

What's the deal with Quantum Computing Part 2

-A little less formalism-

Maksim Levental

January 15, 2015

Tensor product space: equivalence relation on formal sum over cartesian product over individual qubit Hilbert spaces. Denoted $V \otimes W$. Equivalence relations are:

$$(\alpha|\psi\rangle) \otimes |\phi\rangle \sim |\psi\rangle \otimes (\alpha|\phi\rangle) \sim \alpha(|\psi\rangle \otimes |\phi\rangle)$$

$$(|\psi\rangle + |\psi'\rangle) \otimes |\phi\rangle \sim |\psi\rangle \otimes |\phi\rangle + |\psi'\rangle \otimes |\phi\rangle$$

$$|\psi\rangle \otimes (|\phi\rangle + |\phi'\rangle) \sim |\psi\rangle \otimes |\phi\rangle + |\psi\rangle \otimes |\phi'\rangle$$

Basis

$$\begin{aligned} |0\rangle \otimes |0\rangle &= |00\rangle \\ |0\rangle \otimes |1\rangle &= |01\rangle \\ |1\rangle \otimes |0\rangle &= |10\rangle \\ |1\rangle \otimes |1\rangle &= |11\rangle \end{aligned}$$

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}} \sum_{i,j} |ij\rangle$$

Basically \otimes distributes but not commutative. In general $\sqrt{2}$ max states then separable states $(|a\rangle \otimes |b\rangle, |1\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle)$. For $\psi = \alpha|0,0\rangle + \beta|0,1\rangle + \gamma|1,0\rangle$ it must be the case that

$$\begin{aligned} \psi &= \alpha|0\rangle \otimes |0\rangle + \beta|0\rangle \otimes |1\rangle \\ &= |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) \Rightarrow |0\rangle \otimes \psi' \\ &= |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) = \alpha|0,0\rangle + \beta|0,1\rangle \\ &\Rightarrow \alpha\beta = \beta\alpha \Rightarrow \alpha\beta = \beta\alpha \end{aligned}$$



Recap

- A single qubit is a (unit length) linear combination of the basis vectors $|0\rangle, |1\rangle$

$$\psi = \alpha |0\rangle + \beta |1\rangle$$

- Measurement \iff non-deterministic wave function collapse
 \iff all information lost
- Unitary transformations correspond to gates. 1-qubit gates are matrices

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- n -qubit (registers) systems are represented by vectors (tensors) in the tensor product of the vector spaces that each of the individual qubits are elements of

$$\left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right) \otimes \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right) = \frac{1}{2} \left(|0\rangle |0\rangle + |0\rangle |1\rangle + |1\rangle |0\rangle + |1\rangle |1\rangle \right)$$

- Gates on single qubit systems also map to “ n -gates” on n -qubit systems (entrywise)

$$\begin{aligned} H^{\otimes 2} |0\rangle |0\rangle &= (H |0\rangle) \otimes (H |0\rangle) \\ &= \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right) \otimes \left(\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \right) \end{aligned}$$

- Entangled states are important: for $\psi \in V \otimes W$ there **do not exist** $\phi \in V$ and $\varphi \in W$ such that

$$\psi = \frac{|0\rangle |0\rangle + |1\rangle |1\rangle}{\sqrt{2}} = \phi \otimes \varphi$$

Exponential Speed-up

Deutsch's Problem

“Reversible computation can be done **efficiently**, without the production of garbage bits whose values depend on the input to the computation. That is, if there is an irreversible circuit computing a function f , then there is an efficient simulation of this circuit by a reversible [unitary transformation/quantum] circuit with action” [5]

$$|x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle$$

Deutsch's Problem

Let $f : \{0, 1\} \rightarrow \{0, 1\}$ and suppose we are guaranteed that f is either balanced (1 on half of its domain and 0 on the other half) or constant (1 or 0 on the entire domain). How many evaluations classically to discriminate? “Quantumly” you only need to evaluate f once! Let U_f be the quantum circuit such that

