Homework 3

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MAP 4102

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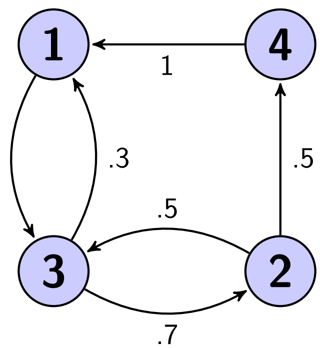
Problem 1a. Prove or disprove the Markov chain



converges to a limiting distribution.

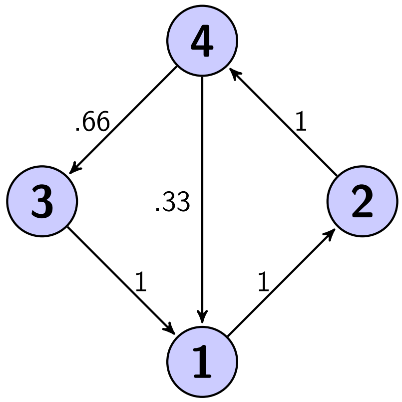
Solution. The Markov of chain does not converge. It is irreducible (to wit $4 \to 1 \to 3 \to 2 \to 4$) and the period of state 1 is 2 (to wit there are only 3 unique pathes from 1 to 1: $1 \to 3 \to 1$, $1 \to 3 \to 2 \to 3 \to 1$, $1 \to 3 \to 2 \to 4 \to 1$ and they have lengths which are multiples of 2) so the chain oscillates between 2 configurations. To wit

$$\begin{pmatrix} 0.481 & 0.518 & & \\ 0.481 & 0.518 & & \\ & & 0.740 & 0.259 \\ & & 0.740 & 0.259 \end{pmatrix}$$

and

$$\begin{pmatrix} 0.740 & 0.259 & & \\ 0.740 & 0.259 & & \\ & & 0.481 & 0.518 \\ & & 0.481 & 0.518 \end{pmatrix}$$

Problem 1b. Prove or disprove the Markov chain

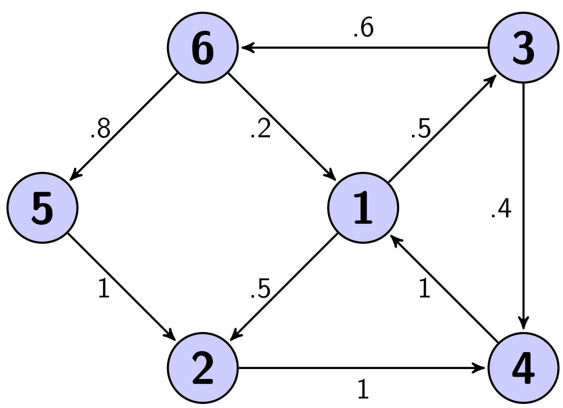


converges to a limiting distribution.

Solution. The Markov chain does converge to a limiting distribution. The chain is obviously irreducible so by Thm. 1.14 it has a unique stationary distribution. The period of state 2 is one because $2 \to 4 \to 1 \to 2$ is a valid traversal with length 3 and so is $2 \to 4 \to 3 \to 1 \to 2$ with length 4 ($gcd(3,4) = 1$). So the chain is aperiodic and by Thm. 1.19 it converges to its stationary distribution. To wit

$$p^n(x, y) = \frac{1}{11}(3, 3, 2, 3)$$

Problem 1c. Prove or disprove the Markov chain



converges to a limiting distribution.

Solution. The Markov chain does not converge to a limiting distribution. The chain is irreducible:

$$6 \to 5 \to 2 \to 4 \to 1 \to 3 \to 6.$$

The period of of state 5 is 6; the only path from 5 to 5 without repetition is of length 6:

$$5 \to 2 \to 4 \to 1 \to 3 \to 6 \to 5.$$

So the period of the Markov chain is three (the period of state 1 is three) and hence the chain oscillates between 3 "configurations". We omit the three 6×6 matrices.

Problem 2. Compute the stationary distribution for

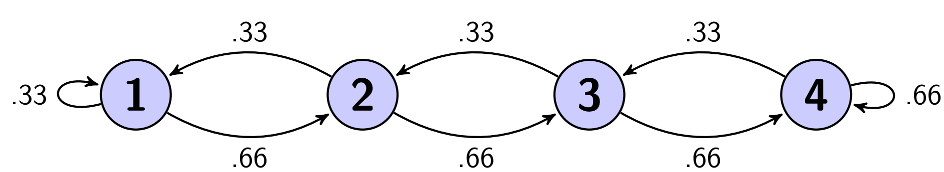
$$\begin{pmatrix}
0 & & & & .5 & & & & & & & & & & .5 \\
1 \\
& 1 \\
& & 1 \\
& & & 1 \\
& & & & 1 \\
& & & & & 1 \\
& & & & & & 1 \\
& & & & & & & 1 \\
& & & & & & & & 1 \\
& & & & & & & & & 1 \\
& & & & & & & & & & 1 \\
& & & & & & & & & & & 1 \\
& & & & & & & & & & & & 1 \\
& & & & & & & & & & & & & 1 & 0
\end{pmatrix}$$

Solution. By the relation $\boldsymbol{\pi} P = \boldsymbol{\pi}$ the Markov chain induces a system of linear equations:

$$\begin{aligned}
\frac{1}{2}\pi_4 + \frac{1}{2}\pi_{14} &= \pi_0 \\
\pi_0 &= \pi_1 \\
\pi_1 &= \pi_2 \\
\vdots &= \vdots \\
\pi_{13} &= \pi_{14}
\end{aligned}$$

So π_{14} is the only free variable. Solving for π_{14} subject to the condition that $\frac{1}{2}\pi_{14} + \frac{1}{2}\pi_{14} = \pi_{14}$ and $\sum_{i=0}^{14} \pi_i = 1$ yields the stationary distribution $\pi_y = \frac{1}{15}$ for all y.

Problem 3a. Find the transition probability matrix representation for the Markov chain



Solution. The transition matrix is straightforward:

$$\begin{pmatrix} .33 & .66 & & \\ .33 & & .66 & \\ & .33 & & .66 \\ & & .33 & .66 \end{pmatrix}$$

Problem 3b. Find the limiting amount of time the chain spends at each site.

Solution. The chain is irreducible (all states clearly communicate), aperiodic ($p(0,0) = .33 > 0$), and has a stationary distribution (because it is irreducible and finite). So by Thm. 1.22 the asymptotic frequency of state y is the y-th entry in the stationary distribution. The stationary distribution is

$$\pi = \frac{1}{15}(1, 2, 4, 8)$$

So the chain spends $\frac{1}{15}$ of its time in state 1, $\frac{2}{15}$ of its time in state 2, $\frac{4}{15}$ of its time in state 3, and $\frac{8}{15}$ of its time in state 4.