When GNNs Met a Word Equations Solver: Learning to Rank Equations

FroCoS 2025

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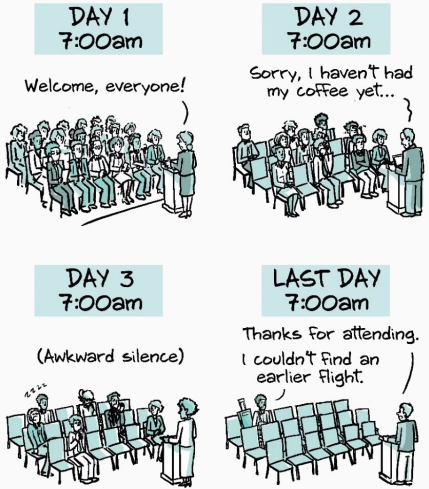
University of Lorraine
CNRS, Inria, LORIA

October 1, 2025

THE CONFERENCE MORNING SESSION



Word Equation

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- Empty string: ϵ
- Letters: a, b, ...
- Variables: X, Y, Z, ...

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- UNSAT

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$$aabb = aabb \wedge aabba = aabba$$

$$X = a,\ Y = a,\ Z = bb$$

Easy, right...?

$$AaAkjhhohhhkokookkkokkookkhkoohohCAYZjhBkjhkkkokhkkEkkoWQB$$
$$= aABBBncYECaa$$

- Variables: A, C, Y, Z, B, E, W, Q
- Letters: a, k, j, h, o, n, c

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- Letters: a, k, j, h, o, n, c

`Z3`, `cvc5`, `Ostrich`, `Z3-noodler`, and `Woorpje` cannot solve it in 300 seconds

Challenges

- Difficult to solve
  - ▷ Decidable
  - ▷ Complexity: NP-Hard and in PSPACE
  - ▷ Practical implementation are incomplete
- Important in modeling string constraints in verification tasks
  - ▷ User inputs, string manipulations, ...

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Can machine learning help to solve conjunctive word equation problem?

Yet Another Talk about ML...

- Is it another “we used ML for [insert problem] and...”?

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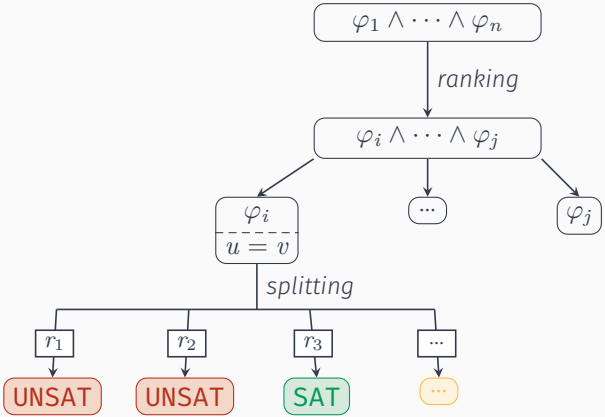
- Is human intelligence over now?
  - ▷ ~~It depends on the human~~
  - ▷ Obviously not!

Big Picture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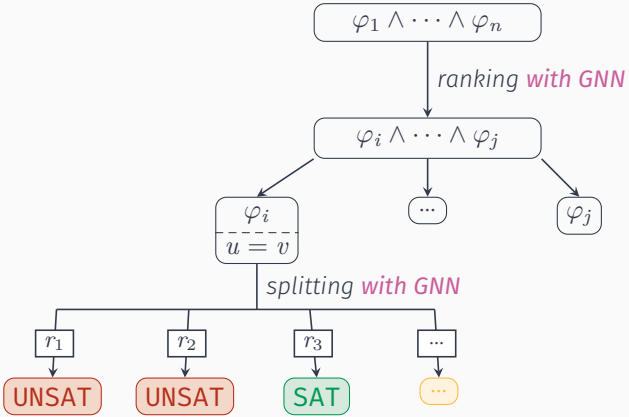
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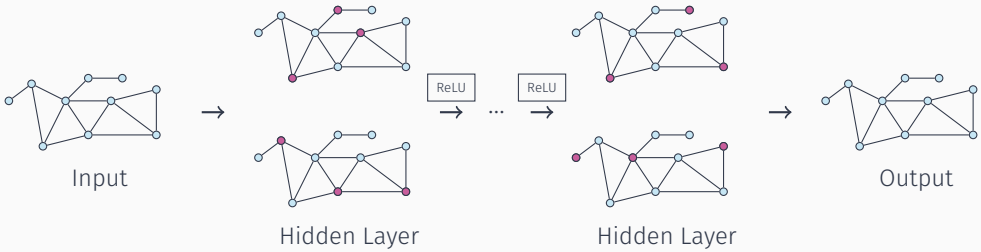
Graph Neural Networks (GNN)

