Homework 1

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Problem 1(a). Compute the transition probability matrix for X.

Solution.

Let X_n be the number of white balls in the left urn at time n. The state space has dimension 6, ranging from 0 white balls in the left urn to all 5 of the white balls in the left urn. We demonstrate how to compute the row entries for rows 0 and 1; the arguments for rows 2, 3 and 4 are similar to that of row 2 and row 5 is similar to that of row 1.

If there are 0 white balls in the left urn then all of the black balls are in the left urn. Hence the left urn will transition to containing 1 white ball w.p. 1 and all other transitions have probability 0. Transitions from state 5 are the obverse. If there is 1 white ball in the left urn then there are 4 black balls in the left urn, 1 black ball in the right urn, and 4 white balls in the right urn. The probability of transitioning to state 0 is that of choosing the one white ball in the left urn and choosing the one black ball in the right urn,

$$\frac{1}{5} \times \frac{1}{5} = \frac{1}{25}.$$

The probability of transitioning to the state wherein the left urn contains 2 balls is the complement of choosing the exceptional ball in each urn,

$$\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}.$$

The probability of returning to state 1, from state 1, is the sum of the probabilities of swapping white balls or black balls between both urns,

$$\frac{4}{5} \times \frac{1}{5} + \frac{1}{5} \times \frac{4}{5} = \frac{8}{25}.$$

Hence the transition matrix is

$$\frac{1}{25}\begin{pmatrix} & 25 & & & & \\ 1 & 8 & 16 & & & \\ & 4 & 12 & 9 & & \\ & & 9 & 12 & 4 & \\ & & & 16 & 8 & 1 \\ & & & & 25 & \end{pmatrix}$$

Problem 1(b). Does this Markov chain have a stationary distribution? If so, compute it.

Solution.

The Markov chain is irreducible because all of the classes communicate (and hence positive recurrent but that's irrelevant presently). It's aperiodic, to wit ergodic, manifestly so because state 2 can transition to state 2 with probability .32 in 1 step, hence it has a unique stationary distribution. To find π we set up the linear system $\pi = \pi\mathbf{P}$ with the added constraint that $\sum_{j\in S}\pi_j = 1$ and solve for π. To effect the constraint we can replace the last column in the transition matrix with 1s and replace the last entry in the output vector also with a 1.

$$\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ 1\end{pmatrix} = \frac{1}{25}\begin{pmatrix}\pi_0\\ \pi_1\\ \pi_2\\ \pi_3\\ \pi_4\\ \pi_5\end{pmatrix}\begin{pmatrix} & 25 & & & & 1\\ 1 & 8 & 16 & & & 1\\ & 4 & 12 & 9 & & 1\\ & & 9 & 12 & 4 & 1\\ & & & 16 & 8 & 1\\ & & & & 25 & 1\end{pmatrix}$$

Using the Mathematica code :

```
P = (1/25 {{0, 25, 0, 0, 0, 25},
           {1, 8, 16, 0, 0, 25},
           {0, 4, 12, 9, 0, 25},
           {0, 0, 9, 12, 4, 25},
           {0, 0, 0, 16, 8, 25},
           {0, 0, 0, 0, 25, 25}})

p = {p0, p1, p2, p3, p4}

Solve[p.P == {p0, p1, p2, p3, 1}]
```

yields $\pi = \left(\frac{1}{252}, \frac{25}{252}, \frac{25}{63}, \frac{25}{63}, \frac{25}{252}, \frac{1}{252}\right)$

Problem 2(a). What is the state space of this Markov Chain and what is the transition matrix P?

Solution.

The state space is $\{0, 1, 2, 3, 4\}$ newspapers in the pile in the evening. It is impossible for there to be five or more papers in in the pile because Mr. Smith would have recycled them in the afternoon. The transition matrix P is

$$\begin{pmatrix} .33 & .66 & & & \\ .33 & & .66 & & \\ .33 & & & .66 & \\ .33 & & & & .66 \\ 1 & & & & \end{pmatrix}$$

Problem 2(b). What is the probability there are 0 papers in the pile if 2 evenings ago there were 3?

Solution.

This is calculated by applying P^2 to the initial distribution $(0,0,0,1,0)$. P^2 is computed using the Mathematica code :

```
P = ({
      {.33, .66,    0,   0,   0},
      {.33,   0,  .66,   0,   0},
      {.33,   0,    0, .66,   0},
      {.33,   0,    0,   0, .66},
      {  1,   0,    0,   0,   0}
     })

MatrixPower[P,2]
```

Hence

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} .33 & .66 & & & \\ .33 & & .66 & & \\ .33 & & & .66 & \\ .33 & & & & .66 \\ 1 & & & & \end{pmatrix}^2 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0.33 & 0.22 & 0.44 & 0. & 0. \\ 0.33 & 0.22 & 0. & 0.44 & 0. \\ 0.33 & 0.22 & 0. & 0. & 0.44 \\ 0.77 & 0.22 & 0. & 0. & 0. \\ 0.33 & 0.66 & 0. & 0. & 0. \end{pmatrix} \\ &= \begin{pmatrix} .77 & .23 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So there is a .77 probability that there 0 papers in the pile on the second evening.

Problem 2(c). Argue, via numerical experimentation, whether there exists a limiting distribution of papers in the pile.

Solution.

