What's the deal with Quantum Computing Part 2
-A little less formalism-

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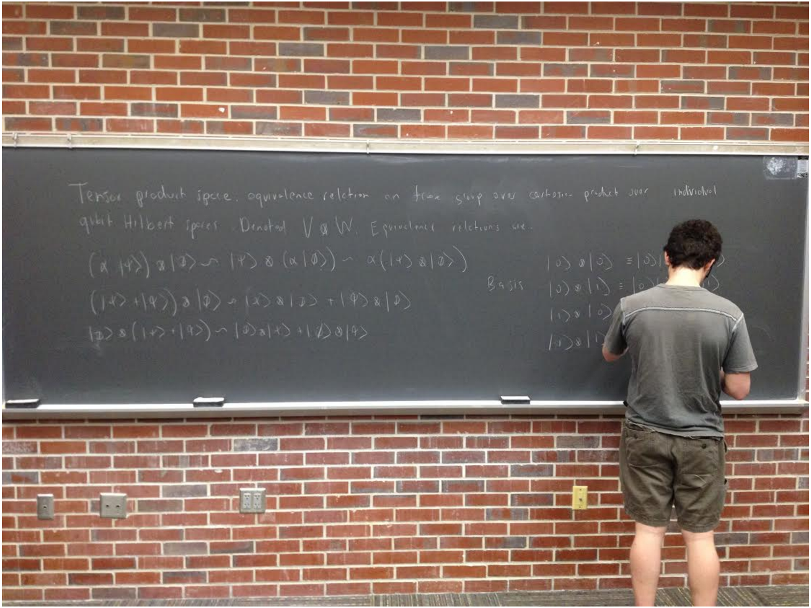
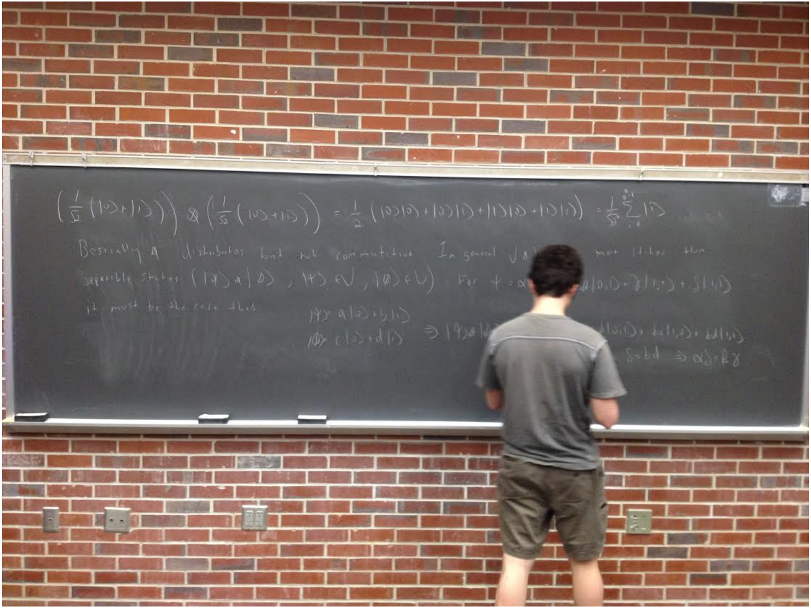

Tensor product space: equivalence relation on free group over cartesian product over individual
qubit Hilbert spaces. Denoted V ⊗ W. Equivalence relations are:
Basis




ARCADE
CCCP
SPIDERMAN

Recap

- A single qubit is a (unit length) linear combination of the basis vectors $|0\rangle, |1\rangle$

$$\psi = \alpha |0\rangle + \beta |1\rangle$$

- Measurement $\iff$ non-deterministic wave function collapse $\iff$ all information lost
- Unitary transformations correspond to gates. 1-qubit gates are matrices

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- n-qubit systems (registers) are represented by vectors (tensors) in the tensor product of the vector spaces that each of the individual qubits are elements of

Recap

$$\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{2}\Big(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle \Big)$$

