

# EIE4512 - Digital Image Processing

## Frequency Domain Operations



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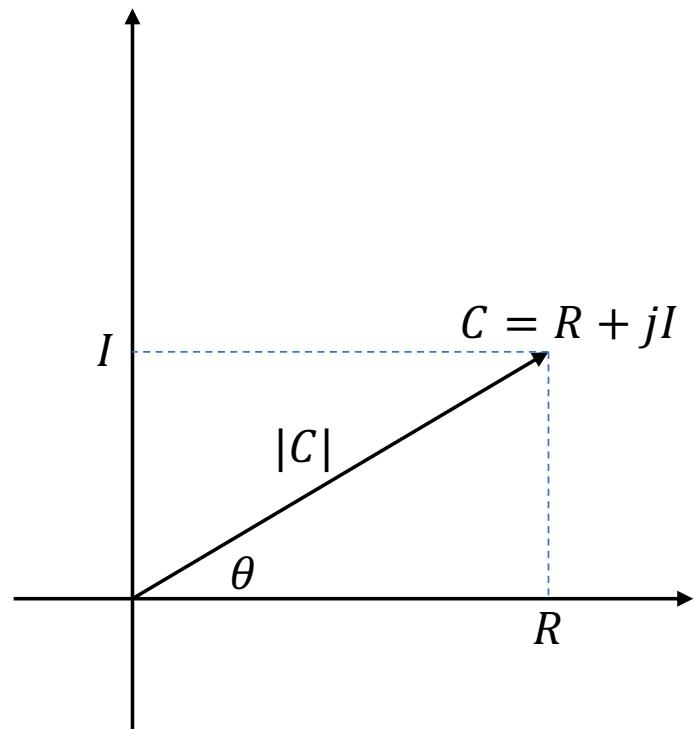
# Basics of Frequency Domain

# Outline

- Preliminary:
  - Complex Numbers
  - Fourier Series & Fourier Transform
  - Impulses
  - Convolution
- Sampling
  - Sampled Function
  - Sampling Theorem

# Revisit of Complex Numbers

- Definition
  - $C = R + j I$
  - Conjugate:  $C^* = R - j I$
- Polar Coordinates
  - $C = |C|(\cos \theta + j \sin \theta)$
  - $|C| = \sqrt{R^2 + I^2}$
  - $\tan \theta = I/R$
- Euler's Formula
  - $e^{j\theta} = \cos \theta + j \sin \theta$
  - $C = |C|e^{j\theta}$



# Fourier Series

- Any periodic function  $f(t)$  with period  $T$  can be represented as a **Fourier Series**

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{T} t}$$

- With

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j \frac{2\pi n}{T} t} dt$$

*Fourier Series for real value function and complex value function*  
<https://www.youtube.com/watch?v=kP02nBNtjrU>

# Unit Impulse

*Unit impulse* of a continuous variable  $t$  located at  $t=0$  :

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

*Sifting property* :

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

(if  $f(t)$  continuous at  $t=0$  )

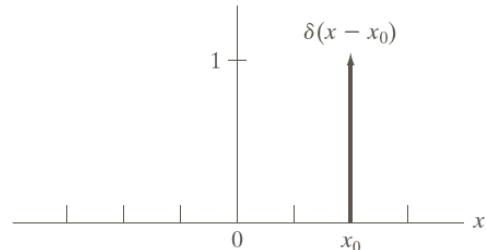
$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

# Discrete Unit Impulse

*Unit discrete impulse located at  $x=0$  :*

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$



**FIGURE 4.2**  
A unit discrete impulse located at  $x = x_0$ . Variable  $x$  is discrete, and  $\delta$  is 0 everywhere except at  $x = x_0$ .

*Sifting property :*

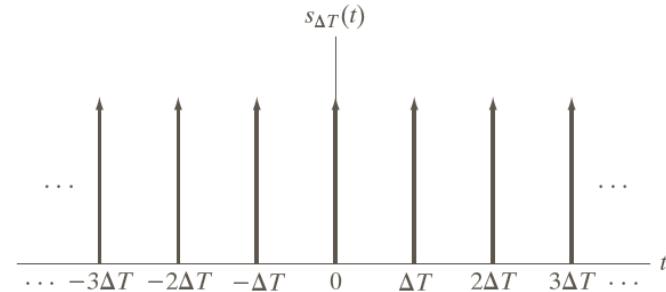
$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

# Impulse Train

*Impulse train* : sum of infinitely many periodic impulses  $\Delta T$  units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$



**FIGURE 4.3** An impulse train.

# Fourier Transform (FT) of Continuous Functions

Fourier Transform of a continuous function  $f(t)$  of a continuous variable  $t$ :

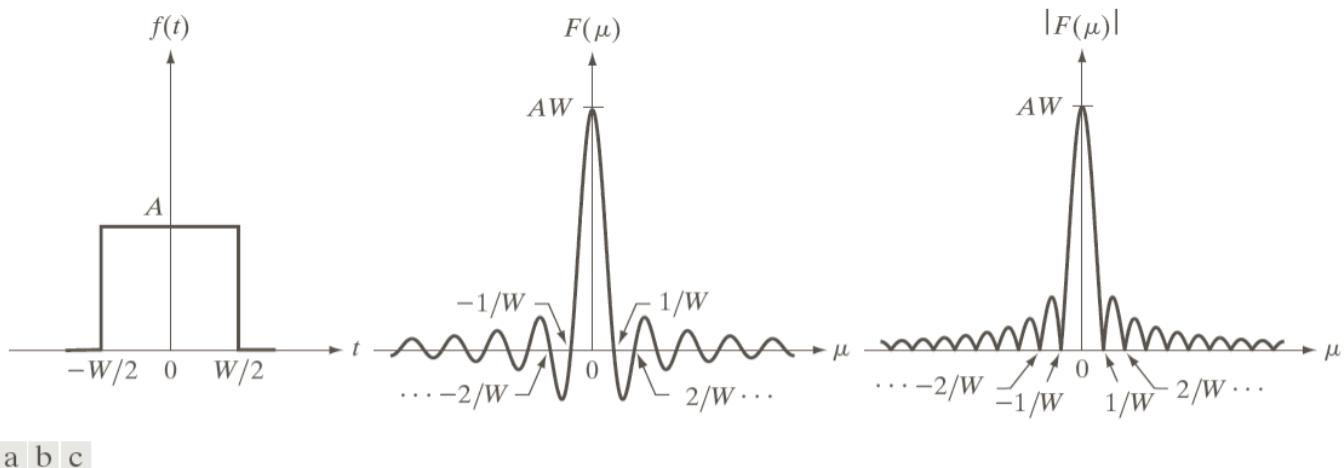
$$FT [f(t)] = F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

Inverse Fourier Transform:

$$FT^{-1} [F(\mu)] = f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

# FT Example

$$\text{Example : } F(\mu) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} Ae^{-j2\pi\mu t} dt \\ = AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} = AW \text{sinc}(\pi\mu W)$$



a b c

**FIGURE 4.4** (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

# FT of Unit Impulse

Example 2 : Fourier Transform of a unit impulse located at the origin:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu 0} = e^0 = 1$$

Fourier Transform of a unit impulse located at  $t = t_0$  :

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu t_0} = \cos(2\pi\mu t_0) - j\sin(2\pi\mu t_0)$$

# FT of Impulse Train

Example 2 : Fourier Transform  $S(\mu)$  of an impulse train with period  $\Delta T$ :

Periodic function with period  $\Delta T \Rightarrow$

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{\Delta T} t}$$

Where :  $c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j \frac{2\pi n}{\Delta T} t} dt = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j \frac{2\pi n}{\Delta T} t} dt = \frac{1}{\Delta T}$

$$\Rightarrow S(\mu) = FT[s_{\Delta T}(t)] = \frac{1}{\Delta T} FT \left[ \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t} \right] = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta \left( \mu - \frac{n}{\Delta T} \right)$$

Also an *impulse train*, with period  $1/\Delta T$

# Continuous-Time Convolution

Convolution of functions  $f(t)$  and  $h(t)$ , of one continuous variable  $t$ :

$$f(t) \star h(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau$$

$$FT \{f(t) \star h(t)\} = H(\mu)F(\mu)$$

Fourier Transform pairs:

$$f(t) \star h(t) \iff H(\mu)F(\mu)$$

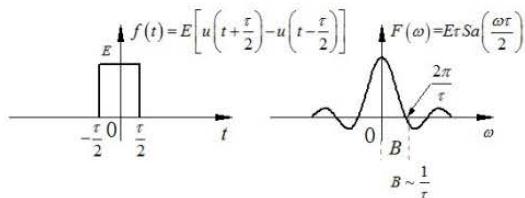
$$f(t)h(t) \iff H(\mu) \star F(\mu)$$

*Spatial Domain Conv   <=> Frequency Domain Multiplication*

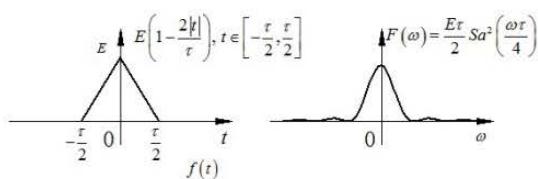
*Spatial Domain Multiplication   <=> Frequency Domain Conv*

