Faster Than Light Travel and Negative Energy in General Relativity

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1 Mathematical Preliminaries

1.1 A motivating example: the Euclidean Metric on $\mathbb{R}^2$.

Let's say we have a smooth curve segment $\gamma : [a, b] \to \mathbb{R}$ in the plane (meaning $\gamma(\lambda) = \begin{pmatrix} \gamma^1(t) \\ \gamma^2(t) \end{pmatrix}$) and we want to know how long it is. If it's a straight line, that's easy. We just use the Pythagorean theorem. Note that the length doesn't depend on the specific endpoints, only the distance between them:

$$L[\gamma] = \sqrt{(\Delta\gamma^1(t))^2 + (\Delta\gamma^2(t))^2} \tag{1}$$

Recall that the tangent vector γ' is given by

$$\gamma'(\lambda) = \begin{pmatrix} (\gamma^1)'(\lambda) \\ (\gamma^2)'(\lambda) \end{pmatrix} \tag{2}$$

Now, λ is an arbitrary parameter, so γ'_x and γ'_y won't necessarily be constant, but their ratio certainly will be if we want γ to be a straight line. If we choose to reparameterize the curve by x, instead of λ, the curve is then:

$$\gamma(x) = \begin{pmatrix} x \\ \gamma^2(x) \end{pmatrix} \tag{3}$$

and its tangent vector at the point x is given by:

$$\gamma'(x) = \begin{pmatrix} 1 \\ \frac{d\gamma^2}{dx} \end{pmatrix} \tag{4}$$

Both terms here are constant (the second is just the slope of the line). So we can rewrite equation 1 as:

$$L[\gamma] = \sqrt{(\Delta x)^2 + \left(\frac{d\gamma^2}{dx}\Delta x\right)^2} = \Delta x\sqrt{1 + \left(\frac{d\gamma^2}{dx}\right)^2} = \Delta x|\gamma'(x)| \tag{5}$$

What if the curve isn't a straight line? What if it's only piecewise straight? That is, γ is a straight line on each $[x_i, x_{i+1}]$ (we'll call each straight line subsegment γ_i)? Well, then you just add up the lengths of each straight segment:

$$L[\gamma] = \sum_{i=1}^{n-1} L[\gamma_i] = \sum_{i=1}^{n-1} (x_{i+1} - x_i)|(\gamma^i)'(x)|$$

What about arbitrary curves? An arbitrary curve is just an infinitesimally piecewise straight curve, so in the limit as $x_{i+1} - x_i \to 0$, we have:

$$L[\gamma] = \int_{x_1}^{x_n} |\gamma'(x)|\, dx \tag{6}$$

which you've probably seen before! It's the definition of arclength. Now, it's important to notice that all this was based on the pythagorean theorem. But not just the pythagorean theorem: we assumed it made sense to "infinitesimally" apply the pythagorean theorem at each point of the curve. But why can we do this? We're taking a dot product of vectors, but all these vectors don't live in the same vector space! So what we need isn't just a dot product, but a dot product on each tangent space! Such a (smoothly varying) set of dot products is called a "Riemannian Metric".

1.2 Riemannian Metrics

The length of a vector is given by $|(v^0, v^1)| = \sqrt{(v^0)^2 + (v^1)^2}$. We can write this as $(v^0, v^1) \cdot (v^0, v^1)$, or as:

$$|(v^0, v^1)| = \sqrt{(v^0)^2 + (v^1)^2} = (v^0, v^1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} = \sum_{i,j=0}^{1} \delta_{ij} v^i v^j \tag{7}$$

where $\delta_{ij} = 0$ if $i \neq j$ and 1 otherwise. The last notation will turn out to be the most useful for our purposes.
It turns out that for a general inner product of two vectors in a vector space V, you can always find a linear operator L such that $\langle u, v \rangle = u^* L(v)$. In coordinates, a matrix A such that:

$$\langle u, v \rangle = (u^0, ...u^n) \begin{pmatrix} A_{0,0} & \dots & A_{0,n} \\ \vdots & \vdots & \vdots \\ A_{n,0} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} v^0 \\ \vdots \\ v^n \end{pmatrix} = \sum_{i,j=0}^{n} u^i A_{ij} v^j = u^i A_{ij} v^j \tag{8}$$

in the future we'll drop the sigma, so if there are repeated indicies, assume we're summing over them. Since the inner product we pick for each tangent space can change, this gives us a very convenient notation for our Riemannian metric: Let M be an n dimensional manifold with a Riemannian metric g. If $m \in M$, then $g(m)$ is an inner product on $T_m M$, so has a matrix representation $g_{ij}(m)$. If $u, v \in T_m(M)$, then $g(u, v) = g_{ij}(m) v^i u^j$. Now, these matricies aren't coordinate independent: depending on the coordinate chart we pick, their values can change. But that's alright! One can verify that the inner product of two vectors is a coordinate independent quantity.

