EIE4512 - Digital Image Processing

Frequency Domain Operations

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Basics of Frequency Domain

Outline

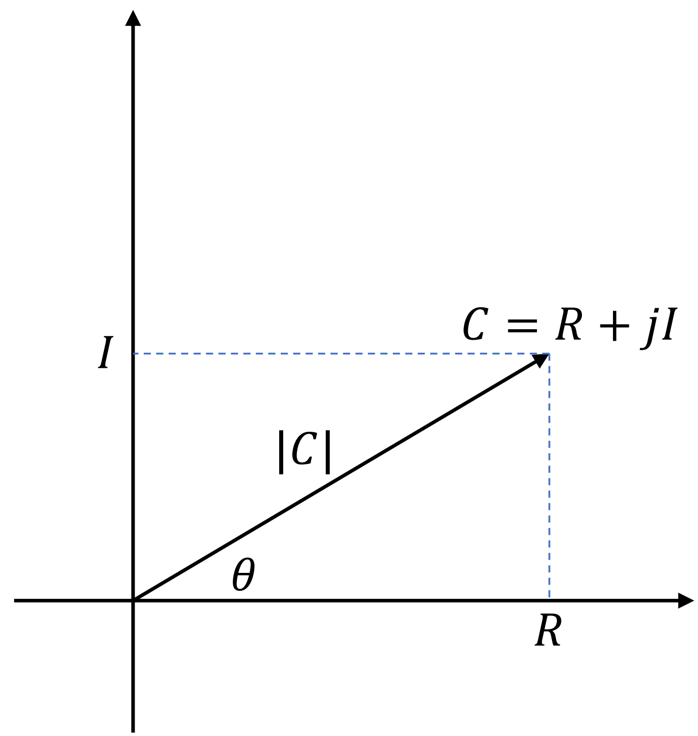
- Preliminary:
  - Complex Numbers
  - Fourier Series & Fourier Transform
  - Impulses
  - Convolution
- Sampling
  - Sampled Function
  - Sampling Theorem

Revisit of Complex Numbers

- Definition
  - $C = R + j\,I$
  - Conjugate: $C^* = R\ - j\,I$

- Polar Coordinates
  - $C = |C|(\cos\theta + j\sin\theta)$
  - $|C| = \sqrt{R^2 + I^2}$
  - $\tan\theta = I/R$

- Euler's Formula
  - $e^{j\theta} = \cos\theta + \mathrm{j}\sin\theta$
  - $C = |C|e^{j\theta}$



Fourier Series

- Any periodic function $f(t)$ with period T can be represented as a **Fourier Series**

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}}$$

- With

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j\frac{2\pi n}{T}t}dt$$

Fourier Series for real value function and conplex value function
https://www.youtube.com/watch?v=kP02nBNtjrU

Unit Impulse

Unit impulse of a continuous variable *t* located at *t=0* :

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Sifting property :

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

(if *f(t)* continuous at *t=0*)

Discrete Unit Impulse

Unit discrete impulse located at *x=0* :

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{\infty} \delta(x) = 1$$

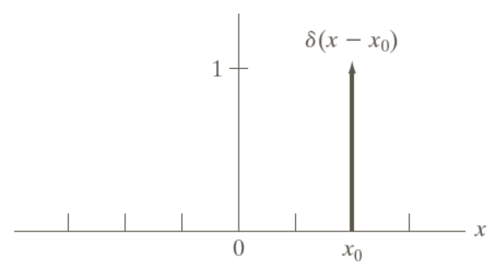


FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Sifting property :

$$\sum_{x=-\infty}^{\infty} f(x)\delta(x) = f(0) \qquad \sum_{x=-\infty}^{\infty} f(x)\delta(x - x_0) = f(x_0)$$

Impulse Train

Impulse train : sum of infinitely many periodic impulses *ΔT* units apart:

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

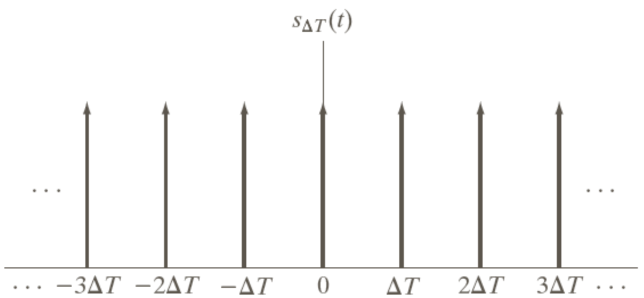


FIGURE 4.3 An impulse train.

Fourier Transform (FT) of Continuous Functions

Fourier Transform of a continuous function *f(t)* of a continuous variable *t* :

$$FT\left[f(t)\right] = \; F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt$$

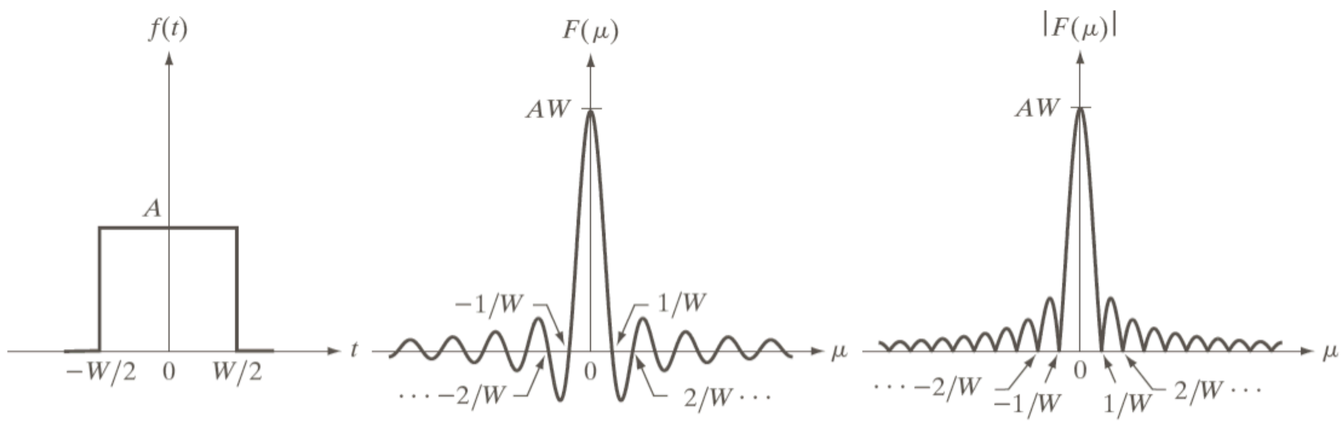
Inverse Fourier Transform:

$$FT^{-1}\left[F(\mu)\right] = \; f(t) = \int_{-\infty}^{\infty} F(\mu) e^{j2\pi\mu t} d\mu$$

FT Example

Example : $$F(\mu) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt$$

$$= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)} \qquad = AW\, sinc(\pi\mu W)$$



a b c

FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

FT of Unit Impulse

Example 2 : Fourier Transform of a unit impulse located at the origin:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi\mu t} dt \quad = e^{-j2\pi\mu 0} = e^{0} = 1$$

Fourier Transform of a unit impulse located at $t = t_0$:

$$F(\mu) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j2\pi\mu t} dt = e^{-j2\pi\mu t_0} \quad = \cos(2\pi\mu t_0) - jsin(2\pi\mu t_0)$$

FT of Impulse Train

Example 2 : Fourier Transform $S(\mu)$ of an impulse train with period ΔT :

Periodic function with period ΔT =>

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\Delta T}}$$

Where : $$c_n = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} s_{\Delta T}(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T}\int_{-\Delta T/2}^{\Delta T/2} \delta(t) e^{-j\frac{2\pi n}{\Delta T}t} dt = \frac{1}{\Delta T}$$

=> $$S(\mu) = FT\left[s_{\Delta t}(t)\right] = \frac{1}{\Delta T} FT\left[\sum_{n=-\infty}^{\infty} e^{j\frac{2\pi n}{T}}\right] = \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} \delta\left(\mu - \frac{n}{\Delta T}\right)$$

Also an *impulse train*, with period $1/\Delta T$

Continuous-Time Convolution

Convolution of functions *f(t)* and *h(t)*, of one continuous variable *t* :

$$f(t)\bigstar h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

$$FT\left\{f(t)\bigstar h(t)\right\} = H(\mu)F(\mu)$$

Fourier Transform pairs:

