Let $U = \{u \in V : u = Pv \text{ for some } v \in V\}$. The fact that this is a subspace of v follows from the linearity of P. We also know $V = U^\perp \oplus U$. First, let's show $Pu = u$ for all $u \in U$.

$$\begin{aligned} Pv &= u \\ PPv &= Pu \\ Pv &= Pu \\ u &= Pu \end{aligned}$$

as desired. Now, take $w \in U^\perp$. Assume by contradiction that $Pw \neq 0$. Then, since $Pw \in U$, $\frac{4||w||^2}{||Pw||^2}Pw \in U$. Then, let $v = \frac{4||w||^2}{||Pw||^2}Pw + w$. Call the first term u. We know that since $u \in U$ that u and w are orthogonal. Since everything involved is positive, $||Pv|| \leq ||v|| \iff ||Pv||^2 \leq ||v||^2$. So, we know

$$\begin{aligned} \langle Pv, Pv\rangle &\leq \langle u+v, u+v\rangle \\ &= u^2 + w^2 \\ \implies ||u||^2 + \langle u, Pw\rangle + \langle Pw, u\rangle + ||Pw||^2 &\leq ||u||^2 + ||w||^2 \\ \frac{4||w||^2}{||Pw||^2}\langle Pw, Pw\rangle + \frac{4||w||^2}{||Pw||^2}\langle Pw, Pw\rangle + ||Pw||^2 &\leq ||w||^2 && \textit{Since } \frac{4||w||^2}{||Pw||^2} \textit{ is positive and real} \\ 4||w||^2 + 4||w||^2 + ||Pw||^2 &\leq ||w||^2 \end{aligned}$$

This is a contradiction, so that means $\nexists w \in U^\perp : Pw = 0$, or, in other words, $\forall w \in U^\perp, Pw = 0$. And, since we know that every $v \in V$ can be written as a sum of a $U \in u$ and a $w \in U^\perp$, we have $Pv = Pw + Pu = 0 + u = u$. This is the definition of P_U. QED.