Intuitionistic BV
(& Friends)

Matteo Acclavio & Lutz Straßburger

TABLEAUX 2025 Reykjavik 28/09/2025

Motivations

Proof theory is born to study **causality** (in a functional sense)

Proof theory is born to study **causality** (in a functional sense)

$$A \Rightarrow B$$

Proof theory is born to study **causality** (in a functional sense)

$$A \Rightarrow B$$

Theorem (Deduction Theorem [in any 'reasonable' logic])

If I can prove B from A, then I can prove $A \Rightarrow B$.

Proof theory is born to study **causality** (in a functional sense)

$$A \Rightarrow B$$

Theorem (Deduction Theorem [in any 'reasonable' logic])

If I can prove B from A, then I can prove $A \Rightarrow B$.

…but what if we want to consider **sequentiality**?

- Concurrent systems

$$a \mid \bar{a}.b \mid \bar{b} \mid c \mid \bar{c}$$

- Imperative programming

$$\boxed{\begin{array}{l} x = 5 \\ x = 2 \\ y = x + 2 \end{array}} \qquad \neq \qquad \boxed{\begin{array}{l} x = 5 \\ y = x + 2 \\ x = 2 \end{array}}$$

- Sequential algorithms
- ...

- Concurrent systems

$$a \mid \bar{a}.b \mid \bar{b} \mid c \mid \bar{c}$$

- Imperative programming

$$\boxed{\begin{array}{l} x = 5 \\ x = 2 \\ y = x + 2 \end{array}} \qquad \neq \qquad \boxed{\begin{array}{l} x = 5 \\ y = x + 2 \\ x = 2 \end{array}}$$

- Sequential algorithms
- ...

How to model these things logically?

- Concurrent systems

$$a \mid \bar{a}.b \mid \bar{b} \mid c \mid \bar{c}$$

- Imperative programming

$$\boxed{\begin{array}{l} x = 5 \\ x = 2 \\ y = x + 2 \end{array}} \quad \neq \quad \boxed{\begin{array}{l} x = 5 \\ y = x + 2 \\ x = 2 \end{array}}$$

- Sequential algorithms
- ...

How to model these things logically?

WHY?

- Concurrent systems

$$a \mid \bar{a}.b \mid \bar{b} \mid c \mid \bar{c}$$

- Imperative programming

$$\boxed{\begin{array}{l} x = 5 \\ x = 2 \\ y = x + 2 \end{array}} \qquad \neq \qquad \boxed{\begin{array}{l} x = 5 \\ y = x + 2 \\ x = 2 \end{array}}$$

- Sequential algorithms
- ...

How to model these things logically?

WHY?

Type systems for imperative programming, (better) type systems for process calculi, logical models of sequential algorithms, ...

IMLL and BV

IMLL

IMLL

$$\text{ax}\ \frac{}{a \vdash a} \qquad \multimap_R\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \multimap_L\ \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C} \qquad \text{cut}\ \frac{\Gamma \vdash A \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

$$\mathbb{I}_R\ \frac{}{\vdash \mathbb{I}} \qquad \mathbb{I}_L\ \frac{\Gamma \vdash A}{\Gamma, \mathbb{I} \vdash A} \qquad \otimes_L\ \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \otimes_R\ \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \text{AX}\ \frac{}{A \vdash A}$$

IMLL

$$\text{ax}\ \frac{}{a \vdash a} \qquad \multimap_R \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \multimap_L \frac{\Gamma \vdash A \quad B, \Delta \vdash C}{\Gamma, A \multimap B, \Delta \vdash C} \qquad \Bigg| \qquad \text{cut}\ \frac{\Gamma \vdash A \quad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

$$\mathbb{I}_R \frac{}{\vdash \mathbb{I}} \qquad \mathbb{I}_L \frac{\Gamma \vdash A}{\Gamma, \mathbb{I} \vdash A} \qquad \otimes_L \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad \otimes_R \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \Bigg| \qquad \text{AX}\ \frac{}{A \vdash A}$$

Trivia about IMLL:

- LJ = IMLL + structural rules
- IMLL type system for linear λ-calculus
- categorical model: symmetric monoidal closed category
- Cut-elimination (β-reduction)

