ESI 6420 : Fundamentals of Mathematical Programming

Jean-Philippe P. Richard

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Homework 4, due in class, Wednesday December 2$^{\text{nd}}$ 2015

Preliminaries

Throughout the semester, I would like you to learn how to use latex (which you will find helpful in the future). In particular, for homework i, I would like you to latex at least $\lfloor \frac{i}{2} \rfloor$ of your answers. For an introduction to latex, refer to `http://www.ctan.org/tex-archive/info/gentle/gentle.pdf`.

When the question you answer involves GAMS, please include your GAMS code, together with the relevant part of the GAMS output. In particular, optimal solution and optimal values should be described clearly.

If you received some help to obtain the solution of a problem, you should acknowledge the source of help you received. In particular, for each question, I would like you to cite, if applicable, any book (other than the textbook) you consulted, any website you searched, or any individual you cooperated with. This information will not be used to adjust your homework score provided that help is limited to a reasonable portion of the homework.

Finally, I would like you to candidly assess the number of hours it took you to complete the homework.

Problem 1: Benders' Reformulation for MIPs

Consider the problem

$$\begin{aligned} \min \quad & c^\intercal x + d^\intercal y \\ s.t. \quad & Ax + Dy \geq b, \\ & x \geq 0, y \geq 0, \\ & x \in X \subseteq \mathbb{Z}^n \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times n'}$, $b \in \mathbb{R}^{m \times 1}$, $c \in \mathbb{R}^{n \times 1}$, $d \in \mathbb{R}^{n' \times 1}$. We can write this problem as:

$$\min_{x \in X} \left\{ c^\intercal x + \min \left\{ d^\intercal y \;\middle|\; Dy \geq b - Ax, y \geq 0 \right\} \right\}.$$

Taking the dual of the inner minimization problem gives:

$$\max_{u \in U} \{ u^\intercal (b - Ax) \}$$

where $U = \{ u \in \mathbb{R}^m | u^\intercal D \leq d^\intercal, u \geq 0 \}$ and we assume that U is nonempty. Let u^p, for $p = 1, \ldots, P$ denote the set of extreme points of U and let v^s, for $s = 1, \ldots, S$ denote the set of extreme rays of U. The Benders' reformulation approach rewrites this problem as:

$$(\mathcal{P}) \qquad \min_{x \in X} \left\{ c^\intercal x + \max_{u \in U} \{ u^\intercal (b - Ax) \} \right\}.$$

1. Write necessary and sufficient conditions on x in terms of the extreme rays $v^s, s = 1, \ldots, S$ that ensure a corresponding feasible y exists for the original mixed integer program.
2. By introducing an additional variable z, show that, after the conditions of Part 1 are enforced, $(\mathcal{P})$ can be reformulated as a linearly constrained minimization problem over $X \times \mathbb{R}$ whose objective is $c^\intercal x + z$.

3. Apply your results of Part 1 and Part 2 to create a reformulation of the problem

$$\min \left\{ -3x_1 - 7x_2 + y_1 + 2y_2 \;\middle|\; \begin{array}{l} -x_1 - x_2 - y_1 - y_2 \geq -4.5 \\ -x_1 + y_1 \geq -1.5 \\ -x_2 + y_2 \geq -0.5 \\ x_1 \in \mathbb{Z}^+, x_2 \in \mathbb{Z}^+ \\ y_1 \geq 0, y_2 \geq 0 \end{array} \right\}$$

without the y variables.

Problem 2: Convex Functions, H-UL, 25, pg 120

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. For $h > 0$ define $f_h : \mathbb{R} \to \mathbb{R}$ as

$$f_h(x) := \frac{1}{2h} \int_{x-h}^{x+h} f(t)dt.$$

Show that f is convex if and only if $f_h(x) \geq f(x)$ for all $x \in \mathbb{R}$ and for all $h > 0$. Give a geometric interpretation of this result.

2. Consider the function $f(X) = -log(det(X))$, where $det(X)$ stands for the determinant of X, defined over the set of symmetric matrices that are positive-definite, *i.e.*, $X \in \mathbb{S}^n_{++}$. For any X in $\mathbb{S}^n_{++}$ and $D \in \mathbb{S}^n$, define $g(t) = f(X + tD)$.

   (a) Show that $g(t) = -log(det(X^{\frac{1}{2}}(I + tX^{-\frac{1}{2}}DX^{-\frac{1}{2}})X^{\frac{1}{2}}))$.

   (b) Using the result of Part 2a, show that $f(X)$ is convex over $\mathbb{S}^n_{++}$ (*you can use the facts that* $det(X) = \Pi_{i=1}^n \lambda_i$ *where* λ_i *are the eigenvalues of* X *and* X *is semi-definite positive iff its eigenvalues are nonnegative*).

