1. Bayes Theorem is

$$\Pr\left(A_n|B\right) = \frac{\Pr\left(B|A_n\right)\Pr\left(A_n\right)}{\Pr\left(B\right)}$$

If $\{A_n : 1, 2, 3, \dots\}$ is finite or countably infinite partition of a sample space (hence partition function) then $\Pr(B) = \sum_n \Pr(B \cap A_n) = \sum_n \Pr(B \mid A_n)\Pr(A_n)$ and then Bayes' says

$$\begin{aligned}\Pr\left(A_n|B\right) &= \frac{\Pr\left(B|A_n\right)\Pr\left(A_n\right)}{\Pr\left(B\right)} \\ &= \frac{\Pr\left(B|A_n\right)\Pr\left(A_n\right)}{\sum_n \Pr(B \mid A_n)\Pr(A_n)}\end{aligned}$$

In our case $\Pr(A_n|B) = \Pr\left(p_h|n,m\right), \Pr(B|A_n) = \Pr\left(n,m|p_h\right) = \binom{n+m}{m}p_h^m\left(1-p_h\right)^n, \Pr(A_n) = p_h^{\alpha-1}\left(1-p_h\right)^{\beta-1}$ and therefore

$$\begin{aligned}\Pr\left(p_h|n,m\right) &= \frac{\binom{n+m}{m}p_h^m\left(1-p_h\right)^n p_h^{\alpha-1}\left(1-p_h\right)^{\beta-1}}{\int_0^1\binom{n+m}{m}p_h^m\left(1-p_h\right)^n p_h^{\alpha-1}\left(1-p_h\right)^{\beta-1}dp_h} \\ &= \frac{p_h^m\left(1-p_h\right)^n p_h^{\alpha-1}\left(1-p_h\right)^{\beta-1}}{\int_0^1 p_h^{m+\alpha-1}\left(1-p_h\right)^{n+\beta-1}dp_h}\end{aligned}$$

So the task is to integrate $\int_0^1 p_h^{m+\alpha-1}\left(1-p_h\right)^{n+\beta-1}dp_h$.

2. If $X \sim \text{gamma}(r,1)$ and $Y \sim \text{gamma}(s,1)$ and $Z_1 = X+Y$ and $Z_2 = X/(X+Y)$ then $0 < z_1 < \infty$ and $0 < z_2 < 1$ and $X = Z_1Z_2$ and $Y = Z_1 - Z_1Z_2$. Hence

$$\|J\| = \left\|\left|\begin{pmatrix} z_2 & z_1 \\ 1-z_2 & -z_1\end{pmatrix}\right|\right\| = \|-z_1z_2 - z_1(1-z_2)\| = \|z_1\left(-z_2-(1-z_2)\right)\| = z_1$$

Then

$$\begin{aligned}f_{Z_1,Z_2}\left(z_1,z_2\right) &= \frac{1}{\Gamma\left(r\right)}\left(z_1z_2\right)^{r-1}e^{-z_1z_2}\frac{1}{\Gamma\left(s\right)}\left(z_1-z_1z_2\right)^{s-1}e^{-z_1+z_1z_2}z_1 \\ &= \frac{1}{\Gamma\left(r\right)}\left(z_1\right)^r e^{-z_1}\left(z_2\right)^{r-1}\frac{1}{\Gamma\left(s\right)}\left(z_1\right)^{s-1}\left(1-z_2\right)^{s-1} \\ &= \frac{1}{\Gamma(r+s)}\left(z_1\right)^{r+s-1}e^{-z_1}\frac{\Gamma(r+s)}{\Gamma\left(r\right)\Gamma\left(s\right)}\left(z_2\right)^{r-1}\left(1-z_2\right)^{s-1}\end{aligned}$$

Now clearly $f_{Z_1}(z) = \frac{1}{\Gamma(r+s)}\left(z_1\right)^{r+s-1}e^{-z_1}$ is the pdf of a $\Gamma\left(r+s,1\right)$ and therefore and $Z_2 \sim \mathrm{B}(r,s)$.

$$\begin{aligned}1 = \int_0^1\int_0^\infty f_{Z_1,Z_2}\left(z_1,z_2\right)dz_1dz_2 &= \int_0^1\int_0^\infty\left(\frac{1}{\Gamma(r+s)}\left(z_1\right)^{r+s-1}e^{-z_1}\frac{\Gamma(r+s)}{\Gamma\left(r\right)\Gamma\left(s\right)}\left(z_2\right)^{r-1}\left(1-z_2\right)^{s-1}\right)dz_1dz_2 \\ 1 &= \int_0^\infty\left(\frac{1}{\Gamma(r+s)}\left(z_1\right)^{r+s-1}e^{-z_1}dz_1\right)\int_0^1\left(\frac{\Gamma(r+s)}{\Gamma\left(r\right)\Gamma\left(s\right)}\left(z_2\right)^{r-1}\left(1-z_2\right)^{s-1}dz_2\right) \\ 1 &= \int_0^1\left(\frac{\Gamma(r+s)}{\Gamma\left(r\right)\Gamma\left(s\right)}\left(z_2\right)^{r-1}\left(1-z_2\right)^{s-1}dz_2\right)\end{aligned}$$

Therefore the normalization constant c such that $1 = \int_0^1 c\left(z_2\right)^{r-1}\left(1-z_2\right)^{s-1}dz_2$ is $\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}$