BV

BV

$$A, B := a \mid \bar{a} \mid A \parr B \mid A \otimes B$$

BV

$$A, B \coloneqq a \mid \bar{a} \mid \mathbb{I} \mid A \parr B \mid A \otimes B$$

BV

$$A, B \coloneqq a \mid \bar{a} \mid \mathbb{I} \mid A \parr B \mid A \otimes B$$

Rules

$$\equiv \frac{A}{B} \dagger \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr \bar{a}}$$

BV

$$A, B \coloneqq a \mid \bar{a} \mid \mathbb{I} \mid A \parr B \mid A \otimes B$$

Rules

$$\equiv \frac{A}{B}\,\dagger \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr \bar{a}}$$

$$\dagger = \boxed{\begin{array}{l} \otimes \text{ is associative and commutative} \\ \\ \parr \text{ is associative and commutative} \\ \mathbb{I} \text{ is a unit for } \otimes, \text{ and } \parr \end{array}} \qquad \boxed{\mathsf{ai}{\uparrow}\,\frac{a \otimes \bar{a}}{\mathbb{I}}}$$

BV

$$A, B \coloneqq a \mid \bar{a} \mid \mathbb{I} \mid A \parr B \mid A \otimes B$$

Rules

$$\equiv \frac{A}{B}\,\dagger \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr \bar{a}}$$

$\dagger =$

$\otimes$ is associative and commutative
$\parr$ is associative and commutative
$\mathbb{I}$ is a unit for $\otimes$, and $\parr$

$$\boxed{\mathsf{ai}{\uparrow}\,\frac{a \otimes \bar{a}}{\mathbb{I}}}$$

Derivations (Deep Inference)

$$\mathcal{D}, \mathcal{D}' \coloneqq A \;\Big|\; \mathcal{D} \parr \mathcal{D}' \;\Big|\; \mathcal{D} \triangleleft \mathcal{D}' \;\Big|\; \mathcal{D} \otimes \mathcal{D}' \;\Big|\; \mathsf{r}\,\frac{\mathcal{D}}{\mathcal{D}'} \;\Big|\; \mathsf{r}\,\genfrac{}{}{0pt}{}{\mathcal{D}}{\mathcal{D}'}$$

BV

$$A, B := a \mid \bar{a} \mid \mathbb{I} \mid A \mathbin{\rotatebox[origin=c]{180}{\&}} B \mid A \otimes B \mid A \triangleleft B$$

Rules

$$\equiv \frac{A}{B}\,\dagger \qquad \mathsf{s}\,\frac{A \otimes (B \mathbin{\rotatebox[origin=c]{180}{\&}} C)}{(A \otimes B) \mathbin{\rotatebox[origin=c]{180}{\&}} C} \qquad \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \mathbin{\rotatebox[origin=c]{180}{\&}} \bar{a}} \qquad \mathsf{q}{\downarrow}\,\frac{(A \mathbin{\rotatebox[origin=c]{180}{\&}} B) \triangleleft (C \mathbin{\rotatebox[origin=c]{180}{\&}} D)}{(A \triangleleft C) \mathbin{\rotatebox[origin=c]{180}{\&}} (B \triangleleft D)}$$

$\dagger =$

$\otimes$ is associative and commutative
$\triangleleft$ is associative
$\mathbin{\rotatebox[origin=c]{180}{\&}}$ is associative and commutative
$\mathbb{I}$ is a unit for $\otimes$, $\triangleleft$, and $\mathbin{\rotatebox[origin=c]{180}{\&}}$

$$\mathsf{ai}{\uparrow}\,\frac{a \otimes \bar{a}}{\mathbb{I}} \qquad \mathsf{q}{\uparrow}\,\frac{(A \triangleleft C) \otimes (B \triangleleft D)}{(A \otimes B) \triangleleft (C \otimes D)}$$

Derivations (Deep Inference)

$$\mathcal{D}, \mathcal{D}' := A \mid \mathcal{D} \mathbin{\rotatebox[origin=c]{180}{\&}} \mathcal{D}' \mid \mathcal{D} \triangleleft \mathcal{D}' \mid \mathcal{D} \otimes \mathcal{D}' \mid \mathsf{r}\,\frac{\mathcal{D}}{\mathcal{D}'} \mid \mathsf{r}\,\frac{\mathcal{D}}{\mathcal{D}'}$$