# Commonly used continuous Fourier transform

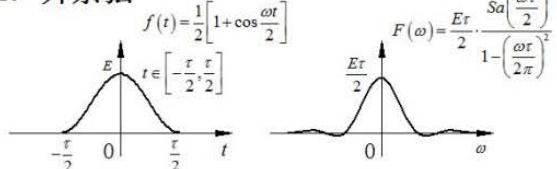
1. 矩形脉冲



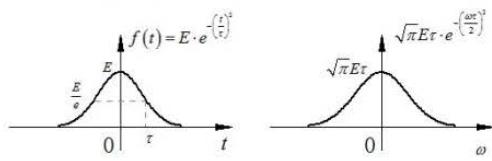
2. 三角脉冲



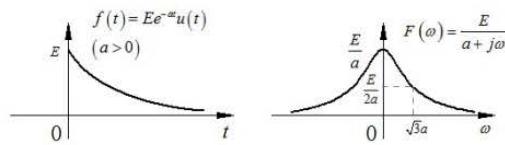
3. 升余弦



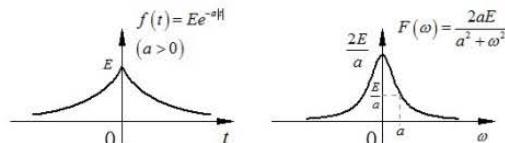
4. 高斯信号



5. 单边指数函数



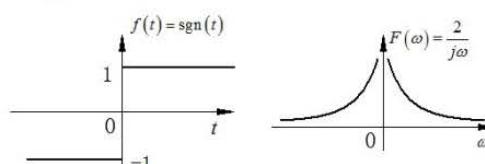
6. 双边指数函数



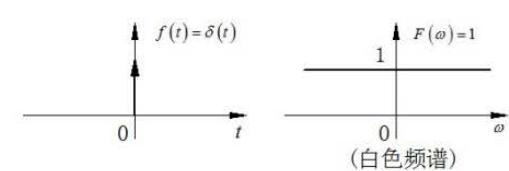
7. 奇对称指数函数



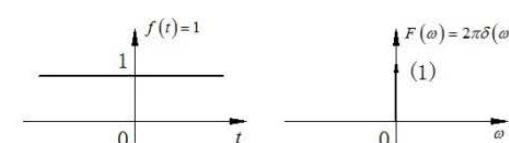
8. 符号函数



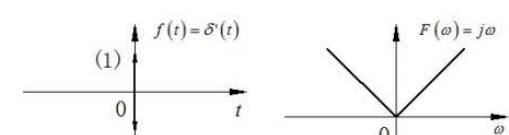
9. 冲击信号



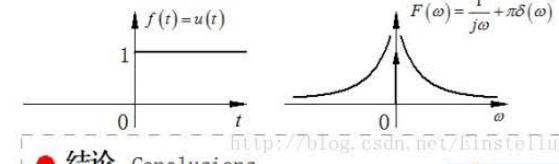
10. 直流信号



11. 冲激偶信号



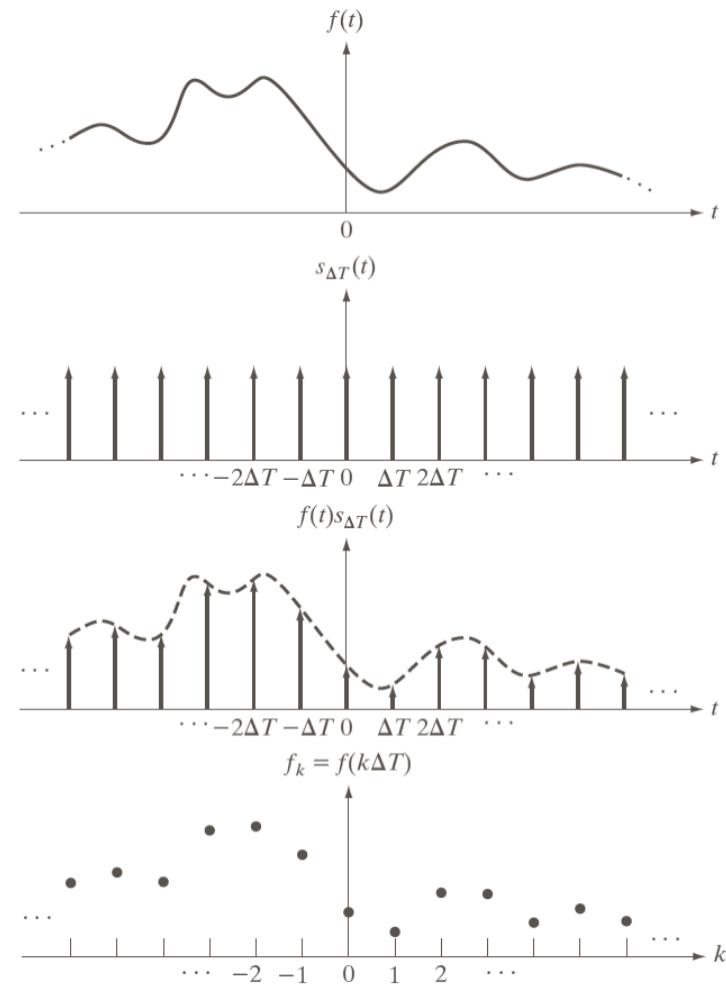
12. 单位阶跃信号



# Sampling

Model of the sampled function:  
 multiplication of  $f(t)$  by a sampling  
 function equal to a train of impulses  $\Delta T$   
 units apart

$$\tilde{f}(t) = f(t)s_{\Delta T} = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$



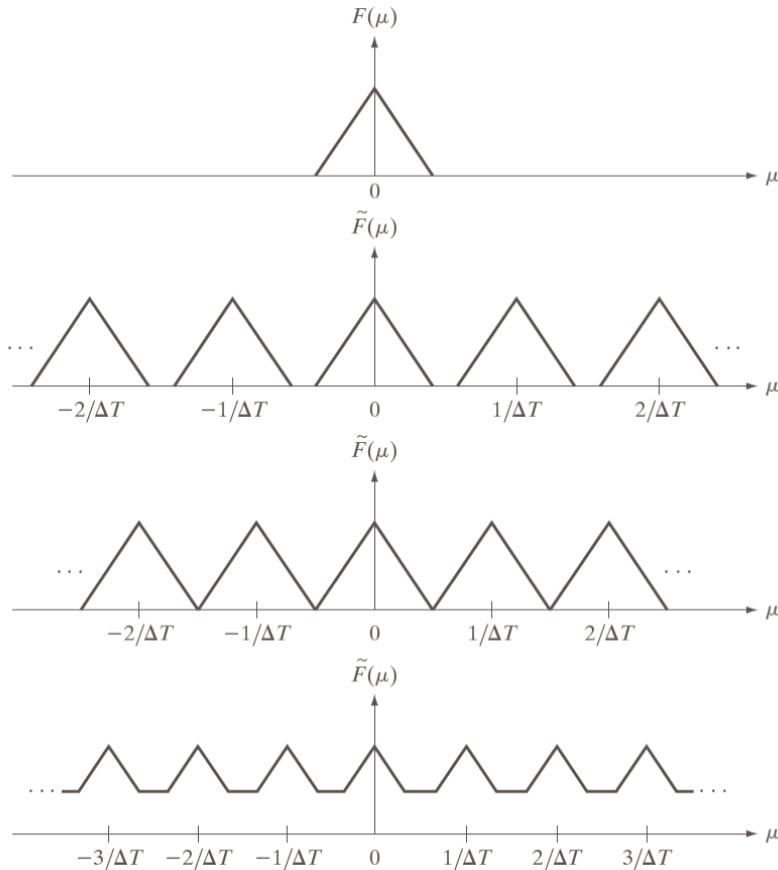
a  
b  
c  
d

**FIGURE 4.5**  
 (a) A continuous function.  
 (b) Train of impulses used to model the sampling process.  
 (c) Sampled function formed as the product of (a) and (b).  
 (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

# FT of Sampled Function

$$\begin{aligned}
 \tilde{F}(\mu) &= FT[\tilde{f}(t)] \\
 &= FT[f(t) \cdot s_{\Delta T}(t)] \\
 &= F(\mu) \star S(\mu) \\
 &= \int_{-\infty}^{\infty} F(\tau) S(\mu - \tau) d\tau \\
 &= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
 &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta\left(\mu - \tau - \frac{n}{\Delta T}\right) d\tau \\
 &= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)
 \end{aligned}$$

Superposition of infinitely many copies of  $F(\mu)$  with an interval  $1/\Delta T$



a  
b  
c  
d

**FIGURE 4.6**  
(a) Fourier transform of a band-limited function.  
(b)–(d)  
Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

# Sampling Theorem

A function  $f(t)$  whose Fourier transform is zero for values of frequencies outside a finite interval (band)  $[-\mu_{\max}, \mu_{\max}]$  about the origin is called a *band-limited* function

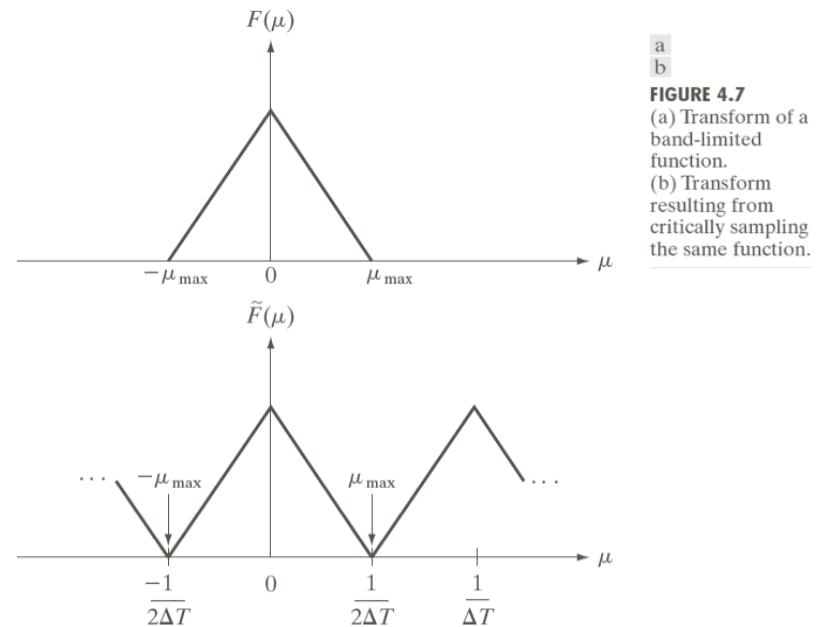
Sufficient separation guaranteed if:

$$\frac{1}{\Delta T} > 2\mu_{\max}$$

*Sampling Theorem:*

A continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function

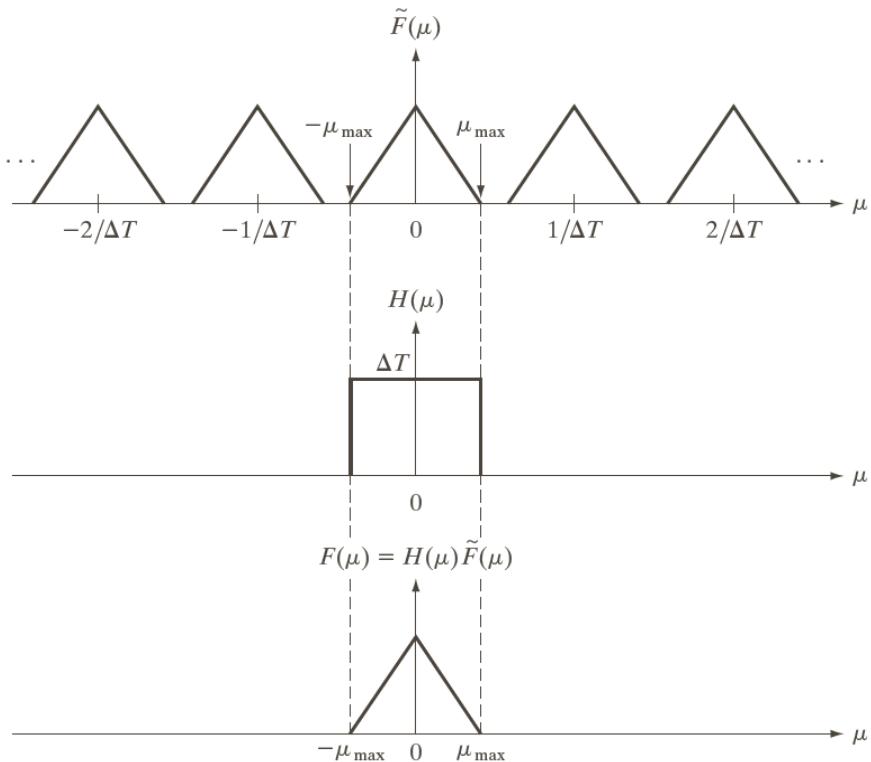
NB: Sampling at:  $\frac{1}{\Delta T} = 2\mu_{\max}$       *Nyquist rate*



a  
b

**FIGURE 4.7**  
(a) Transform of a band-limited function.  
(b) Transform resulting from critically sampling the same function.