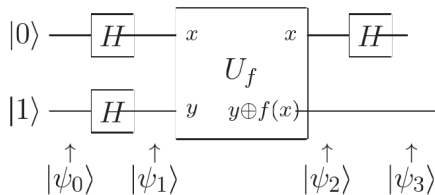
$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$$

Aside: algebraic simplification necessary for following

$$U_f \left(|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) = (-1)^{f(x)} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Deutsch's Algorithm

Construct the quantum circuit



$$|\psi_1\rangle = H^{\otimes 2}(|0\rangle \otimes |1\rangle) = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_2\rangle = U_f |\psi_1\rangle = \begin{cases} \pm \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}} \right) & \text{if } f(0) = f(1) \\ \pm \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}} \right) & \text{if } f(0) \neq f(1) \end{cases}$$

Deutsch's Algorithm

The final Hadamard gate leaves the system in

$$|\psi_3\rangle = \begin{cases} \pm |0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) & \text{if } f(0) = f(1) \\ \pm |1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) & \text{if } f(0) \neq f(1) \end{cases}$$

Now if we measure the first qubit we know whether $f(0) = f(1)$ or $f(0) \neq f(1)$ (depending on whether we get $|0\rangle$ or $|1\rangle$).

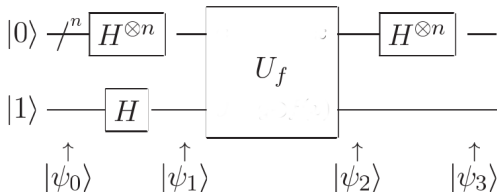
Succinctly stated this allows us to measure a global property: since $f(0) \oplus f(1) = 0$ if $f(0) = f(1)$ and 1 otherwise

$$|\psi_3\rangle = \pm |f(0) \oplus f(1)\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Naive interpretation is that this is a randomized algorithm but in truth interference effects (the final Hadamard gate) are used to discern global properties (H is a generalized DFT).

Deutsch-Jozsa Algorithm

Generalize to $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and still f is either balanced or constant. How many evaluations classically? $2^{n-1} + 1$ but quantumly still 1!



$$|\psi_0\rangle = (|0, 0, \dots, 0\rangle) \otimes |1\rangle$$

then

$$|\psi_1\rangle = \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Deutsch-Jozsa Algorithm

The first register is a superposition of all basis states in the n -qubit computational basis. Using the algebraic simplification above again we have that

$$|\psi_2\rangle = U_f |\psi_1\rangle = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

and the last Hadamard operator (not an index mistake)

$$|\psi_3\rangle = H^{\otimes n} |\psi_2\rangle = \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} \frac{(-1)^{x \cdot z + f(x)} |z\rangle}{2^n} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

where $x \cdot z$ is bitwise inner product mod 2.

Deutsch-Jozsa Algorithm

Let's measure the top register. Note that the amplitude for $|0, 0, \dots, 0\rangle$ is $\sum_x (-1)^{f(x)} / 2^n$. If f is constant then

$$\sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}}{2^n} = \pm 1$$

and because $|\psi_3\rangle$ must be unit length we will certainly measure $|\psi_3\rangle$ to be in the $|0, 0, \dots, 0\rangle$ state. If f is balanced then by definition of balanced ($f(x)$ will be even as often as odd)

$$\sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)}}{2^n} = 0$$

and we will certainly measure something other than $|0, 0, \dots, 0\rangle$.