- Designed to work directly with graph-structured data
- Take graph as input, output node and graph representations
- Capture the structural features of graph



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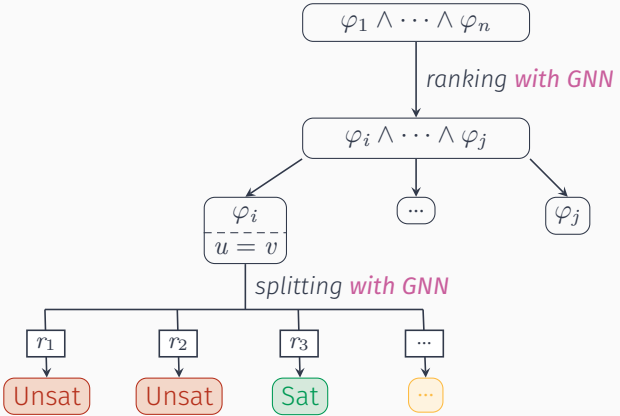
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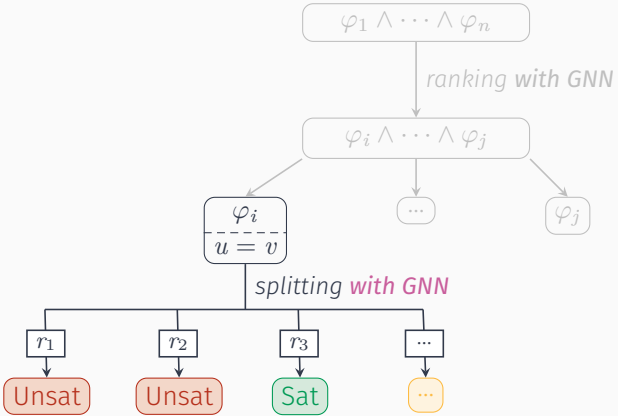
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Previously...[1]

Calculus

- Nielsen transformation
- Splitting rules, iterative deepening & backtracking
- Continuously simplify the first terms of both sides
- `DragonLi`

$$Xabb = YaZ$$

where a is a letter, u and v are terms, and X and Y are variables.

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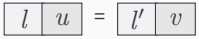

where l, l', a and b are a letters, u and v are terms, X and Y are variables, and X' and Y' are fresh variables.

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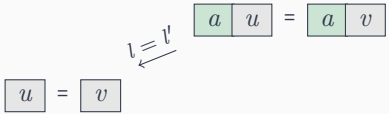

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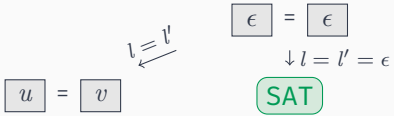

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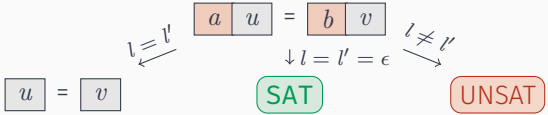

