Pentomino Pathfinding

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September 20, 2024

Contents

1 Introduction

This paper will try to solve the Pentomino Pathfinding problem for various rectangular grids. See [Dec24a] and [Dec24b] for further introduction.

[Dec24a] posed the following problem: Given a rectangular $n \times m$ grid of squares or *cells*, place a subset of the twelve pentominoes (Figure 1), and endpoints A and B on the grid without overlaps such that $\#_{n,m} =$ the length of (the shortest path of nonempty, orthogonal cells from A to B) is maximized.

The maximum length given some placements p is denoted by $\#^p_{n,m}$; independently, when $n = m$, the row and column indices are collapsed: $\#_n$.

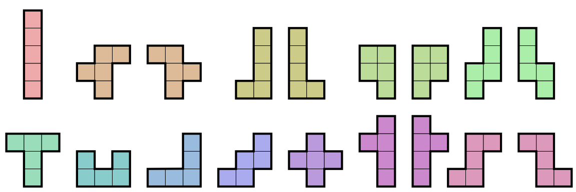

Figure 1: The twelve pentominoes and their reflections [Non08]; from left-to-right they are named I, F, L, P, N, T, U, V, W, X, Y, Z, where F, L, P, N, Y, Z are chiral and have their reflections shown.

2 No pentominoes

For $n = 1$ and 2, $n \times n < 5$, so no pentomino can fit. For $n = 3$, 9 cells minus a pentomino is 4 squares, so the length 5 path is optimal. (Figure 2)

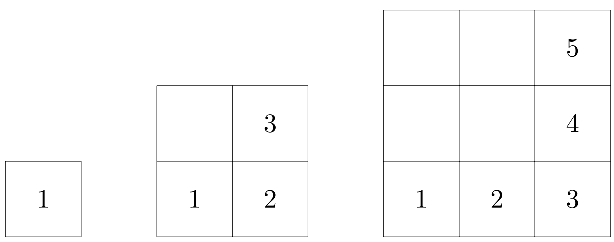

Figure 2: $\#_1 = 1, \#_2 = 3, \#_3 = 5$ [Dec24a]

Similar reasoning holds for $n = 2$, $m \leq 6$: there is a path of length $m + 1 \leq 7$, while $2m - 5 \leq m + 1 \leq 7$, so you cannot do any better than placing nothing. It turns out there is enough room for the I piece, and so for $m = 6$ there are two solutions ignoring symmetry:

					7
1	2	3	4	5	6

					7
1	2	3	4	5	6

And this reasoning also holds for $\#_{1,n} = n$: the path uses all squares, and as each pentomino costs 5 squares the best solution is 0 pentominoes.

1	$\cdots$	n

Figure 3: $\#_{1,n} = n$ (footnote: [1]) [Com24]

3 One pentomino

We'll start with proving some useful things about pentominoes in small grids.

Definition 3.1 (Adjacency)**.** Two squares are *adjacent* iff they are one square diagonally or orthogonally apart. Two squares are *orthogonal* iff they are adjacent but not diagonally so. Two sets of squares S and T are *adjacent* if there is a pair of squares $(s, t) \in S \times T$ (Cartesian product) such that s and t are adjacent. The squares adjacent to a set of squares S is denoted $\text{adj}(S)$.

Definition 3.2 (Subgrid)**.** A subgrid is any connected subgraph of empty squares of the grid. A square is empty iff it is does not have a pentomino.

Definition 3.3 (Platter)**.** A *platter* is a (set of adjacent squares) not adjacent to another square.

I don't use the term "shape" because two squares diagonally adjacent doesn't usually match people's intuitions.

Definition 3.4 (Outside)**.** For a platter P, $\text{outside}(P)$ = All squares reachable from the wall or other platters

Definition 3.5 (Border)**.** A platter's *border* is the squares adjacent to the platter that are outside of it. $\text{border}(P) = \text{adj}(P) \cap \text{outside}(P)$

If a platter cuts the grid into subgrids, we may restrict the outside to squares reachable from just that subgrid s. This is denoted $\text{border}(P)_s$

Lemma 3.6 (Border removal)**.** Say we have a platter p_1 with the following properties:

1. $\text{border}(p_1) = \text{adj}(p_1)$
2. choosing any two squares on its border A and B where there is a path from A to B, the shortest such path and the shortest such path upon removing the platter are the same length.