A Riemannian metric is an example of a *tensor field*. A (k, l) tensor is an object that eats a list of l vectors and k dual vectors, is linear in each entry in the list, and returns a real number. A tensor field is an assignment of a tensor to every point in space. For example, a Riemannian metric is a (2,0) tensor field, since it assigns a bilinear map that eats two vectors and returns a real number to every point in space.
A quick note on notation: an upper index indicates an object that eats a dual vector, and a lower index indicates an object that eats a vector (and multiple upper and lower indicies indicate objects that eat that number of each object). So we'd write a dual vector field (one form) as $\omega_\mu(x)$, and a vector field as $v^\mu(x)$. The metric is written $g_{\mu\nu}$ since it eats two vectors (it's an inner product, after all).

1.3 Derivative Operators and Parallel Transport

Okay, great. So we can now define arclength on an arbitrary manifold. Now what? Well, it would be nice if we could compare vectors at different points in the manifold! In $\mathbb{R}^2$ I can just slide a vector from one point to another, ensuring that at each point along the path it remains parallel to the original vector.

This gets weird when we talk about a sphere. Now the angle between two vectors depends on the paths they were parallel transported along! For example, if we start with two vectors perpendicular to the equator, and parallel transport them together along the equator, they end up parallel. But if we bring them both to the poles, they'll intersect at an angle!
Notice that I'm using words like "angle" and "parallel". It only makes sense to talk about angles between vectors once you can take their inner product, which we now can thanks to our metric!
In order to talk properly about parallel transport, we need a notion of a derivative that measures how much a vector field represents parallel transport along a curve. It turns out that this isn't uniquely determined by the metric. For

the purposes of GR, however, we add properties to it until it becomes unique.

A connection ∇_μ (hereafter referred to as a "derivative operator" to match the standard GR terminology) is a map sending a (k,l) tensor field to a $(k,l+1)$ tensor field, satisfying the following properties:

$$\nabla(S+T) = \nabla S + \nabla T \tag{9}$$
$$\nabla(S \otimes T) = \nabla S \otimes T + S \otimes \nabla T \tag{10}$$
$$\text{Commutes with contraction (not super relevant for this talk)} \tag{11}$$
$$\text{Compatible with tangent vectors as directional derivatives: } v(f) = v^\mu \nabla_\mu f \tag{12}$$
$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T = 0 \tag{13}$$

this set of properties still doesn't uniquely define a derivative operator. However, we can use this to determine if a vector field is parallel transported along a curve γ:

$$\gamma'^\nu \nabla_\nu v^\mu = 0 \tag{14}$$

that is, the directional derivative of the vector field in the direction of the tangent vector of the curve is 0. If you imagine sliding a vector along the curve, this condition says that as long as the vector stays on the curve at all times, it doesn't infinitesimally change angle.

We'll impose one last condition on our derivative operator: it must parallel transport the metric tensor. That is, for any curve γ, we must have:

$$\gamma'^\nu \nabla_\nu g_{\alpha\beta} = 0 \implies \nabla_\nu g_{\alpha\beta} = 0$$

it turns out that this completely characterizes a derivative operator.

1.4 Geodesics

What's so special about a straight line? A couple of things. First, it's the shortest path connecting two points. Second, it parallel transports its own tangent vector! This is the property we'll focus on. It's not obvious, but the two conditions are actually very closely related. On a general manifold, we call a curve a geodesic if it parallel transports its own tangent vector:

$$\gamma'^\mu \nabla_\mu \gamma'^\nu = 0 \tag{15}$$

You can more or less think of geodesics as "the shortest possible path between two points". This is not always true (imagine going around a sphere in one direction, and then the other. Both geodesics, but one is shorter).
These are the most important object I'll talk about today. All the math I just went through was so that I could define these curves. Now, we can talk about general relativity.