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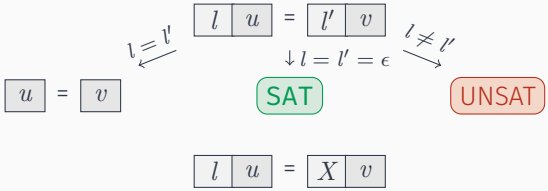

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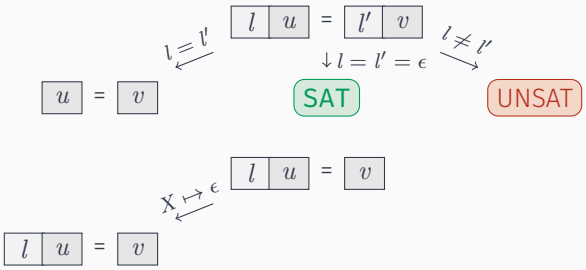

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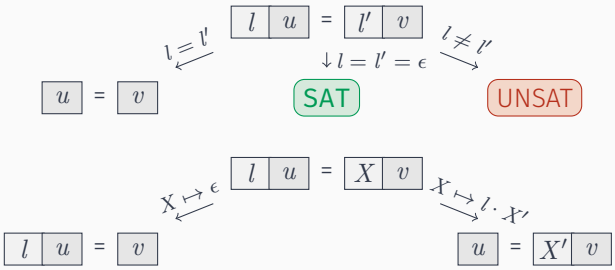

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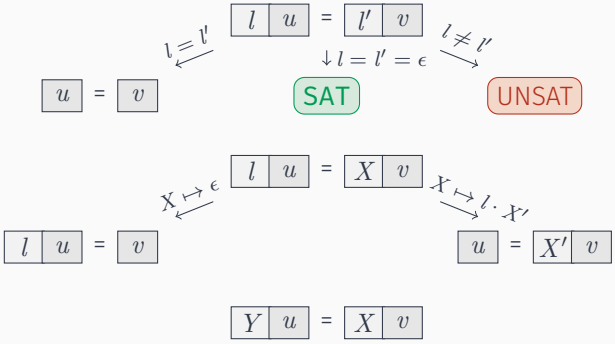

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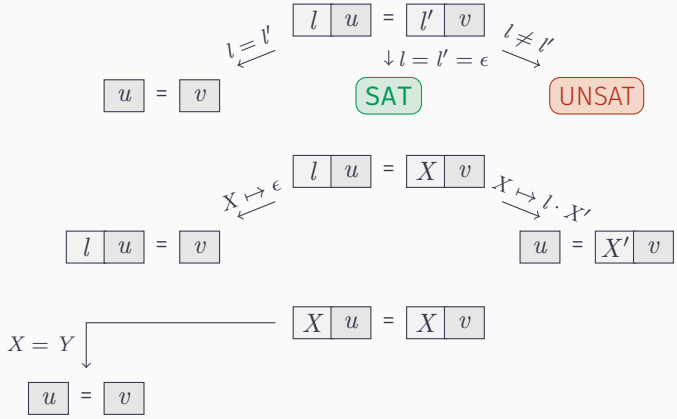

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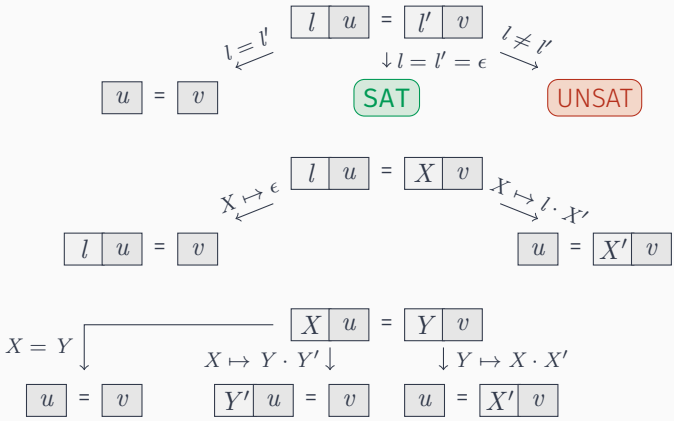

