Test Problem

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Problem . What is the expected time until the shuffler sees the Queen of Hearts on top of the deck when there are 52 cards?

Solution. The transition matrix is

$$P = \begin{pmatrix} \frac{1}{52} & \frac{1}{52} & \frac{1}{52} & \frac{1}{52} & \frac{1}{52} & \cdots & \frac{1}{52} & \frac{1}{52} & \frac{1}{52} \\ \frac{51}{52} & \frac{1}{52} & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & \frac{50}{52} & \frac{2}{52} & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & \frac{49}{52} & \frac{3}{52} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{48}{52} & \frac{4}{52} & & 0 & 0 & 0 \\ & \vdots & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & \frac{2}{52} & \frac{50}{52} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{52} & \frac{51}{52} \end{pmatrix}$$

Observe that $\sum_x p(x,y) = \sum_y p(x,y) = 1$. So the chain is doubly stochastic and therefore has the unique stationary distribution that is the uniform distribution $\pi(x) = \frac{1}{52}$ for all $1 \leqslant x \leqslant 52$. By Thm 1.22 $E_y T_y = \frac{1}{\pi_y} = 52$.