How to break RSA

Reduction to order finding

Pick $x < N$. If x and N have a common factor then $\gcd(x, N)$ can be computed classically in polynomial time using Euclid's algorithm. Otherwise compute the order of x ; the least r such that

$$x^r \equiv 1 \pmod{N}$$

With probability $p > 1 - \left(\frac{1}{2}\right)^{q-1}$, where q is the number of prime factors in N , the order of x will be even. Then

$$x^r - 1 \equiv (x^{r/2} - 1)(x^{r/2} + 1) \equiv 0 \pmod{N}$$

and hence N divides $(x^{r/2} - 1)(x^{r/2} + 1)$. If $1 < x^{r/2} \pmod{N} < N - 1$ then

$$0 < x^{r/2} - 1 \pmod{N} < x^{r/2} + 1 < N$$

and hence $(x^{r/2} - 1), (x^{r/2} + 1)$ must each have a factor of N . Compute $\gcd(x^{r/2} - 1, N)$ and $\gcd(x^{r/2} + 1, N)$.

Order example

For $N = 2013$ it's the case that $8^{20} \equiv 1 \pmod{N} \iff 8^{20} - 1 \equiv 0 \pmod{N}$ and

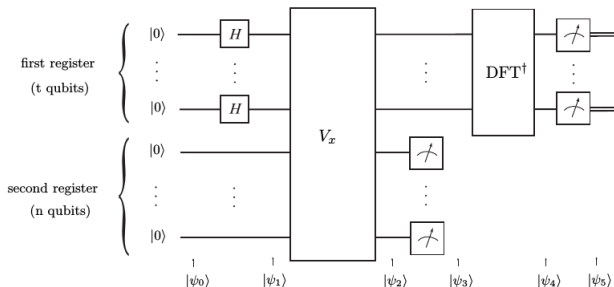
$$\left(8^{\frac{20}{2}} - 1\right) \left(8^{\frac{20}{2}} + 1\right) \equiv 0 \pmod{N}$$

But $\left(8^{\frac{20}{2}} - 1\right) \equiv 1584 \pmod{N}$ and $\left(8^{\frac{20}{2}} + 1\right) \equiv 1586 \pmod{N}$
and $0 < 1584 < 1586 < 2013$ so

$$\gcd(1584, 2013) = 33 \qquad \gcd(1586, 2013) = 61$$

and $61 \times 33 = 2013$

Shor's Algorithm



where

$$V_x (|j\rangle |k\rangle) = |j\rangle |k + x^j\rangle$$

and

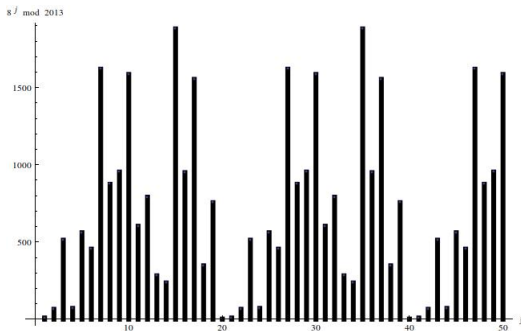
$$DFT (|k\rangle) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2\pi i j k / N} |j\rangle$$

DFT is circuit of size $O((\log N)^2)$

Shor's Algorithm

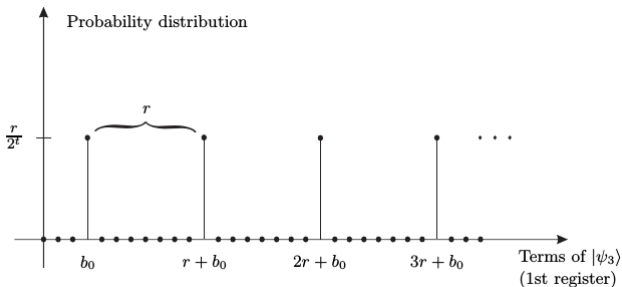
Then

$$|\psi_2\rangle = \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle |x^j\rangle \stackrel{r|2^t}{=} \frac{1}{\sqrt{2^t}} \sum_{b=0}^{r-1} \sum_{a=0}^{\frac{2^t}{r}-1} |ar + b\rangle |x^b\rangle$$



