# Homework 4

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# Problem a

## How many turns on average does it take to complete the game?

We compute the expected hitting time for state “9”. Define the transition matrix P

$$P = \begin{array}{c|c} & \begin{matrix} 1 & 4 & 5 & 7 & 9 \end{matrix} \\ \begin{matrix} 1 \\ 4 \\ 5 \\ 7 \\ 9 \end{matrix} & \begin{pmatrix} 0 & 0 & 0.5` & 0.5` & 0 \\ 0.5` & 0 & 0.5` & 0 & 0 \\ 0.5` & 0 & 0 & 0.5` & 0 \\ 0 & 0.5` & 0 & 0 & 0.5` \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Define $\tilde{P} = P$ with the exception that $p(9,9) = 0$

$$\tilde{P} = \begin{pmatrix} 0 & 0 & 0.5` & 0.5` & 0 \\ 0.5` & 0 & 0.5` & 0 & 0 \\ 0.5` & 0 & 0 & 0.5` & 0 \\ 0 & 0.5` & 0 & 0 & 0.5` \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Define the hitting time vector $h$

$$h = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Then computing the expected hitting time is equivalent to solving the matrix equation

$$\left(I - \tilde{P}\right) \rho = h$$

for the unknown $\rho = (\rho_1, \rho_4, \rho_5, \rho_7, \rho_9)^T$ where $\rho_i$ is the hitting time for hitting state 9 from state i.

```
Solve[(IdentityMatrix[5] - P̃).{ρ1, ρ4, ρ5, ρ7, ρ9} == h, {ρ1, ρ4, ρ5, ρ7, ρ9}]
```

$\{\{\rho_1 \to 7, \rho_4 \to 8, \rho_5 \to 7, \rho_7 \to 5, \rho_9 \to 0\}\}$

And so the expected hitting for hitting state 9 from state 1 is 7.

# Problem b

## What is the probability that a player who has reached the middle

### square will complete the game without slipping back to square 1?

The probability is equivalent to the probability of hitting the states {1,9} with the further conditon that state 1 carries probability of zero of “hitting 9 before 1” and state 9 carries a probability of one of “hitting 9 before 1”

Define $\tilde{P}$ = $P$ with the exception that $p(9,9) = p(1, j) = 0$

$$\tilde{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.5` & 0 & 0.5` & 0 & 0 \\ 0.5` & 0 & 0 & 0.5` & 0 \\ 0 & 0.5` & 0 & 0 & 0.5` \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The state transition diagram is

Then computing the expected hitting time is equivalent to solving the matrix equation

$$\left(I - \tilde{P}\right)\rho = h$$

for the unknown $\rho = (\rho_1, \rho_4, \rho_5, \rho_7, \rho_9)^T$ where $\rho_i$ is the probability of hitting state 1 or 9. It is $h$ that encodes the conditons $\rho_1 = 0$ and $\rho_9 = 1$.

```
Solve[(IdentityMatrix[5] - P̃).{ρ1, ρ4, ρ5, ρ7, ρ9} == h, {ρ1, ρ4, ρ5, ρ7, ρ9}]
```

```
{ρ1 → 7, ρ4 → 8, ρ5 → 7, ρ7 → 5, ρ9 → 0}
```

So

# Problem b

### What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

The probability is equivalent to the probability of hitting the states {1,9} with the further conditon that state 1 carries probability of zero of “hitting 9 before 1” and state 9 carries a probability of one of “hitting 9 before 1”

Define $\tilde{P}$ = $P$ with the exception that $p(9,9) = p(1, j) = 0$

$$\tilde{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.5` & 0 & 0.5` & 0 & 0 \\ 0.5` & 0 & 0 & 0.5` & 0 \\ 0 & 0.5` & 0 & 0 & 0.5` \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

Define the hitting probability vector $h$

$$h = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

Then computing the expected hitting time is equivalent to solving the matrix equation

$$\left(\mathrm{I} - \tilde{\mathrm{P}}\right) \rho = \mathrm{h}$$

for the unknown $\rho = (\rho_1, \rho_4, \rho_5, \rho_7, \rho_9)^T$ where $\rho_i$ is the probability of hitting state 1 or 9. It is $h$ that encodes the conditons $\rho_1 = 0$ and $\rho_9 = 1$.

```
Solve[(IdentityMatrix[5] - P̃).{ρ1, ρ4, ρ5, ρ7, ρ9} == h, {ρ1, ρ4, ρ5, ρ7, ρ9}]
```

```
{{ρ1 → 0., ρ4 → 0.142857, ρ5 → 0.285714, ρ7 → 0.571429, ρ9 → 1.}}
```

So $\rho_5$ = 0.285714 is the probability that starting at state 5 you hit state 9 without hitting state 1 first.

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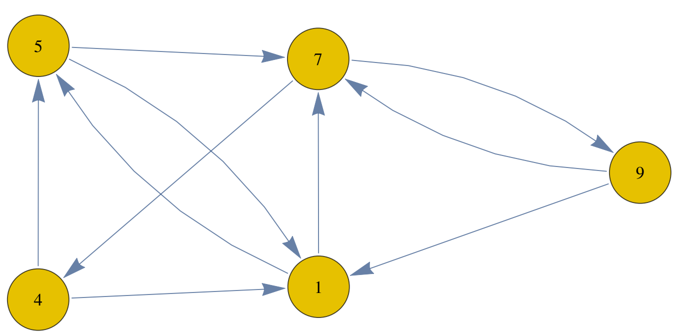
# Problem c

## Does this **new** Markov chain have a unique stationary distribution? Why or why not? In the long-run, in which square will we spend the most time, and what fraction of the time will we be there?

The adjusted transition matrix is

$$\mathrm{P} = \begin{array}{c} \\ 1 \\ 4 \\ 5 \\ 7 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 5 & 7 & 9 \end{array} \\ \begin{pmatrix} 0 & 0 & 0.5` & 0.5` & 0 \\ 0.5` & 0 & 0.5` & 0 & 0 \\ 0.5` & 0 & 0 & 0.5` & 0 \\ 0 & 0.5` & 0 & 0 & 0.5` \\ 0.5 & 0 & 0 & 0.5 & 0 \end{pmatrix} \end{array};$$

The state transition diagram is




The system is irreducible and finite. Hence is has a stationary distribution. Further it's aperiodic because $1 \to 5 \to 1$ and $1 \to 7 \to 4$:->1. Hence it has a limiting distribution and it is equal to the stationarity distribution.

```
MatrixPower[P, 100] // MatrixForm
```

$$\begin{pmatrix} 0.238` & 0.143` & 0.19` & 0.286` & 0.143` \\ 0.238` & 0.143` & 0.19` & 0.286` & 0.143` \\ 0.238` & 0.143` & 0.19` & 0.286` & 0.143` \\ 0.238` & 0.143` & 0.19` & 0.286` & 0.143` \\ 0.238` & 0.143` & 0.19` & 0.286` & 0.143` \end{pmatrix}$$

So in the long run position 7 will have the most visits and $\frac{143}{500}$ of the time will be spent there.

# Problem d

## What is expected return time to square 1? Why same/different as in a?

Since the stationary distribution $\pi = (0.238, 0.143, 0.19, 0.286, 0.143)$ by Thm 1.21 in Durrett the expected return time is $\frac{1}{0.238} = 4.20168$. This is shorter than the 7 steps computing in **a** because you do not have to “cross” state 9 before returning to state 1.