Then the platter may be removed: $\#^{p}_{n,m} \leq \#^{p-p_1}_{n,m}$

Proof. Can any preexisting path get shorter by entering the platter? No. Any preexisting path starts and ends outside the platter, by property 1. If the path wants to enter the platter, it must reach its border, go inside, then exit from another square on the border. But by property 2, going *along* the border is at least as efficient.

So, any preexisting paths determining $\#^{p}_{n,m}$ will never enter the area where the platter disappeared from; i.e, such paths will not get shorter with the removal of the platter. □

Note. We have the possibility of $\#^{p}_{n,m} < \#^{p-p_1}_{n,m}$ because in addition to the preexisting paths, there are new paths that go across or in the removed platter.

Lemma 3.7 (Rectangle cut)**.** Say we have an $n \times m$ grid, where pentominoes form a $k \times m$ rectangle (r) that is not adjacent to any other pentomino.

Then the rectangle may be removed: $\#^{p}_{n,m} \leq \#^{p-r}_{n,m}$

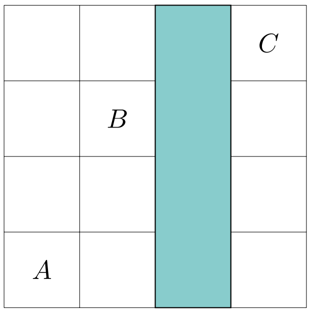

Figure 4: Illustration for Lemma 3.7: The path from A to B cannot be made any shorter by entering the rectangle since it could just go along the border. Meanwhile, removing the rectangle makes new paths A to C and B to C.

Proof. The rectangle satisfies Lemma 3.6, since $\mathrm{adj}(r)$ is zero, one, or two straight lines of squares by construction. □

We know that Lemma 3.6 works for rectangles that span from one side to the other, but this also works for the corner:

Theorem 3.8. A platter P is made of the set of squares S and all squares to the bottom-left of any square in S. Then it satisfies Lemma 3.6.

Proof. Informally, $\mathrm{adj}(P)$ forms a staircase-like shape. So any path along $\mathrm{adj}(P)$ will go in only two cardinal directions, and in fact will have the length of the Manhattan distance between its two endpoints, which cannot be improved upon. □

Note. This construction may be rotated and reflected.

Theorem 3.9 (Cut corner removal)**.** This extends Theorem 3.8 to potentially apply to each "side" (i.e. subgrid border) of a platter that cuts the grid into subgrids.

Proof. For each subgrid s, try applying Theorem 3.8 as if all the other subgrids were part of the platter. Including these other subgrids does not affect $\mathrm{border}(P)_s$, so this is valid for determining whether a path in s can be improved. □

Note. If Theorem 3.8 applies to the entire border, then as a result the whole platter can be removed, since no path within any of the subgrids can be improved.

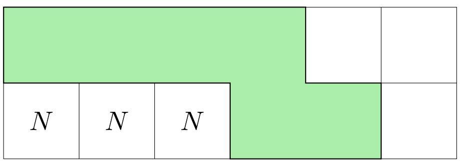

Figure 5: Example for Theorem 3.9: The pentomino N divides the grid into two. For the purposes of the subgrid on the right, N may as well also include the subgrid on the left (marked with three N's). This new platter satisfies Theorem 3.8.

3.1 $2 \times n$ grids

Consider a $2 \times n$ grid. The only pentominoes that fit are I, L, P, N, U, and Y.

By Theorem 3.9 (Cut corner removal), L, P, N, U, and Y can be removed from consideration in $2 \times n$ areas, so only I is left. While placing I on the corner does not help, placing I anywhere else increases $\#^p_{2,n}$ by 1 — the unique way to show $\#_{2,n} = n + 2$ for $n \geq 7$.