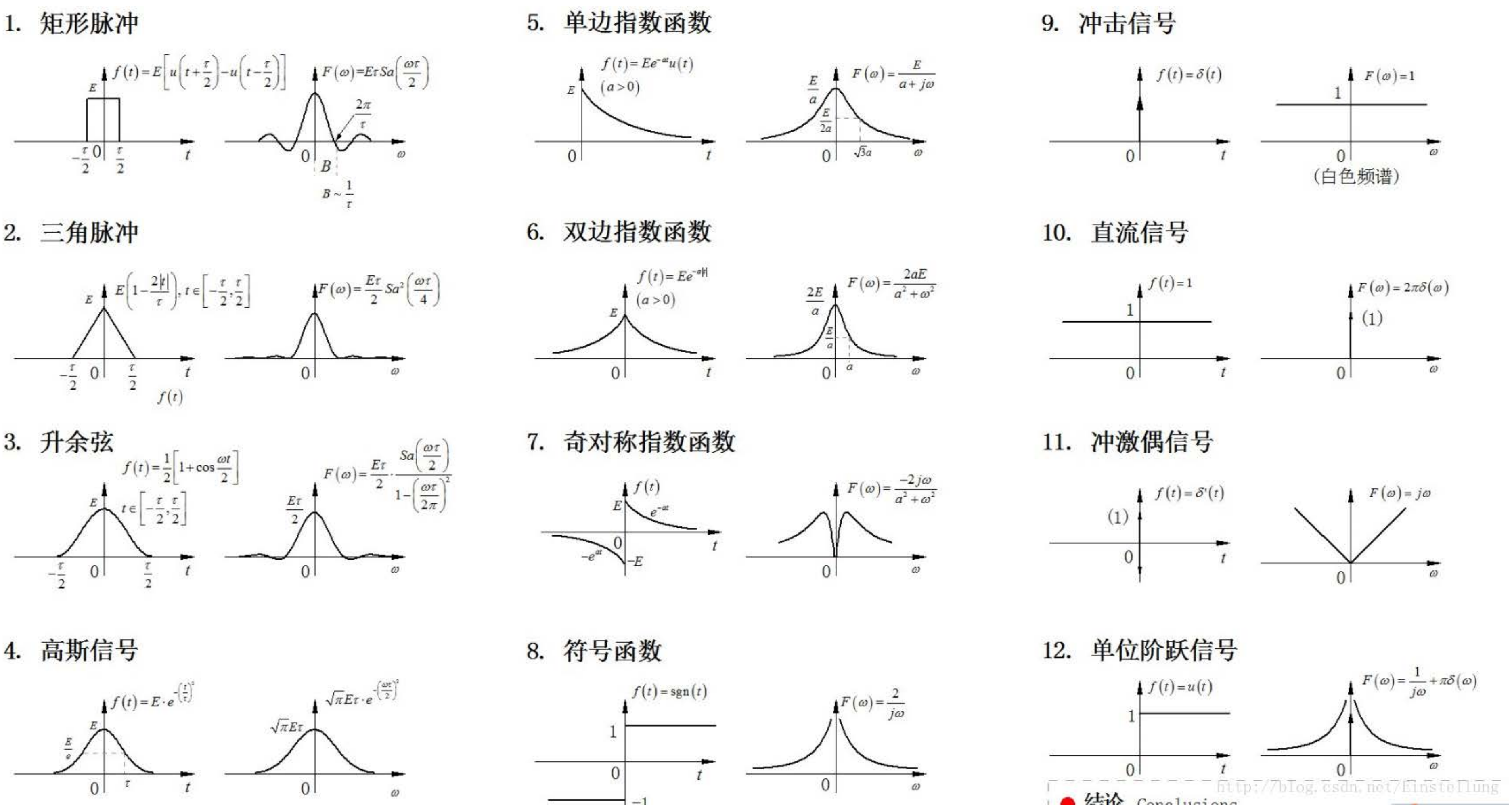
$$f(t)\bigstar h(t) \Longleftrightarrow H(\mu)F(\mu)$$

$$f(t)h(t) \Longleftrightarrow H(\mu)\bigstar F(\mu)$$

Spatial Domain Conv <=> Frequency Domain Multiplication

Spatial Domain Multiplication <=> Frequency Domain Conv

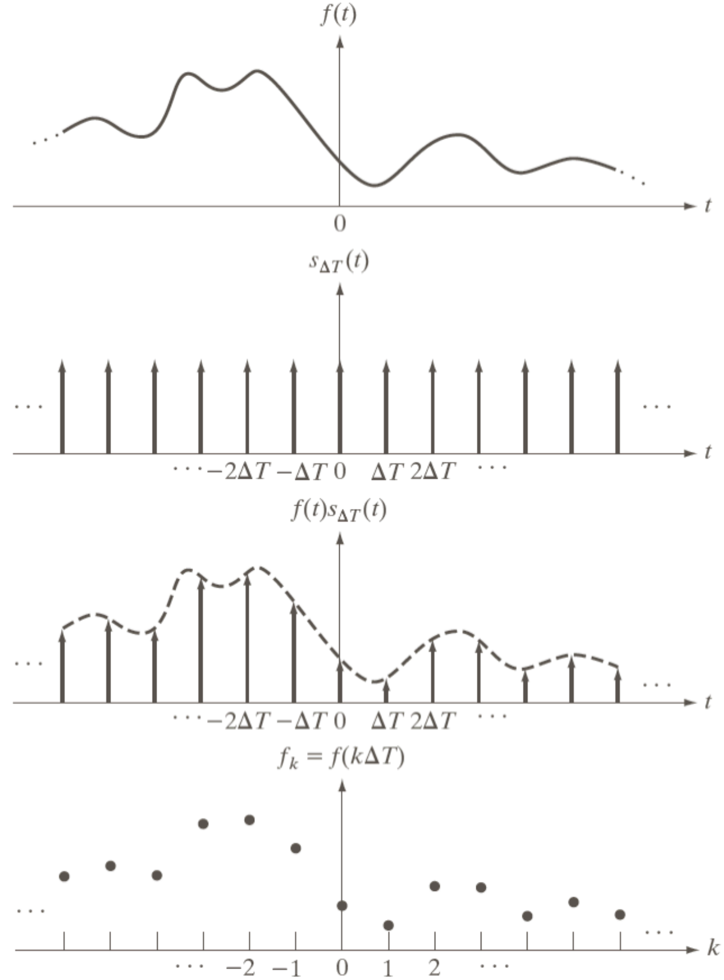
Commonly used continuous Fourier transform



Sampling

Model of the sampled function: multiplication of *f(t)* by a sampling function equal to a train of impulses *ΔT* units apart

$$\tilde{f}(t) = f(t)s_{\Delta T} = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$



a
b
c
d

FIGURE 4.5 (a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

FT of Sampled Function

$$\begin{aligned}\tilde{F}(\mu) &= FT\left[\tilde{f}(t)\right] \\ &= FT\left[f(t)\cdot s_{\Delta T}(t)\right] \\ &= F(\mu)\star S(\mu) \\ &= \int_{-\infty}^{\infty} F(\tau)S(\mu-\tau)d\tau \\ &= \frac{1}{\Delta T}\int_{-\infty}^{\infty} F(\tau)\sum_{n=-\infty}^{\infty}\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)d\tau \\ &= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\tau)\delta\left(\mu-\tau-\frac{n}{\Delta T}\right)d\tau \\ &= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu-\frac{n}{\Delta T}\right)\end{aligned}$$

Superposition of infinitely many copies of $F(\mu)$ with an interval $1/\Delta T$

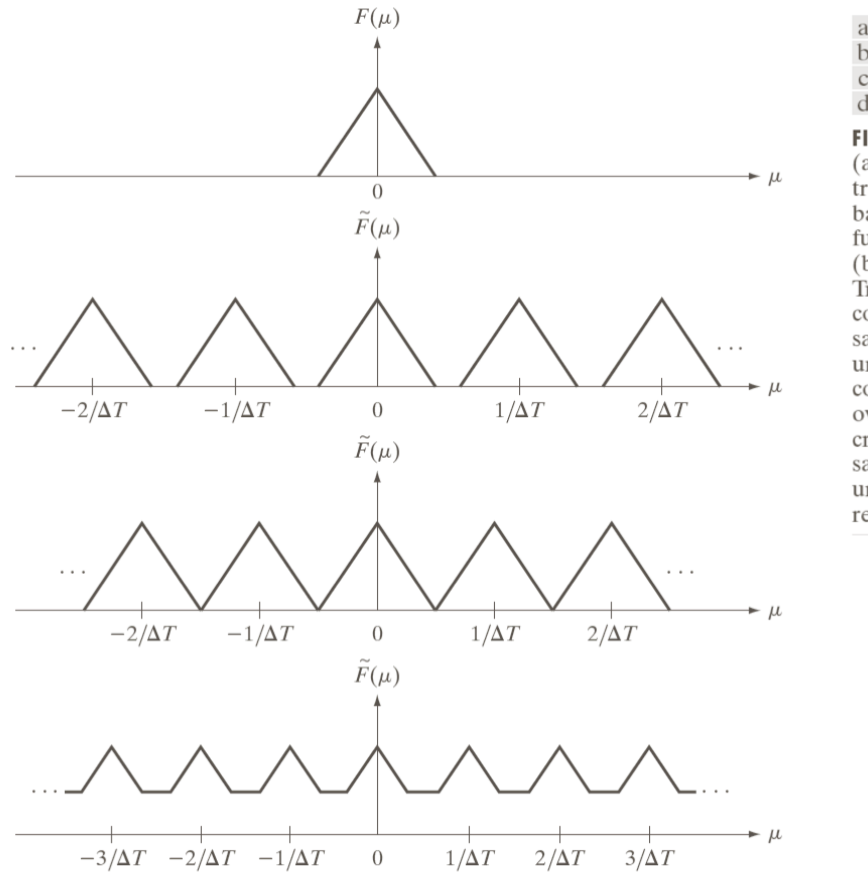
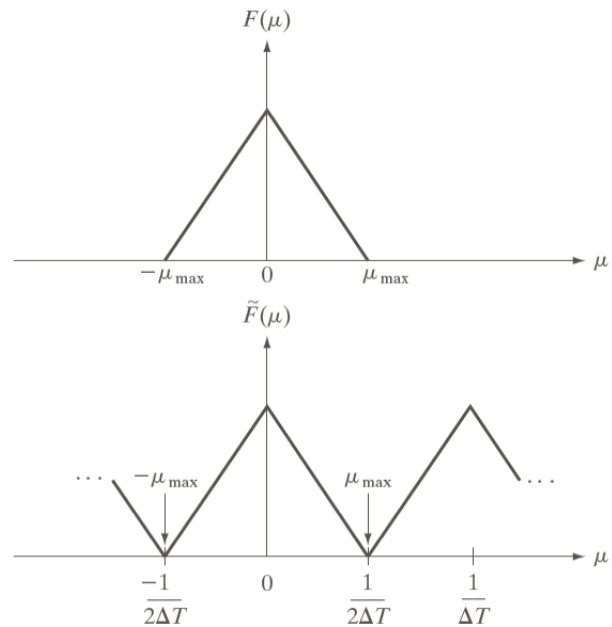


