Homework 1

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Problem 2.31(a). Let $T \sim Exp(\lambda)$. Compute $E(T|T < c)$.

Solution.

By definition

$$E(T|T < c) = \int_0^c t\mathbb{P}(T = t|T < c)dt$$

But $\mathbb{P}(T = t|T < c)$ is the truncated distribution whose probability density function f_T is

$$f_T(t) = \frac{g_T(t)}{F_T(c)}$$

where g_T is the probability density function for T, $\lambda e^{-\lambda t}$, and F_T is the cumulative distribution function for T, $1 - \lambda e^{-\lambda t}$. Hence

$$\begin{aligned} E(T|T < c) &= \int_0^c t\mathbb{P}(T = t|T < c)dt = \int_0^c t\frac{\lambda e^{-\lambda t}}{1 - \lambda e^{-\lambda c}}dt \\ &= \frac{1}{\lambda} + \frac{c}{1 - e^{c\lambda}} \end{aligned}$$

Problem 2.31(b). Let $T \sim Exp(\lambda)$. Compute $E(T|T < c)$.

Solution.

From $ET = \mathbb{P}(T < c)E(T|T < c) + \mathbb{P}(T > c)E(T|T > c)$ we have

$$E(T|T < c) = \frac{ET - \mathbb{P}(T > c)E(T|T > c)}{\mathbb{P}(T < c)}$$

but

$$E(T|T > c) = \int_0^c dt + \int_c^\infty \mathbb{P}(T > t|T > c)dt = c + \int_c^\infty \mathbb{P}(T > t - c)dt$$

by the tail formula for expectation and the memoryless property of the exponential distribution. Finally

$$\int_c^\infty \mathbb{P}(T > t-c)dt = \int_c^\infty e^{-\lambda(t-c)}dt = \frac{1}{\lambda}$$

Hence

$$\begin{aligned} E(T|T<c) &= \frac{\frac{1}{\lambda} - e^{-\lambda c}\left(\frac{1}{\lambda}+c\right)}{1-e^{-\lambda c}} \\ &= \frac{1}{\lambda} - \frac{e^{-\lambda c}}{1-e^{\lambda c}}c = \frac{1}{\lambda} + \frac{c}{1-e^{\lambda c}} \end{aligned}$$

Problem 2.33. Suppose traffic on a road is accurately modeled by a Poisson process with rate parameter $\lambda \frac{cars}{minute}$ and a chicken needs c minutes to cross the road. Show that the expected time for the chicken to cross, including wait, is $(e^{\lambda c}-1)/\lambda$.

Solution.

Let t_i be times between passages of cars and $J = \min\{j : t_j > c\}$. Then t_i is exponentially distributed for all i, and J is geometrically distributed with success probability $p = \mathbb{P}(T|T > c) = e^{-\lambda c}$ and failure probability $1-p = \mathbb{P}(T|T<c) = 1-e^{-\lambda c}$. The expectation value of J is

$$\frac{1}{1-p} = \frac{1}{e^{-\lambda c}} = K$$

The expected passage time for each car that passes in less than c minutes is $E(T|T<c)$ and by the previous problem

$$E(T|T<c) = \frac{1}{\lambda} + \frac{c}{1-e^{\lambda c}}$$

Hence total wait is that of waiting for $K-1$ cars to pass and then crossing:

$$\left(\frac{1}{e^{-\lambda c}} - 1\right)\left(\frac{1}{\lambda} + \frac{c}{1-e^{\lambda c}}\right) = \frac{e^{\lambda c}}{\lambda} - \frac{1+c\lambda}{\lambda} + c = \frac{e^{\lambda c}-1}{\lambda}$$

Problem 2.52(a). How often is the bulb replaced.

Solution.

The janitor replacing the bulb according to when it breaks is a Poission process with rate $\frac{1}{200}\frac{failures}{day}$. The superposition and the handyman-preventitive-maintenance Poisson process with rate $\frac{1}{100}\frac{replacements}{day}$ is again a Poisson process with rate $\frac{1}{200} + \frac{1}{100} = \frac{3}{200}$ which implies that the lightbulb is changed once every $\frac{200}{3} = 66\frac{2}{3}$ days.

Problem 2.52(b). In the long run what fraction of replacements is due to failure?

Solution.

The rate of replacement due to failure is the relative rate $\frac{.005}{.005+.01} = \frac{1}{3}$.

Problem 2.58(a). Compute $\mathbb{P}(N(2) = 5)$.

Solution.

$$\mathbb{P}(N(2) = 5) = \mathbb{P}(N(0+2) - N(0) = 5) = \frac{e^{-2\times 2}(2\times 2)^5}{5!} \sim 0.156$$

Problem 2.58(b). Compute $\mathbb{P}(N(5) = 8|N(2) = 3)$.

Solution.

$$\mathbb{P}(N(5) = 8|N(2) = 3) = \mathbb{P}(N(3) - N(0) = 5)$$

by the memoryless property of the Poisson distribution.

$$\mathbb{P}(N(0+3) - N(0) = 5) = \frac{e^{-2\times 3}(2\times 3)^5}{5!} \sim 0.161$$

Problem 2.58(c). Compute $\mathbb{P}(N(2) = 3|N(5) = 8)$.

Solution.

$$\begin{aligned}\mathbb{P}(N(2) = 3|N(5) = 8) = \binom{8}{3}\left(\frac{2}{5}\right)^3\left(1-\frac{2}{5}\right)^{8-3} \\ \sim 0.279\end{aligned}$$