Weighted Rewriting

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FroCoS'25, Reykjavik

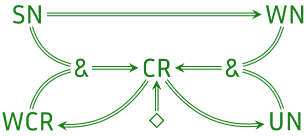
Abstract reduction systems

abstract reduction system (ARS) defined through:

- ⋆ set of objects A (e.g., terms, graphs, configurations, etc)
- ⋆ binary relation $\rightarrow \subseteq A \times A$, the *reduction relation*

give rise to a rich theory of general properties of reduction

- ⋆ termination (aka. strong normalization, **SN**)
- ⋆ confluence (aka. Church-Rosser, **CR**)
- ⋆ unique normal forms (**UN**)
- ⋆ ...



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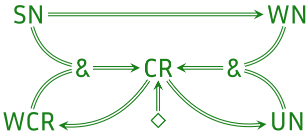

ARSs do not capture quantitative aspects well

- metrical reasoning
- complexity
- probabilistic program properties
- ...

Outline

1. Weighted abstract reduction systems (wARSs)
2. Instances
3. Quantitative *termination-like* properties (boundedness)
4. Embeddings & ranking functions

From qualitative to quantitative ARS

1. equip objects with quantitative information $obs : A \to W$
   - quantitative ARSs à la (Faggian, 2022; Ariola & Blom, 2002)
     - observation capture information
     - assumes CPO structure $(W, \leq)$ compatible with reductions ($a \to b$ implies $obs(a) \leq obs(b)$)
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2. turn relation $\to \subseteq A \times A$, i.e., $\to \equiv A \times A \to \{0, 1\}$, into functions $R : A \times A \to W$
   - quantitative ARSs à la (Gavazzo & Florio, 2023)
     - weights W capture distances
     - given as quantals $(W, 0, +, \leq)$ (monoid + complete lattice), satisfying certain distributivity laws
     - quantal structure enables relation composition $(R; S)(a, c) = \bigvee_{b \in A}(R(a, b) + S(b, c))$

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3. associate step directly with weight $R \subseteq W \times A \times A$
   - many instances in the literature
     - weighted automata for cost analysis, natural language processing, probabilistic systems, etc.
     - monoid measured ARSs abstracting over reduction lengths (van Oostrom & Toyama, 2016)
     - termination & complexity analysis of probabilistic systems (A. et al., 2018,2020,2021,2022)
     - ...
   - most general, few intrinsic constraints on weights

Weighted ARSs

Given monoid $\mathcal{W} = (W, 0, +)$ of weights.

- $\mathcal{W}$-weighted ARS : $R \subseteq W \times A \times A$

$$a \; R^{[w]} \; b :\Longleftrightarrow (w, a, b) \in R$$

- weighted order : $\rightsquigarrow \subseteq W \times A \times A$
  - reflexive : $a \rightsquigarrow^{[0]} a$
  - transitive : $a \rightsquigarrow^{[v]} b \rightsquigarrow^{[w]} c \quad \Longrightarrow \quad a \rightsquigarrow^{[v+w]} c$

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- weighted order $\rightsquigarrow_R$: least weighted order extending R

$$\rightsquigarrow_R^* \triangleq \bigcup_{w \in W} \rightsquigarrow_R^w \qquad \rightsquigarrow_R^+ \triangleq \bigcup_{w \neq 0} \rightsquigarrow_R^w \qquad \mathsf{NF}(R) \triangleq \{a \mid \nexists b. a \rightsquigarrow_R^+ b\}$$

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Example — Uniform weighted ARSs

given ARS $\to \subseteq A \times A$, unitary weighted ARS assigns weight $1 :: \mathbb{N}$ to each step

$$\{1\} \times \to = \{(1, a, b) \mid a \to b\}$$

Observations

- $\to^n = (\{1\} \times \to)^n$
- $\to^+ = (\{1\} \times \to)^+$
- $\to^* = (\{1\} \times \to)^*$
- $\mathrm{NF}(\to) = \mathrm{NF}(\{1\} \times \to)$