- Gates on single qubit systems also map to "*n*-gates" on *n*-qubit systems (entrywise)

$$\begin{aligned} H^{\otimes 2}|0\rangle|0\rangle &= (H|0\rangle) \otimes (H|0\rangle) \\ &= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \end{aligned}$$

- Entangled states are important: for $\psi \in V \otimes W$ there **do not exist** $\phi \in V$ and $\varphi \in W$ such that

$$\psi = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} = \phi \otimes \varphi$$

Deutsch's Problem

"Reversible computation can be done **efficiently**, without the production of garbage bits whose values depend on the input to the computation. That is, if there is an irreversible circuit computing a function f, then there is an efficient simulation of this circuit by a reversible [unitary transformation/quantum] circuit with action" [5]

$$|x\rangle\,|y\rangle \mapsto |x\rangle\,|y \oplus f\,(x)\rangle$$

Deutsch's Problem

Let $f : \{0, 1\} \to \{0, 1\}$ and suppose we are guaranteed that f is either balanced (1 on half of its domain and 0 on the other half) or constant (1 or 0 on the entire domain). How many evaluations classically to discriminate? "Quantumly" you only need to evaluate f once! Let U_f be the quantum circuit such that

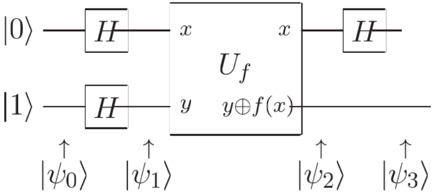
$$U_f \left|x\right\rangle \left|y\right\rangle = \left|x\right\rangle \left|y \oplus f(x)\right\rangle$$

With some algebra (keeping in mind the small-ish domain and range of f)

$$U_f \left(\left|x\right\rangle \left(\frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}} \right) \right) = (-1)^{f(x)} \left|x\right\rangle \left(\frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}} \right)$$

Deutsch's Algorithm

Construct the quantum circuit



$$|\psi_1\rangle = H^{\otimes 2}\left(|0\rangle \otimes |1\rangle\right) = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$\psi_2 = U_f\,|\psi_1\rangle = \begin{cases} \pm \quad \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm \quad \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}$$

Deutsch's Algorithm

The final Hadamard gate leaves the system in

$$\psi_3 = \begin{cases} \pm \quad |0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm \quad |1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}$$

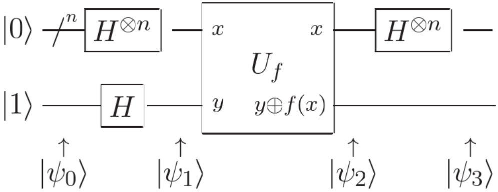
Now if we measure the first qubit we know whether $f(0) = f(1)$ or $f(0) \neq f(1)$ (depending on whether we get $|0\rangle$ or $|1\rangle$). Succinctly stated this allows us to measure a global property: since $f(0) \oplus f(1) = 0$ if $f(0) = f(1)$ and 1 otherwise

$$\psi_3 = \pm |f(0) \oplus f(1)\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Naive interpretation is that this is a randomized algorithm but in truth interference effects (the final Hadmard gate) are used to discern global properties (H is a generalized DFT).

Deutsch-Jozsa Algorithm

Generalize to $f : \{0,1\}^n \to \{0,1\}$ and still f is either balanced or constant. How many evaluations classically? $2^{n-1} + 1$ but quantumly still 1!



$$\psi_0 = \left(|0\rangle^{\otimes n}\right) \otimes |1\rangle$$

then

$$\psi_1 = \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Deutsch-Jozsa Algorithm

The first register is a superposition of all basis states in the n-qubit computational basis. Using the simplification above again we have that

$$\psi_2 = U_f \psi_1 = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

and the last Hadamard operator

$$\psi_3 = H^{\otimes n} \psi_2 = \sum_{z \in \{0,1\}^n} \sum_{x \in \{0,1\}^n} \frac{(-1)^{x \cdot z + f(x)} |z\rangle}{2^n} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

where $x \cdot z$ is bitwise inner product mod 2.

