Pentomino Pathfinding

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1 Introduction

[Dec24a] posed the following problem: Given a rectangular $n \times m$ grid of squares, place a subset of the twelve pentominoes (see Figure 1), and endpoints A and B on the grid without overlaps such that $\#^p_{n,m}$ = the length of (the shortest nonempty, orthogonal path between A and B) is maximized.

The above notation is for the maximum length of any path given some placements p; the maximum length given all possible placements is denoted by $\#_{n,m}$, and when $n = m$, the notation is $\#_n$.

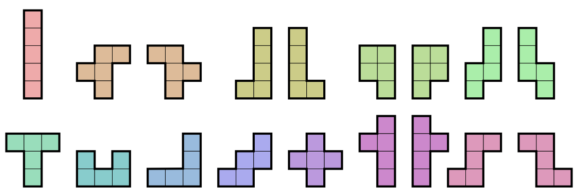


Figure 1: The twelve pentominoes and their reflections [Non08]; from left-to-right they are named I, F, L, P, N, T, U, V, W, X, Y, Z, where F, L, P, N, Y, Z are chiral and have their reflections shown.

2 Small grids

2.1 No pentominoes

For $n = 1$ and 2, $n \times n < 5$, so no pentomino can fit. For $n = 3$, 9 squares minus a pentomino is 4 squares, so the length 5 path is optimal. (Figure 2)

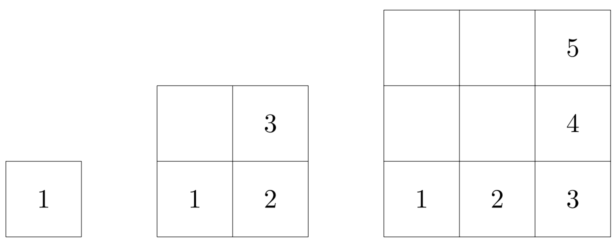
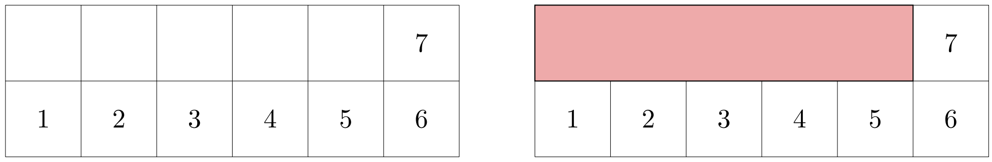


Figure 2: $\#_1 = 1, \#_2 = 3, \#_3 = 5$ [Dec24a]

Similar reasoning holds for $n = 2$, $m \leq 6$: there is a path of length $m + 1 \leq 7$, while $2m - 5 \leq m + 1 \leq 7$, so you cannot do any better than placing nothing. It turns out there is enough room for the I piece, and so there are two solutions ignoring symmetry:



Definition 2.1 (Adjacency)**.** Two squares are *adjacent* iff they are one square diagonally or orthogonally apart. Two squares are *orthogonal* iff they are adjacent but not diagonally so. Two sets of squares S and T are *adjacent* if there is a pair of squares $(s, t) \in S \times T$ (Cartesian product) such that s and t are adjacent. The squares adjacent to a set of squares S is denoted $\mathrm{adj}(S)$.

Definition 2.2 (Platter)**.** A *platter* is a (set of adjacent squares) not adjacent to another square. (I don't use the term "shape" because two squares diagonally adjacent doesn't really fit into most people's intuitions).

Definition 2.3 (Outside)**.** For a platter P, $\mathrm{outside}(P) =$ All squares reachable from the wall or other platters

Definition 2.4 (Border)**.** A platter's *border* is the squares adjacent to the platter that are outside of it. $\mathrm{border}(P) = \mathrm{adj}(P) \cap \mathrm{outside}(P)$

If a platter cuts the grid into subgrids, we may restrict the outside to squares reachable from just that subgrid s. This is denoted $\mathrm{border}(P)_s$

Lemma 2.5 (Border removal)**.** Say we have a platter p_1 with the following properties:

1. $\mathrm{border}(p_1) = \mathrm{adj}(p_1)$
2. choosing any two squares on its border A and B where there is a path from A to B, the shortest such path and the shortest such path upon removing the platter are the same length.

Then the platter may be removed: $\#^{p}_{n,m} \leq \#^{p-p_1}_{n,m}$

Proof. Can any preexisting path get shorter by entering the platter? No. Any preexisting path starts and ends outside the platter, by property 1. If the path wants to enter the platter, it must reach its border, go inside, then exit from another square on the border. But by property 2, going *along* the border is at least as efficient.

So, any preexisting paths determining $\#^{p}_{n,m}$ will never enter the area where the platter disappeared from; i.e, such paths will not get shorter with the removal of the platter. □

Note. We have the possibility of $\#^{p}_{n,m} < \#^{p-p_1}_{n,m}$ because in addition to the preexisting paths, there are new paths that go across or in the removed platter.

