ESI 6420 : Fundamentals of Mathematical Programming

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Homework 1, due in class, Monday September 14$^{\text{th}}$ 2015

Preliminaries

Throughout the semester, I would like you to learn how to use latex (which you will find helpful in the future). In particular, for homework i, I would like you to latex at least $\lfloor \frac{i}{2} \rfloor$ of your answers. For an introduction to latex, refer to `http://www.ctan.org/tex-archive/info/gentle/gentle.pdf`.

If you received some help to obtain the solution of a problem, you should acknowledge the source of help you received. In particular, for each question, I would like you to cite, if applicable, any book (other than the textbook) you consulted, any website you searched, or any individual you cooperated with. This information will not be used to adjust your homework score provided that help is limited to a reasonable portion of the homework.

Finally, I would like you to candidly assess the number of hours it took you to complete the homework.

Problem 1: Minimizing skiing distance, Calafiore & El Ghaoui, 9.2, pg 341

A two-dimensional skier must slalom down a slope, by going through n parallel gates positioned between coordinates $(x_i, y_i - c_i)$ and $(x_i, y_i + c_i)$, for $i = 1, \ldots, n$. The initial position (x_0, y_0) is given, as well as the final one, (x_{n+1}, y_{n+1}). Here, the x-axis represents the direction down the slope, from left to right; see Figure 1.

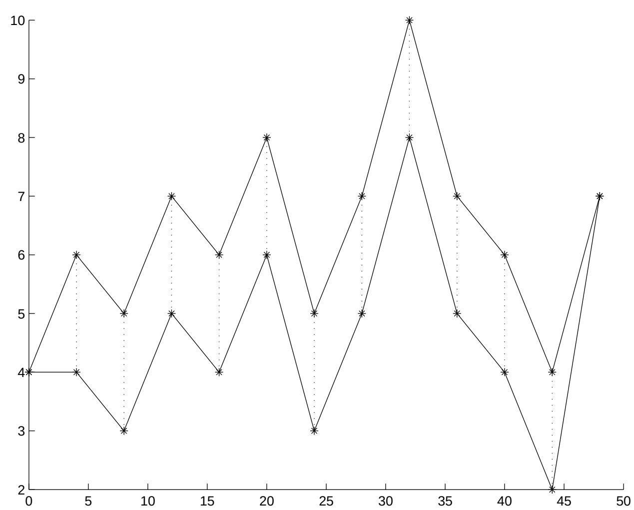

Figure 1: Positions of the gates for Problem 1

1. Create a mathematical program to find the path that minimizes the total distance skied. Your answer should come in the form of an algebraic model.

2. Solve this model using GAMS, with the data given in Table 1 where $n = 11$.

i	x_i	y_i	c_i
0	0	4	NA
1	4	5	1
2	8	4	1
3	12	6	1
4	16	5	1
5	20	7	1
6	24	4	1
7	28	6	1
8	32	9	1
9	36	6	1
10	40	5	1
11	44	3	1
12	48	8	NA

Table 1: Data for Problem 1

3. Represent the solution you obtained graphically.

Problem 2: Inscribing circles in a small rectangle

We would like to inscribe n non-overlapping circles with radii $i = 1, 2, \ldots, n$, respectively, into a rectangle of minimum area.

1. Write down a mathematical program for this problem.

2. Solve this model for $n = 1, 2, \ldots, 10$ and represent the solutions you obtained graphically.

Problem 3: Hub-and-spoke design

An airline is serving n different cities around the country and must decide how to organize flights between them. In the hub-and-spoke design, the company first chooses k different cities to be hubs, where k is typically a small number. Every city that is served by the company is then assigned to a single hub (the hub of a hub-city will always be the hub-city itself). Flights are then be established between each city and its hub and between all pairs of hubs. Therefore, in a hub-and-spoke system, a passenger traveling from city A to B will first travel from city A to its hub (call it $hubA$), then travel from $hubA$ to the hub associated with city B (call it $hubB$) and finally will travel from $hubB$ to B. It follows that, in a hub-and-spoke system, a traveler will never need more than 3 flights to reach her destination.
Consider now the situation where an airline want to establish a hub-and-spoke design between the following nine cities, whose locations in the euclidian plane are given below:

	1	2	3	4	5	6	7	8	9
x	14	15	20	20	39	43	51	52	93
y	73	40	65	61	18	9	94	30	35

Further, the airline company estimates that it will experience the following daily demands between each pair of cities:

	1	2	3	4	5	6	7	8	9
1	0	165	80	125	120	155	195	100	185
2	180	0	110	160	135	185	90	180	90
3	95	155	0	195	175	60	95	90	140
4	55	110	50	0	140	130	55	95	175
5	60	185	90	180	0	85	160	60	145
6	55	110	195	50	155	0	145	125	195
7	155	140	85	130	70	140	0	80	125
8	145	100	70	110	190	60	125	0	180
9	65	180	55	130	125	150	50	140	0

Design a hub-and-spoke system with 3 hubs for this company that will minimize the total distance traveled by all its customers. The distance between one hub-city and one non-hub-city is computed as the euclidian distance between them while the distance between two hub-cities is computed as 1/2 of the euclidian distance between them.

