Q6

Assume there does not exist a direct sum decompostion of V into two invariant subspaces. Since V is a complex vector space, we know T has at least one eigenvalue. We also know

$$V = G(\lambda_1, T) \oplus ... \oplus G(\lambda_n, T)$$

With each $G(\lambda_i, T)$ invariant under T. However, if $n > 1$, then we can write V as the sum of two invariant subspaces (by taking sums of generalized eigenspaces). So, $n = 1$, meaning λ_1 is the only eigenval of T, and $V = G(\lambda_1, T)$, so the minimal polynomial of T is $p(T) = (z - \lambda)^{\dim V}$.
For the other directon, suppose $p(T) = (z - \lambda)^{\dim V}$. We know that λ is the only eigenvalue of T. Assume by contradiction that $V = U_1 \oplus U_2$, with U_1, U_2 invariant under T. Note that $1 \leq dim(U_1), dim(U_2)$ since they're proper subspaces, and $dim(U_1), dim(U_2) < dim(V)$. Now, let $\alpha_1, ...\alpha_n$ be a basis for U_1, and $\beta_1, ..., \beta_m$ be a basis for U_2.
Now, we know that since λ is the only eigenval of T, that

$$p(T|_{U_1}) = (z - \lambda)^{dimU_1}$$

$$p(T|_{U_2}) = (z - \lambda)^{dimU_2}$$

Since U_1, U_2 invariant under T, they're also invariant under polynomials of T. So, if $v \in V$,

$$\begin{aligned} v &= a_1\alpha_1 + ... + a_n\alpha_n + b_1\beta_1 + ... + b_m\beta_m \\ (T - \lambda I)^j v &= (T - \lambda I)^j (a_1\alpha_1 + ... + a_n\alpha_n + b_1\beta_1 + ... + b_m\beta_m) \\ &= (T - \lambda I)^j|_{U_1} (a_1\alpha_1 + ... + a_n\alpha_n) + (T - \lambda I)^j|_{U_2} (b_1\beta_1 + ... + b_m\beta_m) \end{aligned}$$

The first term is 0 for $j \geq dimU_1$, and the second is 0 for $j \geq dimU_2$, so $(T - \lambda I)^j v = 0$ for $j = max\{dimU_1, dimU_2\}$, but $dim(U_1), dim(U_2) < dim(V)$, contradicting $p(T) = (z - \lambda)^{\dim V}$, meaning $V \neq U_1 \oplus U_2$, completing the proof.