Pentomino Pathfinding

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Contents

1 Introduction

This paper will try to solve the Pentomino Pathfinding problem for various rectangular grids. See [Dec24a] and [Dec24b] for further introduction.

[Dec24a] posed the following problem: Given a rectangular $n \times m$ grid of squares or *cells*, place a subset of the twelve pentominoes (Figure 1), and endpoints A and B on the grid without overlaps such that $\#_{n,m}$ = the length of (the shortest path of nonempty, orthogonal cells from A to B) is maximized.

The maximum length given some placements p is denoted by $\#^p_{n,m}$; independently, when $n = m$, the row and column indices are collapsed: $\#_n$.

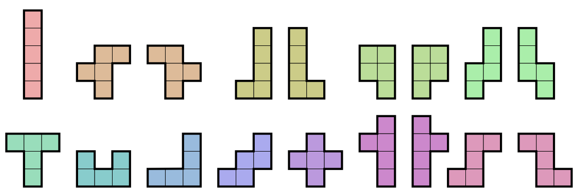


Figure 1: The twelve pentominoes and their reflections [Non08]; from left-to-right they are named I, F, L, P, N, T, U, V, W, X, Y, Z, where F, L, P, N, Y, Z are chiral and have their reflections shown.

2 No pentominoes

For $n = 1$ and 2, $n \times n < 5$, so no pentomino can fit. For $n = 3$, 9 cells minus a pentomino is 4 squares, so the length 5 path is optimal. (Figure 2)

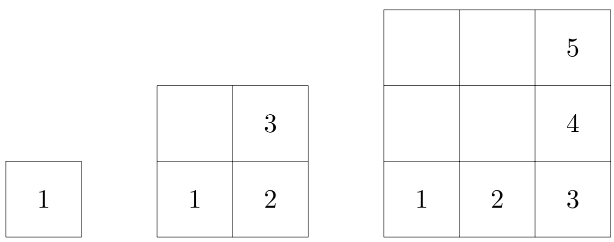
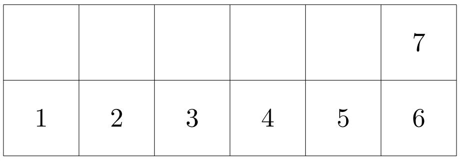
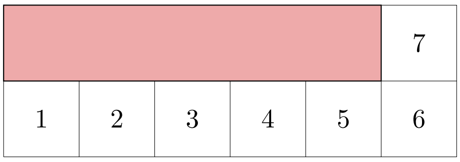


Figure 2: $\#_1 = 1, \#_2 = 3, \#_3 = 5$ [Dec24a]

Similar reasoning holds for $n = 2$, $m \leq 6$: there is a path of length $m + 1 \leq 7$, while $2m - 5 \leq m + 1 \leq 7$, so you cannot do any better than placing nothing. It turns out there is enough room for the I piece, and so for $m = 6$ there are two solutions ignoring symmetry:





And this reasoning also holds for $\#_{1,n} = n$: the path uses all squares, but using at least one pentomino only leaves you with at most one square to work with.

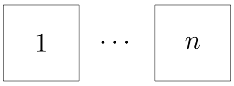


Figure 3: $\#_{1,n} = n$ (footnote: [1]) [Com24]

3 One pentomino

We'll start with proving some useful things about pentominoes in small grids.

Definition 3.1 (Adjacency)**.** Two squares are *adjacent* iff they are one square diagonally or orthogonally apart. Two squares are *orthogonal* iff they are adjacent but not diagonally so. Two sets of squares S and T are *adjacent* if there is a pair of squares $(s, t) \in S \times T$ (Cartesian product) such that s and t are adjacent. The squares adjacent to a set of squares S is denoted $\text{adj}(S)$.

Definition 3.2 (Subgrid)**.** A subgrid is any connected subgraph of empty squares of the grid. A square is empty iff it is does not have a pentomino.

Definition 3.3 (Platter)**.** A *platter* is a (set of adjacent squares) not adjacent to another square.

I don't use the term "shape" because two squares diagonally adjacent doesn't usually match people's intuitions.

