Pentomino Pathfinding

Steven Nguyen (icecream17)

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1 Introduction

[Dec24a] posed the following problem: Given a rectangular $n \times m$ grid of squares, place a subset of the twelve pentominoes (see Figure 1), and endpoints A and B on the grid without overlaps such that $\#^p_{n,m}$ = the length of (the shortest nonempty, orthogonal path between A and B) is maximized.

The above notation is for the maximum length of any path given some placements p; the maximum length given all possible placements is denoted by $\#_{n,m}$, and when $n = m$, the notation is $\#_n$.

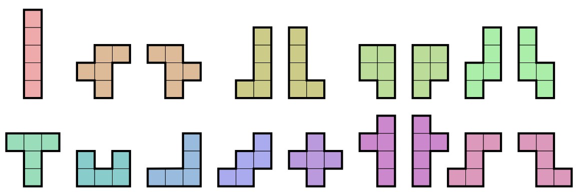
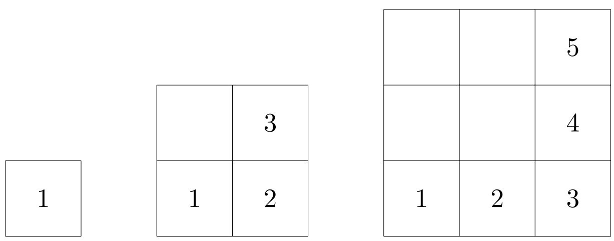


Figure 1: The twelve pentominoes and their reflections [Non08]; from left-to-right they are named I, F, L, P, N, T, U, V, W, X, Y, Z, where F, L, P, N, Y, Z are chiral and have their reflections shown.

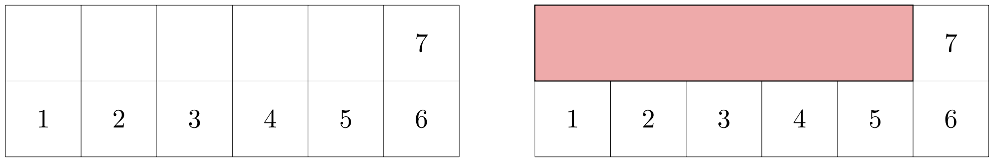
2 Small grids

2.1 No pentominoes

For $n = 1$ and 2, $n \times n < 5$, so no pentomino can fit: $\#_1 = 1, \#_2 = 3$. For $n = 3$, 9 squares minus a pentomino is 4 squares, so the length 5 path is optimal: $\#_3 = 5$.



Similar reasoning holds for $n = 2$, $m \leq 6$: there is a path of length $m + 1 \leq 7$, while $2m - 5 \leq m + 1 \leq 7$, so you cannot do any better than placing nothing. It turns out there is enough room for the I piece, and so there are two solutions ignoring symmetry:



Definition 2.1 (Adjacency)**.** Two squares are *adjacent* iff they are one square diagonally or orthogonally apart. Two squares are *orthogonal* iff they are adjacent but not diagonally so. Two set of squares S and T are *adjacent* if there is a pair of squares $(s, t) \in S \times T$ (footnote [1]) such that s and t are adjacent.

Lemma 2.2 (Rectangle blockage)**.** Say we have an $n \times m$ grid, where pentominoes form a $k \times m$ rectangle (r) that is not adjacent to any other pentomino.

Then the rectangle may be removed: $\#^p_{n,m} \leq \#^{p-r}_{n,m}$

Proof. By construction, on the side of the rectangle facing any path pa there is an empty column adjacent to the rectangle, which for now we'll call its border. The path pa starts and ends on the same subgrid as this border. So if it crosses the border, it must cross back, essentially going to two squares on the border. However, it is always shorter to go *along* the border, cause it's a straight line. So, any preexisting paths determining $\#^p_{n,m}$ will never enter the area where the rectangle disappeared from; i.e, such paths will not get shorter with the removal of the rectangle. □

Note. We have the possibility of $\#^p_{n,m} < \#^{p-r}_{n,m}$ because in addition to the preexisting paths, there are new paths that go across the removed rectangle.

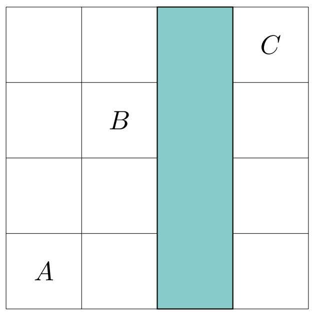


Figure 2: Illustration for Lemma 2.2: The path from A to B cannot be made any shorter by entering the rectangle since it could just go along the border. Meanwhile, removing the rectangle makes new paths A to C and B to C.

Conclusion. And finally, $\#_{1,n} = n$

Proof. The only piece that could possibly fit is the I piece, which is removable thanks to Lemma 2.2. □

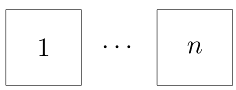


Figure 3: $\#_{1,n} = n$ (where $1 < n$).

[1]Cartesian product

2.2 One pentomino

2.2.1 $2 \times n$ grids

Corollary 2.2.1 (Convex border)**.** Lemma 2.2 extends to any shape, so long as the statement "it is always shorter to go *along* the border" holds. This could be a similar construction to the rectangle blockage, but it extends more generally to include, for example, some placements of pentominoes at the corner. The word "convex" is informal.

