# Analysis of Algorithms Lab 02

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1. Solve the following recurrences. The solution of each exercise includes finding the closed form and then verifying using mathematical induction

1.1 $T(n) = 3T(n/2) + n \qquad T(1) = 0, \ \forall n \in 2k, k > 0$

Base case $T(1)=0$, tell us that we have to find the pattern in order to find the solution, where $T(i)$, $i = 2, 4, 8, 16 \ldots$

1) $T(2) = 3T(\frac{2}{2}) + 2;$
$T(2) = 3T(1) + 2;$
$T(2) = 3(0) + 2;$
$T(2) = 2$

2) $T(4) = 3T(\frac{4}{2}) + 4;$
$T(4) = 3T(2) + 4;$
$T(4) = 3(2) + 4;$
$T(4) = 10$

3) $T(8) = 3T(4) + 8;$
$T(8) = 30 + 8;$
$T(8) = 38$

4) $T(16) = 3T(8) + 16$
$T(16) = 3(38) + 16$
$T(16) = 130$

Thus if we divide n by 2 on the function, we have got:

$$T(\tfrac{n}{2}) = 3T(\tfrac{n}{2}\cdot\tfrac{1}{2}) + \tfrac{n}{2} \Rightarrow T(\tfrac{n}{2}) = 3T(\tfrac{n}{4}) + \tfrac{n}{2} \quad (1)$$

If we substitute (1) in $T(n) = 3T(\frac{n}{2}) + n$, we get

$$T(n) = 3(3T(\tfrac{n}{4}) + \tfrac{n}{2}) + n;$$
$$T(n) = 9T(\tfrac{n}{4}) + \tfrac{3n}{2} + \tfrac{2n}{2};$$
$$T(n) = 9T(\tfrac{n}{4}) + \tfrac{5n}{2}$$

Thus

If we continue with this substituting pattern until $T(8)$, we will get:

$$T(n) = 2T\left(3T\left(\frac{n}{16}\right) + \frac{n}{8}\right) + \frac{9n}{4} + \frac{3n}{2} + n;$$

Therefore

$$T(n) = 3^k \cdot T(1) + n\left[1 + \frac{3}{2}n + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \dots + \left(\frac{3}{2}\right)^{k-1}\right]$$

Thus, since $T(1) = 0$

$$T(n) = n \cdot \sum_{j=0}^{k-1} \left(3/2\right)^j.$$ by inductive hypothesis

1.2 $T(n) = 3T(n-1) + 4T(n-2)$, $T(0) = 0$, $T(1) = 5$

Base case:

It's given that $T(0) = 0 \wedge T(1) = 5 \quad \forall n > 1$

$$T(2) = 3T(2-1) + 4T(2-2);$$
$$T(2) = 3T(1) + 4T(0);$$
$$T(2) = 3(5) + 4(0);$$
$$T(2) = 15$$

$$T(3) = 3T(2) + 4T(1)$$
$$T(3) = 3(15) + 4(5)$$
$$T(3) = 45 + 20 = 65$$
$$T(4) = 3T(3) + 4T(2)$$
$$T(4) = 3(65) + 4(15) = 255$$

Inductive case:

Since the pattern has been found we will use the characteristic equation.

So $T(n) = 3T(n-1) + 4T(n-2)$, becomes

$$r^2 = 3r + 4;$$
$$r^2 - 3r - 4 = 0;$$
$$(r-4)(r+1) = 0;$$

$$r_1 = 4 \wedge r_2 = -1$$

Thus, $T(n) = A * (4^n) + B(-1)^n$

After evaluating on initial condition, we get:

$$A + B = 0$$
$$4A - B = 5$$

Since $A = -B$, then:

$$4(-B) - B = 5$$
$$-5B = 5$$
$$B = -1$$

thus $A = 1$

By inductive hypothesis:

So for a $k > 1$, $k \in \mathbb{Z}$:

asuming $T(k)\ 4^k - (-1)^k$ is valid.

then for a $k+1$:

$$T(k+1) = 3 * T(k+1-1) + 4T\ T(k+1-2)$$

$$T(k+1) = 3T(k) + 4T(k-1)$$

$$T(k+1) = 3[4^k - (-1)^k] + 4[4^{k-1} - (-1)^{k-1}]$$

Simplifying

$$T(k+1) = [3 * 4^k + 4 * 4^{k-1}] - [3 * (-1)^k + 4 * (-1)^{k-1}]$$
$$T(k+1) = 4^{k+1} + (-1)^{k+1} \wedge T(k+1) = 4^{k+1} - (-1)^{k+1}$$

by mathematical induction is correct for all $n > 1$.