Using the Mathematica code

```
P = ({
      {.33, .66,   0,   0,   0},
      {.33,   0, .66,   0,   0},
      {.33,   0,   0, .66,   0},
      {.33,   0,   0,   0, .66},
      {   1,   0,   0,   0,   0}
     })

MatrixPower[P,1000]
```

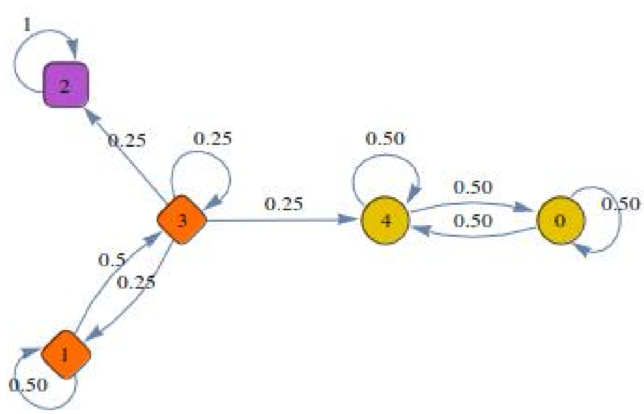
generates the asymptotic transition matrix

$$\begin{pmatrix} 0.38 & 0.26 & 0.17 & 0.11 & 0.08 \\ 0.38 & 0.26 & 0.17 & 0.11 & 0.08 \\ 0.38 & 0.26 & 0.17 & 0.11 & 0.08 \\ 0.38 & 0.26 & 0.17 & 0.11 & 0.08 \\ 0.38 & 0.26 & 0.17 & 0.11 & 0.08 \end{pmatrix}$$

So the limiting distribution is $(0.38, 0.26, 0.17, 0.11, 0.08)$.

Problem 3(a). Draw a flow diagram of this Markov Chain and classify all of its states as stationary or recurrent.

Solution.



States 1 and 3 are transient. To see this note that if $X_0 = 3$ then there is a .25 probability of the chain going to state 2 and being absorbed, so $P_3(T_1 = \infty) \geq p(3,2) = .25 > 0$. Further since state 1 is in the same communicating class as state 3 it too is transient (because transience is a class property). State 3 is trivially recurrent. State 4 and 0 are recurrent because they comprise an irreducible closed class.

Problem 3(b). Find all of its stationary distributions.

Solution.

There are 3 stationary distributions. One for each closed recurrent class and one that is a linear combination of them that represents starting in the transient class, with coefficients being the relative probabilities of transitioning to either recurrent class from the transient class.

We can find the stationary distributions of the recurrent classes by performing eigenvalue decomposition and identifying the eigenvectors corresponding to the eigenvalue 1. To compute the left eigenvectors of the transition matrix we compute the right eigenvectors of the transpose. Using the Mathematica code

```
P = ({
      {.5,    0,     0,    0,   .5},
      { 0,   .5,     0,   .5,    0},
      { 0,    0,     1,    0,    0},
      { 0,  .25,   .25,  .25,  .25},
      {.5,    0,     0,    0,   .5}
     })

Eigensystem[Transpose[P]]
```

we get that the pertinent eigen-row-vectors are $(0, 0, 1, 0, 0)$ and $(0.654, 0, 0.382, 0, 0.654)$. Orthonormalizing yields $(0, 0, 1, 0, 0)$ and $(0.5, 0, 0, 0, 0.5)$. There is one more stationary distribution, that due to starting in the transient class and evolving to one of the recurrent classes. It is the linear combination of both of the 2 aforementioned eigenvectors with weights equal to the probability associated with ending up in that recurrent class, hence by inspection of the diagram, it is

$$\frac{.25}{.25 + .25}(0, 0, 1, 0, 0) + \frac{.25}{.25 + .25}(0.5, 0, 0, 0, 0.5) = (.25, 0, .5, 0, .25)$$

Problem 3(c). Suppose we start in state A. In the long run, what percentage of the time do we spend in state A?

Solution.

50% because A is in a closed recurrent class and the limiting distribution for that recurrent class dictates that transitioning from $A \to A$ and transitioning $A \to D$, and vice-versa, each happen with probability .5 .

Problem 3(d). Suppose we start in state C. In the long run, what percentage of the time do we spend in state A?

Solution.

0% because C is in a closed recurrent class of its own.

Problem 3(e). Suppose we start in state B. In the long run, what percentage of the time do we spend in state A?

Solution.

.25 because there are 2 recurrent classes, that of $\{C\}$ and that of $\{A, E\}$ and the chance of transitioning from B's class to A's is relatively .5 and then the chance of being in A after transitioning to A's class is again .5. So the composite probability is $.5 \times .5$.

Problem 3(f). Why did I have to change the wording of the question in the last item?

Solution.

Because there is not a unique limiting distribution so there is a non-vanishing probability that the Markov chain doesn't end up in A at all. So speaking of the proportion of time spent in A is semantically incorrect.

Problem 5. Prove that if we make the random walk lazy, in the sense that we always give the walker a probability q of not moving, then π is also a stationary distribution of the lazy random walk.

Solution.

Let π be the limiting distribution of the strict random walk, i.e. $\pi\mathbf{P} = \pi$. The "lazy" random walk is related to the strict random walk as such: $\mathbf{P}_\ell = \mathbf{P}(1-q) + \mathbf{I}q$ where I is the identity matrix. This is the case because if the chain does not transition with probability q then it *does* transition with probability $(1-q)$ and whether it transition up or down is conditional on it transitioning at all, hence the "step" probabilities are $p_{ij} \times (1-q)$. That π is a limiting distribution follows immediately: $\pi\mathbf{P}_\ell = \pi\mathbf{P}(1-q) + \pi\mathbf{I}q = \pi(1-q) + \pi q = \pi$.