Automated Static Program Analysis via Constrained Term Rewriting

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Outline

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1. Static Program Analysis
2. Constrained Term Rewriting
3. Proving Termination of Logically Constrained Simply-typed Term Rewriting Systems
4. Proving Termination of Scala Programs by Constrained Term Rewriting

Quality Assurance for Software by Program Analysis

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  — requires good choice of test cases
  — in general no guarantee for absence of errors
- Static analysis:
  Analyse the program text without actually running the program.
  + can prove (verify) correctness of the program
    $\longrightarrow$ important for safety-critical applications
    $\longrightarrow$ motivating example: first flight of Ariane 5 rocket in 1996
      https://www.youtube.com/watch?v=PK_yguLapgA
      https://en.wikipedia.org/wiki/Ariane_5_Flight_501
  — manual static analysis requires high effort and expertise
  $\Rightarrow$ for broad applicability:

Use automatic reasoning for static analysis!

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Note: All these properties are **undecidable**!
$\Rightarrow$ use automatable sufficient criteria in practice

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Early translations in the literature use Term Rewriting Systems (TRSs):

- Prolog [van Raamsdonk, *ICLP '97*; Giesl et al, *PPDP '12*]
- Haskell [Giesl et al, *TOPLAS '11*]
- ...

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- first-order (usually)
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$\Rightarrow$ Programs over user-defined algebraic data types

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Numbers: $\mathsf{0}$, $\mathsf{s}(\mathsf{0})$, $\mathsf{s}(\mathsf{s}(\mathsf{0}))$, $\ldots$

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Rules:

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\begin{aligned}
\mathsf{sum}(\mathsf{0}) &\to \mathsf{0} \\
\mathsf{sum}(\mathsf{s}(x)) &\to \mathsf{plus}(\mathsf{s}(x), \mathsf{sum}(x)) \\
\mathsf{plus}(\mathsf{0}, y) &\to y \\
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Then we can compute, e.g., $1 + 1 = 2$ as

$$\mathsf{plus}(\mathsf{s}(\mathsf{0}), \mathsf{s}(\mathsf{0})) \to_{\mathcal{R}} \mathsf{s}(\mathsf{plus}(\mathsf{0}, \mathsf{s}(\mathsf{0}))) \to_{\mathcal{R}} \mathsf{s}(\mathsf{s}(\mathsf{0}))$$

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Integer arithmetic possible with more complex recursive rules.

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Solution: use **constrained term rewriting**

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- Adoption of LCTRSs by the community: 83 citations so far (Google Scholar, Sep 2025)

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Constrained rewriting by example

Example (Constrained Rewrite System)

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\begin{array}{rcll}
\ell_0(n, r) & \longrightarrow & \ell_1(n, r, \mathsf{Nil}) & \\
\ell_1(n, r, xs) & \longrightarrow & \ell_1(n-1, r+1, \mathsf{Cons}(r, xs)) & [n > 0] \\
\ell_1(n, r, xs) & \longrightarrow & \ell_2(xs) & [n = 0]
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Here 7, 8, $\ldots$ are predefined constants.

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\to \ell_1(1, 8, \mathsf{Cons}(7, \mathsf{Nil})) \\
\to \ell_1(0, 9, \mathsf{Cons}(8, \mathsf{Cons}(7, \mathsf{Nil}))) \\
\to \ell_2(\mathsf{Cons}(8, \mathsf{Cons}(7, \mathsf{Nil})))
\end{array}
$$

Here 7, 8, $\ldots$ are predefined constants.

Termination proof: reuse techniques for TRSs and integer programs

Outline

Why analyse termination?

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2011: PHP and Java issues with floating-point number parser

- http://www.exploringbinary.com/php-hangs-on-numeric-value-2-2250738585072011e-308/
- http://www.exploringbinary.com/java-hangs-when-converting-2-2250738585072012e-308/

Existing translations for termination

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Many translations to (constrained) rewriting in the literature

- Prolog [van Raamsdonk, *ICLP '97*], [Giesl et al, *PPDP '12*]
- Java [Otto et al, *RTA '10*]
- Haskell [Giesl et al, *TOPLAS '11*]
- LLVM [Ströder et al, *JAR '17*]
- C [Fuhs, Kop, Nishida, *TOCL '17*]
- Jinja [Moser, Schaper, *IC '18*]
- Scala [Milovančević, Fuhs, Kunčak, *WPTE '25*]
- ...

