Problem 1 (4-4a)**.**

$$F(z) = \sum_{n=0}^{\infty} F_n z^n$$

Let $F(z) = (0, 1, 1, 2, 3, 5, 8, 13, \dots)$ the coefficients of the monomials. Then multiplication by z

$$zF(z) = (0, 0, 1, 1, 2, 3, 5, 8, \dots)$$

and

$$z^2 F(z) = (0, 0, 0, 1, 1, 2, 3, 5, \dots)$$

Hence

$$\begin{aligned} z = &\quad (0, 1, 0, 0, 0, 0, 0, 0, \dots) \\ zF(z) = &\quad (0, 0, 1, 1, 2, 3, 5, 8, \dots) \\ \underline{+z^2 F(z)} = &\quad (0, 0, 0, 1, 1, 2, 3, 5, \dots) \\ F(z) = &\quad (0, 1, 1, 2, 3, 5, 8, 13, \dots) \end{aligned}$$

Problem 2 (4.4b)**.** Since

$$F(z) = z + zF(z) + z^2 F(z)$$

we have that

$$F(z)\left(1 - z - z^2\right) = z$$

or

$$F(z) = \frac{z}{1 - z - z^2}$$

Factoring the denominator

$$\begin{aligned} F(z) &= \frac{z}{-\left(z + \frac{(1-\sqrt{5})}{2}\right)\left(z + \frac{1+\sqrt{5}}{2}\right)} \\ &= \frac{z}{\left(1 - z\left(\frac{1+\sqrt{5}}{2}\right)\right)\left(1 - z\left(\frac{1-\sqrt{5}}{2}\right)\right)} \\ &= \frac{z}{(1 - \phi z)\left(1 - \hat{\phi} z\right)} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{(1 - \phi z)} - \frac{1}{\left(1 - \hat{\phi} z\right)}\right) \end{aligned}$$

Problem 3 (4.4c)**.** Using the Taylor series

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

we have by above

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{5}}\left(\frac{1}{(1-\phi z)}-\frac{1}{\left(1-\hat{\phi} z\right)}\right) \\ &= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}(\phi z)^n-\sum_{n=0}^{\infty}\left(\hat{\phi} z\right)^n\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left(\phi^n-\hat{\phi}^n\right)z^n \end{aligned}$$

Problem 4 (4.4d)**.** By comparing coefficients in the the original generating function and the re-expression

$$F(z)=\sum_{n=0}^{\infty}F_n z^n=\sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left(\phi^n-\hat{\phi}^n\right)z^n$$

we see that

$$F_n=\frac{1}{\sqrt{5}}\left(\phi^n-\hat{\phi}^n\right)$$

Since $\left|\hat{\phi}\right|<1$ it's the case that $\left|\hat{\phi}^n\right|<1$ and hence is fractional.

Exponentiation by Squaring

Note that

$$x^n=\begin{cases} x\left(x^2\right)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ \left(x^2\right)^{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

This is recursive. For example for x^{10}:

$$\begin{aligned} x^{20} &= \left(x^2\right)^{20/2}=\left(x^2\right)^{10} \\ \left(x^2\right)^{10} &= \left(x^2\times x^2\right)^{10/2}=\left(x^2\times x^2\right)^5 \\ \left(x^2\times x^2\right)^5 &= x^4\left(\left(x^2\times x^2\right)\times\left(x^2\times x^2\right)\right)^{\frac{5-1}{2}}=x^4\left(\left(x^2\times x^2\right)\times\left(x^2\times x^2\right)\right)^2 \\ \left(\left(x^2\times x^2\right)\times\left(x^2\times x^2\right)\right)^2 &= \left(\left[\left(x^2\times x^2\right)\times\left(x^2\times x^2\right)\right]\left[\left(x^2\times x^2\right)\times\left(x^2\times x^2\right)\right]\right) \end{aligned}$$

i.e

$$\begin{aligned} x^{20} &= \left(x^2\right)^{10} \\ &= \left(x^2x^2\right)^5 \\ &= \left(x^2x^2\right)^1\left(x^2x^2\right)^4 \\ &= \left(x^2x^2\right)\left(\left(x^2x^2\right)\left(x^2x^2\right)\right)^2 \\ & \left(x^2x^2\right)\left(\left[\left(x^2x^2\right)\left(x^2x^2\right)\right]\left[\left(x^2x^2\right)\left(x^2x^2\right)\right]\right) \end{aligned}$$

Bottom up though for $n = 20$ we have

$$\begin{aligned} & & y_1 = 1, n = 20 \\ x_1 = x \times x &=& x^2, n = 20/2 = 10 \\ x_2 = (x_1)^2 = x^2 \times x^2 &=& x^4, n = 10/2 = 5 \\ y_2 = x_2 = x^4, x_3 = (x_2)^2 = x^4 \times x^4 &=& x^8, n = (5-1)/2 = 2 \\ x_4 = (x_3)^2 = x^8 \times x^8 &=& x^{16}, n = 2/2 = 1 \end{aligned}$$

You “slide out” the $+1$ of an odd number.