Mortgage Analysis

Nicole Dong | Mohammed Alshehri

Introduction

The interest of a loan is called the annual rate. The frequency with which the rate is applied to the current balance is how often the loan compounds. If a loan is compounded *n* times per year, the value of the loan after *t* years is:

$$y(t) = y_0(1 + \frac{r}{n})^n$$

where y_0 is the original value of the loan, and *r* is the interest rate.

The more frequently that a loan compounds, the higher the value at the end of the year. The limit as *n* goes to infinity models a continuously compounded loan, and has the equation:

$$y(t) = y_0 e^{rt}$$

This can also be expressed as a differential equation, with *p* equal to the monthly payment:

$$y' = ry - 12p$$

$$y(0) = y_0$$

Some mortgages use a fixed annual rate, which means that the value of *r* in the model is constant. Other mortgages may use an adjustable mortgage rate, which results in *r* being a function of time *r(t)* instead of a constant, and the model becomes:

$$y' = r(t)y - 12p$$

$$y(0) = y_0$$

Analysis of Fixed Rate Mortgages

As stated previously, the more frequently that a loan compounds, the higher the value at the end of the year. A loan is continuously compounded when n, the number of times a loan compounds per year, goes to infinity. Since the value of the loan increases the more it is compounded, a continuously compounded loan should have the highest value at the end of the year.

The total cost y of the loan after 5 years for loans compounded n times a year, with a rate of 3% without any payments, is in the following table:

n	1	2	4	12	continuously
y	$862,500	$866,718.75	$868,987.81	$870,565.89	$871,375.68

As expected, when the loan is compounded more times per year, its value at the end is higher. The continuously compounded loan has the highest final value.

Shown below is a graph for the value of the loan at the end of t years, with interest compounded 4 times a year, with an interest rate of 3%. Below it is another graph of the loan, but with interest compounded 12 times a year.

It's quite apparent that the loan increases more when it's compounded 12 times per year, as opposed to 4 times. The last graph shows the value of the loan compounded continuously, which is greater than both graphs before it.

