Finiteness of Symbolic Derivatives in Lean

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Introduction

- *Brzozowski derivatives* of regular expressions

$$\begin{aligned}\mathcal{L}(der(c, R)) &:= \{w \in \Sigma^* \mid c \cdot w \in \mathcal{L}(R)\}\\ null(R) &:= \epsilon \in \mathcal{L}(R)\end{aligned}$$

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- (Brzozowski, 64): *finiteness* of all iterated derivatives

$$\mathcal{D}er(R) = \{der_w^*(R) \mid w \in \boldsymbol{\sigma}^*\}/_{\cong}$$

quotiented by a relation called *ACI*-similarity:

$$\begin{aligned} (L \Cup R) \Cup S &\cong L \Cup (R \Cup S) && \text{Associativity} \\ L \Cup (R \Cup L) &\cong L \Cup R && \text{Commutativity} \\ R \Cup R &\cong R && \text{Idempotence} \end{aligned}$$

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    - `"abab"` vs `"abab"`

Regular Expressions with Lookarounds

$$R, S \ ::= \ \psi \in \alpha \ \mid \ \varepsilon \mid \ R \Cup S \mid \ R \Cap S \mid \ R \cdot S \mid \ R^* \mid \ {\sim}R$$
$$\mid \ \texttt{(?=}R\texttt{)} \mid \ \texttt{(?<=}R\texttt{)} \mid \ \texttt{(?!}R\texttt{)} \mid \ \texttt{(?<!}R\texttt{)}$$

- We work modulo an alphabet theory $\mathcal{A} = (\Sigma, \boldsymbol{\alpha}, \models, \bot, \top, \sqcup, \sqcap, ^{\mathsf{c}})$
  For example, $\psi_{upper} \in \alpha$ and $[\![\psi_{upper}]\!] = [A - Z]$

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  For example, $\psi_{upper} \in \alpha$ and $[\![\psi_{upper}]\!] = [A - Z]$
- Positive lookahead $\texttt{(?=}R\texttt{)}$ and lookbehind $\texttt{(?<=}R\texttt{)}$
  Negative lookahead $\texttt{(?!}R\texttt{)}$ and lookbehind $\texttt{(?<!}R\texttt{)}$

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Semantics

$$(xs, ys, zs) \models \texttt{(?=}R\texttt{)} \iff ys = \epsilon \wedge (xs, zs, \epsilon) \models R \cdot \top *$$

Example: $R = \texttt{(?=}\psi_{upper}\texttt{)}$ and `s = "aAbc"`

Then `("a",`ϵ`,"Abc")` $\models \texttt{(?=}\psi_{upper}\texttt{)}$

since `("a","Abc",`ϵ`)` $\models \psi_{upper} \cdot \top *$ is a valid *future* match

Transition terms and symbolic derivatives

A symbolic derivative is a transition term of the following type:

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inductive TTerm (α : Type) : Type where
| Leaf : RE α → TTerm α
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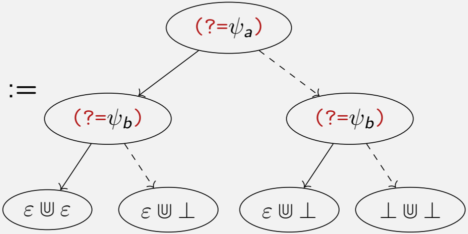
Example

Let ψ_a and ψ_b be atomic predicates.

$\delta\ (\psi_a \Cup \psi_b) : \mathsf{TTerm}\ \alpha :=$



Transition terms and symbolic derivatives

A symbolic derivative is a transition term i.e. trees of regexes

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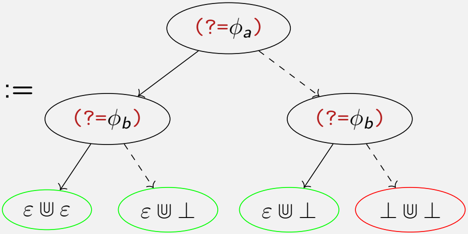

Semantics of transition terms

- Transition term $\approx$ a function from locations to regexes
- ... but it postpones all the nullability tests
- We define the evaluation function of type
  `Loc` σ `→ TTerm` α `→ RE` α

$$L[x] := L \quad (R, f, g)[x] := \begin{cases} f[x], \text{ if } \textit{null } R\ x; \\ g[x], \text{ otherwise.} \end{cases}$$

Symbolic derivatives

Let $\ell \in \text{LA}$, $\psi \in \boldsymbol{\alpha}$ then $\delta\ :\ \ \texttt{RE}\ \alpha \to \texttt{TTerm}\ \alpha$

$$
\begin{aligned}
\delta\ \varepsilon &:= \bot & \delta\ \ell &:= \bot \\
\delta\ \psi &:= (\texttt{(?=}\psi\texttt{)}, \varepsilon, \bot) & \delta\ (L{\cdot}R) &:= (L, \delta\ L{\cdot}R \Cup \delta\ R, \delta\ L{\cdot}R) \\
\delta\ (L \Cap R) &:= \delta\ L \Cap \delta\ R & \delta\ ({\sim}R) &:= {\sim}(\delta\ R) \\
\delta\ (L \Cup R) &:= \delta\ L \Cup \delta\ R & \delta\ (R*) &:= \delta\ R \cdot R*
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We show the equivalence of symbolic and location-based derivatives:

Theorem 1. $\forall x \in \text{Loc}, R \in RE\ \alpha : (\delta\ R)[x] = der\ \ R\ x$

Symbolic derivatives

Let $\ell \in$ LA, $\psi \in \boldsymbol{\alpha}$ then δ : `RE` α `→` `TTerm` α

$$\delta\ \varepsilon := \bot \qquad \delta\ \ell := \bot$$

$$\delta\ \psi := (\texttt{(?=}\psi\texttt{)}, \varepsilon, \bot) \qquad \delta\ (L \cdot R) := (L, \delta\ L \cdot R \Cup \delta\ R, \delta\ L \cdot R)$$

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$\rightarrow$ from now on, we can just work with the symbolic definition.