Shor's Algorithm

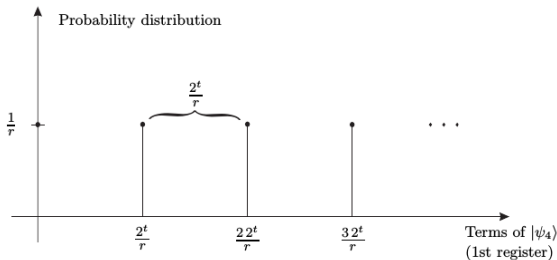
$$|\psi_3\rangle = \sqrt{\frac{r}{2^t}} \sum_{a=0}^{\frac{2^t}{r}-1} |ar + b_0\rangle |x^{b_0}\rangle$$



Shor's Algorithm

$$|\psi_4\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{2\pi i \frac{k}{r} b_0} \left| \frac{k2^t}{r} \right\rangle |x^{b_0}\rangle$$

Assuming the order of x , r is a multiple of 2 (can be generalized), after measuring the first register we have $|\psi_5\rangle = \left| \frac{k_0 2^t}{r} \right\rangle$



If $k_0 = 0$ then we rerun. Otherwise divide $k_0 2^t / r$ by 2^t . If k_0, r are coprime then we can just take the denominator of k_0 / r . Otherwise $r = r_1 r_2$ and we can find the order of x^{r_1} to find r .

Shor's Algorithm Implementation

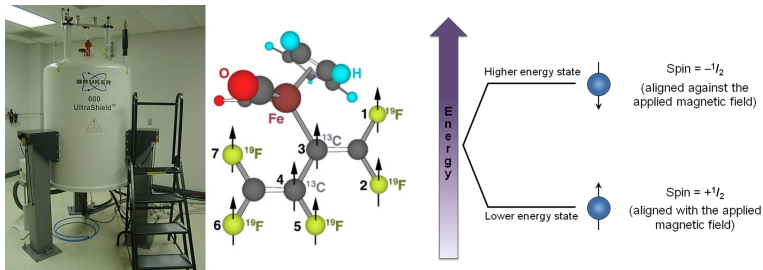


Figure : Experimental realization of Shor's quantum factoring algorithm using nuclear magnetic resonance [7]

Trivia

A **Probabilistic Turing machine** M over an alphabet A is $(Q, A, \delta, q_0, q_a, q_r)$ where

- Q is the set of internal control states
- $q_0, q_a, q_r \in Q$ are initial, accepting, and rejecting states
- $\delta : Q \times A \times Q \times A \times \{-1, 0, 1\} \rightarrow [0, 1]$ is a transition probability function i.e.

$$\sum_{(q_2, a_2, d)} \delta(q_1, a_1, q_2, a_2, d) = 1$$

A **Quantum Turing machine** M over an alphabet A is $(Q, A, \delta, q_0, q_a, q_r)$ where

- Q is the set of internal control states
- $q_0, q_a, q_r \in Q$ are initial, accepting, and rejecting states
- $\delta : Q \times A \times Q \times A \times \{-1, 0, 1\} \rightarrow \mathbb{C}$ is the root of a transition probability function i.e.

$$\sum_{(q_2, a_2, d)} |\delta(q_1, a_1, q_2, a_2, d)|^2 = 1$$

Computability Theorems

- Link between Turing machines and Boolean circuits: A language L has uniformly polynomial circuits iff $L \in \mathbf{P} = \bigcup_k \mathbf{TIME}(n^k)$
- All Boolean circuits can be simulated using reversible Boolean circuits

