COLLARD GREEN'S FUNCTIONS

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1. Motivation

Green's functions are a way to solve certain PDEs. Consider a PDE

$$\frac{\partial u}{\partial t} - \frac{D}{2}\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - \frac{D}{2}\frac{\partial^2}{\partial x^2}\right) \tag{1.1}$$

$$= Lu \tag{1.2}$$

$$u(x,0) = f(x)$$

as a *linear system*[1] L with *input* $f(x)$ and *response* u. This is appropriate because $u(x,t)$ is completely characterized[2] by L and $f(x)$. The *Green's function* $G(x,t)$ of the system is the solution that satisfies

$$\lim_{t\downarrow 0} G(x,t) = \delta(x) \tag{1.3}$$

Note that this is the *impulse response* of the system because it is the response/solution of the system whose input/initial conditions $u(x,0) = G(x,0)$ are in a sense[3] the unit impulse $\delta(x)$. Why is the Green's function useful? For arbitrary[4] input $f(x)$

$$\begin{aligned} L\left(\int_{-\infty}^{\infty} G(s-x,t) f(s)\, ds\right) &= \int_{-\infty}^{\infty} [LG(s-x,t)] f(s)\, ds \\ &= \int_{-\infty}^{\infty} \left[\frac{\partial G}{\partial t} - \frac{D}{2}\frac{\partial^2 G}{\partial x^2}\right] f(s)\, ds \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{t\downarrow 0}\left(\int_{-\infty}^{\infty} G(s-x,t) f(s)\, ds\right) &= \int_{-\infty}^{\infty} \left[\lim_{t\downarrow 0} G(s-x,t)\right] f(s)\, ds \\ &= \int_{-\infty}^{\infty} \delta(s-x) f(s)\, ds \qquad (1.4) \\ &= f(x) \end{aligned}$$

So

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} G(s-x,t) f(s)\, ds \\ &= G(x,t) \star f(x) \end{aligned}$$

is a general solution of the system defined by eqns. 1.2.

2. Green's function for the Diffusion equation

The PDE in eqn. 1.1 is called the *Diffusion* equation and the system I'll call a *Diffusion system*[5]: it diffuses the initial concentration of mass $f(x)$ as time evolves. We seek a general solution to the system and therefore we seek the Green's function $G(x,t)$ for the system. For reasons that will become clearer in the 3rd section

$$G(x,t) = \frac{1}{t^\alpha}\phi\left(\frac{x}{t^\alpha}\right) \tag{2.1}$$

[1] L is a linear differential operator (maps functions to functions).

[2] Kind of obvious because that's the only thing given.

[3] In what sense? In the sense of eqn. 1.3.

[4] In fact $f(x)$ can be very ugly, e.g. unbounded, non-differentiable, etc. All that is required is some regularity conditions that satisfy the hypotheses of Lebesgue's dominated convergence theorem.

[5] This is nonstandard. Typically this is just called a Diffusion boundary value problem with Dirichlet boundary conditions (the values of the solution are specified as opposed to the values of the derivatives, which is a Neumann boundary condition).

for any smooth[6] and integrable function $\phi(y)$ is a good guess for the form of a Green's function. Indeed we will construct G by finding a suitable ϕ. Substituting this G into the diffusion equation

$$\frac{\partial G}{\partial t} - \frac{D}{2}\frac{\partial^2 G}{\partial x^2} = \left(-\frac{\alpha}{t^{\alpha+1}}\phi - \frac{\alpha x}{t^{2\alpha+1}}\phi'\right) - \frac{D}{2}\left(\frac{1}{t^{3\alpha}}\phi''\right) \tag{2.2}$$

