

## Q2

$$\begin{aligned} f((0, 1, 0), (0, 1, 0)) &= 0 \cdot 0 + 0 \cdot 0 \\ &= 0 \end{aligned}$$

However,  $(0, 1, 0) \neq 0$ , so  $f$  does not satisfy the property of definiteness.

## Q3

Let  $S$  be the set of inner products with the positivity condition. Let  $S'$  be the set of inner products with the new condition. To start, let  $f \in S$ . Then,  $f(v, v) > 0 \forall v \in V$  with  $v \neq 0$ , so the new condition is satisfied since  $\exists v \in V$  with  $f(v, v) > 0$ . So,  $f \in S'$ , so  $S \subseteq S'$ . **INCOMPLETE**

## Q8

Let  $u, v \in V$ ,  $\|u\| = \|v\| = 1$ . Then,  $|\langle u, v \rangle| = |1| = 1 = \|u\| \|v\|$ . So, by Cauchy-Schwartz,  $u$  and  $v$  are collinear, and have the same magnitude, so  $u = \alpha v$  for  $|\alpha| = 1$ . However, if  $u = -v$ ,  $\langle u, v \rangle = \langle -v, v \rangle = -\langle v, v \rangle < 0$ . But  $\langle u, v \rangle = 1$ , which is a contradiction, so  $u = v$ .

## Q11

Let  $a, b \in \mathbb{R}^4$  with the inner product defined as the dot product.

$$\begin{aligned} a &= (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) \\ b &= \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right) \\ \text{then } |\langle a, b \rangle| &= |4| = 4 \\ \|a\| &= \sqrt{\langle a, a \rangle} = \sqrt{a + b + c + d} \\ \|b\| &= \sqrt{\langle b, b \rangle} = \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} \end{aligned}$$

So by Cauchy - Schwartz:

$$\begin{aligned} |\langle a, b \rangle| &\leq \|a\| \|b\| \\ 4 &\leq \sqrt{a + b + c + d} \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} \\ 16 &\leq (a + b + c + d) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \end{aligned}$$

**Q19**

$$\begin{aligned}\langle u, v \rangle &= \frac{||u + v||^2 - ||u - v||^2}{4} \\&= \frac{\langle u + v, u + v \rangle - \langle u - v, u - v \rangle}{4} \\&= \frac{\langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle - \langle u, u \rangle + \langle v, u \rangle + \langle u, v \rangle - \langle v, v \rangle}{4} \\&= \frac{4\langle u, v \rangle}{4} \\&= \langle u, v \rangle\end{aligned}$$