Friedrich-Alexander-Universität
Erlangen-Nürnberg

Bounded Inquisitive Logics: Sequent Calculi and Schematic Validity

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Intuition

Inquisitive FOL can be seen as an extension of classical logic by questions.

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Example

Natural Language	**Formula**
Luisa is guilty.	$\mathrm{Guilty}\,(\mathrm{Luisa})$
If Luisa was there, do we know whether Luisa is guilty?	$\mathrm{WasThere}\,(\mathrm{Luisa}) \to ?\,\mathrm{Guilty}\,(\mathrm{Luisa})$
If we knew whether Luisa was there, do we know whether Luisa is guilty?	$?\,\mathrm{WasThere}\,(\mathrm{Luisa}) \to ?\,\mathrm{Guilty}\,(\mathrm{Luisa})$
Is there some person, who is guilty?	$\exists\, x.\ \mathrm{Guilty}\,(x)$

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Was Not There	w_3	w_4

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- We get the following properties regarding the single worlds:

$$w_1 \models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})$$
$$w_2 \models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})$$
$$w_3 \models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})$$
$$w_4 \models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})$$

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w_4 &\models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})
\end{aligned}
$$

- If we look at information states, we get the following support properties:

$$
\begin{aligned}
\{w_1, w_2\} &\not\models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})\\
\{w_1, w_3\} &\models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})\\
\{w_1, w_2, w_3\} &\not\models \mathrm{WasThere}\,(\mathrm{Luisa}) \rightarrow ?\,\mathrm{Guilty}\,(\mathrm{Luisa})
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$$

Syntax [Cia22]

Definition

- We call a set $\Sigma := (\mathsf{P}_\Sigma, \mathsf{F}_\Sigma, \mathrm{ar}_\Sigma, \mathrm{rigid}_\Sigma)$ a signature.
- P_Σ provides predicate symbols.
- F_Σ provides function symbols.
- $\mathrm{ar}_\Sigma \colon \mathsf{P}_\Sigma + \mathsf{F}_\Sigma \to \mathbb{N}$ maps symbols to their arity.
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Definition

Terms and Formulae over a signature Σ are defined as follows:

$$t \in \mathrm{Ter}_\Sigma ::= x \mid f\left(t_1, \dots, t_{\mathrm{ar}_\Sigma(f)}\right) \qquad f \in \mathsf{F}_\Sigma$$

$$\phi, \psi \in \mathcal{F}_\Sigma ::= P\left(t_1, \dots, t_{\mathrm{ar}_\Sigma(P)}\right) \mid \bot \mid \phi \to \psi \mid \phi \wedge \psi \mid \phi \mathbin{\mathbb{V}} \psi \mid \forall x.\, \phi \mid \exists x.\, \phi \qquad P \in \mathsf{P}_\Sigma$$

$$?\phi := \phi \mathbin{\mathbb{V}} \neg\phi$$

Semantics
Models, States

Definition

Let Σ be a signature.

- A tuple $\mathfrak{M} := \left(\mathrm{W}_{\mathfrak{M}}, \mathrm{I}_{\mathfrak{M}}, (\mathfrak{M}_w \llbracket f \rrbracket)_{w \in W, f \in \mathsf{F}_\Sigma}, (\mathfrak{M}_w \llbracket P \rrbracket)_{w \in W, P \in \mathsf{P}_\Sigma}\right)$ is called a model.

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- $\mathrm{W}_{\mathfrak{M}}$ is a set of possible worlds.
- $\mathrm{I}_{\mathfrak{M}}$ is a (non-empty) set of individuals.
- $\mathfrak{M}_w [\![f]\!] : \mathrm{I}_{\mathfrak{M}}^{\mathrm{ar}_\Sigma(f)} \to \mathrm{I}_{\mathfrak{M}}$ is the interpretation of f in a world w.
- $\mathfrak{M}_w [\![P]\!] \subseteq \mathrm{I}_{\mathfrak{M}}^{\mathrm{ar}_\Sigma(P)}$ is the interpretation of P in a world w.

