STA 6326 Homework 5 Solutions

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4.1 (a) Probability of falling within the unit circle is area of the circle divided by area of Ω. Hence $P\left(X^2+Y^2<1\right)=\pi/4$.

(b) $P(2X>Y)$ is the area below the line $y=2x$, divided by the area of Ω. The portion of quadrant one that's above the line is the same as the portion in quadrant 3 that's below the line, and all of quadrant 4 is included. Hence $P(2X>Y)=1/2$.

(c) $P(|X+Y|<2)$ is the area that is comprised of all points p in $\mathbb{R}^2$ such that $\|p\|_1<2$. This is the diamond with vertices $(0,2),(2,0),(-2,0),(0-,2)$ and includes all of Ω. Therefore $P(|X+Y|<2)=1$.

4.2 (a)

$$\begin{aligned}
E\left(ag_1(X,Y)+bg_2(X,Y)+c\right) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(ag_1(x,y)+bg_2(x,y)+c\right)f_{X,Y}(x,y)dxdy\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ag_1(x,y)\,f_{X,Y}(x,y)dxdy+\\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} bg_2(x,y)\,f_{X,Y}(x,y)dxdy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cf_{X,Y}(x,y)dxdy\\
&= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x,y)\,f_{X,Y}(x,y)dxdy\\
&+b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(x,y)\,f_{X,Y}(x,y)dxdy+c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy\\
&= aE\left(g_1(X,Y)\right)+bE\left(g_2(X,Y)\right)+c\cdot 1
\end{aligned}$$

(b)

$$E\left(g(X,Y)\right)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\,f_{X,Y}(x,y)dxdy$$

Now since by assumption $g(x,y)\geq 0$ and by definition $f_{X,Y}(x,y)\geq 0$ it's the case that $g(x,y)\cdot f_{X,Y}(x,y)\geq 0$ and therefore

$$E\left(g(x,y)\right)\geq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(0)\,dxdy=0$$

(c) If $g_1(X,Y)\geq g_2(X,Y)$ then $g_1(X,Y)-g_2(X,Y)\geq 0$ and by part (a) and (b)

$$0\leq E\left(g_1(X,Y)-g_2(X,Y)\right)=E\left(g_1(X,Y)\right)-E\left(g_2(X,Y)\right)$$

Therefore $E\left(g_1(X,Y)\right)\geq E\left(g_2(X,Y)\right)$.

(d) $a\leq g(X,Y)\leq b$ implies $a\leq g(X,Y)$ and $g(X,Y)\leq b$. Then part (a) and (c) implies $a=E(a)\leq E\left(g(X,Y)\right)$ and $E\left(g(X,Y)\right)\leq E(b)=b$.

4.4 $f_{X,Y}(x,y) = C(x+2y)I_{(0,2)\times(0,1)}$

(a)

$$\begin{aligned}
C &= \left[\int_0^1 \int_0^2 (x+2y)dxdy\right]^{-1} \\
&= \left[\int_0^1 \left(\frac{x^2}{2} + 2xy\right)\Big|_0^2 dy\right]^{-1} \\
&= \left[\int_0^1 (2+4y)dy\right]^{-1} \\
&= \left[\left(2y+2y^2\right)\Big|_0^1\right]^{-1} \\
&= [4]^{-1} \\
&= 1/4
\end{aligned}$$

(b)

$$\begin{aligned}
f_X(x) &= \frac{1}{4}\int_0^1 (x+2y)dy \\
&= \frac{1}{4}\left(xy+y^2\right)\Big|_0^1 \\
&= \frac{1}{4}(x+1)I_{(0,2)}
\end{aligned}$$

(c)

$$\begin{aligned}
F_{X,Y}(x,y) &= \frac{1}{4}\int_0^y \int_0^x (x+2y)dxdy \\
&= \frac{1}{4}\int_0^y \left(\frac{x^2}{2} + 2xy\right) dy \\
&= \frac{1}{4}\left(\frac{yx^2}{2} + xy^2\right)
\end{aligned}$$

(d) $Z = g(X) = 9/(X+1)^2$ implies $g^{-1}(Y) = 3/\sqrt{Z}$ since $0 < X < 2$ and

$$\left|g^{-1}(z)\right| = \frac{3}{2}z^{-3/2}$$

and hence

$$f_Z(z) = f_X\left(\frac{3}{2}z^{-3/2}\right) = \frac{1}{4}\left(\frac{3}{2}z^{-3/2}+1\right)I_{(3,9)}$$

4.5 (a) $f_{X,Y}(x,y) = (x+y)I_{(0,1)\times(0,1)}$. Since $Y > 0$ it's the case that $P\left(X > \sqrt{Y}\right) \iff P\left(X^2 > Y\right)$, i.e. Y is below the parabola x^2. Therefore

$$\begin{aligned}
P\left(X^2 > Y\right) &= \int_0^1 \int_0^{x^2} (x+y)dydx \\
&= \int_0^1 \left(x^3 + \frac{x^4}{2}\right) dx \\
&= \frac{1}{4} + \frac{1}{10} = \frac{7}{20}
\end{aligned}$$

(b) $f_{X,Y}(x,y) = 2x \cdot I_{(0,1)\times(0,1)}$. Therefore

$$\begin{aligned} P\left(X^2 < Y < X\right) &= \int_0^1 \int_{x^2}^{x} 2x dy dx \\ &= \int_0^1 2x\left(x - x^2\right) dx \\ &= \int_0^1 \left(2x^2 - 2x^3\right) dx \\ &= \frac{2}{3}x^3 - \frac{1}{2}x^4 \Big|_0^1 = \frac{1}{6} \end{aligned}$$

4.6 Let $A \sim$ uniform$(0,1)$ and similarly B. Then

$$Z = \begin{cases} B - A & \text{if } B \geq A \\ 0 & \text{otherwise} \end{cases}$$

To compute the distribution of Z consider

$$F_Z(z) = P\left(Z\left(\zeta\right) \leq z\right) = P\left(g\left(A,B\right) \leq z\right) = P\left(\left(A,B\right) \in \mathcal{D}_z\right)$$

where $\mathcal{D}_z$ is the subset of the sample space of the product of the samples space of A and B such that $g\left(a,b\right) \leq z$. That is

$$F_Z(z) = P\left(Z\left(\zeta\right) \leq z\right) = P\left(B - A \leq z\right)$$

Since Z is piecewise defined we consider first $F_Z(0) = P\left(B - A \leq 0\right)$. Clearly $F_Z(0) = 1/2$. Otherwise if $z \geq 0$. then $B - A \leq z \iff B \leq A - z$ and $B \geq A$, by definition of Z, which is equivalent to $A \leq B \leq A + z$. There the scheme is to integrate the joint density of A and B where $A \leq B \leq A + z$. This is a portion of the upper left triangle of $(0,1)\times(0,1)$. It's an irregular domain in that for $0 \leq A \leq 1 - z$ the variable B ranges from $A \leq B \leq A + z$ but for $1 - z \leq A \leq 1$ the variable B ranges $A \leq B \leq 1$. Therefore splitting up the integral

$$\begin{aligned} P\left(A \leq B \leq A + z\right) &= \int_0^{1-z} \left(\int_a^{a+z} f_{A,B}(a,b) db\right) da + \int_{1-z}^{1} \left(\int_a^1 f_{A,B}(a,b) db\right) da \\ &= \int_0^{1-z} \left(\int_a^{a+z} db\right) da + \int_{1-z}^{1} \left(\int_a^1 db\right) da \\ &= \int_0^{1-z} z da + \int_{1-z}^{1} \left(1 - a\right) da \\ &= z\left(1 - z\right) + \left(1 - \left(1 - z\right) - \frac{\left(1 - \left(1 - z\right)^2\right)}{2}\right) \\ &= z - z^2 + \frac{z^2}{2} \end{aligned}$$

The simpler way to do this is to compute the regular region and then take the complement, where the total probability of the upper left triangular region is obviously 1/2 similarly to how the lower right triangular region has probability 1/2.

$$
\begin{aligned}
P(A \le B \le A+z) &= \frac{1}{2} - P(B - A \ge z) = 1 - P(1 \ge B \ge A + z) \\
&= \frac{1}{2} - \int_0^{1-z} \left(\int_{a+z}^{1} f_{A,B}(a,b)db \right) da \\
&= \frac{1}{2} - \int_0^{1-z} \left(\int_{a+z}^{1} db \right) da \\
&= \frac{1}{2} - \int_0^{1-z} (1 - a - z)\, da \\
&= \frac{1}{2} - (1-z) - \left(\frac{(1-z)^2}{2} \right) - z(1-z) \\
&= \frac{1}{2} - (1-z)^2 - \left(\frac{(1-z)^2}{2} \right) \\
&= z - \frac{z^2}{2}
\end{aligned}
$$

4.7 Let Y be uniformly distributed on $A = (0, 30)$ and X be uniformly distributed on $B = (40, 50)$ and their joint distribution be

$$
f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{30}\frac{1}{10} I_{B \times A}(y,x)
$$

Then the probability the woman makes it on time is $P(X + Y < 60)$

$$
\begin{aligned}
P(X+Y<60) &= \int_{40}^{50} \int_0^{60-x} \frac{1}{30}\frac{1}{10} dy dx \\
&= \frac{1}{300} \int_{40}^{50} (60 - x)\, dx \\
&= \frac{1}{300} \left(60x - \frac{x^2}{2} \right) \Bigg|_{40}^{50} \\
&= .5
\end{aligned}
$$

4.10 (a) The marginals are

$$
f_X(x) = \begin{cases} 3/12 & x = 1 \\ 1/2 & x = 2 \\ 3/12 & x = 3 \end{cases}
$$

and

$$
f_Y(y) = \begin{cases} 1/3 & y = 2 \\ 1/3 & y = 3 \\ 1/3 & y = 4 \end{cases}
$$

but

$$
f_{X,Y}(x,y) = 0 \ne \frac{1}{12} = f_X(1) f_Y(4)
$$

(b)

$$
f_{X,Y}(x,y) = \begin{cases}
\frac{1}{12} \ \text{if } (x,y) = (1,2) & \frac{1}{6} \ \text{if } (x,y) = (2,2) & \frac{1}{12} \ \text{if } (x,y) = (3,2) \\
\frac{1}{12} \ \text{if } (x,y) = (1,3) & \frac{1}{6} \ \text{if } (x,y) = (2,3) & \frac{1}{12} \ \text{if } (x,y) = (3,3) \\
\frac{1}{12} \ \text{if } (x,y) = (1,4) & \frac{1}{6} \ \text{if } (x,y) = (2,4) & \frac{1}{12} \ \text{if } (x,y) = (3,4)
\end{cases}
$$

4.11 No since since V is necessarily greater than U (the second head must succeed the first head), therefore knowledge of U confers knowledge of V.

4.12 The requirements for 3 lengths to constitute the legs of a triangle is that the sum of any two has to be greater than the third. Consider two uniformly distributed variables: $X \sim \text{uniform}(0,1)$ and $Y \sim \text{uniform}(0,1)$. Then each draws of the tuple (X, Y) correspond to points on a unit length stick to execute the breaks. The 3 resulting stick lengths are $\min\{X,Y\}, \max\{X,Y\} - \min\{X,Y\}, 1 - \max\{X,Y\}$. Then the triangle requirement is that

$$\begin{aligned}
\min\{X,Y\} + (\max\{X,Y\} - \min\{X,Y\}) &> 1 - \max\{X,Y\} \\
&\text{and} \\
\min\{X,Y\} + (1 - \max\{X,Y\}) &> \max\{X,Y\} - \min\{X,Y\} \\
&\text{and} \\
(1 - \max\{X,Y\}) + (\max\{X,Y\} - \min\{X,Y\}) &> \min\{X,Y\}
\end{aligned}$$

and these are equivalent to

$$\begin{aligned}
\max\{X,Y\} &> 1/2 \\
&\text{and} \\
1/2 &> \max\{X,Y\} - \min\{X,Y\} \\
&\text{and} \\
1/2 &> \min\{X,Y\}
\end{aligned}$$

The second inequality states that $|X - Y| < 1/2$ which indicates a strip along the diagonal. Therefore

$$\begin{aligned}
P\left(\left(|X-Y| < \frac{1}{2}\right) \wedge \left(\min\{X,Y\} < \frac{1}{2}\right) \wedge \left(\max\{X,Y\} > \frac{1}{2}\right)\right) &= \int_0^{1/2} \left(\int_{1/2}^{x+1/2} dy\right) dx \\
&+ \int_{1/2}^{1} \left(\int_{x-1/2}^{1/2} dy\right) dx \\
&= \frac{1}{4}
\end{aligned}$$

4.13 (a) Using calculus, setting the first derivative with respect $g(X)$ equal to 0

$$\begin{aligned}
\frac{d}{dg} E\left[(Y - g(X))^2\right] &= \frac{d}{dg} \int (Y - g(X))^2 f(y|x) dy \\
0 &= \int \frac{d}{dg} (Y - g(X))^2 f(y|x) dy \\
&= \int 2(Y - g(X)) f(y|x) dy \\
&\text{therefore} \\
\int y f(y|x) dy &= \int g(X) f(y|x) dy \\
\int y f(y|x) dy &= g(X) \int f(y|x) dy \\
E[Y|X] &= g(X)
\end{aligned}$$

(b) The fact that

$$\min_b E\left[(Y - b)^2\right] = E\left[(Y - E(X))^2\right]$$

follows from part (a) when Y and X are independent, in which case $E[Y|X] = E[Y]$.