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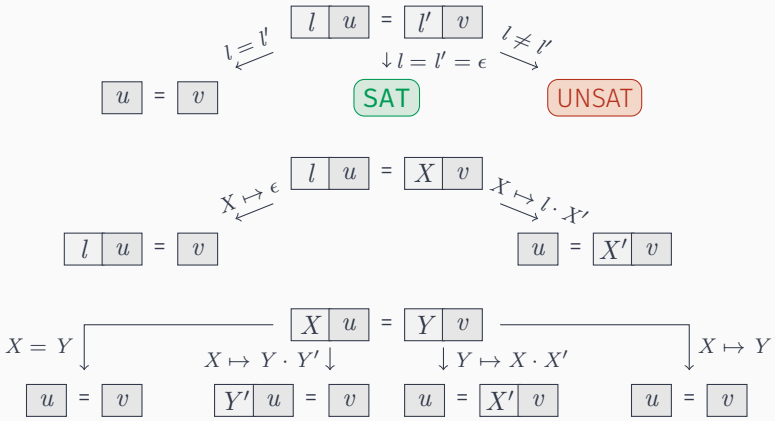

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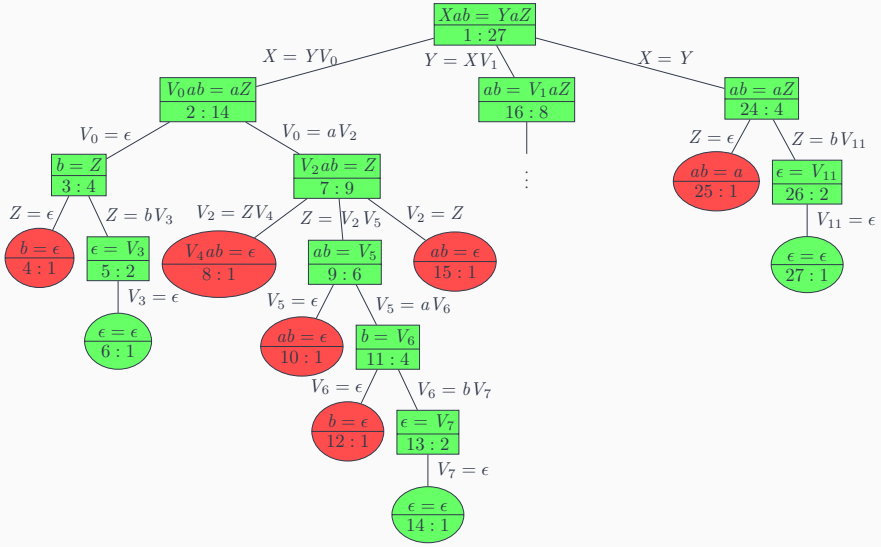
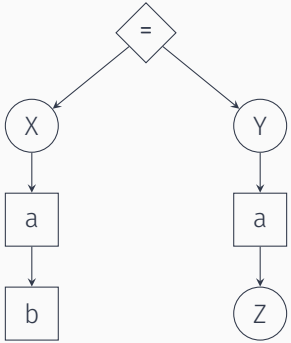


...and Here Is an Equation!

$$Xab = YaZ$$



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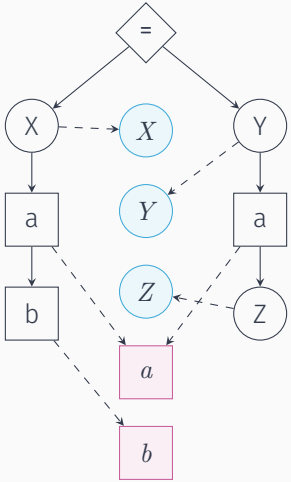