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- for every rigid $f \in \mathsf{F}_\Sigma$ and for all $w_1, w_2 \in \mathrm{W}_{\mathfrak{M}}$ we have $\mathfrak{M}_{w_1} \llbracket f \rrbracket = \mathfrak{M}_{w_2} \llbracket f \rrbracket$.

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Definition

Let Σ be a signature, $\mathfrak{M}$ be a model. A subset $s \subseteq \mathrm{W}_{\mathfrak{M}}$ is called an (information) state.

Semantics
Referent of a Term

Definition

Let Σ be a signature, $\mathfrak{M}$ be a Model, $s \subseteq \mathrm{W}_{\mathfrak{M}}$ an information state and $\eta\colon \mathrm{Var} \to \mathrm{I}_{\mathfrak{M}}$ a variable assignment. The referent of a term $t \in \mathrm{Ter}_{\Sigma}$ is defined as follows:

$$\mathfrak{M}_{w,\eta}\left[\!\left[x\right]\!\right] := \eta\left(x\right)$$
$$\mathfrak{M}_{w,\eta}\left[\!\left[f\left(t_1, \ldots, t_{\mathrm{ar}_{\Sigma}(f)}\right)\right]\!\right] := \mathfrak{M}_{w}\left[\!\left[f\right]\!\right]\left(\mathfrak{M}_{w,\eta}\left[\!\left[t_1\right]\!\right], \ldots, \mathfrak{M}_{w,\eta}\left[\!\left[t_{\mathrm{ar}_{\Sigma}(f)}\right]\!\right]\right)$$

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$$\mathfrak{M}, s, \eta \models P\left(t_1, \ldots, t_{\mathrm{ar}_\Sigma(P)}\right) :\Longleftrightarrow \text{for all } w \in s \text{ we have } \left(\mathfrak{M}_{w,\eta}\left[\!\left[t_1\right]\!\right], \ldots, \mathfrak{M}_{w,\eta}\left[\!\left[t_{\mathrm{ar}_\Sigma(P)}\right]\!\right]\right) \in \mathfrak{M}_w\left[\!\left[P\right]\!\right]$$

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\mathfrak{M}, s, \eta \models \phi \wedge \psi &:\Longleftrightarrow \mathfrak{M}, s, \eta \models \phi \text{ and } \mathfrak{M}, s, \eta \models \psi \\
\mathfrak{M}, s, \eta \models \phi \vee\!\!\!\vee\, \psi &:\Longleftrightarrow \mathfrak{M}, s, \eta \models \phi \text{ or } \mathfrak{M}, s, \eta \models \psi
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\mathfrak{M}, s, \eta \models \forall x.\, \phi &:\Longleftrightarrow \text{for all } i \in \mathrm{I}_{\mathfrak{M}},\ \mathfrak{M}, s, \eta[x \mapsto i] \models \phi \\
\mathfrak{M}, s, \eta \models \exists\!\!\!\exists x.\, \phi &:\Longleftrightarrow \text{there exists } i \in \mathrm{I}_{\mathfrak{M}},\ \mathfrak{M}, s, \eta[x \mapsto i] \models \phi
\end{aligned}
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Semantics
Various properties

Persistency

$$t \subseteq s \text{ and } \mathfrak{M}, s, \eta \models \phi \Longrightarrow \mathfrak{M}, t, \eta, \models \phi$$

Empty State Property

$$\mathfrak{M}, \emptyset, \eta \models \phi$$

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Empty State Property

$$\mathfrak{M}, \emptyset, \eta \models \phi$$

- $\mathfrak{M}|_s := (s \subseteq \mathrm{W}_{\mathfrak{M}}, \mathrm{I}_{\mathfrak{M}}, \ldots)$

Locality

$$\mathfrak{M}, s, \eta \models \phi \Longleftrightarrow \mathfrak{M}|_s, s, \eta \models \phi$$

Semantics

InqFOL

Definition

Define Inquisitive First-Order Logic as follows:

$$\mathsf{InqLog}_{\Sigma} := \{\phi \in \mathcal{F}_{\Sigma} \mid \mathfrak{M}, s, \eta \models \phi \text{ for all models } \mathfrak{M}, s \subseteq \mathrm{W}_{\mathfrak{M}}, \eta\colon \mathrm{Var} \to \mathrm{I}_{\mathfrak{M}}\}$$

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- There exists a ND-System by Ciardelli/Grilletti [CG22] which is sound, but not yet proven to be complete.