4.14 (a) The joint distribution of X and Y is $f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-x^2/2}e^{-y^2/2} = \frac{1}{2\pi}e^{-\left(x^2+-y\right)/2}$. $P\left(X^2+Y^2<1\right)$ is tantamount to the r**v**$=(X,Y)$ falling within the unit circle. Hence using polar coordinates

$$\begin{aligned}
P\left(X^2+Y^2<1\right) &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} e^{-r^2/2}rdrd\theta \\
&= \int_0^1 e^{-r^2/2}rdr \\
&= \int_0^1 e^{-\left(r^2/2\right)}d\left(\frac{r^2}{2}\right) \\
&= -\left(e^{-r^2/2}\Big|_0^1\right) = 1-e^{-1/2}
\end{aligned}$$

(b) $Y=g(X)=X^2$ and so $g^{-1}(Y)=\sqrt{Y}$ on $x\in[0,\infty)$ and $g^{-1}(Y)=-\sqrt{Y}$ on $x\in(-\infty,0)$. Then

$$\left|g^{-1}(y)'\right| = \frac{1}{2\sqrt{y}}$$

and so

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{y}}\left(f_X\left(\sqrt{y}\right)I_{[0,\infty)}\left(\sqrt{y}\right)+f_X\left(-\sqrt{y}\right)I_{(-\infty,0)}\left(-\sqrt{y}\right)\right) \\
&= \frac{2}{2\sqrt{y}}\frac{1}{\sqrt{2\pi}}\left(e^{-y/2}\right)I_{[0,\infty)}(y) \\
&= \frac{1}{2^{1/2}\Gamma\left(\frac{1}{2}\right)}y^{1/2-1}e^{-y/2}I_{[0,\infty)}(y) \\
&= \chi^2_{1/2}
\end{aligned}$$

and then

$$P\left(X^2<1\right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\gamma\left(\frac{1}{2},\frac{1}{2}\right) \approx .317311$$

where $\gamma(x,y)$ is the lower incomplete gamma function.

4.15 (a) Let $Z=X+Y$ then $Z\sim Poisson(\theta+\lambda)$ and

$$\begin{aligned}
P\left(X=x|Z=z\right) &= \frac{P\left(X=x\wedge Z=z\right)}{P(Z=z)} \\
&= \frac{P\left(X=x\wedge Y=z-x\right)}{P(Z=z)} \\
&= \frac{e^{-\theta}\theta^x}{x!}\frac{e^{-\lambda}\lambda^{z-x}}{(z-x)!}\Bigg/\frac{e^{-(\theta+\lambda)}\left(\theta+\lambda\right)^z}{z!} \\
&= \frac{\theta^x}{x!}\frac{\lambda^{z-x}}{(z-x)!}\Bigg/\frac{\left(\theta+\lambda\right)^z}{z!} \\
&= \binom{z}{x}\frac{\theta^x\lambda^{z-x}}{\left(\theta+\lambda\right)^z} \\
&= \binom{z}{x}\left(\frac{\theta}{\theta+\lambda}\right)^x\left(\frac{\lambda}{\theta+\lambda}\right)^{z-x}
\end{aligned}$$

Hence $X|Z=X|\left(X+Y\right)\sim Binomial\left(X+Y,\frac{\theta}{\theta+\lambda}\right)$.

(b) By symmetry $Y|\left(X+Y\right) \sim \textit{Binomial}\left(X+Y, \frac{\lambda}{\theta+\lambda}\right)$.

4.16 (a) $X \sim \text{geom}(p)$ and $Y \sim \text{geom}(p)$. Then $P\left(X=x\right) = (1-p)^{x-1}p$ and $P\left(Y=y\right) = (1-p)^{y-1}p$ on $\mathbb{N}\times\mathbb{N}$ and $P\left(X=x, Y=y\right) = (1-p)^{x-1}p\,(1-p)^{y-1}p$. Then on the upper left triangular region of the plane we have that $U = X$ and $U - V = Y$ and therefore

$$\begin{aligned}
P\left(U=u, V=v\right) &= P\left(X=u, Y=u-v\right) \\
&= (1-p)^{u-1}p\,(1-p)^{u-v-1}p \\
&= (1-p)^{u-1}p\,(1-p)^{u-1}p\,(1-p)^{-v-1} \\
&= \left((1-p)^{u-1}p\right)^2(1-p)^{-v-1} \\
&= \left((1-p)^2\right)^{u-1}p\,(2-p)\,\frac{(1-p)^{-v-1}p}{2-p} \\
&= \left((1-p)^2\right)^{u-1}\left(1-(1-p)^2\right)\frac{(1-p)^{-v-1}p}{2-p} \\
&\text{and since } Y > X \implies V < 0 \\
&= \left((1-p)^2\right)^{u-1}\left(1-(1-p)^2\right)\frac{(1-p)^{|v|-1}p}{2-p}
\end{aligned}$$

Then on the lower right triangular region of the plane we have that $U = Y$ and $U+V = Y$ and therefore

$$\begin{aligned}
P\left(U=u, V=v\right) &= P\left(X=u, Y=u+v\right) \\
&= (1-p)^{u-1}p\,(1-p)^{u+v-1}p \\
&= (1-p)^{u-1}p\,(1-p)^{u-1}p\,(1-p)^{v-1} \\
&= \left((1-p)^{u-1}p\right)^2(1-p)^{v-1} \\
&= \left((1-p)^2\right)^{u-1}p\,(2-p)\,\frac{(1-p)^{v-1}p}{2-p} \\
&= \left((1-p)^2\right)^{u-1}\left(1-(1-p)^2\right)\frac{(1-p)^{v-1}p}{2-p}
\end{aligned}$$

On the line $Y = X$ we have that $U = x = y$ and $V = 0$ and therefore

$$\begin{aligned}
P\left(U=u, V=v\right) &= P\left(X=u, Y=u\right) \\
&= (1-p)^{u-1}p\,(1-p)^{u-1}p \\
&= (1-p)^{u-1}p\,(1-p)^{u-1}p\,(1-p)^{v-1} \\
&= \left((1-p)^{u-1}p\right)^2 \\
&= \left((1-p)^2\right)^{u-1}p\,(2-p)\,\frac{1}{2-p} \\
&= \left((1-p)^2\right)^{u-1}\left(1-(1-p)^2\right)\frac{1}{2-p}
\end{aligned}$$

(b) Let $V = \frac{X}{X+Y}$ and $U = X+Y$. Then $UV = X$ and $U - UV = Y$. Then

$$\begin{aligned}
P\left(U=u, V=v\right) &= P\left(X=uv, Y=u-uv\right) \\
&= (1-p)^{uv-1}p\,(1-p)^{u-uv-1}p \\
&= (1-p)^{u-2}p^2
\end{aligned}$$

So $V \sim$ uniform$(0,1)$ and $U \sim$ neg$(2,p)$.

(c) Let $V = X$ and $U = X + Y$. Then $U - V = Y$ and

$$\begin{aligned} P(U=u, V=v) &= P(X=v, Y=u-v) \\ &= (1-p)^{v-1} p (1-p)^{u-v-1} p \\ &= (1-p)^{u-2} p^2 \end{aligned}$$

and $U \sim$ negb$(2, 1-p)$. This can also be seen from the product of the the MGFs:

$$\begin{aligned} Z \sim X + Y &\iff MGF_Z = MGF_X MGF_Y \\ MGF_Z &= \left(\frac{p}{1-(1-p)e^t}\right)^2 \end{aligned}$$

so $Z \sim$ negb$(2, 1-p)$.

4.17 (a) $X \sim \mathit{exponential}(1)$ and $Y = \lfloor X + 1 \rfloor$. Then

$$\begin{aligned} P(Y = i+1) &= \int_i^{i+1} e^{-x} dx \\ &= -\left(e^{-x}\Big|_i^{i+1}\right) \\ &= e^{-i}\left(1 - \frac{1}{e}\right) \\ &= \left(\frac{1}{e}\right)^i \left(1 - \frac{1}{e}\right) \end{aligned}$$

Therefore $Y \sim \mathit{Geom}\left(1 - \frac{1}{e}\right)$

(b) Let $Z = X - 4$ then $f_Z(z) = f_X(z+4)$ and if $Z \geq 0$ then $Y = X + 1 \geq 5$ hence

$$\begin{aligned} P(Z = z | Y \geq 5) &= \frac{P(Z = z \wedge Y \geq 5)}{P(Y \geq 5)} \\ &= \frac{P(Z=z)}{P(Y \geq 5)} \\ &= \frac{P(Z=z)}{\sum_{i=4}^{\infty} \left(\frac{1}{e}\right)^i \left(1 - \frac{1}{e}\right)} \\ &= \frac{e^{-4}e^{-z}}{1/e^4} = e^{-z} \end{aligned}$$

Therefore $X - 4 \sim \mathit{exponential}(1)$.

4.18 $f(x,y)$ is always positive since $g(x) \geq 0$. Then

$$\begin{aligned} \int_{\mathbb{R}_+^2} f(x,y) dA &= \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} \frac{g\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} dx dy \\ &= \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\infty} \frac{g(r)}{r} r dr d\theta \\ &= \int_0^{\infty} g(r) dr = 1 \end{aligned}$$

4.19 (a) Let $Y = X_1 - X_2 \sim \mathcal{N}(0,2)$. Then $Y/\sqrt{2} \sim \mathcal{N}(0,1)$ and therefore

$$\left(\frac{X_1 - X_2}{\sqrt{2}}\right)^2 \sim \chi_1^2$$

(b) Let $X_1 \sim \Gamma(\alpha_1, 1)$ and $X_2 \sim \Gamma(\alpha_2, 1)$ and $U = \frac{X_!}{X_1+X_2}$ and $V = \frac{X_2}{X_1+X_2}$ then $U + V = 1$

4.20 (a) If $X_1 \sim \mathrm{n}\left(0, \sigma^2\right)$ and $X_2 \sim \mathrm{n}\left(0, \sigma^2\right)$ then $X_1/\sigma \sim \mathrm{n}(0,1)$ and $X_2/\sigma \sim \mathrm{n}(0,1)$. Let $Y = (X_1)^2 + (X_2)^2$ and $V = (X_1/\sigma)^2 + (X_2/\sigma)^2 \sim \chi^2(2)$. Then $\sigma^2 V = Y$. Then

$$
\begin{aligned}
f_Y(y) = f_V\left(\frac{y}{\sigma^2}\right) &= \left(\frac{y}{\sigma^2}\right)^{2/2-1} \frac{e^{-y/2\sigma^2}}{2^{2/2}\Gamma\left(\frac{2}{2}\right)} \\
&= \frac{y^{2/2-1} e^{-y/2\sigma^2}}{\left(\sigma^2\right)^{2/2-1} 2^{2/2}\Gamma\left(\frac{2}{2}\right)} \\
&= \frac{y^{1-1} e^{-y/2\sigma^2}}{\left(2\sigma^2\right)^1 \Gamma(1)}
\end{aligned}
$$