2 General Relativity

Note that for this entire section we'll be taking $c = 1$.

2.1 Geodesics and Proper Time

Nweton's first law states that "Objects do not change their states of motion unless acted upon by an external force". What this means is that objects will travel in straight lines unless a force prevents them from doing so. Unfortunately for Newton, he didn't know what a manifold was. Otherwise, he might have said "Objects travel on geodesics unless acted on by an external force". That would have been much more general.

And that generalization right there is how you get almost all of general relativity (plus the Einstein field equations, which, to be fair, are pretty important). But that's the central idea: we usually think of spacetime as $\mathbb{R}^4$ (or even as a vector bundle over $\mathbb{R}$ where each fiber is $\mathbb{R}^3$, but that's besides the point): three dimensions of space, and one of time. Einstein simply asked "what if we allow spacetime to be a different manifold?".

It's a little bit more complicated than that, of course. We'll take flat spacetime $\mathbb{R}^4$ as a toy example (in usual cartesian coordinates). We'll go ahead and put a metric on it:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

Now, you may notice something right away. $g_{\mu\nu}$ doesn't define an inner product anymore! It's not positive definite. And you'd be right. In general relativity, the metric on the manifold is "pseudo-Riemannian". Luckily, everything we said earlier still works, just remember that the metric isn't positive definite anymore.
Minkowski spacetime has a certain class of observers, called "inertial observers" who all use a coordinate system that puts the metric in form 16. These observers use coordinate systems related by Lorentz transformations.
Why do we do this? So that we can have a notion of causality. We say a vector is "timelike" if it has negative length, "spacelike" if it has positive length, and "lightlike" or "null" if it has zero length. A path is timelike/spacelike/null if its tangent vectors are everywhere timelike/spacelike/null. Particles with mass always travel on timelike paths.

An important thing about Minkowski spacetime is that geodesics are straight lines. It's therefore impossible to travel faster than light on a timelike path. Any null curve is steeper (as measured by an inertial observer) than a timelike curve.
Okay, and how do we deal with arclength? Relativity defines a quantity called the "proper time" for a timelike path,

$$\tau[\gamma] = \int_{\lambda_0}^{\lambda_f} \sqrt{-g_{\mu\nu}\frac{d\gamma^\mu}{d\lambda}\frac{d\gamma^\nu}{d\lambda}}d\lambda$$

It turns out that in relativity, geodesic paths are exactly the ones that *maximize* the proper time.

2.2 Einstein's Equation and Energy Conditions

So we know how things move in space given a metric. How do we find the metric? We'll define a tensor describing the distribution of matter and energy at each point in space: the Stress-Energy tensor, T_{ab}. The metric $g_{\mu\nu}$ satisfies the following equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{17}$$

$R_{\mu\nu}$ and R are the Ricci tensor and scalar respectively, they describe the curvature of the manifold and are themselves dependent on the metric and its first and second derivatives. Basically, this equation sucks to solve. Solutions are only known in certain special cases. We won't solve it in this talk, we'll rather impose a metric, and look at the mass and energy distribution required by Einstein's equation. The energy density observed by someone traveling on a timelike path γ is given by:

$$\rho = \gamma'^\mu \gamma'^\nu T_{\mu\nu}$$

If $\rho < 0$, we say that the weak energy condition has been violated. Basically, there exists an observer who will measure negative energy density at that point. This is generally considered bad, as "negative energy" is a pretty weird thing to have. In QFT it's possible to have small negative energy densities at certain points in space, but not in large quantities. We don't know whether that's an artifact that will go away with a better understanding of physics, either. And if it's not, we don't know if it's possible to store it in large amounts. Basically, if a metric breaks the weak energy condition, it's probably not physically possible to build it in real life.

2.3 Breaking the Light Barrier

What if you're not in Minkowski spacetime? It turns out it's then possible to go faster than light, in a certain sense. If we impose the metric

$$g_{\mu\nu} = \begin{pmatrix} -1 + v(t)^2 f(r(t))^2 & 0 & 0 & -v(t)f(r(t)) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v(t)f(r(t)) & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

on $\mathbb{R}^4$ in cartesian coordinates $((t, x, y, z))$, where $r(t) = \sqrt{x^2 + y^2 + (z - \xi(t))^2}$ (the Euclidean distance to the center of the warp bubble), $\frac{d\xi}{dt}(t) = v(t)$ ($\xi(t)$ is the "center" position of the warp bubble), and $f(r)$ is the function shaping the warp drive (we generally take this to be some sort of "smooth flat top mountain" function, or an analytic approximation thereof).

