STA 6326 Homework 2 Solutions

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1.35 Let $Q(A) = P(A|B)$. Firstly

$$Q(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

by the non-negativity of $P(\cdot)$ and the hypothesis that $P(B) > 0$. Secondly

$$Q(\Omega) = P(\Omega) = \frac{P(\Omega|B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

Finally assume A_i, A_j for all i, j are pairwise disjoint. Then

$$Q\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i \middle| B\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{P(B)} = \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$

and since A_i, A_j for all i, j are pairwise disjoint $(A_i \cap B)$, $(A_j \cap B)$ are also pairwise disjoint for all i, j and by the countable additivity of $P(\cdot)$

$$\frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)} = \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} \frac{P\left(A_i \cap B)\right)}{P(B)} = \sum_{i=1}^{\infty} P(A_i|B) = \sum_{i=1}^{\infty} Q(A_i)$$

1.38 (a) If $P(B) = 1$ then $P(A|B) = \frac{P(A \cap B)}{1}$ but $P(A) = P(A \cap B) + P(A \cap B^c)$ and since $A \cap B^c \subset B^c$ and $P(A \cap B^c) \leq P(B^c) = 1 - P(B) = 0$ it's the case that $P(A) = P(A \cap B)$ so $P(A|B) = P(A)$.

(b) $P(B|A) = P(B \cap A)/P(A)$ but the hypothesis $A \subset B$ implies $B \cap A = A$ so $P(B|A) = P(A)/P(A) = 1$. Then $P(A|B) = P(B|A)P(A)/P(B) = P(A)/P(B)$.

(c)

$$\begin{aligned} P(A|A \cup B) = &\frac{P\left(A \cap (A \cup B)\right)}{P(A \cup B)} \\ &\frac{P(A)}{P(A) + P(B) - P(A \cap B)} && \text{by } A \subset A \cup B \\ &\frac{P(A)}{P(A) + P(B)} && \text{by "mutually exclusive"} \iff A \cap B = \emptyset \end{aligned}$$

(d) $P(A \cap B \cap C) = P(A|B \cap C)P(B \cap C) = P(A|B \cap C)P(B|C)P(C)$.

1.39 (a) If $P(A) > 0$ and$P(B) > 0$ and$P(A \cap B) = 0$ then obviously $P(A) \cdot P(B) \neq P(A \cap B)$.

(b) If $P(A) > 0$ and$P(B) > 0$ and $P(A)P(B) = P(A \cap B)$ then obviously $P(A \cap B) = P(A)P(B) > 0$.

1.44 The number of correct answers is binomially distributed with $p = .25$ and $1 - p = .75$. Then

$$P(X \geq 10) = \sum_{k=10}^{20} \binom{20}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{n-k} = .0138644$$

1.46 There $7^7 = 823,543$ different ways to distribute the 7 balls into the 7 cells. The maximum number of cells that could have 3 balls is 2 and clearly the minimum is 0. Hence $X_3 \in \{0, 1, 2\}$. The only way for $X_3 = 2$ would be 3 balls in a cell, 3 balls in another cell, and the last ball in one cell. There are $\binom{7}{2}$ ways to choose the 2 cells to have the 3 balls each, $\binom{7}{3}$ ways to choose the first set of 3 balls for the first cell, $\binom{4}{3}$ to choose the second set of 3 balls for the second cell, then finally $\binom{5}{1} = 5$ different ways to choose which cell will contain the last balls. Therefore

$$P(X_3 = 2) = \frac{\binom{7}{2}\binom{7}{3}\binom{4}{3}5}{7^7} \approx .0178$$

For $X_3 = 1$ there are 3 different configurations possible:$\{3,1,1,1,1\}, \{3,2,1,1\}, \{3,2,2\}$.