So $(X_1)^2 + (X_2)^2 \sim \Gamma\left(1, 2\sigma^2\right)$. $Y_1 = X_1^2 + X_2^2$ and $Y_2 = \frac{X_1}{\sqrt{X_1^2+X_2^2}} = \frac{X_1}{\sqrt{Y_1}}$. Mapping isn't 1-1 because sign of X_2 is indeterminate from inspection of Y_1 and Y_2. So partition the support of (X_1, X_2) into $\mathcal{A}_1 = \{-\infty < x_1 < \infty, x_2 < 0\}, \mathcal{A}_2 = \{-\infty < x_1 < \infty, x_2 > 0\}, \mathcal{A}_3 = \{-\infty < x_1 < \infty, x_2 = 0\}$. The support of (Y_1, Y_2) is $\mathcal{B} = \{0 \le y_1 < \infty, 0 < y_2 < 1\}$ and the inverse from this support to $\mathcal{A}_1$ is $X_1 = Y_2\sqrt{Y_1}$ and $X_2 = -\sqrt{Y_1 - Y_1 Y_2^2}$, from $\mathcal{B}$ to $\mathcal{A}_2$ is $X_1 = \sqrt{Y_1}$ and $X_2 = 0$, from $\mathcal{B}$ to $\mathcal{A}_3$ is $X_1 = Y_2\sqrt{Y_1}$ and $X_2 = \sqrt{Y_1 - Y_1 Y_2^2}$. The Jacobian on $\mathcal{A}_1$ is

$$
\begin{aligned}
J_1 &= \begin{vmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ -\frac{1-y_2^2}{2\left(y_1 - y_1 y_2^2\right)^{3/2}} & \frac{2y_1 y_2}{2\left(y_1 - y_1 y_2^2\right)^{3/2}} \end{vmatrix} \\
&= \begin{vmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ -\frac{1-y_2^2}{2(y_1)^{1/2}\left(1-y_2^2\right)^{1/2}} & \frac{2y_1 y_2}{2(y_1)^{1/2}\left(1-y_2^2\right)^{1/2}} \end{vmatrix} \\
&= \begin{vmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ -\frac{\sqrt{1-y_2^2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}y_2}{\sqrt{1-y_2^2}} \end{vmatrix} \\
&= \left(\frac{y_2}{2\sqrt{y_1}}\right)\left(\frac{\sqrt{y_1}y_2}{\sqrt{1-y_2^2}}\right) + \sqrt{y_1}\left(\frac{\sqrt{1-y_2^2}}{2\sqrt{y_1}}\right) \\
&= \frac{y_2^2}{2\sqrt{1-y_2^2}} + \frac{\sqrt{1-y_2^2}}{2} = \frac{1}{2\sqrt{1-y_2^2}}
\end{aligned}
$$

The Jacobian on $\mathcal{A}_2$ is

$$
\begin{aligned}
J_3 &= \begin{vmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \frac{1-y_2^2}{2\left(y_1 - y_1 y_2^2\right)^{3/2}} & -\frac{2y_1 y_2}{2\left(y_1 - y_1 y_2^2\right)^{3/2}} \end{vmatrix} \\
&= \begin{vmatrix} \frac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ -\frac{1-y_2^2}{2(y_1)^{1/2}\left(1-y_2^2\right)^{1/2}} & \frac{2y_1 y_2}{2(y_1)^{1/2}\left(1-y_2^2\right)^{1/2}} \end{vmatrix} \\
&= -J_1
\end{aligned}
$$

The Jacobian on $\mathcal{A}_2$ doesn't need to be computed because the event $\mathcal{A}_2$ has measure zero.

Then the joint distribution of Y_1 and Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}\left(y_2\sqrt{y_1},\sqrt{y_1-y_1y_2^2}\right)\frac{1}{2\sqrt{1-y_2^2}}+f_{X_1,X_2}\left(y_2\sqrt{y_1},-\sqrt{y_1-y_1y_2^2}\right)\left|\frac{-1}{2\sqrt{1-y_2^2}}\right| \\ &= \frac{1}{2\pi\sigma^2}\left(e^{-\left(y_2\sqrt{y_1}\right)^2/\sigma^2}e^{-\left(\sqrt{y_1-y_1y_2^2}\right)^2/\sigma^2}+e^{-\left(y_2\sqrt{y_1}\right)^2/\sigma^2}e^{-\left(-\sqrt{y_1-y_1y_2^2}\right)^2/\sigma^2}\right)\frac{1}{2\sqrt{1-y_2^2}} \\ &= \frac{1}{2\pi\sigma^2}e^{-\left(y_2^2y_1+y_1-y_1y_2^2\right)/\sigma^2}\frac{1}{\sqrt{1-y_2^2}} \\ &= \frac{1}{2\pi\sigma^2}e^{-y_1/\sigma^2}\frac{1}{\sqrt{1-y_2^2}} \end{aligned}$$

(b) Independent because density factors.

4.21 If $Y_1 \sim \chi_2^2$ and $R = g(Y_1) = \sqrt{Y_1}$ then $g^{-1}(R) = R^2$ and

$$\left|g^{-1}(r)\right| = 2r$$

Then

$$f_R(r) = f_{Y_1}\left(r^2\right)2r = \frac{\left(r^2\right)^{2/2-1}e^{-r^2/2}}{2^{2/2}\Gamma\left(\frac{2}{2}\right)}2r = re^{-r^2/2}$$

and

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

so

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi}re^{-r^2/2}$$

Solving $X = R\cos(\Theta), Y = R\sin(\Theta)$ simultaneously we get $R = \sqrt{X^2+Y^2}$ and $\Theta = \arctan\left(\frac{Y}{X}\right)$. Then the Jacobian is

$$\|J\| = \left\|\left|\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix}\right|\right\| = \left\|\frac{x^2}{\left(x^2+y^2\right)^{3/2}}+\frac{y^2}{\left(x^2+y^2\right)^{3/2}}\right\| = \frac{x^2+y^2}{\left(x^2+y^2\right)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}}$$

Hence

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\sqrt{x^2+y^2}e^{-\left(x^2+y^2\right)/2}\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{2\pi}e^{-\left(x^2+y^2\right)/2} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}$$

with $-\infty < x, y < \infty$.

4.23 Let

$$f_{X,Y}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{x^{\alpha-1}}{(1-x)^{1-\beta}}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)}\frac{y^{\alpha+\beta-1}}{(1-y)^{1-\gamma}}I_{\{(0,1)\times(0,1)\}}(x,y)$$

(a) $U = XY$ and $V = Y$ implies that $0 < v < 1$ and $0 < u = x \cdot v < v$ since $0 < x < 1$. Therefore $0 < u < v < 1$. Also $U = XY$ and $V = Y$ implies $X = U/V$ hence

$$\|J\| = \left\|\left|\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{pmatrix}\right|\right\| = \left\|\frac{1}{v}-0\right\| = \frac{1}{v}$$

and therefore

$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{u}{v}\right)^{\alpha-1}\left(1-\frac{u}{v}\right)^{\beta-1}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)}v^{\alpha+\beta-1}(1-v)^{\gamma-1}\frac{1}{v}I_{\{0<u<v<1\}}(u,v)$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(\frac{u}{v}\right)^{\alpha-1}\left(1-\frac{u}{v}\right)^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}\frac{1}{v}I_{\{0<u<v<1\}}(u,v)$$

$$\text{let } \Gamma = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$

$$= \Gamma\left(\frac{u}{v}\right)^{\alpha-1}\left(1-\frac{u}{v}\right)^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}\frac{1}{v}I_{\{0<u<v<1\}}(u,v)$$

$$= \Gamma u^{\alpha-1}\left(1-\frac{u}{v}\right)^{\beta-1}v^{\beta}(1-v)^{\gamma-1}\frac{1}{v}I_{\{0<u<v<1\}}(u,v)$$

$$= \Gamma u^{\alpha-1}(v-u)^{\beta-1}(1-v)^{\gamma-1}I_{\{0<u<v<1\}}(u,v)$$

Then

$$f_U(u) = \Gamma u^{\alpha-1}\int_u^1 (v-u)^{\beta-1}(1-v)^{\gamma-1}\,dv$$

$$\text{let } y = \frac{v-u}{1-u} \text{ then } 1-y = \frac{1-v}{1-u} \text{ and } dy = \frac{dv}{1-u}$$

$$= \Gamma u^{\alpha-1}(1-u)^{(\beta-1)+(\gamma-1)+1}\int_u^1\left(\frac{v-u}{1-u}\right)^{\beta-1}\left(\frac{1-v}{1-u}\right)^{\gamma-1}\frac{dv}{1-u}$$

$$= \Gamma u^{\alpha-1}(1-u)^{(\beta-1)+(\gamma-1)+1}\int_0^1 y^{\beta-1}(1-y)^{\gamma-1}\,dy$$

$$= \Gamma u^{\alpha-1}(1-u)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}$$

Therefore $U \sim \text{beta}(\alpha,\beta+\gamma)$.

(b) $U = XY$ and $V = X/Y$ implies $X = \sqrt{U}\sqrt{V}$ and $Y = \sqrt{U}/\sqrt{V}$ and $0 < u < 1$. Since $U = VY^2$ it's the case that $u < v$ and since $X^2/U = V$ it's the case that $v < 1/u$.

$$\|J\| = \left\|\left|\begin{pmatrix} \frac{1}{2}\frac{\sqrt{v}}{\sqrt{u}} & \frac{1}{2}\frac{\sqrt{u}}{\sqrt{v}} \\ \frac{1}{2}\frac{1}{\sqrt{u}\sqrt{v}} & -\frac{1}{2}\frac{\sqrt{u}}{\sqrt{v}v} \end{pmatrix}\right|\right\| = \left\|-\frac{1}{4}\frac{1}{v} - \frac{1}{4}\frac{1}{v}\right\| = \frac{1}{2v}$$

and therefore

$$f_{U,V}(u,v) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(\sqrt{u}\sqrt{v}\right)^{\alpha-1}\left(1-\sqrt{u}\sqrt{v}\right)^{\beta-1}\left(\frac{\sqrt{u}}{\sqrt{v}}\right)^{\alpha+\beta-1}\left(1-\frac{\sqrt{u}}{\sqrt{v}}\right)^{\gamma-1}\frac{1}{2v}I_{\{0<u<v<\infty\}}(u,v)$$

$$\text{let } \Gamma = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$

$$= \Gamma\frac{\left(\sqrt{u}\sqrt{v}\right)^{\alpha-1}}{\left(1-\sqrt{u}\sqrt{v}\right)^{1-\beta}}\frac{\left(\frac{\sqrt{u}}{\sqrt{v}}\right)^{\alpha+\beta-1}}{\left(1-\frac{\sqrt{u}}{\sqrt{v}}\right)^{1-\gamma}}\frac{1}{2v}I_{\{0<u<v<\infty\}}(u,v)$$

Then by

$$f_U(u) = \Gamma\int_u^{1/u}\frac{\left(\sqrt{u}\sqrt{v}\right)^{\alpha-1}}{\left(1-\sqrt{u}\sqrt{v}\right)^{1-\beta}}\frac{\left(\frac{\sqrt{u}}{\sqrt{v}}\right)^{\alpha+\beta-1}}{\left(1-\frac{\sqrt{u}}{\sqrt{v}}\right)^{1-\gamma}}\frac{1}{2v}dv$$

Then... no clue :)

4.24 If $X \sim \text{gamma}(r,1)$ and $Y \sim \text{gamma}(s,1)$ and $Z_1 = X+Y$ and $Z_2 = X/(X+Y)$ then $0 < z_1 < \infty$ and $0 < z_2 < 1$ and $X = Z_1 Z_2$ and $Y = Z_1 - Z_1 Z_2$. Hence

$$\|J\| = \left\|\left|\begin{pmatrix} z_2 & z_1 \\ 1-z_2 & -z_1 \end{pmatrix}\right|\right\| = \|-z_1 z_2 - z_1(1-z_2)\| = \|z_1\left(-z_2-(1-z_2)\right)\| = z_1$$

Then

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= \frac{1}{\Gamma(r)}(z_1 z_2)^{r-1} e^{-z_1 z_2} \frac{1}{\Gamma(s)}(z_1 - z_1 z_2)^{s-1} e^{-z_1+z_1 z_2} z_1 \\ &= \frac{1}{\Gamma(r)}(z_1)^r e^{-z_1}(z_2)^{r-1} \frac{1}{\Gamma(s)}(z_1)^{s-1}(1-z_2)^{s-1} \\ &= \frac{1}{\Gamma(r+s)}(z_1)^{r+s-1} e^{-z_1} \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}(z_2)^{r-1}(1-z_2)^{s-1} \\ &= f_{Z_1}(z_1) f_{Z_2}(z_2) \end{aligned}$$

where $Z_1 \sim \Gamma(r+s,1)$ and $Z_2 \sim \mathrm{B}(r,s)$.