It turns out that in this metric (called the Alcubierre Metric), no matter what v we pick, the path

$$\gamma(t) = \begin{pmatrix} t \\ 0 \\ 0 \\ \xi(t) \end{pmatrix} \tag{19}$$

is a timelike geodesic! Importantly, a timelike geodesic with $\frac{dz}{dt} = v(t)$. So not only can a positive mass object travel along it, but the rate at which it travels in the z direction can be made arbitrarily large! Therefore, relative to an observer far away in flat space, someone sitting in the middle of the warp bubble will be seen to travel faster than light! Unfortunately, this metric breaks the weak energy condition: it requires negative energy to construct. So then we can ask a reasonable question: is this true for *all* FTL metrics?

2.4 Defining FTL travel

The issue is: what do we mean by FTL travel? Consider a spacetime (for simplicity, $(0, 1)_t \times \mathbb{R}$, a $1+1$ dimensional spacetime) with metric

$$g_{\mu\nu} = \begin{pmatrix} -1 + 4t^2x^2 & x(-2t)(1 - t^2) \\ x(-2t)(1 - t^2) & (1 - t^2)^2 \end{pmatrix}$$

then it would seem that this is an FTL metric: in this case, a null ray satisfies

$$\frac{dx}{dt} = \frac{\pm 1 + 2tx}{1 - t^2}$$

this ODE can be solved to give $x(t) = \frac{t}{(1-t^2)}$. This makes it seem like a light ray can travel an arbitrary large distance in a finite amount of time. However, if we make the change of coordinates $x' = x(1 - t^2)$, in the new basis induced by this change of coordinates we find that

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

so this was just flat space all along, in weird coordinates! Note that something stationary in the x coordinate is now moving towards the origin, passing it at $t = 1$ in the x' coordinate. What was nice about the Alcubierre metric was that we had a nice flat background to compare to. So, rather than tackling the hard problem of defining FTL travel (at least, at first) Ken Olum instead choses to solve a better defined problem: proving that any FTL travel in flat-background spacetime requires negative energy.

2.5 Olum's Proof

Consider a spacetime (M, g) with a metric g that is flat outside a closed region $S = \mathbb{R} \times [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$. Inside that region, before some time t_0, the metric is flat. After that time, the metric is somehow different from the flat metric in such a way that we can, by traveling through this region, go along a timelike geodesic connecting (t_1, x_1) and (t_2, x_2), even though $x_2 - x_2 > t_2 - t_2$. This is impossible in usual Minkowski spacetime. If we removed the S region and replaced it with usual flat space, this path would be FTL.

We can show that somewhere in S, the weak energy condition must be violated. Assume that in the region S, the generic condition [1] is obeyed in the region S. Let $\Delta t = t_2 - t_2$.

Now, consider a new spacetime S' with the topology gained by taking the region of S with $x_1 \leq x \leq x_2$ and identifying (t, x_1, y, z) and $(t + \Delta t, x_2, y, z)$ with metric g. In S', the path we can take from (t_1, x_1) to (t_2, x_2) is a closed causal curve. No path only in the region $t \leq t_0$ can be a closed causal curve, as travel is necessarily slower than light in that region, and $\Delta t < x_2 - x_1$.

If there are no singularities in the original spacetime M, there will be none in S'. However, there's a theorem by Tipler that states if the following conditions hold:

1. Null energy condition (which implies WEC) holds
2. Generic condition holds
3. Spacetime is partially asymptotically predictable
4. There are closed timelike curves

then there must be a singularity in spacetime. Since there is no singularity in S', one of the conditions must fail. The generic condition holds by assumption, the spacetime is partially asymptotically predictable from a partial cauchy surface $t = t_0 - \Delta t$, so the condition that fails is the Null energy condition. Since it's a local condition, if it fails in S' it fails in S, and therefore the WEC fails in S too.

[1] A technical condition that is satisfied in non-pathological/trivial spacetimes