

STA 6326 Homework 2 Solutions

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1.35 Let $Q(A) = P(A|B)$. Firstly

$$Q(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

by the non-negativity of $P(\cdot)$ and the hypothesis that $P(B) > 0$. Secondly

$$Q(\Omega) = P(\Omega) = \frac{P(\Omega|B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Finally assume A_i, A_j for all i, j are pairwise disjoint. Then

$$Q\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$

and since A_i, A_j for all i, j are pairwise disjoint $(A_i \cap B), (A_j \cap B)$ are also pairwise disjoint for all i, j and by the countable additivity of $P(\cdot)$

$$\frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)} = \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i|B) = \sum_{i=1}^{\infty} Q(A_i)$$

1.38 (a) If $P(B) = 1$ then $P(A|B) = \frac{P(A \cap B)}{1}$ but $P(A) = P(A \cap B) + P(A \cap B^c)$ and since $A \cap B^c \subset B^c$ and $P(A \cap B^c) \leq P(B^c) = 1 - P(B) = 0$ it's the case that $P(A) = P(A \cap B)$ so $P(A|B) = P(A)$.

(b) $P(B|A) = P(B \cap A)/P(A)$ but the hypothesis $A \subset B$ implies $B \cap A = A$ so $P(B|A) = P(A)/P(A) = 1$. Then $P(A|B) = P(B|A)P(A)/P(B) = P(A)/P(B)$.

(c)

$$\begin{aligned} P(A|A \cup B) &= \frac{P(A \cap (A \cup B))}{P(A \cup B)} \\ &= \frac{P(A)}{P(A) + P(B) - P(A \cap B)} && \text{by } A \subset A \cup B \\ &= \frac{P(A)}{P(A) + P(B)} && \text{by "mutually exclusive" } \iff A \cap B = \emptyset \end{aligned}$$

(d) $P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

1.39 (a) If $P(A) > 0$ and $P(B) > 0$ and $P(A \cap B) = 0$ then obviously $P(A) \cdot P(B) \neq P(A \cap B)$.

- (b) If $P(A) > 0$ and $P(B) > 0$ and $P(A)P(B) = P(A \cap B)$ then obviously $P(A \cap B) = P(A)P(B) > 0$.

1.44 The number of correct answers is binomially distributed with $p = .25$ and $1 - p = .75$. Then

$$P(X \geq 10) = \sum_{k=10}^{20} \binom{20}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{20-k} = .0138644$$

1.46 There $7^7 = 823,543$ different ways to distribute the 7 balls into the 7 cells. The maximum number of cells that could have 3 balls is 2 and clearly the minimum is 0. Hence $X_3 \in \{0, 1, 2\}$. The only way for $X_3 = 2$ would be 3 balls in a cell, 3 balls in another cell, and the last ball in one cell. There are $\binom{7}{2}$ way to choose the 2 cells to have the 3 balls each, $\binom{7}{3}$ ways to choose the first set of 3 balls for the first cell, $\binom{4}{3}$ to choose the second set of 3 balls for the second cell, then finally $\binom{5}{1} = 5$ different ways to choose cell will contain the last balls. Therefore

$$P(X_3 = 2) = \frac{\binom{7}{2} \binom{7}{3} \binom{4}{3} 5}{7^7} \approx .0178$$

For $X_3 = 1$ there are 3 different configurations: $\{3, 1, 1, 1, 1\}, \{3, 2, 1, 1\}, \{3, 2, 2\}$.

$$\#\{3, 1, 1, 1, 1\} = 7 \binom{7}{3} \times \binom{6}{4} \times 4 \times 3 \times 2 \text{ which cell contains 3 balls, which 3 balls,}$$

which cells contain 1 balls, permute the balls

$$\#\{3, 2, 1, 1\} = 7 \binom{7}{3} \times 6 \times \binom{4}{2} \times \binom{5}{2} \times 2 \text{ which cell contains 3 balls, which 3 balls,}$$

which cell contains 2 balls, which 2 balls,

which cells contain 1 each each, permute the balls

$$\#\{3, 2, 2\} = 7 \binom{7}{3} \times \binom{6}{2} \times \binom{4}{2} \text{ which cell contains 3 balls, which 3 balls,}$$

which cells contain 2 balls, which two balls in the first 2 ball cell

which cell contains second set of 2 balls, permute the balls

$$\#\{3, 4\} = 7 \binom{7}{3} \times 6 \text{ which cell contains 3 balls, which 3 balls,}$$

which cell contains 4 balls

1.47 Requirements for being a CDF: (i) right continuous (ii) $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$ (iii) non-decreasing.