Example — Weighted Term Rewriting

- ⋆ weighted term rewrite system : $\mathcal{R} \subseteq W \times \mathcal{T}(F, V) \times \mathcal{T}(F, V)$

$$0 + y \;\mathcal{R}^{[1]}\; y \qquad \mathtt{s}(x) + y \;\mathcal{R}^{[2]}\; \mathtt{s}(x + y) \qquad x + y \;\mathcal{R}^{[0]}\; y + x$$

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- weighted rewrite relation: $\rightarrow \subseteq W \times \mathcal{T}(F, V) \times \mathcal{T}(F, V)$
  - closed under substitutions: $s \rightarrow^{[w]} t \implies s\theta \rightarrow^{[w]} t\theta$
  - closed under contexts: $s \rightarrow^{[w]} t \implies \mathtt{f}(\dots, s, \dots) \rightarrow^{[w]} \mathtt{f}(\dots, t, \dots)$

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- weighted rewrite relation $\to_{\mathcal{R}}$: least weighted rewrite relation extending $\mathcal{R}$

$$\underline{\mathtt{S}(x) + 0} \to_{\mathcal{R}}^{[2]} \mathtt{S}(\underline{x + 0}) \to_{\mathcal{R}}^{[0]} \mathtt{S}(\underline{0 + x}) \to_{\mathcal{R}}^{[1]} \mathtt{S}(x) \qquad \text{i.e.} \qquad \mathtt{S}(x) + 0 \to_{\mathcal{R}}^{3} \mathtt{S}(x)$$

Example — Probabilistic ARSs

⋆ probabilistic ARS : $\mathcal{P} \subseteq A \times \mathsf{Dist}(A)$ (Bournez & Garnier'05)

$$\mathtt{w}(n)\ \mathcal{P}\ \{\mathtt{w}(n-1)^{1/2}, \mathtt{w}(n+1)^{1/2}\} \qquad (\text{for } n > 0)$$

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- probabilistic rewrite relation $\hookrightarrow_{\mathcal{P}}$: least convex closed weighted order (refl., trans.) extending $\mathcal{P}$

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&\hookrightarrow^{3/8}_{\mathcal{P}} \ldots
\end{aligned}$$

Example — Barycentric ARSs

- ⋆ barycentric ARS: $\mathcal{B} \subseteq W \times A \times A$ where monoid W and A are $\mathcal{M}$-abgebras
  - $\mathcal{M}$-abgebra X: X equipped with barycentric operator $\sum_i p_i \cdot x_i$ ($\sum_i p_i = 1$)
  - generalizes convex combinator $+_p : X \times X \to X$ of convex space/barycentric algebra
- ⋆ barycentric rewrite relation : $\rightsquigarrow \subseteq W \times A \times A$
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Instances

- (weighted) probabilistic ARSs $\mathcal{P}^{[\cdot]} \subseteq W \times A \times \mathsf{Dist}(A)$ with $\sum_i p_i \cdot \mu_i \triangleq \biguplus p_i \cdot \mu_i$

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- chemical reaction $\{\mathrm{HCl}^{1/2}, \mathrm{NaOH}^{1/2}\}\ \mathsf{R}^{[56.5]}\ \{\mathrm{NaCl}^{1/2}, \mathrm{H_2O}^{1/2}\}$:

$$\{\mathrm{HCl}^{1/5}, \mathrm{NaOH}^{4/5}\} \rightsquigarrow_{\mathsf{R}}^{[11.3]} \{\mathrm{NaCl}^{1/10}, \mathrm{H_2O}^{1/10}, \mathrm{HCl}^{1/10}, \mathrm{NaOH}^{7/10}\} \rightsquigarrow_{\mathsf{R}}^{[11.3]} \{\mathrm{NaCl}^{1/5}, \mathrm{H_2O}^{1/5}, \mathrm{NaOH}^{3/5}\}$$

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- $x \succ_{\mathbb{R}}^{[w]} y :\Longleftrightarrow x \geq w + y$ is c.c. ($\forall i.\, x_i \geq w_i + y_i \Rightarrow \sum_i p_i \cdot x_i \geq \sum p_i \cdot w_i + \sum p_i \cdot y_i$)
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Qualitative and quantitative termination

weighted ARS $R \subseteq W \times A \times A$ is ...