$$\begin{aligned}
\#\{3,1,1,1,1\} &= 7\binom{7}{3} \times \binom{6}{4} \times 4 \times 3 \times 2 && \text{which cell contains 3 balls} \times \text{which 3 balls} \times \\
& && \text{which cells contain 1 ball} \times \text{permute the balls} \\
\#\{3,2,1,1\} &= 7\binom{7}{3} \times 6 \times \binom{4}{2} \times \binom{5}{2} \times 2 && \text{which cell contains 3 balls} \times \text{which 3 balls} \times \\
& && \text{which cell contains 2 balls, which 2 balls,} \\
& && \text{which cells contain 1 ball each, permute the balls} \\
\#\{3,2,2\} &= 7\binom{7}{3} \times \binom{6}{2} \times \binom{4}{2} && \text{which cell contains 3 balls, which 3 balls,} \\
& && \text{which cells contain 2 balls, which two balls in the first 2-ball cell} \\
& && \text{which cell contains second set of 2 balls, permute the balls} \\
\#\{3,4\} &= 7\binom{7}{3} \times 6 && \text{which cell contains 3 balls, which 3 balls,} \\
& && \text{which cell contains 4 balls} \\
+ & & & \\
\hline
& 288,120 & &
\end{aligned}$$

Hence $P(X_3 = 1) = 1 - 288,120/7^7 \approx .650146$. For $X_3 = 0$ there are very many configurations but we can compute by computing as the complement of $P(X_3 = 1) : P(X_3 = 0) = 1 - P(X_3 = 1) + P(X_3 = 2) = 1 - 0.178 - 0.650 \approx .33$.

1.47 Requirements for being a CDF: (i) right continuous (ii) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ (iii) non-decreasing.

(a) $\frac{1}{2} + \frac{1}{\pi}\arctan(x)$

i. Continuous and hence right-continuous.

ii. $\lim_{x\to-\pi/2} \tan(x) = -\infty$ hence $\lim_{x\to-\infty} \frac{1}{2} + \frac{1}{\pi}\arctan(x) = \frac{1}{2} + \frac{1}{\pi}\frac{-\pi}{2} = \frac{1}{2} - \frac{1}{2} = 0$. $\lim_{x\to\pi/2} \tan(x) = \infty$ hence $\lim_{x\to\infty} \frac{1}{2} + \frac{1}{\pi}\arctan(x) = \frac{1}{2} + \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2} + \frac{1}{2} = 1$.

iii. $\arctan(x)' = \frac{1}{1+x^2} > 0$ is non-decreasing (monotonically increasing) and hence $\frac{1}{2} + \frac{1}{\pi}\arctan(x)$ is non-decreasing.

(b) $(1 + e^{-x})^{-1}$

i. Continuous and hence right-continuous.
ii. $\lim_{x\to-\infty} e^{-x} = \infty$ hence $\lim_{x\to-\infty}(1+e^{-x})^{-1} = 1/\infty = 0$. $\lim_{x\to\infty} e^{-x} = 0$ hence $\lim_{x\to-\infty}(1+e^{-x})^{-1} = 1/1 = 1$.
iii. $\left((1+e^{-x})^{-1}\right)' = -1(1+e^{-x})^{-2}(-1)e^{-x} = (1+e^{-x})e^{-x} > 0$ hence non-decreasing.

(c) $\exp(-e^{-x})$

i. Continuous and hence right-continuous.
ii. $\lim_{x\to-\infty} e^{-x} = \infty$ hence $\lim_{x\to-\infty}\exp(-e^{-x}) = 0$. $\lim_{x\to\infty} e^{-x} = 0$ hence $\lim_{x\to-\infty}\exp(-e^{-x}) = 1$.
iii. $\left(\exp(-e^{-x})\right)' = \exp(-e^{-x})\left(-e^{-x}\right)(-1) = \exp(-e^{-x})e^{-x} > 0$ hence non-decreasing.