Definition 3.4 (Outside)**.** For a platter P, $\text{outside}(P)$ = All squares reachable from the wall or other platters

Definition 3.5 (Border)**.** A platter's *border* is the squares adjacent to the platter that are outside of it. $\text{border}(P) = \text{adj}(P) \cap \text{outside}(P)$

If a platter cuts the grid into subgrids, we may restrict the outside to squares reachable from just that subgrid s. This is denoted $\text{border}(P)_s$

Lemma 3.6 (Border removal)**.** Say we have a platter p_1 with the following properties:

1. $\text{border}(p_1) = \text{adj}(p_1)$
2. choosing any two squares on its border A and B where there is a path from A to B, the shortest such path and the shortest such path upon removing the platter are the same length.

Then the platter may be removed: $\#^{p}_{n,m} \leq \#^{p-p_1}_{n,m}$

Proof. Can any preexisting path get shorter by entering the platter? No. Any preexisting path starts and ends outside the platter, by property 1. If the path wants to enter the platter, it must reach its border, go inside, then exit from another square on the border. But by property 2, going *along* the border is at least as efficient.

So, any preexisting paths determining $\#^{p}_{n,m}$ will never enter the area where the platter disappeared from; i.e, such paths will not get shorter with the removal of the platter. □

Note. We have the possibility of $\#^{p}_{n,m} < \#^{p-p_1}_{n,m}$ because in addition to the preexisting paths, there are new paths that go across or in the removed platter.

Lemma 3.7 (Rectangle cut)**.** Say we have an $n \times m$ grid, where pentominoes form a $k \times m$ rectangle (r) that is not adjacent to any other pentomino.

Then the rectangle may be removed: $\#^{p}_{n,m} \leq \#^{p-r}_{n,m}$

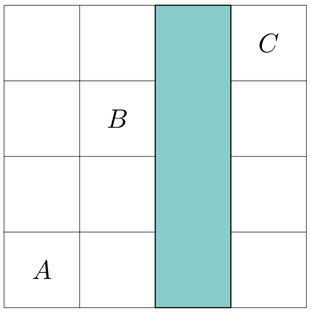


Figure 4: Illustration for Lemma 3.7: The path from A to B cannot be made any shorter by entering the rectangle since it could just go along the border. Meanwhile, removing the rectangle makes new paths A to C and B to C.

Proof. The rectangle satisfies Lemma 3.6, since $\mathrm{adj}(r)$ is zero, one, or two straight lines of squares by construction. □

We know that Lemma 3.6 works for rectangles that span from one side to the other, but this also works for the corner:

Theorem 3.8. A platter P is made of the set of squares S and all squares to the bottom-left of any square in S. Then it satisfies Lemma 3.6.

Proof. Informally, $\mathrm{adj}(P)$ forms a staircase-like shape. So any path along $\mathrm{adj}(P)$ will go in only two cardinal directions, and in fact will have the length of the Manhattan distance between its two endpoints, which cannot be improved upon. □

Note. This construction may be rotated and reflected.

Theorem 3.9 (Cut corner removal)**.** This extends Theorem 3.8 to potentially apply to each "side" (i.e. subgrid border) of a platter that cuts the grid into subgrids.

Proof. For each subgrid s, try applying Theorem 3.8 as if all the other subgrids were part of the platter. Including these other subgrids does not affect $\mathrm{border}(P)_s$, so this is valid for determining whether a path in s can be improved. □

Note. If Theorem 3.8 applies to the entire border, then as a result the whole platter can be removed, since no path within any of the subgrids can be improved.

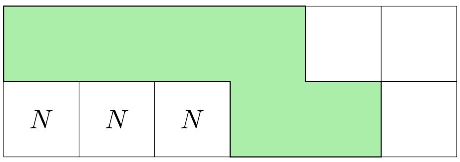


Figure 5: Example for Theorem 3.9: The pentomino N divides the grid into two. For the purposes of the subgrid on the right, N may as well also include the subgrid on the left (marked with three N's). This new platter satisfies Theorem 3.8.

3.1 $2 \times n$ grids

Consider a $2 \times n$ grid. The only pentominoes that fit are I, L, P, N, U, and Y.

By Theorem 3.9 (Cut corner removal), L, P, N, U, and Y can be removed from consideration in $2 \times n$ areas, so only I is left. While placing I on the corner does not help, placing I anywhere else increases $\#^p_{2,n}$ by 1 — the unique way to show $\#_{2,n} = n + 2$ for $n \geq 7$.