4.26 (a) The joint CDF of Z is

$$\begin{aligned} F_{Z,W}(z,w) &= P(Z \le z, W = w) \\ \\ &= \sum_{w \in \mathcal{W}} P(Z \le \zeta, W = w) \\ &= P(Z \le z, W = 0) + P(Z \le z, W = 1) \\ &= P(Y \le z, Y < X) + P(X \le z, X < Y) \\ &= \int_0^z \mu e^{-\mu y} \int_y^\infty \lambda e^{-\lambda x} dx dy + \int_0^z \lambda e^{-\lambda x} \int_x^\infty \mu e^{-\mu y} dy dx \\ &= \frac{1}{\mu+\lambda}\left(\mu\left(1-e^{-(\mu+\lambda)z}\right) + \lambda\left(1-e^{-(\mu+\lambda)z}\right)\right) \\ &= 1 - e^{-(\mu+\lambda)z} \end{aligned}$$

but $P(Z \le z, W = w) = \begin{cases} P(Z \le z, W=0) & \text{if } w = 0 \\ P(Z \le z, W=1) & \text{if } w = 1 \end{cases}$ where

$$\begin{aligned} P(Z \le z, W = 0) &= P(Y \le z, Y < X) \\ &= \int_0^z \mu e^{-\mu y} \int_y^\infty \lambda e^{-\lambda x} dx dy \\ &= \frac{\mu}{\mu+\lambda}\left(1-e^{-(\mu+\lambda)z}\right) \end{aligned}$$

and

$$\begin{aligned} P(Z \le z, W = 1) &= P(X \le z, X < Y) \\ &= \int_0^z \lambda e^{-\lambda x} \int_x^\infty \mu e^{-\mu y} dy dx \\ &= \frac{\lambda}{\mu+\lambda}\left(1-e^{-(\mu+\lambda)z}\right) \end{aligned}$$

Hence

$$F_{Z,W}(z,w) = \begin{cases} \frac{\mu}{\mu+\lambda}\left(1-e^{-(\mu+\lambda)z}\right) & \text{if } w = 0 \\ \frac{\lambda}{\mu+\lambda}\left(1-e^{-(\mu+\lambda)z}\right) & \text{if } w = 1 \end{cases}$$

and therefore the joint PDF is

$$f_{Z,W}(z,w) = \begin{cases} \mu e^{-(\mu+\lambda)z} & \text{if } w=0 \\ \lambda e^{-(\mu+\lambda)z} & \text{if } w=1 \end{cases}$$

(b) The marginal PDF of W is

$$f_W(w) = \int_0^\infty f_{Z,W}(z,w)dz = \begin{cases} \int_0^\infty \mu e^{-(\mu+\lambda)z}dz = \frac{\mu}{\mu+\lambda} & \text{if } w=0 \\ \int_0^\infty \lambda e^{-(\mu+\lambda)z}dz = \frac{\lambda}{\mu+\lambda} & \text{if } w=1 \end{cases}$$

The marginal PDF of Z is

$$f_Z(z) = \sum_{w\in\mathcal{W}} f_{Z,W}(z,w) = \mu e^{-(\mu+\lambda)z} + \lambda e^{-(\mu+\lambda)z} = (\mu+\lambda)\, e^{-(\mu+\lambda)z}$$

Hence Z and W are independent since $f_{Z,W}(z,w) = f_Z(z)f_W(w)$.

4.27 Let $X \sim \mathrm{n}\left(\mu,\sigma^2\right)$ and $Y \sim \mathrm{n}\left(\gamma,\sigma^2\right)$ and $U = X+Y$, $V = X-Y$ and U,V range over the same domain as X,Y. Then $X = \frac{U+V}{2}$ and $Y = \frac{U-V}{2}$ and the Jacobian

$$J = \left|\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\right| = -\frac{1}{2}$$

and therefore

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)\left|-\frac{1}{2}\right| \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(\frac{u+v}{2}-\mu\right)^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(\frac{u-v}{2}-\gamma\right)^2}{2\sigma^2}} \frac{1}{2} \\ &= \frac{1}{\sigma\sqrt{4\pi}} e^{\frac{-(u-(\mu+\gamma))^2}{4\sigma^2}} \frac{1}{\sigma\sqrt{4\pi}} e^{\frac{-(v-(\mu-\gamma))^2}{4\sigma^2}} \end{aligned}$$

4.28 Let $X \sim \mathrm{n}(0,1)$ and $Y \sim \mathrm{n}(0,1)$ and $U = \frac{X}{X+Y}$ and $V = X$, where $0 < u < 1$ and $-\infty < v < \infty$, since for a fixed value of v, say v', u can be any number between 0 and 1 since

$$u = \frac{v'}{v'+y}$$

and y can be any number. Then $Y = \frac{V(1-U)}{U}$ and $X = V$. Then Jacobian

$$J = \left|\begin{pmatrix} 0 & 1 \\ -\frac{v(1-u)}{u^2} - \frac{v}{u} & 1 \end{pmatrix}\right| = \frac{v(1-u)}{u^2} + \frac{v}{u} = \frac{v}{u^2}$$

and therefore

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}\left(v, \frac{v(1-u)}{u}\right)\left|\frac{v}{u^2}\right| \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-v^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{-\left(\frac{v(1-u)}{u}\right)^2}{2}} \left|\frac{v}{u^2}\right| \end{aligned}$$

4.29 (a) If $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$ and R is distributed as a positive random variable and $\Theta \sim \mathrm{uniform}(0,2\pi)$. Then $Z = X/Y = g(\Theta) = \cot(\Theta)$. The inverse of $(\tan(\Theta))^{-1}$ is not 1-1; $-\infty < z < \infty$ maps into both $(0,\pi)$ and $(\pi,2\pi)$. So on $\mathcal{A}_1 = (0,\pi)$ it's the case that $g_1^{-1}(z) = \cot^{-1}(z)$ and on $\mathcal{A}_2 = (\pi,2\pi)$ it's the case that $g_1^{-1}(z) = \pi + \cot^{-1}(z)$.

$$\begin{aligned} f_Z(z) = \sum_{i=1}^{2} f_\Theta\left(g_i^{-1}(z)\right)\left|\frac{d}{dz}\left(g_i^{-1}(z)\right)\right| &= \frac{1}{2\pi}\left|\frac{-1}{1+z^2}\right| + \frac{1}{2\pi}\left|\frac{-1}{1+z^2}\right| \\ &= \frac{1}{\pi}\frac{1}{1+z^2} \end{aligned}$$

(b)

$$\begin{aligned} Z &= \frac{2XY}{\sqrt{X^2+Y^2}} \\ &= \frac{R^2\cos(\Theta)\sin(\Theta)}{R} \\ &= R\cos(\Theta)\sin(\Theta) \\ &= R\sin(2\Theta) \end{aligned}$$

but $\sin(2\Theta) \sim \sin(\Theta)$. This can be seen by considering that $\sin(2\Theta)$ has the density of $\sin(\Theta)$ on each of $(0,\pi)$ and $(\pi, 2\pi)$. Therefore

$$f_{\sin(2\Theta)} = \frac{1}{2} f_{\sin(\Theta)} + \frac{1}{2} f_{\sin(\Theta)} = f_{\sin(\Theta)}$$

and $\sin(\Theta) \sim \cos(\Theta)$. Why? Let $W = \cos(\Theta)$ and $V = \sin(\Theta)$. Since $\cos(\Theta)$ isn't 1-1 on $(0, 2\pi)$ but is 1-1 on $(0,\pi)$ and $(\pi, 2\pi)$ individually. So define $W = g_1(\Theta) = \cos(\Theta)$ and $g_1^{-1}(w) = \arccos(w)$ on $(0,\pi)$ and $W = g_2(\Theta) = \cos(\Theta)$ and $g_2^{-1}(w) = 2\pi - \arccos(w)$ on $(\pi, 2\pi)$. Why? If $\theta \in (\pi, 2\pi)$ then $2\pi - \theta \in (0,\pi)$ but also $\cos(\theta) = \cos(2\pi - \theta)$and so $\arccos(\cos(\theta)) = \arccos(\cos(2\pi - \theta)) = 2\pi - \theta$. Then by the "splitting the original space" theorem:

$$\begin{aligned} f_W(w) &= f_\Theta\left(g_1^{-1}(w)\right)\left|\frac{d}{dw}\left(g_1^{-1}(w)\right)\right| + f_\Theta\left(g_2^{-1}(w)\right)\left|\frac{d}{dw}\left(g_2^{-1}(w)\right)\right| \\ &= \frac{1}{2\pi}\left|\frac{-1}{\sqrt{1-w^2}}\right| + \frac{1}{2\pi}\left|\frac{1}{\sqrt{1-w^2}}\right| = \frac{1}{\pi}\frac{1}{\sqrt{1-w^2}} \end{aligned}$$

Then for V consider the regions $A_1 = \left\{\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)\right\}$ and $A_2 = \left\{\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right\}$, on both of which individually $\arcsin(v)$ is 1-1. The inverse on A_1 is $g_1^{-1} = \arcsin(v)$ and has the same values as $\arcsin(\theta)$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore the distribution of V on A_1 is the same as it would be on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and so the inverse is

$$f_V(v) = f_\Theta(\arcsin(v))\left|\frac{d}{dv}(\arcsin(v))\right| = \frac{1}{\pi}\frac{1}{\sqrt{1-v^2}}$$

If $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ then $\pi - \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\sin(\theta) = \sin(\pi - \theta)$ and so $\arcsin(\sin(\theta)) = \arcsin(\sin(\pi - \theta)) = \pi - \theta$ and so $g_2^{-1} = \pi - \arcsin(v)$. Therefore the distribution of V on A_2 is the same as it would be on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and so the inverse is

$$f_V(v) = f_\Theta(\arcsin(v))\left|\frac{d}{dv}(\pi - \arcsin(v))\right| = \frac{1}{\pi}\frac{1}{\sqrt{1-v^2}}$$

And since there's a probability of 1/2 of landing in each region

$$f_V(v) = \frac{1}{2}\frac{1}{\pi\sqrt{1-v^2}} + \frac{1}{2}\frac{1}{\pi\sqrt{1-v^2}} = \frac{1}{\pi\sqrt{1-v^2}}$$

which is the same distribution as for W.

4.30 $\text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$ and $E(X) = E(E(X|Y))$

(a) Hence

$$\begin{aligned} E(Y) &= E(E(Y|X)) \\ &= E(X) \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned}
\text{Var}(Y) &= E\left(\text{Var}\left(Y|X\right)\right) + \text{Var}\left(E\left(Y|X\right)\right)\\
&= E\left(X^2\right) + \text{Var}\left(X\right)\\
&= \frac{1^2 + 1\cdot 0 + 0^2}{3} + \frac{1}{12}(1-0)^2\\
&= \frac{5}{12}
\end{aligned}$$

and since $E\left(XY\right) = E\left(E\left(XY|X\right)\right) = E\left(X\cdot E\left(Y|X\right)\right)$

$$\begin{aligned}
\text{Cov}(X,Y) &= E\left(XY\right) - \mu_X\mu_Y\\
&= E\left(X\cdot E\left(Y|X\right)\right) - \frac{1}{2}\cdot\frac{1}{2}\\
&= E\left(X\cdot X\right) - \frac{1}{2}\cdot\frac{1}{2}\\
&= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}
\end{aligned}$$

(b) $U = Y/X$ then $-\infty < u < \infty$ and then $V = X$ and $0 < v < 1$ implies

$$\|J\| = \left\|\left|\begin{pmatrix} 0 & 1\\ v & u\end{pmatrix}\right|\right\| = \|0 - v\| = v$$

hence

$$\begin{aligned}
f_{U,V}\left(u,v\right) &= f_{Y,X}(uv,v)\cdot u\\
&= f_{Y|X}(uv|v)f_X(v)\\
&= \frac{1}{v\sqrt{2\pi}}e^{-\frac{(uv-v)^2}{2v^2}}\frac{1}{1-0}v\\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2(u-1)^2}{2v^2}}\\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{(z-1)^2}{2}}
\end{aligned}$$

Therefore $Y/X \sim \mathcal{N}(1,1)$ and $X \sim \text{uniform}(0,1)$.