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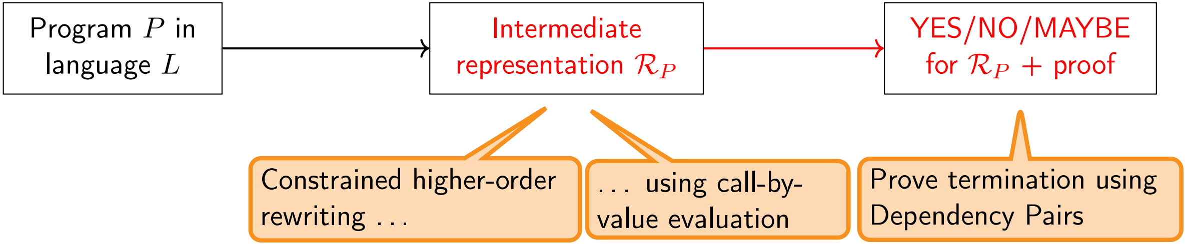

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- be good for automated reasoning (no "lost in encoding")
- express language features directly

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- **Logically Constrained Simply-typed TRSs aka LCSTRSs** [Guo, Kop, *ESOP '24*]:
  LCTRSs + higher-order functions (but no λ)

$$\mathsf{gcdlist} : \mathsf{intlist} \to \mathsf{int}, \quad \mathsf{fold} : (\mathsf{int} \to \mathsf{int} \to \mathsf{int}) \to \mathsf{int} \to \mathsf{intlist} \to \mathsf{int}$$
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Call-by-value (cbv) and innermost rewriting for LCSTRSs

Evaluating with an LCSTRS

$$\begin{aligned} \mathsf{fact}\ 0 &\to 1 \\ \mathsf{fact}\ x &\to x * \mathsf{fact}\ (x-1)\ [x > 0] \\ \mathsf{g}\ x &\to \mathsf{g}\ (\mathsf{fact}\ {-1}) \end{aligned}$$

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Note: transformation enforces cbv only for theory sorts

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Prove termination by Static Dependency Pairs for LCSTRSs [Guo, Hagens, Kop, Vale, *MFCS '24*]

- For LCSTRS $\mathcal{R}$ build dependency pairs $\mathcal{P} = \mathsf{SDP}(\mathcal{R}_{\mathsf{gcd}})$ (∼ function calls)

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- Works on DP problems $(\mathcal{P}, \mathcal{R})$
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while $S = S' \uplus \{(\mathcal{P}, \mathcal{R})\}$
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Existing DP processors for LCSTRSs [Guo, Hagens, Kop, Vale, *MFCS '24*]

- Graph processor
- Subterm criterion processor
- Integer mapping processor

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    $S := S' \cup \rho(\mathcal{P}, \mathcal{R})$ for a DP processor ρ
print "YES"

Existing DP processors for LCSTRSs [Guo, Hagens, Kop, Vale, *MFCS '24*]

- Graph processor
- Subterm criterion processor
- Integer mapping processor

New **innermost** DP processors for LCSTRSs [Fuhs, Guo, Kop, *FSCD '25*]

- Usable rules processor
- Reduction pair processor with usable rules wrt argument filtering
- Chaining processor

Also for **compositional termination analysis** via universal computability!

Existing DP processors for LCSTRSs

$\mathcal{P}$

(1) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{fold}^\sharp\ \mathsf{gcd}\ 0\ l'$

(2) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{gcd}^\sharp\ m'\ n'$

(3) $\mathsf{fold}^\sharp\ f\ y\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{fold}^\sharp\ f\ y\ l\ [x \equiv x \wedge y \equiv y]$

(4) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ (-m)\ n\ [m < 0 \wedge n \equiv n]$

(5) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ m\ (-n)\ [n < 0 \wedge m \equiv m]$

(6) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \wedge n > 0]$

$\mathcal{P}$

(1) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{fold}^\sharp\ \mathsf{gcd}\ 0\ l'$

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(6) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \wedge n > 0]$

$\mathcal{R}$

$\ldots$

$\mathcal{P}$

(1) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{fold}^\sharp\ \mathsf{gcd}\ 0\ l'$

(2) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{gcd}^\sharp\ m'\ n'$

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$\mathcal{R}$

...