FIGURE 4.6 (a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

Sampling Theorem

A function *f(t)* whose Fourier transform is zero for values of frequencies outside a finite interval (band) [- μ_{max}, μ_{max}] about the origin is called a *band-limited* function



a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

Sufficient separation guaranteed if:

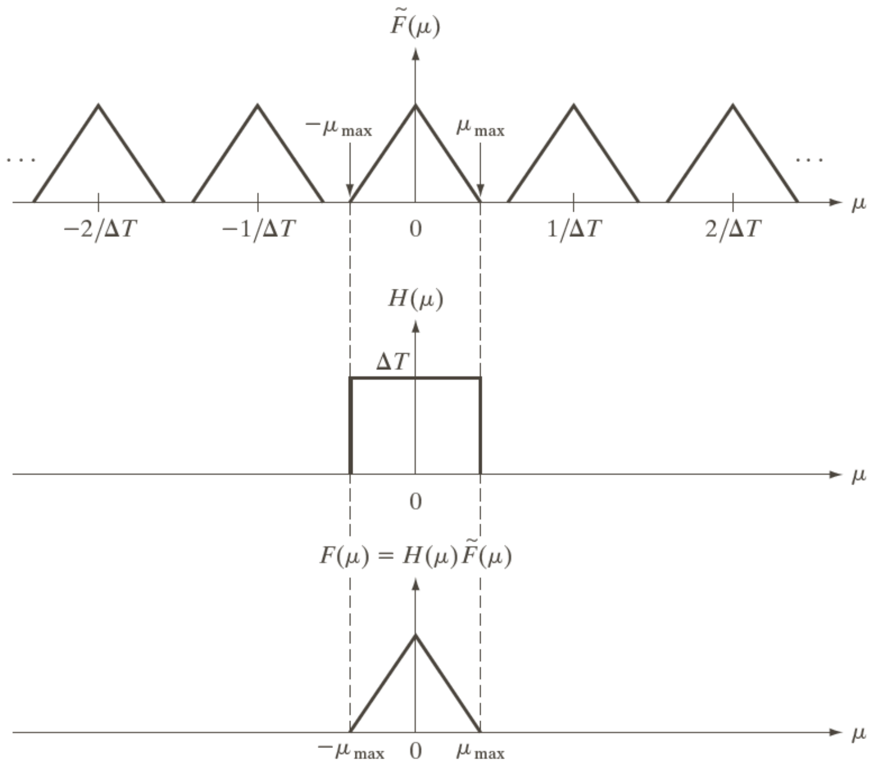
$$\frac{1}{\Delta T} > 2\mu_{max}$$

Sampling Theorem:

A continuous, band-limited function can be recovered completely from a set of its samples if the samples are acquired at a rate exceeding twice the highest frequency content of the function

NB: Sampling at: $\frac{1}{\Delta T} = 2\mu_{max}$ *Nyquist rate*

Recovery with Low-pass Filter



a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.

Example: ideal lowpass filter

=> Recover *f(t)* using the inverse FT:

$$f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

Aliasing

Effect of under-sampling a function

Transform corrupted by frequencies from adjacent periods

NB: the effect of aliasing can be reduced by smoothing the input function to attenuate its higher frequencies: *anti-aliasing.*

Has to be done *before* the sampling

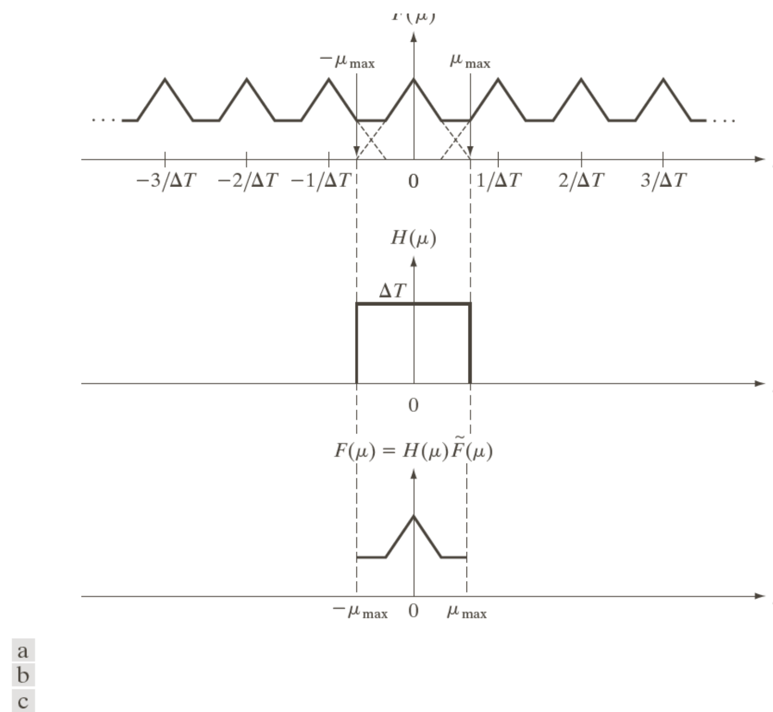


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Aliasing (Illustration)

Illustration: sampling a sine wave of period 2s

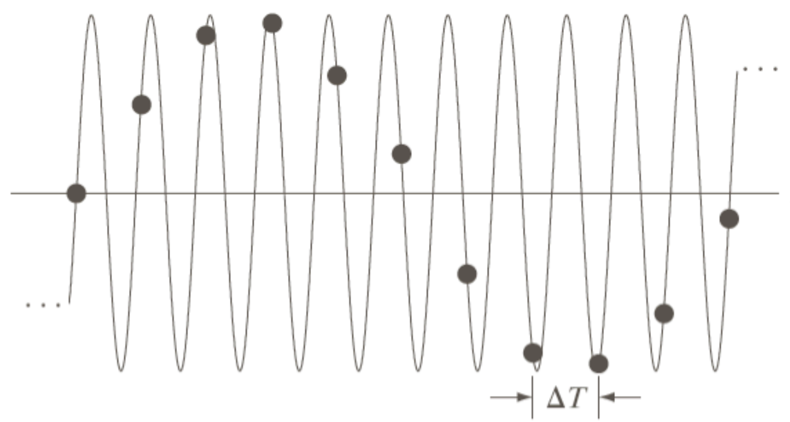


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Function Reconstruction

$$f(t) = FT^{-1}\left[\tilde{F}(\mu)H(\mu)\right]$$
$$= \sum_{n=-\infty}^{\infty} f(n\Delta T) \cdot \mathrm{sinc}\left[\frac{1}{\Delta T}(t - n\Delta T)\right]$$

https://blog.csdn.net/u010592995/article/details/73555425
https://blog.csdn.net/liyuanbhu/article/details/42061089

Discrete Fourier Transform

FT of a sampled function

$$
\begin{aligned}
\tilde{F}(\mu) &= \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\delta(t - n\Delta T) e^{-j2\pi\mu t} dt \\
&= \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi\mu n\Delta T}
\end{aligned}
$$

Consider a finite sequence of samples

DFT: $$F_m = \tilde{F}\left(\frac{m}{M\Delta T}\right) = \sum_{n=0}^{M-1} f_n e^{-j2\pi un/M}$$

DFT and IDFT

Discrete Fourier Transform (DFT)

$$F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}$$

Inverse Discrete Fourier Transform (IDFT)

$$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M}$$

Relation between Sampling & Frequency Interval

- Consider M samples of f, with interval ΔT
  - The duration of the record is
    - $\mathrm{T} = M \cdot \Delta T$
  - The entire frequency range is
    - $\frac{1}{\Delta T}$
  - The spacing in the frequency domain
    - $\Delta\mu = \frac{1}{\mathrm{M}\,\Delta T} = \frac{1}{T}$

Circular Convolution

Both forward and inverse discrete transforms are infinitely periodic, with period *M*

$$F(u) = F(u + kM)$$

$$f(x) = f(x + kM)$$

Where *k* is an integer

Discrete equivalent of the convolution:

$$f(x) \bigstar h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

Extension to Function of Two Variables

Impulse of two continuous variables *t* and *z* :

$$\delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z)\, dt\, dz = 1$$

Impulse of two continuous variables *t* and *z* :

