Lemma 1

Suppose you have a diagonal operator of the form $D = \begin{pmatrix} \lambda & 0 & ... & 0 \\ 0 & \lambda & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & \lambda \end{pmatrix}$ and some nilpotent matrix N with all entries above the diagonal. Then, $DN = ND$.

Proof:

$N_{i,j} = 0$ when $i \geq j$, and is not necessarily 0 when $i < j$. $D_{i,j} = 0$ if $i \neq j$ and equal to λ otherwise. So,

$$\begin{aligned} ND_{i,j} &= \sum_{q=1}^{m} N_{i,q} D_{q,j} \\ &= 0, q \neq j \text{ or } i \geq q \end{aligned}$$

So for it to be nonzero, need $q = j$ and $i < q = j$ So,

$$ND_{i,j} = \begin{cases} i < j & N_{i,j}\lambda \\ otherwise & 0 \end{cases}$$

Now, consider $DN_{i,j}$:

$$\begin{aligned} DN_{i,j} &= \sum_{q=1}^{m} D_{i,q} N_{q,j} \\ &= 0, i \neq q \text{ or } q \geq j \end{aligned}$$

So, we need $q = i$ and $i = q < j$, so:

$$DN_{i,j} = \begin{cases} \lambda N_{i,j} & i < j \\ otherwise & 0 \end{cases}$$

Completing the proof.

Lemma 2

Suppose you have two $m \times m$ block diagonal operators:

$$A = \begin{pmatrix} A_1 & 0... & 0 & \\ 0 & A_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & A_m \end{pmatrix} B = \begin{pmatrix} B_1 & 0... & 0 & \\ 0 & B_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & B_m \end{pmatrix}$$

Where each A_i, B_i is an $a_i \times a_i$ block. Then, letting $P(A_i, B_i)$ represent the block of the products of A_i and B_i as $a_i \times a_i$ matrices, we get:

$$AB = \begin{pmatrix} P(A_1, B_1) & 0... & 0 & \\ 0 & P(A_2, B_2) & ... & 0 \\ ... & ... & ... & ... \\ 0 & ... & ... & P(A_m, B_m) \end{pmatrix}$$

Proof:

For a general block diagonal matrix with m $a_i \times a_i$ diagonal blocks $T_1 ... T_m$, we can write:

$$T_{ij} = \begin{cases} \sum_{p=1}^{k-1} a_p < i \leq \sum_{p=1}^{k} a_p \text{ and } \sum_{p=1}^{k-1} a_p < j \leq \sum_{p=1}^{k} a_p : & (T_k)_{(i-\sum_{p=1}^{k-1} a_p, j-\sum_{p=1}^{k-1} a_p)} \\ \text{otherwise } : & 0 \end{cases}$$