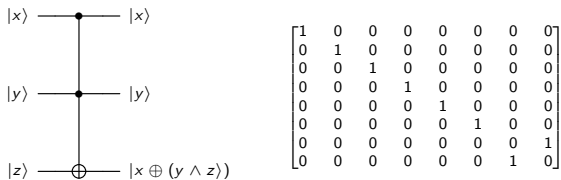
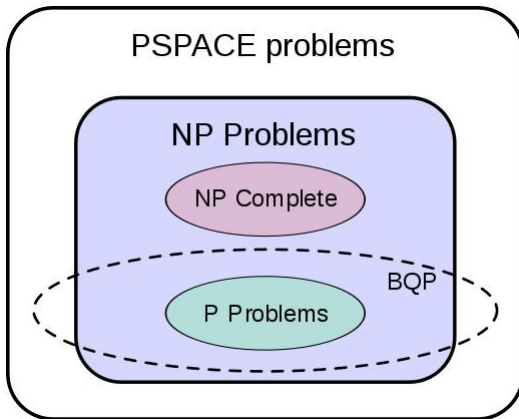


Figure : Toffoli gate

- Toffoli is classically universal but not quantum universal, but $\{TOF, H\}$ are quantum universal and both have successfully implemented [3, 4].

Complexity conjectures



Simon's problem and complexity results

Let $f(x) : \{0,1\}^n \rightarrow \{0,1\}^n$ and we are guaranteed that $\exists s \in \{0,1\}^n$ such that

$$f(y) = f(z) \iff (y = z \vee y \oplus z = s)$$

Find s . Classically $\Omega(2^{n/2})$ while quantumly $O(n)$. Also quantumly optimal; any quantum algorithm needs to make $\Omega(n)$.

Yields an **oracle** separation between **BPP** and **BQP**. Otherwise **BPP** \subset **BQP**

Deutsch-Josza only yields a separation between **P** and **EQP**

Let $f(x) : \{0, 1\}^n \rightarrow \{0, 1\}$ be the PARITY function. Classically how many operations must be performed for f to be computed? Quantumly only $n/2$ queries to the bit string need to be made [1].

Compute the “square-free” part of an integer N , i.e. r such that

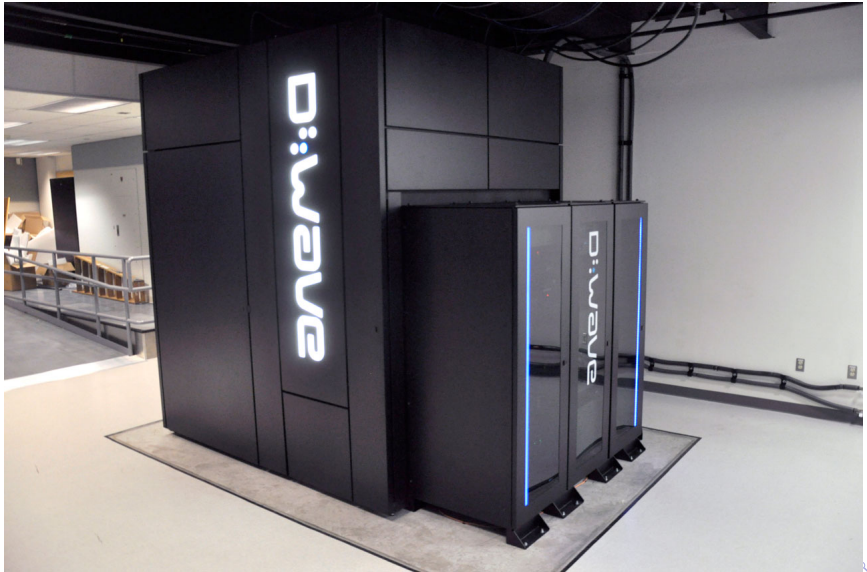
$$N = r \cdot s^2$$

No known polynomial time classical algorithm; “almost” as hard as factorization itself [6]. Quantumly in $O((\log \log N)^2)$ [2].