- **Dependency Graph**:
  which calls may follow one another?

$\mathcal{P}$

(1) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{fold}^\sharp\ \mathsf{gcd}\ 0\ l'$

(2) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{gcd}^\sharp\ m'\ n'$

(3) $\mathsf{fold}^\sharp\ f\ y\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{fold}^\sharp\ f\ y\ l\ [x \equiv x \wedge y \equiv y]$

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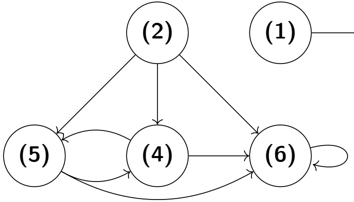
(6) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \wedge n > 0]$

$\mathcal{R}$

$\ldots$

- **Dependency Graph**:
  which calls may follow one another?
- Approximation
  [Guo, Hagens, Kop, Vale, *MFCS '24*]:



$\mathcal{P}$

(1) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{fold}^\sharp\ \mathsf{gcd}\ 0\ l'$
(2) $\mathsf{gcdlist}^\sharp\ l' \Rightarrow \mathsf{gcd}^\sharp\ m'\ n'$
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(4) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ (-m)\ n\ [m < 0 \wedge n \equiv n]$
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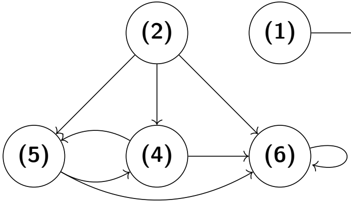
(6) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \wedge n > 0]$

$\mathcal{R}$

. . .

- **Dependency Graph**: which calls may follow one another?
- Approximation [Guo, Hagens, Kop, Vale, *MFCS '24*]:



- **Graph processor**: decompose $\mathcal{P}$ into non-trivial Strongly Connected Components

$\mathcal{P}$

(1) $\text{gcdlist}^\sharp\ l' \Rightarrow \text{fold}^\sharp\ \text{gcd}\ 0\ l'$
(2) $\text{gcdlist}^\sharp\ l' \Rightarrow \text{gcd}^\sharp\ m'\ n'$
(3) $\text{fold}^\sharp\ f\ y\ (\text{cons}\ x\ l) \Rightarrow \text{fold}^\sharp\ f\ y\ l\ [x \equiv x \wedge y \equiv y]$

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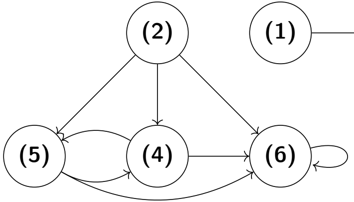
(6) $\text{gcd}^\sharp\ m\ n \Rightarrow \text{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \wedge n > 0]$

$\mathcal{R}$

$\ldots$

- **Dependency Graph**: which calls may follow one another?
- Approximation [Guo, Hagens, Kop, Vale, *MFCS '24*]:



- **Graph processor**: decompose $\mathcal{P}$ into non-trivial Strongly Connected Components
- Here:

$$(\{\mathbf{(3)}\}, \mathcal{R})$$
$$(\{\mathbf{(6)}\}, \mathcal{R})$$
$$(\{\mathbf{(4)}, \mathbf{(5)}\}, \mathcal{R})$$

$\mathcal{P}$

(3) $\text{fold}^\sharp\ f\ y\ (\text{cons}\ x\ l) \Rightarrow \text{fold}^\sharp\ f\ y\ l\ [x \equiv x \wedge y \equiv y]$

$\mathcal{R}$

$\ldots$

$\mathcal{P}$

(3) $\mathsf{fold}^\sharp\ f\ y\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{fold}^\sharp\ f\ y\ l\ [x \equiv x \wedge y \equiv y]$

$\mathcal{R}$

$\ldots$

Subterm criterion processor [Guo, Hagens, Kop, Vale, *MFCS '24*]

- Detect structural decrease in argument
- Use projection $\nu(\mathsf{fold}^\sharp) = 3$
- Get $\mathsf{cons}\ x\ l \rhd l$

$\Rightarrow$ Remove **(3)**

$\Rightarrow$ $(\emptyset, \mathcal{R})$ deleted by graph processor

$\mathcal{P}$

(6) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \land n > 0]$

$\mathcal{R}$

$\ldots$

$\mathcal{P}$

(6) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \wedge n > 0]$

$\mathcal{R}$

$\ldots$

Integer mapping processor [Guo, Hagens, Kop, Vale, *MFCS '24*]

