Q2

$$\begin{aligned} f((0,1,0),(0,1,0)) &= 0 \cdot 0 + 0 \cdot 0 \\ &= 0 \end{aligned}$$

However, $(0,1,0) \neq 0$, so f does not satisfy the property of definiteness.

Q3

Let S be the set of inner products with the positivity condition. Let S' be the set of inner products with the new condition. To start, let $f \in S$. Then, $f(v,v) > 0\ \forall v \in V$ with $v \neq 0$, so the new condition is satisfied since $\exists v \in V$ with $f(v,v) > 0$. So, $f \in S'$, so $S \subseteq S'$. INCMPLETE

Q8

Let $u, v \in V$, $||u|| = ||v|| = 1$. Then, $|\rangle u, v \langle| = |1| = 1 = ||u||||v||$. So, by Cauchy-Schwartz, u and v are collinear, and have the same magnitude, so $u = -v or u = v$. However, if $u = -v$, $\langle u, v\rangle = \langle -v, v\rangle = -\langle v, v\rangle < 0$. But $\langle u, v\rangle = 1$, which is a contradiction, so $u = v$.

Q11

Let $a, b \in \mathbb{R}^4$ with the inner product defined as the dot product.

$$\begin{aligned} a &= (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}) \\ b &= (\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}) \\ \text{then } |\langle a, b\rangle| &= |4| = 4 \\ ||a|| &= \sqrt{\langle a, a\rangle} = \sqrt{a+b+c+d} \\ ||b|| &= \sqrt{\langle b, b\rangle} = \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} \end{aligned}$$

So by Cauchy - Schwartz:

$$\begin{aligned} |\langle a, b\rangle| &\leq ||a||||b|| \\ 4 &\leq \sqrt{a+b+c+d}\sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} \\ 16 &\leq (a+b+c+d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}) \end{aligned}$$

Q19

$$
\begin{aligned}
\langle u, v\rangle &= \frac{||u+v^2|| - ||u-v||^2}{4} \\
&= \frac{\langle u+v, u+v\rangle - \langle u-v, u-v\rangle}{4} \\
&= \frac{\langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle - \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle - \langle v,v\rangle}{4} \\
&= \frac{4\langle u,v\rangle}{4} \\
&= \langle u,v\rangle
\end{aligned}
$$