1.3 $T(n) = 5T(n-1) - 6T(n-2)$, $T(0) = 0$ $T(1) = 1$

$\forall n \in \mathbb{Z}, n > 1.$

Given $T(0) = 0 \wedge T(1) = 1$, we will evaluate on $T(n) = 5T(n-1) - 6T(n-2)$

$T(2) = 5(T(1)) - 6(T(0));$

$T(2) = 5(1) - 6(0);$

$T(2) = 5;$

$T(3) = 5T(2) - 6T(1)$

$T(3) = 25 - 6$

$T(3) = 19$

$T(4) = 5T(3) - 6T(2)$

$T(4) = 5(19) - 6(5)$

$T(4) = 65.$

Base case

$T(2) = 5;$

We will use the characteristic equation as before:

$T(n) - 5T(n-1) - 6T(n-2)$ becomes:

$r^2 = 5r - 6;$

$r^2 - 5r + 6;$

$(r-2)(r-3) = \emptyset$

$r_1 = 2 \wedge r_2 = 3$

Therefore

$$T(n) = A \cdot 2^n + B\,3^n$$

So after evaluating on initial conditions:

$$A + B = 0 \quad (T(0))$$
$$2A + 3B = 1 \quad (T(1))$$

Since $A = -B$, then

$$2(-B) + 3B = 1$$
$$B = 1$$

thus $A = -1$;

The CF will be $3^n - 2^n$

Inductive case:

Assume that $\forall k \in \mathbb{Z}$, $k > 1$ then if $k-1$ or $k-2$, then:

$$T(k-1) = 3^{k-1} - 2^{k-1}$$
$$T(k-2) = 3^{k-2} - 2^{k-2}$$

Since $T(k) = 5T(k-1) + 6T(k-2)$, then:

$$T(k) = 5[3^{k-1} - 2^{k-1}] + 6[3^{k-2} - 2^{k-2}]$$
$$T(k) = 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} + 6 \cdot 3^{k-2} - 6 \cdot 2^{k-2}$$
$$T(k) = 5 \cdot 3^{k-1} + 6 \cdot 3^{k-2} + (5 \cdot 2^{k-1} + 6 \cdot 2^{k-2})$$
$$T(k) = 5 \cdot 3^k \cdot 3^{-1} + 6 \cdot 3^k \cdot 3^{-2} + (5 \cdot 2^k \cdot 2^{-1} + 6 \cdot 2^k \cdot 2^{-2})$$
$$T(k) = 3^k\left(\cancel{\frac{5}{3} + \frac{6}{9}}\right) + \left(2^k\left(\cancel{\frac{5}{2} + \frac{6}{4}}\right)\right)$$
$$T(k) = 3^k - 2^k$$

by inductive hypothesis, $T(n)$ is correct.

1.4 $x(n) = 2x(n-1) + 4x(n-2)$, $x(0) = 1$, $x(1) = 2$

$\forall x > 1$.

As before there is a pattern when viewing $x(i)$ where $i \in \mathbb{Z}$, $i > 1$. Thus using the characteristic equation, then:

$$x(n) = 2x(n-1) + 4x(n-2) = r^2 = 2r + 4$$

where, $r^2 - 2r - 4 = 0$, then by factorization =>

$$r_1 = 2 + \frac{2\sqrt{5}}{2} \wedge r_2 = 2 - \frac{2\sqrt{5}}{2}$$

So $x(n) = A^x(1+\sqrt{5})^n + B(1-\sqrt{5})^n$, after evaluating the initial conditions, we have got:

$$A + B = 1$$
$$A - B = 1/\sqrt{5}$$

Since $A = 1 - B$, then.

$$(1 - B) - B = 1/\sqrt{5};$$

$$1 - 2B = \frac{1}{\sqrt{5}}$$

$$-2B = \frac{1}{\sqrt{5}} - \frac{\sqrt{5}}{\sqrt{5}}$$

$$-2B = \frac{-\sqrt{5}}{\sqrt{5}}$$

$$B = \frac{\cancel{1}\sqrt{5}}{2\cancel{\sqrt{5}}}; \quad \text{thus } A = \frac{\cancel{4}\sqrt{5}}{\cancel{4}\sqrt{5}} - \frac{2\sqrt{5}}{\cancel{4}\sqrt{5}} = \frac{2\sqrt{5}}{4\sqrt{5}} = \frac{-\cancel{1}\sqrt{5}}{2\cancel{\sqrt{5}}}$$

Assuming $x_1(k) = 2x(k-1) + 4x(k-2)$ where $k \in \mathbb{Z} \wedge k = 1$, then:

if $T(k)$ is valid $x(k+1) = 2x(k) + 4x(k-1)$:

$$\Rightarrow x(k+1) = \left[\frac{-\sqrt{5}}{2\sqrt{5}} \cdot (1+\sqrt{5})^{k+1} + \left[\frac{\sqrt{5}}{2\sqrt{5}} \cdot (1-\sqrt{5})^{k+1}\right.\right.$$

By m.i is valid