4.31 $X \sim \text{uniform}(0,1)$ and $Y \sim \text{Binomial}(n,X)$ then $f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|x) = 1\times\binom{n}{y}\left(1-x\right)^y x^{n-y}$. Then

$$f_Y(y) = \binom{n}{y}\int_0^1 \left(1-x\right)^y x^{n-y}dx = \binom{n}{y}\frac{\Gamma\left(n-y+1\right)\Gamma(y+1)}{\Gamma\left(n+2\right)} = \frac{1}{n+1}$$

(a) So

$$E\left(Y\right) = \sum_{y=0}^{n}\frac{y}{n+1} = \frac{1}{n+1}\left(\frac{n\left(n+1\right)}{2}\right) = \frac{n}{2}$$

and

$$E\left(Y^2\right) = \sum_{y=0}^{n}\frac{y^2}{n+1} = \frac{1}{n+1}\frac{n\left(n+1\right)\left(2n+1\right)}{2\cdot 3} = \frac{n\left(2n+1\right)}{6}$$

and finally

$$\text{Var}(Y) = \frac{n\left(2n+1\right)}{6} - \frac{n}{2}$$

Also this can be done as such

$$E\left(Y\right) = E\left(E\left(Y|X\right)\right) = E\left(nX\right) = nE\left(X\right) = \frac{n}{2}$$

and

$$\begin{aligned}
\operatorname{Var}(Y) &= \operatorname{Var}(E(Y|X)) + E(\operatorname{Var}(Y|X)) \\
&= \operatorname{Var}(nX) + E(\operatorname{Var}(Y|X)) \\
&= \operatorname{Var}(nX) + E(nX(1-X)) \\
&= \frac{n}{2}(1-0)^2 + n\left(E(X) - E\left(X^2\right)\right) \\
&= \frac{n^2}{12} + n\left(E(X) - E(X)^2 - \operatorname{Var}(X)\right) \\
&= \frac{n^2}{12} + n\left(\frac{1}{2} - \frac{1}{4} - \frac{1}{12}(1-0)^2\right) \\
&= \frac{n^2}{12} + \frac{n}{6}
\end{aligned}$$

(b) $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y|x) = 1 \times \binom{n}{y}(1-x)^y x^{n-y}$

(c) $f_Y(y) = \frac{1}{n+1}$

4.32 (a) If $Y|\Lambda \sim \text{Poisson}(\Lambda)$ and $\Lambda \sim \Gamma(\alpha, \beta)$ then

$$\begin{aligned}
f_Y(y) = \int_0^\infty f_{Y,\Lambda}(y,\lambda)\, d\lambda &= \int_0^\infty f_{Y|\Lambda}(y|\lambda) f_\Lambda(\lambda)\, d\lambda \\
&= \int_0^\infty \left(\frac{e^{-\lambda}\lambda^y}{y!} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} e^{-\lambda/\beta}\right) d\lambda \\
&= \frac{1}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \left(\lambda^{y+\alpha-1} e^{-\lambda\left(1+\frac{1}{\beta}\right)}\right) d\lambda \\
&= \frac{\Gamma(y+\alpha)\frac{1}{\left(1+\frac{1}{\beta}\right)^{y+\alpha}}}{y!\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(y+\alpha)\frac{1}{\left(1+\frac{1}{\beta}\right)^{y+\alpha}}} \left(\lambda^{y+\alpha-1} e^{-\lambda/\frac{1}{\left(1+\frac{1}{\beta}\right)}}\right) d\lambda \\
&= \frac{\Gamma(y+\alpha)\beta^y}{y!\Gamma(\alpha)\beta^{y+\alpha}\left(1+\frac{1}{\beta}\right)^{y+\alpha}} = \frac{(y+\alpha-1)!}{y!(\alpha-1)!} \frac{\beta^y}{(\beta+1)^{y+\alpha}} \\
&= \frac{(y+\alpha-1)!}{y!(\alpha-1)!} \frac{\beta^y}{(\beta+1)^{y+\alpha}} = \binom{y+\alpha-1}{y} \frac{\beta^y}{(\beta+1)^{y+\alpha}} \\
&= \binom{y+\alpha-1}{y} \left(\frac{\beta}{\beta+1}\right)^y \left(\frac{1}{\beta+1}\right)^\alpha
\end{aligned}$$

Then for the mean

$$\begin{aligned}
E(Y) &= E(E(Y|X)) \\
&= E(\Lambda) = \alpha\beta
\end{aligned}$$

and variance

$$\begin{aligned}
\operatorname{Var}(Y) &= \operatorname{Var}(E(Y|\Lambda)) + E(\operatorname{Var}(Y|\Lambda)) \\
&= \operatorname{Var}(\Lambda) + E(\Lambda) = \alpha\beta^2 + \alpha\beta
\end{aligned}$$

(b) If $Y|N \sim \text{Binomial}(N, p)$ and $N|\Lambda \sim \text{Poisson}(\Lambda)$ and $\Lambda \sim \Gamma(\alpha, \beta)$ then

$$f_N(n) = \binom{n+\alpha-1}{n} \left(\frac{\beta}{\beta+1}\right)^n \left(\frac{1}{\beta+1}\right)^\alpha$$

and since $y \leq n$

$$
\begin{aligned}
f_Y(y) = \sum_{n=0}^{\infty} f_{Y,N}(y,n) &= \sum_{n=y}^{\infty} f_{Y|N}(y|n)\, f_N(n) \\
&= \sum_{n=y}^{\infty} \left(\binom{n}{y} (1-p)^{n-y} p^y \right) \left(\binom{n+\alpha-1}{n} \left(\frac{\beta}{\beta+1} \right)^n \left(\frac{1}{\beta+1} \right)^\alpha \right) \\
&= \sum_{n=y}^{\infty} \left(\binom{n}{y} \binom{n+\alpha-1}{n} \right) \left((1-p)^{n-y} p^y \left(\frac{\beta}{\beta+1} \right)^n \left(\frac{1}{\beta+1} \right)^\alpha \right) \\
&= \sum_{n=y}^{\infty} \left(\left(\frac{n!}{y!\,(n-y!)} \right) \left(\frac{(n+\alpha-1)!}{(n!)\,(\alpha-1)!} \right) \right) \left(\left(\frac{p}{1-p} \right)^y \left(\frac{(1-p)\,\beta}{\beta+1} \right)^n \left(\frac{1}{\beta+1} \right)^\alpha \right) \\
&= \sum_{n=y}^{\infty} \left(\left(\frac{1}{y!\,(\alpha-1)!} \right) \left(\frac{(n+\alpha-1)!}{(n-y!)} \right) \right) \left(\left(\frac{p}{1-p} \right)^y \left(\frac{(1-p)\,\beta}{\beta+1} \right)^n \left(\frac{1}{\beta+1} \right)^\alpha \right) \\
&= \sum_{n=y}^{\infty} \left(\left(\frac{(\alpha+y-1)!}{y!\,(\alpha-1)!} \right) \left(\frac{((n-y)+(\alpha+y)-1)!}{(n-y!)\,(\alpha+y-1)!} \right) \right) \times \\
&\left(\left(\frac{p}{1-p} \right)^y \left(\frac{(1-p)\,\beta}{\beta+1} \right)^n \left(\frac{1}{\beta+1} \right)^\alpha \right) \\
&= \binom{\alpha+y-1}{y} \left(\frac{p}{1-p} \right)^y \left(\frac{(1-p)\,\beta}{\beta+1} \right)^y \left(\frac{1}{\beta+1} \right)^{-y} \frac{1}{(1+p\beta)^{\alpha+y}} \times \\
&\sum_{n=y}^{\infty} \binom{(n-y)+(\alpha+y)-1}{(n-y)} \left(\left(\frac{(1-p)\,\beta}{\beta+1} \right)^{n-y} \left(\frac{1+p\beta}{\beta+1} \right)^{\alpha+y} \right) \\
&= \binom{\alpha+y-1}{y} \left(\frac{p}{1-p} \right)^y \left(\frac{(1-p)\,\beta}{\beta+1} \right)^y \left(\frac{1}{\beta+1} \right)^{-y} \frac{1}{(1+p\beta)^{\alpha+y}} \\
&= \binom{\alpha+y-1}{y} \left(\frac{p\beta}{1+p\beta} \right)^y \left(\frac{1}{1+p\beta} \right)^\alpha
\end{aligned}
$$

Hence $Y \sim \mathrm{NB}\left(\alpha, \frac{1}{1+p\beta}\right)$. An alternative way to perform this calculation is to notice that $f_{Y|\Lambda}(y|\lambda) = \sum_{n=y}^{\infty} f_{Y|N}(y|n)\, f_{N|\Lambda}(n|\lambda)$ and then consider whether $Y|\Lambda \sim \mathrm{Poisson}(\Lambda)$. So

$$
\begin{aligned}
f_{Y|\Lambda}(y|\lambda) &= \sum_{n=y}^{\infty} \left(\binom{n}{y} (1-p)^{n-y} p^y \frac{e^{-\lambda}\lambda^n}{n!} \right) \\
&= \frac{e^{-\lambda} p^y}{y!} \sum_{n=y}^{\infty} \left(\frac{(1-p)^{n-y}}{(n-y!)} \lambda^n \right) \\
&= \frac{e^{-\lambda}(\lambda p)^y}{y!} \sum_{n=y}^{\infty} \left(\frac{((1-p)\,\lambda)^{n-y}}{(n-y!)} \right) \\
&= \frac{e^{-\lambda}(\lambda p)^y}{y!} e^{(1-p)\lambda} \\
&= \frac{e^{-\lambda p}(\lambda p)^y}{y!}
\end{aligned}
$$

and hence $Y|\Lambda \sim \mathrm{Poisson}(p\Lambda)$.

4.33 The MGF of X_i is

$$
\begin{aligned}
E\left(e^{tX_i}\right) &= \sum_{x_i=1}^{\infty} e^{tx_i} \frac{-1}{\log(p)} \frac{(1-p)^{x_i}}{x_i} \\
&= \frac{-1}{\log(p)} \sum_{x_i=1}^{\infty} e^{tx_i} \frac{(1-p)^{x_i}}{x_i} \\
&= \frac{-1}{\log(p)} \sum_{x_i=1}^{\infty} \frac{\left(e^t(1-p)\right)^{x_i}}{x_i} \\
&\text{and by -log}(1-x) = \int_0^x \frac{da}{1-a} = \int_0^x \left(1+a+a^2+a^3\cdots\right) da \\
&= \frac{-1}{\log(p)} \left(-\log\left(1-e^t(1-p)\right)\right)
\end{aligned}
$$

Then the MGF of H is

$$
\begin{aligned}
E\left(e^{tH}\right) &= E\left(E\left(e^{tH}\middle| N\right)\right) \\
&= E\left(E\left(e^{t\sum_{i=1}^{N} X_i}\middle| N\right)\right) \\
&= E\left(\prod_{i=1}^{N} E\left(e^{tX_i}\middle| N\right)\right) \\
&\text{and by i.i.d} \\
&= E\left(\left(E\left(e^{tX_i}\middle| N\right)\right)^N\right) \\
&\text{but } E\left(e^{tX_i}\middle| N\right) = E\left(e^{tX_i}\right) = \frac{\log\left(1-e^t(1-p)\right)}{\log(p)} \\
&\text{since } X_i \text{ are independent of } N \\
&= E_N\left(\left(\frac{\log\left(1-e^t(1-p)\right)}{\log(p)}\right)^N\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{\log\left(1-e^t(1-p)\right)}{\log(p)}\right)^n \frac{e^{-\lambda}\lambda^n}{n!} \\
&= e^{-\lambda} e^{\lambda\left(\frac{\log\left(1-e^t(1-p)\right)}{\log(p)}\right)} \sum_{n=0}^{\infty} \frac{e^{-\lambda\left(\frac{\log\left(1-e^t(1-p)\right)}{\log(p)}\right)} \left(\lambda\left(\frac{\log\left(1-e^t(1-p)\right)}{\log(p)}\right)\right)^n}{n!} \\
&= e^{-\lambda} e^{\lambda\left(\frac{\log\left(1-e^t(1-p)\right)}{\log(p)}\right)} = e^{-\lambda}\left(e^{\log\left(1-e^t(1-p)\right)}\right)^{\frac{\lambda}{\log(p)}} \\
&= \left(e^{\log(p)}\right)^{-\lambda/\log(p)} \left(1-e^t(1-p)\right)^{\frac{\lambda}{\log(p)}} \\
&= (p)^{-\lambda/\log(p)} \left(\frac{1}{1-e^t(1-p)}\right)^{\frac{-\lambda}{\log(p)}} \\
&= \left(\frac{p}{1-e^t(1-p)}\right)^{\frac{-\lambda}{\log(p)}}
\end{aligned}
$$

Therefore $H \sim \text{NB}\left(\frac{-\lambda}{\log(p)}, p\right)$.