Deutsch-Jozsa Algorithm

Let's observe the top register (query register). Note that the amplitude for $|0,0,\ldots,0\rangle$ is $\sum_x (-1)^{f(x)}/2^n$. If f is constant then

$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}}{2^n} = \pm 1$$

and because ψ_3 must be unit length we will certainly measure ψ_3 to be in the $|0,0,\ldots,0\rangle$ state. If f is balanced then by definition of balanced ($f(x)$ will be even as often as odd)

$$\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}}{2^n} = 0$$

and we will certainly measure something other than $|0,0,\ldots,0\rangle$.

Factoring integers - reduction to order finding

Pick $x < N$. If x and N have a common factor then $\gcd(x, N)$ can be computed classically in polynomial time using Euclid's algorithm. Otherwise compute the order of x; the least r such that

$$x^r \equiv 1 \mod N$$

With probability $p > 1 - \left(\frac{1}{2}\right)^{q-1}$, where q is the number of prime factors in N, the order of x will be even. Then

$$x^r - 1 \equiv \left(x^{r/2} - 1\right)\left(x^{r/2} + 1\right) \equiv 0 \mod N$$

and hence N divides $\left(x^{r/2} - 1\right)\left(x^{r/2} + 1\right)$. If $1 < x^{r/2} \mod N < N - 1$ then

$$0 < x^{r/2} - 1 \mod N < x^{r/2} + 1 < N$$

and hence $\left(x^{r/2} - 1\right), \left(x^{r/2} + 1\right)$ must each have a factor of N. Compute $\gcd(x^{r/2} - 1, N)$ and $\gcd(x^{r/2} + 1, N)$

Order example

For $N = 2013$ it's the case that $8^{20} \equiv 1 \mod N \iff 8^{20} - 1 \equiv 0 \mod N$ and

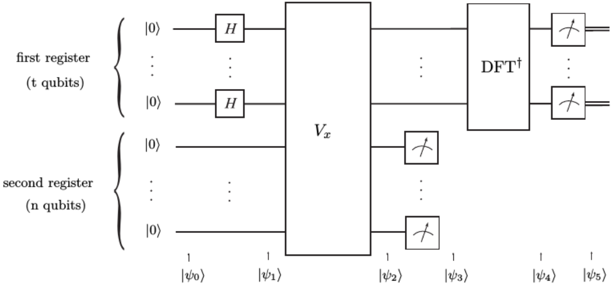
$$\left(8^{\frac{20}{2}} - 1\right)\left(8^{\frac{20}{2}} + 1\right) \equiv 0 \mod N$$

But $\left(8^{\frac{20}{2}} - 1\right) \equiv 1584 \mod N$ and $\left(8^{\frac{20}{2}} + 1\right) \equiv 1586 \mod N$ and $0 < 1584 < 1586 < 2013$ so

$$\gcd(1584, 2013) = 33 \qquad \gcd(1586, 2013) = 61$$

and $61 \times 33 = 2013$

Shor's Algorithm



where

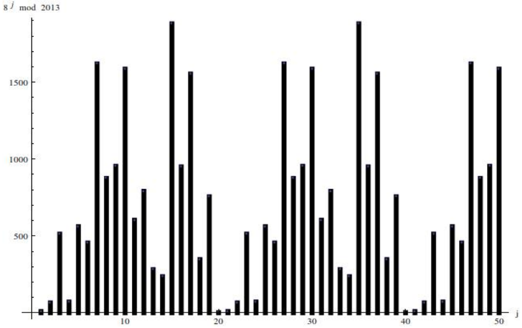
$$V_x\left(|j\rangle\,|k\rangle\right) = |j\rangle\,|k + x^j\rangle$$

and

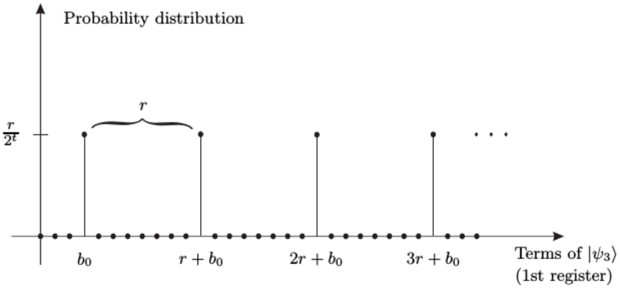
$$DFT\left(|k\rangle\right) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} e^{2\pi ijk/N}\,|j\rangle$$

