Homework 7

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Autocorrelation of a Stochastic Process

Problem 1. Compute $\rho(n, n+1)$ for the i.i.d. model.

Solution.

Since X_i are i.i.d. for all i then $E(X_n X_{n+1}) = E(X_n)E(X_{n+1})$. Hence

$$\rho(n, n+1) = E(X_n X_{n+1}) - E(X_n)E(X_{n+1}) = E(X_n)E(X_{n+1}) - E(X_n)E(X_{n+1}) = 0$$

Problem 2. Compute

$$\hat{\rho}(1) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} X_n X_{n+1} - \left(\frac{1}{N} \sum_{n=0}^{N-1} X_n \right) \left(\frac{1}{N} \sum_{n=1}^{N} X_n \right)$$

for the sequence of random variables $\{X_n\} = (1, 5, 5, 1, 5, 5, 1, 5, 5, 1, \dots)$.

Solution.

We'll rewrite the terms in the sums and then take the limits. The terms in the first sum

$$\sum_{n=0}^{N-1} X_n X_{n+1}$$

are of the form 1×5, 5×5, 5×1. For example

$$\begin{aligned} X_0X_1 + X_3X_4 + X_6X_7 &= 1 \times 5 + 1 \times 5 + 1 \times 5 \\ X_1X_2 + X_4X_5 + X_7X_8 &= 5 \times 5 + 5 \times 5 + 5 \times 5 \\ X_2X_3 + X_5X_6 + X_8X_9 &= 5 \times 1 + 5 \times 1 + 5 \times 1 \end{aligned}$$

So reordering the terms in the sums (taking sums along the columns above) we get

$$\frac{1}{N}\sum_{n=0}^{N-1} X_n X_{n+1} = \frac{1}{3(j+1)}\sum_{i=0}^{j}\Big(X_{3i}X_{3i+1} + X_{3i+1}X_{3i+2} + X_{3i+2}X_{3i+3}\Big)$$

where $j = \lceil N/3 \rceil$. But each term in this sum is simply either 5 or 25. Hence

$$\frac{1}{3(j+1)}\sum_{i=0}^{j}\Big(X_{3i}X_{3i+1}+X_{3i+1}X_{3i+2}+X_{3i+2}X_{3i+3}\Big) = \frac{1}{3(j+1)}(j\cdot 5+j\cdot 25+j\cdot 5) = \frac{35j}{3(j+1)}$$

The second term in $\hat{\rho}(1)$, the product of sums, is simply the product of the means of a deterministic variable that is equal to 1 for $N/3$ instances and equal to 5 for $2N/3$ instances. Hence it, the product, is equal to $(\frac{1}{N}\frac{11\cdot N}{3})^2 = \frac{121}{9}$. Finally

$$\hat{\rho}(1) = \lim_{j\to\infty}\left(\frac{35j}{3(j+1)} - \frac{121}{9}\right) = \frac{105}{9} - \frac{121}{9} = \frac{-16}{3}$$

Problem 3(a). For what value of p will $\{X_n\}$ satisfy the asymptotic frequencies given?

Solution.

Solving

$$\begin{pmatrix}1/3 & 2/3\end{pmatrix}\begin{pmatrix}.1 & .9 \\ p & (1-p)\end{pmatrix} = \begin{pmatrix}1/3 & 2/3\end{pmatrix}$$

yields $p = .45$.

Problem 3(b). Compute the one-step autocovariance function for this process assuming that the initial condition is drawn from the stationary distribution.

Solution.

Given that X_0 is initially drawn from $\{1, 5\}$ with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively then $E(XY)$ is

$$\frac{1}{3}(.1\cdot 1\cdot 1 + .9\cdot 1\cdot 5) + \frac{2}{3}(.45\cdot 5\cdot 1 + .55\cdot 5\cdot 5) = 10.7$$

By the ergodic theorem we can compute $E(X)$; $E(X) = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 5$ and so $E(X_nX_{n+1}) - E(X)^2 = 10.7 - 13.\overline{44} = 2.7\overline{4}$

The rest of the problems...