4.34 (a) $X|P \sim \text{binomial}\,(n, P)$ and $P \sim \text{beta}\,(\alpha, \beta)$ then

$$
\begin{aligned}
f_X(x) &= \int_0^1 f_{X,P}(x,p)\,dp \\
&= \int_0^1 f_{X|P}(x|p)\,f_P(p)\,dp \\
&= \int_0^1 \binom{n}{x}(1-p)^{n-x}\,p^x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{n}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 (1-p)^{n+\beta-x-1}\,p^{x+\alpha-1}dp \\
&= \binom{n}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha+n+\beta)}\int_0^1 \frac{p^{x+\alpha-1}(1-p)^{n+\beta-x-1}}{\frac{\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha+n+\beta)}}dp \\
&= \binom{n}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha+n+\beta)}
\end{aligned}
$$

(b) $X|P \sim \text{NB}\,(r, P)$ and $P \sim \text{beta}\,(\alpha, \beta)$ then

$$
\begin{aligned}
f_X(x) &= \int_0^1 f_{X,P}(x,p)\,dp \\
&= \int_0^1 f_{X|P}(x|p)\,f_P(p)\,dp \\
&= \int_0^1 \binom{x+r-1}{k}(1-p)^{r}\,p^x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{x+r-1}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 (1-p)^{r+\beta-1}\,p^{x+\alpha-1}dp \\
&= \binom{x+r-1}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(r+\beta)}{\Gamma(\alpha+r+x+\beta)}\int_0^1 \frac{(1-p)^{r+\beta-1}\,p^{x+\alpha-1}}{\frac{\Gamma(x+\alpha)\Gamma(r+\beta)}{\Gamma(\alpha+r+x+\beta)}}dp \\
&= \binom{x+r-1}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(r+\beta)}{\Gamma(\alpha+r+x+\beta)}
\end{aligned}
$$

The mean and the variance are

$$
\begin{aligned}
E(E(X|P)) &= E\left(\frac{r(1-P)}{P}\right) \\
&= r\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \left(\frac{1-p}{p}\right)p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= r\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 p^{(\alpha-1)-1}(1-p)^{(\beta+1)-1}\,dp \\
&= r\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}\int_0^1 p^{(\alpha-1)-1}(1-p)^{(\beta+1)-1}\,dp \\
&= r\frac{\Gamma(\alpha-1)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= r\frac{(\alpha-2)!\beta!}{(\alpha-1)!\,(\beta-1)!} = \frac{r\beta}{(\alpha-1)}
\end{aligned}
$$

and

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{Var}(E(X|P)) + E(\mathrm{Var}(X|P)) \\
&= \mathrm{Var}\left(\frac{r(1-P)}{P}\right) + E\left(\frac{r(1-P)}{P^2}\right) \\
&= r^2\mathrm{Var}\left(\frac{(1-P)}{P}\right) + rE\left(\frac{(1-P)}{P^2}\right) \\
&= r^2\left(E\left(\left(\frac{(1-P)}{P}\right)^2\right) - E\left(\frac{(1-P)}{P}\right)^2\right) + rE\left(\frac{(1-P)}{P^2}\right)
\end{aligned}$$

More algebra:

$$\begin{aligned}
E\left(\left(\frac{(1-P)}{P}\right)^2\right) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \left(\frac{(1-p)}{p}\right)^2 p^{\alpha-1}(1-p)^{\beta+2-1}\,dp \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-2)\Gamma(\beta+2)}{\Gamma(\alpha+\beta)}\int_0^1 \frac{p^{(\alpha-2)-1}(1-p)^{(\beta+2)-1}}{\frac{\Gamma(\alpha-2)\Gamma(\beta+2)}{\Gamma(\alpha+\beta)}}dp \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-2)\Gamma(\beta+2)}{1} = \frac{(\alpha-2-1)!(\beta+1)!}{(\alpha-1)!(\beta-1)!} \\
&= \frac{\beta(\beta+1)}{(\alpha-1)(\alpha-2)}
\end{aligned}$$

and

$$\begin{aligned}
E\left(\frac{(1-P)}{P^2}\right) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \left(\frac{(1-p)}{p^2}\right) p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta-1)}\int_0^1 \frac{p^{(\alpha-2)-1}(1-p)^{(\beta+1)-1}}{\frac{\Gamma(\alpha-2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta-1)}}dp \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha-2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta-1)} = \frac{(\alpha+\beta-1)!(\alpha-2-1)!(\beta)!}{(\alpha-1)!(\beta-1)!(\alpha+\beta-2)!} \\
&= \frac{(\alpha+\beta-1)\beta}{(\alpha-1)(\alpha-2)}
\end{aligned}$$

and finally

$$\begin{aligned}
\mathrm{Var}(X) &= r^2\left(E\left(\left(\frac{(1-P)}{P}\right)^2\right) - E\left(\frac{(1-P)}{P}\right)^2\right) + rE\left(\frac{(1-P)}{P^2}\right) \\
&= r^2\left(\frac{\beta(\beta+1)}{(\alpha-1)(\alpha-2)} - \left(\frac{r\beta}{(\alpha-1)}\right)^2\right) + r\frac{(\alpha+\beta-1)\beta}{(\alpha-1)(\alpha-2)}
\end{aligned}$$

4.35 (a) $X|P \sim \mathrm{binomial}(n,P)$ and $P \sim \mathrm{beta}(\alpha,\beta)$ then

$$\begin{aligned}
E(X) &= E(E(X|P)) \\
&= E(nP) \\
&= \frac{n\alpha}{\alpha+\beta}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{Var}(E(X|P)) + E(\mathrm{Var}(X|P)) \\
&= \mathrm{Var}(nP) + E(nP(1-P)) \\
&= \mathrm{Var}(nP) + n\left(E(P) - E(P)^2 - \mathrm{Var}(P)\right) \\
&= n^2\mathrm{Var}(P) - n\mathrm{Var}(P) + nE(P)(1-E(P))
\end{aligned}$$

(b) If $Y|\Lambda \sim \text{Poisson}(\Lambda)$ and $\Lambda \sim \Gamma(\alpha, \beta)$ then

$$\begin{aligned} E(Y) &= E(E(Y|\Lambda)) \\ &= E(\Lambda) \\ &= \alpha\beta \end{aligned}$$

and

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(E(Y|\Lambda)) + E(\text{Var}(Y|\Lambda)) \\ &= \text{Var}(\Lambda) + E(\Lambda) = \alpha\beta^2 + \alpha\beta = \frac{(\alpha\beta)^2}{\alpha} + \alpha\beta \end{aligned}$$

4.36 (a) Let $X_i \big| P_i \sim \text{Bernoulli}(P_i)$ and $P_i \sim \text{beta}(\alpha, \beta)$. Then

$$\begin{aligned} E(Y) &= E\left(E\left(Y \middle| \mathbf{P}\right)\right) \\ &= E\left(E\left(X_1 + X_2 + \cdots + X_n \big| \mathbf{P}\right)\right) \\ &= E\left(\sum_{i=1}^{n} E(X_i|P_i)\right) \\ &= \sum_{i=1}^{n} E(E(X_i|P_i)) \\ &= \sum_{i=1}^{n} E(P_i) \\ &= \sum_{i=1}^{n} \frac{\alpha}{\alpha+\beta} \\ &= \frac{n\alpha}{\alpha+\beta} \end{aligned}$$

(b) The variance is

$$
\begin{aligned}
\operatorname{Var}(Y) &= \operatorname{Var}(E(Y|\mathbf{P})) + E(\operatorname{Var}(Y|\mathbf{P})) \\
&= \operatorname{Var}\left(\sum_{i=1}^{n} E(X_i|P_i)\right) + E\left(\operatorname{Var}\left(\sum_{i=1}^{n} X_i|P_i\right)\right) \\
&= \operatorname{Var}\left(\sum_{i=1}^{n} E(X_i|P_i)\right) + E\left(\sum_{i=1}^{n} \operatorname{Var}(X_i|P_i)\right) \\
&= \sum_{i=1}^{n} \operatorname{Var}(E(X_i|P_i)) + \sum_{i=1}^{n} E(\operatorname{Var}(X_i|P_i)) \\
&= \sum_{i=1}^{n} \operatorname{Var}(P_i) + \sum_{i=1}^{n} E(P_i(1-P_i)) \\
&= \sum_{i=1}^{n} \operatorname{Var}(P_i) + \sum_{i=1}^{n} \left(E(P_i) - E(P_i)^2 - \operatorname{Var}(P_i)\right) \\
&= \sum_{i=1}^{n} \left(E(P_i) - E(P_i)^2\right) \\
&= \sum_{i=1}^{n} \left(\frac{\alpha}{\alpha+\beta} - \left(\frac{\alpha}{\alpha+\beta}\right)^2\right) \\
&= n\left(\frac{\alpha}{\alpha+\beta} - \left(\frac{\alpha}{\alpha+\beta}\right)^2\right) = \frac{n\alpha\beta}{(\alpha+\beta)^2}
\end{aligned}
$$

Using MGFs

$$
\begin{aligned}
E\left(e^{tY}\right) &= E\left(E\left(e^{tY}\middle|\mathbf{P}\right)\right) \\
&= E\left(E\left(e^{t\sum_{i=1}^{n} X_i}\middle|\mathbf{P}\right)\right) \\
&= E\left(\prod_{i=1}^{n} E\left(e^{tX_i}\middle|P_i\right)\right) \\
&= E\left(\prod_{i=1}^{n} \left((1-P_i) + P_i e^t\right)\right) \\
&\text{by independence of } P_i \\
&= \prod_{i=1}^{n} E\left((1-P_i) + P_i e^t\right) \\
&= \prod_{i=1}^{n} \left(\left(1 - \frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha}{\alpha+\beta} e^t\right) \\
&= \left(\left(1 - \frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha}{\alpha+\beta} e^t\right)^n \\
&= \left(\left(1 - \frac{\alpha}{\alpha+\beta}\right) + \frac{\alpha}{\alpha+\beta} e^t\right)^n
\end{aligned}
$$

So $Y \sim \text{Binomial}\left(n, \frac{\alpha}{\alpha+\beta}\right)$.

(c) Let $X_i \big| P_i \sim \text{Binomial}\,(n_i, P_i)$ and $P_i \sim \text{beta}\,(\alpha, \beta)$. Then

$$\begin{aligned}
E\left(Y\right) &= E\left(E\left(Y\middle|\mathbf{n},\mathbf{P}\right)\right) \\
&= E\left(E\left(X_1 + X_2 + \cdots + X_k\big|\mathbf{n},\mathbf{P}\right)\right) \\
&= E\left(\sum_{i=1}^{k} E\left(X_i|n_i, P_i\right)\right) \\
&= \sum_{i=1}^{k} E\left(E\left(X_i|n_i, P_i\right)\right) \\
&= \sum_{i=1}^{k} E\left(n_i P_i\right) \\
&= \sum_{i=1}^{k} n_i \frac{\alpha}{\alpha+\beta} \\
&= \frac{\alpha}{\alpha+\beta} \sum_{i=1}^{k} n_i
\end{aligned}$$

and the variance is

$$\begin{aligned}
\text{Var}\left(Y\right) &= \text{Var}\left(E\left(Y\middle|\mathbf{n},\mathbf{P}\right)\right) + E\left(\text{Var}\left(Y\middle|\mathbf{n},\mathbf{P}\right)\right) \\
&= \text{Var}\left(\sum_{i=1}^{k} E\left(X_i|n_i, P_i\right)\right) + E\left(\text{Var}\left(\sum_{i=1}^{k} X_i|n_i, P_i\right)\right) \\
&= \sum_{i=1}^{k} \text{Var}\left(E\left(X_i|n_i, P_i\right)\right) + E\left(\sum_{i=1}^{k} \text{Var}\left(X_i|n_i, P_i\right)\right) \\
&= \sum_{i=1}^{k} \text{Var}\left(n_i P_i\right) + \sum_{i=1}^{k} E\left(n_i P_i\left(1-P_i\right)\right) \\
&= \sum_{i=1}^{k} n_i^2 \text{Var}\left(P_i\right) + \sum_{i=1}^{k} n_i \left(E\left(P_i\right) - E\left(P_i\right)^2 \text{-Var}\left(n_i P_i\right)\right) \\
&= \sum_{i=1}^{k} n_i \left(E\left(P_i\right) - E\left(P_i\right)^2\right) = \sum_{i=1}^{k} n_i \left(P_i - P_i^2\right) \\
&= \sum_{i=1}^{k} n_i P_i \left(1-P_i\right) = \sum_{i=1}^{k} \text{Var}\left(X_i\right)
\end{aligned}$$

4.37 (a) Let $Y = \sum_{i=1}^{n} X_i$. Then

$$\begin{aligned}
P(Y=k) &= \int_0^1 P(Y=k, C=c)\,dc \\
&= \int_0^1 P(Y=k|C=c)\,f_C(c)\,dc \\
&= \int_0^1 P\left(Y=k|c=\frac{1}{2}(1+p)\right) f_P(p)\,dp \\
&= \int_0^1 \binom{n}{k}\left(\frac{1}{2}(1+p)\right)^k \left(1-\frac{1}{2}(1-p)\right)^{n-k} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{(1+p)^k}{2^k}\left(\frac{1}{2}(1-p)\right)^{n-k} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{(1+p)^k}{2^k}\frac{(1-p)^{n-k}}{2^{n-k}} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{2^n\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1+p)^k(1-p)^{n-k} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{2^n\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \sum_{j=0}^{k}\binom{k}{j} 1^{k-j}p^j(1-p)^{n-k} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{2^n\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{k}\binom{k}{j} \int_0^1 p^j(1-p)^{n-k} p^{\alpha-1}(1-p)^{\beta-1}\,dp \\
&= \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{2^n\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{k}\binom{k}{j} \int_0^1 p^{j+\alpha-1}(1-p)^{n-k+\beta-1}\,dp \\
&= \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{2^n\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{k}\binom{k}{j} \frac{\Gamma(j+\alpha)\Gamma(n-k+\beta)}{\Gamma(j+\alpha+n-k+\beta)} \\
&= \sum_{j=0}^{k}\left(\frac{\binom{k}{j}}{2^n}\right)\binom{n}{k}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mathrm{B}(j+\alpha, n-k+\beta) \\
&= \sum_{j=0}^{k}\left(\frac{\binom{k}{j}}{2^n}\right)\binom{n}{k}\frac{\mathrm{B}(\alpha+j, n-k+\beta)}{\mathrm{B}(\alpha,\beta)}
\end{aligned}$$

(b) The mean

$$\begin{aligned}
E(Y) &= E(E(Y|C=c)) \\
&= E\left(\sum_{i=1}^{n} E(X_i)\middle|C\right) \\
&= E\left(\sum_{i=1}^{n} E\left(X_i\middle|C\right)\right) \\
&= E\left(\sum_{i=1}^{n} E\left(X_i\middle|\frac{1}{2}(1+P)\right)\right) \\
&= E\left(\sum_{i=1}^{n} \frac{1}{2}(1+E(P))\right) \\
&= E\left(\sum_{i=1}^{n} \frac{1}{2}\left(1+\frac{\alpha}{\alpha+\beta}\right)\right) \\
&= \frac{n}{2}\left(1+\frac{\alpha}{\alpha+\beta}\right)
\end{aligned}$$

and the variance just uses the trick from 4.35(a) and more algebra.

