Pentomino Pathfinding

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1 Introduction

[Dec24a] posed the following problem: Given a rectangular $n \times m$ grid of squares, place a subset of the twelve pentominoes (Figure 1), and endpoints A and B on the grid without overlaps such that $\#^p_{n,m}$ = the length of (the shortest nonempty, orthogonal path between A and B) is maximized.

The above notation is for the maximum length of any path given some placements p; the maximum length given all possible placements is denoted by $\#_{n,m}$, and when $n = m$, the notation is $\#_n$.

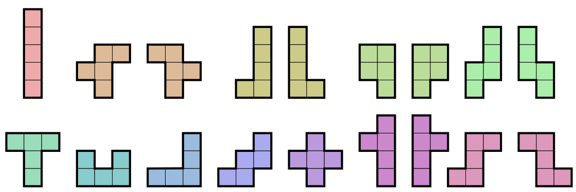


Figure 1: The twelve pentominoes and their reflections [Non08]; from left-to-right they are named I, F, L, P, N, T, U, V, W, X, Y, Z, where F, L, P, N, Y, Z are chiral and have their reflections shown.

2 No pentominoes

For $n = 1$ and 2, $n \times n < 5$, so no pentomino can fit. For $n = 3$, 9 squares minus a pentomino is 4 squares, so the length 5 path is optimal. (Figure 2)

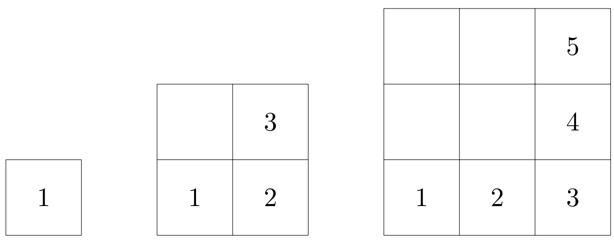
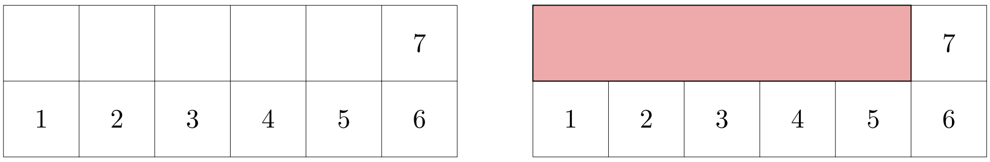


Figure 2: $\#_1 = 1, \#_2 = 3, \#_3 = 5$ [Dec24a]

Similar reasoning holds for $n = 2$, $m \leq 6$: there is a path of length $m + 1 \leq 7$, while $2m - 5 \leq m + 1 \leq 7$, so you cannot do any better than placing nothing. It turns out there is enough room for the I piece, and so there are two solutions ignoring symmetry:



Definition 2.1 (Adjacency)**.** Two squares are *adjacent* iff they are one square diagonally or orthogonally apart. Two squares are *orthogonal* iff they are adjacent but not diagonally so. Two sets of squares S and T are *adjacent* if there is a pair of squares $(s, t) \in S \times T$ (Cartesian product) such that s and t are adjacent. The squares adjacent to a set of squares S is denoted $\operatorname{adj}(S)$.

Definition 2.2 (Subgrid)**.** A subgrid is any connected subgraph of squares of the grid.

Definition 2.3 (Platter)**.** A *platter* is a (set of adjacent squares) not adjacent to another square. (I don't use the term "shape" because two squares diagonally adjacent doesn't really fit into most people's intuitions).

Definition 2.4 (Outside)**.** For a platter P, $\operatorname{outside}(P) =$ All squares reachable from the wall or other platters

Definition 2.5 (Border)**.** A platter's *border* is the squares adjacent to the platter that are outside of it. $\operatorname{border}(P) = \operatorname{adj}(P) \cap \operatorname{outside}(P)$

If a platter cuts the grid into subgrids, we may restrict the outside to squares reachable from just that subgrid s. This is denoted $\operatorname{border}(P)_s$

Lemma 2.6 (Border removal)**.** Say we have a platter p_1 with the following properties:

1. $\operatorname{border}(p_1) = \operatorname{adj}(p_1)$
2. choosing any two squares on its border A and B where there is a path from A to B, the shortest such path and the shortest such path upon removing the platter are the same length.

