MEEN 673
Assignment 1

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Problem 1

Consider the nonlinear differential equation

$$-\frac{d}{dx}\left[\left(u+\sqrt{2}\right)\frac{du}{dx}\right]=1, 0<x<1, \frac{du}{dx}(0)=0, \mathrm{u}(1)=0$$

Develop weak form and finite element model of the equation over an element. Compute the tangent coefficient matrix of the model.

$$a=u+\sqrt{2}, b=c=0,\ f=1$$

(1) Weak form:

$$0=\int_{x_a}^{x_b}\left[-w_i\frac{d}{dx}\left(\left(u+\sqrt{2}\right)\frac{du_h}{dx}\right)-w_i\right]dx=\int_{x_a}^{x_b}\left[\left(u_h+\sqrt{2}\right)\frac{dw_i}{dx}\frac{du_h}{dx}-w_i\right]dx-w_i\left(x_a\right)Q_a-w_i\left(x_b\right)Q_b$$

$$=\int_{x_a}^{x_b}\left[\left(u_h+\sqrt{2}\right)\frac{dw_i}{dx}\frac{du_h}{dx}\right]dx-\left[\int_{x_a}^{x_b}w_i dx+w_i\left(x_a\right)Q_a+w_i\left(x_b\right)Q_b\right]$$

where $Q_a=-\left(u+\sqrt{2}\right)\frac{du}{dx}\Big|_{x=x_a}$, $Q_b=\left(u+\sqrt{2}\right)\frac{du}{dx}\Big|_{x=x_b}$.

(2) Finite element model:

Let $u_h^e\left(x\right)=\sum_{j=1}^{n}u_j^e\varphi_j^e\left(x\right), w_i=\varphi_i^e\left(x\right)$.

$$0=\int_{x_a}^{x_b}\left[\left(u_h+\sqrt{2}\right)\frac{dw_i}{dx}\frac{du_h}{dx}\right]dx-\left[\int_{x_a}^{x_b}w_i dx+w_i\left(x_a\right)Q_a+w_i\left(x_b\right)Q_b\right]$$

$$=\sum_{j=1}^{n}u_j^e\int_{x_a}^{x_b}\left[\left(\sum_{j=1}^{n}u_j^e\varphi_j^e+\sqrt{2}\right)\frac{d\varphi_i^e}{dx}\frac{d\varphi_j^e}{dx}\right]dx-\left[\int_{x_a}^{x_b}\varphi_i dx+\varphi_i\left(x_a\right)Q_1^e+\varphi_i\left(x_b\right)Q_n^e\right]$$

$$=\sum_{j=1}^{n}K_{ij}^e u_j^e-F_i^e$$

Thus,

$$K_{ij}^e=\int_{x_a}^{x_b}\left[\left(\sum_{j=1}^{n}u_j^e\varphi_j^e+\sqrt{2}\right)\frac{d\varphi_i^e}{dx}\frac{d\varphi_j^e}{dx}\right]dx=\int_{x_a}^{x_b}\sqrt{2}\frac{d\varphi_i^e}{dx}\frac{d\varphi_j^e}{dx}dx+\sum_{k=1}^{n}u_k^e\int_{x_a}^{x_b}\varphi_k^e\frac{d\varphi_i^e}{dx}\frac{d\varphi_j^e}{dx}dx$$

$$F_i^e=\int_{x_a}^{x_b}\varphi_i dx+\varphi_i\left(x_a\right)Q_1^e+\varphi_i\left(x_b\right)Q_n^e$$

(3) Tangent coefficient matrix

$$T_{ij}^e=K_{ij}^e+\sum_{m=1}^{n}\frac{\partial K_{im}^e}{\partial u_j^e}u_m^e=K_{ij}^e+\sum_{m=1}^{n}\frac{\partial}{\partial u_j^e}\left(\int_{x_a}^{x_b}\left(u_h+\sqrt{2}\right)\frac{d\varphi_i^e}{dx}\frac{d\varphi_m^e}{dx}dx\right)u_m^e$$

$$=K_{ij}^e+\int_{x_a}^{x_b}\frac{\partial\left(u_h^e+\sqrt{2}\right)}{\partial u_j^e}\frac{d\varphi_i^e}{dx}\left(\sum_{m=1}^{n}u_m^e\frac{d\varphi_m^e}{dx}\right)dx=K_{ij}^e+\int_{x_a}^{x_b}\varphi_j^e\frac{d\varphi_i^e}{dx}\frac{du_h}{dx}dx$$

Program 1

(1) $-\frac{d}{dx}\left[u\frac{du}{dx}\right]=-1, 0<x<1, \frac{du}{dx}(0)=0, \mathrm{u}(1)=\sqrt{2}$

Direct iteration: NONLIN = 1, Newton iteration: NONLIN = 2.