To simplify the notation a little define $\eta(x) = \phi\left(\frac{x}{t^\alpha}\right)$. Then

$$\begin{aligned} \eta'(x) &= \frac{1}{t^\alpha}\phi'(x) \\ \eta''(x) &= \frac{1}{t^{2\alpha}}\phi''(x) \end{aligned}$$

and eqn. 2.2 becomes

$$\left(-\frac{\alpha}{t^{\alpha+1}}\eta(x) - \frac{\alpha x}{t^{\alpha+1}}\eta'(x)\right) - \frac{D}{2}\left(\frac{1}{t^\alpha}\eta''(x)\right) = 0$$

or (by multiplying both sides by $-t^\alpha/(D/2)$ and moving $\eta''(x)$ over)

$$\frac{\alpha}{(D/2)\,t}\eta(x) + \frac{\alpha x}{(D/2)\,t}\eta'(x) = -\eta''(x) \tag{2.3}$$

This is now a linear second order ordinary differential equation that's easy to solve. The first trick is recognizing the left side is an exact differential, i.e.

$$\frac{\alpha}{(D/2)\,t}\eta(x) + \frac{\alpha x}{(D/2)\,t}\eta'(x) = \frac{\alpha}{(D/2)\,t}\frac{d}{dx}(x\eta(x))$$

and hence eqn. 2.3 can be integrated once easily

$$\begin{aligned} \frac{\alpha}{(D/2)\,t}\int\frac{d}{dx}(x\eta(x)) &= -\int\eta''(x)\,dx \\ \frac{\alpha}{(D/2)\,t}x\eta(x) &= -n'(x) + c_1 \end{aligned}$$

This again is a linear first order ordinary differential equation more commonly written

$$n'(x) + \frac{\alpha x}{(D/2)\,t}\eta(x) = c_1$$

which is solved by a similar sort of trick. The left side is almost an exact differential[7] except the x spoils it. We can hack it to indeed be an exact differential by multiplying both sides by some function $h(x)$

$$[h(x)]\,n'(x) + \left[h(x)\frac{\alpha x}{(D/2)\,t}\right]\eta(x) = h(x)\,c_1 \tag{2.4}$$

such that the second term becomes the first derivative of h, i.e.

$$h(x)\,n'(x) + h'(x)\,\eta(x) = \frac{d}{dx}(h(x)\,\eta(x))$$

Which function has the property that it's first derivative is equal to itself times $\alpha x/(D/2)\,t$? Well that's just another[8] differential equation in disguise!

$$\frac{dh}{dx} = h\cdot\frac{\alpha x}{(D/2)\,t} \Rightarrow \frac{dh}{h} = dx\frac{\alpha x}{(D/2)\,t} \Rightarrow \log(h) = \frac{\alpha}{Dt}x^2$$

or $h(x) = e^{\alpha x^2/Dt}$. So substituting h into eqn. 2.4

$$\frac{d}{dx}\left(e^{\alpha x^2/Dt}\eta(x)\right) = e^{\alpha x^2/Dt}c_1$$

and finally

$$\begin{aligned} e^{\frac{\alpha x^2}{Dt}}\eta(x) &= c_1\int e^{\alpha x^2/Dt}dx + c_2 \\ &\text{or} \\ \phi\left(\frac{x}{t^\alpha}\right) = \eta(x) &= c_1 e^{-\frac{\alpha x^2}{Dt}}\int e^{\alpha x^2/Dt}dx + c_2 e^{-\frac{\alpha x^2}{Dt}} \end{aligned}$$

Now to reconcile that $\phi\left(\frac{x}{t^\alpha}\right)$ should be a function of only $\frac{x}{t^\alpha}$ we need to pick the appropriate α. Inspect that for $\alpha = 1/2$

$$\phi\left(\frac{x}{\sqrt{t}}\right) = c_1 e^{-\frac{1}{2D}\left(\frac{x}{\sqrt{t}}\right)^2}\int e^{\frac{1}{2D}\left(\frac{x}{\sqrt{t}}\right)^2}dx + c_2 e^{-\frac{1}{2D}\left(\frac{x}{\sqrt{t}}\right)^2}$$

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[6]At least C^1, i.e. the first derivative exists.

[7]Notice that the first term has a first derivative of η and the second has just an η.

[8]It's DEs all the way down.