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\delta(t, z)\, dt\, dz = f(0, 0)$$

Sifting property: $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\delta(t, z)\, dt\, dz = f(0, 0)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z)\delta(t - t_0, z - z_0)\, dt\, dz = f(t_0, z_0)$$

Extension of Discrete Impulse

2-D discrete impulse:
$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y = 0 \\ 0 & \text{otherwise} \end{cases}$$

Sifting property:
$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x,y) = f(0,0)$$

$$\sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f(x,y)\delta(x-x_0, y-y_0) = f(x_0, y_0)$$

Extension of Fourier Transform

f(t,z) continuous function of two continuous variables *t* and *z*, the 2-D continuous Fourier transform pair is given by:

$$F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} \, dt \, dz$$

and

$$f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} \, d\mu \, d\nu$$

2D FT: Example

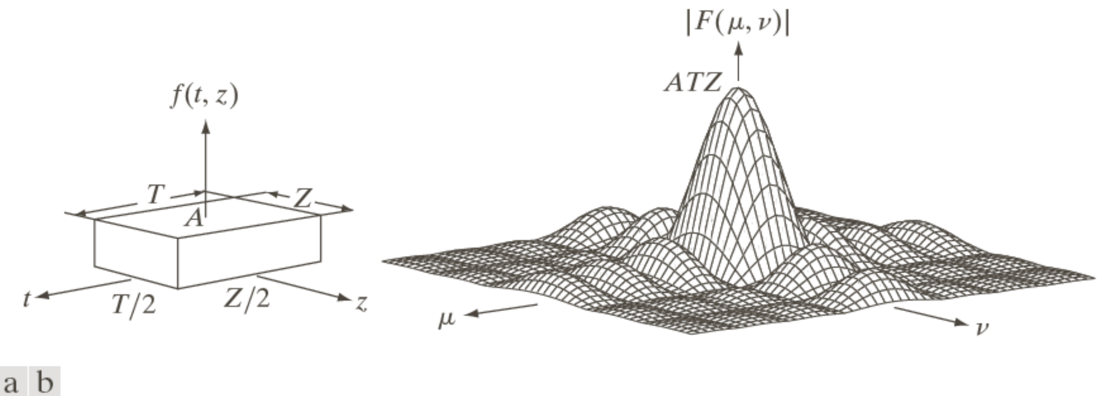


FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

$$F(\mu,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,z)e^{-j2\pi(\mu t+\nu z)}\,dt\,dz$$

$$= ATZ\left[\frac{\sin(\pi\mu T)}{(\pi\mu T)}\right]\left[\frac{\sin(\pi\nu Z)}{(\pi\nu Z)}\right]$$

2D Sampling

Sampling in 2-D can be modeled using the sampling function (2-D impulse train):

$$s_{\Delta T \Delta Z}(t, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(t - m\Delta T, z - n\Delta Z)$$

Multiplying *f(t,z)* by $s_{\Delta T \Delta Z}(t, z)$ yields the sampled function

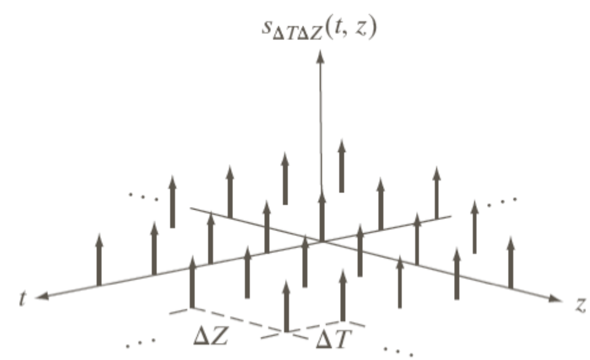


FIGURE 4.14 Two-dimensional impulse train.

2D Sampling Theorem

f(t,z) is said to be *band-limited* if its Fourier transform is 0 outside of a rectangle $[-\mu_{\max}, \mu_{\max}]$ and $[-\nu_{\max}, \nu_{\max}]$

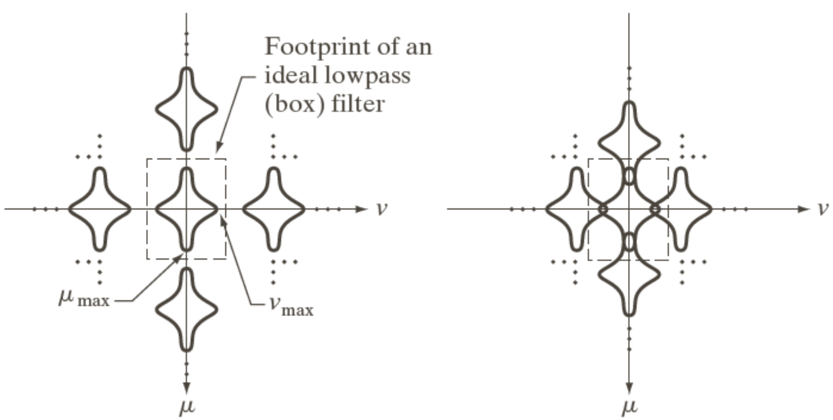
2-D sampling theorem:

A continuous, band-limited function *f(t,z)* can be recovered with no error from a set of its sample if the sampling intervals are

$$\Delta T < \frac{1}{2\mu_{max}}$$

and

$$\Delta Z < \frac{1}{2\nu_{max}}$$



a b

FIGURE 4.15 Two-dimensional Fourier transforms of (a) an over-sampled, and (b) under-sampled band-limited function.

2D Discrete Fourier Transform

2-D *Discrete Fourier Transform* (DFT) of a digital image *f(x,y)* of size *M*x*N*:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

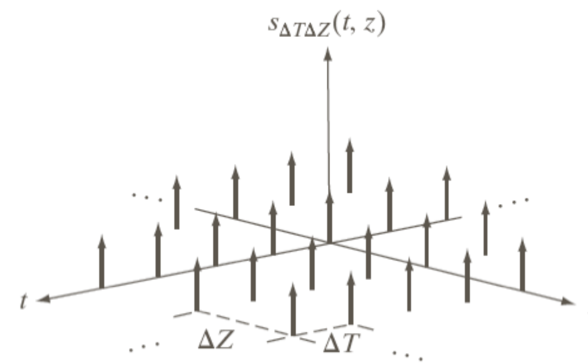
Inverse Discrete Fourier Transform (IDFT):

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

Relations between Spatial & Frequency Domains

If ΔT and ΔZ are the separations between samples, the separations between the corresponding discrete, frequency domain variables are:

$$\Delta u = \frac{1}{M\Delta T} \quad \text{and} \quad \Delta v = \frac{1}{N\Delta Z}$$



Filtering
in Frequency Domain

2D Convolution

2-D *circular convolution* :

$$f(x,y)\bigstar h(x,y)\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$$

2-D convolution theorem :

$$f(x,y)\bigstar h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

and conversely:

$$f(x,y)h(x,y) \Leftrightarrow F(u,v)\bigstar H(u,v)$$

Wraparound Error & Padding

Data from adjacent periods produce wraparound error
=> Need to append zeros to both functions

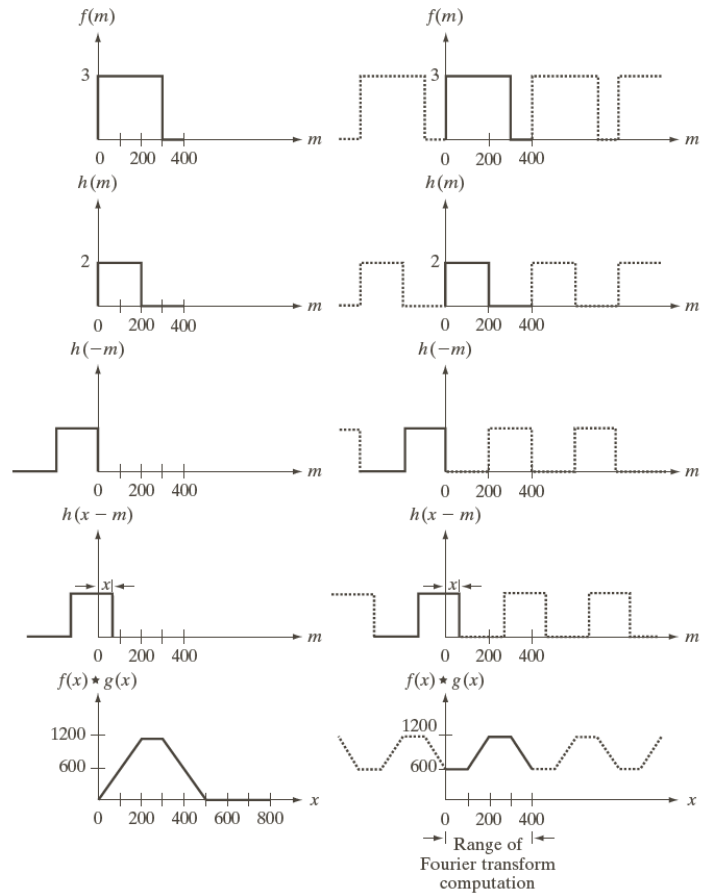
In 2-D, padd the two images array of size MxN by zeros

