Let $L(f) = \sum_{n\in\mathbb{Z}} f(n)$. First, show L is linear:

$$\begin{aligned} L(f+g) &= \sum_{n\in\mathbb{Z}} (f+g)(n) \\ &= \sum_{n\in\mathbb{Z}} f(n) + \sum_{n\in\mathbb{Z}} g(n) \\ &= L(f) + L(g) && \textit{So, additive. Now show homogenous:} \\ L(\lambda f) &= \sum_{n\in\mathbb{Z}} \lambda f(n) \\ &= \lambda L(f) \end{aligned}$$

So L is a linear map from $V \to \mathbb{C}$. Now suppose, by contradiction that $\exists v \in V : L(w) = \langle w, v\rangle \forall w$. Then, have $L(w) = \sum_n w(n)\overline{v(n)}$. Now, there are finitely many $n : \overline{v(n)} \neq 0$. Let $s = \{n \in \mathbb{Z} : \overline{V(n)}\} =\neq 0$. Let $M = \mathbb{Z}/s$. Define take some finite subset of M with all positive entries, call it m. Define

$$g : \mathbb{Z} \to C : g(x) = \begin{cases} x \in m & x \\ x \notin m & 0 \end{cases}$$

Then g is nonzero at only the cardinality of m points, which is finite, so $g \in V$, but $L(g) = \sum(m_i \in m)$, but $\langle g, v\rangle = 0$, since v is 0 everywhere g is nonzero, contradicting our assumption that v worked for all $f \in V$, as desired.