Final

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Problem 1(a). Compute the transition probabilities for this Markov chain.

Solution.

$$\begin{pmatrix} 2/5 & 3/5 & 0 & 0 \\ 3/25 & 2/25 + 12/25 & 8/25 & 0 \\ 0 & 8/25 & 2/25 + 12/25 & 3/25 \\ 0 & 0 & 3/5 & 2/5 \end{pmatrix}$$

Problem 1(b). Argue whether this chain has a stationary distribution π.

Solution.

The Markov chain is finite, irreducible, and every state has period 1. Hence there exists a stationary distribution. MATLAB computes $\pi = (1/12, 5/12, 5/12, 1/12)$

Problem 1(c). In the long run, how many purple balls should we expect to see in the first urn?

Solution.

$$\arg\max_i \pi(i) = 1 \wedge 2$$

hence we're most often, and equally likely, to find 1 or 2 balls in the first urn.

Problem 1(d). Compute the hitting time for state 3.

Solution.

The reduced transition matrix

$$\tilde{P} = \begin{pmatrix} 2/5 & 3/5 & 0 \\ 3/25 & 2/25 + 12/25 & 8/25 \\ 0 & 8/25 & 2/25 + 12/25 \end{pmatrix}$$

Solving $(I - \tilde{P})\vec{u} = \vec{1}$ for $\vec{u}$ yields $\vec{u} = (95/4, 265/12, 55/3)$ hence it will take on average $95/3 = 23.75$ turns before there are 3 purple balls in the first urn.

Problem 1(e). Is X_n a martingale? If so, does the MCT apply?

Solution.

Yes because the expectation of the fraction of purple balls converges to the ensemble average by the ergodic theorem (so $\mathbb{E}[M_{n+1}] = M_n$). Furthermore the MCT applies because the fraction of purple balls is bounded above and below.

Problem 2(a). What is the stationary distribution and what fraction of the time will all chairs be full?

Solution.

$$\begin{aligned} q(i, i-1) &= 4 \quad \text{for } i = 1, 2, 3, 4 \\ q(i, i+1) &= 5 \quad \text{for } i = 0, 1, 2, 3 \end{aligned}$$

The detailed balance conditions say $5\pi(i-1) = 4\pi(i)$. Setting $\pi(0) = c$ and solving, we have

$$\pi(1) = \frac{5}{4}c \quad \pi(2) = \frac{25}{16}c \quad \pi(3) = \frac{125}{48}c \quad \pi(4) = \frac{625}{192}c$$

The sum of the πs is $2101/256$ so $c = 256/2101$ and

$$\pi(0) = \frac{256}{2101} \quad \pi(1) = \frac{320}{2101} \quad \pi(2) = \frac{400}{2101} \quad \pi(3) = \frac{500}{2101} \quad \pi(4) = \frac{625}{2101}$$

From this we see that $625/2101 = .297$ of the time all four of the chairs are full.

Problem 2(b). What fraction of the time is the barber losing business because there are no empty seats?

Solution.

From part (a) we see that 29.7% of the time he's turning away potential customers hence 29.7% of his potential business is lost.

Problem 2(c). In the long run, how many customers does the barber serve per hour?

Solution.

By the ergodic theorem $\sum_{i=0}^{4} i \cdot \pi(i)$ is the average number of customers in the barber's shop and hence the average number of customers being served: $\sum_{i=0}^{4} i \cdot \pi(i) = 2.44$

Problem 2(d). How long does the average customer wait?

Solution.

Using Little's formula the average waiting time W is the average number of customers in the shop, part (c), divided by long-run average rate at which customers arrive and are able to get any seat at all $\lambda(\pi(0) + \pi(1) + \pi(2) + \pi(3))$. Hence

$$\begin{aligned} W &= \frac{2.44}{5\left(\frac{256}{2101} + \frac{320}{2101} + \frac{400}{2101} + \frac{500}{2101}\right)} \\ &= .3424 \text{ hours} \end{aligned}$$

Problem 3(a). Compute $\mathbb{P}(\tau > \sigma)$

Solution.

