MAP4102, Spring 2014, Homework 1 *due Wed, January 22*

These problems are due at the beginning of the class next Wednesday. Several problems will require simulation. A template program in `R` has been provided on the course web page.

Students enrolled in MAP4102 can choose to do just one of the $\diamond$ problems. Students enrolled in MAT6932 should do all problems.

1. (Durrett 1.2+) Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n.

   (a) Compute the transition probability matrix for X.

   (b) Does this Markov chain have a stationary distribution? If so, compute it.

2. (Lawler 1.1, 1.3) The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3, someone takes all the papers in the pile and puts them in the recycling bin. If there are ever at least 5 papers in the pile, Mr. Smith (with probability 1) takes the papers to the recycling bin. Let X_n be the number of papers in the pile *in the evening.*

   (a) What is the state space of this Markov Chain and what is the transition matrix P?

   (b) Suppose there are 3 papers in the pile tonight. What is the probability that 2 evenings from now there are 0 papers in the pile?

   (c) Argue, via numerical experimentation, whether there exists a limiting distribution of papers in the pile.

   (d) In the long run, what is the probability there is just 1 paper in the pile in the evening?

   (e) In the long run, what is the expected number of papers in the pile in the evening?

3. Consider the following transition matrrix for a Markov Chain with state space $\{A, B, C, D, E\}$.

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

   (a) Draw a flow diagram of this Markov Chain and classify all of its states as stationary or recurrent.

   (b) Find all of its stationary distributions. (Be sure to argue how you know that you have not left any out.)

   (c) Suppose we start in state A. In the long run, what percentage of the time do we spend in state A?

   (d) Suppose we start in state C. In the long run, what percentage of the time do we spend in state A?

(e) Suppose we start in state B. In the long run, what is the probability that we are in state A at any given time?

(f) Why did I have to change the wording of the question in the last item?

4. ⋄ (Durrett 1.22) In a test paper, the questions are arranged so that 3/4's of the time True is followed by a True, while 2/3's of the time a False is followed by a False. You are confronted with a 100 question text. Approximately what fraction of the answers will be True?

5. ⋄ Let $\{X_n\}_{n\geq 0}$ be a simple random walk on some graph with transition matrix P and suppose that $\bar{\pi}$ is a stationary distribution of X. Prove that if we make the random walk "lazy", in the sense that we always give the walker a probability q of not moving, then $\bar{\pi}$ is also a stationary distribution of the lazy random walk.

   Hint: Write the transition matrix of the lazy random walk $\tilde{P}$ as a linear combination of P and the identity matrix I. Then use the definition of a stationary distribution.