Boundedness
Introduction

- Restricting the set of worlds to be finite yields Bounded Inquisitive FOL.

$$\begin{aligned}\mathsf{InqLogB}_{\Sigma,n} &:= \{\phi \in \mathcal{F}_\Sigma \mid \mathfrak{M}, s, \eta \models \phi \text{ for all models } \mathfrak{M} \text{ with } |\mathrm{W}_{\mathfrak{M}}| < n, s \subseteq \mathrm{W}_{\mathfrak{M}}, \eta\colon \mathrm{Var} \to \mathrm{I}_{\mathfrak{M}}\} \\ \mathsf{InqLogB}_{\Sigma} &:= \bigcap_{n\in\mathbb{N}} \mathsf{InqLogB}_{\Sigma,n} \\ &= \{\phi \in \mathcal{F}_\Sigma \mid \mathfrak{M}, s, \eta \models \phi \text{ for all models } \mathfrak{M}, s \subseteq_{\mathsf{fin}} \mathrm{W}_{\mathfrak{M}}, \eta\colon \mathrm{Var} \to \mathrm{I}_{\mathfrak{M}}\}\end{aligned}$$

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$$\mathsf{InqLogB}_\Sigma := \bigcap_{n\in\mathbb{N}} \mathsf{InqLogB}_{\Sigma,n}$$

$$= \{\phi \in \mathcal{F}_\Sigma \mid \mathfrak{M}, s, \eta \models \phi \text{ for all models } \mathfrak{M}, s \subseteq_{\mathsf{fin}} \mathrm{W}_\mathfrak{M}, \eta\colon \mathrm{Var} \to \mathrm{I}_\mathfrak{M}\}$$

- Ciardelli/Griletti [CG22] extended their ND-System for $\mathsf{InqLogB}_{\Sigma,n}$ and it proved the resulting extensions to be complete *(for most signatures)*.
- Added axiom: Cardinality Formula, which depends on the concrete signature.
- Apart from signature-dependency, such axioms seem to destroy most desirable proof-theoretic properties of a ND system ...

Cardinality Formulae[CG22]

Only One Predicate

$$C_0^{\{P\}} := \bot$$

$$C_1^{\{P\}} := \forall x ?Px$$

$$C_{n+1}^{\{P\}} := \exists x \mathbb{V}_{i=1}^{n} \Big[(Px \to C_i^{\{P\}}) \land (\neg Px \to C_{n+1-i}^{\{P\}}) \Big]$$

Cardinality Formulae [CG22]

Assuming all function symbols are rigid

$$
\begin{aligned}
C_0^{\Sigma} &:= \bot \\
C_1^{\Sigma} &:= \forall \overline{x}_1 ?R_1(\overline{x}_1) \wedge \ldots \wedge \forall \overline{x}_l ?R_l(\overline{x}_l) \\
C_{n+1}^{\Sigma} &:= \exists \overline{x}_1 \mathop{\bigvee\!\!\!\!\bigvee}_{i=1}^{n} \left[(R_1(\overline{x}_1) \to C_i^{\Sigma}) \wedge (\neg R_1(\overline{x}_1) \to C_{n+1-i}^{\Sigma}) \right] \;\mathbin{\vee\!\!\!\vee} \ldots \\
&\quad \ldots \mathbin{\vee\!\!\!\vee} \exists \overline{x}_l \mathop{\bigvee\!\!\!\!\bigvee}_{i=1}^{n} \left[(R_l(\overline{x}_l) \to C_i^{\Sigma}) \wedge (\neg R_l(\overline{x}_l) \to C_{n+1-i}^{\Sigma}) \right]
\end{aligned}
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Cardinality Formulae [CG22]