Problem 3: Maximizing probability of satisfying a linear inequality, VdB

Let c be a random variable in $\mathbb{R}^n$ that is normally distributed with mean μ and covariance matrix Σ, where we assume that Σ is positive definite. We wish to solve the problem

$$(\mathcal{P}) \quad \begin{array}{ll} \max & \Pr[c^\intercal x \geq \alpha] \\ s.t. & Fx \leq g \\ & Ax = b, \end{array}$$

where $c \in \mathbb{R}^n$, $F \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$, $A \in \mathbb{R}^{p\times n}$, and $b \in \mathbb{R}^p$. Assuming that there exists a vector $x \in \mathbb{R}^n$ such that $\Pr[c^\intercal x \geq \alpha] \geq \frac{1}{2}$, reformulate $(\mathcal{P})$ as a convex optimization program with quadratic objective and linear constraints. (*Hint: a transformation similar to the one we used to convert certain fractional programs into LPs might come in handy here.*)

Problem 4: Simple optimization problem, dKRT, 2.7, pg 40

We wish to design a cylindrical can with height h and radius r such that the volume is at least V units and the total surface area is minimal. We can formulate this problem as the following mathematical program:

$$p^* = \min 2\pi r^2 + 2\pi rh$$

subject to

$$\pi r^2 h \geq V, r > 0, h > 0.$$

1. Show that we can rewrite the above problem as the following mathematical program:

$$p^* = \min 2\pi(e^{2x_1} + e^{x_1+x_2})$$

subject to

$$\ln\left(\frac{V}{\pi}\right) - 2x_1 - x_2 \leq 0,\ x_1 \in \mathbb{R}, x_2 \in \mathbb{R}.$$

2. Prove that the new problem is a convex optimization problem.

3. Prove that an optimal design is $r = \frac{1}{2}h = \left(\frac{V}{2\pi}\right)^{1/3}$.

Problem 5: KKT points for bilinear programs, BSS, 4.25 pg 226, modified

Consider the bilinear program

$$\begin{aligned} \min \quad & c^\intercal x + d^\intercal y + x^\intercal H y \\ s.t. \quad & x \in X, y \in Y, \end{aligned}$$

where X and Y are bounded polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $\hat{x}$ and $\hat{y}$ be extreme points of the sets X and Y, respectively.

1. Prove that there exists an extreme point $(\bar{x}, \bar{y})$ that solves the bilinear program.

2. Prove that the point $(\hat{x}, \hat{y})$ is a local minimum of the bilinear program if and only if the following are true: (i) $(c^\intercal + \hat{y}^\intercal H)(x - \hat{x}) \geq 0$ and $(d^\intercal + \hat{x}^\intercal H)(y - \hat{y}) \geq 0$ for each $x \in X$ and $y \in Y$; (ii) $(c^\intercal + \hat{y}^\intercal H)(x - \hat{x}) + (d^\intercal + \hat{x}^\intercal H)^\intercal(y - \hat{y}) > 0$ whenever $(x - \hat{x})^\intercal H(y - \hat{y}) < 0$.

3. Show that the point $(\hat{x}, \hat{y})$ is a KKT point if and only if $(c^\intercal + \hat{y}^\intercal H)(x - \hat{x}) \geq 0$ for each $x \in X$ and $(d^\intercal + \hat{x}^\intercal H)(y - \hat{y}) \geq 0$ for each $y \in Y$.

4. Consider the problem of minimizing $x_2 + y_1 + x_2y_1 - x_1y_2 + x_2y_2$ subject to $(x_1, x_2) \in X$ and $(y_1, y_2) \in Y$, where X is the polyhedral set defined by its extreme points $(0, 0)$, $(0, 1)$, $(1, 4)$, $(2, 4)$ and $(3, 0)$, and Y is the polyhedral set defined by its extreme points $(0, 0)$, $(0, 1)$, $(1, 5)$, $(3, 5)$, $(4, 4)$ and $(3, 0)$. Verify that the point $(x_1, x_2, y_1, y_2) = (0, 0, 0, 0)$ is a KKT point but not a local minimum. Verify that the point $(x_1, x_2, y_1, y_2) = (3, 0, 1, 5)$ is both a KKT point and a local minimum. What is the global minimum of the problem?

Problem 6: Kantorovich Inequality, VdB

1. Suppose $a \in \mathbb{R}^n$ with $a_1 \geq a_2 \geq \ldots \geq a_n > 0$ and $b \in \mathbb{R}^n$ with $b_k = \frac{1}{a_k}$. Derive KKT conditions for the convex optimization problem

$$\begin{aligned} z^* = \quad \min \quad & -\ln(a^\intercal x) - \ln(b^\intercal x) \\ s.t. \quad & 1^\intercal x = 1, x \geq 0. \end{aligned}$$

2. Show that $x^* = (0.5, 0, \ldots, 0, 0.5)$ is an optimal solution for this problem. Compute the value of z^*.

3. Suppose that A is a symmetric, positive definite matrix with eigenvalues λ_k sorted in decreasing order. Then apply the result of Part 1 with $a_k = \lambda_k$ to prove Kantorovich inequality:

$$2(u^\intercal Au)^{1/2}(u^\intercal A^{-1}u)^{1/2} \leq \sqrt{\frac{\lambda_1}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_1}}$$

for all u with $||u||_2 = 1$. *Hint: Using the spectral decomposition theorem might come in handy here.*