Then

$$|\psi_2\rangle = \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle\,|x^j\rangle \underset{r|2^t}{=} \frac{1}{\sqrt{2^t}} \sum_{b=0}^{r-1} \sum_{a=0}^{\frac{2^t}{r}-1} |ar+b\rangle\,|x^b\rangle$$

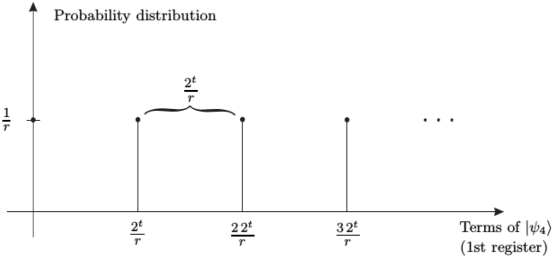


$$|\psi_3\rangle = \sqrt{\frac{r}{2^t}} \sum_{a=0}^{\frac{2^t}{r}-1} |ar + b_0\rangle \, |x^{b_0}\rangle$$



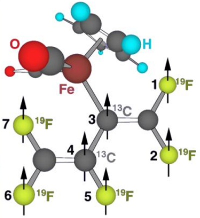
$$|\psi_4\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{2\pi \frac{k}{r} b_0} \left|\frac{k2^t}{r}\right\rangle |x^{b_0}\rangle$$

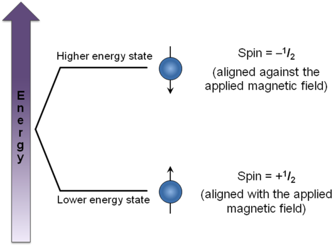
Assuming the order of x, r is a multiple of 2 (can be generalized), after measuring the first register we have $|\psi_5\rangle = |\frac{k_0 2^t}{r}\rangle$



If $k_0 = 0$ then we rerun. Otherwise divide $k_0 2^t/r$ by 2^t. If k_0, r are coprime then we can just take the denominator of k_0/r. Otherwise $r = r_1 r_2$ and we can find the order of x^{r_1} to find r.

Shor's Algorithm Implementation





Experimental realization of Shor's quantum factoring algorithm using nuclear magnetic resonance [7]

A **Probabilistic Turing machine** M over an alphabet A is $(Q, A, \delta, q_0, q_a, q_r)$ where

- Q is the set of internal control states
- $q_0, q_a, q_r \in Q$ are initial,accepting, and rejecting states
- $\delta : Q \times A \times Q \times A \times \{-1, 0, 1\} \to [0, 1]$ is a transition probability function i.e.

$$\sum_{(q_2, a_2, d)} \delta(q_1, a_1, q_2, a_2, d) = 1$$

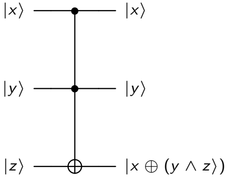
A **Quantum Turing machine** M over an alphabet A is $(Q, A, \delta, q_0, q_a, q_r)$ where

- Q is the set of internal control states
- $q_0, q_a, q_r \in Q$ are initial,accepting, and rejecting states
- $\delta : Q \times A \times Q \times A \times \{-1, 0, 1\} \to \mathbb{C}$ is the root of a transition probability function i.e.

$$\sum_{(q_2, a_2, d)} |\delta(q_1, a_1, q_2, a_2, d)|^2 = 1$$

Computability Theorems

- A language L has uniformly polynomial circuits iff $L \in \mathbf{P} = \bigcup_k \mathbf{TIME}\left(n^k\right)$
- All Boolean circuits can be simulated using reversible Boolean circuits

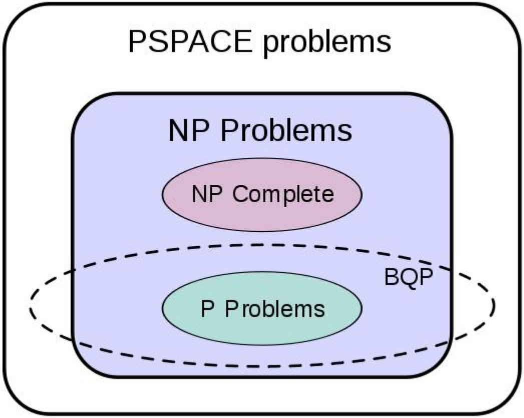


$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Toffoli gate

- Toffoli is classically universal but not quantum universal, but $\{TOF, H\}$ are quantum universal and both have successfully implemented [3, 4].

