Waves

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Descriptors

What is a Wave?

- Quasi-periodic
- Transfer energy and momentum at large distances without transfer of fluid particles

Free/Forced

- Free waves are resonant normal modes, excited by weak random motions in the atmosphere
- e.g., Hirota and Hirooka (1984), '5-Day Wave'
- Forced waves require continuous forcing at compatible space ant time scales
- e.g., Tides

Internal/External

- Internal: wave amplitude maximizes in the interior
- Sinusoidal form for phase variation $\sim e^{im_r z}$ which indicates sinusoidal variation in ther vertical
- e.g., synoptic Rossby waves, internal gravity waves

- External: wave amplitude maximizes at the edge ('evanescent waves' $\sim$ amplitude decays in space)
- wave energy decreases away from the boundary $\sim e^{-m_i z}$
- e.g., surface water waves (wave impact dampens away from the density gradient)

Stationary/Travelling

- Stationary $c_r = 0$
- e.g. 'standing wave' (fixed node seiche, guitar string, etc)
- Travelling $c_r \neq 0$

Steady/Transient

- Steady $c_i = 0$, fixed amplitude
- Transient $c_i \neq 0$, amplitude varies in time (growth or decay)

Linear/Non-linear

- Linear (infinitesimal, theoretical)
- Nonlinear (finite amplitude, observable)

Take the Eulerian and advective form of the Lagrangian time rate of change of u. The flow is 'self-steepening' in that over faster flow will 'catch up to' slower flow and, subsequently, break. This process is fundamental to chaos theory. Note, however, that if $\overline{u}$ is constant to first order, the total derivative is linearized ($u = \overline{u} + u'$).

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \text{ nonlinear advection, self-steepening with a tendency to break}$$

$$\text{If} \frac{|u'|}{|\overline{u}|} \ll 1 \text{ where } u = \overline{u} + u' \text{ then}$$

$$\frac{\partial u'}{\partial t} + \overline{u}\frac{\partial u'}{\partial x} = 0 \text{ advection linear in eddy amplitude } u'\text{, linearized operator}$$

Dispersive/Non-dispersive

- Dispersive: propagation speed depends on wavelength
- e.g., Rossby waves ($c = -\frac{\beta}{k^2} \propto L_x^2$)
- Non-dispersive: all waves travel at the same speed, regardless of wavelength
- e.g., linear shallow water waves, $c = \sqrt{gh}$; linear sound waves, $c = \sqrt{\gamma RT}$

Dispersion relation

- A *dispersion relation* relates the time and space scales of a given wve type to physical parameters
- Expressed as the functional dependence of frequency ω on size and physics ($\omega = \omega(\vec{k}$, physics))
- If $\omega(k)$, waves travel at different speeds
- Can also show how energy and wave crests move relative to one another
- A *critical surface* is defined where $c = \overline{u}$. This determines where wave energy emerges and cannot pass
- *phase speed* ($c = \frac{\omega}{k}$) indicates how fast a wave crest or trough moves
- *group speed* ($G = \frac{\partial \omega}{\partial k}$) indicates how fast energy is moving

Wave Definitions

Phase

phase is defined as $\theta = kx + ly + mz - \omega t$, which is the definition of a plane in 3D place with a temporal dependence (wave on a given phase can travel relative to the three axis). Phase increases in the direction of propagation...or phase decreases in time as a wave crest passes (hence negative sign on ωt term.

Disturbance

Take a generalized disturbance: $\psi = \mathrm{Re}\left[Ce^{i\theta}\right] = \mathrm{Re}\left[(C_r + iC_i)(\cos\theta + i\sin\theta)\right] = C_r \cos\theta - C_i \sin\theta = A\sin\theta + B\sin\theta$.

Wavenumber Vector

$$\vec{k} = \nabla\theta = \hat{i}\frac{\partial\theta}{\partial x} + \hat{j}\frac{\partial\theta}{\partial y} + \hat{k}\frac{\partial\theta}{\partial z} \tag{1}$$

Recall over one wavelength $\partial\theta = 2\pi$, thus, $k = \frac{2\pi}{k}$ (same for meridional and vertical wave numbers)

$$|\vec{k}| = \sqrt{k^2 + l^2 + m^2}$$

Since $\vec{k} = \nabla\theta$, $k \perp \theta$ pointing towards increasing θ.

$\therefore$ phase decreases with time at a point

Frequency

$\omega = -\frac{\partial\theta}{\partial t}$ 'how fast θ varies in time. When $\omega > 0$ the wave is travelling to the east. When $\omega < 0$ the wave is travelling to the west. Though, we don't really talk about the sign of wave periodicity.

In τ (one period), $\partial\theta = 2\pi$, thus, $|\omega| = \frac{2\pi}{\tau}$

Conservation of Wave Crests

The local rate of change of the wavenumber vector and the frequency gradient sum to zero, or,

$$\frac{\partial\vec{k}}{\partial t} + \nabla\omega = 0$$

$$\frac{\partial}{\partial t}(\frac{\partial\theta}{\partial x}) + \frac{\partial}{\partial x}(-\frac{\partial\theta}{\partial t}) = 0$$

This is easily interpreted through an example. If ω is decreasing to the east ($\nabla\omega > 0$) then $\vec{k}$ must decrease in time (longer waves reaching the point).

Phase Velocity

This is the rate that a point on a phase surface travels in the direction of $\vec{k}$.

$$\vec{c_p} = \frac{\omega}{|\vec{k}|}\frac{\vec{k}}{|\vec{k}|} = \frac{\omega}{k^2 + l^2 + m^2}(k\hat{i} + l\hat{j} + m\hat{k})$$

$\vec{c_p} \perp \theta$ and $\vec{c_p} \parallel \vec{k}$