Problem 2(a). Compute $\mathbb{E}[X \mid A]$

Solution.

$$\mathbb{E}[X \mid A] = \frac{1/2}{1/2 + 1/3} \times 5 + \frac{1/3}{1/2 + 1/3} \times 2 = 3.8$$

Problem 2(b). Compute $\mathbb{E}[X 1_A]$

Solution.

$$\mathbb{E}[X \mid A] = \frac{1}{2} \times 5 + \frac{1}{3} \times 2 + 0 \times 1 = 3\frac{1}{6}$$

Problem 2(c). Let Y be distributed like X and let $Z := X + Y$. Compute $\mathbb{E}[Z \mid A]$.

Solution.

$$\begin{aligned}
\mathbb{E}[Z \mid A] = {} & \frac{(1/2)^2}{(1/2)^2 + (1/6)^2 + (1/2)(1/3) + (1/3)(1/2)} \times 10 + \\
& \frac{(1/3)^2}{(1/2)^2 + (1/6)^2 + (1/2)(1/3) + (1/3)(1/2)} \times 4 + \\
& \frac{(1/2)(1/3)}{(1/2)^2 + (1/6)^2 + (1/2)(1/3) + (1/3)(1/2)} \times 7 + \\
& \frac{(1/3)(1/2)}{(1/2)^2 + (1/6)^2 + (1/2)(1/3) + (1/3)(1/2)} \times 7 \\
= {} & 8.\overline{63}
\end{aligned}$$

Problem 2(c). Compute $\mathbb{E}[Z \mid X]$.

Solution.

$$\begin{aligned}
\mathbb{E}[Z \mid X] = {} & \frac{1}{2}\left(\frac{1}{2}(5+5) + \frac{1}{3}(5+2) + \frac{1}{6}(5+1)\right) + \\
& \frac{1}{3}\left(\frac{1}{2}(2+5) + \frac{1}{3}(2+2) + \frac{1}{6}(2+1)\right) + \\
& \frac{1}{6}\left(\frac{1}{2}(1+5) + \frac{1}{3}(1+2) + \frac{1}{6}(1+1)\right) \\
= {} & 6\frac{2}{3}
\end{aligned}$$

Problem 3. Show that M_n is a martingale with respect to $\mathcal{F}_n$.

Solution.

$$M_n = \frac{e^{sS_n}}{\mathbb{E}[e^{sX_1}]^n} = \frac{e^{s\sum_{i=1}^n X_i}}{\mathbb{E}[e^{sX_1}]^n}$$

Hence

$$\begin{aligned}
\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] &= \mathbb{E}\left[\frac{e^{s\sum_{i=1}^{n+1} X_i}}{\mathbb{E}[e^{sX_1}]^{n+1}} \,\middle|\, \mathcal{F}_n \right] \\
&= \mathbb{E}\left[\frac{e^{sX_{n+1}}}{\mathbb{E}[e^{sX_1}]} \frac{e^{s\sum_{i=1}^{n} X_i}}{\mathbb{E}[e^{sX_1}]^{n}} \,\middle|\, \mathcal{F}_n \right]
\end{aligned}$$

But X_{n+1} is independent of $\mathcal{F}_n$ (and hence so is $g(X_{n+1})$) and X_n is $\mathcal{F}_n$ measurable (and hence so is $f(X_n)$) so

$$\begin{aligned}
\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] &= \mathbb{E}\left[\frac{e^{sX_{n+1}}}{\mathbb{E}[e^{sX_1}]} \,\middle|\, \mathcal{F}_n \right] \mathbb{E}\left[\frac{e^{s\sum_{i=1}^{n} X_i}}{\mathbb{E}[e^{sX_1}]^{n}} \,\middle|\, \mathcal{F}_n \right] \\
&= \mathbb{E}\left[\frac{e^{sX_{n+1}}}{\mathbb{E}[e^{sX_1}]} \,\middle|\, \mathcal{F}_n \right] \frac{e^{s\sum_{i=1}^{n} X_i}}{\mathbb{E}[e^{sX_1}]^{n}} \\
&= \frac{\mathbb{E}[e^{sX_{n+1}}]}{\mathbb{E}[e^{sX_1}]} \frac{e^{s\sum_{i=1}^{n} X_i}}{\mathbb{E}[e^{sX_1}]^{n}}
\end{aligned}$$

But X_i are i.i.d so $\mathbb{E}[e^{sX_{n+1}}] = \mathbb{E}[e^{sX_1}]$ and hence finally

$$\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] = 1 \times \frac{e^{s\sum_{i=1}^{n} X_i}}{\mathbb{E}[e^{sX_1}]^{n}} = M_n$$