Then the platter may be removed: $\#^{p}_{n,m} \leq \#^{p-p_1}_{n,m}$

Proof. Can any preexisting path get shorter by entering the platter? No. Any preexisting path starts and ends outside the platter, by property 1. If the path wants to enter the platter, it must reach its border, go inside, then exit from another square on the border. But by property 2, going *along* the border is at least as efficient.

So, any preexisting paths determining $\#^{p}_{n,m}$ will never enter the area where the platter disappeared from; i.e, such paths will not get shorter with the removal of the platter. □

Note. We have the possibility of $\#^{p}_{n,m} < \#^{p-p_1}_{n,m}$ because in addition to the preexisting paths, there are new paths that go across or in the removed platter.

Lemma 2.7 (Rectangle cut)**.** Say we have an $n \times m$ grid, where pentominoes form a $k \times m$ rectangle (r) that is not adjacent to any other pentomino.

Then the rectangle may be removed: $\#^{p}_{n,m} \leq \#^{p-r}_{n,m}$

Proof. The rectangle satisfies 2.6, since $\operatorname{adj}(r)$ is zero, one, or two straight lines of squares by construction. □

Conclusion. And so, $\#_{1,n} = n$

Proof. The only piece that could possibly fit is the I piece, which is removable thanks to Lemma 2.7. □

3 One pentomino

We know that Lemma 2.6 works for rectangles that span from one side to the other, but this also works for the corner.

Theorem 3.1. A platter P is made of the set of squares S and all squares to the bottom-left of any square in S. Then it satisfies 2.6.

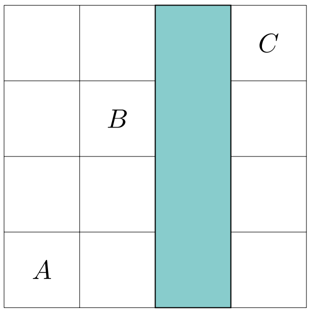


Figure 3: Illustration for Lemma 2.7: The path from A to B cannot be made any shorter by entering the rectangle since it could just go along the border. Meanwhile, removing the rectangle makes new paths A to C and B to C.

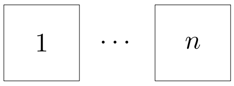


Figure 4: $\#_{1,n} = n$ (where $1 < n$). [Com24]

Proof. Informally, $\mathrm{adj}(P)$ forms a staircase-like shape. So any path along $\mathrm{adj}(P)$ will go in only two cardinal directions, and in fact will have the length of the Manhattan distance between its two endpoints, which cannot be improved upon. □

Note. This construction may be rotated and reflected.

Theorem 3.2 (Cut corner removal)**.** This extends Theorem 3.1 to potentially apply to each "side" (i.e. subgrid border) of a platter that cuts the grid into subgrids.

Proof. For each subgrid s, try applying Theorem 3.1 as if all the other subgrids were part of the platter. Including these other subgrids does not affect $\mathrm{border}(P)_s$, so this is valid for determining whether a path in s can be improved. □

Note. If Theorem 3.1 applies to the entire border, then as a result the whole platter can be removed, since no path within any of the subgrids can be improved.

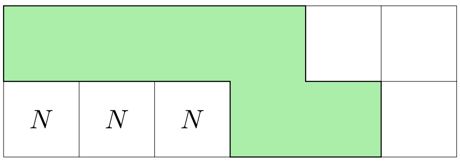


Figure 5: Example for Theorem 3.2: The pentomino N divides the grid into two. For the purposes of the subgrid on the right, N may as well also include the subgrid on the left (marked with three N's). This new platter satisfies Theorem 3.1.

3.1 $2 \times n$ grids

Consider a $2 \times n$ grid. The only pentominoes that fit are I, L, P, N, U, and Y.

By Theorem 3.2 (Cut corner removal), L, P, N, U, and Y can be removed from consideration in $2 \times n$ areas, so only I is left. While placing I on the corner does not help, placing the I anywhere else increases $\#^p_{2,n}$ by 1 — the unique way to show $\#_{2,n} = n + 2$ for $n \geq 7$.

3.2 $3 \times n$ grids

Definition 3.3 (Cave)**.** A subgrid is a *cave* iff it is only connected to the rest of the subgrid its in by one square. That square is called a *separating vertex* [Wik21].

