STA 6326 Homework 3 Solutions

Maksim Levental

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2.11 $X \sim \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

(a)

$$\begin{aligned}
E\left(X^2\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x \cdot x \cdot e^{-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}\left(\left(x\int x\cdot e^{-x^2/2}dx\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\int x\cdot e^{-y^2/2}dy\right)dx\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(x\left(-e^{-x^2/2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}dx\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(0 + \int_{-\infty}^{\infty} e^{-x^2/2}dx\right) \\
&= 1
\end{aligned}$$

By example 2.1.7

$$f_Y(y) = \frac{1}{2\sqrt{2\pi y}}\left(e^{-y/2} + e^{-y/2}\right) = \frac{1}{\sqrt{2\pi y}}e^{-y/2}$$

and since $0 < y < \infty$

$$\begin{aligned}
E\left(Y\right) &= \int_0^{\infty} y f_Y(y)dy \\
&= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{e^{-y/2}}{\sqrt{y}}dy \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-\infty}^{\infty} 2e^{-\left(\sqrt{y}\right)^2/2}d\left(\sqrt{y}\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{y}\right)^2/2}d\left(\sqrt{y}\right) \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} \\
&= 1
\end{aligned}$$

(b) The support of Y is $0 < y < \infty$. If $-\infty < x < 0$ then $Y = -X$ and $g(Y)^{-1} = -Y$, else if $0 \leq x < \infty$ then $Y = X$ and $g(Y)^{-1} = Y$. Then

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\left(e^{-(-y)^2/2}\left|-1\right| + e^{-y^2/2}\left|1\right|\right) = \frac{2e^{-y^2/2}}{\sqrt{2\pi}}$$

and therefore

$$
\begin{aligned}
E(Y) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty y e^{-y^2/2} dy \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-(y^2/2)} d\left(y^2/2\right) \\
&= \frac{2}{\sqrt{2\pi}}
\end{aligned}
$$

(c) $\operatorname{Var}(Y) = E\left(Y^2\right) - \left(E\left(Y\right)\right)^2$

$$
\begin{aligned}
E\left(Y^2\right) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty y^2 e^{-y^2/2} dy \\
&= 2\frac{1}{2} \int_{-\infty}^\infty y^2 \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\
&= 1 \text{ by part (a)}
\end{aligned}
$$

Hence $\operatorname{Var}\left(Y\right) = 1 - \frac{2}{\pi}$.

2.12 $Y = g(X) = d\tan(X)$. $g(X)$ is increasing for $0 < x < \pi/2$ and $g^{-1}(Y) = \arctan(Y)$. Hence

$$
\left|\left(\arctan(Y/d)\right)'\right| = \frac{1}{1+(Y/d)^2}\frac{1}{d}
$$

and therefore

$$
f_Y(y) = \frac{2}{\pi}\frac{1}{1+(y/d)^2}\frac{1}{d}
$$

with support $y \in (0, \infty)$. This is the Cauchy distribution hence $E(Y) = \infty$.

2.13 The probability that there are k heads, given the first flip lands heads, is geometrically distributed "flips until first tail": $P_H(X = k) = p^k(1-p)$ but restricted to $k = 1, 2, 3, \ldots$ and the probability that there are k tails, given the first flip lands tails, is also geometrically distributed "flips until first head": $P_T(X = k) = (1-p)^k p$, but also restricted to $k = 1, 2, 3, \ldots$. Therefore the probability that there's either a run of k heads or tails is

$$
P_{H\vee T}(X = k) = P_H + P_T = p^k(1-p) + (1-p)^k p
$$

and

$$
\begin{aligned}
E\left(X\right) &= \sum_{k=1}^\infty k\left(p^k(1-p) + (1-p)^k p\right) \\
&= \sum_{k=1}^\infty k p^k(1-p) + \sum_{k=1}^\infty k(1-p)^k p \\
&= E\left(H\right) - (1-p) + E\left(T\right) - p \\
&= \frac{1-(1-p)}{1-p} + \frac{1-p}{p} \\
&= \frac{1}{p} + \frac{1}{1-p} - 2
\end{aligned}
$$

2.14 (a)

$$
\begin{aligned}
E\left(X\right) &= \int_0^\infty x \cdot f_X(x) dx \text{ let } u = F_X(x) \text{ and since } F_X \text{ strictly monotonic} \\
&= \int_0^1 F_X^{-1}(u) du \\
&= \int_0^\infty \left(1 - F_X(x)\right) dx
\end{aligned}
$$

(b) First note that $x = \sum_{k=1}^{x} 1$

$$\begin{aligned}
E\left(X\right) &= \sum_{x=0}^{\infty} x \cdot f_X(x) \\
&= \sum_{x=1}^{\infty} x \cdot f_X(x) \\
&= \sum_{x=1}^{\infty} \sum_{k=1}^{x} f_X(x) \\
&= \sum_{k=1}^{\infty} \sum_{x=k}^{\infty} f_X(x) \text{ since } 0 < k < x \text{ and } 0 < x < \infty \iff k < x < \infty \text{ and } 0 < k < \infty \\
&= \sum_{k=1}^{\infty} (1 - F_X(k)) \\
&= \sum_{k=0}^{\infty} (1 - F_X(k))
\end{aligned}$$

2.16

$$\begin{aligned}
E\left(X\right) &= \int_0^{\infty} \left(1 - F_X(t)\right) dt \\
&= \int_0^{\infty} P(T > t) dt \\
&= \int_0^{\infty} \left(a e^{-\lambda t} + (1-a) e^{-\mu t}\right) dt \\
&= \frac{a}{\lambda} + \frac{\lambda - a}{\mu}
\end{aligned}$$

2.17 m is such that $m = F_X^{-1}(1/2)$

(a) $3 \int_0^m x^2 dx = m^3$ and therefore $m = \sqrt[3]{1/2}$.