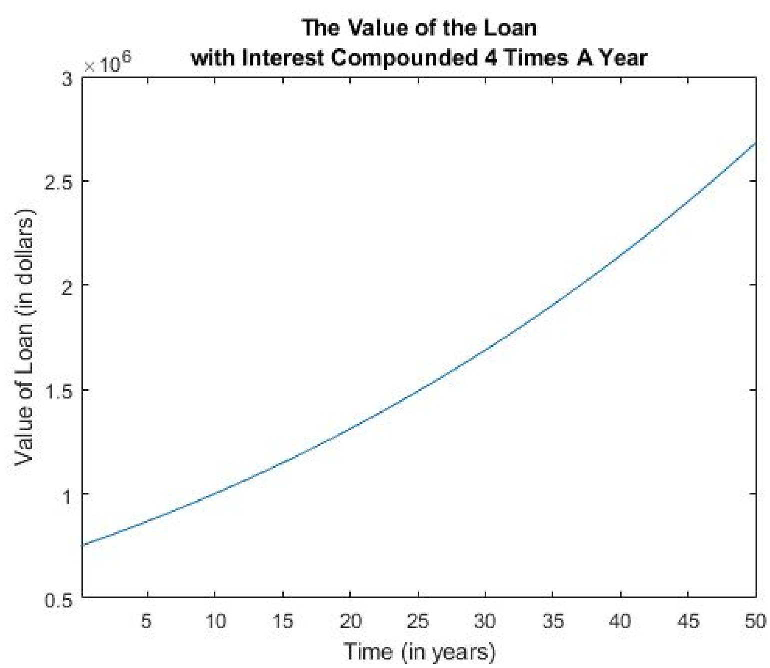

Figure 1: The value of the loan with interest rate 3%, compounded 4 times a year

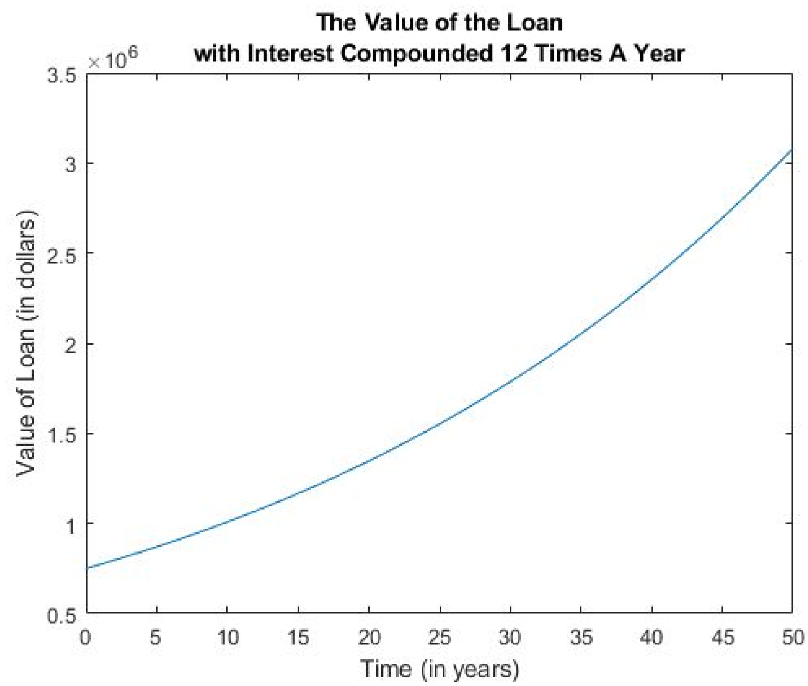

Figure 2: The value of the loan with interest rate 3%, compounded 12 times a year

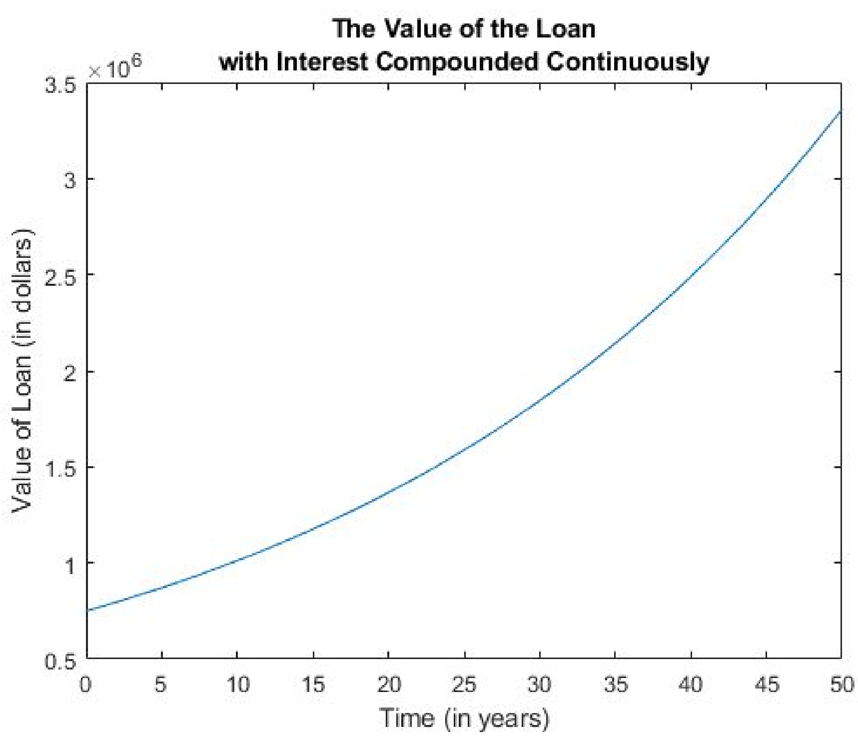

Figure 3: The value of the loan with interest rate 3%, compounded continuously

The differential equation that represents a continuously compounded loan, with p equal to the monthly payment, is as follows:

$$y' = ry - 12p$$

$$y(0) = y_0$$

Solving this equation using separation of variables yields:

$$y(t) = (y_0 - \frac{12p}{r})e^{rt} + \frac{12p}{r}$$

The equilibrium solution for this equation occurs at $y = \frac{12p}{r}$. Since y represents the amount of money owed at any given time, y' represents the change in money owed with respect to time.

At $y = \frac{12p}{r}$, the change in money owed over the years is constant. Below that equilibrium, the change in money owed decreases, which means that you will eventually be able to pay back your loan. Above that equilibrium, the change in money owed increases as time passes, and you will never be able to pay back your loan.

So, make sure that you do not borrow more than $\frac{12p}{r}$ in order to stay below the equilibrium!