- $\star$ weakly normalizing on $S \subseteq A$ if — $\mathrm{WN}_R(S)$
  every $a \in S$ has b with $a \rightsquigarrow_R^w b \in \mathrm{NF}(\rightarrow)$
- $\star$ strongly normalizing on $S \subseteq A$ if — $\mathrm{SN}_R(S)$
  all reduction sequences from $a \in S$ are *terminating*

$$\mathrm{WN} \Longleftarrow \mathrm{SN}$$

Note: reduction sequence $a_0 \rightsquigarrow_R^{[w_1]} a_1 \rightsquigarrow_R^{[w_2]} a_2 \rightsquigarrow_R^{[w_3]} \cdots$ is

- terminating if finite or $w_i = w_i + 1 = \cdots = 0$ for some $i \in \mathbb{N}$

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if weights W partially ordered, then R is ...

- $\star$ weakly bounded on $S \subseteq A$ if $\qquad \mathrm{WB}_R(S)$
  all reduction sequences from $a \in S$ are bounded

$$\begin{tikzcd} \mathrm{WN} & \mathrm{WB} \\ \mathrm{SN} \arrow[u, Rightarrow] \arrow[ur, Rightarrow] & \end{tikzcd}$$

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- bounded (by $u \in W$) if $\sum_i w_i \leq u$

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- $\star$ weakly bounded on $S \subseteq A$ if $\quad \mathsf{WB}_R(S)$
  all reduction sequences from $a \in S$ are bounded
- $\star$ strongly bounded on $S \subseteq A$ if $\quad \mathsf{SB}_R(S)$
  all reduction sequences from $a \in S$ are bounded by fixed u_a

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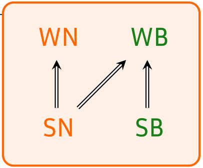
Qualitative and quantitative termination



⋆ WN $\not\Rightarrow$ SN: 

Qualitative and quantitative termination

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- ⋆ WN $\not\Rightarrow$ SN: $\begin{tikzcd} \circledcirc \arrow[loop left, "1"] \arrow[r, "1"] & \circ \end{tikzcd}$

- ⋆ WB $\not\Rightarrow$ WN: $\circledcirc \xrightarrow{1} \circ \xrightarrow{\frac{1}{2}} \circ \xrightarrow{\frac{1}{4}} \circ \xrightarrow{\frac{1}{8}} \cdots \Big| \, 2$ Zeno sequence

Qualitative and quantitative termination

$$\begin{tikzcd} \mathsf{WN} & \mathsf{WB} \\ \mathsf{SN} \arrow[u, Rightarrow] \arrow[ur, Rightarrow] & \mathsf{SB} \arrow[u, Rightarrow] \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{WN} \\ \mathsf{SN} = \mathsf{WB} \arrow[u, Rightarrow] \\ \mathsf{SB} \arrow[u, Rightarrow] \end{tikzcd}$$

non-Zeno

⋆ WN $\not\Rightarrow$ SN: $\begin{tikzcd} \circledcirc \arrow[loop left, "1"] \arrow[r, "1"] & \circ \end{tikzcd}$

⋆ WB $\not\Rightarrow$ WN: $\circledcirc \xrightarrow{1} \circ \xrightarrow{\frac{1}{2}} \circ \xrightarrow{\frac{1}{4}} \circ \xrightarrow{\frac{1}{8}} \cdots \Big|\, 2$

Zeno sequence

Qualitative and quantitative termination

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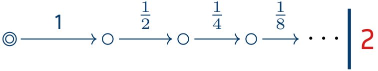

Zeno sequence

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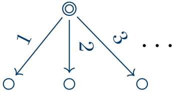

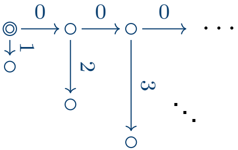

not (strongly) finitely branching

$$\begin{array}{ccc} \text{WN} & & \text{WB} \\ \Uparrow & \nearrow & \Uparrow \\ \text{SN} & & \text{SB} \end{array}$$

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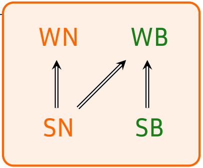