(d) $1 - e^{-x}$

i. Continuous and hence right-continuous.
ii. $e^{-0} = 1$ hence $\lim_{x\to 0} 1-e^{-x} = 1-1 = 0$. $\lim_{x\to\infty} e^{-x} = 0$ hence $\lim_{x\to\infty} 1-e^{-x} = 1$.
iii. $\left(1 - e^{-x}\right)' = 1 + e^{-x} > 0$ hence non-decreasing.

(e) If $0 < \epsilon < 1$ then $F_Y(y) = \begin{cases} \frac{1-\epsilon}{1+e^{-y}} & \text{if } y < 0 \\ \epsilon + \frac{1-\epsilon}{1+e^{-y}} & \text{if } y \geq 0 \end{cases}$

i. Since each piece of the piecewise definition of F_Y is continuous and the domain is defined with equality from the right ($y \geq 0$) F_Y is right-continuous.
ii. $\lim_{y\to-\infty} \frac{1-\epsilon}{1+e^{-y}} = 0$ similarly to (b.ii) and $\lim_{y\to\infty} \frac{1-\epsilon}{1+e^{-y}} = 1-\epsilon$ hence $\lim_{y\to\infty} \epsilon + \frac{1-\epsilon}{1+e^{-y}} = \epsilon + 1 - \epsilon = 1$.
iii. $\frac{1-\epsilon}{1+e^{-y}}$ and $\epsilon + \frac{1-\epsilon}{1+e^{-y}}$ are non-decreasing by (b.ii) and $\lim_{y\uparrow 0} F_Y(y) = (1-\epsilon)/2 < \epsilon + (1-\epsilon)/2 = \lim_{y\downarrow 0} F_Y(y)$ hence non-decreasing.

1.49 Assume $F_X(t) \leq F_Y(t)$ for all t and $F_X(t) < F_Y(t)$ for some t_0. By definition $F_X(t) = P(X \leq t) = 1 - P(X > t)$. Similarly $F_Y(t) = P(Y \leq t) = 1 - P(Y > t)$. Then

$$1 - P(X > t) \leq 1 - P(Y > t)$$

and so $P(X > t) \geq P(Y > t)$. Similarly for t_0 it's the case $P(X > t_0) > P(Y > t_0)$.

1.50 Let $S = \sum_{i=1}^{n} t^{k-1}$. Then

$$S(1-t) = S - St = \sum_{i=1}^{n} t^{k-1} - \sum_{i=1}^{n} t^{k} = \sum_{i=1}^{n}(t^k - t^{k-1})$$

But this last sum is telescoping hence

$$S(1-t) = 1 - t^{k-1} \implies S = \frac{1-t^{k-1}}{1-t}$$

1.51 X is distributed Hypergeometrically, i.e. there are $\binom{30}{4}$ ways to draw a sample of 4 from 30, $\binom{5}{x}$ ways to draw x microwaves from the subset of microwaves that is defective, and then finally per draw from the defective subset there are $\binom{25}{4-x}$ to draw the remainder from the subset of functional microwaves. Hence for $x = 0, 1, 2, 3, 4$

$$P(X = x) = \frac{\binom{5}{x}\binom{25}{4-x}}{\binom{30}{4}}$$

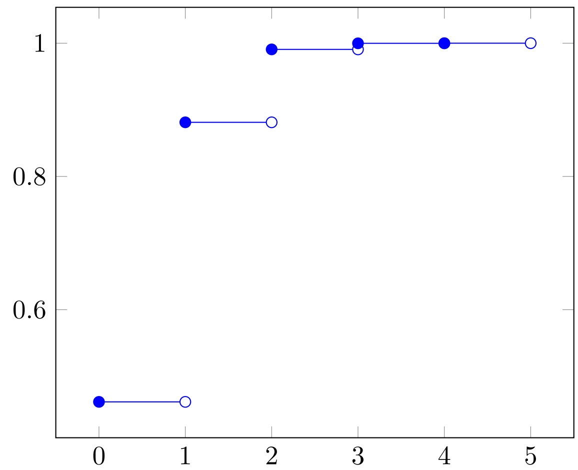

Figure 1: F_X for problem 1.51 (note at $F_X(4)$ there is overlap)