Using the solution for the differential equation, we can calculate the monthly payment p needed to pay off a t year fixed rate mortgage with rate r and initial debt y_0:

Let's say you begin with an initial debt of $750,000. I calculated the difference between taking on a 10 year fixed rate mortgage with rate 3%, and a 30 year fixed rate mortgage with rate 5%.

To pay off a 10 year fixed rate mortgage with rate 3% the monthly payment p is $7234.30. The total amount paid in 10 years is $868,116, which means that the total interest paid was $118,116.

To pay off a 30 year fixed rate mortgage with rate 5%, the monthly payment p is $4022.55/ The total amount paid in 30 years is $1,448,118, which means that the total interest paid was $698,118.

Let's say that you pay $100,000 right away, so that your debt is now only $650,000. What is the new difference between taking on a 10 year fixed rate mortgage with rate 3%, and a 30 year fixed rate mortgage with rate 5%?

To pay off a 10 year fixed rate mortgage with rate 3%, the monthly payment p is $6269.73. The total amount paid in 10 years is $752,367.60, which means that the total interest paid was $102,367.60.

To pay off a 30 year fixed rate mortgage with rate 5%, the monthly payment p is $3486.21. The total amount paid in 30 years is $1,255,035.60, which means that the total interest paid was $605,035.60.

If you want to pay less total interest over the years, I definitely recommend paying $100,000 right away if you are able to. The interest paid for a starting debt of $650,000 is thousands of dollars less than the interest paid for a starting debt of $750,000, so it is much more preferable.

Next, I recommend choosing the 10 year fixed rate mortgage with rate 3%, instead of the 30 year fixed rate mortgage with rate 5%. The 30 year plan has a much lower monthly payment compared to the 10 year plan, so it is easier to afford in the short term, yet you will end up paying hundreds of thousands of dollars more as interest!

Numerical Solution

Using Euler's method with small step sizes such as 0.01 will definitely get a pretty good result compared to the true solution. However, that is so much work if you try to do it by hand. Using computer programs is really helpful in this case. Remember, that Euler's method is only an estimation and not the exact value.

With $4,000 as the monthly payment, fixed interest rate 5% (0.05), and a mortgage of $750,000 you would expect it to be paid off in about 30.41 years. I agree, that is too many years, but it is the typical duration for such a mortgage and monthly payment in these days. Figure (4) illustrates the behavior for this case.

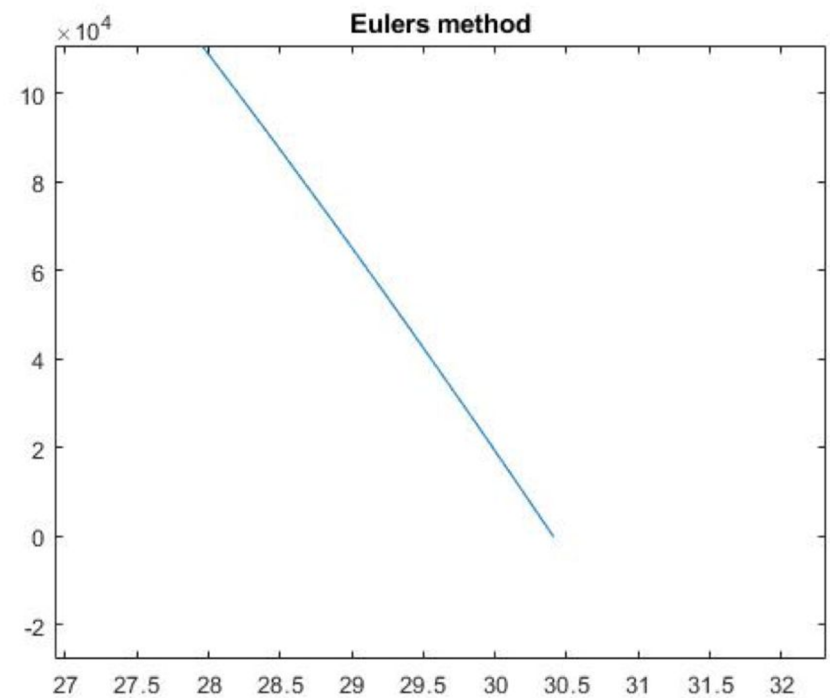

Figure 4: plot of Euler's method using step size of 0.01

Comparing the true solution to Euler's method estimation (numerical solution) could really vary depending on how small is your step size. Using step size equal to 0.01 will give very close estimation as I mentioned before.

Figure (5) shows the difference between the two solutions.