# Recovery with Low-pass Filter



a  
b  
c

**FIGURE 4.8**  
Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

=> Recover  $f(t)$  using the inverse FT:

$$f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

Example: ideal lowpass filter

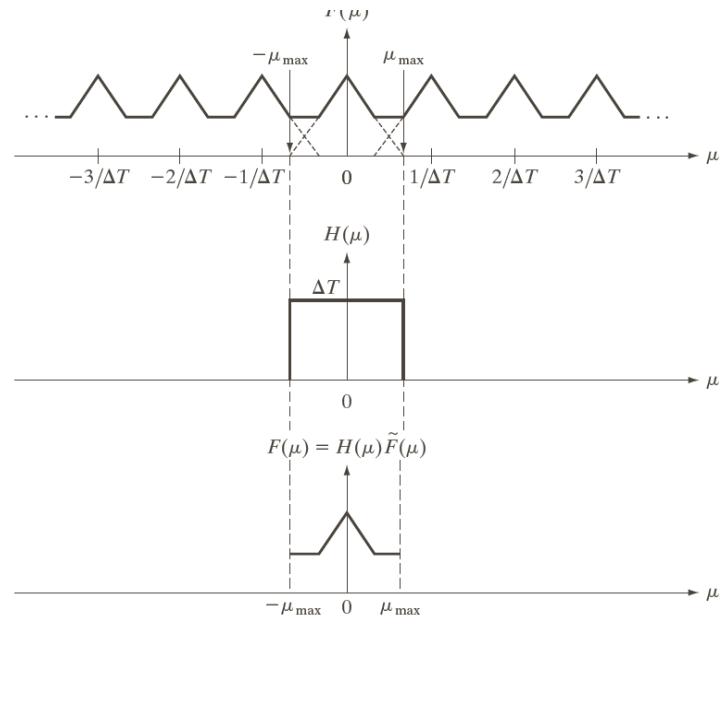
# Aliasing

Effect of under-sampling a function

Transform corrupted by frequencies from adjacent periods

NB: the effect of aliasing can be reduced by smoothing the input function to attenuate its higher frequencies: *anti-aliasing*.

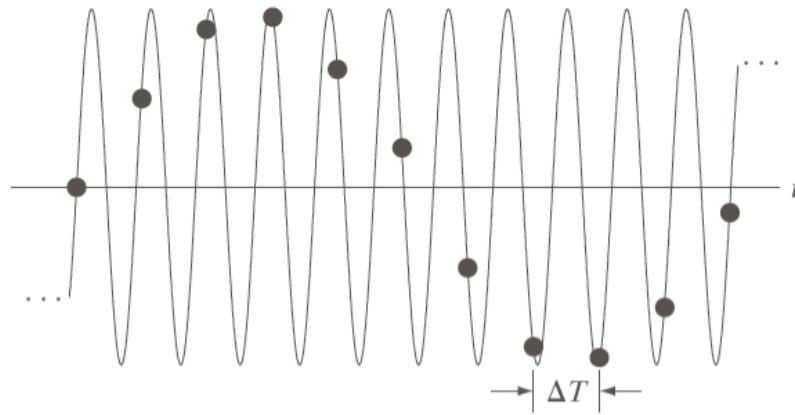
Has to be done *before* the sampling



**FIGURE 4.9** (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of  $F(\mu)$  and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

# Aliasing (Illustration)

Illustration: sampling a sine wave of period 2s



**FIGURE 4.10** Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second.  $\Delta T$  is the separation between samples.

# Function Reconstruction

$$\begin{aligned} f(t) &= FT^{-1} \left[ \tilde{F}(\mu)H(\mu) \right] \\ &= \sum_{n=-\infty}^{\infty} f(n\Delta T) \cdot \text{sinc} \left[ \frac{1}{\Delta T}(t - n\Delta T) \right] \end{aligned}$$

<https://blog.csdn.net/u010592995/article/details/73555425>

<https://blog.csdn.net/liyuanchu/article/details/42061089>

# Discrete Fourier Transform

FT of a sampled function

$$\begin{aligned}\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi\mu t} dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi\mu t} dt \\ &= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}\end{aligned}$$

Consider a finite sequence of samples

**DFT:**  $F_m = \tilde{F} \left( \frac{m}{M\Delta T} \right) = \sum_{n=0}^{M-1} f_n e^{-j2\pi u n m / M}$

# DFT and IDFT

Discrete Fourier Transform (DFT)

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}$$

Inverse Discrete Fourier Transform (IDFT)

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}$$

# Relation between Sampling & Frequency Interval

- Consider  $M$  samples of  $f$ , with interval  $\Delta T$ 
  - The duration of the record is
    - $T = M \cdot \Delta T$
  - The entire frequency range is
    - $\frac{1}{\Delta T}$
  - The spacing in the frequency domain
    - $\Delta\mu = \frac{1}{M \Delta T} = \frac{1}{T}$

# Circular Convolution

Both forward and inverse discrete transforms are infinitely periodic, with period  $M$

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM) \quad \text{Where } k \text{ is an integer}$$

Discrete equivalent of the convolution:

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x - m)$$

# Extension to Function of Two Variables

Impulse of two continuous variables  $t$  and  $z$  :

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) dt dz = 1$$

Impulse of two continuous variables  $t$  and  $z$  :

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \delta(t, z) dt dz = f(0, 0)$$

Sifting property:  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \delta(t, z) dt dz = f(0, 0)$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \delta(t - t_0, z - z_0) dt dz = f(t_0, z_0)$$

# Extension of Discrete Impulse

2-D discrete impulse:

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Sifting property:

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y) \delta(x, y) = f(0, 0)$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x, y) \delta(x - x_0, y - y_0) = f(x_0, y_0)$$

# Extension of Fourier Transform

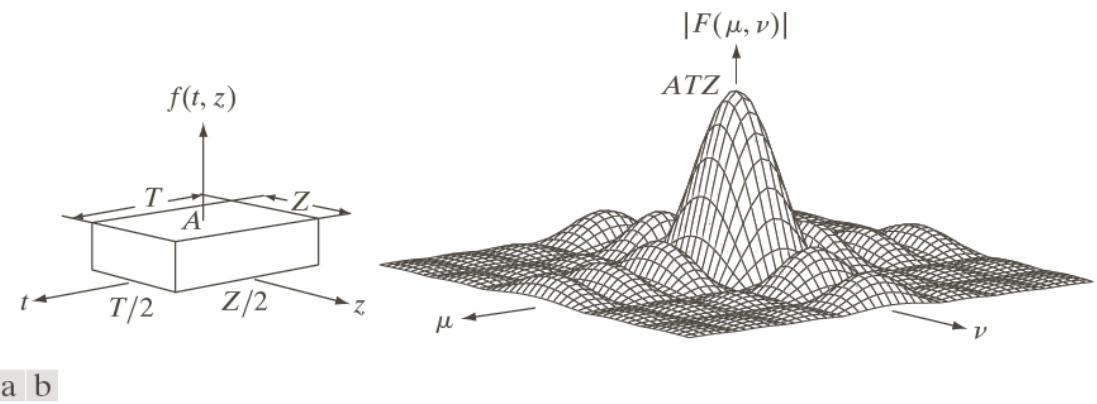
$f(t,z)$  continuous function of two continuous variables  $t$  and  $z$ , the 2-D continuous Fourier transform pair is given by:

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt dz$$

and

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} d\mu d\nu$$

# 2D FT: Example



a b

**FIGURE 4.13** (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the  $t$ -axis, so the spectrum is more “contracted” along the  $\mu$ -axis. Compare with Fig. 4.4.

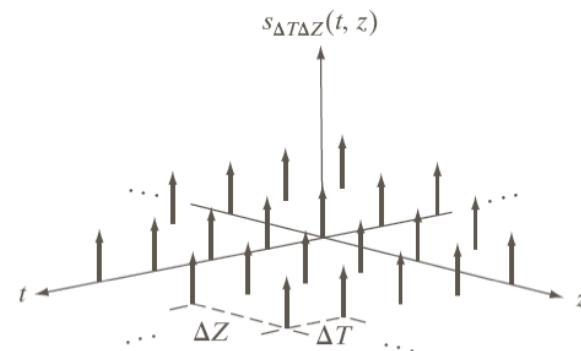
$$\begin{aligned} F(\mu, \nu) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} dt dz \\ &= ATZ \left[ \frac{\sin(\pi\mu T)}{(\pi\mu T)} \right] \left[ \frac{\sin(\pi\nu Z)}{(\pi\nu Z)} \right] \end{aligned}$$

# 2D Sampling

Sampling in 2-D can be modeled using the sampling function (2-D impulse train):

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

Multiplying  $f(t, z)$  by  $s_{\Delta T \Delta Z}(t, z)$  yields the sampled function



**FIGURE 4.14**  
Two-dimensional  
impulse train.

# 2D Sampling Theorem

$f(t,z)$  is said to be *band-limited* if its Fourier transform is 0 outside of a rectangle  $[-\mu_{\max}, \mu_{\max}]$  and  $[-v_{\max}, v_{\max}]$

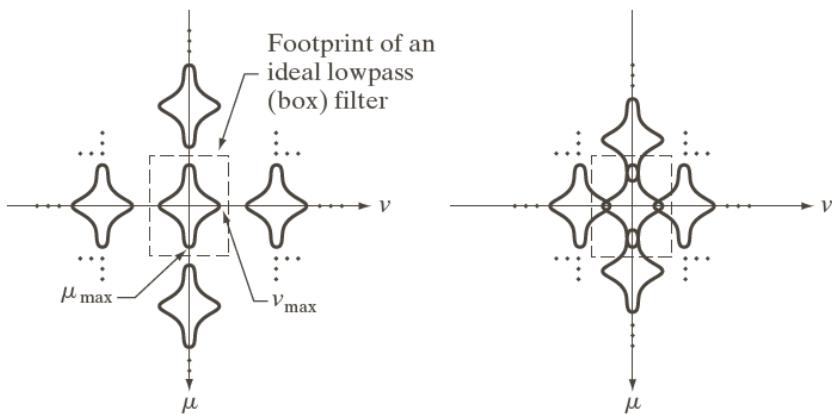
*2-D sampling theorem:*

A continuous, band-limited function  $f(t,z)$  can be recovered with no error from a set of its samples if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{\max}}$$