3.2 $3 \times n$ grids

Definition 3.10 (Cave)**.** A subgrid is a *cave* iff it is only connected to the rest of the subgrid its in by one square. That square is called a *separating vertex.* [Wik21]

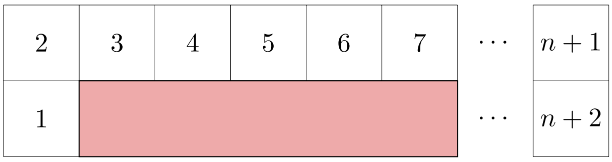


Figure 6: $\#_{2,n} = n + 2$ (where $7 \leq n$). [Com24]

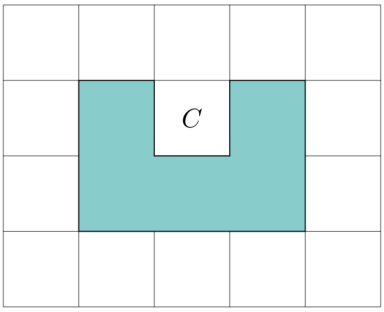


Figure 7: U is the unique pentomino with a cave square.

Theorem 3.11 (Ways to orient a pentomino)**.** A square has 4 symmetries, so a platter has at most 8 orientations, ignoring location within a grid. However, some pentominoes have one or more symmetries, which reduce the possible orientations (Table 1).

Table 1: Number of ways to orient each pentomino. [Dec24a, 0:41]

Pentomino	Symmetries	Orientations
X	4	1
I	2	2
T	1	4
U	1	4
V	1	4
W	1	4
Z	1	4
F	0	8
L	0	8
P	0	8
N	0	8
Y	0	8

Corollary 3.12. In a $3 \times n$, Theorem 3.9 works for V , W , Z , the vertical orientations of P , T , and U , and one side of either orientation of F . Note that we may reflect the entire grid itself along both a vertical and horizontal axis, since Theorem 3.9 works after reflection. This reduces the number of orientations a pentomino could have by a factor of four; the maximum orientations goes from eight to two. (Figure 8)

Note. U satisfies a general version of Theorem 3.9 where both corners are covered.

Property 3.13 (Cycle inefficiency)**.** If there is a cycle of empty squares in a grid, the path will not use all empty squares.

Property 3.14 (Zero pentomino result)**.** With nothing placed, we have $\#^{\emptyset}_{n,m} = n + m - 1$

Property 3.15 (One-pentomino solution)**.** Say we place a pentomino and $n + m - 1, nm - 10 < \#^{p}_{n,m}$. Then any solution must contain one pentomino.

Definition 3.16 (Brimming solution)**.** A solution is *brimming* when every square in the grid is used, i.e, every square corresponds to either a pentomino or the path.

[1]The figure assumes $n > 1$ but that's not necessary of course.

