STA 6326 Homework 1 Solutions

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1.1 (a) All 4 bit sequences of 0s and 1s, where 0 represents tails and 1 represents heads, i.e. $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$where $\omega_i \in \{0, 1\}$.

(b) All n bit sequences of 0s and 1s, where 0 represents undamaged and 1 represents damaged and n is the number of leaves on the plant. Then the number of damaged leaves is the number of 1 bits in the sequence (a RV).

(c) All $x \in (0, \infty)$.

(d) All $x \in (0, k)$where k is some reasonable maximum weight for a 10-day old rat (probably like 100 pounds).

(e) All $x \in [0, 1]$.

1.2 (a) $A \backslash (A \cap B) = A \cap (A \cap B)^c = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = \emptyset \cup (A \cap B^c) = A \cap B^c$.

(b) $(B \cap A) \cup (B \cap A^c) = B \cap (A \cup A^c) = B \cap \Omega = B$.

(c) $B \backslash A = B \cap A^c$ by definition.

(d) $A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c) = (A \cup B) \cap \Omega = A \cup B$.

1.4 (a) Probability of $A \vee B \iff P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(b) Probability of $(A \vee B) \wedge \neg(A \wedge B) \iff P(A \cup B) - P(A \cap B)$ and

$$\begin{aligned} P(A \cup B) - P(A \cap B) &= P(A) + P(B) - P(A \cap B) - P(A \cap B) \\ &= P(A) + P(B) - 2 \cdot P(A \cap B) \end{aligned}$$

(c) Same as (a).

(d) Same as (b).

1.13 $A \cap B = \emptyset \implies A \subset B^c \implies P(A) \leq P(B^c) \implies 1/3 \leq 1/4$ which is a contradiction.

1.14 Each subset involves making n binary choices, one choice per element in the set S, of whether to include the element in that subset or exclude that element from the subset. Therefore by **Thm 1.2.14** in Casella there are $\underbrace{2 \times 2 \times \cdots \times 2}_{\text{n times}} = 2^n$ different ways to pick which elements to include in a subset.

1.16 (a) Choose 3 from 26 with replacement hence $26^3 = 17576$.

(b) Choose 3 from 26 with replacement or (therefore the $\cup$) choose 2 from 26 with replacement hence $26^3 + 26^2$.

(c) Choose 4 from 26 with replacement or (therefore the $\cup$) choose 3 from 26 with replacement or choose 2 from 26 with replacement hence $26^4 + 26^3 + 26^2$.

1.18 Imagine laying out the n balls in a particular order then going through one by one and assigning a cell to each ball. This induces an implicit order on the cell assignments since the balls have a fixed ordering. The sample space is all such assignments, i.e. all n-tuples with each entry being an integer from 1 to n. There are n^n different ways to assign the n cells to the n balls. A successful outcome is then one where exactly one cell is assigned twice, i.e. exactly two repetitions in the assignment list. There are n ways to choose which cell will be assigned twice and $\binom{n}{2}$ to choose which balls will bear the reptition. Then there are $n-1$ ways to choose a cell that will remain empty. Finally there are $(n-2)!$ way to assign the remaining $n-2$ cells to the $n-2$ balls. Hence the probability is $n\binom{n}{2}(n-1)(n-2)!/n^n = \binom{n}{2}n!/n^n$.

1.21 The "experiment" is choosing $2r$ shoes from $2n$ shoes without replacement. There are $\binom{2n}{2r}$ ways to choose $2r$ shoes from $2n$ shoes, i.e. n pairs of shoes. There are $\binom{n}{2r}$ different ways to choose $2r$ different pairs of shoes from n pairs of shoes, i.e. since we'll only be choosing one shoe from each pair we must choose which pairs exactly it is we'll be taking a single shoe from. Then finally there are 2^{2r} different ways to choose either the left shoe or right shoe from each pair of previously $2r$ chosen shoes. Hence the fraction of choices of $2r$ shoes from $2n$ shoes is

$$\frac{\text{number of choices of 2r pairs} \times \text{number of ways of choosing either left or right}}{\text{number of ways of choosing 2r shoes}} = \frac{\binom{n}{2r}2^{2r}}{\binom{2n}{2r}}$$

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1.22 (a) The "experiment" is to draw 180 days from the year. The sample spaces is all 180-element subsets of the 366 days, so there are $\binom{366}{180}$ ways to draw 180 days from anywhere in the year for the 180 lottery tickets. In order that the days be evenly distributed there must be $180/12 = 15$ days chosen from each month. Then there are $\underbrace{\binom{30}{15} \times \binom{31}{15} \times \cdots \times \binom{29}{15}}_{12} =$ $\binom{30}{15}^4 \times \binom{31}{15}^7 \times \binom{29}{15}$ ways to assign exactly 15 lottery tickets to each of the 12 months. Hence the probability of having all 180 lottery tickets evenly distributed is $\frac{\binom{30}{15}^4 \times \binom{31}{15}^7 \times \binom{29}{15}}{\binom{366}{180}} =$ $.167 \times 10^{-8}$

(b) The experiment is again to draws days from the year. There are $\binom{336}{30}$ ways to choose 30 days from all the days in the year that are not in September $(366-30=336)$ and $\binom{366}{30}$ ways to choose 30 days from any of the days of the year. Hence the probability of not selecting any days from September is $\binom{336}{30}/\binom{366}{30} = 0.0686905$.