Lemma 2.6 (Rectangle cut)**.** Say we have an $n \times m$ grid, where pentominoes form a $k \times m$ rectangle (r) that is not adjacent to any other pentomino.

Then the rectangle may be removed: $\#^{p}_{n,m} \leq \#^{p-r}_{n,m}$

Proof. The rectangle satisfies 2.5, since $adj(r)$ is zero, one, or two straight lines of squares by construction. □

Conclusion. And so, $\#_{1,n} = n$

Proof. The only piece that could possibly fit is the I piece, which is removable thanks to Lemma 2.6. □

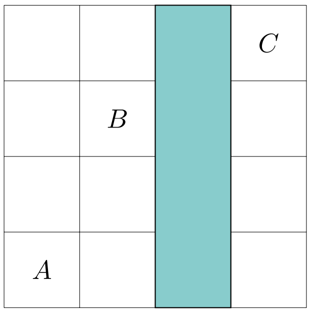


Figure 3: Illustration for Lemma 2.6: The path from A to B cannot be made any shorter by entering the rectangle since it could just go along the border. Meanwhile, removing the rectangle makes new paths A to C and B to C.

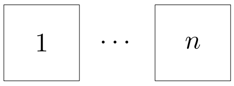


Figure 4: $\#_{1,n} = n$ (where $1 < n$). [Com24]

2.2 One pentomino

We know that Lemma 2.5 works for rectangles that span from one side to the other, but this also works for the corner.

Theorem 2.7. A platter P is made of the set of squares S and all squares to the bottom-left of any square in S. Then it satisfies 2.5.

Proof. Informally, $adj(P)$ forms a staircase-like shape. So any path along $adj(P)$ will go in only two cardinal directions, and in fact will have the length of the Manhattan distance between its two endpoints, which cannot be improved upon. □

Note. This construction may be rotated and reflected.

2.2.1 $2 \times n$ grids

Consider a $2 \times n$ grid. The only pentominoes that fit are I, L, P, N, U, and Y.

Corollary 2.8 (2-wide platters in $2 \times n$ areas are eliminable)**.** L, P, N, U, and Y can be removed from consideration in $2 \times n$ areas.

Proof. These pentominoes have one or more 2×1 areas, which will be converted into walls. In each subgrid formed, the remainder of the pentomino's squares satisfy Theorem 2.7.[1] □

Only the pentomino I is left. While placing I on the corner does not help, placing the I anywhere else increases $\#^{p}_{2,n}$ by 1 – the unique way to show $\#_{2,n} = n + 2$ for $n \geq 7$.

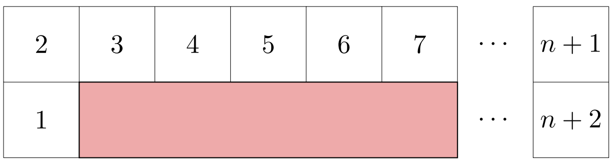


Figure 5: $\#_{2,n} = n + 2$ (where $7 \leq n$). [Com24]

2.2.2 $3 \times n$ grids

Property 2.9 (Cycle inefficiency)**.** If there is a cycle of empty squares in a grid, the path will not use all empty squares.

[1] Proving the statement for platters in general is left as an exercise to the reader. Also, since we are subtracting a set of squares from the pentomino, one horrible way to describe this is a "difference of squares".

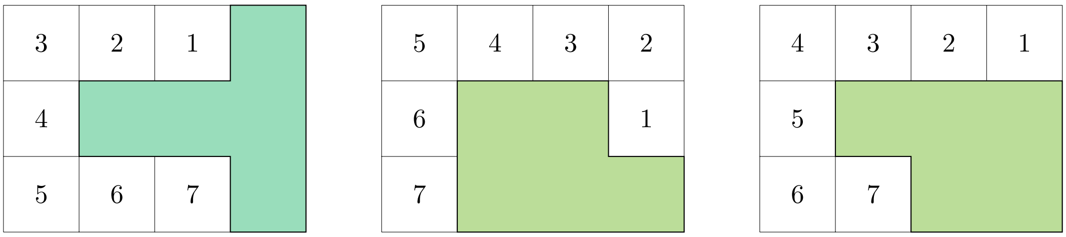


Figure 6: $\#_{3,4} = 7$. [Dec24b, 6:04]

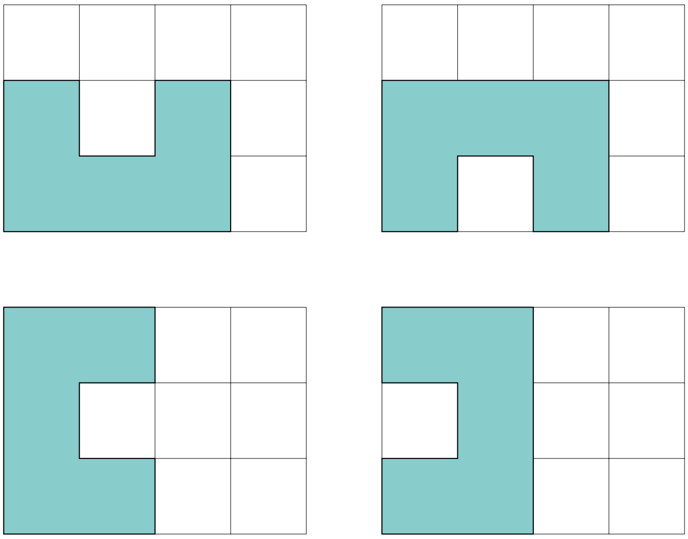


Figure 7: Orientations of U in a 3×4 (footnote: [2])

