Q2

$$
\begin{aligned}
f((0,1,0),(0,1,0)) &= 0\cdot 0 + 0\cdot 0 \\
&= 0
\end{aligned}
$$

However, $(0,1,0) \neq 0$, so f does not satisfy the property of definiteness.

Q3

Let S be the set of inner products with the positivity condition. Let S' be the set of inner products with the new condition. To start, let $f \in S$. Then, $f(v,v) > 0\ \forall v \in V$ with $v \neq 0$, so the new condition is satisfied since $\exists v \in V$ with $f(v,v) > 0$. So, $f \in S'$, so $S \subseteq S'$. Now consider the other direction: take a function $g \in S'$. We know that by the new inner product definition, $\exists v \in V$ such that $g(v,v) > 0$. This v must be nonzero, as we still have the definiteness axiom. Now, by contradiction, consider a nonzero vector k orthogonal to v. Then,

$$
\begin{aligned}
\langle k, k\rangle &< 0 \\
\exists a : a\langle k, k\rangle &= -\langle v, v\rangle \\
\langle \sqrt{a}k, \sqrt{a}k\rangle + \langle v, v\rangle &= 0
\end{aligned}
$$

So,

$$\langle \sqrt{a}k + v, \sqrt{a}k + v\rangle = \langle \sqrt{a}k, \sqrt{a}k\rangle + \langle v, v\rangle = 0$$

Thus $\sqrt{a}k + v = 0 \Rightarrow k = -\frac{v}{\sqrt{a}}$, contradicting k being orthogonal to v. So, $g(k,k) = 0$ for all k orthogonal to v. Then, $\forall w \in V$, we can write w as a sum of a scalar multiple of c plus a vector k orthogonal to v.

$$
\begin{aligned}
w &= cv + k \\
\langle w, w\rangle &= \langle cv + k, cv + k\rangle \\
&= \langle cv, cv\rangle + \langle k, k\rangle
\end{aligned}
$$

And since both terms of this sum are positive, we know that the inner product of w with itself is positive, so $g(w,w) > 0\ \forall$ nonzero $w \in V$. That means $g \in S$, so $S' \subseteq S$, completing the proof.

Q8

Let $u, v \in V$, $||u|| = ||v|| = 1$. Then, $|\rangle u, v\langle| = |1| = 1 = ||u||||v||$. So, by Cauchy-Schwartz, u and v are collinear, and have the same magnitude, so $u = -v or u = v$. However, if $u = -v$, $\langle u, v\rangle = \langle -v, v\rangle = -\langle v, v\rangle < 0$. But $\langle u, v\rangle = 1$, which is a contradiction, so $u = v$.

Q11

Let $a, b \in \mathbb{R}^4$ with the inner product defined as the dot product.

$$
\begin{aligned}
a &= (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) \\
b &= (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}) \\
\text{then } |\langle a, b\rangle| &= |4| = 4 \\
||a|| &= \sqrt{\langle a, a\rangle} = \sqrt{a+b+c+d} \\
||b|| &= \sqrt{\langle b, b\rangle} = \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}
\end{aligned}
$$

So by Cauchy - Schwartz:

$$
\begin{aligned}
|\langle a, b\rangle| &\leq ||a||||b|| \\
4 &\leq \sqrt{a+b+c+d}\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}} \\
16 &\leq (a+b+c+d)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d})
\end{aligned}
$$

Q19

$$
\begin{aligned}
\langle u, v\rangle &= \frac{||u+v^2|| - ||u-v||^2}{4} \\
&= \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle}{4} \\
&= \frac{\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle - \langle u, u\rangle + \langle v, u\rangle + \langle u, v\rangle - \langle v, v\rangle}{4} \\
&= \frac{4\langle u, v\rangle}{4} \\
&= \langle u, v\rangle
\end{aligned}
$$