Barotropic Rossby Wave Dispersion Relation

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Assumptions

- Conservation of absolute vorticity, that is, $\frac{d}{dt}(\zeta + f) = 0$
- Ignore stretching term, $\frac{dw}{dz} = 0$
- Uniform zonal flow ($\bar{\zeta} = 0$, $\bar{v} = 0$, $\bar{w} = 0$)
- Linearization (ignore $u\prime\frac{\partial\zeta\prime}{\partial x}$, $v\prime\frac{\partial\zeta\prime}{\partial y}$)

Linearization of the conservation of absolute vorticity

We begin with the conservation of absolute vorticity and expand the total derivative into its Eulerian and advective components.

$$\frac{d}{dt}(\zeta + f) = 0 \tag{1}$$

$$(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})(\zeta + f) = 0 \tag{2}$$

Recall that under uniform zonal flow we may drop $\bar{\zeta} = 0$ and $\bar{v} = 0$ when linearizing.

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + u\prime\frac{\partial}{\partial x} + v\prime\frac{\partial}{\partial y})(\zeta\prime + f) = 0 \tag{3}$$

Ignore $u\prime\frac{\partial\zeta\prime}{\partial x}$ and $v\prime\frac{\partial\zeta\prime}{\partial y}$ and recall that f only varies with latitude (β) whilst distributing the linearized derivative operand into the perturbation absolute vorticity.

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x})\zeta\prime + v\prime\beta = 0 \tag{4}$$

Spatial and temporal derivatives of the perturbation streamfunction

We now pause our evaluation of the consevation of absolute vorticity, and recall that $\psi\prime = \psi_0 e^{i(kx-\omega t)}$, which assumes a sinusoidal wave pattern due to the streamfunction perturbation.

Taking the first derivative of the streamfunction perturbation, $\psi\prime$

$$\frac{\partial \psi\prime}{\partial x} = \frac{\partial(\psi_0 e^{i(kx-\omega t)})}{\partial x} \tag{5}$$

$$= \psi_0 e^{i(kx-\omega t)}) \cdot ik \tag{6}$$

$$= \psi\prime\, ik \tag{7}$$

Additionally, recall that the zonal derivative of the wavefunction perturbation is equal to the meridional flow perturbation, or

$$\psi\prime_x = v\prime \tag{8}$$

$$v\prime = \psi\prime\, ik \tag{9}$$

Taking the second derivative of the perturbation streamfunction with respect to x,

$$\psi\prime_{xx} = \psi\prime ik \cdot ik \tag{10}$$

$$= -k^2 \psi\prime \tag{11}$$

Recall that $\zeta\prime = \frac{\partial v\prime}{\partial x} - \frac{\partial u}{\partial y}$, or,

$$\zeta\prime = \psi\prime_{xx} - \psi\prime_{yy} \tag{12}$$

However, recall that $\psi\prime = \psi_0 e^{i(kx-wt)}$ and it is obvious that $\psi\prime$ does not vary in y ($\psi\prime_{yy} = 0$). Thus,

$$\zeta\prime = \psi\prime_{xx} \tag{13}$$

Finally, we take the derivative of $\psi\prime$ with respect to time,

$$\frac{\partial \psi\prime}{\partial t} = \frac{\partial(\psi_0 e^{i(kx-\omega t)})}{\partial t} \tag{14}$$

$$= \psi\prime \cdot -i\omega \tag{15}$$

Thus, the linearized total derivative operand, when applied to $\psi\prime$ is,

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x})\psi\prime = (\frac{\partial \psi\prime}{\partial t} + \bar{u}\frac{\partial \psi\prime}{\partial x}) \tag{16}$$

$$= (-i\omega\psi\prime + ik\psi\prime) \tag{17}$$

$$= (-i\omega + ik)\psi\prime \tag{18}$$

Return to conservation of absolute vorticity

We now have substitutions (see (11/13), (9), and (18) for many of the terms seen in the linearized conservation of absolute vorticity (4). As a reminder, we begin with (4).

$$(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x})\zeta\prime + v\prime\beta = 0$$

$$-(-i\omega + i\bar{u}k)k^2\psi\prime + ik\beta\psi\prime = 0 \tag{19}$$

Frequency, Phase Speed and Group Velocity

Isolating ω

$$-(-i\omega + i\bar{u}k)k^2\psi\prime + ik\beta\psi\prime = 0 \tag{20}$$

$$(-i\omega + i\bar{u}k)k^2\psi\prime = ik\beta\psi\prime \tag{21}$$

$$(-i\omega + i\bar{u}k)k^2 = ik\beta \tag{22}$$

$$(-i\omega + i\bar{u}k)k = i\beta \tag{23}$$

$$-i\omega k + i\bar{u}k^2 = i\beta \tag{24}$$

$$-\omega k + \bar{u}k^2 = \beta \tag{25}$$

$$-\omega k = \beta - \bar{u}k^2 \tag{26}$$

$$-\omega = \frac{\beta}{k} - \bar{u}k \tag{27}$$

$$\boxed{\omega = \bar{u}k - \frac{\beta}{k}} \tag{28}$$

Divide (28) by k in order to derive the celerity of an individual Rossby wave, c_x

$$\frac{\omega}{k} = \frac{\bar{u}k}{k} - \frac{\beta}{k^2} \tag{29}$$

$$\boxed{c_x = \bar{u} - \frac{\beta}{k^2}} \tag{30}$$

Recall that the group velocity, G_x is the derivative of ω with respect to k, thus,

$$\frac{\partial\omega}{\partial k} = \bar{u} + \frac{\beta}{k^2} \tag{31}$$

$$\boxed{G_x = \bar{u} + \frac{\beta}{k^2}} \tag{32}$$

Implications

- Rossby waves are dispersive in k (ω explicitely depends on k)
- $-\frac{\beta}{k^2}$ is the Rossby wave propagation speed, thus, individual transverse waves travel upstream
- $\frac{\beta}{k^2}$ is the Rossby *packet energy* propagation speed, thus, energy propagates downstream
- High wave-number waves (large k) travel westward quickly ($-\frac{\beta}{k^2} << \bar{u}$) (e.g. short-wave troughs can round the base of long-wave troughs)
- Low wave-number waves (small k) can be quasi-stationary or even travel upstream ($-\frac{\beta}{k^2} \approx \bar{u}$) (e.g. 500 hPa 'blocks' can retrograde within the flow)
- The above is an alternative view of the quasi-geostrophic equation for heigh falls (χ equation). For example, the forcing for height falls associated with a cut-off low (or blocking) high is usually dominated by planetary vorticity advection within the meridional flow associated with such a feature.