*Note: The blue curve is the numerical solution, while the orange curve is the true solution.

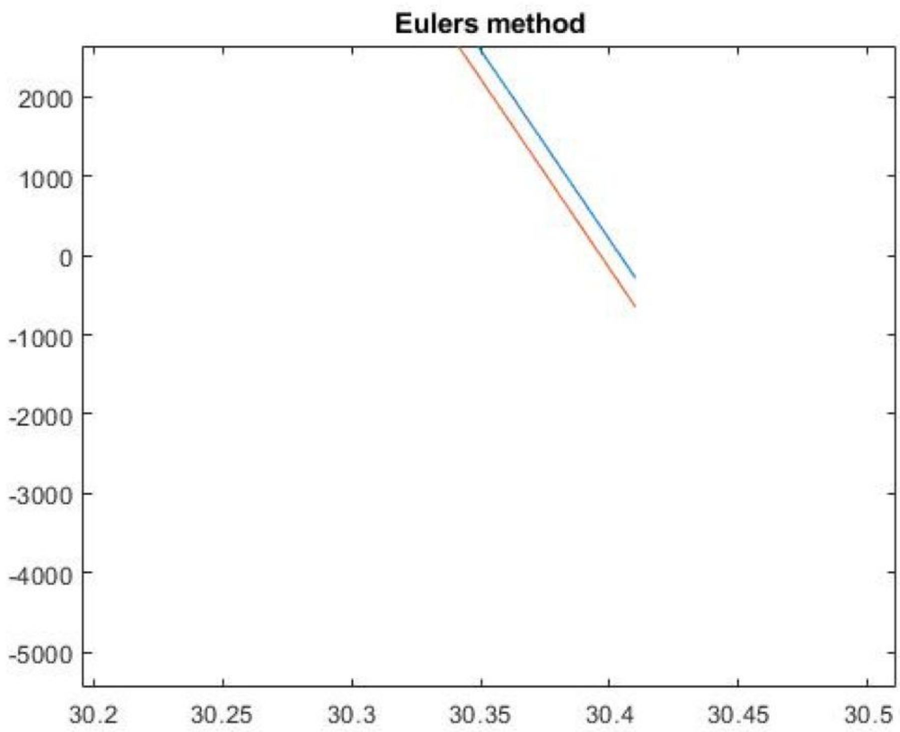

Figure 5: plot of true solution and numerical solution with h=0.01

As you can tell from the scale, I had to zoom in a lot to see the difference between the two curves. Which indicates that both curves are almost identical since we used a small step size.

Also, we can do the math to compare. Using the true solution equation, it will be paid off in 30.39. Which is really close to 30.41 that we got using small step size in Euler's method.

While, on the other hand, if you choose to use a bigger step size such as 0.5, you will easily see the difference between the numerical and true solutions without even zooming in.

Which makes sense! Because the numerical solution is not very accurate anymore compared to what we have before when we used 0.01.

Figure (6) shows the difference between the two solutions.

*Note: The blue curve is the numerical solution, while the orange curve is the true solution.