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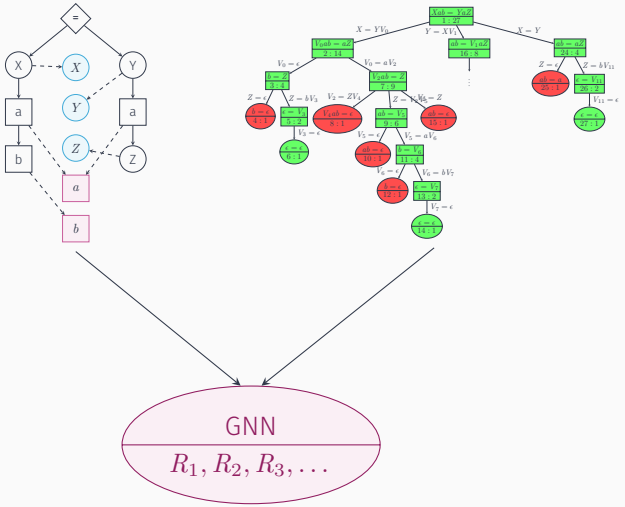

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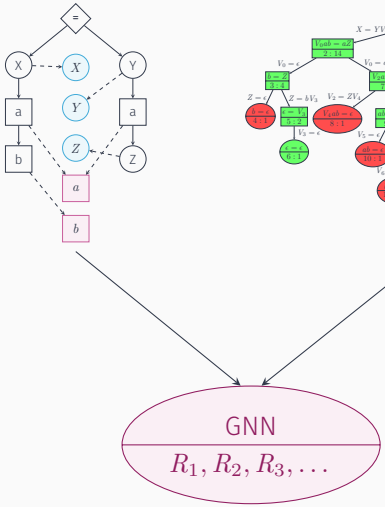

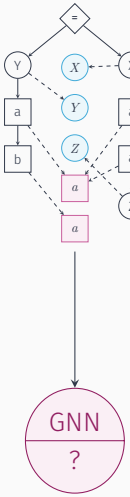

DragonLi, I Choose You!

- Python
- Focus on SAT
- Results comparable to state-of-the-art solvers

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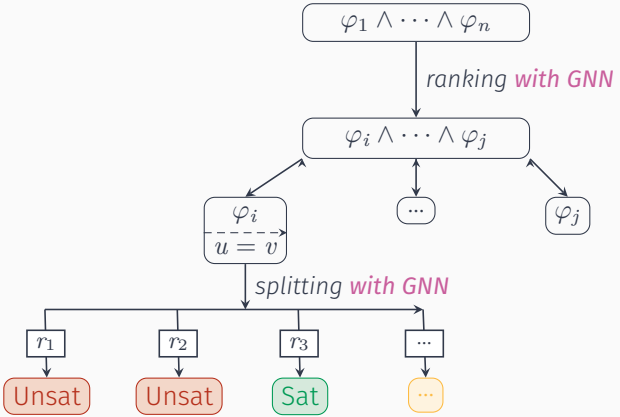
How to deal with more than one equation?

How to improve on UNSAT problems?

Big Picture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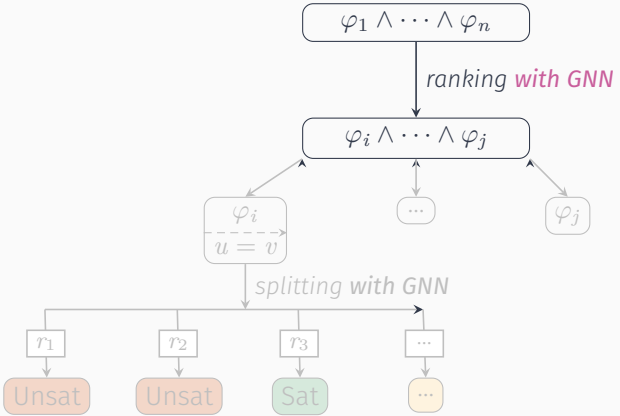
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Order Matters

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  1. $\epsilon = \epsilon$
  2. $\epsilon = u \cdot v$ or $u \cdot v = \epsilon$
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  4. $a \cdot u = a \cdot v$
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Can a GNN do better?