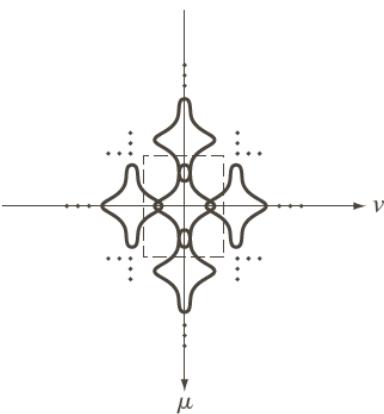
and

$$\Delta Z < \frac{1}{2v_{\max}}$$



a

**FIGURE 4.15**  
Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.



b

# 2D Discrete Fourier Transform (DFT)

2-D *Discrete Fourier Transform* (DFT) of a digital image  $f(x,y)$  of size  $M \times N$ :

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

*Inverse Discrete Fourier Transform* (IDFT):

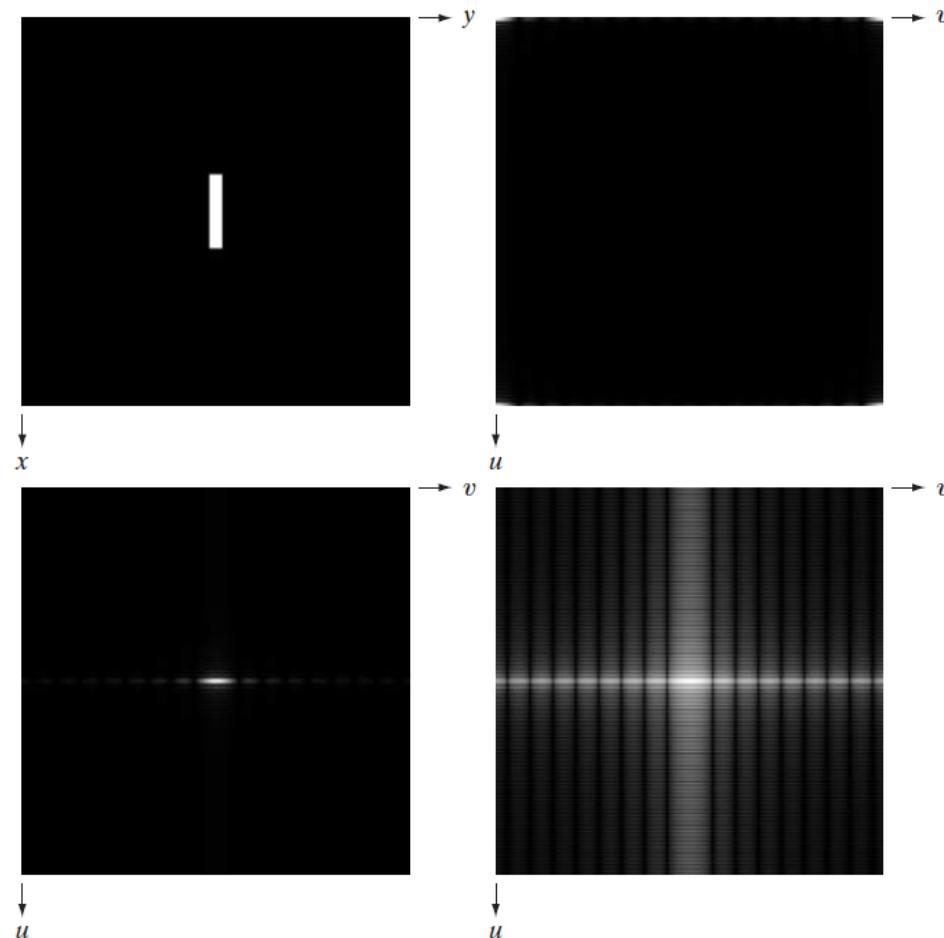
$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

# 2D DFT example

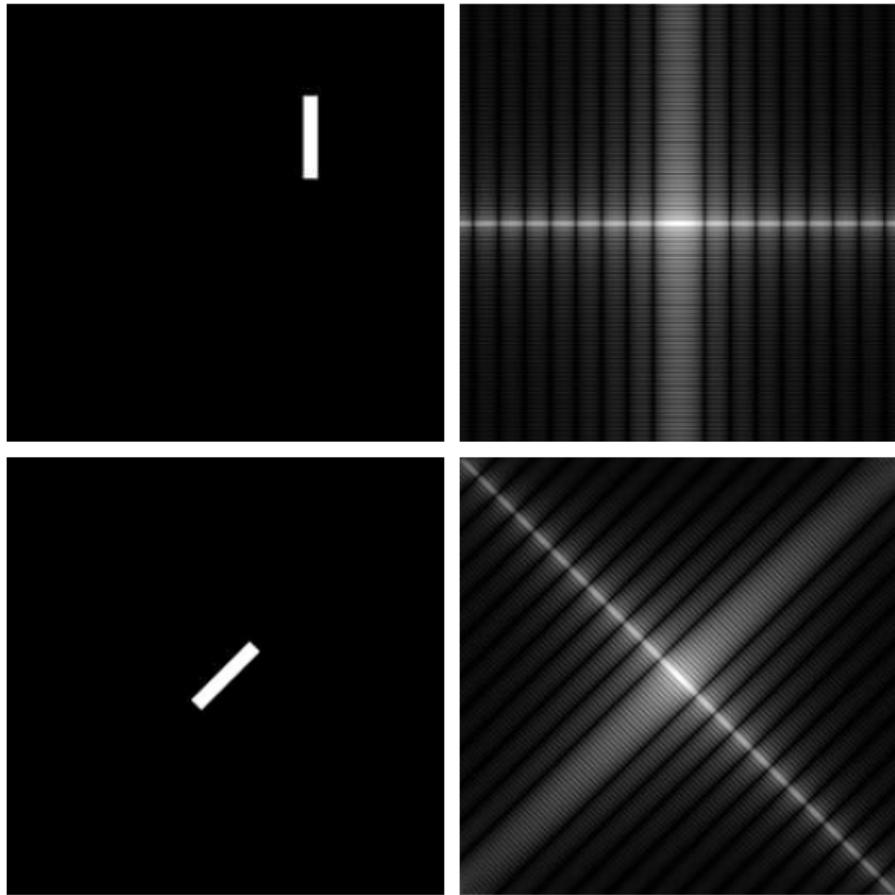
a	b
c	d

**FIGURE 4.24**

- (a) Image.
- (b) Spectrum showing bright spots in the four corners.
- (c) Centered spectrum.
- (d) Result showing increased detail after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The coordinate convention used throughout the book places the origin of the spatial and frequency domains at the top left.



# 2D DFT example



a	b
c	d

**FIGURE 4.25**

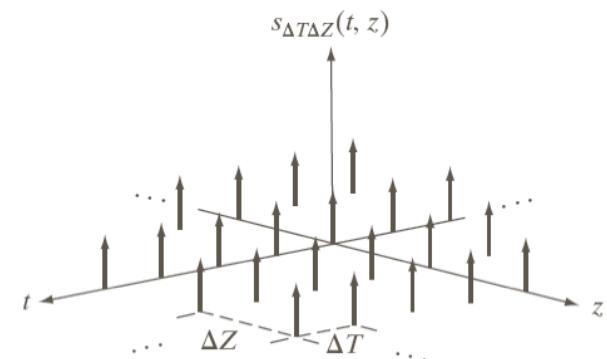
(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

translation  
insensitive,  
but rotation  
sensitive

# Relations between Spatial & Frequency Domains

If  $\Delta T$  and  $\Delta Z$  are the separations between samples, the separations between the corresponding discrete, frequency domain variables are:

$$\Delta u = \frac{1}{M\Delta T} \quad \text{and} \quad \Delta v = \frac{1}{N\Delta Z}$$



# Filtering in Frequency Domain

# 2D Convolution

2-D *circular convolution* :

$$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$$

2-D convolution theorem :

$$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$$

and conversely:

$$f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$$

# Wraparound Error & Padding

Data from adjacent periods produce wraparound error  
 => Need to append zeros to both functions

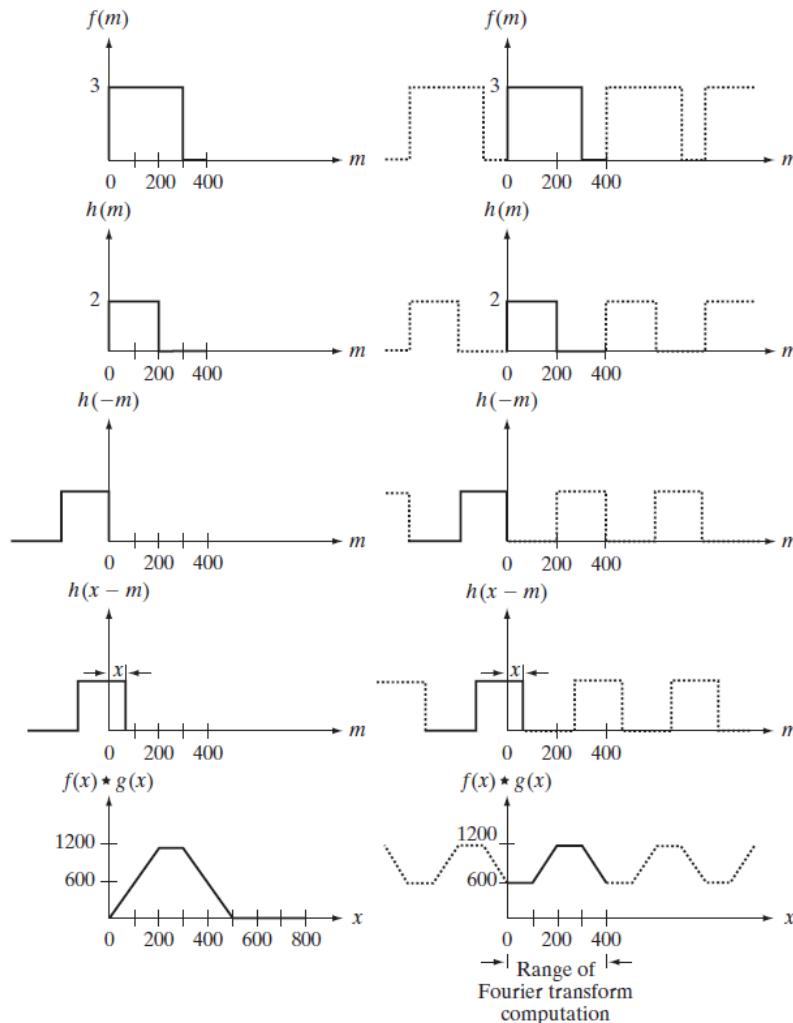
In 2-D, padd the two images array of size  $M \times N$  by zeros  
 Padded images of size  $P \times Q$ , with:

$$P \geq 2M-1$$

and

$$Q \geq 2N-1$$

**More 2D wraparound/padding**  
 example see Page 281 Figure 4.32



a	f
b	g
c	h
d	i
e	j

**FIGURE 4.28** Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

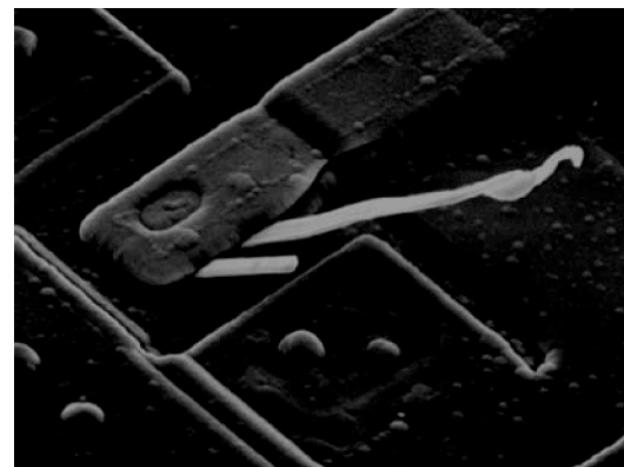
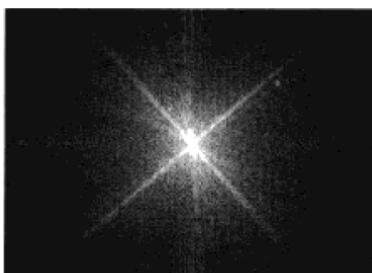
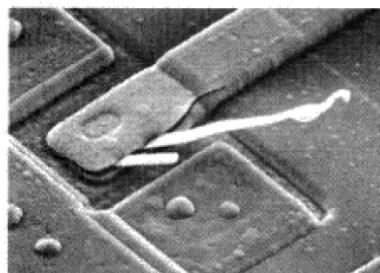
# Frequency Domain Filtering: Fundamentals

Given a digital image  $f(x,y)$  of size  $M \times N$ , the basic filtering equation is:

$$g(x, y) = IDFT [H(u, v)F(u, v)]$$

Filtered (output) image      Filter function (or filter transfer function)      DFT if the input image      NB: product = array multiplication

NB: For simplification, use functions  $H(u, v)$  that are centered symmetric about their centre  
+ centre  $F(u, v)$  multiplying  $f(x, y)$  by  $(-1)^{x+y}$



**FIGURE 4.30**  
Result of filtering the image in Fig. 4.29(a) by setting to 0 the term  $F(M/2, N/2)$  in the Fourier transform.

# Shifting 2D Frequency Spectrum

$$f(x) \Leftrightarrow F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$

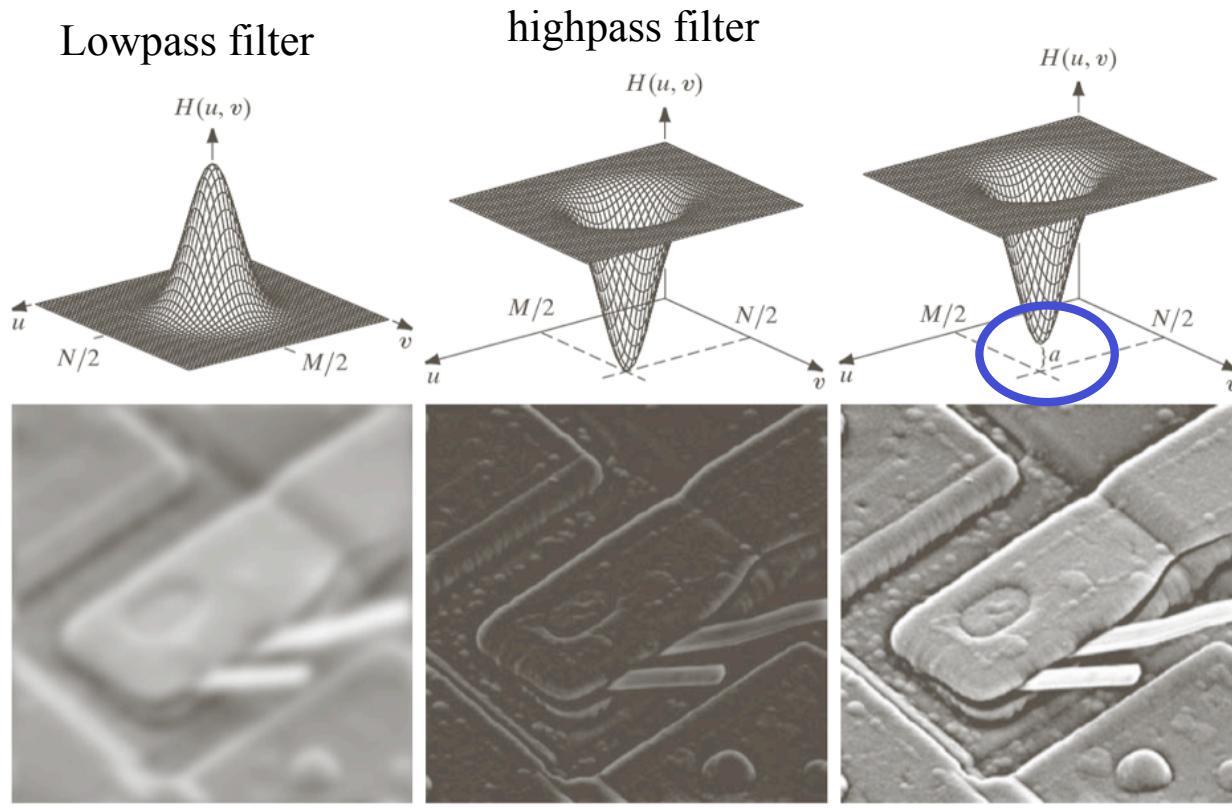
$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u - u_0) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi(u-u_0)x/M}$$

$$f(x)(-1)^x \Leftrightarrow F(u - M/2) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi(u-M/2)x/M}$$

Extension to 2D:

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$

# Filtering in Frequency Domain

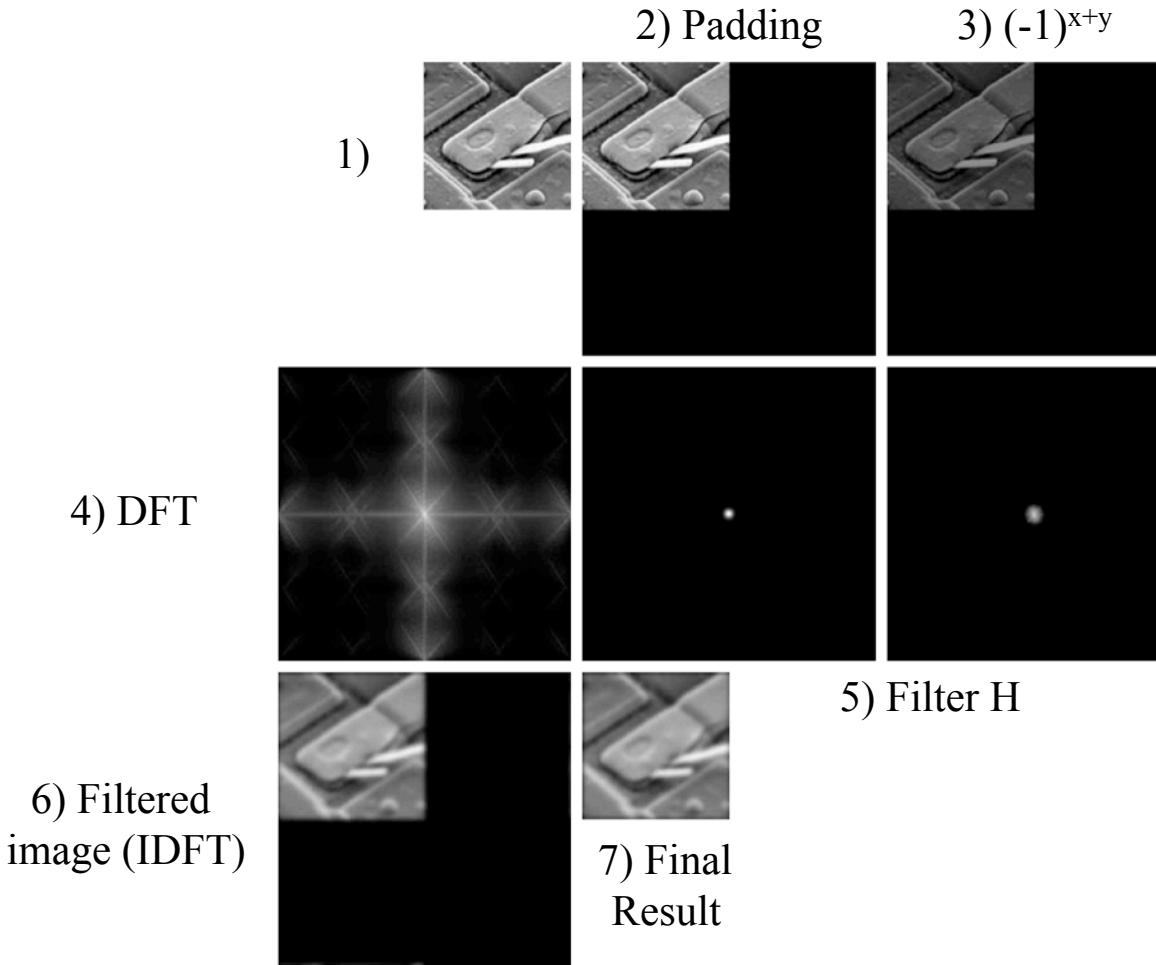


**FIGURE 4.31** Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq.(4.7-1). We used  $a = 0.85$  in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

# Steps for Frequency Domain Filtering

1. Given an input image  $f(x,y)$  of size  $M \times N$ , obtain padding parameters  $P$  and  $Q$ . Typically,  $P=2M$  and  $Q=2N$ .
2. Form a padded image  $f_p(x,y)$  of size  $P \times Q$  by appending the necessary number of zeros to  $f(x,y)$ .
3. Multiply  $f_p(x,y)$  by  $(-1)^{x+y}$  to centre its transform.
4. Compute the DFT,  $F(u,v)$ , of the image from step 3.
5. Generate a real, symmetric filter function,  $H(u,v)$ , of size  $P \times Q$  with centre at coordinates  $(P/2, Q/2)$ . Form the product  $G(u,v)=H(u,v)F(u,v)$  using array multiplication.
6. Obtain the processed image:  $g_p(x,y) = \text{real} [IDFT [G(u,v)]] (-1)^{x+y}$
7. Obtain the final processed result,  $g(x,y)$ , by extracting the  $M \times N$  region from the top, left quadrant of  $g_p(x,y)$

# Steps for Frequency Domain Filtering



a	b	c
d	e	f
g	h	

**FIGURE 4.36**

- (a) An  $M \times N$  image,  $f$ .
- (b) Padded image,  $f_p$  of size  $P \times Q$ .
- (c) Result of multiplying  $f_p$  by  $(-1)^{x+y}$ .
- (d) Spectrum of  $F_p$ .
- (e) Centered Gaussian lowpass filter,  $H$ , of size  $P \times Q$ .
- (f) Spectrum of the product  $HF_p$ .
- (g)  $g_p$ , the product of  $(-1)^{x+y}$  and the real part of the IDFT of  $HF_p$ .
- (h) Final result,  $g$ , obtained by cropping the first  $M$  rows and  $N$  columns of  $g_p$ .