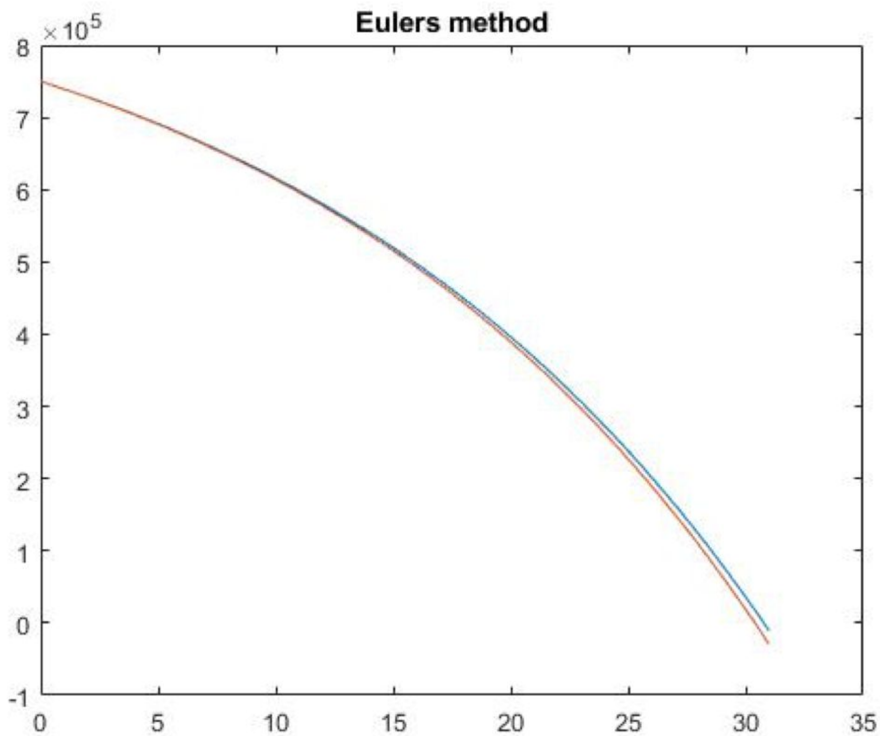

Figure 6: plot of true solution and numerical solution with h=0.5

Now, using this estimation, you would expect to get the mortgage paid off in about 31 years. Now the difference between the true solution and the estimation got bigger. Some may say that is not a big deal, or there is not a huge difference between 30.39 and 31 years. Well, I agree, but if you do the math, it will be a different story since you will almost pay 6 months more.

Thinking of the other offer from the bank, and using the adjustable rate mortgage would affect the period of paying off the mortgage. It could be more years or even less depending on factors like monthly payment and/or the interest rate.

In the adjustable rate mortgage, all the idea is that the rate now is changing over the time and it is not fixed anymore.

You may start with lower rate, initially, compared to the fixed rate, but it changes after that and eventually it would be more than the fixed rate just like our case.

If we suppose that the borrower pays $4,000 per month with the same exact mortgage, then it will be paid off in about 34.9 years if the borrower chooses to pick the adjustable rate mortgage.

Which makes sense, because you now will pay more interest and thus it will pay off in a longer period of time. However, if the borrower decided to increase the monthly payment to $ 4,500, we will definitely expect less years. Figure (7) shows the difference between paying 4,000 and 4,500 per month in the adjustable rate mortgage.

*Note: The blue curve represents monthly payment as $4,000, while the orange curve represent monthly payment as $4,500.

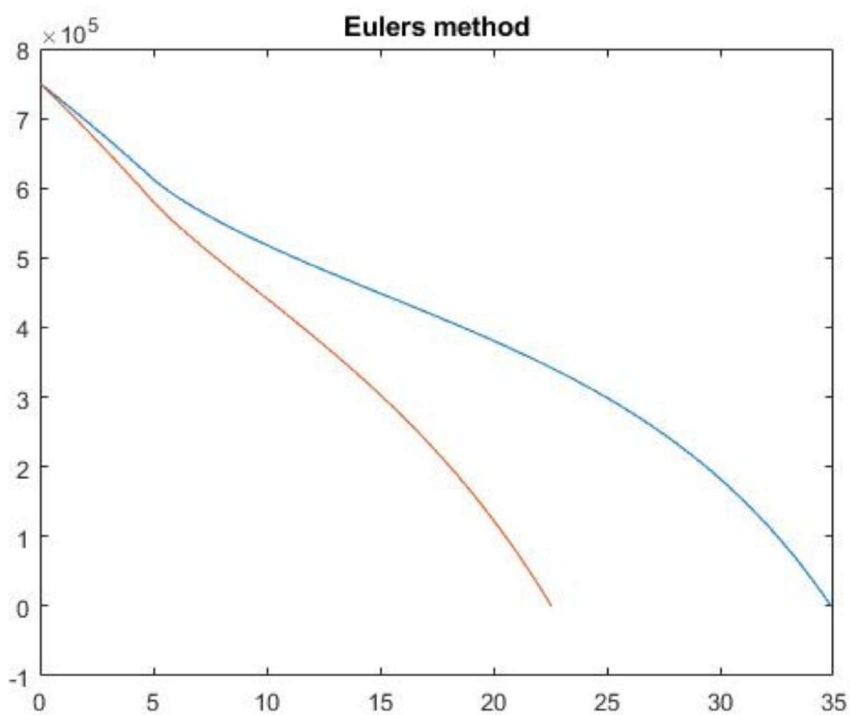

Figure 7: plot of the numerical solution for both scenarios

As you can see in Figure (7), when you pay more every month, you will definitely get it paid off in less amount of years. The orange curve decreases faster than the blue curve, which is exactly what we expected. The plot illustrates that it could be paid in about 22.6 years if the borrower choose to pay $4,500 monthly.

