ESI 6420 : Fundamentals of Mathematical Programming

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Fall 2015

Homework 3, due in class, Wednesday October 21$^{\text{th}}$ 2015

Preliminaries

Throughout the semester, I would like you to learn how to use latex (which you will find helpful in the future). In particular, for homework i, I would like you to latex at least $\lfloor \frac{i}{2} \rfloor$ of your answers. For an introduction to latex, refer to `http://www.ctan.org/tex-archive/info/gentle/gentle.pdf`.

When the question you answer involves GAMS, please include your GAMS code, together with the relevant part of the GAMS output. In particular, optimal solutiona and optimal values should be described clearly.

If you received some help to obtain the solution of a problem, you should acknowledge the source of help you received. In particular, for each question, I would like you to cite, if applicable, any book (other than the textbook) you consulted, any website you searched, or any individual you cooperated with. This information will not be used to adjust your homework score provided that help is limited to a reasonable portion of the homework.

Finally, I would like you to candidly assess the number of hours it took you to complete the homework.

Problem 1: About convex sets, BV 2.10 and 2.12, pg61-62, modified

Which of the following sets are convex? Prove or disprove.

1. The set of points closer to one set than another, i.e.
$$\left\{ x \in \mathbb{R}^n \;\middle|\; dist(x,S) \le dist(x,T) \right\}$$
where $S, T \subseteq \mathbb{R}^n$, and
$$dist(x,S) = \inf \left\{ ||x - z||_2 \;\middle|\; z \in S \right\}.$$

2. The set $\left\{ x \in \mathbb{R}^n \;\middle|\; x + S_2 \subseteq S_1 \right\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

3. The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, i.e., the set
$$\left\{ x \in \mathbb{R}^n \;\middle|\; ||x - a||_2 \le \theta ||x - b||_2 \right\}.$$
You can assume $a \ne b$ and $0 \le \theta \le 1$.

4. The set of points
$$\left\{ x \in \mathbb{R}^n \;\middle|\; x^\intercal A x + b^\intercal x + c \le 0, g^\intercal x + h = 0 \right\}$$
where $A + \lambda g g^\intercal \succeq 0$ for some $\lambda \in \mathbb{R}$.

5. The set of points
$$\left\{ y \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} B_i y_i - A \succeq 0 \right\}$$
where A, B_1, B_2, $\ldots$, $B_n \in \mathcal{S}^{p \times p}$.

Problem 2: Convex sets and cones, BNO 1.15

1. Let C be a nonempty convex subset of $\mathbb{R}^n$. Show that
$$cone(C) = \cup_{x\in C}\{\gamma x \mid \gamma \geq 0\}.$$
2. Let C_1 and C_2 be two convex cones containing the origin. Show that
$$C_1 + C_2 = \text{conv}(C_1 \cup C_2),$$
$$C_1 \cap C_2 = \cup_{\alpha\in[0,1]}(\alpha C_1 \cap (1-\alpha)C_2).$$

Problem 3: Distance between sets and projection, B. 6.21, pg 105, modified

Let A and B be non-empty closed convex subsets of $\mathbb{R}^n$. For each $a \in A$, define
$$d(a, B) = \inf_{b\in B} ||a - b||_2,$$
and then define
$$d(A, B) = \inf_{a\in A} d(a, B).$$
Let
$$E = \{a \in A \mid d(a, B) = d(A, B)\}.$$
and
$$F = \{b \in B \mid d(b, A) = d(B, A)\}$$
assume that both E and F are nonempty. The displacement vector is $v = P_K(0)$, where K is the closure of the set $B - A$. For any transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^n$, denote by $\text{Fix}(T)$ the set of all $x \in \mathbb{R}^n$ such that $Tx = x$.

1. Describe two sets A and B that satisfy the assumption of the theorem, are such that $A \cap B = \emptyset$, and yet $d(A, B) = 0$.
2. Define $\mathcal{C} = \{x \in \mathbb{R}^2 \mid ||x||_\infty \leq 1\}$. Assuming that $A = (-2, 0) + \mathcal{C}$ and $B = (2, 0) + \mathcal{C}$, compute v, E, F, $A \cap (B - v)$, and $\text{Fix}(p_A p_B)$.

In general, prove that

3. $||v||_2 = d(A, B)$
4. $E + v = F$
5. $E = \text{Fix}(p_A p_B) = A \cap (B - v)$
6. $p_A f = p_E f = f - v$, for all $f \in F$.