(b)

$$\begin{aligned}
\frac{1}{2} &= \frac{1}{\pi} \int_{-\infty}^{m} \frac{1}{1+x^2} \\
&= \frac{1}{\pi} \left(\arctan(m) - \arctan(-\infty)\right) \\
&= \frac{1}{\pi} \left(\arctan(m) + \frac{\pi}{2}\right)
\end{aligned}$$

Therefore $m = \tan\left(0\right) = 0$

2.22 (a) Note that $\int_0^\infty e^{-\alpha x^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$. Let $\alpha = 1/\beta^2$ under the integral, then

$$\begin{aligned}
\int_0^\infty f_X(x)dx &= \frac{4}{\beta^3\sqrt{\pi}}\int_0^\infty x^2 e^{-x^2/\beta^2}dx \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\int_0^\infty \frac{\partial}{\partial\alpha}e^{-\alpha x^2}dx \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\frac{d}{d\alpha}\int_0^\infty e^{-\alpha x^2}dx \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\frac{d}{d\alpha}\alpha^{-1/2} \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\frac{-1}{2}\alpha^{-3/2} \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\frac{-1}{2}\left(\frac{1}{\beta^2}\right)^{-3/2} \\
&= \frac{1}{\beta^3}\frac{1}{\left(\frac{1}{\beta^2}\right)^{3/2}} = \frac{1}{\beta^3}\frac{\beta^3}{1} = 1
\end{aligned}$$

(b) Note that $\int_0^\infty xe^{-\alpha x^2} = \frac{1}{2\alpha}$ (u substitution). Let $\alpha = 1/\beta^2$ under the integral, then

$$\begin{aligned}
E(X) &= \frac{4}{\beta^3\sqrt{\pi}}\int_0^\infty x^3 e^{-x^2/\beta^2}dx \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\int_0^\infty \frac{\partial}{\partial\alpha}\left(xe^{-\alpha x^2}\right)dx \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\frac{d}{d\alpha}\int_0^\infty xe^{-\alpha x^2}dx \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\frac{d}{d\alpha}\frac{1}{2}\alpha^{-1} \\
&= \frac{2}{\beta^3\sqrt{\pi}}\alpha^{-2} = \frac{2\beta^4}{\beta^3\sqrt{\pi}} = \frac{2\beta}{\sqrt{\pi}}
\end{aligned}$$

The second moment is

$$\begin{aligned}
E\left(X^2\right) &= \frac{4}{\beta^3\sqrt{\pi}}\int_0^\infty x^4 e^{-x^2/\beta^2}dx \\
&= \frac{4}{\beta^3\sqrt{\pi}}\int_0^\infty \frac{\partial^2}{\partial\alpha^2}\left(e^{-\alpha x^2}\right)dx \\
&= \frac{4}{\beta^3\sqrt{\pi}}\frac{d^2}{d\alpha^2}\int_0^\infty e^{-\alpha x^2}dx \\
&= \frac{4}{\beta^3\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\frac{d^2}{d\alpha^2}\alpha^{-1/2} \\
&= \frac{-4}{\beta^3\sqrt{\pi}}\frac{\sqrt{\pi}}{4}\frac{d}{d\alpha}\alpha^{-3/2} = \frac{6}{\beta^3\sqrt{\pi}}\frac{\sqrt{\pi}}{4}\frac{d}{d\alpha}\alpha^{-5/2} \\
&= \frac{6}{\beta^3\sqrt{\pi}}\frac{\sqrt{\pi}}{4}\left(\frac{1}{\beta^2}\right)^{-5/2} = \frac{3}{2}\beta^2
\end{aligned}$$

Hence $\mathrm{Var}(X) = \frac{3}{2}\beta^2 - \left(\frac{2\beta}{\sqrt{\pi}}\right)^2 = \beta^2\left(\frac{3}{2} - \frac{4}{\pi}\right)$.

2.23 (a) $f_Y(y) = \frac{1}{2\sqrt{y}}\left(f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right) = \frac{1}{4\sqrt{y}}\left(1+\sqrt{y}+1-\sqrt{y}\right) = \frac{1}{2\sqrt{y}}$.

(b) $E\left(Y\right) = \frac{1}{2}\int_0^1 \frac{y}{\sqrt{y}}dy = \frac{1}{2}\int_0^1 \sqrt{y}dy = \frac{1}{3}$ and $E\left(Y^2\right) = \frac{1}{2}\int_0^1 y^{3/2}dy = \frac{1}{5}$ therefore $\mathrm{Var}(X) = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$.

2.26 (a) $\mathcal{N}(0,1)$, Standard Cauchy, Student's t.

(b)

$$\begin{aligned}
1 &= \lim_{\epsilon\to\infty} \int_{a-\epsilon}^{a+\epsilon} f_X(x)dx \\
&= \lim_{\epsilon\to\infty} \left(\int_{a-\epsilon}^{a} f_X(x)dx + \int_{a}^{a+\epsilon} f_X(x)dx\right) \\
&= \lim_{\epsilon\to\infty} \left(\int_{a-\epsilon}^{a} f_X(x)dx + \int_{a-\epsilon}^{a} f_X(x-\epsilon)d(x-\epsilon)\right) \\
&= \lim_{\epsilon\to\infty} \left(\int_{a-\epsilon}^{a} f_X(x)dx + \int_{a-\epsilon}^{a} f_X((x-\epsilon)+\epsilon)d(x-\epsilon)\right)
\end{aligned}$$

2.30 (a) For $t \in \mathbb{R}$

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_0^c \frac{1}{c}e^{tx}dx \\
&= \frac{1}{tc}\left(e^{tc}-1\right)
\end{aligned}$$

(b) For $t \in \mathbb{R}$

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \int_0^c \frac{2x}{c^2}e^{tx}dx \\
&= \frac{2}{c^2}\int_0^c xe^{tx}dx \\
&= \frac{2}{c^2}\left(\frac{x}{t}e^{tx}\Big|_0^c - \int_0^c e^{tx}dx\right) \\
&= \frac{2}{c^2}\left(\frac{c}{t}e^{tc} - \frac{1}{tc}\left(e^{tc}-1\right)\right)
\end{aligned}$$