By Eqn. 2.9 in Durrett

$$\mathbb{P}(\sigma < \tau) = \frac{3}{2+3} = \frac{3}{5}$$

Problem 3(b). Compute $\mathbb{P}(\tau > \sigma \,|\, \sigma > 5)$

Solution.

By Eqn. 2.9 in Durrett

$$\begin{aligned} \mathbb{P}(\tau > \sigma \,|\, \sigma > 5) &= \frac{\mathbb{P}(\tau > \sigma, \sigma > 5)}{\mathbb{P}(\sigma > 5)} \\ &= \frac{\int_5^\infty 3e^{-5s} ds}{1 - e^{-3 \cdot 5}} \\ &= \frac{\frac{3}{5e^2 5}}{1 - e^{-15}} \\ &= \frac{3}{5e^{10}(e^{15} - 1)} \end{aligned}$$

Problem 3(c). Compute $\mathbb{E}[M]$.

Solution.

By Durrett Chapter 2 summary

$$\mathbb{E}[M] = \frac{1}{3} + \frac{1}{2} - \frac{1}{2+3} = \frac{1}{3} + \frac{1}{2} - \frac{1}{5} = \frac{19}{30}$$

Problem 3(d). Compute $\mathbb{E}[S \,|\, m]$.

Solution.

$$S = \tau + \sigma = M + m$$

hence

$$\begin{aligned}
\mathbb{E}[S \,|\, m] &= \mathbb{E}[M + m \,|\, m] \\
&= \mathbb{E}[M \,|\, m] + \mathbb{E}[m \,|\, m] \\
&= \mathbb{E}[M \,|\, m] + m \\
&= \mathbb{E}[\sigma]\Big(\mathbb{P}(M = \sigma \,|\, m = \sigma) + \mathbb{P}(M = \sigma \,|\, m = \tau)\Big) + \\
&\quad \mathbb{E}[\tau]\Big(\mathbb{P}(M = \tau \,|\, m = \sigma) + \mathbb{P}(M = \tau \,|\, m = \tau)\Big) + m \\
&= {}^1\!/\!_3(0+1) + {}^1\!/\!_2(1+0) + m \\
&= \frac{5}{6} + m
\end{aligned}$$

Problem 4(a). Compute $\mathbb{E}[N(1) \,|\, N(2)]$.

Solution.

By Durrett Thm. 2.15

$$\mathbb{P}(N(1) = m \,|\, N(2) = n) = \binom{n}{m}\left(\frac{1}{2}\right)^m\left(1 - \frac{1}{2}\right)^{n-m} = \binom{n}{m}\left(\frac{1}{2}\right)^n$$

Let $\eta = N(2)$ then

$$\mathbb{E}[N(1) \,|\, N(2)] = \sum_{i=0}^{\eta} i \cdot \binom{\eta}{i}\left(\frac{1}{2}\right)^\eta = \frac{\eta}{2} = \frac{N(2)}{2}$$

Problem 4(b). Compute $\mathbb{E}[N(2)\,|\,N(1)]$.

Solution.

$$N(2) = \Big(N(1+1) - N(1)\Big) + N(1)$$

Therefore

$$\mathbb{E}[N(2)\,|\,N(1)] = \mathbb{E}\Bigg[\Big(N(1{+}1){-}N(1)\Big){+}N(1)\,\Bigg|\,N(1)\Bigg] = \mathbb{E}\Bigg[\Big(N(1{+}1){-}N(1)\Big)\,\Bigg|\,N(1)\Bigg]{+}N(1)$$

But by Durrett Lemma 2.5 $N(1+1) - N(1)$ is independent of $N(r)$ for $0 \le r \le 1$ and furthermore distributed $Poisson(\lambda \cdot 1)$. Let $\eta = N(1+1) - N(1)$ with $\eta \sim Poisson(\lambda)$. Hence $\mathbb{E}[N(2)\,|\,N(1)] = \eta + N(1)$.

Problem 5(a). Show that W_n is a martingale.