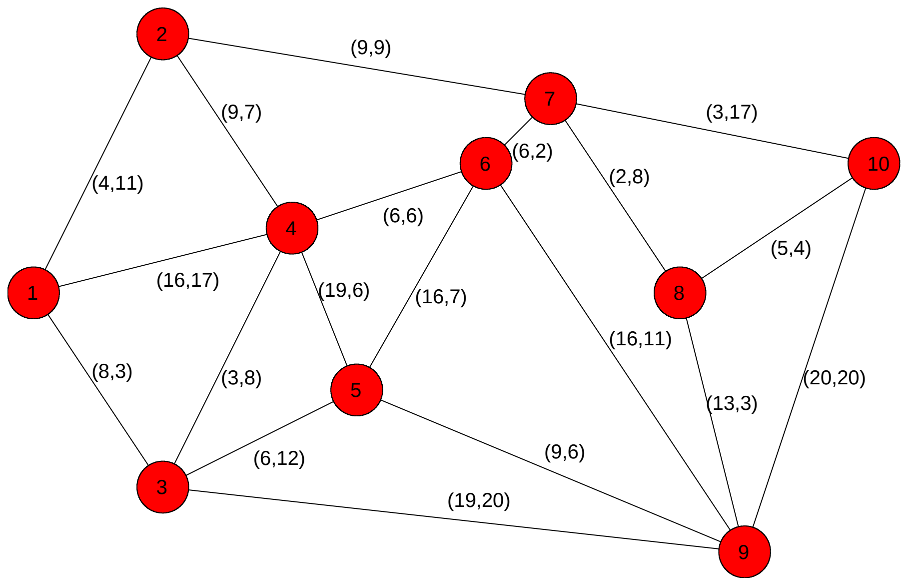
In the above expressions, we let p_A be the (usual) metric projection over the set A.

Problem 4: Alternative theorems, F. & LV, pg 33

1. Using Farkas Lemma, prove the following alternative result. The system $Ax = 0$, $x > 0$ is unsolvable if and only if the system $y^\intercal A \geq 0$, $y^\intercal A \neq 0$ is solvable.
2. Let K be a skew-symmetric $(n\times n)$-matrix, *i.e.*, $K^\intercal = -K$. Then the system of linear inequalities with a nonnegativity constraint $Kx \geq 0$, $x \geq 0$ has a solution verifying $x + Kx > 0$ (all the components of $x+Kx$ are positive). (*Hint: There are various ways of proving this result, one uses several applications of alternative theorems.*)

Problem 5: Maximum flow interdiction model, JPR

An international criminal organization has established different pathways to smuggle drugs from country A to country B. The resulting network is represented in the following picture:



where the arcs are directed from left to right. Each of the arcs of the network represent one particular highway, waterway, airplane connection that smugglers have used to move drugs around. Further, depending on the type of transportation used, the amount of drugs that can be carried is different (it is easier to hide large amounts of drugs in a truck than in a suitcase). The capacity of each one of the arcs (per day) is represented on the figure as the first number on the corresponding arc.

A drug fighting agency is looking to disrupt the activities of this criminal organization by introducing checkpoints on some of the arcs that have been identified to belong to the network. The cost of installing a checkpoint on arc is represented as the second number on the corresponding arc.

It is assumed that the criminal organization will not be able to send any more drugs on an arc on which a checkpoint has been installed. The drug fighting agency seeks to locate its check-point so as to minimize the maximum amount of drugs that can be sent from country A (node 1) to country B (node 10). Further, its budget for establishing check points is 15.

1. Formulate a mathematical model for this problem. (*Hint: Express this problem as a min-max problem and take the dual of the maximization problem to convert into a min-min problem*).
2. Obtain an optimal solution of this problem with GAMS.

Problem 6: About recession cones, dKRT, 1.27, pg 20

Assume that C is a closed convex set that is not bounded. For $x \in C$, define $R(x) = \{z \in \mathbb{R}^n \,|\, x + \lambda z \in C, \forall \lambda \geq 0\}$. Let $\bar{x}$ be a point of C, define $\mathcal{R} = R(\bar{x})$.
Show that

1. for each $x \in C$, $R(x) \neq \emptyset$.

2. $R(x)$ is a closed convex cone (called the *recession cone* of C at x).

3. $R(x) = \mathcal{R}$ for all $x \in C$.

The above results establish that we can indeed talk about "the" recession cone of C. Conclude the above study of the recession cone of C by proving that

4. a nonempty closed convex set C is bounded if and only if its recession cone $\mathcal{R}$ consists of the zero vector alone.