notes

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The first isomorphism theorem for groups is:

Let G and H be groups, and let $\phi : G \to H$ be a homomorphism. Then the kernel of ϕ is a normal subgroup of G, the image of ϕ is a subgroup of H, and the image of ϕ is isomorphic to the quotient group $G/\ker(\phi)$

Proof. Let $K = \ker(\phi)$ and $\tilde{\phi} : G/K \to \mathrm{im}(\phi)$ be defined in terms of the right cosets of K, i.e. $gK \mapsto \phi(g)$. Then the $\tilde{\phi}$ is well-defined, i.e. different representatives of the same coset map to the same element, because if $g'K = gK$ then $g' = gk$ for some $k \in K$ and so

$$\phi(g') = \phi(gk) = \phi(g)\phi(k) = \phi(g) \cdot e = \phi(g)$$

.
$\tilde{\phi}$ is a homomorphism because ϕ is:

$$\begin{aligned}
\tilde{\phi}(gKg'K) &= \tilde{\phi}(gg'K) && \text{by coset multiplication} \\
&= \phi(gg') && \text{by definition of } \tilde{\phi} \\
&= \phi(g)\phi(g') && \phi \text{ is a homomorphism} \\
&= \tilde{\phi}(gK)\tilde{\phi}(g'K) && \text{by definition of } \tilde{\phi}
\end{aligned}$$

$\tilde{\phi}$ is injective because $\ker(\tilde{\phi})$ is only K, which is the 0 in $G/\ker(\phi)$: If $\tilde{\phi}(gK) = e$ then $\phi(g) = e$ and so $g \in \ker(\phi)$ making $gK = K$. And $\tilde{\phi}$ is surjective because $\mathrm{im}(\tilde{\phi}) = \mathrm{im}(\phi)$. □

A good example is the rank-nullity theorem from Linear Algebra. It's stated:
If A is a matrix with m rows and n columns (it maps from R^n to R^m because m dot products with the n length column vector are carried out) then the dimensions of the column space and the row space of A are the same, collectively called the rank r, and the dimension of the nullspace is $n - r$ and the dimension of the nullspace of A^T is $m - r$.

Proof. Let T be the homomorphism that corresponds to A, that maps between $V = R^n$ and $W = R^m$. The theorem says that $V/\ker(T)$ is isomorphic to $\mathrm{im}(T)$. But $\mathrm{im}(T)$ is of course the column space and $V/\ker(T)$ is (the only thing i can say so far is that the basis for this space maps back to a basis for V that is linearly independent from the basis for the kernel. need to think about dual space to figure out orthogonal complement and see if the algebra will give me that too - but i suspect the algebra is agnostic to orthogonality without incorporating inner products, which is exactly what the dual space will do for me) $\square$

5/29/2014

Fibonacci numbers and generating functions (CLRS 4-4):

Let $\mathcal{F}(z) = \sum_{i=0}^{\infty} F_i z^i$ where F_i is the ith Fibonacci number.

Prove that $\mathcal{F}(z) = z + z\mathcal{F}(z) + z^2\mathcal{F}(z)$.

Proof. You can think of the equation as a recurrence relation; multiplying by z and z^2 "aligns" ith and $i+1$th coefficients and then you sum them. So

$$\begin{aligned} z\mathcal{F}(z) &= z^2 + z^3 + 2z^4 + 3z^5 + 5z^6 + 8z^7 + 13z^8 + 21z^9 \\ z^2\mathcal{F}(z) &= 0 + z^3 + z^4 + 2z^5 + 3z^6 + 5z^7 + 8z^8 + 13z^9 + 21z^{10} \end{aligned}$$

$\square$

Show that

$$\mathcal{F}(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right)$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\hat{\phi} = \frac{1-\sqrt{5}}{2}$.

Proof. Using just prior results

$$\begin{aligned} \mathcal{F}(z) &= \frac{z}{1-z-z^2} \\ &= \frac{z}{(1-\phi z)(1-\hat{\phi} z)} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right) \end{aligned}$$

$\square$

Show that

$$\mathcal{F}(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}}(\phi^i - \hat{\phi}^i) z^i$$

Proof. It's basically asking you to recompute the coefficients of the generating function. You can do this from the analytic formula for $\mathcal{F}(z)$ derived just prior. The trick is to compute derivatives of $\mathcal{F}(z)$ at $z = 0$. Think about it. For example evaluate both the analytic formula and the generating function at $z = 0$. What do you get? Well first of all the agree but more importantly you get the constant coefficient in the power-series/generating function (which happens to be 0). Then take the first derivative of the power-series/generating function and evaluate it at 0. What do you get? You get the only term in left in the differentiated power-series that doesn't have a z factor, which will be 1! times the coefficient of that term. Now take the second derivative and evaluate at $z = 0$. What do you get? The coefficient of the second term in the power-series multiplied by 2!. And so on. In general you terms of the form

$$\frac{1}{\sqrt{5}}(i!\phi^i - i!\hat{\phi}^i)$$

So just dividing by $i!$ gives you the ith coefficient of the power-series/generating function, i.e. $\frac{1}{\sqrt{5}}(\phi^i - \hat{\phi}^i)$. The implication being that a closed form for the ith Fibonacci number is $\frac{1}{\sqrt{5}}(\phi^i - \hat{\phi}^i)$. Notice that for large i, since $|\hat{\phi}| < 1$, the ith Fibonacci number is just $\frac{1}{\sqrt{5}}(\phi^i)$. $\square$