

## Homework 4

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### Problem a

How many turns on average does it take to complete the game?

We compute the expected hitting time for state "9". Define the transition matrix  $P$

$$P = \begin{matrix} & \begin{matrix} 1 & 4 & 5 & 7 & 9 \end{matrix} \\ \begin{matrix} 1 \\ 4 \\ 5 \\ 7 \\ 9 \end{matrix} & \begin{pmatrix} 0 & 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

Define  $\tilde{P} = P$  with the exception that  $p(9,9) = 0$

$$\tilde{P} = \begin{pmatrix} 0 & 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Define the hitting time vector  $h$

$$h = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Then computing the expected hitting time is equivalent to solving the matrix equation

$$(I - \tilde{P}) \rho = h$$

for the unknown  $\rho = (\rho_1, \rho_4, \rho_5, \rho_7, \rho_9)^T$  where  $\rho_i$  is the hitting time for hitting state 9 from state  $i$ .

$$\text{Solve}[(\text{IdentityMatrix}[5] - \tilde{P}) \cdot \{\rho_1, \rho_4, \rho_5, \rho_7, \rho_9\} = h, \{\rho_1, \rho_4, \rho_5, \rho_7, \rho_9\}]$$

$$\{\{\rho_1 \rightarrow 0, \rho_4 \rightarrow 0.142857, \rho_5 \rightarrow 0.285714, \rho_7 \rightarrow 0.571429, \rho_9 \rightarrow 1\}\}$$

And so the expected hitting for hitting state 9 from state 1 is  $\rho_5 = 0.285714$ .

### Problem b

What is the probability that a player who has reached the middle

## square will complete the game without slipping back to square 1?

This probability is equivalent to the probability of hitting the states  $\{1,9\}$  with the further condition that state 1 carries probability of zero of “hitting 9 before 1” and state 9 carries a probability of one of “hitting 9 before 1”.

Define  $\tilde{P} = P$  with the exception that  $p(9,9) = p(1,j) = 0$

$$\tilde{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

Define the hitting probability vector  $h$

$$h = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

Then computing the expected hitting time is equivalent to solving the matrix equation

$$(I - \tilde{P}) \rho = h$$

for the unknown  $\rho = (\rho_1, \rho_4, \rho_5, \rho_7, \rho_9)^T$  where  $\rho_i$  is the probability of hitting state 1 or 9. It is  $h$  that encodes the conditions  $\rho_1 = 0$  and  $\rho_9 = 1$ .

`Solve[(IdentityMatrix[5] -  $\tilde{P}$ ).{ $\rho_1, \rho_4, \rho_5, \rho_7, \rho_9$ } == h, { $\rho_1, \rho_4, \rho_5, \rho_7, \rho_9$ }]`

`{{ $\rho_1 \rightarrow 0.$ ,  $\rho_4 \rightarrow 0.142857$ ,  $\rho_5 \rightarrow 0.285714$ ,  $\rho_7 \rightarrow 0.571429$ ,  $\rho_9 \rightarrow 1.$ }}`

So  $\rho_5 = 0.285714$  is the probability that starting at state 5 you hit state 9 without hitting state 1 first.

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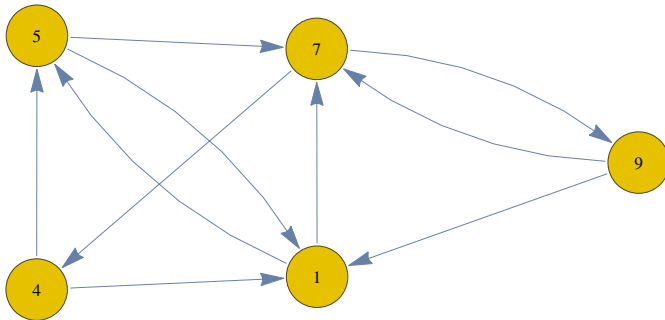
## Problem c

Does this **new** Markov chain have a unique stationary distribution? Why or why not? In the long-run, in which square will we spend the most time, and what fraction of the time will we be there?

The adjusted transition matrix is

$$P = \begin{matrix} & \begin{matrix} 1 & 4 & 5 & 7 & 9 \end{matrix} \\ \begin{matrix} 1 \\ 4 \\ 5 \\ 7 \\ 9 \end{matrix} & \begin{pmatrix} 0 & 0 & 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \end{pmatrix} \end{matrix};$$

The state transition diagram is



The system is irreducible and finite. Hence it has a stationary distribution. Further it's aperiodic because  $1 \rightarrow 5 \rightarrow 1$  and  $1 \rightarrow 7 \rightarrow 4 \rightarrow 1$ . Hence it has a limiting distribution and it is equal to the stationary distribution.

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MatrixPower[P, 100] // MatrixForm
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$$\begin{pmatrix} 0.238 & 0.143 & 0.19 & 0.286 & 0.143 \\ 0.238 & 0.143 & 0.19 & 0.286 & 0.143 \\ 0.238 & 0.143 & 0.19 & 0.286 & 0.143 \\ 0.238 & 0.143 & 0.19 & 0.286 & 0.143 \\ 0.238 & 0.143 & 0.19 & 0.286 & 0.143 \end{pmatrix}$$

So in the long run position 7 will have the most visits and  $\frac{143}{500}$  of the time will be spent there.

## Problem d

What is expected return time to square 1? Why same/different as in a?

Since the stationary distribution  $\pi = (0.238, 0.143, 0.19, 0.286, 0.143)$  by Thm 1.21 in Durrett the expected return time is  $\frac{1}{0.238} = 4.20168$ . This is shorter than the 7 steps computing in a because you do not have to “cross” state 9 before returning to state 1.