BV

Trivia about BV:

- it extends $\mathsf{MLL} \cup \{\mathsf{mix}\}$ with a non-commutative connective $\triangleleft$;

BV

Trivia about BV:

- it extends $\mathsf{MLL} \cup \{\mathsf{mix}\}$ with a non-commutative connective $\triangleleft$;
- executions of processes can be interpreted as proof search in BV;

BV

Trivia about BV:

- it extends MLL $\cup$ $\{\text{mix}\}$ with a non-commutative connective $\triangleleft$;
- executions of processes can be interpreted as proof search in BV;
- categorical model: isomix category with a degenerate linear functor;

BV

Trivia about BV:

- it extends MLL $\cup$ {mix} with a non-commutative connective $\triangleleft$;
- executions of processes can be interpreted as proof search in BV;
- categorical model: isomix category with a degenerate linear functor;
- BV $\subsetneq$ Pomset Logic (coherent spaces);

BV

Trivia about BV:

- it extends $\mathsf{MLL} \cup \{\mathsf{mix}\}$ with a non-commutative connective $\triangleleft$;
- executions of processes can be interpreted as proof search in BV;
- categorical model: isomix category with a degenerate linear functor;
- BV $\subsetneq$ Pomset Logic (coherent spaces);
- "Cut-elimination" (which implies transitivity of $\multimap$);

BV

Trivia about BV:

- it extends MLL $\cup$ {mix} with a non-commutative connective $\triangleleft$;
- executions of processes can be interpreted as proof search in BV;
- categorical model: isomix category with a degenerate linear functor;
- BV $\subsetneq$ Pomset Logic (coherent spaces);
- "Cut-elimination" (which implies transitivity of $\multimap$);
- since few years trending in the quantum community;

BV

Trivia about BV:

- it extends MLL $\cup$ {mix} with a non-commutative connective $\triangleleft$;
- executions of processes can be interpreted as proof search in BV;
- categorical model: isomix category with a degenerate linear functor;
- BV $\subsetneq$ Pomset Logic (coherent spaces);
- "Cut-elimination" (which implies transitivity of $\multimap$);
- since few years trending in the quantum community;
- recent new applications to concurrent PLs;

BV

Trivia about BV:

- it extends MLL $\cup$ {mix} with a non-commutative connective $\triangleleft$;
- executions of processes can be interpreted as proof search in BV;
- categorical model: isomix category with a degenerate linear functor;
- BV $\subsetneq$ Pomset Logic (coherent spaces);
- "Cut-elimination" (which implies transitivity of $\multimap$);
- since few years trending in the quantum community;
- recent new applications to concurrent PLs;
- ...

BV

Theorem

The rule $\mathsf{cut}\,\dfrac{A \otimes \bar{A}}{\mathbb{I}}$ *is admissible in* BV.

Proof.

- Splitting lemma ($\odot \in \{\triangleleft, \otimes\}$):

$$\vdash_{\mathsf{BV}} (A \odot B) \mathbin{⅋} K \quad \Rightarrow \quad \begin{array}{c} K_A \bar{\odot} K_B \\ \| \\ K \end{array} \quad \text{and} \quad \begin{array}{c} \| \\ K_A \mathbin{⅋} A \end{array} \quad \text{and} \quad \begin{array}{c} \| \\ K_B \mathbin{⅋} B \end{array} .$$

- Context reduction:

$$\vdash_{\mathsf{BV}} C[A] \quad \Rightarrow \quad \begin{array}{c} K \mathbin{⅋} X \\ \mathcal{P}_X \| \\ P[X] \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{P}_A \| \\ K \mathbin{⅋} A \end{array} \quad \text{for any formula } X.$$

- up-rules elimination:

$$\vdash_{\mathsf{BV}} C\left[\mathsf{r}\!\uparrow \frac{A}{B}\right] \quad \Rightarrow \quad \begin{array}{c} \|\mathsf{BV} \\ C[B] \end{array}$$

□

IBV

From MLL to IMLL

Where to start?

Take MLL

$$\mathsf{ax}\,\frac{}{\vdash a, \bar{a}} \qquad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

From MLL to IMLL

Where to start?

