First, note that if $||u|| \leq \varepsilon ||v||, \forall \varepsilon > 0$, that's the same as saying $||u|| \leq \varepsilon, \forall \varepsilon > 0$. Also, since both $||u||, \varepsilon \geq 0$, $||u|| \leq \varepsilon \iff ||u||^2 \leq \varepsilon, \forall \varepsilon$.

It is therefore sufficient to show that $\exists m : \forall \varepsilon > 0, ||T^m v||^2 \leq \varepsilon$.

Since $\mathbb{F} = \mathbb{C}$, there exists an orthonormal basis of V such that T is upper triangular. We'll work in that basis. Let λ_i be the ith diagonal element of T. Each λ_i corresponds to an eigenvalue of T, so the absolute value of each $|\lambda_i$ is less than 1. Also, we know that if T is upper triangular, so is T^m. Any subspace of U is invariant under T, so any subspace of U is also invariant under T^m. We'll prove using induction on the dimension of U.

Base case: $dimU = 1$. (Since it's trivially true for $dimU = 0$) Then, $v = a_1 e_1$. So, $Tv = a_1 \lambda_1^m e_1$. So, $||T^m v||^2 = |a_1 \lambda_1^m||^2 = |a_1|^2 |\lambda_1|^{2m}$, and since $|\lambda_1| < 1$ picking $m : |\lambda|^{2m} < \frac{\varepsilon}{|a_1|}$ suffices.

Inductive Step: Assume if U =span$(e_1, ...e_k)$, $dimU = k$ there exists an m that works for all vectors in U. Note that, in fact, if m works for a subspace U, any integer multiple of m does, since $T^m u \in U$. For simplicity, let U' refer to the k dimensional subspace given by span$(e_1, ...e_k)$. This will make the rest of the proof a lot easier to read.

Now, we need to prove for a $k+1$ dimensional subspace of V. Let U =span$(e_1, ...e_{k+1})$. Then, $v = a_1 e_1 + ... + a_k e_k + a_{k+1} e_{k+1}$. So, $T^m v = T^m(a_1 e_1 + ... + a_k e_k) + T^m(a_{k+1} e_{k+1})$. Let $T^m(a_1 e_1 + ... + a_k e_k) = L$, and note that since T^m is upper triangular, $T^m a_{k+1} e_{k+1} = a_{k+1} T^m e_{k+1}$. $T^m e_{k+1}$ is equal to $\lambda_{k+1}^m e_{k+1}$ plus some linear combination of $e_1, ...e_k$. Call this linear combination v'. So, we have $T^m v = L + v' + a_{k+1} \lambda_{k+1}^m e_{k+1}$. Both $L, v' \in U'$.

Since the power of T can be whatever we want, apply T^m to both sides again, getting:

$$T^m T^m v = T^m(L + v') + a_{k+1} \lambda_{k+1}^m T^m e_{k+1}$$

And, again, for readability, let's let $p = 2m$. Then, we have:

$$T^p v = T^m(L + v') + a_{k+1} \lambda_{k+1}^m (\lambda_{k+1}^m e_{k+1} + v').$$

The claim is that $\forall \varepsilon > 0 \exists p$ so that $||T^p v|| \leq \varepsilon$. Now we expand the left side:

$$\begin{aligned}
||T^p v||^2 &= \langle T^p v, T^p v \rangle \\
&= \langle T^m(L+v') + a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1}, T^m(L+v') + a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle \\
&= \langle T^m(L+v'), T^m(L+v') + a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle \\
&+ \langle a_{k+1}\lambda_{k+1}^m v', T^m(L+v') + a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle \\
&+ \langle a_{k+1}\lambda_{k+1}^p e_{k+1}, T^m(L+v') + a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle \\
&= \langle T^m(L+v'), T^m(L+v') \rangle \\
&+ \langle T^m(L+v'), a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle \\
&+ a_{k+1}\lambda_{k+1}^m \langle v', T^m(L+v') \rangle \\
&+ a_{k+1}\lambda_{k+1}^m \langle v', a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle \\
&+ a_{k+1}\lambda_{k+1}^p \langle e_{k+1}, T^m(L+v') + a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle \\
&= ||T^p v||^2 \\
&+ \overline{a_{k+1}\lambda_{k+1}^m} \langle T^m(L+v'), v' + \lambda_{k+1}^m e_{k+1} \rangle \\
&+ a_{k+1}\lambda_{k+1}^m \langle v', T^m(L+v') \rangle \\
&+ a_{k+1}\lambda_{k+1}^m \langle v', a_{k+1}\lambda_{k+1}^m v' + \lambda_{k+1}^m e_{k+1} \rangle \\
&+ a_{k+1}\lambda_{k+1}^p \langle e_{k+1}, T^m(L+v') + a_{k+1}\lambda_{k+1}^m v' + a_{k+1}\lambda_{k+1}^p e_{k+1} \rangle
\end{aligned}$$

$$= ||T^p v||^2 + \overline{a_{k+1}\lambda_{k+1}^m}\langle T^m(L+v'), v'\rangle + a_{k+1}\lambda_{k+1}^m\langle v', T^m(L+v')\rangle + a_{k+1}\lambda_{k+1}^m\langle v', a_{k+1}\lambda_{k+1}^m v'\rangle$$
$$+ a_{k+1}\lambda_{k+1}^p\langle e_{k+1}, a_{k+1}\lambda_{k+1}^p e_{k+1}\rangle$$
$$= ||T^p v||^2 + Re(\overline{a_{k+1}\lambda_{k+1}^m}\langle T^m(L+v'), v'\rangle) + |a_{k+1}|^2|\lambda_{k+1}^m|^2||v'||^2 + ||a_{k+1}|^2|\lambda_{k+1}^p|^2$$
$$= ||T^p v||^2 + Re(\overline{a_{k+1}\lambda_{k+1}^m}\langle T^m(L+v'), v'\rangle) + |a_{k+1}|^2|\lambda_{k+1}|^p||v'||^2 + |a_{k+1}|^2|\lambda_{k+1}|^{2p}$$

Don't worry, almost done. That was gross, but what's important is that in front of every term of that sum there's a factor of either λ_{k+1}^m or λ_{k+1}^p, other than the first term $||T^p v||^2$, and the term that takes the real part of some complex number. Let's address the $||T^p v||^2$ first. Since by induction we know that there's a number k that lets us make the first term as small as we want, we can pick a value of m that's an integer multiple of k that makes $|\lambda_{k+1}^p|$ as small as we want. And of course, if $|\lambda_{k+1}|^p$ is as as small as we want, $|\lambda_{k+1}|^{2p}$ is smaller still. Similarly, $Re(\overline{a_{k+1}\lambda_{k+1}^m}\langle T^m(L+v'), v'\rangle)$ scales with λ_{k+1}, so we can make that as small as we want by exponentiating λ_{k+1} as well. So, we can pick an m that makes each term as small as we want. For a given ε, we just pick an m that makes each term less or equal to $\frac{\varepsilon}{4}$, and we're done.