Padded images of size PxQ, with:

$$P \geq 2M\text{-}1$$

and

$$Q \geq 2N\text{-}1$$



a f
b g
c h
d i
e j

FIGURE 4.28 Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

Frequency Domain Filtering: Fundamentals

Given a digital image *f(x,y)* of size *M*x*N*, the basic filtering equation is:

$$g(x,y) = IDFT\left[H(u,v)F(u,v)\right]$$

NB: product = array multiplication

Filtered (output) image

Filter function
(or filter transfer
function)

DFT if the input image

NB: For simplification, use functions *H(u,v)* that are centered symmetric about their centre
+ centre *F(u,v)* multiplying *f(x,y)* by $(-1)^{x+y}$

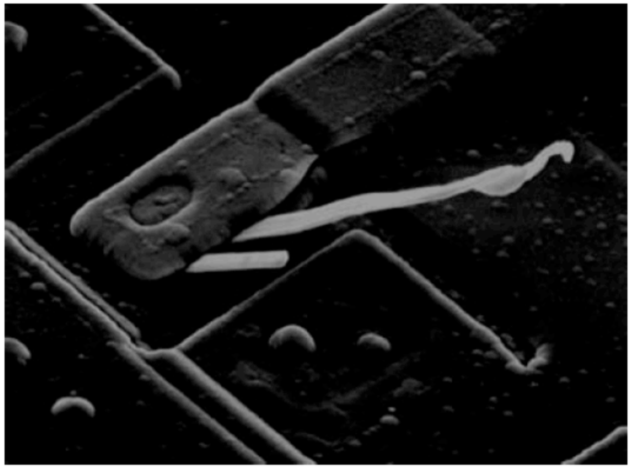


FIGURE 4.30 Result of filtering the image in Fig. 4.29(a) by setting to 0 the term $F(M/2, N/2)$ in the Fourier transform.

Shifting 2D Frequency Spectrum

$$f(x) \Leftrightarrow F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$

$$f(x) e^{j2\pi(u_0 x/M)} \Leftrightarrow F(u - u_0) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi(u-u_0)x/M}$$

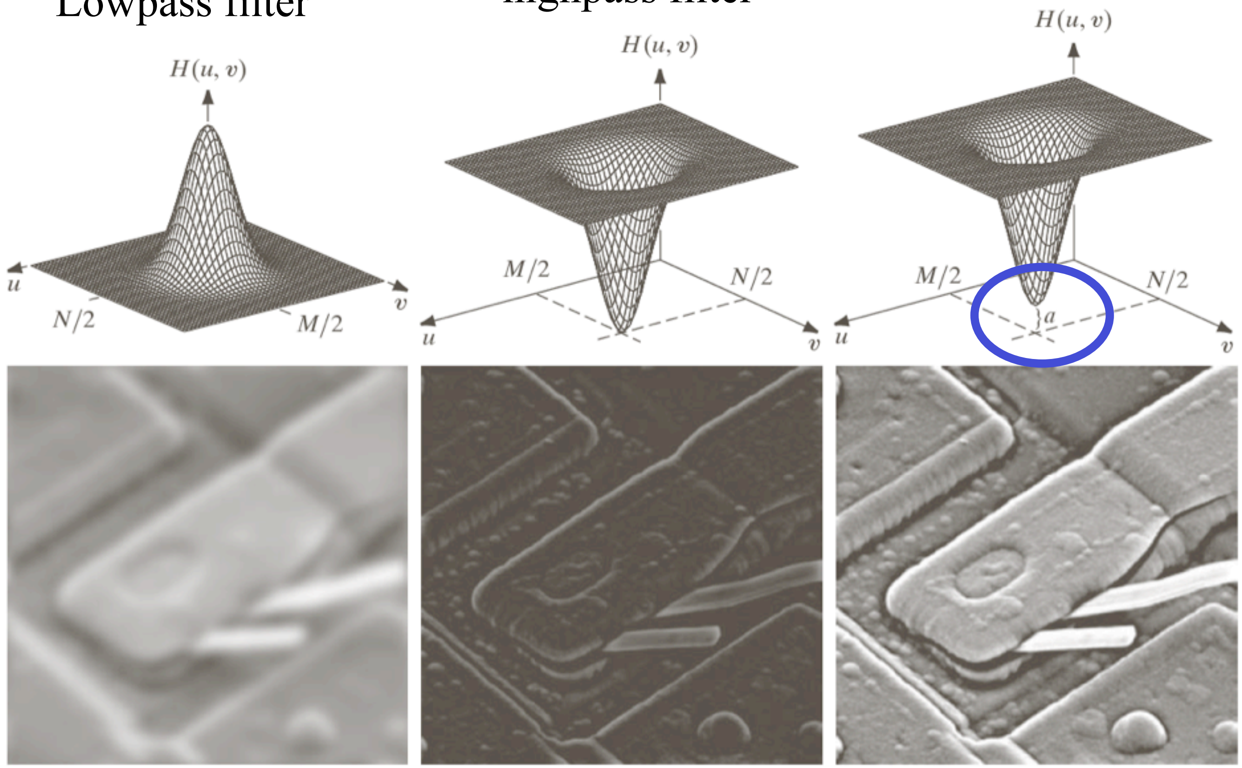
$$f(x)(-1)^x \Leftrightarrow F(u - M/2) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi(u-M/2)x/M}$$

Extension to 2D:

$$f(x,y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$

Filtering in Frequency Domain

Lowpass filter highpass filter



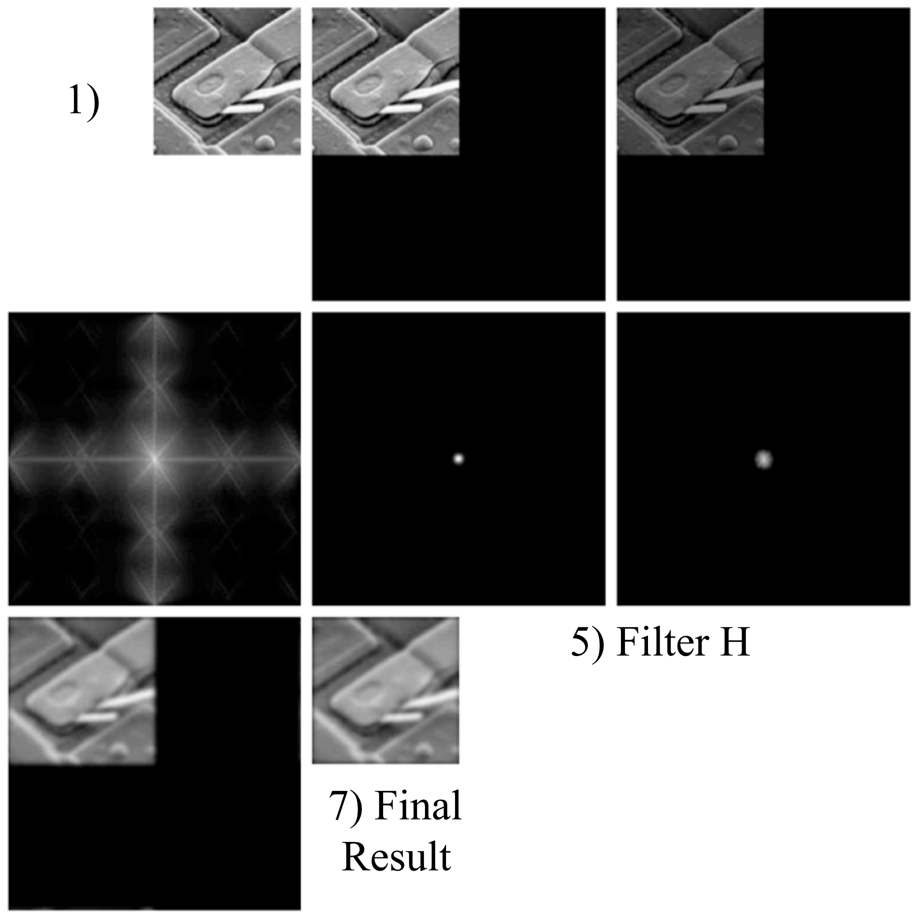
a b c
d e f

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Steps for Frequency Domain Filtering

1. Given an input image $f(x,y)$ of size MxN, obtain padding parameters P and Q. Typically, $P=2M$ and $Q=2N$.
2. Form a padded image $f_p(x,y)$ of size PxQ by appending the necessary number of zeros to f(x,y).
3. Multiply $f_p(x,y)$ by $(-1)^{x+y}$ to centre its transform.
4. Compute the DFT, $F(u,v)$, of the image from step 3.
5. Generate a real, symmetric filter function, $H(u,v)$, of size PxQ with centre at coordinates $(P/2,Q/2)$. Form the product $G(u,v)=H(u,v)F(u,v)$ using array multiplication.
6. Obtain the processed image: $g_p(x,y) = \text{real}\left[IDFT\left[G(u,v)\right]\right](-1)^{x+y}$
7. Obtain the final processed result, $g(x,y)$, by extracting the MxN region from the top, left quadrant of $g_p(x,y)$

Steps for Frequency Domain Filtering



a b c
d e f
g h

FIGURE 4.36 (a) An $M \times N$ image, f. (b) Padded image, f_p of size $P \times Q$. (c) Result of multiplying f_p by $(-1)^{x+y}$. (d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$. (f) Spectrum of the product HF_p. (g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p. (h) Final result, g, obtained by cropping the first M rows and N columns of g_p.