Take MLL

$$\text{ax}\,\frac{}{\vdash a, \bar{a}} \qquad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

Polarize formulas

positive: $A^\circ, B^\circ \coloneqq a \mid A^\bullet \parr B^\circ \mid A^\circ \otimes B^\circ$

negative: $A^\bullet, B^\bullet \coloneqq \bar{a} \mid A^\circ \otimes B^\bullet \mid A^\bullet \parr B^\bullet$

From MLL to IMLL

Where to start?

Take MLL

$$\mathsf{ax}\,\frac{}{\vdash a, \bar{a}} \qquad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

Polarize formulas

positive: $A^\circ, B^\circ \coloneqq a \mid A^\bullet \parr B^\circ \mid A^\circ \otimes B^\circ$

negative: $A^\bullet, B^\bullet \coloneqq \bar{a} \mid A^\circ \otimes B^\bullet \mid A^\bullet \parr B^\bullet$

IMLL = positive MLL

From MLL to IMLL

Where to start?

Take MLL

$$\mathsf{ax}\,\frac{}{\vdash a, \bar{a}} \qquad \parr\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B} \qquad \otimes\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad \mathsf{mix}\,\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \Delta}$$

Polarize formulas

positive: $A^\circ, B^\circ \coloneqq a \mid A^\bullet \parr B^\circ \mid A^\circ \otimes B^\circ$

negative: $A^\bullet, B^\bullet \coloneqq \bar{a} \mid A^\circ \otimes B^\bullet \mid A^\bullet \parr B^\bullet$

$\text{IMLL} = \text{positive } \text{MLL}_{\mathsf{mix}}$

BV from IBV

Take BV

$$\equiv \frac{A}{B} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr \bar{a}} \qquad \mathsf{q}{\downarrow}\,\frac{(A \parr B) \triangleleft (C \parr D)}{(A \triangleleft C) \parr (B \triangleleft D)}$$

BV from IBV

Take BV

$$\equiv \frac{A}{B} \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr \bar{a}} \qquad \mathsf{q}{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

Polarize formulas

$$\begin{array}{llll} \text{positive:} & A^\circ, B^\circ & := & \mathbb{I} \mid a \mid A^\bullet \parr B^\circ \mid A^\circ \otimes B^\circ \mid A^\circ \lhd B^\circ \\ \text{negative:} & A^\bullet, B^\bullet & := & \mathbb{I} \mid \bar{a} \mid A^\circ \otimes B^\bullet \mid A^\bullet \parr B^\bullet \mid A^\bullet \lhd B^\bullet \end{array}$$

. . . fine tune units (otherwise $\otimes$ and $\triangleleft$ collapse)

…fine tune units (otherwise $\otimes$ and $\triangleleft$ collapse)

$$\mathsf{ai}^{\circ}_{\downarrow}\ \dfrac{\mathbb{I}}{a \multimap a}\circ$$

$$\mathsf{s}^{\circ}_{\mathsf{L}}\ \dfrac{A \otimes (B \multimap C)}{(A \multimap B) \multimap C}\circ \qquad \mathsf{s}^{\circ}_{\mathsf{R}}\ \dfrac{(A \multimap B) \otimes C}{A \multimap (B \otimes C)}\circ \qquad \mathsf{s}^{\bullet}_{\mathsf{L}}\ \dfrac{(A \multimap B) \multimap C}{A \otimes (B \multimap C)}\bullet \qquad \mathsf{s}^{\bullet}_{\mathsf{R}}\ \dfrac{A \multimap (B \otimes C)}{(A \multimap B) \otimes C}\bullet$$

$$\mathsf{com}^{\otimes}\ \dfrac{A \otimes B}{B \otimes A} \qquad \mathsf{asso}^{\otimes}\ \dfrac{(A \otimes B) \otimes C}{A \otimes (B \otimes C)}$$

$$\mathsf{u}^{\otimes}_{\downarrow}\ \dfrac{A}{\mathbb{I} \otimes A} \qquad \mathsf{u}^{\multimap}_{\downarrow}\ \dfrac{A}{\mathbb{I} \multimap A} \qquad \mathsf{cur}\ \dfrac{(A \otimes B) \multimap C}{A \multimap (B \multimap C)} \qquad \mathsf{ruc}\ \dfrac{A \multimap (B \multimap C)}{(A \otimes B) \multimap C}$$