Theorem 3.4 (Ways to orient a pentomino)**.** A square has 4 symmetries, so a platter has at most 8 orientations, ignoring location within a grid. However, some pentominoes have one or more symmetries, which reduce the possible orientations (Table 1).

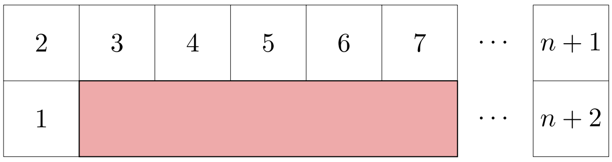


Figure 6: $\#_{2,n} = n + 2$ (where $7 \leq n$). [Com24]

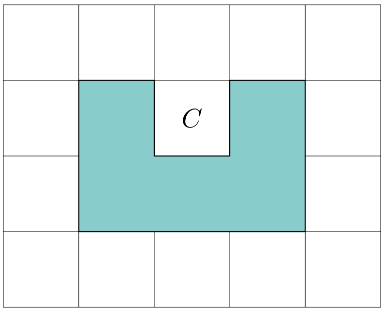


Figure 7: U is the unique pentomino with a cave square.

Table 1: Number of ways to orient each pentomino. [Dec24a, 0:41]

Pentomino	Symmetries	Orientations
X	4	1
I	2	2
T	1	4
U	1	4
V	1	4
W	1	4
Z	1	4
F	0	8
L	0	8
P	0	8
N	0	8
Y	0	8

