Thermal Wind

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The thermal wind describes how the geostrophic winds change with height. This vague definition allows two similar yet different mathematical representations, each useful in their own right.

Differentiation of the Geostrophic Wind

We begin with the zonal geostrophic wind, and subsequently substitute hydrostatic balance in order to define u_g in terms of height.

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \tag{1}$$

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y} \tag{2}$$

Differentiating both sides with respect to pressure,

$$\frac{\partial}{\partial p}u_g = \frac{\partial}{\partial p}(-\frac{g}{f}\frac{\partial z}{\partial y}) \tag{3}$$

$$\frac{\partial u_g}{\partial p} = -\frac{g}{f}\frac{\partial}{\partial y}\frac{\partial z}{\partial p} \tag{4}$$

Recall that hydrostatic balance can also be written $\frac{\partial p}{\partial z} = -\frac{pg}{R_d T}$ using the IGL. Substituting into (4),

$$\frac{du_g}{dp} = \frac{g}{f}\frac{\partial}{\partial y}\frac{R_d T}{pg} \tag{5}$$

$$\frac{\partial u_g}{\partial p} = \frac{R_d}{fp}\frac{\partial T}{\partial y} \tag{6}$$

Alternatively, we can express the LHS derivative in terms of z with another substitution of hydrostatic balance.

$$\frac{R_d T}{gp}\frac{\partial u_g}{\partial z} = \frac{R_d}{fp}\frac{\partial T}{\partial y} \tag{7}$$

$$\frac{\partial u_g}{\partial z} = \frac{g}{fT}\frac{\partial T}{\partial y} \tag{8}$$

Our final formulations include

$$\boxed{\frac{\partial u_g}{\partial p} = \frac{R_d}{fp}\frac{\partial T}{\partial y}} \tag{9}$$

$$\boxed{\frac{\partial u_g}{\partial z} = \frac{g}{fT}\frac{\partial T}{\partial y}} \tag{10}$$

$$\boxed{\frac{\partial v_g}{\partial p} = -\frac{R_d}{fp}\frac{\partial T}{\partial x}} \tag{11}$$

$$\boxed{\frac{\partial v_g}{\partial z} = -\frac{g}{fT}\frac{\partial T}{\partial x}} \tag{12}$$

Note that via this definition, the thermal wind does not have typical wind units.

Vector Analysis

The thermal wind can be alternatively be defined using vectors, where $\vec{V_T} = \vec{V_{upper}} - \vec{V_{lower}}$. While the formulations (9-12) explicitely defined the vertical speed shear of the environment, $\vec{V_T}$ describes the vector difference providing a useful window into vertical directional shear. Before delving into the implications of this view-point, let's expand upon the proposed vector difference.

Recalling that the geostrophic wind is $V_g = \frac{1}{f}\hat{k} \times \nabla_p \Phi$

$$\vec{V_T} = \vec{V_2} - \vec{V_1} = \frac{1}{f}\hat{k} \times \nabla_p(\Phi_2 - \Phi_1) \tag{13}$$

The hypsometric equation may be written $R_d T \ln \frac{p_2}{p_1} = (\Phi_2 - \Phi_1)$. Substituting into (13),

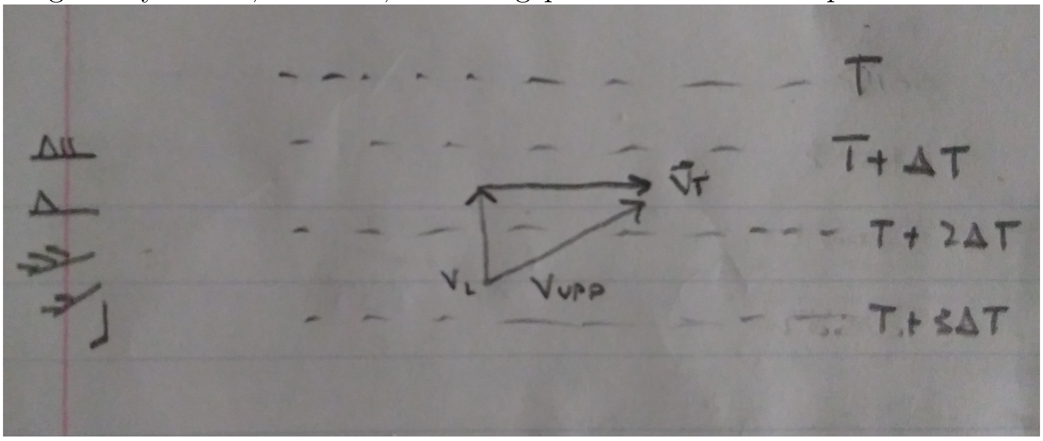
$$\vec{V_T} = \frac{1}{f}\hat{k} \times \nabla_p \left[R_d T \ln \frac{p_2}{p_1} \right] \tag{14}$$

$$\vec{V_T} = \left[R_d \ln \frac{p_2}{p_1} \right] \frac{1}{f}\hat{k} \times \nabla_p T \tag{15}$$

While this is a result we could have assumed from (9-12), note that $\vec{V_T} \propto \nabla_p T \propto \nabla_p \Delta\Phi$, or that the thermal wind is proportional to the temperature/thickness gradient. Additionally, the right hand rule tells us that the thermal wind is always perpendicular to the temperature/thickness gradient with cold air/low thickness to the left.

The Thermal Wind and Temperature/Thickness Advection

Since the thermal wind is not a real wind, it cannot be responsible for physical advections such as temperature advection, nor should it due to orthogonality. Take, however, a veering profile such as that pictured below.



The thermal wind is cleary indicating the prescence of westerly shear in a north/south temperature gradient and blows with cold air to its left. The low-level geostrophic wind vector is scaled up for viewing, though clearly its southerly direction suggests warm air advection in the prescence of such a temperature gradient. The vertical wind profile is drawn to the left, confirming the prescence of low-level warm air advection and, by necesity, veering. This example illustrates the ubiquity of veering/backing with WAA/CAA in the mid-latitudes. If veering/backing does not exist, there can be no WAA/CAA (draw wind vectors and subtract them, none of them will cross isotherms). While this is a simplified example, be sure to take note of regions of vertical directional shear in atmospheric soundings when discussing the level at which temperature advections are actually occuring.