1. Present a mathematical model for this problem.
2. Solve this model with GAMS.
3. Represent the solution obtained on a picture.

Problem 4: Supply chain problem, JB, 18 pg 355

A company wants an aggregate production plan for the next six months. Projected sales for its product are listed in Table 2.

Month	Sales goals (units)	Production cost ($/unit)
1	1300	100
2	1400	105
3	1000	110
4	800	115
5	1700	110
6	1900	110

Table 2: Data for Problem 4

Production that exceeds sales in each month may be put in inventory and sold in some future month. Because of seasonal factors, the production costs vary from month to month. An additional charge of $4 per unit per month is incurred for units in inventory.

The major costs center on production but the company also wishes to account for failures to meet monthly sales, production, and inventory goals by penalizing deviations from either predetermined targets or policies. The inventory goal is to have 100 units in stock at the end of each month. The production goal is to minimize labor and raw material fluctuations from one month to the next. To account for these goals, square penalty terms of the following form are added to the objective function:

$$w_s(\text{actual sales-sales goal})^2 + w_i(\text{actual inventory-inventory goal})^2 + \\ w_p(\text{production-production in previous month})^2$$

where w_s, w_i and w_p are predetermined weights (that you can assume are all equal to 1). Assume in your model that the initial and final inventories are 100 and assume the production prior to month 1 is 1000.

1. Set up a mathematical program that minimizes production, inventory and goal penalty cost.
2. Solve this model with GAMS.

Problem 5: Capacity planning, Winston, modified

A power company faces different demands throughout the day. The company has divided the day into n periods, and has decided to set the price of electricity differently depending on the period. The company has determined that customers will demand $a_j - b_j p$ kwh of power during period j, if the company sets the price of the khw to $\$p$. In this formulas, $a_j > 0$, and $b_j > 0$ for $j = 1, \ldots, n$. The power company must have sufficient capacity to meet demand during all of the periods, and will maintain that capacity constant throughout the day. It costs $\$f$ per day to maintain each kwh of capacity.

1. Formulate a mathematical program for the power company to determine how it can maximize daily revenue less operating costs.
2. Solve this model using GAMS for the data given in Table 3 and for the following values of $f = 5$, $f = 10$, and $f = 20$. In particular, report installed capacity together with price and realized demand

i	a_i	b_i
1	40	6
2	39	3
3	36	3
4	32	6
5	25	3
6	23	4
7	21	9
8	14	8
9	13	2
10	6	6

Table 3: Data for Problem 5

   for each time period.
3. Assume that the day periods are ordered so that $a_i \geq a_{i+1}$ for $i = 1, \ldots, n-1$. Propose a (linear time) algorithm to solve the above problem. *(Hint: How easy is the problem to solve if the installed capacity is known?)*

Problem 6: Quickest take-off, Boyd and Vandenberghe, 14.7, pg 122

This problem concerns the braking and thrust profiles for an airplane during take-off. For simplicity we will use a discrete-time model. The position (down the runway) and the velocity in time period t are p_t and v_t, respectively, for $t = 0, 1, \ldots$. These satisfy $p_0 = 0$, $v_0 = 0$, and $p_{t+1} = p_t + hv_t$, $t = 0, 1, \ldots$, where $h > 0$ is the sampling time period. The velocity updates as

$$v_{t+1} = (1 - \eta)v_t + h(f_t - b_t), \quad t = 0, 1, \ldots,$$

where $\eta \in (0, 1)$ is a friction or drag parameter, f_t is the engine thrust, and b_t is the braking force, at time period t. These must satisfy

$$0 \leq b_t \leq \min\{B_{max}, f_t\}, 0 \leq f_t \leq F_{max}, \quad t = 0, 1, \ldots,$$

as well as a constraint on how fast the engine thrust can be changed,

$$|f_{t+1} - f_t| \leq S, \quad t = 0, 1, \ldots.$$

Here B_{max}, F_{max}, and S are given parameters. The initial thrust is $f_0 = 0$. The take-off time is $T^{to} = min\{t | v_t \geq V^{to}\}$, where V^{to} is a given take-off velocity. The take-off position is $P^{to} = p_{T^{to}}$, the position of the aircraft at the take-off time. The length of the runway is $L > 0$, so we must have $P^{to} \leq L$.

1. Explain how to find the thrust and braking profiles that minimize the take-off time T^{to}, respecting all constraints. Your solution can involve solving more than one mathematical program, if necessary.

2. Solve the quickest take-off problem with data

$$h = 1, \eta = 0.05, B_{max} = 0.5, F_{max} = 4, S = 0.8, V^{to} = 40, L = 300.$$

   Plot p_t, v_t, f_t, and b_t versus t. Comment on what you see. Report the take-off time and take-off position for the profile you find.