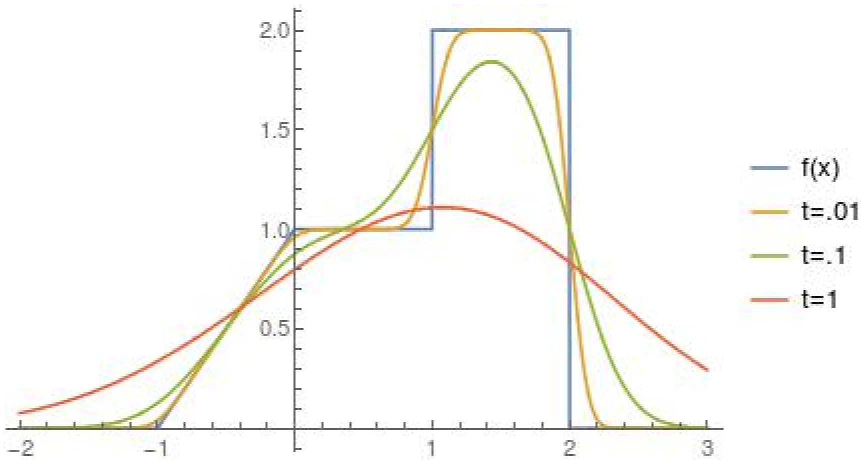

FIGURE 3.1. Smoothing

So $\phi\left(\frac{x}{\sqrt{t}}\right)$ is the Green's function of the diffusion equation. Well almost. The regularity conditions mentioned in footnote 4 require that $\phi \to 0$ as $x \to \infty$ and for the calculation in eqns. 1.4 to work G should be normalized to integrate to 1. To satisfy the first requirement it's clear that c_1 should be 0. To meet the second requirement we set c_2:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} G(x,t)\,dx \\
&= c_2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{1}{2D}\left(\frac{x}{\sqrt{t}}\right)^2} dx \\
&= c_2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{1}{2}\left(\frac{x}{\sqrt{Dt}}\right)^2} dx \\
&\quad \text{let } u = x/\sqrt{Dt} \\
&= c_2\sqrt{D} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du \\
&= c_2\sqrt{2D\pi}
\end{aligned}$$

Therefore the normalization factor $c_2 = 1/\sqrt{D\pi}$ and the complete Green's function is

$$G(x,t) = \frac{1}{\sqrt{2\pi D t}} e^{-\frac{1}{2}\frac{x^2}{Dt}} \tag{2.5}$$

Indeed a very recognizable function! And hence the general solution to the diffusion equation is

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} G(s-x,t)\, f(s)\,ds \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{1}{2}\frac{(s-x)^2}{Dt}} f(s)\,ds
\end{aligned}$$

3. SMOOTHING

A Green's function is a smoother[9] and $u(x,t)$ is the smoothed version of $f(x)$. Figure 3.1 shows initial conditions $f(x)$ for

$$f(x) = \begin{cases} 0 & x < -1 \\ x+1 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \\ 2 & 1 \le x < 2 \\ 0 & 2 \le x \end{cases}$$

and $G(x,t)$ convolved with $f(x)$ for $t = .01, .1, 1$, i.e. the solution $u(x,t)$ to the diffusion equation at those times. As you can see as t increases the initial distribution of mass $f(x)$ is diffused out **and** the points where $f(x)$ is nondifferentiable[10] vanish, i.e. $u(x,t)$ is differentiable at those points. Actually $u(x,t)$ is C^∞ for any $t > 0$, so $f(x)$ is instaneously smoothed out. How

[9]And convolution is a smoothing process.

[10]$x = -1, 0, 1, 2$.