IBV

. . . fine tune units (otherwise $\otimes$ and $\lhd$ collapse)

$$\mathsf{ai}_{\downarrow}^{\circ}\,\frac{\mathbb{I}}{a \multimap a} \qquad \mathsf{u}_{\downarrow}^{\lhd}\,\frac{A}{\mathbb{I} \lhd A} \qquad \mathsf{u}_{\downarrow}^{\rhd}\,\frac{A}{A \lhd \mathbb{I}} \qquad \mathsf{ref}^{\circ}\,\frac{A \otimes B}{A \lhd B} \qquad \mathsf{ref}^{\bullet}\,\frac{A \lhd B}{A \otimes B}$$

$$\mathsf{s}_{\mathsf{L}}^{\circ}\,\frac{A \otimes (B \multimap C)}{(A \multimap B) \multimap C} \qquad \mathsf{s}_{\mathsf{R}}^{\circ}\,\frac{(A \multimap B) \otimes C}{A \multimap (B \otimes C)} \qquad \mathsf{s}_{\mathsf{L}}^{\bullet}\,\frac{(A \multimap B) \multimap C}{A \otimes (B \multimap C)} \qquad \mathsf{s}_{\mathsf{R}}^{\bullet}\,\frac{A \multimap (B \otimes C)}{(A \multimap B) \otimes C}$$

$$\mathsf{sq}_{\mathsf{L}}^{\circ}\,\frac{(A \multimap B) \lhd C}{A \multimap (B \lhd C)} \qquad \mathsf{sq}_{\mathsf{R}}^{\circ}\,\frac{B \lhd (A \multimap C)}{A \multimap (B \lhd C)} \qquad \mathsf{sq}_{\mathsf{L}}^{\bullet}\,\frac{(A \otimes B) \lhd C}{A \otimes (B \lhd C)} \qquad \mathsf{sq}_{\mathsf{R}}^{\bullet}\,\frac{B \lhd (A \otimes C)}{A \otimes (B \lhd C)}$$

$$\mathsf{q}_{\downarrow}^{\circ}\,\frac{(A \multimap B) \lhd (C \multimap D)}{(A \lhd C) \multimap (B \lhd D)} \qquad \mathsf{q}_{\downarrow}^{\bullet}\,\frac{(A \otimes B) \lhd (C \otimes D)}{(A \lhd C) \otimes (B \lhd D)}$$

- -

$$\mathsf{com}^{\otimes}\,\frac{A \otimes B}{B \otimes A} \qquad \mathsf{asso}^{\otimes}\,\frac{(A \otimes B) \otimes C}{A \otimes (B \otimes C)} \qquad \mathsf{asso}_{\mathsf{L}}^{\lhd}\,\frac{(A \lhd B) \lhd C}{A \lhd (B \lhd C)} \qquad \mathsf{asso}_{\mathsf{R}}^{\lhd}\,\frac{A \lhd (B \lhd C)}{(A \lhd B) \lhd C}$$

$$\mathsf{u}_{\downarrow}^{\otimes}\,\frac{A}{\mathbb{I} \otimes A} \qquad \mathsf{u}_{\downarrow}^{\multimap}\,\frac{A}{\mathbb{I} \multimap A} \qquad \mathsf{cur}\,\frac{(A \otimes B) \multimap C}{A \multimap (B \multimap C)} \qquad \mathsf{ruc}\,\frac{A \multimap (B \multimap C)}{(A \otimes B) \multimap C}$$

IBV (Categorically)

IMLL	BV
$\langle \otimes, \parr, \mathbb{I} \rangle$ symmetric monoidal closed category	$\langle \otimes, \parr, \mathbb{I} \rangle$ isomix category $+$ degenerate linear functor $\triangleleft$ $\big((A \triangleleft C) \otimes (B \triangleleft D)\big) \Rightarrow \big((A \otimes B) \triangleleft (C \otimes D)\big)$

IBV (Categorically)