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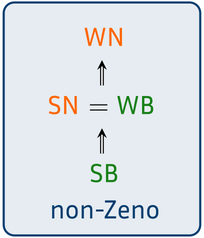
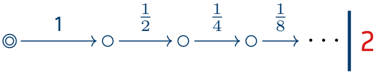
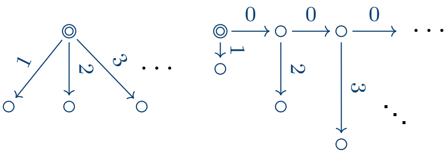


- $\star$ WB $\not\Rightarrow$ WN: 

Zeno sequence

- $\star$ WB $\not\Rightarrow$ SB : 

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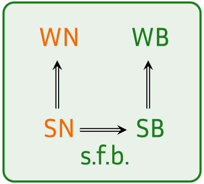
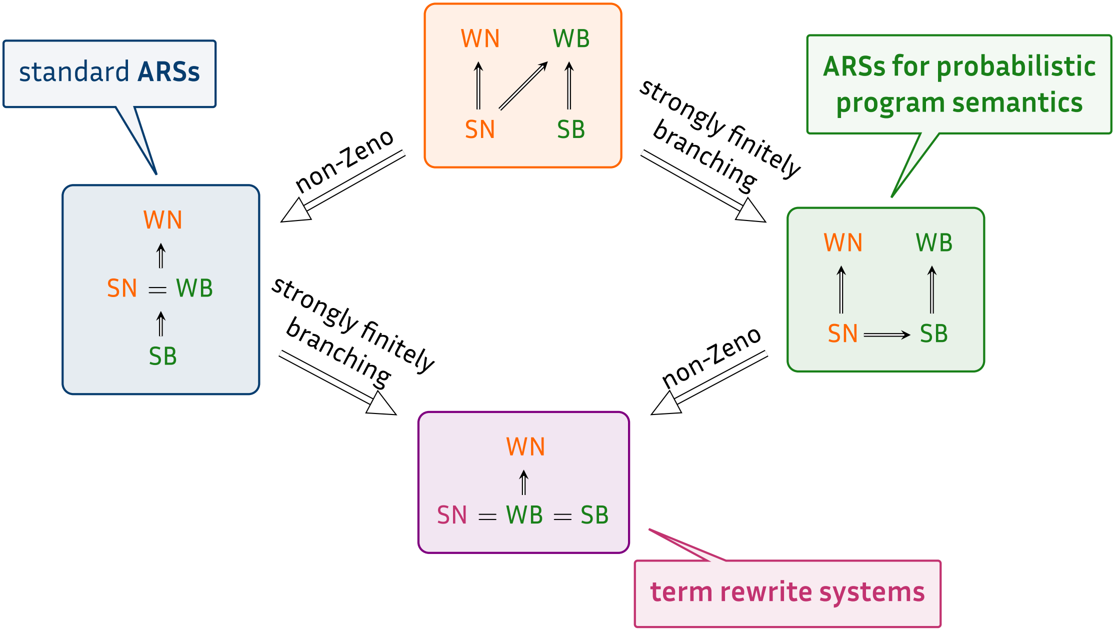


Summary



standard **ARSs**
WN
SN = WB
SB
WN
WB
SN
SB
ARSs for probabilistic program semantics
WN
WB
SN
SB
WN
SN = WB = SB
term rewrite systems
non-Zeno
strongly finitely branching
strongly finitely branching
non-Zeno

Outline

1. Weighted abstract reduction systems (wARSs)
2. Instances
3. Quantitative *termination-like* properties (boundedness)
4. Embeddings & ranking functions

Warm-up — Monotone embeddings and termination

mapping $\eta : A \to X$ is (monotone) embedding of $\to \subseteq A \times A$ into $\succ \subseteq X \times X$ if

$$a \to b \quad \Longrightarrow \quad \eta(a) \succ \eta(b) \qquad \text{(i.e., } \eta(\to) \subseteq \succ\text{)}$$

Theorem: finitely branching ARS $\to$ is SN $\iff$ $\eta(\to) \subseteq >_{\mathbb{N}}$ for some $\eta : A \to \mathbb{N}$