Solution.

$$\begin{aligned}
\mathbb{E}[W_{n+1}\,|\,\mathcal{F}_n] &= \mathbb{E}\Bigg[\sum_{k=1}^{n+1} H_k X_k \,\Bigg|\, \mathcal{F}_n\Bigg] \\
&= \mathbb{E}[H_{k+1}X_{k+1} + W_n|\mathcal{F}_n] \\
&= \mathbb{E}[H_{k+1}X_{k+1}|\mathcal{F}_n] + W_n \\
&= H_{k+1}\mathbb{E}[X_{k+1}|\mathcal{F}_n] + W_n \\
&= H_{k+1}\cdot 0 + W_n = W_n
\end{aligned}$$

The fourth line follows because H_i is independent of $\mathcal{F}_j$ $\forall i, j$ and the last line follows since $\mathbb{E}[X_i] = 0$.

Problem 5(b). Show that $\mathbb{E}[W_n^2] = \sigma^2 \sum_{k=1}^{n} \mathbb{E}[H_k^2]$.

Solution.

$$W_n^2 = \left(\sum_{k=1}^{n} H_k X_k\right)^2$$

Now by the multinomial theorem

$$W_n^2 = \sum_{i_1+i_2+\cdots i_n=2} \binom{2}{i_1, i_2, \cdots, i_n}\Big((H_1X_1)^{i_1}(H_2X_2)^{i_2}\cdots(H_nX_n)^{i_n}\Big)$$

But because there are only binomial terms in the sum

$$W_n^2 = \sum_{i=1}^{n} (H_i X_i)^2 + 2\sum_{i \neq j} (H_i X_i)(H_j X_j) = \sum_{i=1}^{n} H_i^2 X_i^2 + 2\sum_{i \neq j} (H_i X_i)(H_j X_j)$$

Then

$$\begin{aligned}
\mathbb{E}[W_n^2] &= \mathbb{E}\left[\sum_{i=1}^{n} H_i^2 X_i^2 + 2\sum_{i \neq j} (H_i X_i)(H_j X_j)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} H_i^2 X_i^2\right] + \mathbb{E}\left[2\sum_{i \neq j} (H_i X_i)(H_j X_j)\right]
\end{aligned}$$

But by independence

$$\mathbb{E}[W_n^2] = \sum_{i=1}^{n} \mathbb{E}[H_i^2]\mathbb{E}[X_i^2] + 2\sum_{i \neq j} \mathbb{E}[H_i X_i H_j]\mathbb{E}[X_j]$$

And by the hypothesis $\mathbb{E}[X_i] = 0$

$$\begin{aligned}
\mathbb{E}[W_n^2] &= \sum_{i=1}^{n} \mathbb{E}[H_i^2]\mathbb{E}[X_i^2] + 2\sum_{i \neq j} \mathbb{E}[H_i X_i H_j] \cdot 0 \\
&= \sum_{i=1}^{n} \mathbb{E}[H_i^2]\mathbb{E}[X_i^2] \\
&= \sum_{i=1}^{n} \mathbb{E}[H_i^2]\sigma^2 \\
&= \sigma^2 \sum_{i=1}^{n} \mathbb{E}[H_i^2]
\end{aligned}$$

Problem 6(a). Show that $M_n := S_n^2 - n$ is a martingale.

Solution.

Let $H_k = 1$ then by the just prior result, with $\sigma^2 = var(X_i)^2 = 1^2$

$$\mathbb{E}[M_n] = \mathbb{E}[S_n^2 - n] = \mathbb{E}[S_n^2] - n = 1^2 \sum_{1}^{n} \mathbb{E}[H_k^2] - n$$

But $\mathbb{E}[H_k^2] = 1$ so $\mathbb{E}[M_n] = 0$. Finally

$$\mathbb{E}[M_{n+1} - M_n|\mathcal{F}] = \mathbb{E}[M_{n+1}|\mathcal{F}] - \mathbb{E}[M_n|\mathcal{F}] = 0 - 0 = 0$$

.
Hence by definition 5.4 in Durrett S_n^2 is a martingale.

Problem 6(b). Compute $\mathbb{E}[\tau]$.

Solution.

Using Example 5.11 in Durett $\mathbb{E}_0[\tau] = -(-N) \cdot N = N^2$.