For $3 \times n$ grids, all pentominoes fit, but V, W, and Z can be removed in a generalization of Corollary 2.8. In the case of a 3×4 grid, we have solutions with 1 pentomino and all empty squares filled ($\#_{3,4} = 7$, Figure 6). Placing zero pentominoes gets a length of 6, while with two pentominoes there are only 2 leftover squares, so the only solutions have 1 pentomino and all empty squares in the path.

Pentominoes must be placed so that the grid is *not* split into two. This is impossible for F and X, which span a 1×3 area with two squares padding each side of the 1×3, forcing the 1×3 into the center. Pentominoes that span a 1×4 area (L, N, Y) cannot have their 1×4 in the middle, but if it is on the edge of the grid, there are two disjoint 2×2 cycles which cannot be prevented by the remaining square of the pentomino. So by 2.9, L, N, and Y are eliminated. U has 4 orientations (translations can be produced by rotation and reflection, there is 1 relevant non-reflection for U, and 1 relevant non-rotational symmetry for the 3×4), which all do not work for various reasons.

We are left with the pentominoes P and T. If P does not touch the corner, after placing the 2×2, the 5th square must divide the grid in two. There are four ways to touch the corner, and two work. For T, there is only one orientation that does not split the grid in two. So the three solutions given by [Dec24b, 6:04] are the only ones.

A polynomial's contribution is informally how much longer it makes a given path. But 2.2.2 shows a formal definition is still a ways away. In such examples, we must consider entire platters.

References

[Com24] Community. *Pentomino Pathfinding*. Sept. 3, 2024. URL: https://docs.google.com/spreadsheets/d/1NrbqWmnBLMtHH253q_v89bMYSuoPE7hDFr8g5VTbGMI/edit.

[Dec24a] Deckard. *Pentomino Facts*. Aug. 2, 2024. URL: https://youtu.be/LPDAzHpSyAo?t=700.

[Dec24b] Deckard. *More Pentomino Pathfinding*. Sept. 2, 2024. URL: https://youtu.be/39YYZcwCuv0.

[2]film2860 on discord used gpt-4o to try to remove pixel artifacts. It generated equivalent code, which did not fix the problem, but did illustrate the usage of scopes which was helpful as a beginner. So congrats on contributing!

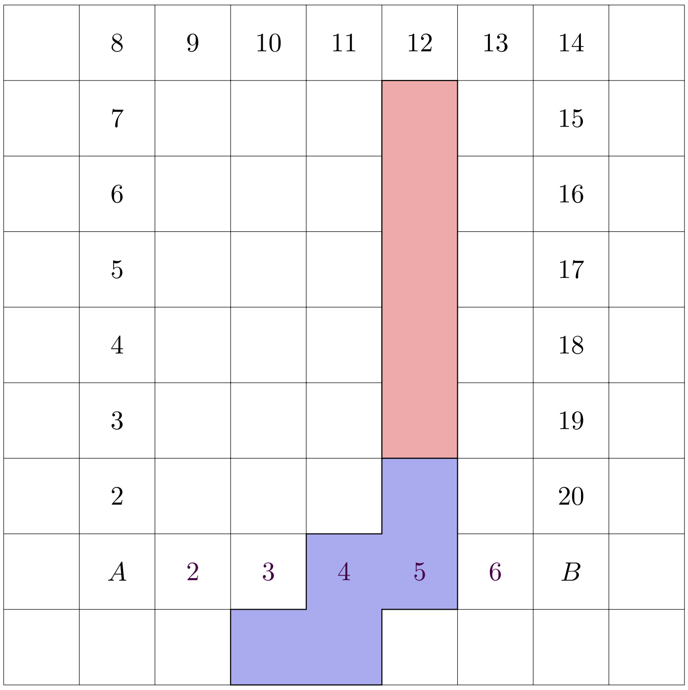


Figure 8: If the I pentomino was already placed, then adding the W changes the path's length from 7 to 21. Nevertheless, W's contribution $C(\mathrm{W}) = 4$. Additionally, W helps by blocking the bottom row. So some squares are spent on setup, and some on lengthening the path.

[Non08] R. A. Nonenmacher. *All 18 Pentominoes*. CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons; latest uploaded on day (13:43 UTC). July 21, 2008. URL: https://commons.wikimedia.org/wiki/File:All_18_Pentominoes.svg.