4.38 Let $X \sim \text{Gamma}(r, \lambda)$ Then

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

but with $u = \frac{x}{\nu} - \frac{x}{\lambda}$ and $\nu = \frac{x}{u+x/\lambda} = \frac{\lambda x}{\lambda u + x}$ $du = -\frac{x}{\nu^2} d\nu \implies d\nu = -\frac{1}{x}\left(\frac{\lambda u+x}{\lambda}\right)^2$

$$\begin{aligned}
\frac{1}{\Gamma(r)\Gamma(1-r)} \int_0^\lambda \frac{\nu^{r-1}}{(\lambda-\nu)} \frac{1}{\nu} e^{-x/\nu} d\nu &= \frac{1}{\Gamma(r)\Gamma(1-r)} \int_\infty^0 \frac{\left(\frac{\lambda x}{\lambda u+x}\right)^{r-1}}{\left(\lambda - \frac{\lambda x}{\lambda u+x}\right)^r} \frac{1}{\frac{\lambda x}{\lambda u+x}} e^{-x/\frac{\lambda x}{\lambda u+x}} \left(-\frac{1}{x}\left(\frac{\lambda u+x}{\lambda}\right)^2\right) du \\
&= \frac{1}{\Gamma(r)\Gamma(1-r)} \int_\infty^0 \frac{\left(\frac{\lambda x}{\lambda u+x}\right)^{r-1}}{\left(\lambda - \frac{\lambda x}{\lambda u+x}\right)^r} \frac{1}{\frac{\lambda x}{\lambda u+x}} e^{-x/\frac{\lambda x}{\lambda u+x}} \left(-\frac{1}{x}\left(\frac{\lambda x}{\lambda u+x}\right)^{-2}\right) du
\end{aligned}$$

4.39 Let $\mathbf{X} = (X_1, \ldots, X_n) \sim \text{Multi}(p_1, \ldots, p_n)$ then

$$P(X_1 = x_1, \cdots, X_n = x_n) = \binom{m}{x_1, \ldots, x_n} \prod_{i=1}^{n} p_i^{x_i}$$

WLOG let $X_i = X_1$ and $X_j = X_2$ then

$$P(X_1 = a | X_2 = b) = \frac{P(X_1 = a, X_2 = b)}{P(X_2 = b)}$$

Piece by piece:

$$
\begin{aligned}
P\left(X_1=a, X_2=b\right) &= \sum_{\substack{(x_3,\ldots,x_n) \\ \sum_{j=3}^n x_j = m-a-b}} \binom{m}{a,b,x_3\ldots,x_n} p_1^a p_2^b \prod_{i=3}^n p_i^{x_i} \\
&= p_1^a p_2^b \left(1-p_1-p_2\right)^{m-a-b} \sum_{\substack{\mathbf{x}=(x_3,\ldots,x_n) \\ |\mathbf{x}|=m-a-b}} \frac{m!}{a!b!x_3!\cdots x_n!} \frac{1}{\left(1-p_1-p_2\right)^{m-a-b}} \prod_{i=3}^n p_i^{x_i} \\
&= \frac{p_1^a p_2^b}{a!b!} \left(1-p_1-p_2\right)^{m-a-b} \sum_{\substack{\mathbf{x}=(x_3,\ldots,x_n) \\ |\mathbf{x}|=m-a-b}} \frac{m!\,(m-a-b)!}{(m-a-b)!x_3!\cdots x_n!} \prod_{i=3}^n \left(\frac{p_i}{1-p_1-p_2}\right)^{x_i} \\
&= \frac{m!}{(m-a-b)!} \frac{p_1^a p_2^b}{a!b!} \left(1-p_1-p_2\right)^{m-a-b} \sum_{\substack{\mathbf{x}=(x_3,\ldots,x_n) \\ |\mathbf{x}|=m-a-b}} \frac{(m-a-b)!}{x_3!\cdots x_n!} \prod_{i=3}^n p_i^{x_i} \\
&= \frac{m!}{(m-a-b)!} \frac{p_1^a p_2^b}{a!b!} \left(1-p_1-p_2\right)^{m-a-b} = \binom{m}{a,b,m-a-b} p_1^a p_2^b \left(1-p_1-p_2\right)^{m-a-b}
\end{aligned}
$$

and

$$
\begin{aligned}
P\left(X_2=b\right) &= \sum_{\substack{\mathbf{x}=(x_1,x_3,\ldots,x_n) \\ |\mathbf{x}|=m-b}} \binom{m}{x_1,b,x_3\ldots,x_n} p_1^{x_1} p_2^b \prod_{i=3}^n p_i^{x_i} \\
&= \sum_{\substack{\mathbf{x}=(x_1,x_3,\ldots,x_n) \\ |\mathbf{x}|=m-b}} \frac{m!}{x_1!x_3!\cdots x_n!} p_1^{x_1} \prod_{i=3}^n p_i^{x_i} \\
&= \frac{m!}{(m-b)!} \frac{p_2^b}{b!} \left(1-p_2\right)^{m-b} \sum_{\substack{\mathbf{x}=(x_1,x_3,\ldots,x_n) \\ |\mathbf{x}|=m-b}} \frac{(m-b)!}{x_1!x_3!\cdots x_n!} \left(\frac{p_1}{1-p_2}\right)^{x_1} \prod_{i=3}^n \left(\frac{p_i}{1-p_2}\right)^{x_i} \\
&= \frac{m!}{(m-b)!} \frac{p_2^b}{b!} = \binom{m}{b} p_2^b \left(1-p_2\right)^{m-b}
\end{aligned}
$$

Therefore $X_2 \sim \text{Binomial}\left(m, p_2\right)$ and by

$$
\begin{aligned}
P\left(X_1=a \middle| X_2=b\right) &= \frac{\binom{m}{a,b,m-a-b} p_1^a p_2^b \left(1-p_1-p_2\right)^{m-a-b}}{\binom{m}{b} p_2^b \left(1-p_2\right)^{m-b}} \\
&= \frac{\frac{m!}{a!b!(m-a-b)!} p_1^a \left(1-p_1-p_2\right)^{m-a-b}}{\frac{m!}{b!(m-b)!} \left(1-p_2\right)^{m-b}} \\
&= \binom{m-b}{a} \left(\frac{p_1}{1-p_2}\right)^a \frac{\left(1-p_1-p_2\right)^{m-b-a}}{\left(1-p_2\right)^{m-b-a}} \\
&= \binom{m-b}{a} \left(\frac{p_1}{1-p_2}\right)^a \left(1-\frac{p_1}{1-p_2}\right)^{m-b-a}
\end{aligned}
$$

$X_1|X_2 \sim \text{Binomial}\left(m-b, \frac{p_1}{1-p_2}\right)$. Then the covariance

$$\begin{aligned}
\text{Cov}(X_i, X_j) &= E(X_iX_j) - E(X_i)E(X_j) \\
&= E(E(X_iX_j|X_j)) - mp_imp_j \\
&= E(X_jE(X_i|X_j)) - mp_imp_j \\
&= E\left(X_j(m-X_j)\left(\frac{p_i}{1-p_j}\right)\right) - mp_imp_j \\
&= \left(\frac{p_i}{1-p_j}\right)E(X_j(m-X_j)) - mp_imp_j \\
&= \left(\frac{p_i}{1-p_j}\right)\left(mE(X_j) - (E(X_j))^2 - \text{Var}(X_j)\right) - mp_imp_j \\
&= \left(\frac{p_i}{1-p_j}\right)\left(mm(p_j) - (mp_j)^2 - mp_j(1-p_j)\right) - mp_imp_j \\
&= -mp_ip_j
\end{aligned}$$

4.40 $f_{X,Y}(x,y) = Cx^{a-1}y^{b-1}(1-x-y)^{c-1}$ on $0 < x < 1, 0 < y < 1, 0 < y < 1-x < 1$.

(a) By $\frac{1-x-y}{1-y} = 1 - \frac{x}{1-y}$ we have

$$\begin{aligned}
1 = \int_0^1\int_0^{1-y}\left(Cx^{a-1}y^{b-1}(1-x-y)^{c-1}\right)dxdy &= C\int_0^1\int_0^{1-y}\left(x^{a-1}y^{b-1}(1-y)^{c-1}\left(\frac{1-x-y}{1-y}\right)^{c-1}\right)dxdy \\
&= C\int_0^1\int_0^{1-y}\left(x^{a-1}y^{b-1}(1-y)^{c-1}\left(1-\frac{x}{1-y}\right)^{c-1}\right)dxdy \\
\text{let } u &= \frac{x}{1-y}, x = u(1-y), dx = (1-y)\,du \\
&= C\int_0^1\int_0^1\left((u(1-y))^{a-1}y^{b-1}(1-y)^{c-1}(1-u)^{c-1}\right)(1-y \\
&= C\int_0^1\left((1-y)^{a+c-1}y^{b-1}\int_0^1\left(u^{a-1}(1-u)^{c-1}\right)du\right)dy \\
&= C\int_0^1\left((1-y)^{a+c-1}y^{b-1}\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}\right)dy \\
&= C\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}\int_0^1\left(y^{b-1}(1-y)^{a+c-1}\right)dy \\
&= C\frac{\Gamma(a)\Gamma(c)}{\Gamma(a+c)}\frac{\Gamma(b)\Gamma(a+c)}{\Gamma(a+b+c)}
\end{aligned}$$

Hence $C = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$

(b) Doing essentially the same integration again:

$$\begin{aligned}
f_Y(y) &= \int_0^{1-y}\left(Cx^{a-1}y^{b-1}(1-x-y)^{c-1}\right)dx \\
&= \frac{\Gamma(a+b+c)}{\Gamma(b)\Gamma(a+c)}(1-y)^{a+c-1}y^{b-1}
\end{aligned}$$

and

$$\begin{aligned}
f_X(x) &= \int_0^{1-x} \left(C x^{a-1} y^{b-1} (1-x-y)^{c-1} \right) dy \\
&\text{let } u = \frac{y}{1-x}, y = u(1-x), dy = (1-x)\,du \\
&= C x^{a-1} (1-x)^{b+c-1} \int_0^1 \left(u^{b-1} (1-u)^{c-1} \right) du \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)} x^{a-1} (1-x)^{b+c-1}
\end{aligned}$$

(c)

$$\begin{aligned}
f_{Y|X}(y|x) &= \frac{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}}{\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b+c)}} \frac{x^{a-1} y^{b-1} (1-x-y)^{c-1}}{x^{a-1} (1-x)^{b+c-1}} \\
&= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} (1-x) \frac{y^{b-1}}{(1-x)^{b-1}} \left(1 - \frac{y}{1-x}\right)^{c-1} \\
&= (1-x) \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \left(\frac{y}{1-x}\right)^{b-1} \left(1 - \frac{y}{1-x}\right)^{c-1}
\end{aligned}$$

and hence $Y/(1-x) \sim \text{Beta}(b, c)$.