1.23 Every "draw" has 4 possibilities: $\{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$. So there are 4^n different outcomes of the n "draws". Then there are $\binom{n}{i}\binom{n}{i}$ different each person could flip i heads, for $i = 0, 1, 2, \ldots, n$. Hence the probability of each person flipping the same number of heads is $\sum_{i=0}^{n}\binom{n}{i}^2/4^n$. To see that this equals $\binom{2n}{n}/4^n$ we prove Vandermonde's identity, namely that

$$\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$$

To see that this is the case take 2 urns, one with m red balls and one with n green balls and consider how many collections of size r can be drawn from both urns. Certainly it's $\binom{m+n}{r}$. But it's also the sum, for all k, of the number of collections with k red balls and $n-k$ green balls. For fixed k this is $\binom{m}{k}\binom{n}{r-k}$ and thence follows Vandermonde's identity. Finally letting $m = r = n$ and using the fact that $\binom{n}{n-i} = \binom{n}{i}$ we have

$$\frac{1}{4^n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{i} = \frac{1}{4^n}\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \frac{1}{4^n}\binom{n+n}{n} = \frac{1}{4^n}\binom{2n}{n}$$

1.24 (a) Number of flips until heads is geometrically distributed, i.e. the probability that the kth flip is a head $P(X=k)=(1-p)^{k-1}p$ with $p=1/2$. The probability that A wins is tantamount to the probability that k is odd (since A flips first). Therefore

$$P(\text{A wins})=P(X=k=2i-1|i=1,2,3,\cdots)=\sum_{i=1}^{\infty}(1-p)^{(2i-1)-1}p=-\frac{(p-1)^2}{(1-p)^2(p-2)}$$

For $p=1/2$ this is $-\frac{(1/2-1)^2}{(1-1/2)^2(1/2-2)}=\frac{2}{3}$.

(b) Simplifying the result from (a) we get that

$$\begin{aligned}P(\text{A wins})&=-\frac{(p-1)^2}{(1-p)^2(p-2)}\\&=-\frac{(1-p)^2}{(1-p)^2(p-2)}\\&=\frac{1}{2-p}\\&=\frac{p}{2p-p^2}\\&=\frac{p}{1-(1-2p+p^2)}\\&=\frac{p}{1-(1-p)^2}\end{aligned}$$

(c) Since $P(\text{A wins})=\frac{1}{2-p}$ we see that if $0<p<1$ then $\frac{1}{2}<P(\text{A wins})=\frac{1}{2-p}<1$.

1.27 (a) Using the binomial theorem

$$0=(1-1)^n=\sum_{k=0}^{n}\binom{n}{k}1^{n-k}(-1)^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}$$

(b) Using the binomial theorem

$$\begin{aligned}\sum_{k=1}^{n}k\binom{n}{k}&=\sum_{k=1}^{n}k\frac{n!}{k!(n-k)!}\\&=\sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}\\&=\sum_{k=1}^{n}\frac{n\cdot(n-1)!}{(k-1)!\big(n-1-(k-1)\big)!}\\&=n\sum_{k=1}^{n}\binom{n-1}{k-1} && \text{let } j=k-1\\&=n\sum_{j=0}^{n-1}\binom{n-1}{j}\\&=n(1+1)^{n-1}\\&=n2^{n-1}\end{aligned}$$

(c) Using essentially the same calculation (steps 1-4) from part (a) we have that

$$\sum_{k=1}^{n}(-1)^{k+1}k\binom{n}{k}=n\sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k-1}$$

. Let $j = k - 1$ then

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n-1}{k-1} = (-1)^2 n \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} = 0$$

by part (a).

1.33 The probability is

$$\begin{aligned}
P(\text{male}|\text{colorblind}) &= \frac{P(\text{colorblind}|\text{male})P(\text{male})}{\sum_{i \in sex} P(\text{colorblind}|\text{i})P(\text{i})} \\
&= \frac{P(\text{colorblind}|\text{male})P(\text{male})}{P(\text{colorblind}|\text{male})P(\text{male}) + P(\text{colorblind}|\text{female})P(\text{female})} \\
&= \frac{.05 \cdot .5}{.05 \cdot .5 + .25 \cdot .5} \\
&= \frac{1}{6}
\end{aligned}$$