(c) For $|t| < 1/\beta$

$$\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \frac{1}{2\beta}\int e^{-|x-\alpha|/\beta}e^{tx}dx \\
&= \frac{1}{2\beta}\left(\int_\alpha^\infty e^{-(x-\alpha)/\beta}e^{tx}dx + \int_{-\infty}^\alpha e^{-(\alpha-x)/\beta}e^{tx}dx\right) \\
&= \frac{1}{2\beta}\left(e^{\alpha/\beta}\int_\alpha^\infty e^{x(t-1/\beta)}dx + e^{-\alpha/\beta}\int_{-\infty}^\alpha e^{x(t+1/\beta)}dx\right) \\
& t < 1/\beta \implies (t-1/\beta) < 0 \implies \\
&= \frac{1}{2\beta}\left(\frac{e^{\alpha/\beta}}{t-1/\beta}\left(0 - e^{\alpha(t-1/\beta)}\right) + \frac{e^{-\alpha/\beta}}{t+1/\beta}\left(e^{\alpha(t+1/\beta)} - 0\right)\right) \\
&= \frac{1}{2\beta}\left(\frac{e^{\alpha/\beta}}{t-1/\beta}\left(0 - e^{\alpha(t-1/\beta)}\right) + \frac{e^{-\alpha/\beta}}{t+1/\beta}\left(e^{\alpha(t+1/\beta)} - 0\right)\right) \\
&= \frac{e^{\alpha t}}{1-(\beta t)^2}
\end{aligned}$$

(d) For $e^t(1-p)<1 \iff |t|<-\log(1-p)$ (permitted since $p<1 \implies \log(1-p)>0$)

$$
\begin{aligned}
M_X(t) &= E(e^{tX}) \\
&= \sum_{x=0}^{\infty} e^{tx}\binom{r+x-1}{x}p^r(1-p)^x \\
&= \sum_{x=0}^{\infty} \binom{r+x-1}{x}p^r\left(e^t(1-p)\right)^x \\
&\text{$e^t(1-p)<1 \therefore$ let $1-p'=e^t(1-p)$} \\
&= \sum_{x=0}^{\infty} \binom{r+x-1}{x}\left(1-\frac{1-p'}{e^t}\right)^r(1-p')^x \\
&= e^{-tr}\sum_{x=0}^{\infty} \binom{r+x-1}{x}\left((e^t-1)+p'\right)^r(1-p')^x \\
&= e^{-tr}\sum_{x=0}^{\infty} \binom{r+x-1}{x}\left(\sum_{k=0}^{r}\binom{r}{k}(p')^{r-k}\left(e^t-1\right)^k\right)(1-p')^x \\
&= e^{-tr}\sum_{x=0}^{\infty} \binom{r+x-1}{x}(p')^r\left(\sum_{k=0}^{r}\binom{r}{k}\left(\frac{e^t-1}{p'}\right)^k\right)(1-p')^x \\
&= e^{-tr}\left(\sum_{k=0}^{r}\binom{r}{k}\left(\frac{e^t-1}{p'}\right)^k\right)\sum_{x=0}^{\infty} \binom{r+x-1}{x}(p')^r(1-p')^x \\
&\text{since $P(X)$ is a pdf} \\
&= e^{-tr}\sum_{k=0}^{r}\binom{r}{k}\left(\frac{e^t-1}{p'}\right)^k \\
&= e^{-tr}\sum_{k=0}^{r}\binom{r}{k}1^{r-k}\left(\frac{e^t-1}{p'}\right)^k \\
&= e^{-tr}\left(1+\frac{e^t-1}{p'}\right)^r \\
&= \left(e^{-t}+\frac{1-e^{-t}}{1-e^t(1-p)}\right)^r = \frac{p^r}{1+e^t(p-1)}
\end{aligned}
$$

2.31 $t/(1-t)$ cannot be a moment generating function for any probability distribution:

$$
\frac{0}{1-0} = 0 = M_X(0) = E(1) = \int_\Omega f_X d\omega = 1
$$

which is a contradiction.

2.33 $S(t) = \log(M_X(t))$. Then

$$\begin{aligned}
\left.\frac{d}{dt}S(t)\right|_{t=0} &= \left.\frac{d}{dt}\log(M_X(t))\right|_{t=0} \\
&= \left.\frac{1}{M_X(t)}\frac{d}{dt}M_X(t)\right|_{t=0} \\
&= \left.\frac{1}{M_X(t)}\frac{d}{dt}\int_\Omega e^{tx}f_X\right|_{t=0} \\
&= \left.\frac{1}{M_X(t)}\int_\Omega \frac{\partial}{\partial t}e^{tx}f_X\right|_{t=0} \\
&= \left.\frac{1}{M_X(t)}\int_\Omega xe^{tx}f_X\right|_{t=0} \\
&= \frac{1}{M_X(0)}\int_\Omega xe^{0x}f_X = \int_\Omega xf_X = E(X)
\end{aligned}$$

and

$$\begin{aligned}
\left.\frac{d^2}{dt^2}S(t)\right|_{t=0} &= \left.\frac{d}{dt}\frac{1}{M_X(t)}\int_\Omega xe^{tx}f_X\right|_{t=0} \\
&= \left.\frac{d}{dt}\frac{1}{M_X(t)}\int_\Omega xe^{tx}f_X\right|_{t=0} \\
&= \left.\left[\frac{-1}{M_X^2(t)}\left(\int_\Omega xe^{tx}f_X\right)^2 + \frac{1}{M_X(t)}\int_\Omega \frac{\partial}{dt}xe^{tx}f_X\right]\right|_{t=0} \\
&= \left.\left[\frac{-1}{M_X^2(t)}\left(\int_\Omega xe^{tx}f_X\right)^2 + \frac{1}{M_X(t)}\int_\Omega x^2e^{tx}f_X\right]\right|_{t=0} \\
&= \left.\left[\frac{-1}{M_X^2(0)}\left(\int_\Omega xe^{0x}f_X\right)^2 + \frac{1}{M_X(0)}\int_\Omega x^2e^{0x}f_X\right]\right|_{t=0} \\
&= -\left(\int_\Omega xf_X\right)^2 + \int_\Omega x^2 f_X = -(E(X)) + E\left(X^2\right) = \mathrm{Var}(x)
\end{aligned}$$

2.38 (a) From **2.30(d)**

$$M_X(t) = \frac{p^r}{1+e^t(p-1)}$$

(b)

$$M_Y(t) = M_X(2pt) = \left(\frac{p}{1+e^{2pt}(p-1)}\right)^r$$

Then L'Hospitale's rule implies

$$\lim_{p\downarrow 0}\frac{p}{1+e^{2pt}(p-1)} = \lim_{p\downarrow 0}\frac{1}{e^{2pt}+2te^{2pt}(p-1)} = \frac{1}{1-2t}$$

and hence

$$\lim_{p\downarrow 0}M_Y(t) = \left(\frac{1}{1-2t}\right)^r$$

3.2 (a) The probability that 0 items in k draws are defective if 6 are defective in 100 is

$$\begin{aligned}
P(X=0) &= \frac{\binom{6}{0}\binom{94}{k}}{\binom{100}{k}} \\
&= \frac{\frac{94!}{k!(94-k)!}}{\frac{100!}{k!(100-k)!}} \\
&= \frac{(100-k)(99-k)(98-k)(97-k)(96-k)(95-k)}{100\cdot 99\cdot 98\cdot 96\cdot 95} \leq .10
\end{aligned}$$

Then solving $P(X=0)\leq .10$ numerically yields $k\geq 32$. So to detect 6 defectives in a batch of 100 you need at least 32 draws, but as the number of defectives goes up this number will decrease hence you need at most 32 draws.