# Ideal Lowpass Filter

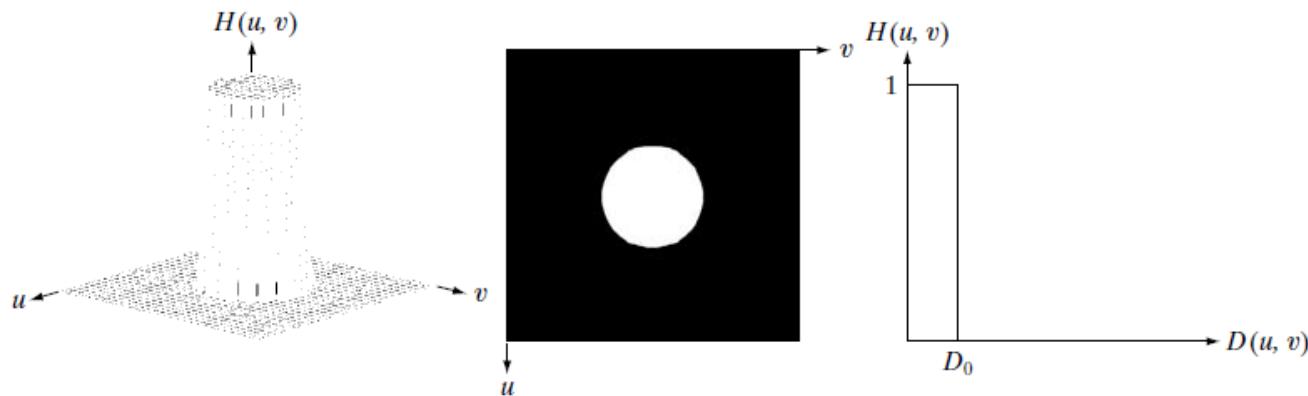
*Ideal Lowpass Filter (ILPF)* = 2-D lowpass filter that passes without attenuation all frequencies within a circle of radius  $D_0$  from the origin and “cuts off” all frequencies outside this circle

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{if } D(u, v) > D_0 \end{cases}$$

Where:  $D_0 \geq 0$

And  $D(u, v)$  is the distance between a point  $(u, v)$  and the centre of the frequency rectangle:

$$D(u, v) = [(u - P/2)^2 + (v - Q/2)^2]^{1/2}$$

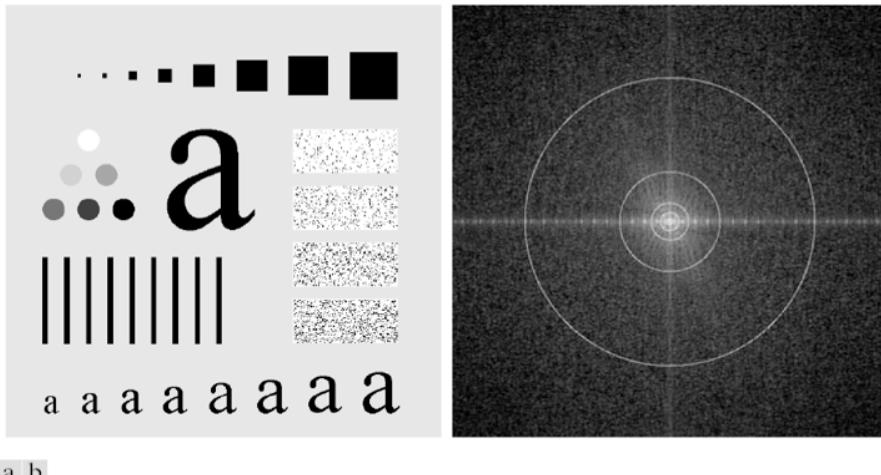


a b c

**FIGURE 4.40** (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

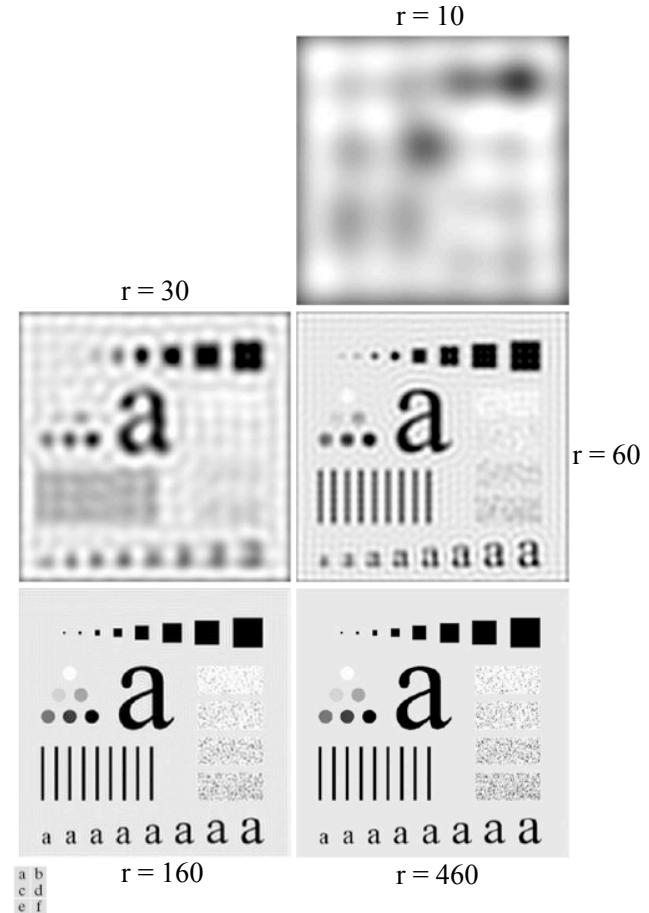
# Ideal Lowpass Filter (Effect)

Example:



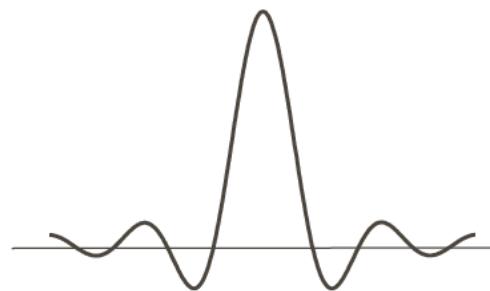
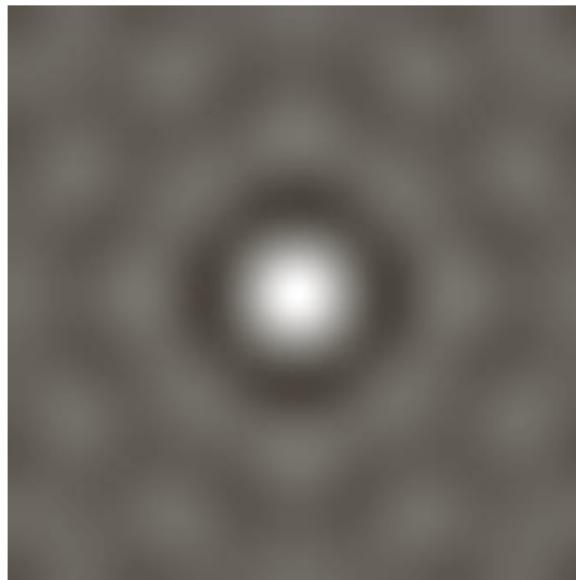
a b

**FIGURE 4.41** (a) Test pattern of size  $688 \times 688$  pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



**FIGURE 4.42** (a) Original image. (b)-(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

# Ideal Lowpass: Blurring and Ringing



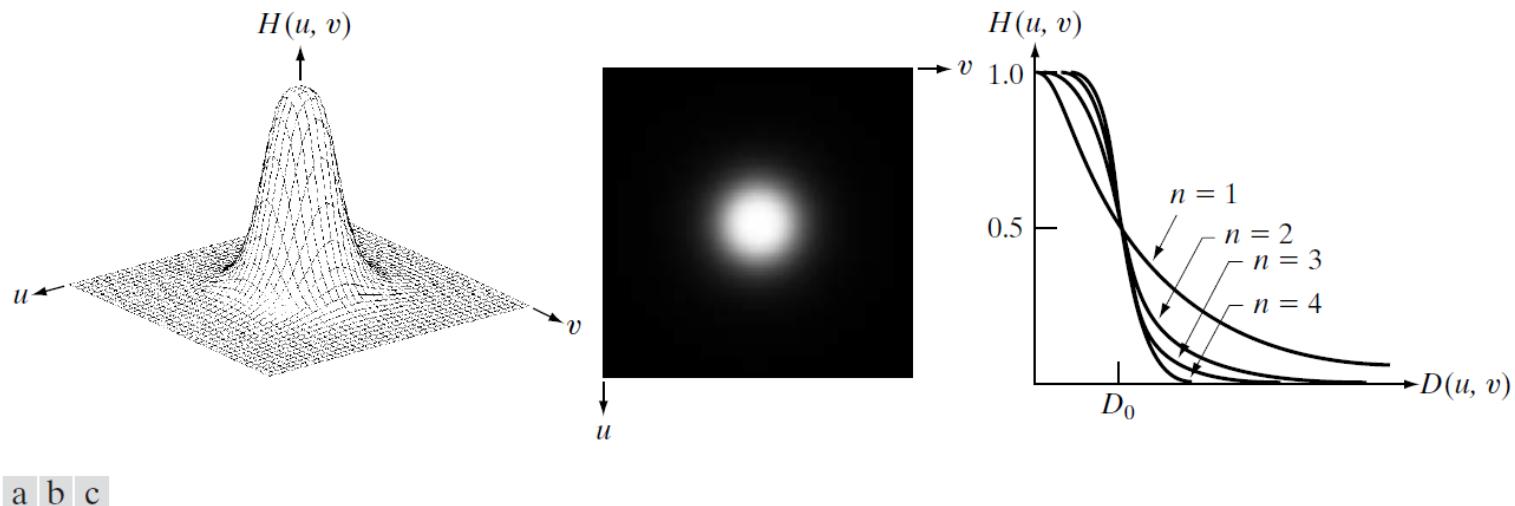
a b

**FIGURE 4.43**  
(a) Representation in the spatial domain of an ILPF of radius 5 and size  $1000 \times 1000$ .  
(b) Intensity profile of a horizontal line passing through the center of the image.

