MAD6406 HOMEWORK FALL 2015

1.3 Claim: R upper triangular and nonsingular iff R^{-1} upper triangular.

Proof. R nonsingular implies all diagonal entries of R are non-zero. Therefore for $n < m$ the span of $\{r_1, \ldots, r_n\}$, where r_i are the columns of R, is $\mathbb{R}^n$. Therefore there exist ρ_{jn} for $j = 1, \ldots, n$ such that

$$r_1\rho_{1n} + r_2\rho_{2n} + \cdots + r_n\rho_{nn} = e_n$$

Let $\rho_{jn} = 0$ for $n < j <\leq m$ and hence

$$r_1\rho_{1n} + r_2\rho_{2n} + \cdots + r_m\rho_{mn} = e_n \tag{0.1}$$

Then let

$$\rho_i = \begin{bmatrix} \rho_{1i} \\ \vdots \\ \rho_{ii} \\ \vdots \\ \rho_{mi} \end{bmatrix} = \begin{bmatrix} \rho_{1i} \\ \vdots \\ \rho_{ii} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $P = \begin{bmatrix} \rho_1 & \cdots & \rho_m \end{bmatrix}$. Note that P is upper-triangular since for all i it's the case that $\rho_{jn} = 0$ for $j = n+1, \cdots, m$. Furthermore by **??** we have that $R \cdot P = I$ and by uniqueness of inverses $P = R^{-1}$. □

2.3 Claim: for self-adjoint $A \in \mathbb{C}^{m\times m}$
(a) All eigenvalues of A are real.
(b) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. (a) Let $x \neq 0$ and λ such that $Ax = \lambda x$. Then by self-adjointness

$$\left(x^\dagger Ax\right)^\dagger = \left((Ax)^\dagger \left(x^\dagger\right)^\dagger\right) = \left(x^\dagger A^\dagger x\right) = \left(x^\dagger Ax\right)$$

but

$$x^\dagger Ax = x^\dagger \lambda x = \lambda \left\|x\right\|^2$$

and $\lambda \left\|x\right\|^2 = \left(\lambda \left\|x\right\|^2\right)^\dagger = \lambda^* \left\|x\right\|^2$. Therefore $\lambda = \lambda^*$ hence $\lambda \in \mathbb{R}$.

(b) Let $x \neq y \neq 0$ and λ, λ' such that$Ax = \lambda x$ and $Ay = \lambda' y$. Then by self-adjointness

$$x^\dagger Ay = (Ax)^\dagger y$$

and so since $\lambda, \lambda' \in \mathbb{R}$

$$0 = x^\dagger Ay - (Ax)^\dagger y = \lambda' x^\dagger y - \lambda x^\dagger y = \left(\lambda' - \lambda\right) x^\dagger y$$

Therefore since $\lambda \neq \lambda'$ it's the case that $\lambda' - \lambda \neq 0$ and hence $x^\dagger y = 0$.

□

3.5 Claim: $\left\|A\right\|_F = \left\|uv^*\right\|_F = \left\|u\right\|_F \left\|v\right\|_F$.

Proof. Since the Frobenius norm of a vector is just the 2-norm so it distributes over the product. To wit if $A = uv^*$

$$\left\|A\right\|_F = \left\|uv^*\right\|_F = \sqrt{\sum_{i,j} \left|u_j v_i^*\right|^2} = \sqrt{\sum_j \left|u_j\right|^2 \sum_i \left|v_i^*\right|^2} = \sqrt{\sum_j \left|u_j\right|^2}\sqrt{\sum_i \left|v_i^*\right|^2}$$

Now since $|x^*| = |x|$

$$\sqrt{\sum_j \left|u_j\right|^2}\sqrt{\sum_i \left|v_i^*\right|^2} = \sqrt{\sum_j \left|u_j\right|^2}\sqrt{\sum_i \left|v_i\right|^2} = \left\|u\right\|_F \left\|v\right\|_F$$

□

4.4 Claim: it is **false** that $A, B \in \mathbb{C}^m$ are unitary equivalent iff they have the same singular values.

Proof. A and $-A$ have the same singular values: let $A = U\Sigma V^{-1}$, then

$$-A = U\Sigma\left(-V^{-1}\right)$$

But A and $-A$ cannot be unitarily equivalent since then

$$\begin{aligned}\det(A) &= \det\left(Q(-A)Q^{-1}\right)\\ &= \det(Q)\det(-A)\det\left(Q^{-1}\right)\\ &= \det(Q)\det\left(Q^{-1}\right)\det(-A)\\ &= (-1)^m\det(A)\end{aligned}$$

which is only true if m is even or $\det(A) = 0$. Alternatively

$$\begin{aligned}\operatorname{tr}(A) &= \operatorname{tr}\left(Q(-A)Q^{-1}\right)\\ &= \operatorname{tr}\left(Q^{-1}Q(-A)\right)\\ &= -\operatorname{tr}(A)\end{aligned}$$

which is only true if $(A)_{ii} = 0$ for all i. □

5.4 Claim: if $A \in \mathbb{C}^{m\times m}$, with $A = U\Sigma V^{-1} = U\Sigma V^{\dagger}$, and

$$B = \begin{pmatrix} 0 & A^{\dagger} \\ A & 0 \end{pmatrix}$$

then there exists X, Λ such that $B = X\Lambda X^{-1}$ is an eigenvalue decomposition.