Cherry Picking

Minimal Unsatisfiable Set (MUS)

A set of equations U of a conjunctive word equation is a MUS iff the conjunction of U is unsatisfiable, and for all equations $e \in U$, the conjunction of $U \setminus \{e\}$ is satisfiable.

$$a = X \wedge b = Y \wedge aa = Xa \wedge aab = XaY \wedge X = b$$

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$$\{a = X, X = b\}$$

How to (m)Use Them?

Training Data

- Problems not solved by `DragonLi` are given to `Z3`, `Z3-Noodler`, `cvc5`, `Ostrich` and `Woorpje`
- Extraction of MUS

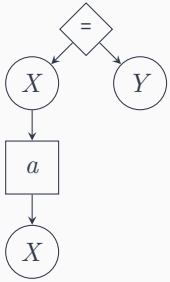
Labelling

For a conjunctive word equation $\phi = e_1 \wedge \cdots \wedge e_n$ and U its MUS:

$$\text{label}(e_i) = \begin{cases} 1 & \text{if } e_i \in U \text{ and } |e_i| = \min\left(\{|e| \mid e \in U\}\right) \\ 0 & \text{otherwise.} \end{cases}$$

I Wouldn’t Count on It…

$XaX = Y$



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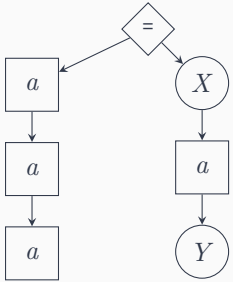

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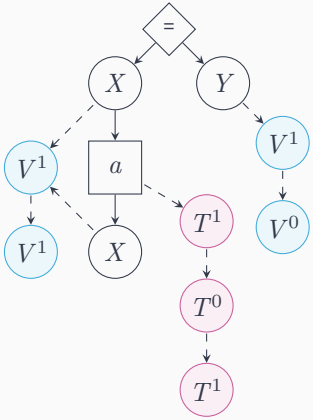