Consider a $2 \times n$ grid. The only pentominoes that fit are I, L, N, and Y. However, by Corollary 2.2.1, L, N, and Y can be removed. So, can the pentomino I be used to increase $\#^p_{2,n}$? Yes. While placing I on the corner does not help, placing the I anywhere else increases $\#^p_{2,n}$ by 1 – the unique way to show $\#_{2,n} = n + 2$ for $n \geq 7$.

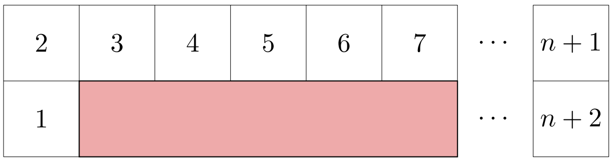


Figure 4: $\#_{2,n} = n + 2$ (where $7 \leq n$). [Com24]

2.2.2 $3 \times n$ grids

Property 2.3 (Cycle inefficiency)**.** If there is a cycle of empty squares in a grid, the path will not use all empty squares.

For $3 \times n$ grids, all pentominoes fit, but V, W, and Z can be removed by Corollary **??**. In the case of a 3×4 grid, we have solutions with 1 pentomino and all empty squares filled ($\#_{3,4} = 7$, Figure 5). Placing zero pentominoes gets a length of 6, while with two pentominoes there are only 2 leftover squares, so the only solutions have 1 pentomino and all empty squares in the path.

Pentominoes must be placed so that the grid is *not* split into two. This is impossible for F and X, which span a 1×3 area with two squares padding each side of the 1×3. Pentominoes that span a 1×4 area (L, N, Y) cannot have their 1×4 in the middle, but if it is on the edge of the grid, there are two disjoint 2×2 cycles which cannot be prevented by the remaining square of the pentomino. So by 2.3, L, N, and Y are eliminated. U has 4 orientations (translations can be produced by rotation and reflection, there is 1 relevant non-reflection for U, and 1 relevant non-rotational symmetry for the 3×4), which all do not work for various reasons.

We are left with the pentominoes P and T. If P does not touch the corner, after placing the 2×2, the 5th square must divide the grid in two. There are four ways to touch the corner, and two work. For T, there is only one orientation that does not split the grid in two. So the three solutions given by [Dec24b, 6:04] are the only ones.

References

[Com24] Community. *Pentomino Pathfinding*. Sept. 3, 2024. URL: https://docs.google.com/spreadsheets/d/1NrbqWmnBLMtHH253q_v89bMYSuoPE7hDFr8g5VTbGMI/edit.

[2]film2860 on discord used gpt-4o to try to remove pixel artifacts. It generated equivalent code, which did not fix the problem, but did illustrate the usage of scopes which was helpful as a beginner. So congrats on contributing!

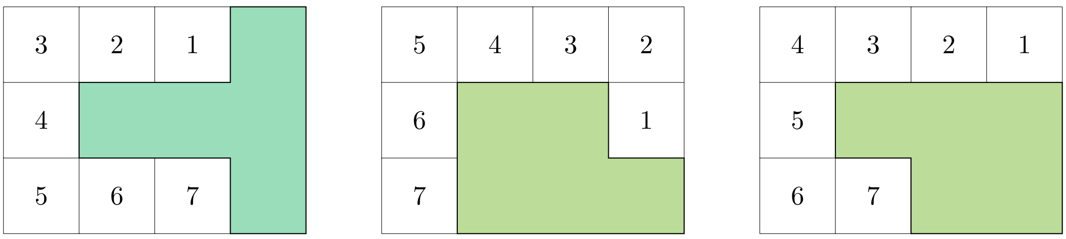


Figure 5: $\#_{3,4} = 7$. [Dec24b, 6:04]

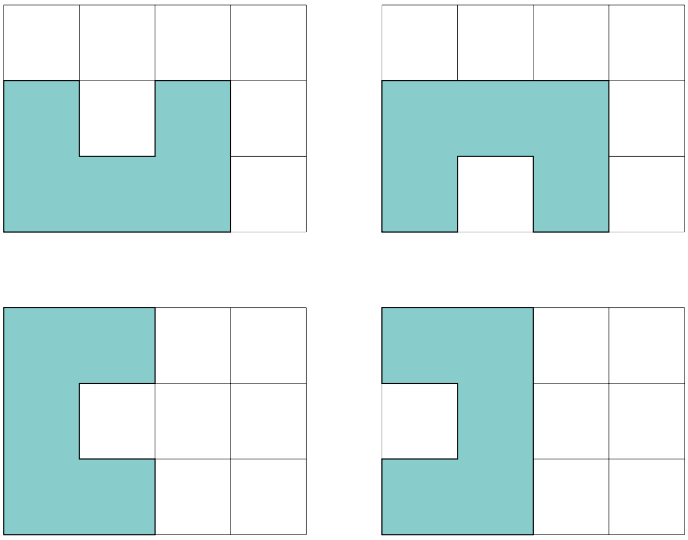


Figure 6: Orientations of U in a 3×4 (footnote: [2])

[Dec24a] Deckard. *Pentomino Facts*. Aug. 2, 2024. URL: https://youtu.be/LPDAzHpSyAo?t=700.

[Dec24b] Deckard. *More Pentomino Pathfinding*. Sept. 2, 2024. URL: https://youtu.be/39YYZcwCuv0.

[Non08] R. A. Nonenmacher. *All 18 Pentominoes*. CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons; latest uploaded on day (13:43 UTC). July 21, 2008. URL: https://commons.wikimedia.org/wiki/File:All_18_Pentominoes.svg.