Complexity conjectures



Simon's problem and complexity results

Let $f(x) : \{0,1\}^n \to \{0,1\}^n$ and we are guaranteed that $\exists\ s \in \{0,1\}^n$ such that

$$f(y) = f(z) \iff (y = z \vee y \oplus z = s)$$

Find s. Classically $\Omega\left(2^{n/2}\right)$ while quantumly $O(n)$. Also quantumly optimal; any quantum algorithm needs to make $\Omega(n)$.

Yields an **oracle** separation between **BPP** and **BQP**. Otherwise **BPP** $\subset$ **BQP**

Deutsch-Josza only yields a separation between **P** and **EQP**

More EQP

Let $f(x) : \{0,1\}^n \rightarrow \{0,1\}$ be the PARITY function. Classically how many operations must be performed for f to be computed? Quantumly only $n/2$ queries to the bit string need to be made [1].

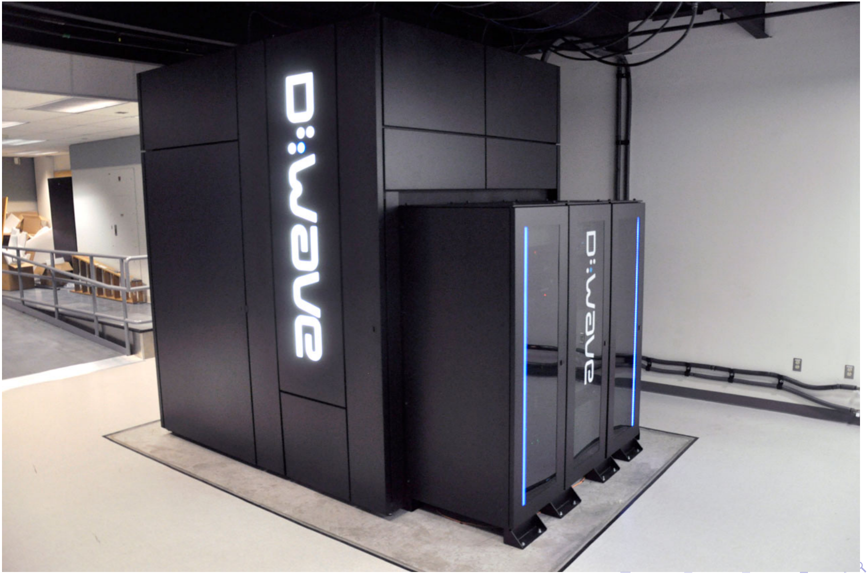
Compute the "square-free" part of an integer N, i.e. r such that

$$N = r \cdot s^2$$

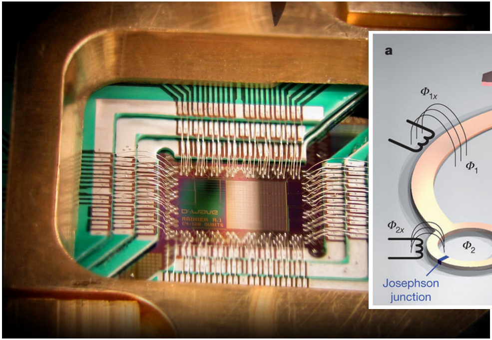
No known polynomial time classical algorithm; "almost" as hard as factorization itself [6]. Quantumly in $O\left((\log \log N)^2\right)$ [2].

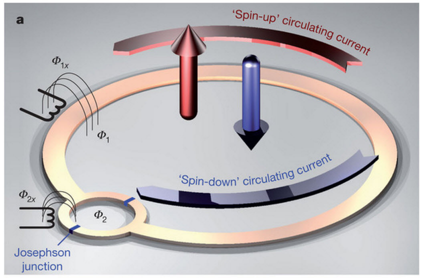
Interestingly, while this algorithm uses the Fourier transform it is exact (as opposed to Shor's).