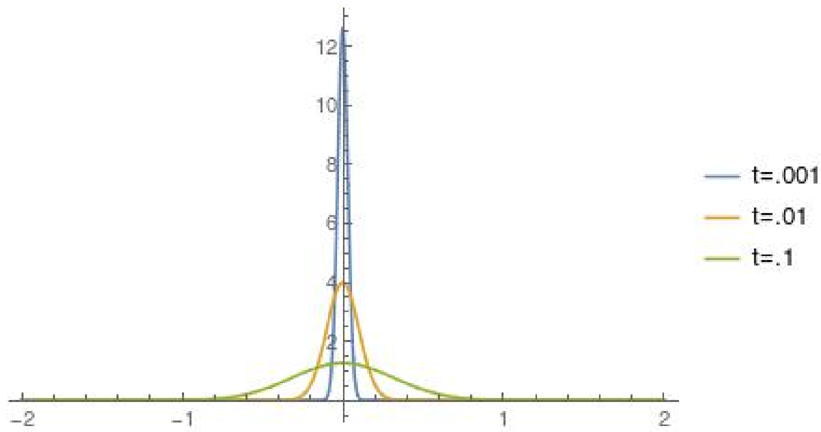

FIGURE 3.2. Gaussian progresion for $t = .001, .01, .1$

smooth? Perfectly smooth:

$$\begin{aligned} \frac{\partial^n}{\partial x^n} u(x,t) &= \int_{-\infty}^{\infty} \left(\frac{\partial^n}{\partial x^n} G(s-x,t) \right) f(s)\, ds \\ &= \int_{-\infty}^{\infty} (-1)^n G^{(n)}(s-x,t) f(s)\, ds \end{aligned}$$

where $G^{(n)}$ denotes the nth partial deriviative of G with respect to its first argument. And since $G(x,t)$ is C^{∞}[11] this integral converges for all n. Even more shockingly $u(x,t)$ is non-zero everywhere on $\mathbb{R}$ for any $t > 0$, so $f(x)$ is diffused everywhere instaneously.

Why is that eqn. 2.1

$$G(x,t) = \frac{1}{t^{\alpha}} \phi\left(\frac{x}{t^{\alpha}}\right)$$

is a good guess for the form a Green's function? First of all first of all since ϕ is integrable we can normalize it such that

$$\int_{-\infty}^{\infty} \phi(y)\, dy = 1$$

and then by a change of variables

$$\int_{-\infty}^{\infty} \frac{1}{t^{\alpha}} \phi\left(\frac{x}{t^{\alpha}}\right) dx = 1$$

which as already mentioned is necessary for the calcuation in eqns. 1.4 to work.

But more intuitively eqn. 2.1 is the right form for a Green's function because it has the behavior of a smoother as $t \to 0$ and as $t \to \infty$. As $t \to 0$ the the factor of $1/t^{\alpha}$ increases the value of $G(x,t)$ around $x = 0$ and shrinks the base because the $1/t^{\alpha}$ in the argument of ϕ functions as a scale parameter[12]. So initially (at $t \approx 0$) a G of this form will preserve the initial distribution of mass $f(x)$. As $t \to \infty$ the inverse effect on $G(x,t)$ is observed: the factor $1/t^{\alpha}$ will flatten $G(x,t)$ and therefore spread/smear/smooth out $f(x)$. It's also critical that $G(x,t)$ integrates to 1 because otherwise it would add mass to the initial distribution[13]and that's not. My point here is that if you wanted to construct a smoothing function de-noveau you would want these properties and the $1/t^{\alpha}$ trick would be an easy way to effect them.. Figure 3.2 shows what the Green's function for the diffusion equation, eqn. 2.5, looks like as t increases.

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[11]The nth derivative of $G(x,t)$ is

$$\left(\frac{-1}{\sqrt{2Dt}}\right)^n H_n\left(\frac{x}{\sqrt{2Dt}}\right) G(x,t)$$

where H_n is the nth Hermite polynomial defined by

$$H_n(x) = \left(2x - \frac{d}{dx}\right)^n \cdot 1$$

The 1 is necessary because the differential operator must be applied to a function.

[12]And the scale increases, is made coarser, as $t \to 0$.

[13]You have to look at G in Fourier space to rigorously define and prove this notion. Suffice it to say that if G didn't integrate to 1, by Parseval's theorem, it wouldn't just redistribute power amongst the frequency components of $f(x)$, it would add power too.