Iterated derivatives

- We can now compute the immediate derivatives of R:

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lvs :  TTerm α → RE α
step (R : RE α) :  List (RE α) := lvs (δ R)
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- The `step` function is well-behaved wrt operations on `RE` α:

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- We compute the n-th derivatives (words of length n):

  `steps : RE` α `→ Nat → List (RE` α`)`

Finiteness of the state space

- **Classical approach** (Brzozowski/DFA construction)
  - $\mathcal{D}er(R) = \{der_w(R) \mid w \in \Sigma^*\}/_{\cong}$
  - $\cong$ is the equivalence induced by ACI for union $\mathbb{U}$

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- **Our approach**
  - Follow Antimirov's strategy for finiteness
  - While dealing with the extended class of expressions

Similarity

We define helpers to reason *up-to* a relation R

- List membership
  `x ∈[ R ] ys` $:=$ `∃ y, R x y ∧ y ∈ ys`
- List inclusion
  `xs ⊆[ R ] ys` $:=$ `∀ x ∈ xs, x ∈[ R ] ys`
- List equality
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Our relation: the ADI-similarity relation used for quotienting:

$(L \Cup R) \Cup S \cong L \Cup (R \Cup S)$	Associativity
$L \Cup (R \Cup L) \cong L \Cup R$	right Deduplication
$R \Cup R \cong R$	Idempotence

Finiteness of Antimirov derivatives

- Why is proving finiteness easy for Antimirov derivatives?

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\begin{array}{rcl}
\mathit{support}(\bot) & := & \emptyset \\
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- ACI is built into the set representation
- Can we use a similar strategy for Brzozowski-style derivatives?

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$$\begin{tikzcd}
a \Cup b \cdot c \arrow[r, "\mathrm{der}_a"] \arrow[d, "\texttt{pieces}"'] & \varepsilon \Cup \bot \cdot c \arrow[dl, "\ni"] \\
{[\bot, \varepsilon, a] \mathbin{+\!\!+} [\bot, \varepsilon, c, \bot \cdot c, \varepsilon \cdot c, b \cdot c]} &
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\end{tikzcd}$$

- One step: ε and $\bot \cdot c$ are contained in *pieces* $(a \Cup b \cdot c)$
- **Key idea:** all derivatives can be given as union of `pieces`

Pieces

- We don't have commutativity of union so we have to consider all permutations of a list:
  `⊕[a, b] = [a, a ⋓ b, b ⋓ a, b]`

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```
def pieces : RE α → List (RE α)
  | ε       => [ε, Pred ⊥]
  | Pred φ  => [Pred φ, ε, Pred ⊥]
  | ?= r    => [?= r, ε, Pred ⊥] | ...
  | l ⋓ r   => pieces l ++ pieces r
  | l ⋒ r   => productWith (· ⋒ ·) ⊕(pieces l) ⊕(pieces r)
  | ~r      => map (~ ·) ⊕(pieces r)
  | l · r   => map (· · r) ⊕(pieces l) ++ pieces r
  | r*      => r* :: map (· · r*) ⊕(pieces r)
```

Main theorem

1. **Reflexivity**:
```
∀ r,
∃ xs, toSum xs ≅ r ∧ xs ∈ neSublists (pieces r)
```
2. **Transitivity**:
```
   e ∈ pieces f
 → f ∈ pieces g
 → e ∈[ (· ≅ ·) ] pieces g
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3. **One-step reconstruction**:
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∀ r d, d ∈ step r
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theorem finiteness [DecidableEq α] {r : RE α} :
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- The witness is `xs := ⊕(pieces R)`

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- (Moreira et al., 2012) avoid the need for normalisation modulo ACI by using Antimirov derivatives
- We take inspiration from the Antimirov finiteness proof, but adapt it to handle intersection and negation

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Allowed simplifications

```
⊥ ⋓ s ⇝ s     ~⊥ ⋓ s ⇝ ~⊥     ε · s ⇝ s
r ⋓ ⊥ ⇝ r     r ⋓ ~⊥ ⇝ ~⊥     r · ⊥ ⇝ ⊥

(r ⋓ s) · t ⇝ r · t ⋓ s · t
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- `r · s ⇝ s` and `r ⋓ s ⇝ r` are allowed

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Allowed simplifications

$$
\begin{array}{lll}
\bot \Cup s \rightsquigarrow s & {\sim}\bot \Cup s \rightsquigarrow {\sim}\bot & \varepsilon \cdot s \rightsquigarrow s \\
r \Cup \bot \rightsquigarrow r & r \Cup {\sim}\bot \rightsquigarrow {\sim}\bot & r \cdot \bot \rightsquigarrow \bot \\
\end{array}
$$

$$(r \Cup s) \cdot t \rightsquigarrow r \cdot t \Cup s \cdot t$$

- $r \cdot s \rightsquigarrow s$ and $r \Cup s \rightsquigarrow r$ are allowed
- $r \cdot s \rightsquigarrow r$ is **not** allowed

Conclusion

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Thank you!

`github.com/ezhuchko/finiteness-derivatives`