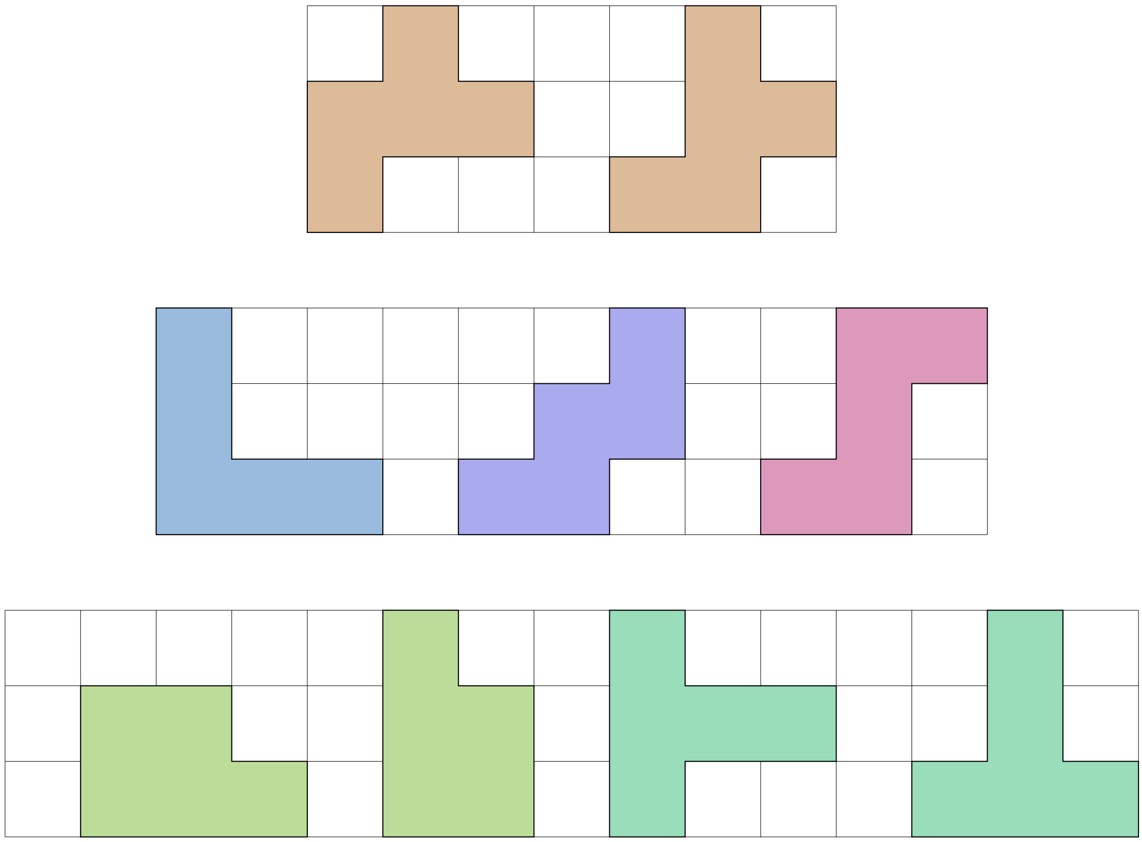


Figure 8: All orientations of F, V, W, Z, P, and T in a $3 \times n$ after reducing by symmetry. The left side of both F's are ignorable, as well as both sides of the right P and T. Additionally, the second F is strictly worse than the first F, since after filling in the ignorable subgrid, the first F practically replaces a wall with an empty square when compared to the second F.

Corollary 3.5. Theorem 3.2 works for V, W, Z, the vertical orientations of P and T, and one side of either orientation of F. Note that we may reflect the entire grid itself along both a vertical and horizontal axis, so we can reduce the number of orientations by a factor of four; the maximum number of orientations a pentomino could have goes from eight to two. (Figure 8)

Property 3.6 (Cycle inefficiency)**.** If there is a cycle of empty squares in a grid, the path will not use all empty squares.

Property 3.7 (Zero pentomino result)**.** With nothing placed, we have $\#^{\emptyset}_{n,m} = n + m - 1$

Property 3.8 (One-pentomino solution)**.** Say we place a pentomino and $n+m-1, nm-10 < \#^{p}_{n,m}$. Then any solution must contain one pentomino.

Lemma 3.9 (Maximal solutions in $3 \times n$ grids)**.** Say we know how many pentominoes are in a solution, and that all squares are used. Then:

- F, X, and U are not part of the solution.
- L, N, and Y must have their 1×4 in the middle row, if they are not adjacent to other pentominoes.

Proof. If a path enters or exits a cave square (enters without loss of generality), it cannot exit since there is only one entrance.

- Pentominoes must be placed so that the grid is *not* split into two. This is impossible for F and X, which span a 1×3 area with two squares padding each side of the 1×3, forcing the 1×3 into the center. For U, say a path ends in U's cave square. If U is placed horizontally (2 rows by 3 columns, Figure 9), it is at least as good to just go around the U (removing any obstacles). But in a vertical placement, removing the U allows any path to go two steps further into the U.

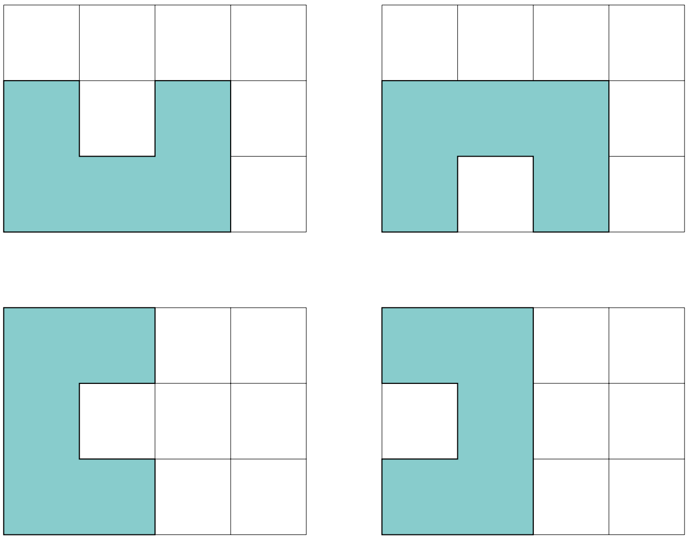


Figure 9: Orientations of U in a 3×4 (footnote: [1])

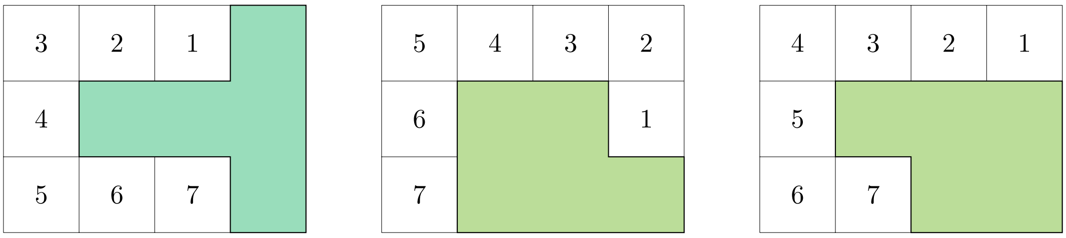


Figure 10: $\#_{3,4} = 7$. [Dec24b, 6:04]