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proof idea

$\Longleftarrow$: $a_0 \to a_1 \to a_2 \to \cdots \implies \eta(a_0) >_{\mathbb{N}} \eta(a_1) >_{\mathbb{N}} \eta(a_2) >_{\mathbb{N}} \cdots$

$\Longrightarrow$: derivation height $\mathrm{dh}(a) \triangleq \max\{n \mid a \to^n b\}$ defines embedding

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Notes

- $\eta : A \to \mathbb{N}$ essentially ranking function
- embeddings can prove boundedness, actually

Potential

potential(s) of $a \in A$ w.r.t weighted ARS $\rightsquigarrow \subseteq W \times A \times A$ on partially ordered W is

$$\mathsf{Pots}_{\rightsquigarrow}(a) \triangleq \{w \mid \exists b.a \rightsquigarrow^{w}_{R} b\} \qquad \mathsf{pot}_{\rightsquigarrow}(a) \triangleq \sup \mathsf{Pots}_{\rightsquigarrow}(a)$$

Notes

- ⋆ potential $\mathsf{pot}_{\rightsquigarrow} : A \rightharpoonup W$ is a partial function
- ⋆ $\mathsf{pot}_{\{1\}\times\to}(a) = \mathsf{dh}_{\to}(a)$ for ARSs $\to \subseteq A \times A$
- ⋆ $\mathsf{pot}_{\hookrightarrow_{\mathcal{P}}}(\mu) = \mathbb{E}(\mathsf{dh}_{\mathcal{P}}(a))$ for probabilistic ARS $\mathcal{P} \subseteq A \times \mathsf{Dist}(A)$

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Proposition: $\mathsf{SB}_{\rightsquigarrow}(S) \iff \mathsf{Pots}_{\rightsquigarrow}(S)$ is bounded

Embeddings and strong boundedness

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what are choices for $\succ$?

Ranking functions

- for partially ordered $\mathcal{W}$, define weighted ARS $\succ_{\mathcal{W}} \subseteq W \times W^\infty \times W^\infty$

$$x \succ_{\mathcal{W}}^{[w]} y \quad :\Longleftrightarrow \quad x \geq w + y$$

- embeddings into $\succ_{\mathcal{W}}$ give notion of $\mathcal{W}$-valued ranking function

$$\eta(\rightsquigarrow) \subseteq \succ_{\mathcal{W}} \quad \text{means} \quad a \rightsquigarrow_R^{[w]} b \quad \Longrightarrow \quad \eta(a) \geq w + \eta(b)$$

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- if $\mathcal{W}$ is *positive* ($w \geq 0$) then $\mathrm{pot}_{\succ_{\mathcal{W}}}(w) = w$, thus by previous theorem:

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Theorem: for $\mathcal{W}$ positive, bounded complete & continuous

$$\mathsf{SB}_{\rightsquigarrow}(S) \quad \Longleftrightarrow \quad \text{wARS} \rightsquigarrow \text{ admits ranking function on } S \ (\textit{i.e.}, \eta(S) \subseteq W)$$

- bounded complete: $\sup X \in W$ for bounded $X \subseteq W$
- continuous: $\sup\{x + w \mid x \in X\} = \sup X + w$

For weighted TRSs...

recall: closed under contexts & substitutions

Rewrite orders:

for weighted TRS $\mathcal{R}$ and rewrite order $\succ \subseteq W \times \mathcal{T}(F,V) \times \mathcal{T}(F,V)$

$$\mathcal{R} \subseteq \succ \quad \Longrightarrow \quad \mathsf{Pots}_{\to_{\mathcal{R}}}(t) \subseteq \mathsf{Pots}_{\succ}(t)$$

Theorem: $\mathsf{SB}_{\to_{\mathcal{R}}}(S) \iff \mathcal{R} \subseteq \succ$ for some rewrite order $\succ$ with $\mathsf{SB}_{\succ}(S)$

For weighted TRSs...