(a) $\frac{1}{2} + \frac{1}{\pi} \arctan(x)$

i. Continuous and hence right-continuous.

ii. $\lim_{x \rightarrow -\pi/2} \tan(x) = -\infty$ hence $\lim_{x \rightarrow -\infty} \frac{1}{2} + \frac{1}{\pi} \arctan(x) = \frac{1}{2} + \frac{1}{\pi} \frac{-\pi}{2} = \frac{1}{2} - \frac{1}{2} = 0$.
 $\lim_{x \rightarrow \pi/2} \tan(x) = \infty$ hence $\lim_{x \rightarrow \infty} \frac{1}{2} + \frac{1}{\pi} \arctan(x) = \frac{1}{2} + \frac{1}{\pi} \frac{\pi}{2} = \frac{1}{2} + \frac{1}{2} = 1$.

iii. $\arctan(x)$ is non-decreasing (monotonically increasing) and hence $\frac{1}{2} + \frac{1}{\pi} \arctan(x)$ is non-decreasing.

(b) $(1 + e^{-x})^{-1}$

i. Continuous and hence right-continuous.

ii. $\lim_{x \rightarrow -\infty} e^{-x} = \infty$ hence $\lim_{x \rightarrow -\infty} (1 + e^{-x})^{-1} = 1/\infty = 0$. $\lim_{x \rightarrow \infty} e^{-x} = 0$ hence $\lim_{x \rightarrow \infty} (1 + e^{-x})^{-1} = 1/1 = 1$.

iii. $\left((1 + e^{-x})^{-1} \right)' = -1(1 + e^{-x})^{-2}(-1)e^{-x} = (1 + e^{-x})e^{-x} > 0$ hence non-decreasing.

(c) $\exp(-e^{-x})$

i. Continuous and hence right-continuous.

ii. $\lim_{x \rightarrow -\infty} e^{-x} = \infty$ hence $\lim_{x \rightarrow -\infty} \exp(-e^{-x}) = 0$. $\lim_{x \rightarrow \infty} e^{-x} = 0$ hence $\lim_{x \rightarrow \infty} \exp(-e^{-x}) = 1$.

iii. $\left(\exp(-e^{-x})\right)' = \exp(-e^{-x})(-e^{-x})(-1) = \exp(-e^{-x})e^{-x} > 0$ hence non-decreasing.

(d) $1 - e^{-x}$

i. Continuous and hence right-continuous.

ii. $e^{-0} = 1$ hence $\lim_{x \rightarrow 0} 1 - e^{-x} = 1 - 1 = 0$. $\lim_{x \rightarrow \infty} e^{-x} = 0$ hence $\lim_{x \rightarrow \infty} 1 - e^{-x} = 1$.

iii. $\left(1 - e^{-x}\right)' = 1 + e^{-x} > 0$ hence non-decreasing.

(e) If $0 < \epsilon < 1$ then $F_Y(y) = \begin{cases} \frac{1-\epsilon}{1+e^{-y}} & \text{if } y < 0 \\ \epsilon + \frac{1-\epsilon}{1+e^{-y}} & \text{if } y \geq 0 \end{cases}$

i. Since each piece of the piecewise definition of F_Y is continuous and the domain is defined with equality from the right ($y \geq 0$) F_Y is right-continuous.

ii. $\lim_{y \rightarrow -\infty} \frac{1-\epsilon}{1+e^{-y}} = 0$ similarly to (b.ii) and $\lim_{y \rightarrow \infty} \frac{1-\epsilon}{1+e^{-y}} = 1 - \epsilon$ hence $\lim_{y \rightarrow \infty} \epsilon + \frac{1-\epsilon}{1+e^{-y}} = \epsilon + 1 - \epsilon = 1$.

iii. $\frac{1-\epsilon}{1+e^{-y}}$ and $\epsilon + \frac{1-\epsilon}{1+e^{-y}}$ are non-decreasing by (b.ii) and $\lim_{y \uparrow 0} F_Y(y) = (1 - \epsilon)/2 < \epsilon + (1 - \epsilon)/2 = \lim_{y \downarrow 0} F_Y(y)$ hence non-decreasing.

1.49 Assume $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t_0 . By definition $F_X(t) = P(X \leq t) = 1 - P(X > t)$. Similarly $F_Y(t) = P(Y \leq t) = 1 - P(Y > t)$. Then $1 - P(X > t) \leq 1 - P(Y > t)$ and so $P(X > t) \geq P(Y > t)$. Similarly for t_0 it's the case $P(X > t_0) > P(Y > t_0)$.