(b)

$$\begin{aligned}
P(X=0) &= \frac{\binom{1}{0}\binom{99}{k}}{\binom{100}{k}} \\
&= \frac{\frac{99!}{k!(99-k)!}}{\frac{100!}{k!(100-k)!}} \\
&= \frac{(100-k)}{100} \leq .10
\end{aligned}$$

Therefore $k\geq 90$.

3.4 (a) The number of "flips" until success (finding the right key) is geometrically distributed with success probability $1/n$ and failure probability $(n-1)/n$. Therefore the mean number of trials is

$$\frac{1}{1/n}=n$$

(b) There are $n!$ permutations of the keys (assuming they're all distinct) and n different positions in any permutation that the correct key could be in. There are $\binom{n-1}{k-1}(k-1)!$ different permutations of keys that could precede the correct key and $\binom{n-k}{n-k}(n-k)!$ permutations of keys that could succeed the correct key. Therefore the probability the correct key is in the kth position is

$$P(X=k)=\frac{\binom{n-1}{k-1}(k-1)!\binom{n-k}{n-k}(n-k)!}{n!}=\frac{1}{n}$$

and then $E(X)=n+1/2$, i.e. in the middle, as you'd expect.

3.7 $P(X=k)=e^{-\lambda}\lambda^k/k!$ implies

$$\begin{aligned}
P(X\geq 2) &= e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^k}{k!} \\
&= e^{-\lambda}\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}-\lambda-1\right) \\
&= e^{-\lambda}\left(e^{\lambda}-\lambda-1\right) \\
&= 1-e^{-\lambda}\lambda-e^{-\lambda}
\end{aligned}$$

Therefore $P(X\geq 2)\geq .99 \iff 1-e^{-\lambda}\lambda-e^{-\lambda}\geq .99 \iff \lambda\approx 6.63835$

3.10 (a) The probability of choosing 4 packets of cocaine out all 496 packets is

$$\frac{\binom{N}{4}}{\binom{N+M}{4}}$$

The probability of choosing 2 non-cocaine packets out all the rest is

$$\frac{\binom{M}{2}}{\binom{N+M-4}{2}}$$

Therefore, by independence, the probability of choosing 4 packets of cocaine and then 2 packets of non-cocaine is

$$\frac{\binom{N}{4}}{\binom{N+M}{4}} \frac{\binom{M}{2}}{\binom{N+M-4}{2}}$$

(b)

$$\frac{\binom{N}{4}}{\binom{N+M}{4}} \frac{\binom{M}{2}}{\binom{N+M-4}{2}} = \frac{m(m-1)n(n-1)(n-2)(n-3)}{(m+n)(m+n-1)(m+n-2)(m+n-3)(m+n-4)(m+n-5)}$$

3.13 (a) $P(X > 0) = \sum_{k=1}^{\infty} e^{-\lambda}\lambda^k/k! = \sum_{k=0}^{\infty} e^{-\lambda}\lambda^k/k! - e^{-\lambda} = 1 - e^{-\lambda}$ hence

$$P(X_T = k) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \frac{\lambda^k}{k!} I_{\{1,2,\dots\}}$$

Then

$$\begin{aligned} E(X_T) &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left(\sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} - 0 \right) \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \lambda \end{aligned}$$

and

$$\begin{aligned} E(X_T^2) &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \sum_{k=1}^{\infty} k^2 \frac{\lambda^k}{k!} \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left(\frac{\partial^2}{\partial t^2} MGF(X) \Big|_{t=0} - 0 \right) \\ &= \frac{e^{-\lambda}}{1-e^{-\lambda}} \left(\frac{\partial^2}{\partial t^2} e^{\lambda e^t - 1} \Big|_{t=0} \right) \\ &= \frac{\lambda e^{-\lambda-1}}{1-e^{-\lambda}} \left(\frac{\partial}{\partial t} e^t e^{\lambda e^t} \Big|_{t=0} \right) \\ &= \frac{\lambda e^{-\lambda-1}}{1-e^{-\lambda}} \left(e^t e^{\lambda e^t} + \lambda e^{2t} e^{\lambda e^t} \Big|_{t=0} \right) \\ &= \frac{\lambda e^{-\lambda-1}}{1-e^{-\lambda}} \left(e^{\lambda} + \lambda e^{\lambda} \right) \\ &= \frac{\lambda (1+\lambda)}{e (1-e^{-\lambda})} \end{aligned}$$

and finally

$$\mathrm{Var}(X_T) = E(X_T^2) - (E(X_T))^2 = \frac{\lambda(1+\lambda)}{e(1-e^{-\lambda})} - \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} = \frac{\lambda}{1-e^{-\lambda}} \left(\frac{(1+\lambda)}{e} - e^{-\lambda} \right)$$

(b) $P(X=k) = \binom{k+r-1}{k}(1-p)^k(p)^r$. Note this definition is obverse from the book - $p = 1-p'$. Firstly

$$\begin{aligned}
P(X > k) &= \sum_{i=1}^{\infty} \binom{k+r-1}{k}(1-p)^k(p)^r \\
&= \sum_{i=0}^{\infty} \binom{k+r-1}{k}(1-p)^k(p)^r - \binom{0+r-1}{0}(1-p)^0(p)^r \\
&= 1 - p^r
\end{aligned}$$