3.2 $3 \times n$ grids

Definition 3.10 (Cave)**.** A subgrid is a *cave* iff it is only connected to the rest of the subgrid its in by one square. That square is called a *separating vertex*. [Wik21]

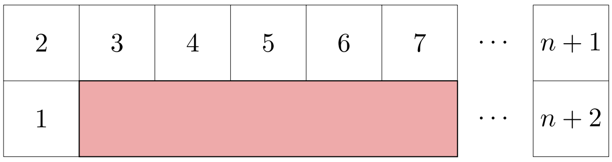

Figure 6: $\#_{2,n} = n + 2$ (where $7 \leq n$). [Com24]

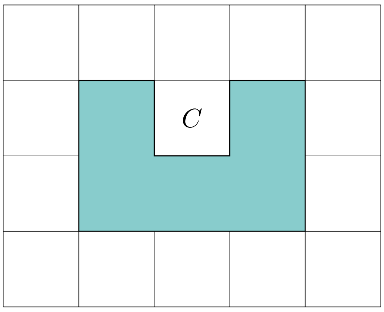

Figure 7: U is the unique pentomino with a cave square.

Property 3.11 (Cycle inefficiency)**.** If there is a cycle of empty squares in a grid, the path will not use all empty squares.

Property 3.12 (Zero pentomino result)**.** With nothing placed, we have $\#^{\emptyset}_{n,m} = n + m - 1$

Property 3.13 (One-pentomino solution)**.** Say we place a pentomino and $n + m - 1, nm - 10 < \#^{p}_{n,m}$. Then any solution must contain one pentomino.

Definition 3.14 (Brimming solution)**.** A *solution* (set of placements) is *brimming* when every square in the grid is used, i.e, every square corresponds to either a pentomino or the path. In graph theory terms, the path is a Hamiltonian path among the empty squares.

Property 3.15 (A brimming solution gives a strict upper bound for the number of pentominoes)**.** If a solution is brimming, any further placements can only decrease the path length.

Theorem 3.16. When a solution is brimming with only one pentomino, that pentomino is not U.

Proof. In a brimming solution, there are no empty cells adjacent to three other cells. This is because the graph of a brimming solution must be isomorphic to a straight line, as otherwise, either one end of the line became adjacent to some earlier square in the graph, forming a loop, or the graph has something that branches off like a fork in the road, and so any path will not include all squares.

U has a cave square. If the cave square is adjacent to some empty square, then a rectangular grid must include two more empty squares adjacent to it, violating the three other cells property. Therefore, the empty square is isolated. If it is the only empty square, the grid is a 2×3 so this holds. If it isn't, there are two subgrids of empty squares, so the placement is not brimming. □

Lemma 3.17. When a solution is brimming with only one pentomino, the pentomino must include the center two squares[2] of any (3×4)-subgrid.

Proof. Without loss of generality, say the left center square is not part of the pentomino.

If we also do not include the right center square (the C in Figure 8), then within the 3×4 there is no way to include the four 2×2 regions containing its corner squares. However, even starting from C the only ways to reach the left hand side 2×2's is to use the U pentomino, which is not valid by Theorem 3.16. □

Definition 3.18 (Edge)**.** The *edge* of the grid are the squares adjacent to the wall.

Corollary 3.19. Lemma 3.17 is equivalent to saying all non-edge squares are part of the pentomino. Additionally, the squares of the pentomino on the edge must be orthogonally adjacent, since the edges of the grid form a loop that cannot be cut into two separate subgrids.

[1]The figure assumes $n > 1$ but that's not necessary of course.
[2]the two squares diagonally adjacent to a corner

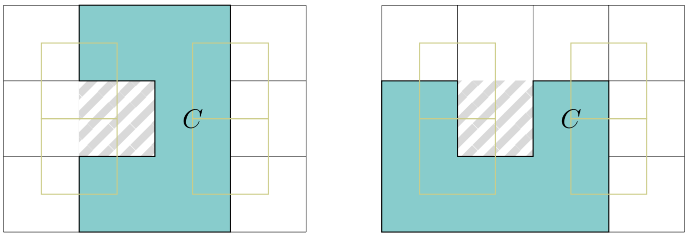

Figure 8: Illustration for Lemma 3.17: the only ways to occlude all 2×2 cycles in a 3×4, given that the left center cell is kept empty.