- Detect integer value decrease in argument
- Use projection $\nu(\mathsf{gcd}^\sharp) = 2$
- Get $m \geq 0 \wedge n > 0 \models n > m \bmod n$
  and $m \geq 0 \wedge n > 0 \models n \geq 0$

$\Rightarrow$ Remove **(6)**

$\Rightarrow$ $(\emptyset, \mathcal{R})$ deleted by graph processor

$\mathcal{P}$

(6) $\mathsf{gcd}^\sharp\ m\ n \Rightarrow \mathsf{gcd}^\sharp\ n\ (m \bmod n)\ [m \geq 0 \wedge n > 0]$

$\mathcal{R}$

$\ldots$

Integer mapping processor [Guo, Hagens, Kop, Vale, *MFCS '24*]

- Detect integer value decrease in argument
- Use projection $\nu(\mathsf{gcd}^\sharp) = 2$
- Get $m \geq 0 \wedge n > 0 \models n > m \bmod n$
  and $m \geq 0 \wedge n > 0 \models n \geq 0$

$\Rightarrow$ Remove **(6)**

$\Rightarrow$ $(\emptyset, \mathcal{R})$ deleted by graph processor

$(\{\mathbf{(4)}, \mathbf{(5)}\}, \mathcal{R})$ handled by integer mapping processor + graph processor

$\Rightarrow$ termination of $\mathcal{R}_{\mathsf{gcd}}$ proved!

New DP processors for LCSTRSs

Usable rules processor

$\mathcal{R}_{\mathsf{dfoldr}}$

drop : int → alist → alist
dfoldr : (a → b → b) → b → int → alist → b

$$
\begin{array}{lcll}
\mathsf{drop}\ n\ l & \to & l & [n \leq 0] \\
\mathsf{drop}\ n\ \mathsf{nil} & \to & \mathsf{nil} & [n \equiv n] \\
\mathsf{drop}\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{drop}\ (n-1)\ l & [n > 0]
\end{array}
$$

$$
\begin{array}{lcll}
\mathsf{dfoldr}\ f\ y\ n\ \mathsf{nil} & \to & y & [n \equiv n] \\
\mathsf{dfoldr}\ f\ y\ n\ (\mathsf{cons}\ x\ l) & \to & f\ x\ (\mathsf{dfoldr}\ f\ y\ n\ (\mathsf{drop}\ n\ l)) & [n \equiv n]
\end{array}
$$

Usable rules processor

$\mathcal{R}_{\text{dfoldr}}$

drop : int → alist → alist
dfoldr : (a → b → b) → b → int → alist → b

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\begin{array}{lcll}
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\text{dfoldr}\ f\ y\ n\ \text{nil} & \to & y & [n \equiv n] \\
\text{dfoldr}\ f\ y\ n\ (\text{cons}\ x\ l) & \to & f\ x\ (\text{dfoldr}\ f\ y\ n\ (\text{drop}\ n\ l)) & [n \equiv n]
\end{array}
$$

- Troublesome DP problem:

$$(\ \{ \ \text{dfoldr}^\sharp\ f\ y\ n\ (\text{cons}\ x\ l) \Rightarrow \text{dfoldr}^\sharp\ f\ y\ n\ (\text{drop}\ n\ l)\ [n \equiv n] \ \},\ \mathcal{R}_{\text{dfoldr}} \ )$$

Usable rules processor

$\mathcal{R}_{\text{dfoldr}}$

drop : int → alist → alist
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- Troublesome DP problem:

$$(\ \{ \ \text{dfoldr}^\sharp\ f\ y\ n\ (\text{cons}\ x\ l) \Rightarrow \text{dfoldr}^\sharp\ f\ y\ n\ (\text{drop}\ n\ l)\ [n \equiv n] \ \}, \ \mathcal{R}_{\text{dfoldr}} \ )$$

- Reduction pair processor can show $\text{cons}\ x\ l \succ \text{drop}\ n\ l$
- But cannot show $\text{dfoldr}\ f\ y\ n\ (\text{cons}\ x\ l) \succsim f\ x\ (\text{dfoldr}\ f\ y\ n\ (\text{drop}\ n\ l))[n \equiv n]$

