Homework 2

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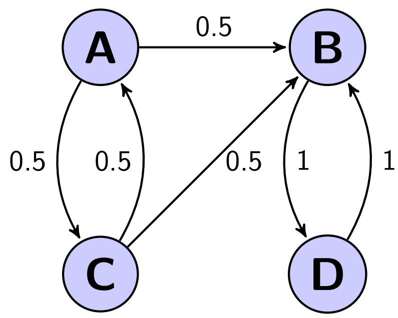
Problem 1. Using one-step analysis, compute ρ_{AA} for an unbiased random walk on the following graph:



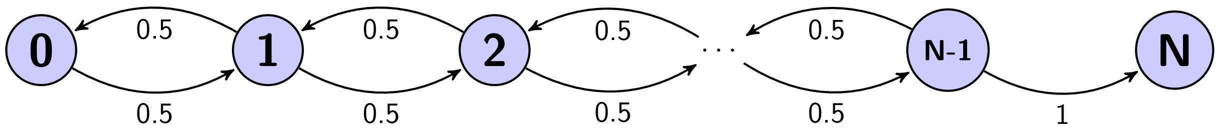
Solution.

$$\rho_{AA} = \rho_{BA} \cdot p(A,B) + \rho_{CA} \cdot p(A,C) = \rho_{BA} \cdot \frac{1}{2} + \rho_{CA} \cdot \frac{1}{2}$$
$$\rho_{CA} = \rho_{AA'} \cdot p(C,A) + \rho_{BA} \cdot p(C,B) = 1 \cdot \frac{1}{2} + \rho_{BA} \cdot \frac{1}{2}$$
$$\rho_{BA} = \rho_{DA} \cdot p(B,D) = 0 \cdot 1 = 0$$

Note that ρ_{AA} and $\rho_{AA'}$ are different; ρ_{AA} is the probability of hitting A for some time $n \geq 1$ and $\rho_{AA'}$ is the probability of hitting A for sometime $n \geq i$ given that $X_i = A$. Hence

$$\rho_{CA} = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}$$
$$\rho_{AA} = 0 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Problem 2. Compute ρ_{00} for an unbiased random walk on the following graph



Solution. We claim that

$$\rho_{00} = 1 - \frac{1}{M-1}$$

where M is the number of nodes, or $1 - \frac{1}{N}$ if we abide by the numbering scheme above.

Proof. For arbitrary state x, with the exception of ρ_{00} and ρ_{N0},

$$\rho_{x0} = \rho_{(x-1)0} \cdot \frac{1}{2} + \rho_{(x+1)0} \cdot \frac{1}{2}.$$

Summing ρ_{x0} and $\rho_{(x+1)0}$ we get

$$\rho_{x0} + \rho_{(x+1)0} = \rho_{(x-1)0} \cdot \frac{1}{2} + \rho_{(x+1)0} \cdot \frac{1}{2} + \rho_{(x)0} \cdot \frac{1}{2} + \rho_{(x+2)0} \cdot \frac{1}{2}.$$

Rearranging, combining like terms, and cancelling the common factor of $\frac{1}{2}$ yields

$$\rho_{x0} - \rho_{(x-1)0} = \rho_{(x+2)0} - \rho_{(x+1)0} \ .$$

Hence

$$\rho_{20} - \rho_{10} = \rho_{40} \ - \rho_{30} = \ldots = \rho_{N0} - \rho_{(N-1)0}.$$

But $\rho_{N0} = 0$, because it's an absorbing state. Hence

$$\rho_{20} - \rho_{10} = \rho_{40} \ - \rho_{30} = \ldots = -\rho_{(N-1)0}$$

$$\rho_{10} - \rho_{20} = \rho_{30} - \rho_{40} = \ldots = \rho_{(N-1)0}.$$

This shows that ρ_{x0} decreases to zero in increments of $\rho_{(N-1)0}$. Let $\rho_{(N-1)0} = K$ for some yet to be determined constant K and M be the number of nodes. Hence $\rho_{10} = (M-2) \cdot K$ ($M-2$ incremental "subtractions" occur state 1 and state M). But then

$$\rho_{10} = (M-2) \cdot K = \rho_{00'} \cdot \frac{1}{2} + \rho_{20} \cdot \frac{1}{2} = 1 \cdot \frac{1}{2} + \rho_{20} \cdot \frac{1}{2}$$

where $\rho_{00'} = 1$ because $\rho_{00'}$ is the probability of hitting 0 for sometime $n \geq i$ given that $X_i = 0$. Since similarly $\rho_{20} = (M-3) \cdot K$ it follows that

$$(M-2) \cdot K = 1 \cdot \frac{1}{2} + \frac{1}{2}(M-3) \cdot K$$

$$K = \frac{1}{M-1}$$

Hence $\rho_{10} = (M-2)\frac{1}{M-1} = 1 - \frac{1}{M-1}$ and

$$\rho_{00} = \rho_{10} = 1 - \frac{1}{M-1} = 1 - \frac{1}{(N+1)-1} = 1 - \frac{1}{N}$$

. $\square$

For the instances where $M = 2, 3, 4$ (corresponding to $N = 1, 2, 3$) $\rho_{00} = 0, \frac{1}{2}, \frac{2}{3}$. In the limit as $N \to \infty$ the probability of returning is clearly 1.

Numerical computation confirms this:

`randomwalk.py`

```
import sys
from random import choice

direction = [1,-1]
returns = 0.
k = int(sys.argv[2])
n = int(sys.argv[1])
for i in range(k):
  step = 1
  while step != 0 and step !=n:
    step += choice(direction)
  if(step == 0):
    returns += 1

print returns/k
```

```
  $ python randomwalk.py 1 10000
  $ 0.0
  $ python randomwalk.py 2 10000
  $ 0.4986
  $ python randomwalk.py 3 10000
  $ 0.6733
  $ python randomwalk.py 10 10000
  $ 0.9017
```