Then $P(X=k) = \frac{1}{p^r}\binom{k+r-1}{k}(1-p)^k(p)^r I_{\{1,2,\dots\}}$ and

$$\begin{aligned}
E(X_T) &= \frac{1}{p^r}\sum_{k=1}^{\infty} k\binom{k+r-1}{k}p^k(1-p)^r \\
&= \frac{1}{p^r}\sum_{k=0}^{\infty} k\binom{k+r-1}{k}p^k(1-p)^r \\
&= \frac{r(1-p)}{p^{r+1}}
\end{aligned}$$

and

$$\begin{aligned}
E(X_T^2) &= E(X_T(X_T-1)) + E(X_T) \\
&= \frac{1}{p^r}\sum_{k=1}^{\infty} k(k-1)\binom{k+r-1}{k}p^k(1-p)^r + E(X_T) \\
&= \frac{1}{p^r}\sum_{k=0}^{\infty} \frac{(k+r-1)!}{(k-2)!(r-1)!}p^k(1-p)^r + E(X_T) \\
&= \frac{1}{p^r}\sum_{k=0}^{\infty} \frac{((k-2)+(r+2)-1)!}{(k-2)!((r+2)-2-1)!}p^k(1-p)^r + E(X_T) \\
&= \frac{p^2p^{-2}}{p^r((r+2)-3)((r+2)-2)}\sum_{k=0}^{\infty} \frac{((k-2)+(r+2)-1)!}{(k-2)!((r+2)-1)!}p^{k-2}(1-p)^{r+2} + E(X_T) \\
&= \frac{1}{p^r(r-1)r}\sum_{k=0}^{\infty} \binom{(k-2)+(r+2)-1}{k-2}p^{k-2}(1-p)^{r+2} + E(X_T) \\
&= \frac{1}{p^r(r-1)r} + \frac{r(1-p)}{p^{r+1}} = \frac{1+p^{r-1}r(1-p)(r-1)}{p^r(r-1)r}
\end{aligned}$$

Finally

$$\mathrm{Var}(X_T) = \frac{1+p^{r-1}r(1-p)(r-1)}{p^r(r-1)r} - \left(\frac{r(1-p)}{p^{r+1}}\right)^2 = 1-(p-1)p - \frac{(r(p-1))^2}{p^{2(r+1)}}$$

3.14 (a) $f_X > 0$ since $p < 1$ and therefore $\log(p) < 0$. Furthermore

$$\begin{aligned}
\sum_{x=1}^{\infty} \frac{-(1-p)^x}{x\log(p)} &= \frac{1}{\log(p)}\sum_{x=1}^{\infty}\frac{-(1-p)^x}{x} \\
&= \frac{1}{\log(p)}\log(1-(1-p)) \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
E(X) &= \frac{1}{\log(p)} \sum_{x=1}^{\infty} x \frac{-(1-p)^x}{x} \\
&= \frac{-1}{\log(p)} \sum_{x=1}^{\infty} (1-p)^x \\
&= \frac{-1}{\log(p)} \frac{1-p}{1-(1-p)} \\
&= \frac{p-1}{p\log(p)}
\end{aligned}$$

Then

$$\begin{aligned}
E(X^2) &= \frac{1}{\log(p)} \sum_{x=1}^{\infty} x^2 \frac{-(1-p)^x}{x} \\
&= \frac{-1}{\log(p)} \sum_{x=1}^{\infty} x(1-p)^x
\end{aligned}$$

Now since $\sum(1-p)^x$ absolutely converges to $(1-p)/p$ we can differentiate under the sum

$$\begin{aligned}
\frac{d}{dp}\frac{1-p}{p} &= \frac{\partial}{\partial p} \sum_{x=1}^{\infty} (1-p)^x \\
\frac{p-1}{p^2} - \frac{1}{p} &= \sum_{x=1}^{\infty} \frac{\partial}{\partial p} (1-p)^x \\
&= \sum_{x=1}^{\infty} x(1-p)^{x-1} \\
&= \frac{1}{1-p} \sum_{x=1}^{\infty} x(1-p)^x \implies \\
\frac{(1-p)(p-1)}{p^2} - \frac{1-p}{p} &= \sum_{x=1}^{\infty} x(1-p)^x
\end{aligned}$$

and therefore

$$\begin{aligned}
E(X^2) &= \frac{-1}{\log(p)} \left(\frac{(1-p)(p-1)}{p^2} - \frac{1-p}{p} \right) \\
&= \frac{1-p}{p^2\log(p)}
\end{aligned}$$

and finally

$$\operatorname{Var}(X) = \frac{1-p}{p^2\log(p)} - \left(\frac{p-1}{p\log(p)}\right)^2 = \frac{\log(1-p)-(p-1)^2}{p^2\log^2(p)}$$

3.19 $Z \sim \Gamma(\alpha, 1) \implies P(Z = z) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}$. Therefore

$$\begin{aligned}
\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz &= \frac{1}{\Gamma(\alpha)} \int_x^\infty z^{\alpha-1} e^{-z} dz \\
&= \frac{1}{\Gamma(\alpha)} \left(- \left(z^{\alpha-1} e^{-z} \right) \Big|_x^\infty - (\alpha - 1) \int_x^\infty z^{\alpha-2} \left(-e^{-z} \right) dx \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(- \left(0 - x^{\alpha-1} e^{-x} \right) + (\alpha - 1) \int_x^\infty z^{\alpha-2} e^{-z} dx \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} e^{-x} + (\alpha - 1) \int_x^\infty z^{\alpha-2} e^{-z} dx \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} e^{-x} + (\alpha - 1) \int_x^\infty z^{\alpha-2} e^{-z} dx \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} e^{-x} + (\alpha - 1) \left(- \left(z^{\alpha-2} e^{-z} \right) \Big|_x^\infty - (\alpha - 2) \int_x^\infty z^{\alpha-3} \left(-e^{-z} \right) dx \right) \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} e^{-x} + (\alpha - 1) \left(x^{\alpha-2} e^{-x} + (\alpha - 2) \int_x^\infty z^{\alpha-3} e^{-z} dx \right) \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} e^{-x} + (\alpha - 1) x^{\alpha-2} e^{-x} + (\alpha - 1)(\alpha - 2) \int_x^\infty z^{\alpha-3} e^{-z} dx \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} e^{-x} + (\alpha - 1) x^{\alpha-2} e^{-x} + (\alpha - 1)(\alpha - 2) x^{\alpha-3} e^{-x} + \cdots + (\alpha - 1)! \int_x^\infty e^{-z} dx \right) \\
&= \frac{1}{\Gamma(\alpha)} \left(x^{\alpha-1} e^{-x} + (\alpha - 1) x^{\alpha-2} e^{-x} + (\alpha - 1)(\alpha - 2) x^{\alpha-3} e^{-x} + \cdots + (\alpha - 1)! e^{-x} \right) \\
&\text{but } \Gamma(\alpha) = (\alpha - 1)! \\
&= \frac{x^{\alpha-1} e^{-x}}{(\alpha - 1)!} + \frac{(\alpha - 1) x^{\alpha-2} e^{-x}}{(\alpha - 1)!} + \frac{(\alpha - 1)(\alpha - 2) x^{\alpha-3} e^{-x}}{(\alpha - 1)!} + \cdots + \frac{(\alpha - 1)! e^{-x}}{(\alpha - 1)!} \\
&= \frac{x^{\alpha-1} e^{-x}}{(\alpha - 1)!} + \frac{x^{\alpha-2} e^{-x}}{(\alpha - 2)!} + \frac{x^{\alpha-3} e^{-x}}{(\alpha - 3)!} + \cdots + e^{-x} \\
&= \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}
\end{aligned}$$

