

Outline of psdmul

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We have

$$(FG)(B) = \sum_{b \leq B} F(b)G(B - b) \quad (\text{mul})$$

$$\frac{\partial F}{\partial X} = (k \mapsto (k(X) + 1) F(k + X)) \quad (\text{psd})$$

where F and G are multivariate polynomials (represented as functions from bags of variables to coefficients) and X is a variable.

We prove

$$\frac{\partial(FG)}{\partial X} = \left(\frac{\partial F}{\partial X} G \right) + \left(F \frac{\partial G}{\partial X} \right)$$

from

$$\frac{\partial(FG)}{\partial X}(k) = \left(\frac{\partial F}{\partial X} G \right)(k) + \left(F \frac{\partial G}{\partial X} \right)(k) \quad (1 = 2 + 3)$$

by expanding expressions 1, 2, and 3 as follows:

$$\begin{aligned} \frac{\partial(FG)}{\partial X}(k) &= (k(X) + 1) (FG)(k + X) \\ &= (k(X) + 1) \sum_{b \leq k+X} F(b)G((k + X) - b) \\ &= \sum_{b \leq k+X} (k(X) + 1) F(b)G((k + X) - b) \end{aligned} \quad (1)$$

$$\begin{aligned} \left(\frac{\partial F}{\partial X} G \right)(k) &= \sum_{b \leq k} \left(\frac{\partial F}{\partial X} \right)(b) G(k - b) \\ &\xrightarrow{b=b'-X} \sum_{X \leq b' \leq k+X} \left(\frac{\partial F}{\partial X} \right)(b' - X) G(k - (b' - X)) \\ &= \sum_{X \leq b' \leq k+X} ((b' - X)(X) + 1) F((b' - X) + X) G(k - (b' - X)) \\ &= \sum_{X \leq b \leq k+X} b(X) F(b) G((k + X) - b) \end{aligned} \quad (2)$$

$$\begin{aligned} \left(F \frac{\partial G}{\partial X} \right)(k) &= \sum_{b \leq k} F(b) \left(\frac{\partial G}{\partial X} \right)(k - b) \\ &= \sum_{b \leq k} F(b) ((k - b)(X) + 1) G((k - b) + X) \\ &= \sum_{b \leq k} ((k - b)(X) + 1) F(b) G((k - b) + X) \\ &= \sum_{b \leq k} ((k - b)(X) + 1) F(b) G((k + X) - b) \end{aligned} \quad (3)$$

Then we split each result, a sum, into summations over certain ranges, as shown in Table 1.

Table 1: Final step: $2 + 3 = 1$

Reference	$b < X^*$	$X \leq b \leq k$	$k < b \leq k + X$
(2)		$b(X)F(b)G((k + X) - b)$	$b(X)F(b)G((k + X) - b)$
(3)	$((k - b)(X) + 1)F(b)G((k + X) - b)$	$((k - b)(X) + 1)F(b)G((k + X) - b)$	
(1)	$(k(X) + 1)F(b)G((k + X) - b)$	$(k(X) + 1)F(b)G((k + X) - b)$	$(k(X) + 1)F(b)G((k + X) - b)$

* bags without the variable X

For the first column,

$$b < X$$

means

$$b(X) = 0$$

, so

$$(k - b)(X) = k(X) - b(X) = k(X)$$

.

For the last column,

$$k < b \leq k + X$$

implies

$$b(X) = (k + X)(X) = k(X) + 1$$

.