Outline of psdmul

Steven Nguyen (icecream17)

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We have

$$(FG)(B) = \sum_{b \le B} F(b)G(B-b) \tag{mul}$$

$$\frac{\partial F}{\partial X} = (k \mapsto (k(X)+1)\, F(k+X)) \tag{psd}$$

where F and G are multivariate polynomials (represented as functions from bags of variables to coefficients) and X is a variable. $F(b)$ is the coefficient corresponding to a bag, and $k(X)$ is the amount of the variable X in the bag. $b+k$ or $b-k$ is the sum or difference of two bags, whereas $b+X$ or $b-X$ adds or removes a variable to or from a bag. (A bag can be thought of as a set that may have duplicates, which is represented by mapping variables to nonnegative integers).

We prove

$$\frac{\partial (FG)}{\partial X} = \left(\frac{\partial F}{\partial X} G\right) + \left(F \frac{\partial G}{\partial X}\right)$$

from

$$\frac{\partial (FG)}{\partial X}(k) = \left(\frac{\partial F}{\partial X} G\right)(k) + \left(F \frac{\partial G}{\partial X}\right)(k) \tag{1 = 2 + 3}$$

by expanding expressions 1, 2, and 3 as follows:

$$\begin{aligned}
\frac{\partial (FG)}{\partial X}(k) &= (k(X)+1)\,(FG)(k+X) \\
&= (k(X)+1) \sum_{b \le k+X} F(b)G((k+X)-b) \\
&= \sum_{b \le k+X} (k(X)+1)F(b)G((k+X)-b) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial F}{\partial X} G\right)(k) &= \sum_{b \le k} \left(\frac{\partial F}{\partial X}\right)(b) G(k-b) \\
\xrightarrow[b=b'-X]{} & \sum_{X \le b' \le k+X} \left(\frac{\partial F}{\partial X}\right)(b'-X) G(k-(b'-X)) \\
&= \sum_{X \le b' \le k+X} ((b'-X)(X)+1)F((b'-X)+X)G(k-(b'-X)) \\
&= \sum_{X \le b \le k+X} b(X)F(b)G((k+X)-b) \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\left(F \frac{\partial G}{\partial X}\right)(k) &= \sum_{b \le k} F(b) \left(\frac{\partial G}{\partial X}\right)(k-b) \\
&= \sum_{b \le k} F(b)((k-b)(X)+1)G((k-b)+X) \\
&= \sum_{b \le k} ((k-b)(X)+1)F(b)G((k-b)+X) \\
&= \sum_{b \le k} ((k-b)(X)+1)F(b)G((k+X)-b) \qquad (3)
\end{aligned}$$

Then we split each result, a sum, into summations over certain ranges, as shown in Table 1.

Table 1: Final step: 2 + 3 = 1

Reference	$b < X^*$	$X \le b \le k$	$k < b \le k + X$
(2)		$b(X)F(b)G((k + X) - b)$	$b(X)F(b)G((k + X) - b)$
(3)	$((k - b)(X) + 1)F(b)G((k + X) - b)$	$((k - b)(X) + 1)F(b)G((k + X) - b)$	
(1)	$(k(X) + 1)F(b)G((k + X) - b)$	$(k(X) + 1)F(b)G((k + X) - b)$	$(k(X) + 1)F(b)G((k + X) - b)$

* bags without the variable X

For the first column,

$$b < X$$

means

$$b(X) = 0$$

, so

$$(k - b)(X) = k(X) - b(X) = k(X)$$

.

For the last column,

$$k < b \le k + X$$

implies

$$b(X) = (k + X)(X) = k(X) + 1$$

.