IMLL	BV
$\langle \otimes, \parr, \mathbb{I} \rangle$ symmetric monoidal closed category	$\langle \otimes, \parr, \mathbb{I} \rangle$ isomix category $+$ degenerate linear functor $\triangleleft$ $\big((A \triangleleft C) \otimes (B \triangleleft D)\big) \Rightarrow \big((A \otimes B) \triangleleft (C \otimes D)\big)$

$$\langle \otimes, \multimap, \mathbb{I} \rangle + \text{degenerate linear functor-ish } \triangleleft + \left(\begin{array}{l} A \multimap (A \triangleleft \mathbb{I}) \\ A \multimap (\mathbb{I} \triangleleft A) \end{array} \right)$$

Theorem

The rule $\mathsf{cut}\,\dfrac{A \multimap \bar{A}}{\mathbb{I}}$ *is admissible in* IBV*.*

Theorem

The rule $\mathsf{cut}\,\dfrac{A \multimap \bar{A}}{\mathbb{I}}$ *is admissible in* IBV.

Theorem

These rules are admissible in IBV*:*

$$\mathsf{ai}^{\bullet}_{\uparrow}\,\frac{a \multimap a}{\mathbb{I}}\bullet \qquad \mathsf{u}^{\triangleleft}_{\uparrow}\,\frac{\mathbb{I} \triangleleft A}{A}\bullet \qquad \mathsf{u}^{\triangleright}_{\uparrow}\,\frac{A \triangleleft \mathbb{I}}{A}\bullet \qquad \mathsf{u}^{\otimes}_{\uparrow}\,\frac{\mathbb{I} \otimes A}{A} \qquad \mathsf{u}^{\multimap}_{\uparrow}\,\frac{\mathbb{I} \multimap A}{A}$$

$$\mathsf{q}^{\circ}_{\uparrow}\,\frac{(A \triangleleft C) \otimes (B \triangleleft D)}{(A \otimes B) \triangleleft (C \otimes D)}\circ \qquad \mathsf{q}^{\bullet}_{\uparrow}\,\frac{(A \triangleleft C) \multimap (B \triangleleft D)}{(A \multimap B) \triangleleft (C \multimap D)}\bullet$$

Lemma (splitting)

- *If* $\vdash_{\mathsf{IBV}} K \multimap (A \otimes B)$, *then there are formulas* K_A *and* K_B *such that*

$$\begin{array}{c} K_A \otimes K_B \\ \Vert \\ K \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ K_A \multimap A \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ K_B \multimap B \end{array} .$$

- *If* $\vdash_{\mathsf{IBV}} (A \multimap B) \multimap K$, *then there are formulas* K_A *and* K_B *such that*

$$\begin{array}{c} K_A \multimap K_B \\ \Vert \\ K \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ K_A \multimap A \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ B \multimap K_B \end{array} .$$

- *If* $\vdash_{\mathsf{IBV}} K \multimap (A \triangleleft B)$, *then there are formulas* K_A *and* K_B *such that*

$$\begin{array}{c} K_A \triangleleft K_B \\ \Vert \\ K \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ K_A \multimap A \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ K_B \multimap B \end{array} .$$

- *If* $\vdash_{\mathsf{IBV}} (A \triangleleft B) \multimap K$, *then there are formulas* K_A *and* K_B *such that*

$$\begin{array}{c} K_A \triangleleft K_B \\ \Vert \\ K \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ A \multimap K_A \end{array} \quad \text{and} \quad \begin{array}{c} \prod \\ B \multimap K_B \end{array} .$$

Lemma (Atomic Splitting)

1. *if* $\vdash_{\mathsf{IBV}} K \multimap a$, *then there is a negative derivation* $\begin{array}{c} a \\ \Vert\!\!\bullet \\ K \end{array}$;

2. *if* $\vdash_{\mathsf{IBV}} a \multimap K$, *then there is a positive derivation* $\begin{array}{c} a \\ \Vert \\ K \end{array}$.

