Q3

For the first direction, suppose range T^m = range T^{m+1}. Now, suppose $v \in$ null T^m. Then, $T^m v = 0$, so $T^{m+1}v = 0$. Therefore, null $T^m \subset$ null T^{m+1}. By the rank-nullity theorem, dim range T^m + dim null T^m = dim range T^{m+1}+dim null T^{m+1}, and from our assumption range T^m = range T^{m+1}, so dim null T^m = dim null T^{m+1} = k. Let $\beta = \{e_1, ...e_k\}$ be a basis for range T^m. Since null $T^m \subset$ null T^{m+1}, $\{e_1, ...e_k\} \in$ range T^{m+1}. But, since dim null $T^{m+1} = m$, β is also a basis for null T^{m+1}, so null T^m = null T^{m+1}, completing the proof in one direction.

Now, for the other direction, suppose null T^m = null T^{m+1}. We know from the rank-nullity theorem that dim range T^m = dim range $T^{m+1} = k$. So, it only remains to prove that either range $T^m \subset$ range T^{m+1}, or vise-verse, since the same argument as above will apply. We will prove the latter (range $T^{m+1} \subset$ range T^m): Consider a nonzero $v \in$ range T^{m+1} (because it's trivially true that both ranges contain 0) Then, $\exists k \in V : T^{m+1}k = v$. So, $T^{m+1}k = TT^m k = v$, so $v \in$ range T^m, meaning range $T^{m+1} \subset$ range T^m, and the rest follows from the same argument as above.