Ideal Lowpass Filter

Ideal Lowpass Filter (ILPF) = 2-D lowpass filter that passes without attenuation all frequencies within a circle of radius D_0 from the origin and "cuts off" all frequencies outside this circle

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

Where: $D_0 \ge 0$

And $D(u,v)$ is the distance between a point (u,v) and the centre of the frequency rectangle:

$$D(u,v) = \left[(u - P/2)^2 + (v - Q/2)^2\right]^{1/2}$$

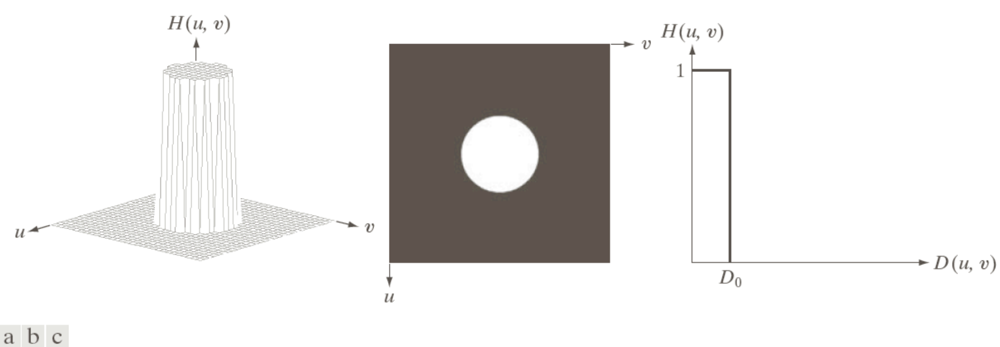
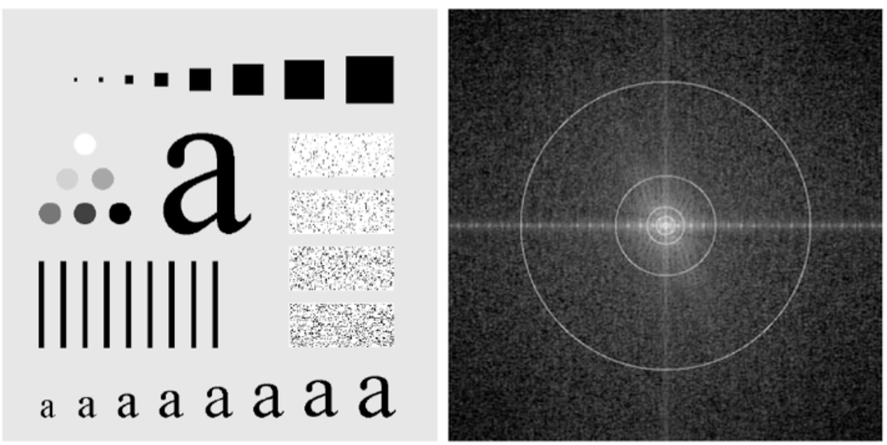


FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

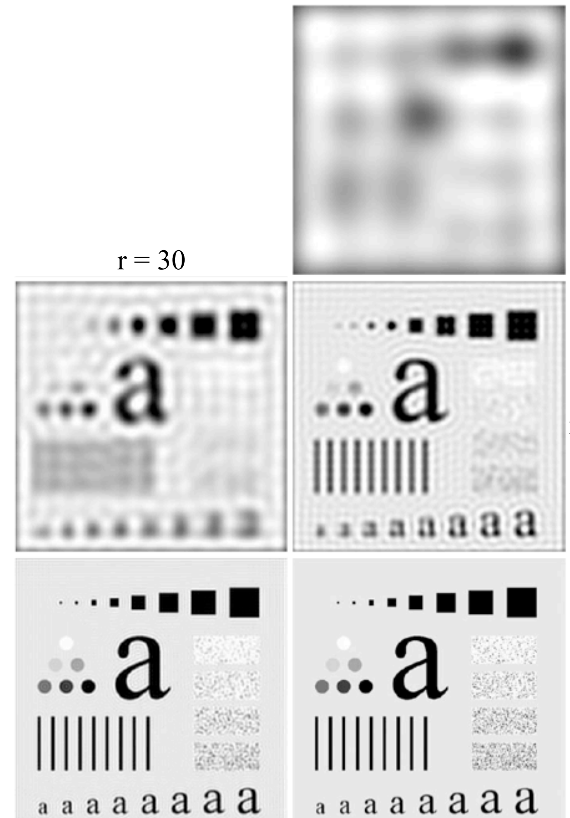
Ideal Lowpass Filter (Effect)

Example:



a b

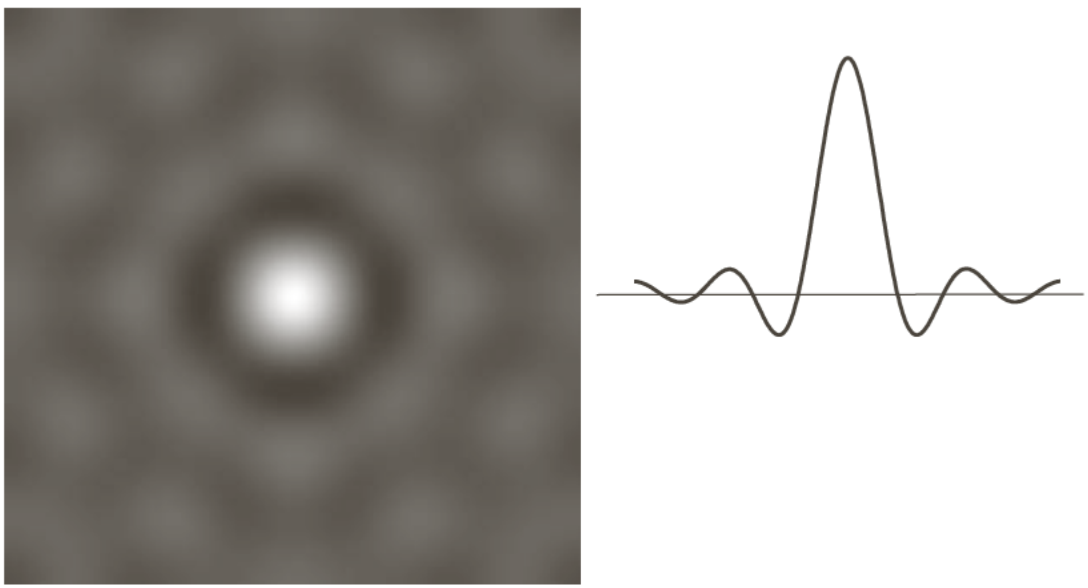
FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Ideal Lowpass: Blurring and Ringing



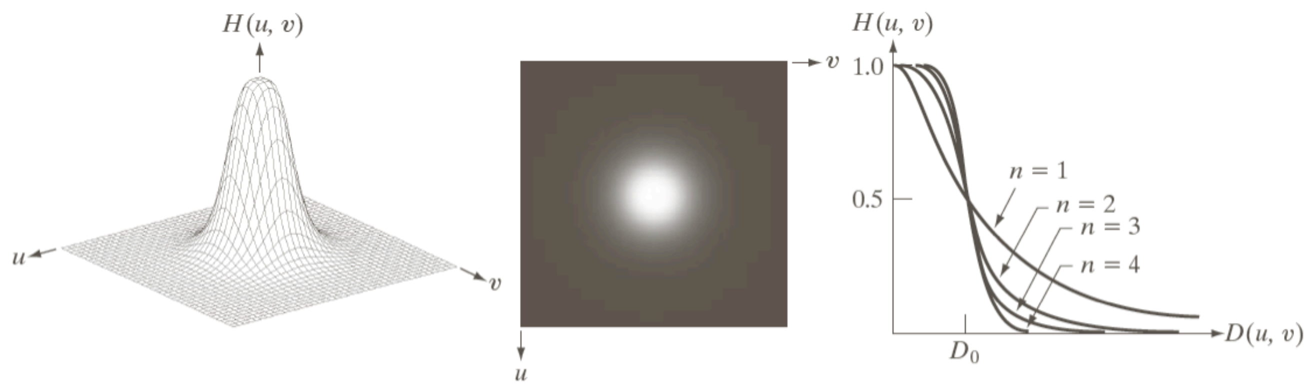
a b

FIGURE 4.43 (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000×1000. (b) Intensity profile of a horizontal line passing through the center of the image.