1.51 X is distributed Hypergeometrically, i.e. there are $\binom{30}{4}$ ways to draw a sample of 4 from 30, $\binom{5}{x}$ ways to draw x microwaves from the subset of microwaves that is defective, and then finally per draw from the defective subset there are $\binom{25}{4-x}$ to draw the remainder from the subset of functional microwaves. Hence for $x = 0, 1, 2, 3, 4$

$$P(X = x) = \frac{\binom{5}{x} \binom{25}{4-x}}{\binom{30}{4}}$$

So

$$P(X = x) = \begin{cases} \frac{2530}{5481} & \text{for } x = 0 \\ \frac{2300}{5481} & \text{for } x = 1 \\ \frac{600}{5481} & \text{for } x = 2 \\ \frac{50}{5481} & \text{for } x = 3 \\ \frac{1}{5481} & \text{for } x = 4 \end{cases}$$

Then

$$F_X(x) = \begin{cases} \frac{2530}{5481} & \text{for } x = 0 \\ \frac{4830}{5481} & \text{for } x = 1 \\ \frac{5430}{5481} & \text{for } x = 2 \\ \frac{5480}{5481} & \text{for } x = 3 \\ \frac{5481}{5481} & \text{for } x = 4 \end{cases}$$

The plot of the CDF is Figure 1.

1.52 Let $f(x)$ be a pdf with cdf $F(x)$, $F(x_0) < 1$, and

$$g(x) = \begin{cases} f(x)/(1 - F(x_0)) & \text{if } x \geq x_0 \\ 0 & \text{if } x < x_0 \end{cases}$$

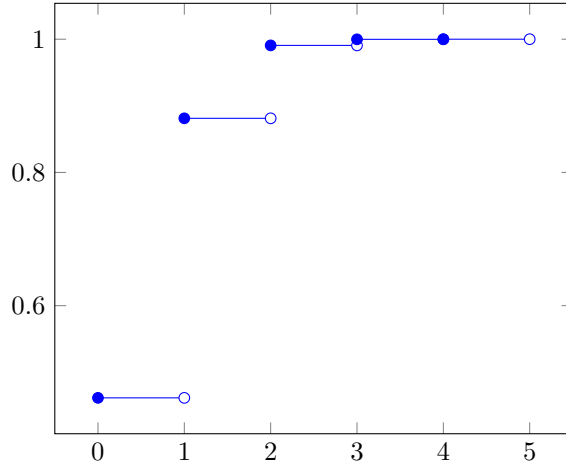


Figure 1: F_X for problem 1.51

Then since $f(x) \geq 0$ and $1 > F(x_0)$ (and since F is a cdf $F(x_0) \geq 0$) it's the case that $g(x) \geq 0$ for all x . Finally

$$\int_{-\infty}^{\infty} g(x) = \int_{-\infty}^{x_0} g(x) + \int_{x_0}^{\infty} \frac{f(x)}{1 - F(x_0)} dx = 0 + \frac{1}{1 - F(x_0)} \int_{x_0}^{\infty} f(x) dx = \frac{1 - \int_{-\infty}^{x_0} f(x)}{1 - F(x_0)} = \frac{1 - F(x_0)}{1 - F(x_0)} = 1$$

1.54 C is a normalization constant.

(a) $\int_0^{\pi/2} \sin(x) dx = 1$ hence $c = 1 \implies \int_0^{\pi/2} f(x) = 1$.

(b) $\int_{-\infty}^{\infty} e^{-|x|} dx = 2$ hence $c = 1/2 \implies \int_{-\infty}^{\infty} ce^{-|x|} dx = 1$.

2.1 (a) If $Y = g(X) = X^3$ then $g^{-1}(Y) = \sqrt[3]{Y}$ and $\left(g^{-1}(Y)\right)' = \frac{1}{3}y^{-2/3}$ and

$$f_Y(y) = 42(\sqrt[3]{y})^5(1 - \sqrt[3]{y}) \left| \frac{1}{3}y^{-2/3} \right| = 14(\sqrt[3]{y} - 1)y$$

Hence

$$\int_0^1 f_Y(y) dy = 14 \int_0^1 y(\sqrt[3]{y} - 1) dy = 14 \frac{1}{14} = 1$$

(b) If $Y = g(X) = 4X + 3$ then $g^{-1}(Y) = (Y - 3)/4$ and $\left(g^{-1}(Y)\right)' = \frac{1}{4}$ and

$$f_Y(y) = 7e^{-\frac{7}{4}(y-3)} \left| \frac{1}{4} \right|$$

Hence, since $g(0) = 4(0) + 3 < Y < \infty$

$$\int_0^{\infty} f_Y(y) dy = \frac{7}{4} e^{\frac{21}{4}} \int_3^{\infty} e^{-\frac{7y}{4}} dy = e^{\frac{21}{4}} e^{-\frac{7 \cdot 3}{4}} = 1$$