Box 1. An example of input file for the validation case (Ex4.4.1)

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Example 4.4.1: Nonlinear analysis of a problem (DI)
  0.0   0.0   1.0   0.0   0.0   0.0        AX0,AX1,AU1,AU2,AUX1,AUX2
  0.0   0.0   0.0   0.0   0.0   0.0        BX0,BX1,BU1,BU2,BUX1,BUX2
  0.0   0.0   0.0   0.0   0.0   0.0        CX0,CX1,CU1,CU2,CUX1,CUX2
 -1.0   0.0   0.0   0.0   0.0   0.0        FX0,FX1,FX2
  0.0   1.0                                X0,AL
  1     2                                  IEL, NEM
  1     0     0                            NSPV, NSSV, NSMB
  3     1     1.4142                       ISPV(I,J),VSPV(I)
  1     0                                  NONLIN, NPRNT
  1     10     1.0E-03     0.0             NLS, ITMAX, EPS, GAMA
  1.0                                      DP(I)
  1.0   1.0   1.4142                       GP1(I)
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(2) $-\frac{d^2u}{dx^2}-2u\frac{du}{dx}=0, 0<x<1, u(0)=1, \mathrm{u}(1)=0.5$

Box 2. An example of input file for Problem 4.5

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Example 4.5: Nonlinear analysis of a problem
  1.0   0.0   0.0   0.0   0.0   0.0        AX0,AX1,AU1,AU2,AUX1,AUX2
  0.0   0.0  -2.0    0.0   0.0   0.0        BX0,BX1,BU1,BU2,BUX1,BUX2
  0.0   0.0   0.0   0.0   0.0   0.0        CX0,CX1,CU1,CU2,CUX1,CUX2
  0.0   0.0   0.0   0.0   0.0   0.0        FX0,FX1,FX2
  0.0   1.0                                X0,AL
  1     2                                  IEL, NEM
  2     0     0                            NSPV, NSSV, NSMB
  1     1     1.0                          ISPV(I,J),VSPV(I)
  3     1     0.5
  1     0                                  NONLIN, NPRNT
  1     10     1.0E-03     0.0             NLS, ITMAX, EPS, GAMA
  1.0                                      DP(I)
  1.0   1.0   0.5                          GP1(I)
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Table 1. Numerical results of the validation case (Ex4.4.1)

X(m)	Direct Iteration						Newton Iteration					
	2L	4L	8L	1Q	2Q	4Q	2L	4L	8L	1Q	2Q	4Q
0.000	0.99992	0.99987	0.99982	0.99993	0.99987	0.99980	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998
0.125	-	-	1.00760	-	-	1.00760	-	-	1.00780	-	-	1.00780
0.250	-	1.03070	1.03060	-	1.03080	1.03060	-	1.03080	1.03080	-	1.03080	1.03080
0.375	-	-	1.06790	-	-	1.06790	-	-	1.06800	-	-	1.06800
0.500	1.11800	1.11800	1.11800	1.11870	1.11800	1.11800	1.11800	1.11800	1.11800	1.11800	1.11800	1.11800
0.625	-	-	1.17920	-	-	1.17920	-	-	1.17920	-	-	1.17920
0.750	-	1.25000	1.25000	-	1.25000	1.25000	-	1.25000	1.25000	-	1.25000	1.25000
0.875	-	-	1.32880	-	-	1.32880	-	-	1.32880	-	-	1.32880
1.000	1.41420	1.41420	1.41420	1.41420	1.41420	1.41420	1.41420	1.41420	1.41420	1.41420	1.41420	1.41420

Table 2. Numerical results of Problem 4.5

X(m)	Direct Iteration						Newton Iteration					
	2L	4L	8L	1Q	2Q	4Q	2L	4L	8L	1Q	2Q	4Q
0.000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.125	-	-	0.88857	-	-	0.88883	-	-	0.88855	-	-	0.88881
0.250	-	0.79821	0.79958	-	0.79924	0.80003	-	0.79819	0.79956	-	0.79921	0.80001
0.375	-	-	0.72686	-	-	0.72728	-	-	0.72683	-	-	0.72725
0.500	0.66002	0.66514	0.66631	0.66076	0.66675	0.66670	0.66000	0.66512	0.66629	0.66071	0.66673	0.66667
0.625	-	-	0.61511	-	-	0.61540	-	-	0.61509	-	-	0.61537
0.750	-	0.57063	0.57124	-	0.57132	0.57145	-	0.57062	0.57123	-	0.57131	0.57143
0.875	-	-	0.53324	-	-	0.53334	-	-	0.53323	-	-	0.53333
1.000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.5000	0.50000