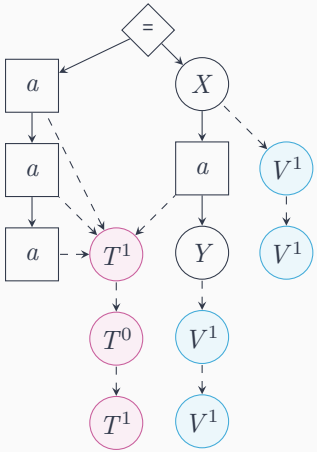
$\wedge$

$Occ(X) = 3$
$Enc(X) = 11$

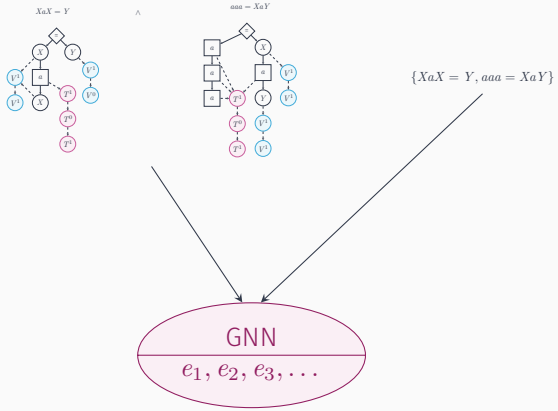
$Occ(Y) = 2$
$Enc(Y) = 10$

$Occ(a) = 5$
$Enc(a) = 101$

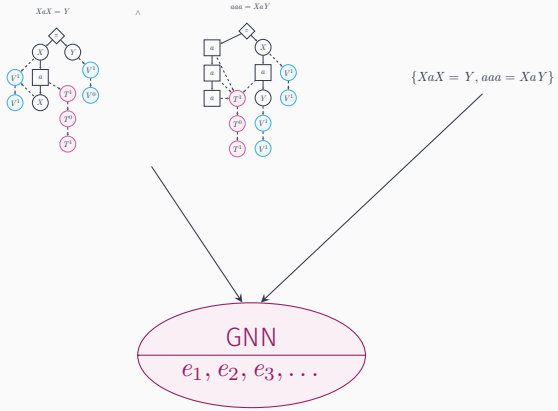
$aaa = XaY$

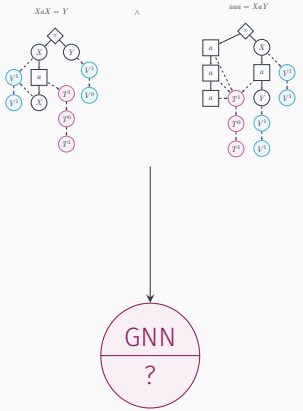


Better Together (bis repetita)



Better Together (bis repetita)





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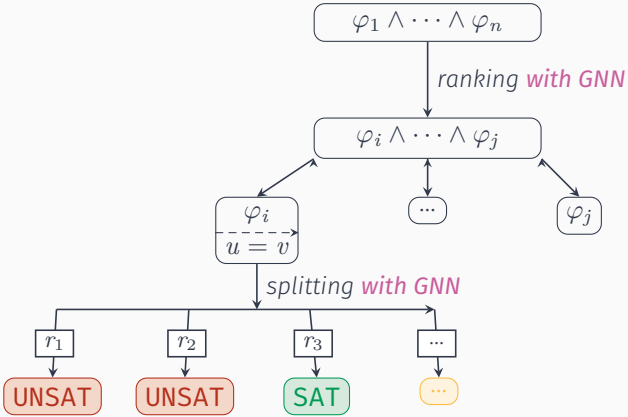
  - ▷ Once at the beginning with 5 equation of priority level 5
  - ▷ Limit overhead

Big Picture


$\varphi_1 \wedge \cdots \wedge \varphi_n$
ranking with GNN
$\varphi_i \wedge \cdots \wedge \varphi_j$
φ_i
$u = v$
...
φ_j
splitting with GNN
r_1
r_2
r_3
...
UNSAT
UNSAT
SAT
...

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  - Remove length constraint, boolean operator, and regex
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  - ▷ SMT-LIB and Zaligvinder
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- Artificial benchmarks:
  - ▷ $s = s$
  - ▷ At most 60 letters (among a set T)
  - ▷ Replace n times substrings on both side by m variables
  - ▷ Between 0 and 100 equations

Linearity and Benchmarks

Linearity

A conjunctive word equation is called *linear* iff each variable appear **at most once** in each equation.

$X = a \wedge Y = b$ ✅ $X = a \wedge X = b$ ✅ $XaX = aaa \wedge YaX = YbZ$ ❌

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- **A1** : fresh variable, $T = 6$, $n \in [0, 5]$, $m \in [0, 5]$ (*linear*)
- **A2** : A1 + $T = 26$, $n \in [0, 16]$, and $m = 1$ (*linear*)
- **B** : A1 + No fresh variable (*non-linear*)
- **C** : $X_n a X_n b X_{n-1} \cdots b X_1 = a X_n X_{n-1} X_{n-1} b \cdots X_1 X_1 baa$ + $b \mapsto A_1$ (*highly non-linear*)

Experimental Setup

- `DragonLi`: basic, random and GNN-based
- `Z3`, `Z3-Noodler`, `cvc5`, `Ostrich`, and `Woopje`
- 300 seconds timeout
- 2 Intel Xeon E5 2630 v4 at 2.20 GHz/core and 128GB memory
- GNNs trained on NVIDIA A100 GPUs
- 60 000 problems per benchmark for training
- 1000 problems per benchmark for experiments