Adding equality to the syntax

$$C_0^{\Sigma} := \bot$$

$$C_1^{\Sigma} := \bigwedge_{j=1}^{l} \forall \overline{x}_j ?R_j(\overline{x}_j) \ \wedge \ \bigwedge_{j=1}^{h} \forall \overline{y}_j \exists z (f_j(\overline{y}_j) = z)$$

$$C_{n+1}^{\Sigma} := \mathbin{\rotatebox[origin=c]{180}{$\Cap$}}_{j=1}^{l} \exists \overline{x}_j \mathbin{\rotatebox[origin=c]{180}{$\Cap$}}_{i=1}^{n} [\ (R_j(\overline{x}_j) \to C_i^{\Sigma}) \wedge (\neg R_j(\overline{x}_j) \to C_{n+1-i}^{\Sigma})\ ] \ \mathbin{\rotatebox[origin=c]{180}{$\Cap$}}$$

$$\mathbin{\rotatebox[origin=c]{180}{$\Cap$}} \ \mathbin{\rotatebox[origin=c]{180}{$\Cap$}}_{j=1}^{h} \exists \overline{y}_j z \mathbin{\rotatebox[origin=c]{180}{$\Cap$}}_{i=1}^{n} [\ (f_j(\overline{y}_j) = z \ \to \ C_i^{\Sigma}) \ \wedge \ (f_j(\overline{y}_j) \neq z \ \to \ C_{n+1-i}^{\Sigma})\ ]$$

Schematic validity

- Proof theorists developed various criteria for well-designed ND and sequent systems
- The cardinality/coherence axioms we've seen above have a brutally Hilbertian flavour
- By its very nature, inquisitive logic cannot even meet a standard Hilbert-style criterion: closure under uniform substitution

- This naturally leads to the question of schematic validity in inquisitive logic:
- What is its schematic core/fragment, i.e., the largest standard superintuitionistic logic (closed under substition) contained in it?
- For the propositional inquisitive logic InqL, Ciardelli [Cia09] established that its schematic fragment is exactly Medvedev's logic ML of finite problems or finite (topless) boolean cubes.
- Conversely, InqL can be obtained as the negative counterpart of ML, i.e., the collection of formulas whose negatively substituted variants (replacing each atom with its negation) belong to ML.
- To the best of our knowledge, corresponding first-order characterizations do not exist.
- (This even after going through the exercise of formulating a suitable predicate notion of schematic validity and uniform substitution of formulas for predicates: Ono [Ono73], Church [Chu58, Ch. III], Gabbay, Shehtman and Skvortsov [GSS09, § 2.2–2.5], Kleene [Kle52, § VII.34, pp. 155–162] . . .)

A Sequent Calculus

- We provide a sequent calculus for $\mathsf{InqLogB}_{\Sigma}$ which we prove to be sound and complete for each $\mathsf{InqLogB}_{\Sigma,n}$ (with a corresponding restriction on labels)

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- Semantics of a labelled formula (X, ϕ) are given by a mapping $f \colon \mathbb{N} \to \mathrm{W}_{\mathfrak{M}}$.

- Semantics of a sequent $\Gamma \Rightarrow \Delta$:

$$\begin{array}{rll} \text{If } \mathfrak{M}, f, \eta \models (X, \phi) & \text{for all} & (X, \phi) \in \Gamma, \\ \text{then } \mathfrak{M}, f, \eta \models (X, \psi) & \text{for some} & (Y, \phi) \in \Delta \end{array}$$

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$$\begin{array}{rll} \text{If } \mathfrak{M}, f, \eta \models (X, \phi) & \text{for all} & (X, \phi) \in \Gamma, \\ \text{then } \mathfrak{M}, f, \eta \models (X, \psi) & \text{for some} & (Y, \phi) \in \Delta \end{array}$$