# Butterworth Lowpass Filter

Transfer function of a Butterworth lowpass filter (BLPF) of order  $n$  and with cutoff frequency at a distance  $D_0$  from the origin:

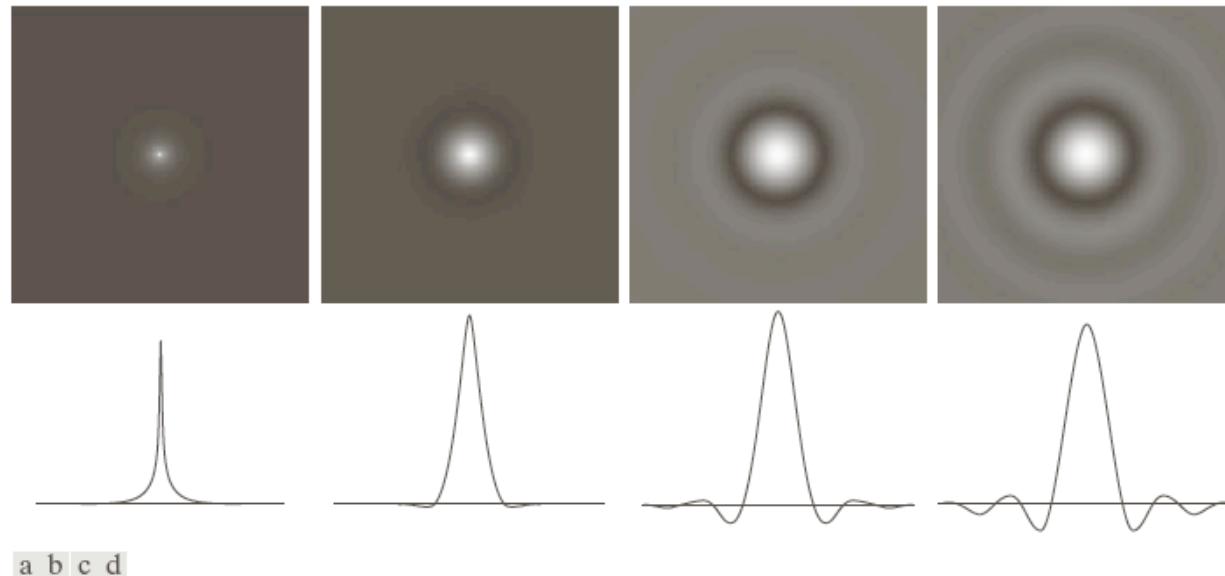
$$H(u, v) = \frac{1}{1 + [D(u, v)/D_0]^{2n}}$$



a b c

**FIGURE 4.44** (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

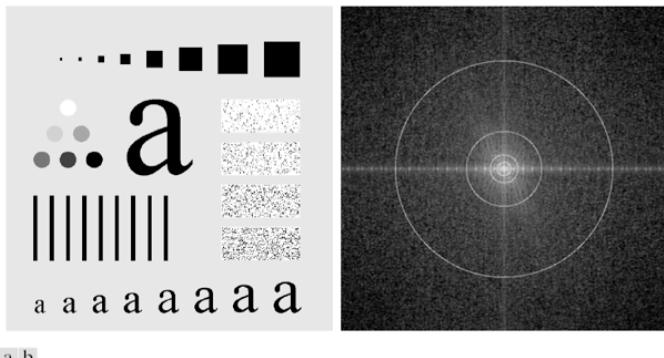
# Butterworth Lowpass Filter (Spatial)



**FIGURE 4.46** (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is  $1000 \times 1000$  and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

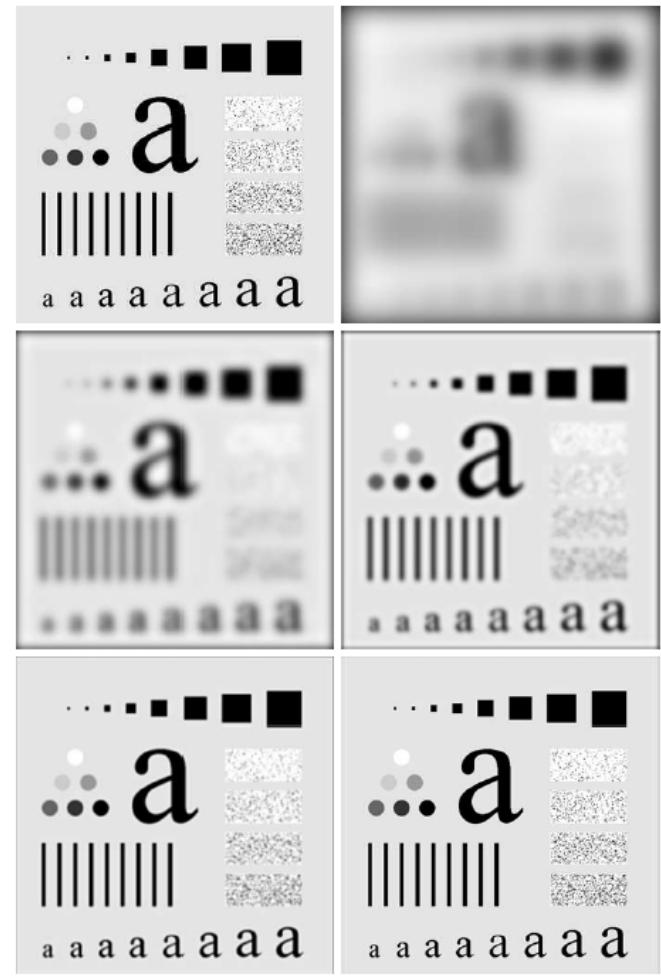
# Butterworth Lowpass Filter (Effect)

## 4.8.2 Butterworth Lowpass Filters



a b

**FIGURE 4.41** (a) Test pattern of size  $688 \times 688$  pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



a b  
c d  
e f

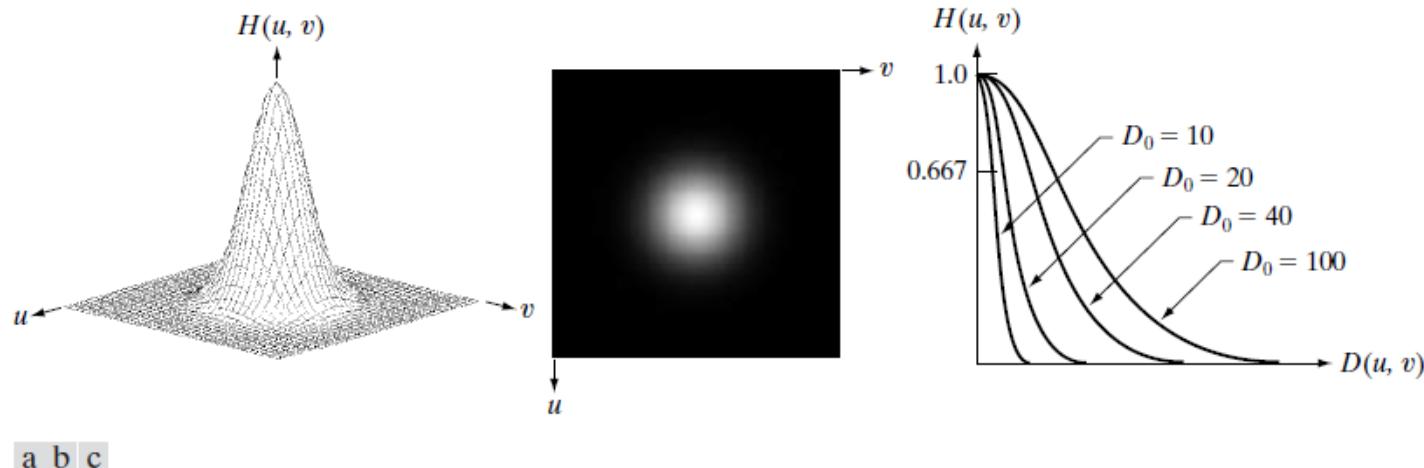
**FIGURE 4.45** (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

# Gaussian Lowpass Filter

Gaussian Lowpass Filters (GLPFs) in two-dimensions:

$$H(u, v) = e^{-D^2(u,v)/2\sigma^2} \quad (\sigma = \text{measure of spread about the centre})$$

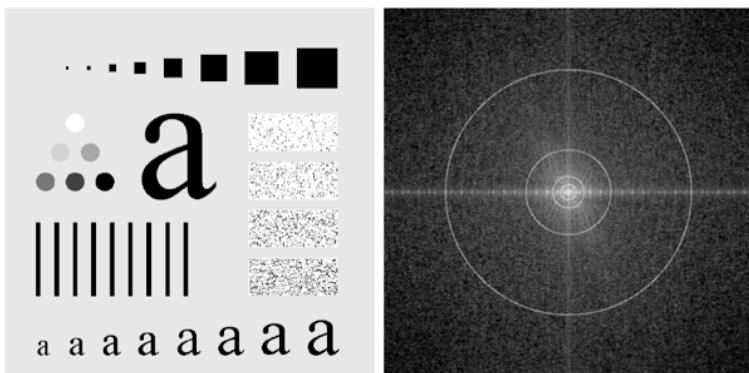
$$\sigma = D_0 \Rightarrow H(u, v) = e^{-D^2(u,v)/2D_0^2} \quad (D_0 = \text{cutoff frequency})$$



a b c

**FIGURE 4.47** (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of  $D_0$ .

# Gaussian Lowpass Filter (Effect)



a

b

**FIGURE 4.41** (a) Test pattern of size  $688 \times 688$  pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



**FIGURE 4.48** (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

# Highpass Filter

*Highpass filtering:* attenuation of the high-frequency components of the Fourier transform of the image

As before :

- Radially symmetric filters
- All filter functions are assumed to be discrete functions of size  $P \times Q$

A highpass  $H_{HP}$  filter can be obtained from a given lowpass  $H_{LP}$  filter by:

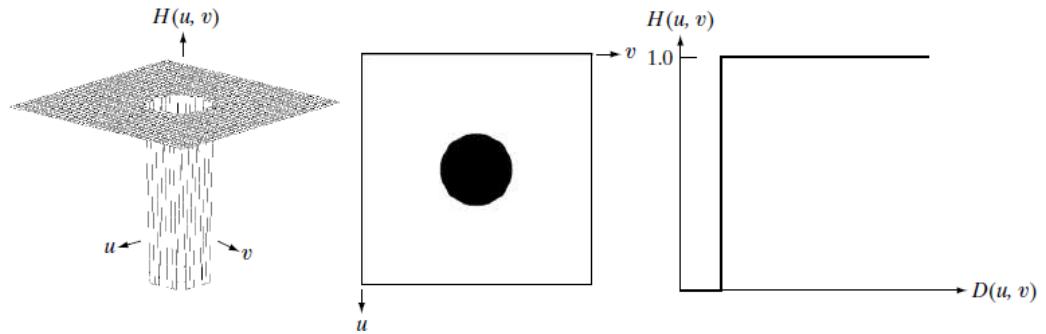
$$H_{HP}(u, v) = 1 - H_{LP}(u, v)$$

A 2-D *Ideal Highpass Filter* (IHPF) is defined as:

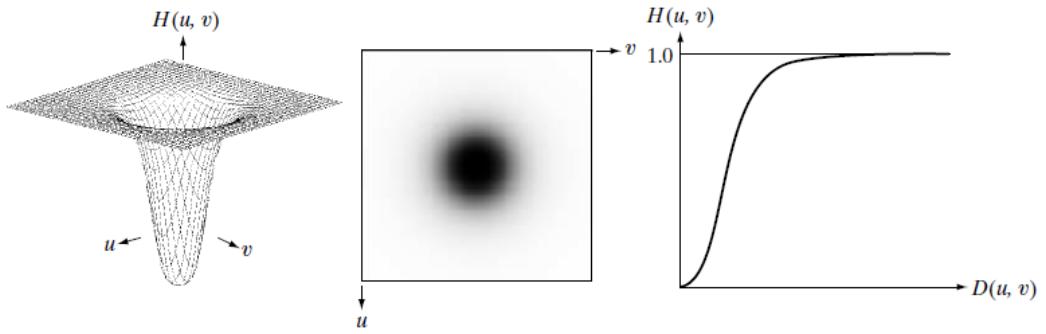
$$H(u, v) = \begin{cases} 0 & \text{if } D(u, v) \leq D_0 \\ 1 & \text{if } D(u, v) > D_0 \end{cases}$$

# Highpass Filters (in Frequency Domain)

Ideal Highpass Filter:



Butterworth Highpass Filter



Gaussian Highpass Filter:

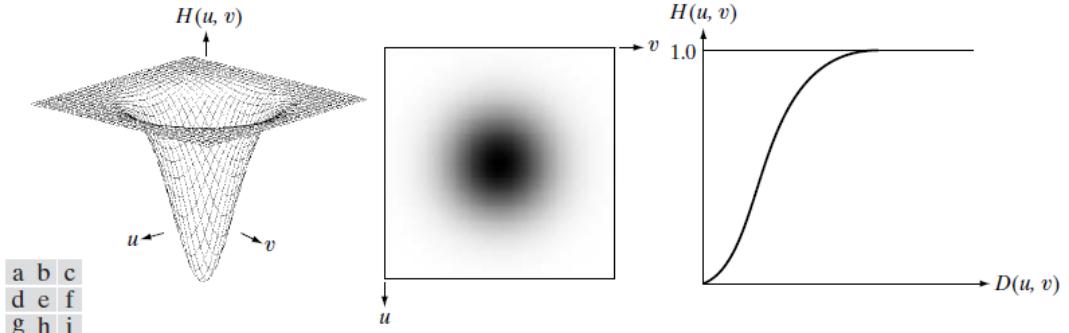
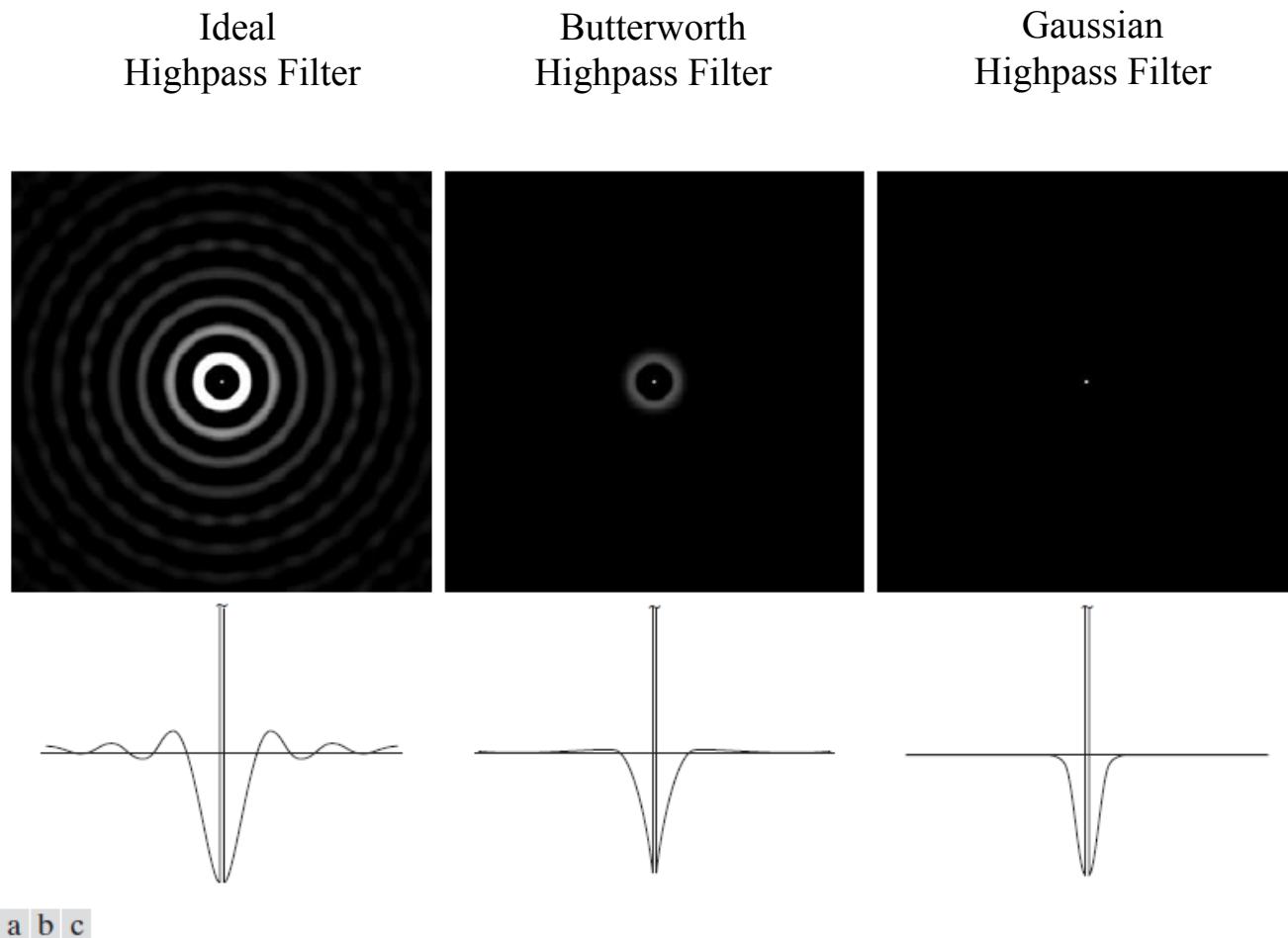


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

a b c  
d e f  
g h i

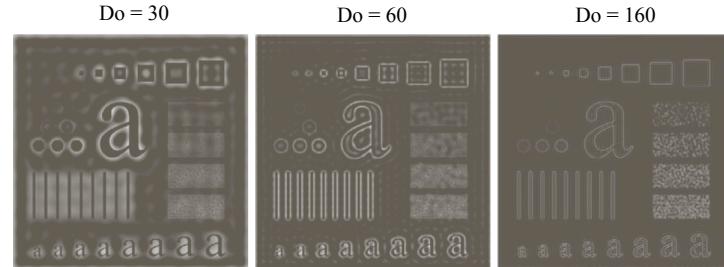
# Highpass Filters (in Spatial Domain)



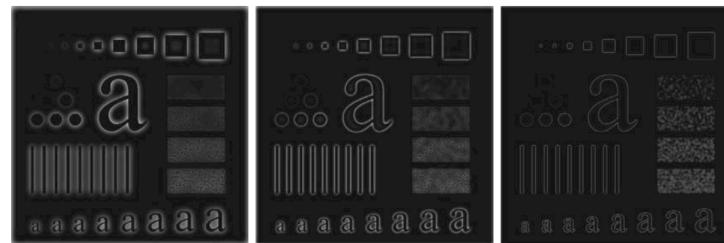
**FIGURE 4.53** Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

# Highpass Filters (Effect)

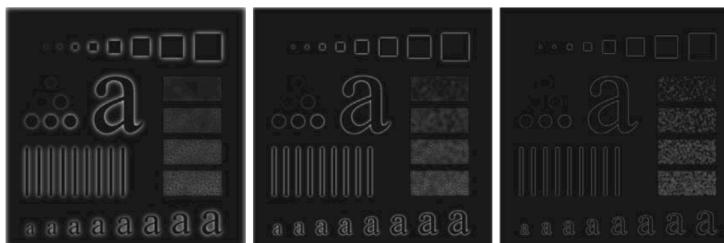
Ideal HPF



Butterfly HPF

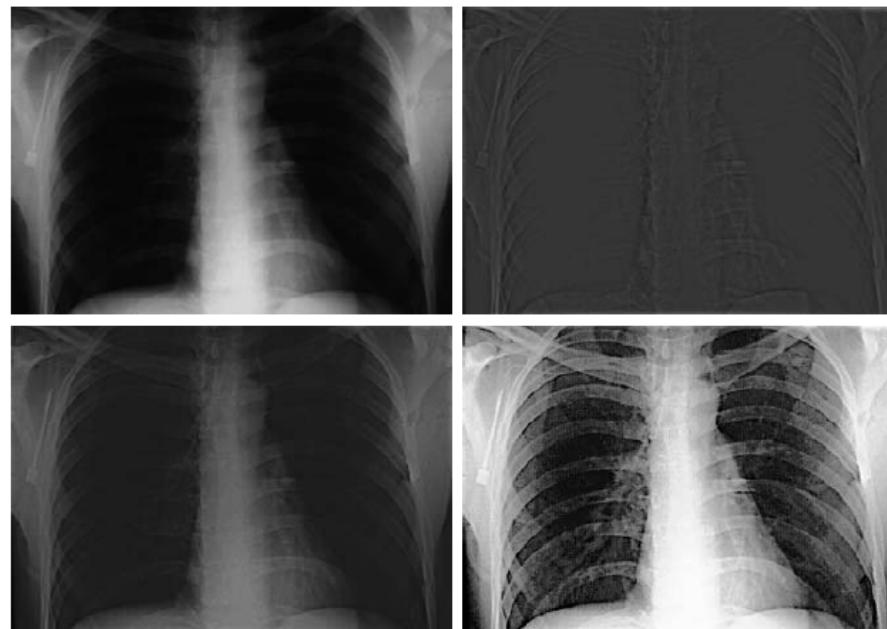


Gaussian HPF



# High Frequency Emphasis Filter

$$g(x, y) = IDFT \left\{ \underbrace{[1 + k * H_{HP}(u, v)]}_{\text{High-frequency-emphasis filter}} F(u, v) \right\}$$



High-freqe  
emphasis filt

Gaussian highpass  
filtering

Histogram  
Equalisation

a b  
c d

**FIGURE 4.59** (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)