(c) If $Y = g(X) = X^2$ then $g^{-1}(Y) = \pm\sqrt{Y}$ and $\left(g^{-1}(Y)\right)' = 1/2\sqrt{Y}$ and

$$f_Y(y) = \frac{30}{4} y^2 \left(1 - \frac{1}{2\sqrt{y}}\right)$$

Hence

$$\int_0^1 f_Y(y) dy = \frac{30}{4} \int_0^1 y^2 \left(1 - \frac{1}{2\sqrt{y}}\right) dy = \frac{30}{4} \frac{4}{30} = 1$$

2.2 (a) If $Y = g(X) = X^2$ then $g(Y)^{-1} = \pm\sqrt{Y}$ and $\left| \left(g^{-1}(Y) \right)' \right| = 1/2\sqrt{Y}$ and

$$f_Y(y) = 1 \cdot \frac{1}{2\sqrt{y}}$$

(b) If $Y = g(X) = -\log(X)$ then $g^{-1}(Y) = e^{-Y}$ and $\left| \left(g^{-1}(Y) \right)' \right| = e^{-Y}$ and

$$f_Y(y) = \binom{n+m+1}{n, m, 1} e^{-ny} (1 - e^{-y})^m e^{-y}$$

With domain $-\log(1) = 0 < y < -\log(0) = \infty$.

(c) If $Y = g(X) = e^X$ then $g(Y)^{-1} = \log Y$ and $\left| \left(g^{-1}(Y) \right)' \right| = 1/Y$ and

$$f_Y(y) = \frac{1}{\sigma^2} \frac{1}{y^2} e^{-(1/y\sigma)^2/2} = \frac{1}{(\sigma y)^2} e^{-(1/y\sigma)^2/2}$$

2.3 If $Y = g(X) = X/(X+1)$ then $g(Y)^{-1} = 1/(1-Y)$ and $\left| \left(\frac{1}{1-y} \right)' \right| = \frac{1}{(1-y)^2}$ Hence for $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

$$f_Y(y) = \frac{1}{3} \left(\frac{2}{3} \right)^{\frac{1}{1-y}} \frac{1}{(1-y)^2}$$

$$2.4 \quad f(x) = \begin{cases} \frac{1}{2} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ \frac{1}{2} \lambda e^{\lambda x} & \text{if } x < 0 \end{cases}$$

(a) $e^{\lambda x} > 0$ for all $x \in (-\infty, \infty)$ hence $f(x) \geq 0$. Further

$$\int_{-\infty}^{\infty} f(x) dx = \frac{\lambda}{2} \int_{-\infty}^0 e^{\lambda x} dx + \frac{\lambda}{2} \int_0^{\infty} e^{-\lambda x} dx$$

Then by $-u = x$

$$\frac{\lambda}{2} \int_{-\infty}^0 e^{\lambda x} dx = \frac{\lambda}{2} \int_{\infty}^0 e^{-\lambda u} d(-u) = (-1)(-1) \frac{\lambda}{2} \int_0^{\infty} e^{-\lambda u} du = \frac{\lambda}{2} \int_0^{\infty} e^{-\lambda x} dx$$

and hence

$$\int_{-\infty}^{\infty} f(x) dx = 2 \frac{\lambda}{2} \int_0^{\infty} e^{-\lambda x} dx = \lambda \frac{1}{\lambda} (1 - 0) = 1.$$

(b) If $x < 0$ then

$$F_X(x) = \frac{\lambda}{2} \int_{-\infty}^x e^{\lambda x} dx = \frac{\lambda}{2} \frac{1}{\lambda} e^{\lambda x} = \frac{1}{2} e^{\lambda x}$$

If $x \geq 0$ then

$$F_X(x) = \frac{\lambda}{2} \int_{-\infty}^0 e^{\lambda x} dx + \frac{\lambda}{2} \int_0^x e^{-\lambda x} dx = \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2} e^{-\lambda x} \right)$$

(c) $P(|X| < t) = \int_{-t}^t f(x) dx$. Arguments from part (a) imply

$$\int_{-t}^t f(x) dx = 2 \frac{\lambda}{2} \int_0^t e^{-\lambda x} dx = 2 \left(\frac{1}{2} - \frac{1}{2} e^{-\lambda t} \right)$$