- The TABLEAUX setup for simplicity restricted to purely relational signatures; no function symbols, no complex terms, no discussion of rigidity
- A slightly adapted version suggested by my former FAU student MO Elliger covers , e.g., nontrivial rigid terms.
- Note that such extensions may break down metatheory, like completeness or cut elimination (although no evidence for that so far)
- Soundness of the extended variants of the calculi already formalized in Coq

Table 2 Sequent Calculus $\mathbf{G}(\mathsf{FBInqBQ})$

$$\frac{}{X : P(\overline{x}), \Gamma \Rightarrow \Delta, Y : P(\overline{x})}\ (\mathtt{id}) \text{ where } X \supseteq Y \qquad \frac{}{X : \bot, \Gamma \Rightarrow \Delta}\ (\bot \Rightarrow)$$

$$\frac{\{\Gamma \Rightarrow \Delta, \{k\} : P(\overline{x}) \mid k \in X\}}{\Gamma \Rightarrow \Delta, X : P(\overline{x})}\ (\Rightarrow \mathtt{at})$$

$$\frac{\Gamma \Rightarrow \Delta, X : \varphi \quad \Gamma \Rightarrow \Delta, X : \psi}{\Gamma \Rightarrow \Delta, X : \varphi \wedge \psi}\ (\Rightarrow \wedge) \qquad \frac{X : \varphi, X : \psi, \Gamma \Rightarrow \Delta}{X : \varphi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, X : \varphi, X : \psi}{\Gamma \Rightarrow \Delta, X : \varphi \vee\!\!\!\vee \psi}\ (\Rightarrow \vee\!\!\!\vee) \qquad \frac{X : \varphi, \Gamma \Rightarrow \Delta \quad X : \psi, \Gamma \Rightarrow \Delta}{X : \varphi \vee\!\!\!\vee \psi, \Gamma \Rightarrow \Delta}\ (\vee\!\!\!\vee \Rightarrow)$$

$$\frac{\{Y : \varphi, \Gamma \Rightarrow \Delta, Y : \psi \mid X \supseteq Y\}}{\Gamma \Rightarrow \Delta, X : \varphi \to \psi}\ (\Rightarrow \to)$$

$$\frac{X : \varphi \to \psi, \Gamma \Rightarrow \Delta, Y : \varphi \quad Y : \psi, X : \varphi \to \psi, \Gamma \Rightarrow \Delta}{X : \varphi \to \psi, \Gamma \Rightarrow \Delta}\ (\to \Rightarrow) \text{ where } X \supseteq Y$$

$$\frac{\Gamma \Rightarrow \Delta, X : \varphi[z/x]}{\Gamma \Rightarrow \Delta, X : \forall x.\varphi}\ (\Rightarrow \forall)\dagger \qquad \frac{X : \varphi[y/x], X : \forall x.\varphi, \Gamma \Rightarrow \Delta}{X : \forall x.\varphi, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, X : \exists\!\!\exists\, x.\varphi, X : \varphi[y/x]}{\Gamma \Rightarrow \Delta, X : \exists\!\!\exists\, x.\varphi}\ (\Rightarrow \exists\!\!\exists) \qquad \frac{X : \varphi[z/x], \Gamma \Rightarrow \Delta}{X : \exists\!\!\exists\, x.\varphi, \Gamma \Rightarrow \Delta}\ (\exists\!\!\exists \Rightarrow)\dagger$$

where $\dagger$ is the eigenvariable condition: z does not occur in the conclusion.

- Note: it is enough to be provable without the atomic rule to be schematically valid!
- The converse remains an open question

Truth Semantics

- Defined via support of singleton states:

$$\mathfrak{M}, w, \eta \models_{\mathsf{truth}} \phi :\Longleftrightarrow \mathfrak{M}, \{w\}, \eta \models \phi$$

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Example

$$\begin{array}{cll} \neg\neg P(x) \to P(x) & \in \mathsf{InqLog}_{\Sigma} & \\ \neg\neg\phi \to \phi & \in \mathsf{InqLogB}_{\Sigma,1} & \\ \neg\neg(P(x) \vee\!\!\!\vee \neg P(x)) \to (P(x) \vee\!\!\!\vee \neg P(x)) & \notin \mathsf{InqLogB}_{\Sigma,2} & \supsetneq \mathsf{InqLogB}_{\Sigma} \end{array}$$

