< e^\pi \end{aligned}$$

c).

$$\begin{aligned} x^y &= y^x \\ \frac{log(y)}{y} &= \frac{log(x)}{x} \end{aligned}$$

This means that $x^y = y^x \iff f(x) = f(y)$. Since for $x \leq 1$ f is increasing, this equation is true iff $y = x$. If $x = e$, f is at a maximum, so there are no other solutions than $y = x$.

If $x > 1$, f is decreasing for $x > e$ and decreasing for $x < e$, meaning there if there is a solution for $f(y) = f(x)$ with $x \neq y$, there is exactly 1, and if $1 < x < e$, then $e < y$ (or vice-versa). Now, since f is continuous on $(0, \infty)$, and on $(1, e), f \in (0, \frac{1}{e}$, and similarly, $(e, \infty), f \in (\frac{1}{e}, \infty)$, we know by IVT that for each $f(x), x \in (1, e)$ there's a corresponding $y \in (e, \infty)$ such that $f(y) = f(x)$.

d).

Suppose x, y are natural numbers with $f(y) = f(x)$ and $x \neq y$. We know that either x or y must be in $(1, e)$, which means that either $x = 2$ or $y = 2$. Suppose, wlog, that $x = 2$(since we can just swap x and y below if not). Then, have

$$\begin{aligned} \frac{log(2)}{2} &= \frac{y}{y} \\ ylog(2) &= 2log(y) \\ 2^y &= y^2 \end{aligned}$$

It's obvious that 4 satisfies this equation, and by c. we know there's only 1 y satisfying this equation, so $y = 4$ is the only solution, and we're done.