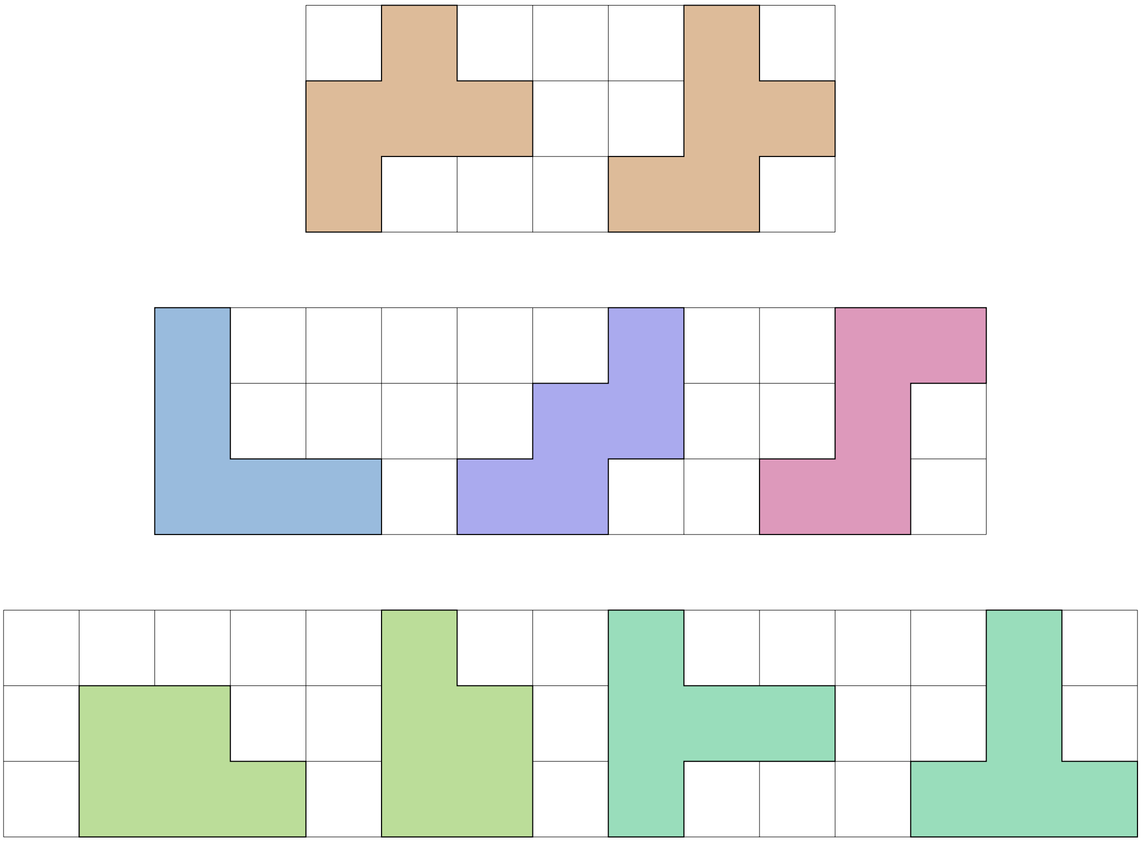


Figure 8: All orientations of F, V, W, Z, P, and T in a $3 \times n$ after reducing by symmetry. The left side of both F's are ignorable, as well as both sides of the right P and T. Additionally, the second F is strictly worse than the first F, since after filling in the ignorable subgrid, the first F practically replaces a wall with an empty square when compared to the second F.

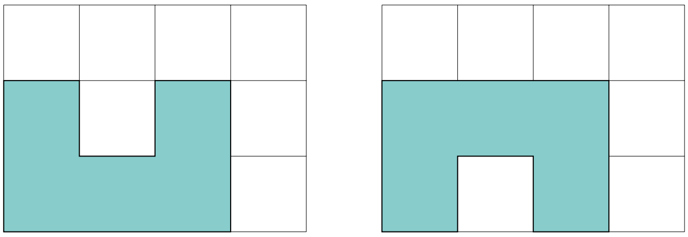


Figure 9: Horizontal orientations of U in a 3×4 (footnote: [2])

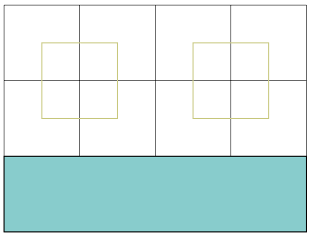


Figure 10: Two disjoint 2×2 cycles existing after placing a 1×4 on the edge of a 3×4.

Lemma 3.17 (Brimming solutions in $3 \times n$ grids)**.** Say we know how many pentominoes are in a solution, and that all squares are used. Then we know any solution must be brimming, and so:

- F, X, and U are not part of the solution.
- L, N, and Y must have their 1×4 in the middle row, if none of the columns the pentomino is in contains any part of another pentomino.

Proof. First, we take U from the first bullet point. If a path enters or exits a cave square (enters without loss of generality), it cannot exit since there is only one entrance. Since U has a cave square, we can say a path must end there. Placing U vertically is eliminated by Corollary 3.12. But if U is placed horizontally (2 rows by 3 columns, Figure 9), it is at least as good to just go around the U (removing any obstacles).

- Pentominoes must be placed so that the grid is *not* split into two. This is impossible for F and X, which span a 1×3 area with two squares padding each side of the 1×3, forcing the 1×3 into the center.
- If a pentomino that spans a 1×4 area (L, N, Y) has their 1×4 on the edge of the grid, there are two disjoint 2×2 cycles (Figure 10) which cannot be prevented by the remaining square of the pentomino and no help from other pentominoes. But this is not allowed by Property 3.13. □

Theorem 3.18. $\#_{3,4} = 7$

Proof. By Figure 11, we have brimming solutions with 1 pentomino, and since $3+4-1, 12-10 < 7$, Property 3.15 and therefore Lemma 3.17 applies. L, N, and Y can be eliminated since a 1×4 in the center row would divide the grid into two subgrids (this is not allowed; remember that every nonempty square must be in the path).