Butterworth Lowpass Filter

Transfer function of a Butterworth lowpass filter (BLPF) of order n and with cutoff frequency at a distance D_0 from the origin:

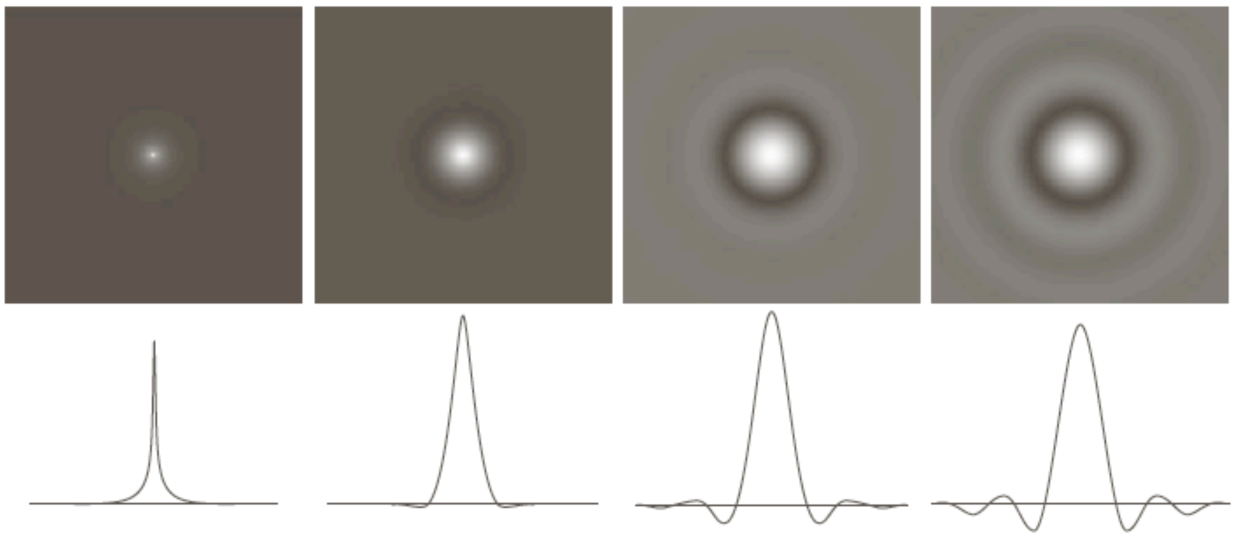
$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$



a b c

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth Lowpass Filter (Spatial)

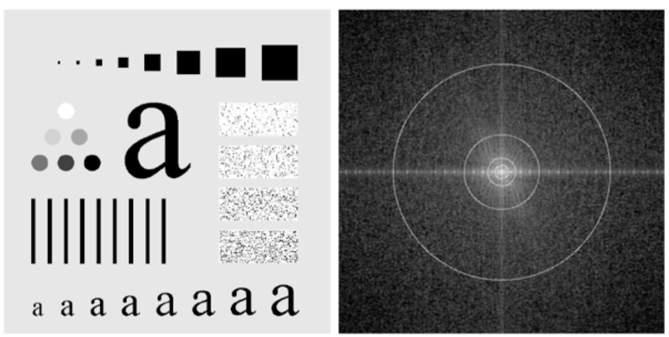


a b c d

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000×1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

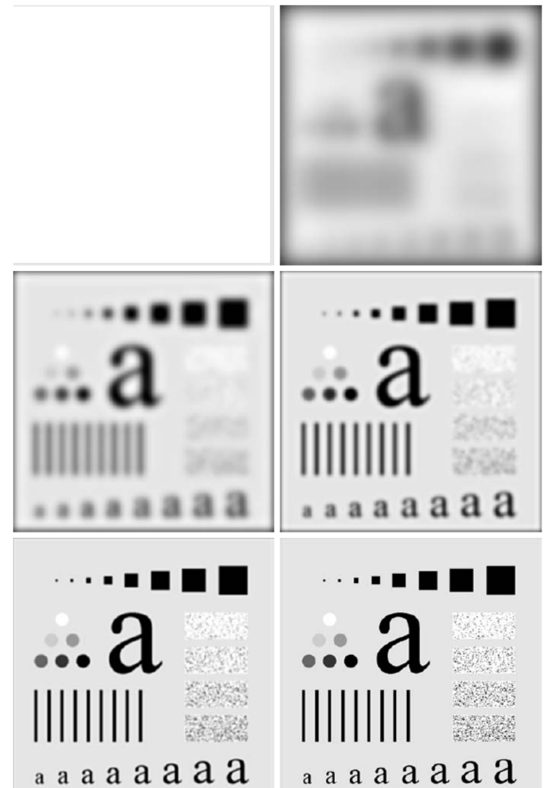
Butterworth Lowpass Filter (Effect)

4.8.2 Butterworth Lowpass Filters



a b

FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



a b
c d
e f

FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

Gaussian Lowpass Filter

Gaussian Lowpass Filters (GLPFs) in two-dimensions:

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

(σ = measure of spread about the centre)

$$\sigma = D_0 \;\Rightarrow\; H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

(D_0 = cutoff frequency)

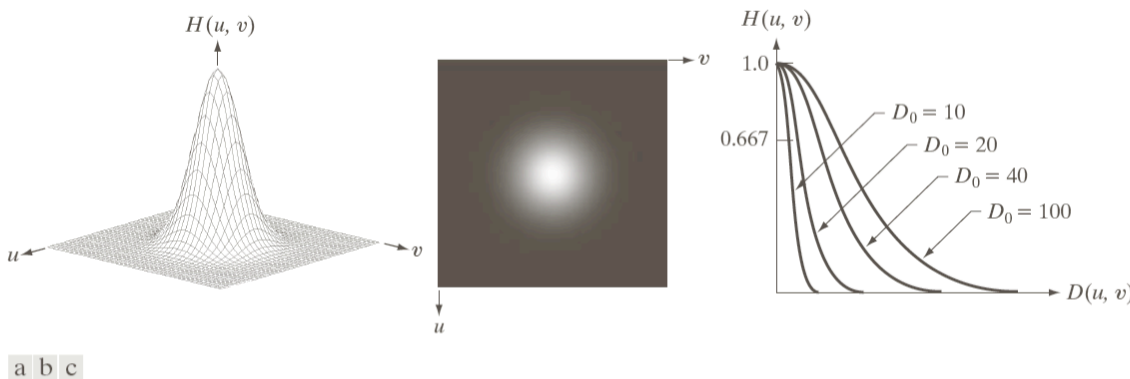
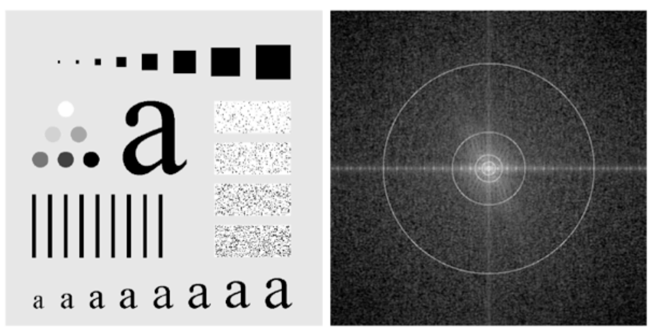


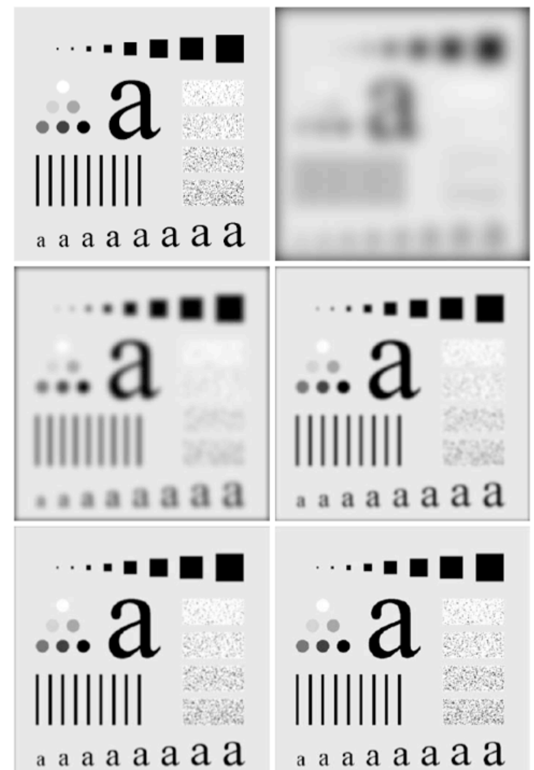
FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Gaussian Lowpass Filter (Effect)



a b

FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



a b
c d
e f

FIGURE 4.48 (a) Original image. (b)–(f) Results of filtering using GLPFs with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Figs. 4.42 and 4.45.