Moreover, you can notice that both curves behave differently in the first couple of years, specifically in the first 5 years. That's because of the function of rate. The function says that in the first 5 years, the interest rate is 3%, but after that rate is tied to the credit market. That is exactly what we see here, in the first 5 years, both curves start with almost a straight line. After that, it starts changing depending on the market.

The interest rate paid in each case will be different since the monthly payments are different. If the borrower chooses to pay $4,000 only, the interest rate paid would be about:

($4,000)*(12)*(34.9 years)-($750,000) = $925,200

But if the borrower chooses to pay a little bit more like $4,500 monthly, the interest rate paid would be about:

($4,500)*(12)*(22.6 years)-($750,000) = $470,400

Conclusion

To conclude, I recommend the %3 fixed rate mortgage plan with a down payment of $100,000 in 10 years since it is the lowest interest option you would pay. The monthly payment is $6269.73, which results in a total interest of $102,367.60. Using this plan, you will save much more money in the long run compared to starting with a $750,000 debt or using a 30 year fixed rate mortgage with rate 5%. The 30 year mortgage plan with a rate of 5% has the advantage that you will pay less per month, but your total interest rate will be hundreds of thousands of dollars more! In the 10 year plan starting with a debt of $650,000, although you will have a higher monthly payment and you will need to pay $100,000 upfront, you will pay much less in total interest by the end, which will save you quite a bit of money.

However, if the borrower decided to choose an adjustable rate mortgage with a debt of $750,000, then $4,500 monthly payment would be a lot better than $4,000. Because if you pay $4,500 monthly, the total interest would be $470,400 in 22.6 years. This is a lot better than total interest of $925,200 in 34.9 years with $4,000 monthly payment. Even though that $4,500 is a bit more than $4,000 as a monthly payment, you will save a lot more money and get it paid off faster at the same time. Basically, in long term mortgage, you will have a lower monthly payment but more interest paid over all. On the other hand, short term mortgage, you will have higher monthly payment but lower interest rate paid over all.

Appendix

Figure 1

```
y = 750000*((1+0.0075*x).^4);
plot(x,y)
```

Figure 2

```
y = 750000*((1+0.0025*x).^12);
plot(x,y)
```

Figure 3

```
y = 750000*exp(0.03*t);
plot(x,y)
```

Figure 4

```
t(1)=0;
y(1)=750000;
for n=1:10000
  t(n+1)=t(n)+0.01;
  y(n+1)=y(n)+0.01.*(0.05*(y(n))-12*4000);
  if y(n+1)<0
    break
  end
end
plot(t,y)
title('Eulers method')
```

Figure 5

```
t(1)=0;
y(1)=750000;
for n=1:10000
t(n+1)=t(n)+0.01;
  y(n+1)=y(n)+0.01.*(0.05*(y(n))-12*4000);
  if y(n+1)<0
  break
  end
end
plot(t,y)
title('Eulers method')

yt=-210000*exp(0.05*t)+960000;
hold on
plot(t,yt)
```

Figure 6

```
t(1)=0;
y(1)=750000;
for n=1:10000
   t(n+1)=t(n)+0.5;
   y(n+1)=y(n)+0.5.*(0.05*(y(n))-12*4000);
   if y(n+1)<0
      break
   end
end
plot(t,y)
title('Eulers method')

yt=-210000*exp(0.05*t)+960000;
hold on
plot(t,yt)
```

Figure 7

```
%y=zeros(10000,1);
%t=zeros(10000,1);
t(1,1)=0;
y(1,1)=750000;
for n=1:10000
t(n+1,1)=t(n,1)+0.01;
if (t(n,1)<=5)
   r=0.03;
else (t(n,1)>5)
  r=0.03+0.015*sqrt(t(n,1)-5);
end
   y(n+1,1)=y(n,1)+0.01.*(r*(y(n,1))-12*4000);
if y(n+1)<0
   break
   end
end
plot(t,y)
title('Eulers method')
hold on
t2(1,1)=0;
y2(1,1)=750000;
for n=1:10000
t2(n+1,1)=t2(n,1)+0.01;
if (t2(n,1)<=5)
   r2=0.03;
else (t2(n,1)>5)
   r2=0.03+0.015*sqrt(t2(n,1)-5);
end
   y2(n+1,1)=y2(n,1)+0.01.*(r2*(y2(n,1))-12*4500);
if y2(n+1)<0
   break
   end
end

plot(t2,y2)
```