- If a pentomino that spans a 1×4 area (L, N, Y) has their 1×4 on the edge of the grid, there are two disjoint 2×2 cycles which cannot be prevented by the remaining square of the pentomino and no help from other pentominoes. But this is not allowed by 3.6. □

Theorem 3.10. $\#_{3,4} = 7$

Proof. By Figure 10, we have maximal solutions with 1 pentomino, and since $3+4-1, 12-10 < 7$, 3.9 applies. L, N, and Y can be eliminated since a 1×4 in the center row would divide the grid into two subgrids (this is not allowed; remember that every nonempty square must be in the path).

We are left with the pentominoes P and T. If P does not touch the corner, after placing the 2×2, the 5th square must divide the grid in two. There are four ways to touch the corner, and two work. For T, there is only one orientation that does not cut the grid. So the three solutions given by [Dec24b, 6:04] are the only ones. □

A polynomial's contribution is informally how much longer it makes a given path. But 3.2 shows a formal definition is still a ways away. In such examples, we must consider entire platters.

References

[Com24] Community. *Pentomino Pathfinding.* Sept. 3, 2024. URL: `https://docs.google.com/spreadsheets/d/1NrbqWmnBLMtHH253q_v89bMYSuoPE7hDFr8g5VTbGMI/edit`.

[Dec24a] Deckard. *Pentomino Facts.* Aug. 2, 2024. URL: `https://youtu.be/LPDAzHpSyAo?t=700`.

[Dec24b] Deckard. *More Pentomino Pathfinding.* Sept. 2, 2024. URL: `https://youtu.be/39YYZcwCuv0`.

[1] film2860 on discord used gpt-4o to try to remove pixel artifacts. It generated equivalent code, which did not fix the problem, but did illustrate the usage of scopes which was helpful as a beginner. So congrats on contributing!

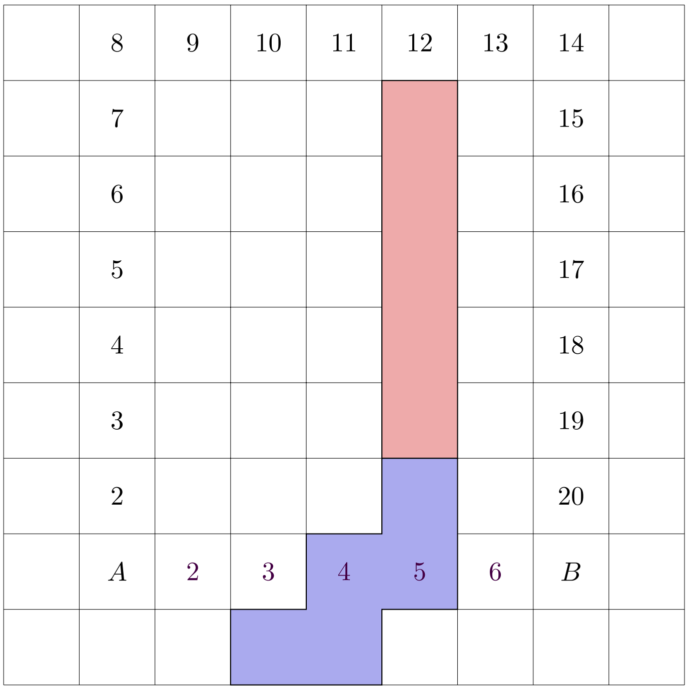


Figure 11: If the I pentomino was already placed, then adding the W changes the path's length from 7 to 21. Nevertheless, W's contribution $C(\mathrm{W}) = 4$. Additionally, W helps by blocking the bottom row. So some squares are spent on setup, and some on lengthening the path.

[Non08] R. A. Nonenmacher. *All 18 Pentominoes*. CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons; latest uploaded on day (13:43 UTC). July 21, 2008. URL: https://commons.wikimedia.org/wiki/File:All_18_Pentominoes.svg.

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