Interpretation method:

- ⋆ term algebra $\mathcal{A}$ interprets terms into carrier A (e.g., $\mathbb{N}$)

$$\mathtt{0}_{\mathcal{A}} \triangleq 1 \qquad \mathtt{s}_{\mathcal{A}}(x) \triangleq x + 1 \qquad \mathtt{+}_{\mathcal{A}}(x, y) = 3(x + y)$$

- ⋆ gives rise to interpretation of terms $[\![t]\!]_{\mathcal{A}}$ within A

$$[\![\mathtt{s}(x) + y]\!]_{\mathcal{A}} = 3(x + 1 + y)$$

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- $\star$ interpretation lifts $\succ \subseteq W \times A \times A$ (e.g., $\succ_{\mathbb{N}}$) to $\succ_{\mathcal{A}} \subseteq W \times \mathcal{T}(F,V) \times \mathcal{T}(F,V)$
  - $\succ_{\mathcal{A}}$ is rewrite order when $\mathcal{A}$ monotone wrt. $\succ$

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$$0 + y \, \mathcal{R}^{[1]} \, y \qquad 1+y \succ_{\mathbb{N}}^{[1]} y$$

$$\mathsf{s}(x) + y \, \mathcal{R}^{[2]} \, \mathsf{s}(x+y) \qquad 3(x+1+y) \succ_{\mathbb{N}}^{[2]} 3(x+y)+1$$

$$x + y \, \mathcal{R}^{[0]} \, y + x \qquad 3(x+y) \succ_{\mathbb{N}}^{[0]} 3(x+y)$$

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$$\mathsf{s}(x) + y \; \mathcal{R}^{[2]} \; \mathsf{s}(x + y) \qquad 3(x + 1 + y) \succ_{\mathbb{N}}^{[2]} 3(x + y) + 1$$

$$x + y \; \mathcal{R}^{[0]} \; y + x \qquad 3(x + y) \succ_{\mathbb{N}}^{[0]} 3(x + y)$$

Theorem: $\mathsf{SB}_{\to_{\mathcal{R}}}(S) \iff \mathcal{R} \subseteq \succ_{\mathcal{A}}$ for monotone term algebra $(\mathcal{A}, \succ)$ with $\mathsf{SB}_{\succ}(S)$

For barycentric ARSs...

$\eta(\sum_i^A p_i \cdot a_i) = \sum_i^X p_i \cdot \eta(a_i)$

Affine embeddings for barycentric ARSs:

for barycentric ARS $\mathcal{B}$ and affine mapping $\eta : A \to X$ and convex closed $\succ$ (e.g., $\succ_{\mathbb{R}}$),

$$\eta(\mathcal{B}) \subseteq \succ \implies \mathrm{Pots}_{\rightsquigarrow_{\mathcal{B}}}(a) \subseteq \mathrm{Pots}_{\succ}(\eta(a))$$

Theorem: $\mathrm{SB}_{\rightsquigarrow_{\mathcal{B}}}(S) \impliedby \eta(\mathcal{B}) \subseteq \succ$ for affine embedding η into convex closed $\succ$

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Theorem: $\mathrm{SB}_{\rightsquigarrow_{\mathcal{B}}}(S) \quad \Longleftarrow \quad \eta(\mathcal{B}) \subseteq \succ$ for affine embedding η into convex closed $\succ$

Notes:

- ⋆ encompasses Lyapunov ranking functions $\eta : A \to [0, \infty)$ (Bournez & Garnier'05)

$$a \; \mathcal{P} \; \{b_i^{p_i}\} \Rightarrow \eta(a) \geq \epsilon + \sum_i p_i \cdot \eta(b_i)$$

- ⋆ complete ($\Longrightarrow$) for probabilistic ARSs
- ⋆ affineness sufficient but not necessary in general

Conclusion

- ⋆ abstract study of weighted ARSs $R \subseteq W \times A \times A$
  - conservative extension of ARSs
  - weighted TRSs
  - probabilistic / barycentric ARSs
- ⋆ weak & strong boundedness & their relationship to termination
- ⋆ ranking functions as a means to prove strong boundedness
  - rewrite orders & interpretation method
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Future work

- further quantitative versions of ARS properties (e.g., confluence, strategies) and methods (e.g., Newmans lemma)
- applications (e.g., Newmans Lemma for probabilistic ARSs)
- formalisation