If $X \sim Poisson(1)$

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} dz = P(Z > x) = P(X < \alpha) = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}$$

3.20 (a) $f_X(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$ implies

$$\begin{aligned}
E(X) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty x e^{-x^2/2} dx \\
&= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-\left(x^2/2\right)} d\left(x^2/2\right) \\
&= \frac{-2}{\sqrt{2\pi}} (0 - 1) = \frac{2}{\sqrt{2\pi}}
\end{aligned}$$

and by **2.22(a)**

$$\begin{aligned}
E(X^2) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty x^2 e^{-x^2/2} dx \\
&= \frac{2}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{2} = \frac{1}{2}
\end{aligned}$$

Hence
$$\mathrm{Var}(X) = \frac{1}{2} - \frac{2}{\pi}$$

(b) Let $Y = g(X) = X^2$ then $g^{-1}(Y) = \sqrt{y}$ and
$$\left| g^{-1}(y)' \right| = y^{1/2-1}$$

and then
$$\begin{aligned} f_Y(y) &= f_X(\sqrt{y})\, y^{1/2-1} \\ &= \frac{1}{\sqrt{2\pi}} y^{1/2-1} e^{-y/2} \\ &= \frac{y^{1/2-1}}{\sqrt{\pi} 2^{1/2}} e^{-y/2} \\ &= \frac{y^{1/2-1}}{\Gamma(1/2) 2^{1/2}} e^{-y/\beta} \end{aligned}$$

Hence $Y \sim \mathrm{Gamma}(1/2, 2)$.

3.23 $f_X = \beta\alpha^\beta / x^{\beta+1}$

(a) $0 < \alpha < x < \infty$ implies $x^{\beta+1} > 0$ and hence $f_X > 0$. Furthermore
$$\begin{aligned} \int_\alpha^\infty f_X(x)dx &= \int_\alpha^\infty f_X(x)dx \\ &= \int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}} dx \\ &= \frac{\beta\alpha^\beta}{-\beta} \left(\frac{1}{x^\beta} \Big|_\alpha^\infty \right) \\ &= -\alpha^\beta \left(0 - \frac{1}{\alpha^\beta} \right) = 1 \end{aligned}$$

(b)
$$\begin{aligned} E(X) &= \int_\alpha^\infty x \frac{\beta\alpha^\beta}{x^{\beta+1}} dx \\ &= \beta\alpha^\beta \int_\alpha^\infty \frac{1}{x^\beta} dx \\ &= \frac{\beta\alpha^\beta}{-(\beta+1)} \left(\frac{1}{x^{\beta-1}} \Big|_\alpha^\infty \right) \\ &= \frac{\beta\alpha^\beta}{-(\beta+1)} \left(0 - \frac{1}{\alpha^{\beta-1}} \right) = \frac{\beta\alpha}{(\beta+1)} \end{aligned}$$

and
$$\begin{aligned} E(X^2) &= \int_\alpha^\infty x^2 \frac{\beta\alpha^\beta}{x^{\beta+1}} dx \\ &= \beta\alpha^\beta \int_\alpha^\infty \frac{1}{x^{\beta-1}} dx \\ &= \frac{\beta\alpha^\beta}{-(\beta+2)} \left(\frac{1}{x^{\beta-2}} \Big|_\alpha^\infty \right) \\ &= \frac{\beta\alpha^\beta}{-(\beta+2)} \left(0 - \frac{1}{\alpha^{\beta-2}} \right) = \frac{\beta\alpha^2}{(\beta+2)} \end{aligned}$$

Therefore

$$\mathrm{Var}(X) = \frac{\beta\alpha^2}{(\beta+2)} - \left(\frac{\beta\alpha}{(\beta+1)}\right)^2 = \alpha^2\left(\frac{\beta}{(\beta+2)} - \frac{\beta^2}{(\beta+1)}\right) = \alpha^2\beta\left(\frac{1}{(\beta+2)} + \frac{1}{(\beta+1)} - 1\right)$$

3.24 (a) Let $Y = g(X) = X^{1/\gamma}$ then $g^{-1}(Y) = y^\gamma$ and

$$\left|g^{-1}(y)'\right| = \gamma y^{\gamma-1}$$

and then

$$\begin{aligned} f_Y(y) &= f_X(y^\gamma)\,\gamma y^{\gamma-1} \\ &= \gamma\beta y^{\gamma-1} e^{-\beta y^\gamma} \end{aligned}$$

which is positive on $0 < y < \infty$ and

$$\begin{aligned} \int_0^\infty f_Y(y) &= \int_0^\infty f_X(y^\gamma)\,\gamma y^{\gamma-1} dy \\ &= \int_0^\infty \gamma\beta y^{\gamma-1} e^{-\beta y^\gamma} dy \\ &= \int_0^\infty e^{-(\beta y^\gamma)} d(\beta y^\gamma) \\ &= -\left(e^{-\beta y^\gamma}\Big|_0^\infty\right) \\ &= -(0-1) = 1 \end{aligned}$$