Usable rules processor

$\mathcal{R}_{\mathsf{dfoldr}}$

drop : int → alist → alist
dfoldr : (a → b → b) → b → int → alist → b

$$
\begin{array}{llll}
\mathsf{drop}\ n\ l & \to & l & [n \le 0] \\
\mathsf{drop}\ n\ \mathsf{nil} & \to & \mathsf{nil} & [n \equiv n] \\
\mathsf{drop}\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{drop}\ (n-1)\ l & [n > 0] \\
\mathsf{dfoldr}\ f\ y\ n\ \mathsf{nil} & \to & y & [n \equiv n] \\
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\end{array}
$$

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$$(\ \{ \ \mathsf{dfoldr}^\sharp\ f\ y\ n\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{dfoldr}^\sharp\ f\ y\ n\ (\mathsf{drop}\ n\ l)\ [n \equiv n] \ \}, \ \mathcal{R}_{\mathsf{dfoldr}} \ )$$

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- But cannot show $\mathsf{dfoldr}\ f\ y\ n\ (\mathsf{cons}\ x\ l) \succsim f\ x\ (\mathsf{dfoldr}\ f\ y\ n\ (\mathsf{drop}\ n\ l))[n \equiv n]$
- **Usable rules processor**: keep only **usable** rules, called from DPs
- Here: rules for drop

Usable rules processor

$\mathcal{R}$

drop : int → alist → alist
dfoldr : (a → b → b) → b → int → alist → b

$$\begin{array}{lcll} \mathsf{drop}\ n\ l & \to & l & [n \le 0] \\ \mathsf{drop}\ n\ \mathsf{nil} & \to & \mathsf{nil} & [n \equiv n] \\ \mathsf{drop}\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{drop}\ (n-1)\ l & [n > 0] \end{array}$$

- Troublesome DP problem:

$$(\ \{\ \ \mathsf{dfoldr}^\sharp\ f\ y\ n\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{dfoldr}^\sharp\ f\ y\ n\ (\mathsf{drop}\ n\ l)\ [n \equiv n]\ \ \},\ \ \mathcal{R}_{\mathsf{dfoldr}}\ \ )$$

- Reduction pair processor can show $\mathsf{cons}\ x\ l \succ \mathsf{drop}\ n\ l$
- But cannot show $\mathsf{dfoldr}\ f\ y\ n\ (\mathsf{cons}\ x\ l) \succsim f\ x\ (\mathsf{dfoldr}\ f\ y\ n\ (\mathsf{drop}\ n\ l))[n \equiv n]$
- **Usable rules processor**: keep only **usable** rules, called from DPs
- Here: rules for drop

Reduction pair processor with usable rules wrt an argument filtering

$\mathcal{R}_{\mathsf{dfoldl}}$

$$
\begin{array}{lcll}
\mathsf{drop}\ n\ l & \to & l & [n \leq 0] \\
\mathsf{drop}\ n\ \mathsf{nil} & \to & \mathsf{nil} & [n \equiv n] \\
\mathsf{drop}\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{drop}\ (n-1)\ l & [n > 0] \\
\\
\mathsf{dfoldl}\ f\ y\ n\ \mathsf{nil} & \to & y & [n \equiv n] \\
\mathsf{dfoldl}\ f\ y\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{dfoldl}\ f\ (f\ y\ x)\ n\ (\mathsf{drop}\ n\ l) & [n \equiv n]
\end{array}
$$

Reduction pair processor with usable rules wrt an argument filtering

$\mathcal{R}_{\mathsf{dfoldl}}$

$$
\begin{array}{lcll}
\mathsf{drop}\ n\ l & \to & l & [n \leq 0] \\
\mathsf{drop}\ n\ \mathsf{nil} & \to & \mathsf{nil} & [n \equiv n] \\
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\mathsf{dfoldl}\ f\ y\ n\ \mathsf{nil} & \to & y & [n \equiv n] \\
\mathsf{dfoldl}\ f\ y\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{dfoldl}\ f\ (f\ y\ x)\ n\ (\mathsf{drop}\ n\ l) & [n \equiv n]
\end{array}
$$