Theorem 3.20. $\#_{3,4} = 7, \#_{3,5} = 10, \#_{3,6} = 13$

Proof. By Figure 9, Figure 10, and Figure 11 respectively, we have brimming solutions with 1 pentomino, which is optimal by Property 3.13 as $3 + 4 - 1, 12 - 10 < 7$. Therefore Corollary 3.19 applies, and it turns out all possibilities with a pentomino containing all non-edge squares and some orthogonally-adjacent edge squares work out. $\square$

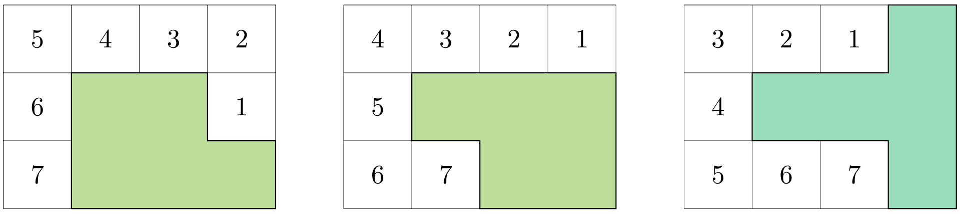

Figure 9: $\#_{3,4} = 7$. [Dec24b, 6:04]

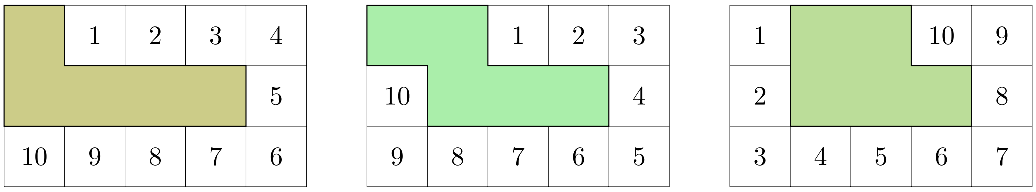

Figure 10: $\#_{3,5} = 10$. [Dec24b, 6:10]

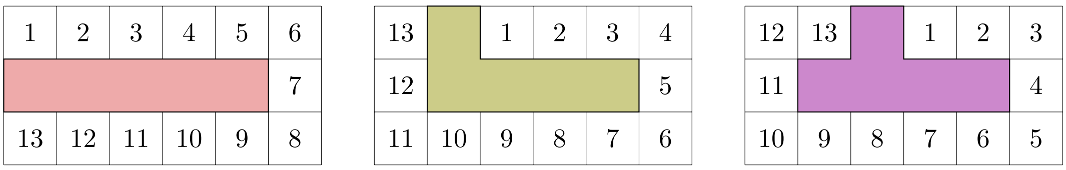

Figure 11: $\#_{3,6} = 13$. [Dec24b, 6:16]

Table 1: Number of ways to orient each pentomino. [Dec24a, 0:41]

Pentomino	Symmetries	Orientations
X	4	1
I	2	2
T	1	4
U	1	4
V	1	4
W	1	4
Z	1	4
F	0	8
L	0	8
P	0	8
N	0	8
Y	0	8

Theorem 3.21 (Ways to orient a pentomino)**.** A square has 4 symmetries, so a platter has at most 8 orientations, ignoring location within a grid. However, some pentominoes have one or more symmetries, which reduce the possible orientations (Table 1).

Corollary 3.22. The caption for Figure 12 holds. This is partly thanks to the entire grid itself being symmetrical along both a vertical and horizontal axis: since Theorem 3.9 works after reflection, this reduces the number of orientations a pentomino could have by a factor of four; the maximum orientations goes from eight to two.

Note. U satisfies a general version of Theorem 3.9 where both corners are covered.