Highpass Filter

Highpass filtering: attenuation of the high-frequency components of the Fourier transform of the image
As before :
- Radially symmetric filters
- All filter functions are assumed to be discrete functions of size *P*x*Q*

A highpass H_{HP} filter can be obtained from a given lowpass H_{LP} filter by:

$$H_{HP}(u,v) = 1 - H_{LP}(u,v)$$

A 2-D *Ideal Highpass Filter* (IHPF) is defined as:

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \leq D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Highpass Filters (in Frequency Domain)

Ideal Highpass Filter:

Butterworth Highpass Filter:

Gaussian Highpass Filter:

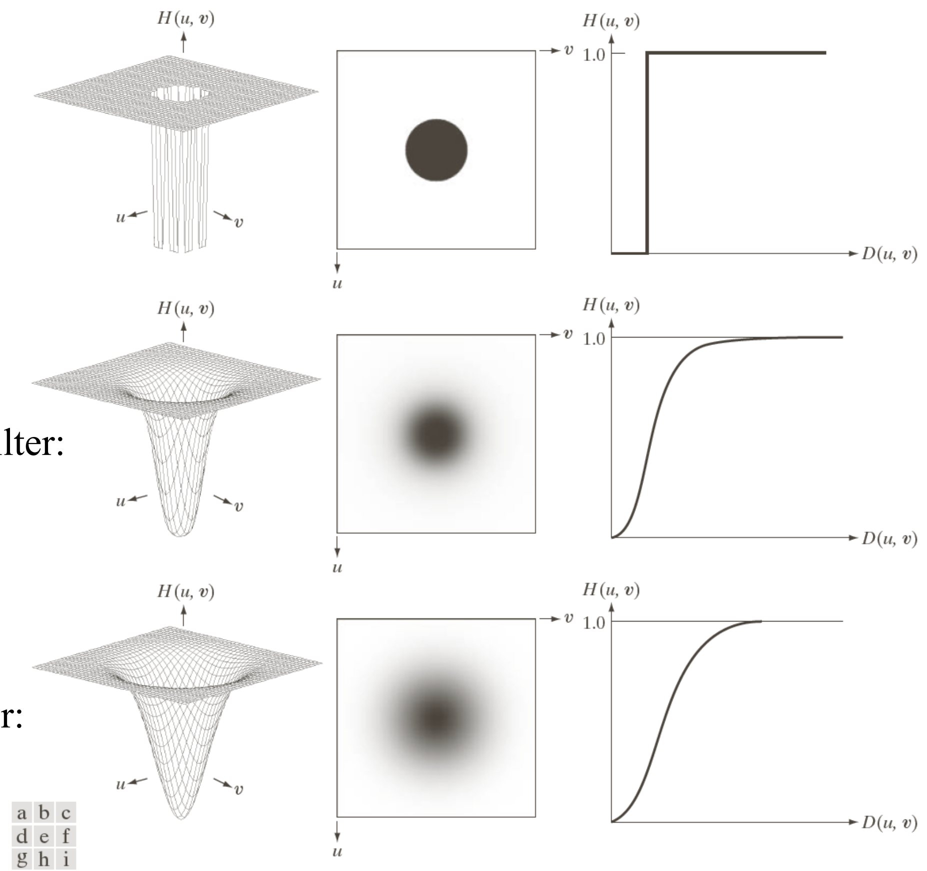


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Highpass Filters (in Spatial Domain)

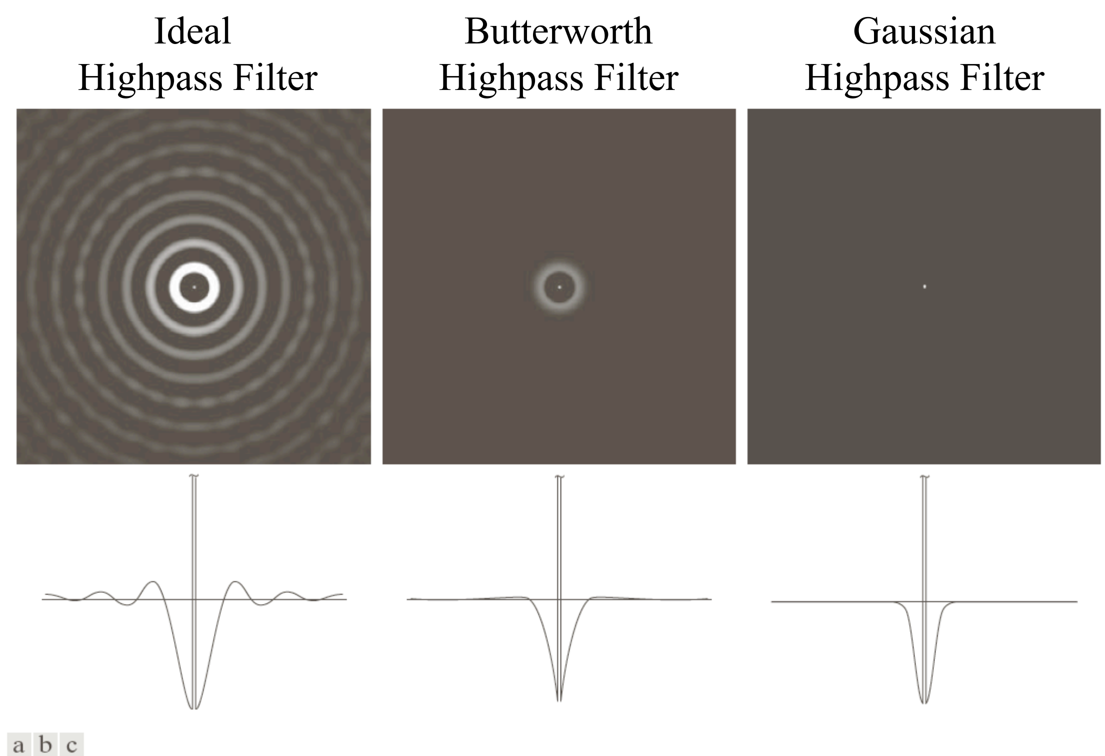
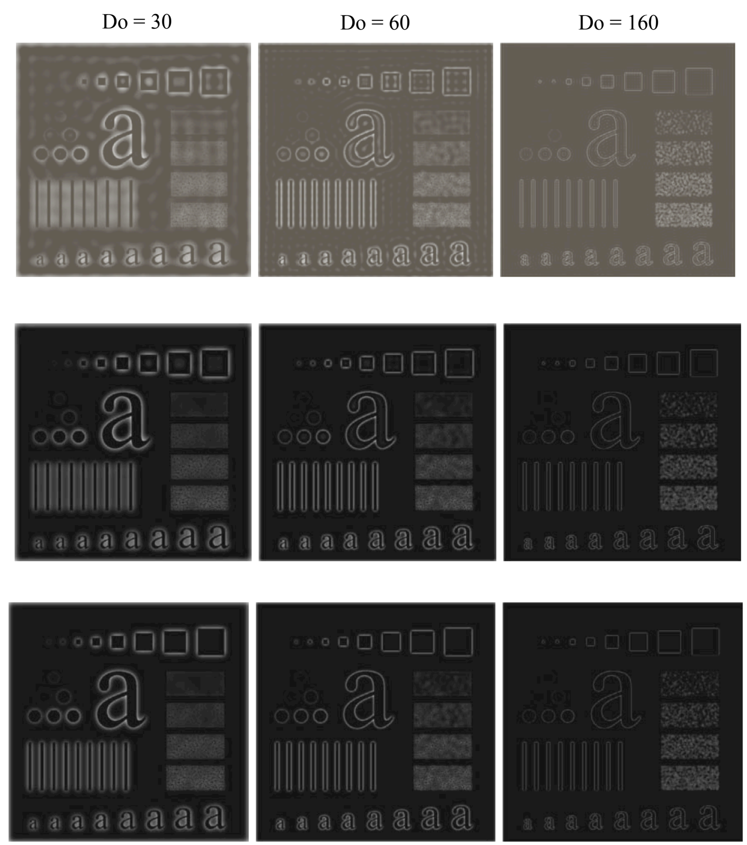


FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

Highpass Filters (Effect)



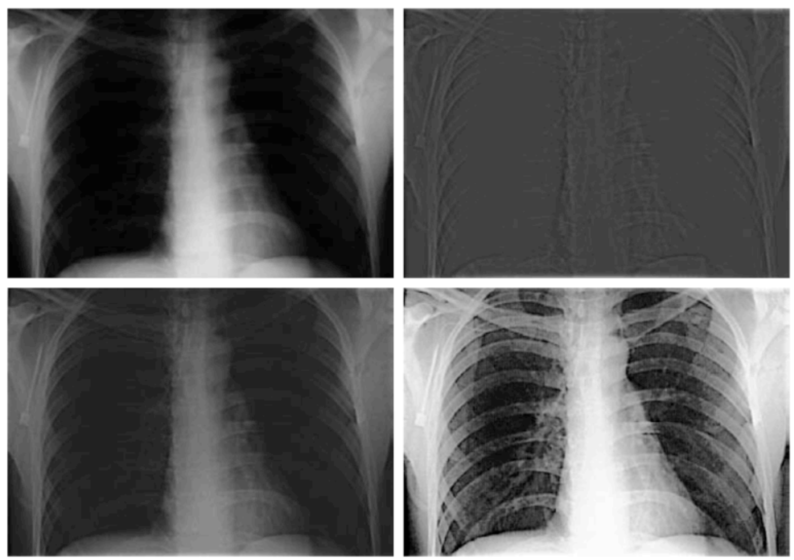
Ideal HPF

Butterfly HPF

Gaussian HPF

High Frequency Emphasis Filter

$$g(x,y) = IDFT\left\{ \underbrace{\left[1 + k * H_{HP}(u,v)\right]}_{\textit{High-frequency-emphasis filter}} F(u,v) \right\}$$



Gaussian highpass filtering

High-frequency emphasis filtering

Histogram Equalisation

a b
c d

FIGURE 4.59 (a) A chest X-ray image. (b) Result of highpass filtering with a Gaussian filter. (c) Result of high-frequency-emphasis filtering using the same filter. (d) Result of performing histogram equalization on (c). (Original image courtesy of Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)