Uncountable a.e. Convergence

Let

$$A_m = \bigcup_{k=0}^{3^{m-1}-1} \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right)$$

For $n \geq 1$ set $C_n = [0,1] \setminus \bigcup_{m=1}^{n} A_m$ and $C = \lim_{n\to\infty} C_n$. For $\omega \in [0,1]$ let X_n be the random variable defined on $\Omega = [0,1]$ by

$$X_n(\omega) = \omega + I_{C_n}(\omega)$$

and let X be the random variable defined on $\Omega = [0,1]$ by $X(\omega) = \omega$.

Claim: X_n converges almost surely to X, i.e.

$$P\left(\lim_{n\to\infty} X_n = X \right) = 1$$

Proof: Since A_m are not disjoint we cannot compute the measure of C easily. We reason intuitively that each successive A_m removes from $[0,1]$ an additional 2^{m-1} intervals of length 3^m and therefore

$$\begin{aligned} P([0,1] \setminus C) &= \sum_{m=1}^{\infty} \frac{2^{m-1}}{3^m} \\ &= \frac{1}{2} \sum_{m=1}^{\infty} \left(\frac{2}{3} \right)^m = \frac{1}{2}(2) = 1 \end{aligned}$$

and so $P(C)$ must be 0. By definition of the indicator $X_n \neq X$ only on C_n. But $P(C)$ being 0 is tantamount to the set on which $X_n \neq X$ for infinitely many n having measure 0. Therefore

$$P\left(\lim_{n\to\infty} X_n \neq X \right) = 0$$

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Claim: C is uncountable.

Proof: Express all the numbers in the interval $[0,1]$ in their ternary representation. "Removing middle thirds" from each successive C_n corresponds to removing all numbers which have a digit of 1 in the nth place after the decimal, in their ternary expansion. Therefore the only numbers remaining in C are those which admit a ternary representation consisting solely of 0s and 2s (terminating decimals, which have 2 ternary representations, are not a contradiction). The mapping "replace all 2s with 1s" maps C onto the interval $[0,1]$.

Note the mapping is not into since

$$\frac{7}{9} = .20222\ldots_3 \to 0.10111\ldots_2 = 0.11_2 \leftarrow .220000\ldots = \frac{8}{9}$$

but by Cantor-Berstein we have $|[0,1]| \leq |C| \leq |[0,1]|$ and the claims follows .
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