- Troublesome DP problem:

$$
(\ \{\ \mathsf{dfoldl}^{\sharp}\ f\ y\ n\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{dfoldl}^{\sharp}\ f\ (f\ y\ x)\ n\ (\mathsf{drop}\ n\ l)\ [n \equiv n]\ \},\ \mathcal{R}_{\mathsf{dfoldl}}\ )
$$

Reduction pair processor with usable rules wrt an argument filtering

$\mathcal{R}_{\mathsf{dfoldl}}$

$$
\begin{array}{lcll}
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\mathsf{drop}\ n\ \mathsf{nil} & \to & \mathsf{nil} & [n \equiv n] \\
\mathsf{drop}\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{drop}\ (n-1)\ l & [n > 0] \\
\mathsf{dfoldl}\ f\ y\ n\ \mathsf{nil} & \to & y & [n \equiv n] \\
\mathsf{dfoldl}\ f\ y\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{dfoldl}\ f\ (f\ y\ x)\ n\ (\mathsf{drop}\ n\ l) & [n \equiv n]
\end{array}
$$

- Troublesome DP problem:

$$
(\ \{\ \mathsf{dfoldl}^\sharp\ f\ y\ n\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{dfoldl}^\sharp\ f\ (f\ y\ x)\ n\ (\mathsf{drop}\ n\ l)\ [n \equiv n]\ \},\ \mathcal{R}_{\mathsf{dfoldl}}\ )
$$

- All rules are usable!

Reduction pair processor with usable rules wrt an argument filtering

$\mathcal{R}_{\mathsf{dfoldl}}$

$$
\begin{array}{lcll}
\mathsf{drop}\ n\ l & \to & l & [n \leq 0] \\
\mathsf{drop}\ n\ \mathsf{nil} & \to & \mathsf{nil} & [n \equiv n] \\
\mathsf{drop}\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{drop}\ (n-1)\ l & [n > 0] \\
\mathsf{dfoldl}\ f\ y\ n\ \mathsf{nil} & \to & y & [n \equiv n] \\
\mathsf{dfoldl}\ f\ y\ n\ (\mathsf{cons}\ x\ l) & \to & \mathsf{dfoldl}\ f\ (f\ y\ x)\ n\ (\mathsf{drop}\ n\ l) & [n \equiv n]
\end{array}
$$

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$$(\ \{ \ \mathsf{dfoldl}^{\sharp}\ f\ y\ n\ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{dfoldl}^{\sharp}\ f\ (f\ y\ x)\ n\ (\mathsf{drop}\ n\ l)\ [n \equiv n] \ \},\ \mathcal{R}_{\mathsf{dfoldl}} \ )$$

- All rules are usable!
- **Reduction pair processor with usable rules wrt argument filtering**:

  temporarily disregard arguments, calculate usable rules, use reduction pair (HORPO, ...)

Reduction pair processor with usable rules wrt an argument filtering

$\mathcal{R}_{\text{dfoldl}}$

$$
\begin{array}{lcll}
\text{drop}\ n\ l & \to & l & [n \leq 0] \\
\text{drop}\ n\ \text{nil} & \to & \text{nil} & [n \equiv n] \\
\text{drop}\ n\ (\text{cons}\ x\ l) & \to & \text{drop}\ (n-1)\ l & [n > 0] \\
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\end{array}
$$

- Troublesome DP problem:

$$(\ \{ \ \text{dfoldl}^\sharp\ f\ y\ n\ (\text{cons}\ x\ l) \Rightarrow \text{dfoldl}^\sharp\ f\ (f\ y\ x)\ n\ (\text{drop}\ n\ l)\ [n \equiv n] \ \}, \ \mathcal{R}_{\text{dfoldl}} \ )$$

- All rules are usable!
- **Reduction pair processor with usable rules wrt argument filtering**:

  temporarily disregard arguments, calculate usable rules, use reduction pair (HORPO, ...)
- $\mathrm{regard}(\text{dfoldl}^\sharp) = \{4\}$ $\quad\Rightarrow\quad$ use first-order RPO!

