

Hypsometric Equation

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Hydrostatic Balance

Recall that the vertical pressure gradient force balances with the gravitational acceleration such that vertical motions are extremely limited over large-scales. This is known as hydrostatic balance.

$$\frac{dw}{dt} = 0 = \rho g + \frac{\partial p}{\partial z} \quad (1)$$

$$\boxed{-\rho g = \frac{\partial p}{\partial z}} \quad (2)$$

Hypsometric Equation

Begin by re-writing hydrostatic balance (2),

$$-\rho g \partial z = \partial p \quad (3)$$

Recall that the ideal gas law can be written $\rho = \frac{p}{R_d T}$, thus (3) becomes

$$-\frac{p g}{R_d T} \partial z = \partial p \quad (4)$$

$$-\frac{g}{R_d T} \partial z = \frac{1}{p} \partial p \quad (5)$$

$$-\frac{g}{R_d T} \int_{z_1}^{z_2} \partial z = \int_{p_1}^{p_2} \frac{1}{p} \partial p \quad (6)$$

$$-\frac{g}{R_d T} \Delta z = \ln p_2 - \ln p_1 \quad (7)$$

$$\boxed{\Delta z = \frac{R_d \overline{T_v}}{g} \ln \frac{p_1}{p_2}} \quad (8)$$

Where $p_1 > p_2$

Virtual Temperature

Notice that in (8) T appears in the numerator. This suggests that the thickness of a layer directly proportional to the temperature of the layer. Deciding on a value for T , however, is easier said than done. (8) is typically evaluated using a mean layer virtual temperature, $\overline{T_v}$. Referenced equations in the derivation below are from Petty (2008).

We begin with equation (3.43), the total pressure of moist air.

$$p = p_d + e = (\rho_d R_d + \rho_v R_v) T \quad (9)$$

Next, we multiply equation (1) by $\frac{\rho R_d}{\rho R_d}$,

$$p \frac{\rho R_d}{\rho R_d} = (\rho_d R_d + \rho_v R_v) T \frac{\rho R_d}{\rho R_d} \quad (10)$$

$$p = \rho R_d \frac{(\rho_d R_d + \rho_v R_v)}{\rho R_d} T \quad (11)$$

$$p = \rho R_d \left[\frac{\rho_d R_d}{\rho R_d} + \frac{\rho_v R_v}{\rho R_d} \right] T \quad (12)$$

$$p = \rho R_d \left[\frac{\rho_d}{\rho} + \frac{\rho_v R_v}{\rho R_d} \right] T \quad (13)$$

Substituting $\frac{R_v}{R_d}$ for $\frac{1}{\epsilon}$ and $\frac{\rho_v}{\rho}$ for q within equation (13),

$$p = \rho R_d \left[\frac{\rho_d}{\rho} + \frac{1}{\epsilon} q \right] T \quad (14)$$

$\frac{\rho_d}{\rho} = 1 - \frac{\rho_v}{\rho}$, and this may be substituted into equation (14),

$$p = \rho R_d \left[1 - \frac{\rho_v}{\rho} + \frac{1}{\epsilon} q \right] T \quad (15)$$

$\frac{\rho_v}{\rho} = q$ may then be substituted into equation (15),

$$p = \rho R_d \left[1 - q + \frac{1}{\epsilon} q \right] T \quad (16)$$

A simple rearrangement of the sum within the bracketed term yields,

$$p = \rho R_d \left[1 + \frac{1}{\epsilon} q - q \right] T \quad (17)$$

q can then be factored from the second and third terms of the bracketed term,

$$p = \rho R_d \left[1 + \left(\frac{1}{\epsilon} - 1 \right) q \right] T \quad (18)$$

(18) is identical to the Ideal Gas Law for dry air, except for the bracketed term scaling R_d . It turns out that the bracketed term is used to scale R_d in order to adjust the gas constant for the prescence of water vapor. We can, however, use the scaling term on T , producing a 'virtual' temperature which accounts for the prescence of water vapor. Thus the virtual temperature is defined as

$$\boxed{T_v = \left[1 + \left(\frac{1}{\epsilon} - 1 \right) q \right] T} \quad (19)$$

where q is the specific humidity (recall specific humidity \approx mixing ratio) and $\epsilon = \frac{R_d}{R_v} \approx .622$. A good approximation for (18) is Petty (3.50),

$$T_v \approx (1 + .061q)T \quad (20)$$