Results — A1

Solver	Number of solved problems					Average solving time (split number)			
	SAT	UNSAT	UNI	CS	CU	SAT	UNSAT	CS	CU
DragonLi	**24**	955	0	13	678	5.6 (244.8)	6.5 (1085.3)	5.0 (94.4)	5.7 (126.3)
Random-DragonLi	22	944	0			5.6 (198.8)	6.3 (932.6)	5.6 (137.6)	5.7 (180.5)
GNN-DragonLi	**24**	**961**	0			6.1 (164.7)	7.5 (1974.8)	6.1 (96.4)	6.3 (**60.5**)
cvc5	**24**	952	1			0.5	0.6	0.1	0.3
Z3	17	960	0			8.7	0.4	1.1	0.1
Z3-Noodler	22	939	2			5.7	0.3	4.8	0.1
Ostrich	17	931	0			15.0	5.5	8.0	4.7
Woorpje	23	744	0			3.0	12.5	0.1	12.2

Results — A2

Solver	Number of solved problems					Average solving time (split number)			
	SAT	UNSAT	UNI	CS	CU	SAT	UNSAT	CS	CU
DragonLi	59	824	0	3	0	8.5 (4233.4)	11.8 (1231.3)	4.7 (27.3)	-
Random-DragonLi	44	806	1			24.7 (29779.6)	6.2 (210.9)	4.6 (27.3)	-
GNN-DragonLi	59	836	4			8.4 (1330.6)	11.6 (1074.1)	5.9 (27.3)	-
cvc5	**67**	142	15			0.6	56.0	0.1	-
Z3	8	**870**	10			1.1	0.6	0.1	-
Z3-Noodler	22	7	1			15.4	3.8	0.4	-
Ostrich	13	18	2			24.8	38.8	8.6	-
Woorpje	0	0	0	-	-	-	-	-	-

Results — B

Solver	Number of solved problems					Average solving time (split number)			
	SAT	UNSAT	UNI	CS	CU	SAT	UNSAT	CS	CU
DragonLi	11	805	0	4	294	4.9 (62.5)	5.2 (81.5)	4.9 (29.2)	5.3 (82.4)
Random-DragonLi	10	894	0			5.0 (58.7)	5.8 (295.2)	5.0 (27.25)	5.2 (73.1)
GNN-DragonLi	11	821	0			6.5 (65.1)	6.8 (70.0)	6.5 (28.25)	6.8 (**60.2**)
cvc5	12	915	0			0.1	0.6	0.1	0.7
Z3	11	859	3			0.1	0.2	0.1	0.1
Z3-Noodler	**24**	911	1			4.9	0.4	1.3	0.4
Ostrich	12	**917**	2			6.9	3.7	3.3	4.2
Woorpje	19	330	1			29.5	6.0	0.2	5.0
	3	62	0	1		65.0	28.4	0.2	223.1

Results — C

Solver	Number of solved problems					Average solving time (split number)			
	SAT	UNSAT	UNI	CS	CU	SAT	UNSAT	CS	CU
DragonLi	2	0	0	-	-	5.1 (85.5)	-	-	-
Random-DragonLi	2	0	0	-	-	5.0 (85.5)	-	-	-
GNN-DragonLi	-	-	-	-	-	-	-	-	-
cvc5	0	**909**	17	-	1	-	46.9	-	17.3
Z3	1	821	12	1		0.8	1.7	0.8	0.1
Z3-Noodler	**7**	657	4	1		0.2	94.1	0.1	1.0
Ostrich	0	61	0	-		-	77.2	-	27.1
Woorpje	3	62	0	1		65.0	28.4	0.2	223.1

“Why Is Your Work Good?”

- `DragonLi` performs well on linear problems
- Fewer splits
- No improvement on non-linear problems
- Need for dedicated strategies (equation length, letter count, …)
- Re-implementation can offer improvements
- Increasing training dataset increases the number of problems solved

This is the End!

Contributions

- Extension of `DragonLi` to handle conjunctive word equations
- Multi-classification task to handle variable number of inputs
- MUS & new graph representation
- Good results on linear problems

What's Next?

- Optimize GNN overhead
- Call timing
- Length constraint and regular expressions

Thank you! ☺

https://github.com/ChenchengLiang/DragonLi