D-Wave



D-Wave




a
'Spin-up' circulating current
Φ_{1x}
Φ_1
'Spin-down' circulating current
Φ_{2x}
Φ_2
Josephson junction

Timeline

- 1951 - EDVAC (first binary computer, Vacuum tubes)
- 1956 - John Bardeen invents the transistor
- 1958 - Jack Kilby invents ICs
- 1964 - IBM System/360
- 1968 - Intel founded by Robert Noyce
- 1971 - Intel 4004 (first commercially available processor, 4bit @ 740 kHz)
- 1975 - MITS Altair 8800 (first commercially successful hobby computer @ \$397 $\approx$ \$1700, uses Intel 8080)

Appendix U_f

$$U_f\left(|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) = U_f\left(\frac{|x\rangle|0\rangle-|x\rangle|1\rangle}{\sqrt{2}}\right) = \frac{|x\rangle|0\oplus f(x)\rangle-|x\rangle|1\oplus f(x)\rangle}{\sqrt{2}}$$

Now if $f(x)=0$ then

$$\frac{|x\rangle|0\oplus f(x)\rangle-|x\rangle|1\oplus f(x)\rangle}{\sqrt{2}} = \frac{|x\rangle|0\rangle-|x\rangle|1\rangle}{\sqrt{2}}$$

and if $f(x)=1$ then because $\oplus$ is mod 2

$$\frac{|x\rangle|0\oplus f(x)\rangle-|x\rangle|1\oplus f(x)\rangle}{\sqrt{2}} = \frac{|x\rangle|1\rangle-|x\rangle|0\rangle}{\sqrt{2}}$$

Appendix U_f

and so

$$
\begin{aligned}
U_f\left(|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) &= U_f\left(\frac{|x\rangle|0\rangle-|x\rangle|1\rangle}{\sqrt{2}}\right) \\
&= \frac{|x\rangle|0\oplus f(x)\rangle-|x\rangle|1\oplus f(x)\rangle}{\sqrt{2}}
\end{aligned}
$$

Succintly put

$$
U_f\left(|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) = (-1)^{f(x)}|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)
$$

Appendix Deutsch

$$
\begin{aligned}
|\psi_1\rangle &= H^{\otimes 2}\left(|0\rangle \otimes |1\rangle\right) \\
&= \left(H|0\rangle\right) \otimes \left(H|1\rangle\right) \\
&= \left(\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\right) \otimes \left(\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\right) \\
&= |0\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} + |1\rangle \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)
\end{aligned}
$$

Then

$$
\psi_2 = U_f|\psi_1\rangle = \begin{cases} \pm \quad \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm \quad \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}
$$

The final Hadamard gate on the first qubit gives

$$
\psi_3 = \begin{cases} \pm \quad |0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) = f(1) \\ \pm \quad |1\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) & \text{if } f(0) \neq f(1) \end{cases}
$$

Appendix Josza

$$\psi_2 = U_f \psi_1 = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^n}} \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Now extrapolating from $H|0\rangle = \sum_{z\in\{0,1\}} (-1)^{0\cdot z} |z\rangle / \sqrt{2}$ and $H|1\rangle = \sum_{z\in\{0,1\}} (-1)^{1\cdot z} |z\rangle / \sqrt{2}$ applying to

$$\begin{aligned} H^{\otimes n} |x_1, \ldots, x_n\rangle_{x_i \in \{0,1\}} &= \bigotimes_{i=1}^{n} \left(H |x_i\rangle \right)_{x_i \in \{0,1\}} \\ &= \left(\sum_{z\in\{0,1\}} \frac{(-1)^{x_i \cdot z}}{\sqrt{2}} |z\rangle \right)^{\otimes n}_{x_i \in \{0,1\}} \\ &= \sum_{z_1,\ldots,z_n} \frac{(-1)^{x_1 z_1 + \cdots + x_n z_n}}{\sqrt{2^n}} |z_1, \ldots, z_n\rangle \end{aligned}$$

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