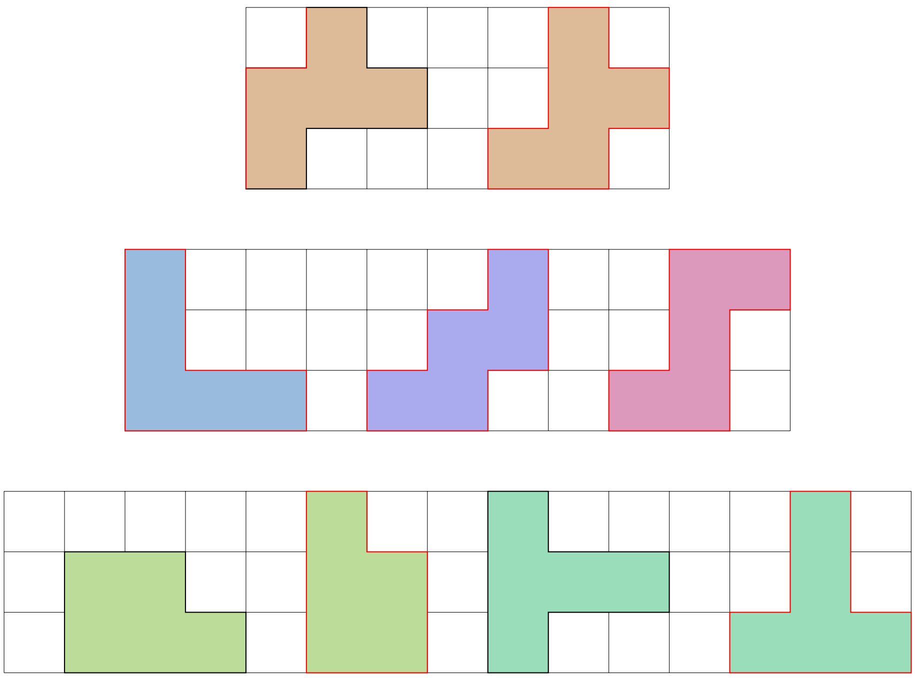

Figure 12: All orientations of F, V, W, Z, P, and T in a $3 \times n$ after reducing by symmetry. The following satisfy Theorem 3.9: V, W, Z, the right P and T, and the left side of both F's. Additionally, the second F is strictly worse than the first F, since after filling in the ignorable subgrid, the first F practically replaces a wall with an empty square when compared to the second F.

A polynomial's contribution is informally how much longer it makes a given path. But Figure 13 shows a formal definition is still a ways away. In such examples, we must consider entire platters.

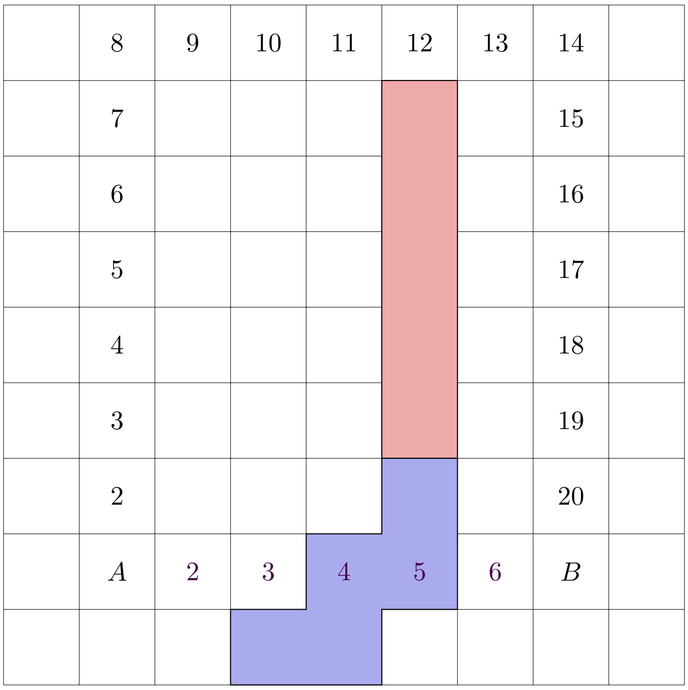

Figure 13: If the I pentomino was already placed, then adding the W changes the path's length from 7 to 21. Nevertheless, W 's contribution $C($ W $) = 4$. Additionally, W helps by blocking the bottom row. So some squares are spent on setup, and some on lengthening the path.

References

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