Furthermore

$$\begin{aligned} E(Y) &= \int_0^\infty y\gamma\beta y^{\gamma-1} e^{-\beta y^\gamma} dy \\ &= \int_0^\infty \beta y^\gamma e^{-\beta y^\gamma} dy \\ u = \beta y^\gamma &\implies \frac{1}{\gamma\beta}\left(\frac{u}{\beta}\right)^{\frac{1}{\gamma}-1} du = dy \\ &= \int_0^\infty \frac{\gamma}{\gamma\beta}\left(\frac{u}{\beta}\right)^{\frac{1}{\gamma}-1} u e^{-u} dy \\ &= \int_0^\infty \left(\frac{u}{\beta}\right)^{\frac{1}{\gamma}-1} \frac{u}{\beta} e^{-u} dy \\ &= \beta^{-1/\gamma} \int_0^\infty u^{\frac{1}{\gamma}} e^{-u} dy \\ &= \beta^{-1/\gamma}\Gamma\left(1 + \frac{1}{\gamma}\right) \end{aligned}$$

and

$$
\begin{aligned}
E(Y^2) &= \int_0^\infty y^2 \gamma\beta y^{\gamma-1} e^{-\beta y^\gamma}\, dy \\
&= \int_0^\infty \beta y^{\gamma+1} e^{-\beta y^\gamma}\, dy \\
u = \beta y^\gamma &\implies \frac{1}{\gamma\beta}\left(\frac{u}{\beta}\right)^{\frac{1}{\gamma}-1} du = dy \\
&= \int_0^\infty \left(\frac{u}{\beta}\right)^{\frac{1}{\gamma}-1}\left(\frac{u}{\beta}\right)^{1+\frac{1}{\gamma}} e^{-u} dy \\
&= \int_0^\infty \left(\frac{u}{\beta}\right)^{\frac{2}{\gamma}} e^{-u} dy \\
&= \beta^{-2/\gamma}\int_0^\infty u^{\frac{2}{\gamma}} e^{-u} dy \\
&= \beta^{-2/\gamma}\Gamma\left(1+\frac{2}{\gamma}\right)
\end{aligned}
$$

Therefore

$$
\mathrm{Var}(Y) = \beta^{-2/\gamma}\Gamma\left(1+\frac{2}{\gamma}\right) - \left(\beta^{-1/\gamma}\Gamma\left(1+\frac{1}{\gamma}\right)\right)^2 = \beta^{-2/\gamma}\left(\Gamma\left(1+\frac{2}{\gamma}\right) - \Gamma^2\left(1+\frac{1}{\gamma}\right)\right)
$$

(b) Let $X \sim \mathrm{Exp}(\beta)$ and $W = X^{1/2}$ and $Y = g(W) = 2^{1/2}W/\beta^{1/2}$. Then $W \sim \mathrm{Weibull}(2, \beta)$ and $g^{-1}(Y) = \beta^{1/2}W/2^{1/2}$

$$
\left|g^{-1}(y)'\right| = \beta^{1/2}/2^{1/2}
$$

and then

$$
\begin{aligned}
f_Y(y) &= f_W\left(\sqrt{\frac{\beta}{2}}y\right)\sqrt{\frac{\beta}{2}} \\
&= 2\beta\left(\sqrt{\frac{\beta}{2}}y\right) e^{-\beta\left(\sqrt{\frac{\beta}{2}}y\right)^2}\sqrt{\frac{\beta}{2}} \\
&= 2\left(\frac{\beta^2}{2}y\right) e^{-\frac{\beta^2}{2}y^2}
\end{aligned}
$$

Hence $Y \sim \mathrm{Weibull}(2, \frac{\beta^2}{2})$. Therefore immediately $\int_0^\infty f_Y(y)dy = 1$. Furthermore

$$
E(Y) = \left(\frac{\beta^2}{2}\right)^{-1/2}\Gamma\left(1+\frac{1}{2}\right)
$$

and

$$
\mathrm{Var}(Y) = \frac{2}{\beta^2}\left(\Gamma(2) - \Gamma^2\left(1+\frac{1}{2}\right)\right)
$$

(c) Let $Y = g(X) = X^{-1}$ then $g^{-1}(Y) = y^{-1}$ and

$$
\left|g^{-1}(y)'\right| = \frac{1}{y^2}
$$

and then

$$
\begin{aligned}
f_Y(y) &= \frac{f_X\left(y^{-1}\right)}{y^2} \\
&= \frac{\left(y^{-1}\right)^{\alpha-1} e^{-y^{-1}/\beta}}{y^2\beta^\alpha\Gamma(\alpha)} \\
&= \frac{y^{-1-\alpha}}{\beta^\alpha\Gamma(\alpha)} e^{-y^{-1}/\beta}
\end{aligned}
$$

which is positive on $0 < y < \infty$ and

$$
\begin{aligned}
\int_0^\infty f_Y(y) &= \int_0^\infty \frac{y^{-1-\alpha}}{\beta^\alpha\Gamma(\alpha)} e^{-y^{-1}/\beta} dy \\
u &= \frac{1}{y\beta} \implies du = -\frac{dy}{\beta y^2} \\
&= \int_\infty^0 \frac{-\left(u\beta\right)^{\alpha-1}}{\beta^{\alpha-1}\Gamma(\alpha)} e^{-u} du \\
&= \int_0^\infty \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-u} dy \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
E(X) &= \int_0^\infty y \frac{y^{-1-\alpha}}{\beta^\alpha\Gamma(\alpha)} e^{-y^{-1}/\beta} dy \\
&= \frac{\Gamma(\alpha-1)}{\beta\Gamma(\alpha)} \int_0^\infty \frac{y^{-1-(\alpha-)}}{\beta^{\alpha-1}\Gamma(\alpha-1)} e^{-y^{-1}/\beta} dy \\
&\text{integrand is kernel of IG}(\alpha\text{-}1,\beta) \\
&= \frac{\Gamma(\alpha-1)}{\beta\Gamma(\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
E(X^2) &= \int_0^\infty y^2 \frac{y^{-1-\alpha}}{\beta^\alpha\Gamma(\alpha)} e^{-y^{-1}/\beta} dy \\
&= \frac{\Gamma(\alpha-2)}{\beta^2\Gamma(\alpha)} \int_0^\infty \frac{y^{-1-(\alpha-2)}}{\beta^{\alpha-2}\Gamma(\alpha-2)} e^{-y^{-1}/\beta} dy \\
&\text{integrand is kernel of IG}(\alpha\text{-}2,\beta) \\
&= \frac{\Gamma(\alpha-2)}{\beta^2\Gamma(\alpha)}
\end{aligned}
$$