Reduction pair processor with usable rules wrt an argument filtering

$\mathcal{R}_{\text{dfoldl}}$

$$
\begin{array}{lcll}
\text{drop}\ n\ l & \to & l & [n \leq 0] \\
\text{drop}\ n\ \text{nil} & \to & \text{nil} & [n \equiv n] \\
\text{drop}\ n\ (\text{cons}\ x\ l) & \to & \text{drop}\ (n-1)\ l & [n > 0] \\
\text{dfoldl}\ f\ y\ n\ \text{nil} & \to & y & [n \equiv n] \\
\text{dfoldl}\ f\ y\ n\ (\text{cons}\ x\ l) & \to & \text{dfoldl}\ f\ (f\ y\ x)\ n\ (\text{drop}\ n\ l) & [n \equiv n]
\end{array}
$$

- Troublesome DP problem:

$$(\ \{ \ \text{dfoldl}^\sharp \quad (\text{cons}\ x\ l) \Rightarrow \text{dfoldl}^\sharp \quad (\text{drop}\ n\ l)\ [n \equiv n] \ \},\ \mathcal{R}_{\text{dfoldl}} \ )$$

- All rules are usable!
- **Reduction pair processor with usable rules wrt argument filtering**:

  temporarily disregard arguments, calculate usable rules, use reduction pair (HORPO, . . .)
- $\mathrm{regard}(\text{dfoldl}^\sharp) = \{4\} \quad \Rightarrow$ use first-order RPO!

Reduction pair processor with usable rules wrt an argument filtering

$\mathcal{R}_{\mathsf{dfoldl}}$

$$
\begin{array}{lll}
\mathsf{drop}\ n\ l & \to\ l & [n \leq 0] \\
\mathsf{drop}\ n\ \mathsf{nil} & \to\ \mathsf{nil} & [n \equiv n] \\
\mathsf{drop}\ n\ (\mathsf{cons}\ x\ l) & \to\ \mathsf{drop}\ (n-1)\ l & [n > 0]
\end{array}
$$

- Troublesome DP problem:

$$(\ \{ \ \mathsf{dfoldl}^\sharp \ (\mathsf{cons}\ x\ l) \Rightarrow \mathsf{dfoldl}^\sharp \ (\mathsf{drop}\ n\ l)\ [n \equiv n] \ \},\ \mathcal{R}_{\mathsf{dfoldl}} \ )$$

- All rules are usable!
- **Reduction pair processor with usable rules wrt argument filtering**:

  temporarily disregard arguments, calculate usable rules, use reduction pair (HORPO, . . .)
- $\mathrm{regard}(\mathsf{dfoldl}^\sharp) = \{4\} \quad \Rightarrow$ use first-order RPO!

Chaining processor

(S)DPs from imperative program [Fuhs, Kop, Nishida, *TOCL '17*]

```
def fact(x):
  z = 1            # fact
  i = 1            # u1
  while i <= x:    # u2
    z = z * i      # u3
    i = i + 1      # u4
                   # u5
```

Chaining processor

(S)DPs from imperative program [Fuhs, Kop, Nishida, *TOCL '17*]

```
def fact(x):
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  while i <= x:    # u2
    z = z * i      # u3
    i = i + 1      # u4
                   # u5
```

$$
\begin{array}{lcll}
\mathsf{fact}^\sharp\ x & \Rightarrow & \mathsf{u_1}^\sharp\ x\ 1 & [x \equiv x] \\
\mathsf{u_1}^\sharp\ x\ z & \Rightarrow & \mathsf{u_2}^\sharp\ x\ z\ 1 & [x \equiv x \wedge z \equiv z] \\
\mathsf{u_2}^\sharp\ x\ z\ i & \Rightarrow & \mathsf{u_3}^\sharp\ x\ z\ i & [i \leq x \wedge z \equiv z] \\
\mathsf{u_3}^\sharp\ x\ z\ i & \Rightarrow & \mathsf{u_4}^\sharp\ x\ (z * i)\ i & [x \equiv x \wedge z \equiv z \wedge i \equiv i] \\
\mathsf{u_4}^\sharp\ x\ z\ i & \Rightarrow & \mathsf{u_2}^\sharp\ x\ z\ (i+1) & [x \equiv x \wedge z \equiv z \wedge i \equiv i] \\
\mathsf{u_2}^\sharp\ x\ z\ i & \Rightarrow & \mathsf{u_5}^\sharp\ x\ z & [\neg(i \leq x) \wedge z \equiv z]
\end{array}
$$

- Automated translations $\Rightarrow$ DPs with many small steps
- Can be hard to analyse!