So

$$P(X=x)=\begin{cases}\frac{2530}{5481} & \text{for } x=0\\ \frac{2300}{5481} & \text{for } x=1\\ \frac{600}{5481} & \text{for } x=2\\ \frac{50}{5481} & \text{for } x=3\\ \frac{1}{5481} & \text{for } x=4\end{cases}$$

Then

$$F_X(x)=\begin{cases}\frac{2530}{5481} & \text{for } x=0\\ \frac{4830}{5481} & \text{for } x=1\\ \frac{5430}{5481} & \text{for } x=2\\ \frac{5480}{5481} & \text{for } x=3\\ \frac{5481}{5481} & \text{for } x=4\end{cases}$$

The plot of the CDF is Figure 1.

1.52 Let $f(x)$ be a pdf with cdf $F(x)$, $F(x_0) < 1$, and

$$g(x)=\begin{cases} f(x)/(1-F(x_0)) & \text{if } x \geq x_0\\ 0 & \text{if } x < x_0\end{cases}$$

Then since $f(x) \geq 0$ and $1 > F(x_0)$ (and since F is a cdf $F(x_0) \geq 0$) it's the case that $g(x) \geq 0$ for all x. Finally

$$\int_{-\infty}^{\infty} g(x) = \int_{-\infty}^{x_0} g(x) + \int_{x_0}^{\infty} \frac{f(x)}{1-F(x_0)} dx = 0 + \frac{1}{1-F(x_0)} \int_{x_0}^{\infty} f(x)dx = \frac{1-\int_{-\infty}^{x_0} f(x)}{1-F(x_0)} = \frac{1-F(x_0)}{1-F(x_0)} = 1$$

1.54 Let c be a normalization constant.

(a) $\int_0^{\pi/2} \sin(x)dx = 1$ hence $c = 1 \implies \int_0^{\pi/2} cf(x) = 1$.

(b) $\int_{-\infty}^{\infty} e^{-|x|}dx = 2$ hence $c = 1/2 \implies \int_{-\infty}^{\infty} ce^{-|x|}dx = 1$.

2.1 (a) If $Y = g(X) = X^3$ then $g^{-1}(Y) = \sqrt[3]{Y}$ and $\left(g^{-1}(Y)\right)' = \frac{1}{3}y^{-2/3}$ and

$$f_Y(y) = 42(\sqrt[3]{y})^5(1-\sqrt[3]{y})\left|\frac{1}{3}y^{-2/3}\right| = 14(\sqrt[3]{y}-1)y$$

Hence

$$\int_0^1 f_Y(y)dy = 14\int_0^1 y(\sqrt[3]{y}-1)dy = 14\frac{1}{14} = 1$$

(b) If $Y = g(X) = 4X+3$ then $g^{-1}(Y) = (Y-3)/4$ and $\left(g^{-1}(Y)\right)^{'} = \frac{1}{4}$ and

$$f_Y(y) = 7e^{-\frac{7}{4}(y-3)}\left|\frac{1}{4}\right|$$

Hence, since $g(0) = 4(0)+3 < Y < \infty$

$$\int_0^\infty f_Y(y)dy = \frac{7}{4}e^{\frac{21}{4}}\int_3^\infty e^{\frac{-7y}{4}}dy = e^{\frac{21}{4}}e^{-\frac{7\cdot 3}{4}} = 1$$

(c) If $Y = g(X) = X^2$ then $g(Y)^{-1} = \pm\sqrt{Y}$ and $\left|\left(g^{-1}(Y)\right)^{'}\right| = 1/2\sqrt{Y}$ and

$$f_Y(y) = \frac{30}{4}y^2\left(1-\frac{1}{2\sqrt{y}}\right)$$

Hence

$$\int_0^1 f_Y(y)dy = \frac{30}{4}\int_0^1 y^2\left(1-\frac{1}{2\sqrt{y}}\right)dy = \frac{30}{4}\frac{4}{30} = 1$$