Therefore

$$
\mathrm{Var}(Y) = \frac{\Gamma(\alpha-2)}{\beta^2\Gamma(\alpha)} - \left(\frac{\Gamma(\alpha-1)}{\beta\Gamma(\alpha)}\right)^2 = \frac{1}{\beta^2}\left(\frac{\Gamma(\alpha-2)}{\Gamma(\alpha)} - \frac{\Gamma^2(\alpha-1)}{\Gamma^2(\alpha)}\right)
$$

(d) Let $Y = g(X) = (X/\beta)^{1/2}$ then $g^{-1}(Y) = \beta y^2$ and

$$
\left|g^{-1}(y)^{'}\right| = 2\beta y
$$

and then

$$
\begin{aligned}
f_Y(y) &= f_X\left(\beta y^2\right) 2\beta y \\
&= \frac{\left(\beta y^2\right)^{\frac{3}{2}-1} e^{-\beta y^2/\beta}}{\beta^{3/2}\Gamma(3/2)} 2\beta y \\
&= \frac{2y^2 e^{-y^2}}{\Gamma(3/2)}
\end{aligned}
$$

which is positive on $0 < y < \infty$ and

$$
\begin{aligned}
\int_0^\infty f_Y(y)dy &= \int_0^\infty \frac{2y^2 e^{-y^2}}{\Gamma(3/2)} dy \\
&= \frac{2}{\Gamma(3/2)} \int_0^\infty y^2 e^{-y^2} dy \\
&= \frac{2}{\Gamma(3/2)} \frac{\sqrt{\pi}}{4} = 1
\end{aligned}
$$

Then

$$
\begin{aligned}
E(X) &= \int_0^\infty \frac{2y^3 e^{-y^2}}{\Gamma(3/2)} dy \\
&= \frac{2}{\Gamma(3/2)} \int_0^\infty y^3 e^{-y^2} dy \\
&\text{using the trick from } \mathbf{2.22(a)} \\
&= \frac{2}{\Gamma(3/2)} \frac{1}{2} = \frac{2}{\sqrt{\pi}}
\end{aligned}
$$

and

$$
\begin{aligned}
E(X^2) &= \int_0^\infty \frac{2y^4 e^{-y^2}}{\Gamma(3/2)} dy \\
&= \frac{2}{\Gamma(3/2)} \int_0^\infty y^4 e^{-y^2} dy \\
&\text{using the trick from } \mathbf{2.22(a)} \\
&= \frac{2}{\Gamma(3/2)} \frac{3\sqrt{\pi}}{8} = \frac{3}{4}
\end{aligned}
$$

Finally

$$
\mathrm{Var}(X) = \frac{3}{4} - \frac{4}{\pi}
$$

(e) Let $Y = g(X) = \alpha - \xi \log(X)$ then $g^{-1}(Y) = e^{(\alpha-y)/\xi}$ and

$$
\left| g^{-1}(y)' \right| = \frac{1}{\xi} e^{(\alpha-y)/\xi}
$$

and then

$$
\begin{aligned}
f_Y(y) &= f_X\left(e^{(\alpha-y)/\xi}\right) \frac{1}{\xi} e^{(\alpha-y)/\xi} \\
&= e^{-e^{(\alpha-y)/\xi}} \frac{1}{\xi} e^{(\alpha-y)/\xi}
\end{aligned}
$$

When $x \to 0$ then $y \to \infty$ and when $x \to \infty$ then $y \to -\infty$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} f_Y(y)dy &= \int_{-\infty}^{\infty} e^{-e^{(\alpha-y)/\xi}} \frac{1}{\xi} e^{(\alpha-y)/\xi} dy \\
&= \int_{\infty}^{0} e^{-\left(e^{(\alpha-y)/\xi}\right)} d\left(e^{(\alpha-y)/\xi}\right) \\
&= -\int_{0}^{\infty} e^{-\left(e^{(\alpha-y)/\xi}\right)} d\left(e^{(\alpha-y)/\xi}\right) \\
&= -\left(e^{-u}\Big|_0^{\infty}\right) = -(0-1) = 1
\end{aligned}$$

Then

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} y e^{-e^{(\alpha-y)/\xi}} \frac{1}{\xi} e^{(\alpha-y)/\xi} dy \\
&\text{let } u = e^{(\alpha-y)/\xi} \\
&= \int_{0}^{\infty} \left(\alpha - \xi \log(u)\right) e^{-u} du \\
&= \int_{0}^{\infty} \alpha e^{-u} du + \xi \int_{0}^{\infty} \log(u) e^{-u} du \\
&= -\alpha - \xi\gamma
\end{aligned}$$

where γ is the Euler-Mascheroni constant and $\gamma \approx 0.57721$. Then

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} y^2 e^{-e^{(\alpha-y)/\xi}} \frac{1}{\xi} e^{(\alpha-y)/\xi} dy \\
&\text{let } u = e^{(\alpha-y)/\xi} \\
&= \int_{0}^{\infty} \left(\alpha - \xi \log(u)\right)^2 e^{-u} du \\
&= \int_{0}^{\infty} \alpha^2 e^{-u} du - 2\alpha\xi \int_{0}^{\infty} \log(u) e^{-u} du + \xi^2 \int_{0}^{\infty} \log^2(u) e^{-u} du \\
&= -\alpha^2 - 2\alpha\xi\gamma + \xi^2 \left(\gamma^2 + \frac{\pi^2}{6}\right)
\end{aligned}$$

Finally

$$\mathrm{Var}(X) = -\alpha^2 - 2\alpha\xi\gamma + \xi^2\left(\gamma^2 + \frac{\pi^2}{6}\right) - \alpha^2 - \xi^2\gamma^2 + 2\alpha\xi\gamma = \xi^2\left(\frac{\pi^2}{6}\right) - 2\alpha^2$$