(d) The expectation

$$\begin{aligned}
E(XY) &= \int_0^1 \int_0^{1-y} xy \left(C x^{a-1} y^{b-1} (1-x-y)^{c-1} \right) dxdy \\
&= C \int_0^1 \int_0^{1-y} \left(x^{(a+1)-1} y^{(b+1)-1} (1-x-y)^{c-1} \right) dxdy \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \frac{\Gamma(a+1)\Gamma(b+1)\Gamma(c)}{\Gamma(a+b+c+2)} \\
&= \frac{ab}{(a+b+c+1)(a+b+c)\Gamma(a+b+c)}
\end{aligned}$$

and variance

$$\text{Var}(XY) = E\left((XY)^2\right) - (E(XY))^2$$

where

$$\begin{aligned}
E\left(X^2Y^2\right) &= \int_0^1 \int_0^{1-y} x^2 y^2 \left(C x^{a-1} y^{b-1} (1-x-y)^{c-1} \right) dxdy \\
&= C \int_0^1 \int_0^{1-y} \left(x^{(a+2)-1} y^{(b+2)-1} (1-x-y)^{c-1} \right) dxdy \\
&= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \frac{\Gamma(a+2)\Gamma(b+2)\Gamma(c)}{\Gamma(a+b+c+4)} \\
&= \frac{ab(a+1)(b+1)}{(a+b+c+3)(a+b+c+2)(a+b+c+1)\Gamma(a+b+c)}
\end{aligned}$$

and so

$$\begin{aligned}
\text{Var}(XY) &= \frac{ab(a+1)(b+1)}{(a+b+c+3)(a+b+c+2)(a+b+c+1)\Gamma(a+b+c)} \\
&\quad - \left(\frac{ab}{(a+b+c+1)(a+b+c)\Gamma(a+b+c)} \right)^2
\end{aligned}$$

The covariance is

$$\operatorname{Cov}(X,Y) = \frac{ab}{(a+b+c+1)(a+b+c)\Gamma(a+b+c)} - (a(b+c))(b(a+c))$$

4.41 $\operatorname{Cov}(X,k) = E(Xk) - E(X)E(k) = kE(X) - E(X)k = 0$

4.42 First the covariance

$$\begin{aligned}\operatorname{Cov}(XY,Y) &= E\left(XY^2\right) - E(XY)E(Y) \\ &= E(X)E\left(Y^2\right) - E(X)E(Y)^2 \\ &= E(X)\operatorname{Var}(Y) \\ &= \mu_X\sigma_Y^2\end{aligned}$$

Then the standard deviation of XY

$$\begin{aligned}\operatorname{Var}(XY) &= E\left((XY)^2\right) - (E(XY))^2 \\ &= E\left(X^2\right)E\left(Y^2\right) - (E(X))^2(E(Y))^2 \\ &= \left((E(X))^2 + \operatorname{Var}(X)\right)\left((E(Y))^2 + \operatorname{Var}(Y)\right) - (E(X))^2(E(Y))^2 \\ &= \left(\mu_X^2 + \sigma_X^2\right)\left(\mu_Y^2 + \sigma_Y^2\right) - \mu_X^2\mu_Y^2 \\ &= \mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \sigma_X^2\sigma_Y^2\end{aligned}$$

Therefore

$$\begin{aligned}\rho_{XY,Y} &= \frac{\operatorname{Cov}(XY,Y)}{\sqrt{\sigma_{XY}^2}\sqrt{\sigma_Y^2}} \\ &= \frac{\mu_X\sigma_Y^2}{\sqrt{\sigma_{XY}^2}\sqrt{\sigma_Y^2}} \\ &= \frac{\mu_X\sigma_Y}{\sqrt{\mu_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \sigma_X^2\sigma_Y^2}}\end{aligned}$$

4.43

$$\begin{aligned}\operatorname{Cov}(X_1+X_2, X_2+X_3) &= \operatorname{Cov}(X_1,X_2) + \operatorname{Cov}(X_1,X_3) + \operatorname{Cov}(X_2,X_2) + \operatorname{Cov}(X_2,X_3) \\ &= \sigma^2\end{aligned}$$

and

$$\begin{aligned}\operatorname{Cov}(X_1+X_2, X_1-X_2) &= \operatorname{Cov}(X_1,X_1) - \operatorname{Cov}(X_1,X_2) + \operatorname{Cov}(X_2,X_1) - \operatorname{Cov}(X_2,X_2) \\ &= \sigma^2 - \sigma^2 = 0\end{aligned}$$

4.44 By induction - the base is thm 4.5.6. Assume that for $k < n$

$$\operatorname{Var}\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k}\operatorname{Var}(X_i) + 2\sum_{1\le i<j\le k}\operatorname{Cov}(X_i,X_j)$$

Then let $Y = \sum_{i=1}^{n-1} X_i$. By thm 4.5.6

$$\operatorname{Var}(Y+X_n) = \operatorname{Var}(Y) + \operatorname{Var}(X_n) + 2\operatorname{Cov}(Y,X_n)$$

then by the induction hypothesis

$$\operatorname{Var}(Y) = \sum_{i=1}^{n-1}\operatorname{Var}(X_i) + 2\sum_{1\le i<j\le k}\operatorname{Cov}(X_i,X_j)$$

and by linearity of covariance

$$\operatorname{Cov}(Y, X_n) = \sum_{i=1}^{n-1} \operatorname{Cov}(X_i, X_n)$$

hence

$$\begin{aligned}
\operatorname{Var}(Y + X_n) &= \sum_{i=1}^{n-1} \operatorname{Var}(X_i) + 2 \sum_{1 \le i < j \le k} \operatorname{Cov}(X_i, X_j) \\
&+ \operatorname{Var}(X_n) + 2 \left(\sum_{i=1}^{n-1} \operatorname{Cov}(X_i, X_n) \right) \\
&= \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \operatorname{Cov}(X_i, X_j)
\end{aligned}$$

since for $1 < j < n-1$ it's the case that $j < n$.

4.45 (a) Let $(X, Y) = \mathbf{V} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then

$$f_{\mathbf{V}}(\mathbf{v}) = \frac{1}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} e^{-\frac{1}{2}(\mathbf{v}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{v}-\boldsymbol{\mu})}$$

where $\boldsymbol{\mu}$ is the mean vector, i.e. $\boldsymbol{\mu} = (\mu_x, \mu_y)^T$, $\boldsymbol{\Sigma}$ is the covariance matrix and in this instance

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_x^2 & \rho_{x,y}\sigma_x\sigma_y \\ \rho_{y,x}\sigma_y\sigma_x & \sigma_y^2 \end{pmatrix}$$

and hence $|\boldsymbol{\Sigma}| = \sigma_x^2\sigma_y^2 - \sigma_x^2\sigma_y^2(\rho_{y,x} + \rho_{x,y}) = \sigma_x^2\sigma_y^2\left(1 - \rho_{y,x}^2\right)$. Then

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{\mathbf{V}}(\mathbf{v})\, dy \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} e^{-\frac{1}{2}(\mathbf{v}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{v}-\boldsymbol{\mu})} \right) dy
\end{aligned}$$

First of all

$$\begin{aligned}
(\mathbf{v}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{v}-\boldsymbol{\mu}) &= (x-\mu_x, y-\mu_y) \begin{pmatrix} \sigma_x^2 & \rho_{x,y}\sigma_x\sigma_y \\ \rho_{y,x}\sigma_y\sigma_x & \sigma_y^2 \end{pmatrix}^{-1} \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix} \\
&= (x-\mu_x, y-\mu_y) \begin{pmatrix} \sigma_y^2 & -\rho_{x,y}\sigma_x\sigma_y \\ -\rho_{y,x}\sigma_y\sigma_x & \sigma_x^2 \end{pmatrix}^{-1} \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix} \\
&= \frac{1}{1-\rho_{x,y}^2} \left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - 2\rho_{x,y} \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} \right)
\end{aligned}$$

so

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{\mathbf{V}}(\mathbf{v})\, dy \\
&= \frac{1}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \int_{-\infty}^{\infty} \left(e^{-\frac{1}{2}\frac{1}{1-\rho_{x,y}^2}\left(\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-2\rho_{x,y}\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)} \right) dy \\
&\text{let } a = \left(\frac{x-\mu_x}{\sigma_x}\right), b = \left(\frac{y-\mu_y}{\sigma_y}\right), \sigma_y db = dy \\
&= \frac{\sigma_y}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \int_{-\infty}^{\infty} \left(e^{-\frac{1}{2}\frac{1}{1-\rho_{x,y}^2}\left(a^2+b^2-2\rho_{x,y}ab\right)} \right) db \\
&= \frac{\sigma_y e^{-\frac{1}{2}\frac{a^2}{1-\rho_{x,y}^2}}}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \int_{-\infty}^{\infty} \left(e^{-\frac{1}{2}\frac{1}{1-\rho_{x,y}^2}\left(b^2-2\rho_{x,y}ab\right)} \right) db \\
&\text{then } \left(b^2 - 2\rho_{x,y}ab + \rho_{x,y}^2 a^2\right) - \rho_{x,y}^2 a^2 = (b-\rho_{x,y}a)^2 - \rho_{x,y}^2 a^2 \\
&= \frac{\sigma_y e^{-\frac{1}{2}\left(\frac{a^2}{1-\rho_{x,y}^2}-\frac{\rho_{x,y}^2 a^2}{1-\rho_{x,y}^2}\right)}}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \int_{-\infty}^{\infty} \left(e^{-\frac{1}{2}\frac{(b-\rho_{x,y}a)^2}{1-\rho_{x,y}^2}} \right) db \\
&\text{the integrand is the kernel of } \mathcal{N}\left(\rho_{x,y}a, 1-\rho_{x,y}^2\right) \\
&\text{and } -\frac{1}{2}\left(\frac{a^2}{1-\rho_{x,y}^2} - \frac{\rho_{x,y}^2 a^2}{1-\rho_{x,y}^2}\right) = -\frac{a^2}{2} \\
&= \frac{\sigma_y e^{-\frac{a^2}{2}}}{\sqrt{(2\pi)^2 |\boldsymbol{\Sigma}|}} \sqrt{2\pi}\sqrt{1-\rho_{x,y}^2} \\
&= \frac{\sigma_y e^{-\frac{a^2}{2}}}{\sqrt{(2\pi)^2 \left(\sigma_x^2\sigma_y^2\left(1-\rho_{y,x}^2\right)\right)}} \sqrt{2\pi}\sqrt{1-\rho_{x,y}^2} \\
&= \frac{e^{-\frac{a^2}{2}}}{\sigma_x\sqrt{(2\pi)}} = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\sigma_x\sqrt{(2\pi)}}
\end{aligned}
$$

and by symmetry $f_Y(y) = \frac{e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}{\sigma_y\sqrt{(2\pi)}}$.

(b) TODO

4.46 TODO

4.47 (a) If $X \sim Y \sim \mathcal{N}(0,1)$ then $XY > 0$ in the first and third quadrant of $\mathbb{R}^2$ and $XY < 0$ in the second and fourth quadrant of $\mathbb{R}^2$. Therefore

$$
\begin{aligned}
f_Z(z) &= P(XY>0)\, f_X(z) \left| \frac{d}{dz} z \right| + P(XY<0)\, f_X(-z) \left| \frac{d}{dz}(-z) \right| \\
&= P(XY>0)\, f_X(z) + P(XY<0)\, f_X(-z) \\
&\text{but } f_X(x) \text{ is symmetric so} \\
&= \left(P(XY>0) + P(XY<0)\right) f_X(z) \\
&= \left(\left(P(X>0,Y>0) + P(X<0,Y<0)\right) + \left(P(X<0,Y>0) + P(X<0,Y>0)\right)\right) f_X(z) \\
&\text{but } X \text{ and } Y \text{ are i.i.d. so} \\
&= \left(\left(P(X>0)\,P(Y>0) + P(X<0)\,P(Y<0)\right) + \left(P(X<0)\,P(Y>0) + P(X<0)\,P(Y>0)\right) \right. \\
&= \left(\left(\frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2}\right) + \left(\frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2}\right)\right) f_X(z) \\
&= f_X(z) = \frac{e^{-\frac{1}{2}\left(\frac{z-\mu_x}{\sigma_x}\right)^2}}{\sigma_x \sqrt{(2\pi)}}
\end{aligned}
$$

(b) If $Y>0$ then if $X>0$ immediately $Z=X>0$. If $Y>0$ then if $X<0$ immediately $Z=-X>0$. If $Y<0$ then if $X<0$ immediately $Z=X<0$. If $Y<0$ then if $X>0$ immediately $Z=-X<0$. Any vector (Z,Y) is bivariate normal iff $aZ+bY \sim \mathcal{N}(\alpha,\beta)$ for all a,b and some α,β. But

$$P(Z-Y=0) = P(Z=Y) = 1$$

while if $Z+Y$ were normally distributed then $P(Z+Y=0)$ would equal 0.