Proof. First note that $A^{\dagger} = \left(U\Sigma V^{\dagger}\right)^{\dagger} = V\Sigma^{\dagger}U^{\dagger} = V\Sigma U^{\dagger}$ since singular values are always real[1]. Since A is square

$$\begin{aligned}A^{\dagger}u_j &= \sigma_j v_j\\ Av_j &= \sigma_j u_j\end{aligned}$$

Hence

$$B\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} 0 & A^{\dagger} \\ A & 0 \end{pmatrix}\begin{pmatrix} v_j \\ u_j \end{pmatrix} = \begin{pmatrix} A^{\dagger}u_j \\ Av_j \end{pmatrix} = \begin{pmatrix} \sigma_j v_j \\ \sigma_j u_j \end{pmatrix} = \sigma_j\begin{pmatrix} v_j \\ u_j \end{pmatrix}$$

and

$$B\begin{pmatrix} v_j \\ -u_j \end{pmatrix} = \begin{pmatrix} 0 & A^{\dagger} \\ A & 0 \end{pmatrix}\begin{pmatrix} v_j \\ -u_j \end{pmatrix} = \begin{pmatrix} -A^{\dagger}u_j \\ Av_j \end{pmatrix} = \begin{pmatrix} -\sigma_j v_j \\ \sigma_j u_j \end{pmatrix} = -\sigma_j\begin{pmatrix} v_j \\ -u_j \end{pmatrix}$$

Therefore the eigenvectors of B are $\left\{\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_m \\ u_m \end{pmatrix}, \begin{pmatrix} v_1 \\ -u_1 \end{pmatrix}, \dots, \begin{pmatrix} v_m \\ -u_m \end{pmatrix}\right\}$ with eigenvalues $\{\sigma_1, \dots, \sigma_m, -\sigma_1, \dots, -\sigma_m\}$ and hence

$$B = \begin{pmatrix} V & V \\ U & -U \end{pmatrix}\begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix}\begin{pmatrix} V & V \\ U & -U \end{pmatrix}^{-1}$$

□

6.1 Claim: if P is an orthogonal projector, the $I - 2P$ is unitary.

Proof. Since P is an orthogonal projector $P = P^{\dagger}$ and hence

$$\begin{aligned}(I-2P)^{\dagger}(I-2P) &= \left(I^{\dagger} - 2P^{\dagger}\right)(I-2P)\\ &= (I-2P)(I-2P)\\ &= I - 4P - 4P^2\\ &= I - 4P - 4P\\ &= I\end{aligned}$$

Hence P is unitary. The geometric intuition is that since $I - 2P = (I-P) - P$ it's the case that

$$(I-2P)x = (I-P)x - Px$$

Since in general $x = (I-P)x + Px$ it's obvious that $(I-P)x + (-Px)$ is the reflection of x across range $(I-P)$, which is what unitary transformations are (reflections and/or rotations). □

7.4 Let $P_1 = \left[x^{(1)}, y^{(1)}\right]$ and $P_2 = \left[x^{(2)}, y^{(2)}\right]$ be matrices. Then compute the full QR factorizations $P_1 = Q_1R_1$ and $P_2 = Q_2R_2$. The third columns $z^{(1)}, z^{(2)}$of Q_1, Q_2 are orthogonal to $P^{(1)}, P^{(2)}$ respectively. Then let matrix $P_3 = \left[z^{(1)}, z^{(2)}\right]$ and $P_3 = Q_3R_3$. Then the third column $z^{(3)}$ of Q_3 is orthogonal to $\left\langle z^{(1)}, z^{(2)}\right\rangle$, i.e. in both $P^{(1)}$ and $P^{(2)}$.

[1] Thm. 4.1 in Trefethen: $\sigma_1 = \|A\|$

8.2 QR factorization in MATLAB

```
function [ Q,R ] = mgs( A )

n=size(A,2);
V = A;
R = zeros(n);
Q = zeros(size(A));
for i=1:n
    R(i,i) = norm(V(:,i));
    Q(:,i) = V(:,i)/R(i,i);
    for j=i+1:n
        R(i,j) = dot(conj(Q(:,i)),V(:,j));
        V(:,j) = V(:,j) - R(i,j)*Q(:,i);
    end
end
```

10.2 (a) QR factorization by Householder orthogonal triangularization in MATLAB

```
function [W,R] = house( A )
%HOUSE computes implicit representation of QR using householder
%orthogonal trianglurization
[m,n]=size(A);
W=zeros([m,n]);
R=A;
for k=1:n
    x = R(k:m,k);
    vk = sign(x(1))*norm(x)*vertcat(1,zeros(m-k,1)) + x;
    vk = vk/norm(vk);
    W(k:m,k) = vk;
    R(k:m,k:n) = R(k:m,k:n) - 2*vk*(vk'*R(k:m,k:n));
end
end
```

(b) Reconstruction of Q from Householder decomposition

```
function Q=formQ(W)
%formQ reconstructs Q from W by computing Qe_i for i=1,..,m
[m,n]=size(W);
Q=eye(m);
for j=1:m
    for k=n:-1:1
        Q(k:m,j) = Q(k:m,j) - 2*W(k:m,k)*(W(k:m,k)'*Q(k:m,j));
    end
end
end
```