Lemma (Context Reduction)

1. *If* $\vdash_{\mathsf{IBV}} P[A]$ *with* $P[\cdot]$ *a positive context*

$$\begin{array}{c} K \multimap X \\ \mathcal{P}_X \Vert \\ P[X] \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{P}_A \Vert \\ K \multimap A \end{array} \quad \text{for any formula } X.$$

2. *If* $\vdash_{\mathsf{IBV}} N[A]$ *with* $N[\cdot]$ *a negative context*

$$\begin{array}{c} X \multimap K \\ \mathcal{P}_X \Vert \\ N[X] \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{P}_A \Vert \\ A \multimap K \end{array} \quad \text{for any formula } X.$$

INML

$$\mathsf{INML} = \mathsf{IMLL} \cup \left\{ \lhd \frac{\Gamma, A_1, \ldots, A_n \vdash A \quad \Delta, B_1, \ldots, B_n \vdash B}{\Gamma, \Delta, A_1 \lhd B_1, \ldots, A_n \lhd B_n \vdash A \lhd B} \; n \geq 0 \right\}$$

$$\mathsf{INML} = \mathsf{IMLL} \cup \left\{ \triangleleft \frac{\Gamma, A_1, \ldots, A_n \vdash A \quad \Delta, B_1, \ldots, B_n \vdash B}{\Gamma, \Delta, A_1 \triangleleft B_1, \ldots, A_n \triangleleft B_n \vdash A \triangleleft B} \; n \geq 0 \right\}$$

Theorem (Cut Elimination)

The cut-rule is admissible in INML.

$$\mathsf{INML} = \mathsf{IMLL} \cup \left\{ \lhd \frac{\Gamma, A_1, \ldots, A_n \vdash A \quad \Delta, B_1, \ldots, B_n \vdash B}{\Gamma, \Delta, A_1 \lhd B_1, \ldots, A_n \lhd B_n \vdash A \lhd B} \; n \geq 0 \right\}$$

Theorem (Cut Elimination)

The cut-rule is admissible in INML.

Theorem

INML *is a conservative extension of* IMLL.

$$\mathsf{INML} = \mathsf{IMLL} \cup \left\{ \triangleleft \frac{\Gamma, A_1, \ldots, A_n \vdash A \quad \Delta, B_1, \ldots, B_n \vdash B}{\Gamma, \Delta, A_1 \triangleleft B_1, \ldots, A_n \triangleleft B_n \vdash A \triangleleft B} \; n \geq 0 \right\}$$

Theorem (Cut Elimination)

The cut-rule is admissible in INML.

Theorem

INML *is a conservative extension of* IMLL.

Theorem

$$\mathsf{IBV} = \mathsf{INML} \cup \left\{ \begin{array}{c} \text{a-cut}_{\mathsf{L}} \dfrac{\Gamma \vdash (A \triangleleft B) \triangleleft C \quad A \triangleleft (B \triangleleft C), \Delta \vdash D}{\Gamma, \Delta \vdash D} \\ \text{a-cut}_{\mathsf{R}} \dfrac{\Gamma \vdash A \triangleleft (B \triangleleft C) \quad (A \triangleleft B) \triangleleft C, \Delta \vdash D}{\Gamma, \Delta \vdash D} \end{array} \right\}$$

Conclusion

Now

$$
\begin{tikzcd}
\mathsf{MLL}_{\mathsf{mix}} \arrow[r] & \mathsf{NML} \arrow[r] & \mathsf{BV} \\
\mathsf{IMLL} \arrow[u] \arrow[r] & \boxed{\mathsf{INML}} \arrow[u] \arrow[r] & \boxed{\mathsf{IBV}} \arrow[u]
\end{tikzcd}
$$

Now

$$
\begin{tikzcd}
\mathsf{MLL}_{\mathsf{mix}} \arrow[r] & \mathsf{NML} \arrow[r] & \mathsf{BV} \\
\mathsf{IMLL} \arrow[u] \arrow[r] & \boxed{\mathsf{INML}} \arrow[u] \arrow[r] & \boxed{\mathsf{IBV}} \arrow[u]
\end{tikzcd}
$$

Next

$$
\begin{tikzcd}
\mathsf{LJ} \arrow[r] & \boxed{\mathsf{NLJ}} \arrow[r] & \boxed{\mathsf{BLJ}} \\
\mathsf{IMLL} \arrow[u] \arrow[r] & \mathsf{INML} \arrow[u] \arrow[r] & \mathsf{IBV} \arrow[u]
\end{tikzcd}
$$

Thanks

Thanks

Questions?