[2]film2860 on discord used gpt-4o to try to remove pixel artifacts. It generated equivalent code, which did not fix the problem, but did illustrate the usage of scopes which was helpful as a beginner. So congrats on contributing!

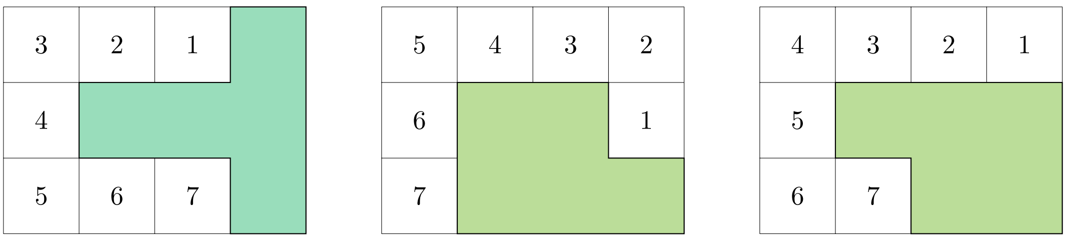


Figure 11: $\#_{3,4} = 7$. [Dec24b, 6:04]

We are left with the pentominoes P and T. If P does not touch the corner, after placing the 2×2, the 5th square must divide the grid in two. There are four ways to touch a specific corner including symmetry, and two (the horizontal placements) work. For T, there is only one orientation that does not cut the grid. So the three solutions given by [Dec24b, 6:04] are the only ones. □

Theorem 3.19. When a solution is brimming with only one pentomino, for grids with at least 3 columns and 4 rows, for any corner square, the pentomino must include the square diagonally adjacent to that corner square.

Proof. □

A polynomial's contribution is informally how much longer it makes a given path. But Figure 12 shows a formal definition is still a ways away. In such examples, we must consider entire platters.

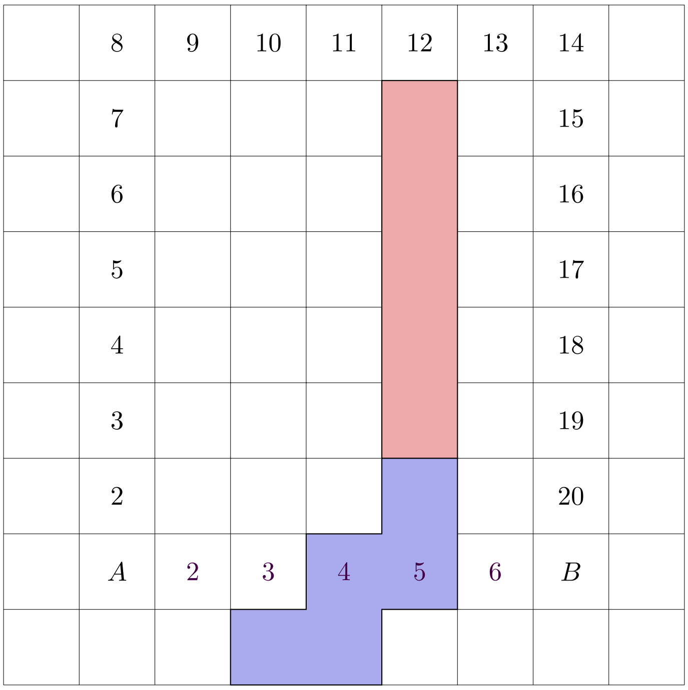


Figure 12: If the I pentomino was already placed, then adding the W changes the path's length from 7 to 21. Nevertheless, W 's contribution $C($ W $) = 4$. Additionally, W helps by blocking the bottom row. So some squares are spent on setup, and some on lengthening the path.

References

[Com24] Community. *Pentomino Pathfinding*. Sept. 3, 2024. URL: https://docs.google.com/spreadsheets/d/1NrbqWmnBLMtHH253q_v89bMYSuoPE7hDFr8g5VTbGMI/edit.

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[Dec24b] Deckard. *More Pentomino Pathfinding*. Sept. 2, 2024. URL: https://youtu.be/39YYZcwCuv0.

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