2.2 (a) If $Y = g(X) = X^2$ then $g(Y)^{-1} = \pm\sqrt{Y}$ and $\left|\left(g^{-1}(Y)\right)^{'}\right| = 1/2\sqrt{Y}$ and

$$f_Y(y) = 1\cdot\frac{1}{2\sqrt{y}}$$

(b) If $Y = g(X) = -\log(X)$ then $g^{-1}(Y) = e^{-Y}$ and $\left|\left(g^{-1}(Y)\right)\right|^{'} = e^{-Y}$ and

$$f_Y(y) = \binom{n+m+1}{n,m,1}e^{-ny}(1-e^{-y})^m e^{-y}$$

With domain $-\log(1) = 0 < y < -\log(0) = \infty$.

(c) If $Y = g(X) = e^X$ then $g(Y)^{-1} = \log Y$ and $\left|\left(g^{-1}(Y)\right)^{'}\right| = 1/Y$ and

$$f_Y(y) = \frac{1}{\sigma^2}\frac{1}{y^2}e^{-(1/y\sigma)^2/2} = \frac{1}{(\sigma y)^2}e^{-(1/y\sigma)^2/2}$$

2.3 If $Y = g(X) = X/(X+1)$ then $g(Y)^{-1} = 1/(1-Y)$ and $\left|\left(\frac{1}{1-y}\right)^{'}\right| = \frac{1}{(1-y)^2}$ Hence for $y = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$$f_Y(y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{1}{1-y}}\frac{1}{(1-y)^2}$$

2.4 $f(x) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda x} & \text{if } x \geq 0 \\ \frac{1}{2}\lambda e^{\lambda x} & \text{if } x < 0 \end{cases}$

(a) $e^{\lambda x} > 0$ for all $x \in (-\infty, \infty)$ hence $f(x) \geq 0$. Furthermore

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\lambda}{2}\int_{-\infty}^{0} e^{\lambda x}dx + \frac{\lambda}{2}\int_{0}^{\infty} e^{-\lambda x}dx$$

Then by $-u = x$

$$\frac{\lambda}{2}\int_{-\infty}^{0} e^{\lambda x}dx = \frac{\lambda}{2}\int_{\infty}^{0} e^{-\lambda u}d(-u) = (-1)(-1)\frac{\lambda}{2}\int_{0}^{\infty} e^{-\lambda u}du = \frac{\lambda}{2}\int_{0}^{\infty} e^{-\lambda x}dx$$

and hence

$$\int_{-\infty}^{\infty} f(x)dx = 2\frac{\lambda}{2}\int_{0}^{\infty} e^{-\lambda x}dx = \lambda\frac{1}{\lambda}(1-0) = 1.$$

(b) If $x < 0$ then

$$F_X(x) = \frac{\lambda}{2}\int_{-\infty}^{x} e^{\lambda x}dx = \frac{\lambda}{2}\frac{1}{\lambda}e^{\lambda x} = \frac{1}{2}e^{\lambda x}$$

If $x \geq 0$ then

$$F_X(x) = \frac{\lambda}{2}\int_{-\infty}^{0} e^{\lambda x}dx + \frac{\lambda}{2}\int_{0}^{x} e^{-\lambda x}dx = \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2}e^{-\lambda x}\right)$$

(c) $P(|X| < t) = \int_{-t}^{t} f(x)dx$. Arguments from part (a) imply

$$\int_{-t}^{t} f(x)dx = 2\frac{\lambda}{2}\int_{0}^{t} e^{-\lambda x}dx = 2\left(\frac{1}{2} - \frac{1}{2}e^{-\lambda t}\right)$$