The Casari Scheme

- Consider the following scheme:

$$\text{Casari} := (\forall x.\ (\phi\,(x) \to \forall x.\,\phi\,(x)) \to \forall x.\,\phi\,(x)) \to \forall x.\,\phi\,(x)$$

- We get the following properties:

$$(\forall x.\ (P\,(x) \to \forall x.\,P\,(x)) \to \forall x.\,P\,(x)) \to \forall x.\,P\,(x) \in \mathsf{InqLog}_{\Sigma}$$

$$(\forall x.\ (\phi\,(x) \to \forall x.\,\phi\,(x)) \to \forall x.\,\phi\,(x)) \to \forall x.\,\phi\,(x) \in \mathsf{InqLogB}_{\Sigma}$$

$$(\forall x.\ ((\mathbin{\rotatebox[origin=c]{180}{E}}\, y.R(x,y)) \to \forall x.\ \mathbin{\rotatebox[origin=c]{180}{E}}\, y.R(x,y)) \to \forall x.\ \mathbin{\rotatebox[origin=c]{180}{E}}\, y.R(x,y)) \to \forall x.\ \mathbin{\rotatebox[origin=c]{180}{E}}\, y.R(x,y) \notin \mathsf{InqLog}_{\Sigma}$$

The Casari Scheme

Regarding Schematic Bounded Validity

Theorem

The Casari Scheme is schematically bounded valid.[1]

Proof.

1. Prove that for every label X, the sequent $\Rightarrow (X, \mathrm{Casari})$ is derivable in the given sequent calculus.
2. By the rule $(\Rightarrow\rightarrow)$), it suffices to show for every $Y \subseteq X$ the derivability of the following sequent:

$$(Y, \forall x.\ (\phi(x) \rightarrow \forall x.\ \phi(x)) \rightarrow \forall x.\ \phi(x)) \Rightarrow (Y, \forall x.\ \phi(x))$$

3. Use wellfounded induction on Y to proceed. Proof uses the rule of cut.

□

[1] [LS25]

The Casari Scheme
Regarding Schematic Validity

Theorem
The Casari Scheme is not schematically valid, e.g. Casari *instantiated with* $\phi := \exists y.R(x, y)$ *is not schematically valid.*

Proof Sketch.
By a suitable counterexample. . . whose formalization took MO Elliger quite a while, and then a TABLEAUX referee proposed a dramatically simplified version. We're still recovering from the shock. □

3. Conclusions & Future Work

Conclusions & Future Work

Conclusions

- We introduced cut-free labelled sequent calculi complete for n-bounded inquisitive logics.
- We illustrate the intricacies of schematic validity in such systems by showing that
  - the well-known Casari formula is *atomically* valid in (a weak sublogic of) predicate inquisitive logic InqBQ,
  - fails to be schematically valid in it, and yet
  - is schematically valid under the finite boundedness assumption.
- The derivations in our calculi, however, are guaranteed to be schematically valid whenever a single specific rule is not used.
- (not discussed here, see a remark in the paper) We can capture entailments with so-called rex conclusions without additional rules
- We are also seeing the benefits of working with a (nascent) Coq/Rocq formalization: more about it in the afternoon at the Rocqshop!

Conclusions & Future Work

Future Work

- The Craig interpolation property for logics considered herein?
- Relationship with papers concerning model theory and correspondence theory of extensions of CD [MTO90; Ono73]?
- Resolving Ciardelli and Grilletti's challenge of algorithmically identifying formulas coherent for a fixed cardinality?
- Extend existing computational interpretations of sequent calculi to this setting?
- Potential database connections, e.g., the discussion of Armstrong relations by Abramsky and Väänänen [AV09] or Ciardelli's perspective on mention-some questions [**Ciardelli2016**]?

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