Outline of psdmul

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We have

$$(FG)(k) = \sum_{b \le k} F(b)G(k-b) \tag{mul}$$

$$\frac{\partial F}{\partial X} = (k \mapsto (k(X)+1)\, F(k+X)) \tag{psd}$$

where F and G are multivariate polynomials (represented as functions from bags of variables to coefficients) and X is a variable. A bag (aka multiset) can be thought of as a set that may have duplicates, which is represented by mapping variables to nonnegative integers.

Expression	Meaning
$F(b)$	Coefficient corresponding to a bag
$k(X)$	Amount of variable X in the bag
$b+k$, $b-k$	Sum or difference of two bags
$b+X$, $b-X$	Add or remove variable X to/from a bag

We prove

$$\frac{\partial (FG)}{\partial X} = \left(\frac{\partial F}{\partial X} G\right) + \left(F \frac{\partial G}{\partial X}\right)$$

from

$$\frac{\partial (FG)}{\partial X}(k) = \left(\frac{\partial F}{\partial X} G\right)(k) + \left(F \frac{\partial G}{\partial X}\right)(k) \tag{1 = 2 + 3}$$

by expanding expressions 1, 2, and 3 as follows:

$$\begin{aligned}
\frac{\partial (FG)}{\partial X}(k) &= (k(X)+1)\,(FG)(k+X) \\
&= (k(X)+1) \sum_{b \le k+X} F(b)G((k+X)-b) \\
&= \sum_{b \le k+X} (k(X)+1)F(b)G((k+X)-b)
\end{aligned} \tag{1}$$

$$\begin{aligned}
\left(\frac{\partial F}{\partial X} G\right)(k) &= \sum_{b' \le k} \left(\frac{\partial F}{\partial X}\right)(b')G(k-b') \\
&\xrightarrow[b'=b-X]{} \sum_{X \le b \le k+X} \left(\frac{\partial F}{\partial X}\right)(b-X)G(k-(b-X)) \\
&= \sum_{X \le b \le k+X} ((b-X)(X)+1)F((b-X)+X)G(k-(b-X)) \\
&= \sum_{X \le b \le k+X} b(X)F(b)G((k+X)-b)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\left(F \frac{\partial G}{\partial X}\right)(k) &= \sum_{b \le k} F(b) \left(\frac{\partial G}{\partial X}\right)(k-b) \\
&= \sum_{b \le k} F(b)((k-b)(X)+1)G((k-b)+X) \\
&= \sum_{b \le k} ((k-b)(X)+1)F(b)G((k-b)+X) \\
&= \sum_{b \le k} ((k-b)(X)+1)F(b)G((k+X)-b)
\end{aligned} \tag{3}$$

Table 2: Final step: 2 + 3 = 1

Reference	$b < X$*	$X \leq b \leq k$	$k < b \leq k + X$
(2)		$b(X)F(b)G((k+X)-b)$	$b(X)F(b)G((k+X)-b)$
(3)	$((k-b)(X)+1)F(b)G((k+X)-b)$	$((k-b)(X)+1)F(b)G((k+X)-b)$	
(1)	$(k(X)+1)F(b)G((k+X)-b)$	$(k(X)+1)F(b)G((k+X)-b)$	$(k(X)+1)F(b)G((k+X)-b)$

* bags without the variable X

Then we split each result, a sum, into summations over certain ranges, as shown in Table 2.

Over each range, we have $2 + 3 = 1$:

For the first column,

$$b < X$$

means

$$b(X) = 0$$

, so

$$(k-b)(X) = k(X) - b(X) = k(X)$$

.

For the last column,

$$k < b \leq k + X$$

implies

$$\begin{aligned} b(X) &= (k+X)(X) \\ &= k(X) + 1 \end{aligned}$$

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