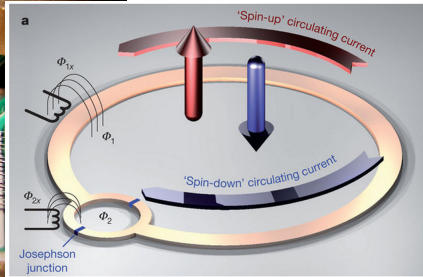
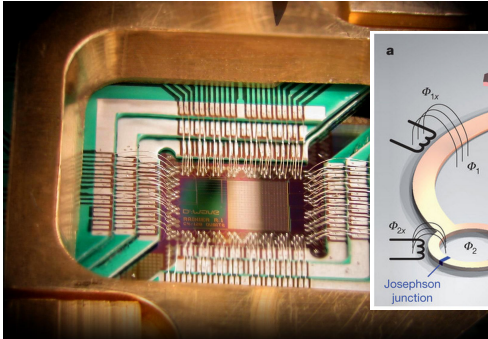
Interestingly, while this algorithm uses the Fourier transform it is exact (as opposed to Shor’s).

So where can I get one (so I
can start mining Bitcoin)

D-Wave



D-Wave



- 1951 - EDVAC (first binary computer, Vacuum tubes)
- 1956 - John Bardeen invents the transistor
- 1958 - Jack Kilby invents ICs
- 1964 - IBM System/360
- 1968 - Intel founded by Robert Noyce
- 1971 - Intel 4004 (first commercially available processor, 4bit @ 740 kHz)
- 1975 - MITS Altair 8800 (first commercially successful hobby computer @ \$397 \approx \$1700, uses Intel 8080)

$$\begin{aligned} U_f \left(|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) &= U_f \left(\frac{|x\rangle|0\rangle - |x\rangle|1\rangle}{\sqrt{2}} \right) \\ &= \frac{|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle}{\sqrt{2}} \end{aligned}$$

Now if $f(x) = 0$ then

$$\frac{|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle}{\sqrt{2}} = \frac{|x\rangle|0\rangle - |x\rangle|1\rangle}{\sqrt{2}}$$

and if $f(x) = 1$ then because \oplus is mod 2

$$\frac{|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle}{\sqrt{2}} = \frac{|x\rangle|1\rangle - |x\rangle|0\rangle}{\sqrt{2}}$$

and so

$$\begin{aligned} U_f \left(|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) &= U_f \left(\frac{|x\rangle|0\rangle - |x\rangle|1\rangle}{\sqrt{2}} \right) \\ &= \frac{|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle}{\sqrt{2}} \end{aligned}$$

Succintly put

$$U_f \left(|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) = (-1)^{f(x)} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$\begin{aligned} |\psi_1\rangle &= H^{\otimes 2}(|0\rangle \otimes |1\rangle) \\ &= (H|0\rangle) \otimes (H|1\rangle) \\ &= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) \\ &= |0\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} + |1\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \end{aligned}$$

Then

$$\psi_2 = U_f |\psi_1\rangle = \begin{cases} \pm \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}$$

The final Hadamard gate on the first qubit gives

$$\psi_3 = \begin{cases} \pm |0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm |1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}$$

$$\psi_2 = U_f \psi_1 = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Now extrapolating from $H|0\rangle = \sum_{z \in \{0,1\}} (-1)^{0 \cdot z} |z\rangle / \sqrt{2}$ and $H|1\rangle = \sum_{z \in \{0,1\}} (-1)^{1 \cdot z} |z\rangle / \sqrt{2}$ applying to

$$\begin{aligned} H^{\otimes n} |x_1, \dots, x_n\rangle_{x_i \in \{0,1\}} &= \bigotimes_{i=1}^n (H |x_i\rangle)_{x_i \in \{0,1\}} \\ &= \left(\sum_{z \in \{0,1\}} \frac{(-1)^{x_i \cdot z}}{\sqrt{2}} |z\rangle \right)_{x_i \in \{0,1\}}^{\otimes n} \\ &= \sum_{z_1, \dots, z_n} \frac{(-1)^{x_1 z_1 + \dots + x_n z_n}}{\sqrt{2^n}} |z_1, \dots, z_n\rangle \end{aligned}$$



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