Chaining processor

(S)DPs from imperative program [Fuhs, Kop, Nishida, *TOCL '17*]

```
def fact(x):
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                   # u5
```

$$
\begin{array}{lcll}
\mathsf{fact}^\sharp\ x & \Rightarrow & \mathsf{u_1}^\sharp\ x\ 1 & [x \equiv x] \\
\mathsf{u_1}^\sharp\ x\ z & \Rightarrow & \mathsf{u_2}^\sharp\ x\ z\ 1 & [x \equiv x \wedge z \equiv z] \\
\mathsf{u_2}^\sharp\ x\ z\ i & \Rightarrow & \mathsf{u_3}^\sharp\ x\ z\ i & [i \leq x \wedge z \equiv z] \\
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\end{array}
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- Automated translations $\Rightarrow$ DPs with many small steps
- Can be hard to analyse!
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Chaining processor

(S)DPs from imperative program [Fuhs, Kop, Nishida, *TOCL '17*]

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- Integer mapping processor + graph processor prove termination

Compositional analysis from Universal Computability

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- Analyse LCSTRS for use in context of larger program
- Usable rules + reduction pair processor available for innermost (and cbv) rewriting!

Implementation

- Implementation in open-source tool Cora: https://github.com/hezzel/cora/
- HORPO as reduction pair
- Z3 as SMT solver

Experiments (1/3)

Experiments using 60 seconds timeout

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275 inputs: integer TRSs + λ-free HO-TRSs from TPDB + own benchmarks

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Cora (innermost/cbv) v Cora (full) [Guo, Hagens, Kop, Vale, *MFCS '24*]

	Termination			**Universal Computability**		
	Full	Innermost	Call-by-value	Full	Innermost	Call-by-value
Total yes	171	179	182	155	179	182

Experiments (2/3)

117 integer TRSs: Cora v AProVE [Giesl et al, *JAR '17*] [Fuhs et al, *RTA '09*]

	Cora innermost	Cora call-by-value	AProVE innermost
Total yes	72	73	102

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Total yes	72	73	102

- AProVE has strong reduction pair processor with polynomial interpretations and usable rules
- AProVE can handle rules $\mathsf{f}(x) \to \mathsf{g}(x > 0, x)$, $\mathsf{g}(\mathfrak{t}, x) \to r_1$, $\mathsf{g}(\mathfrak{f}, x) \to r_2$ well

Experiments (3/3)

140 λ-free HO-TRSs: Cora v WANDA [Kop, *FSCD '20*]

	Cora innermost / call-by-value	WANDA full termination
Total yes	79	105

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- WANDA: Polynomial interpretations, dynamic DPs, delegation to first-order termination tool, ...

Conclusion: call-by-value termination analysis of LCSTRSs

- Transformation for analysis of LCSTRSs with call-by-value via innermost strategy
- Three new processors: usable rules, reduction pair with temporary argument filtering, chaining
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- FSCD 2025 paper:

An Innermost DP Framework for Constrained Higher-Order Rewriting

Carsten Fuhs
Birkbeck, University of London, UK

Liye Guo
Radboud University, Nijmegen, The Netherlands

Cynthia Kop
Radboud University, Nijmegen, The Netherlands

Outline

1. Static Program Analysis
2. Constrained Term Rewriting
3. Proving Termination of Logically Constrained Simply-typed Term Rewriting Systems
4. Proving Termination of Scala Programs by Constrained Term Rewriting

Background: Stainless verifier for Scala programs

- Open-source deductive verifier for Scala: https://stainless.epfl.ch
- Tutorial: https://epfl-lara.github.io/asplos2022tutorial/
- Verification approach requires functions to be **terminating**

Termination proving in Stainless

```scala
// Scala data type
sealed abstract class Formula
case object True extends Formula
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  $\Rightarrow$ (simplified) LCTRS:

$$
\begin{array}{rll|rll}
\mathsf{f}(\mathsf{True}) & \to \mathfrak{t} & & \mathsf{f}(\mathsf{False}) & \to \mathfrak{f} & \\
\mathsf{f}(\mathsf{Not}(n)) & \to \mathsf{f}_1(\mathsf{Not}(n), \mathsf{f}(n)) & & \mathsf{f}(\mathsf{Imply}(l,r)) & \to \mathsf{f}_2(\mathsf{Imply}(l,r), \mathsf{f}(\mathsf{Not}(l))) & \\
\mathsf{f}_1(\mathsf{Not}(n), tmp_1) & \to \mathfrak{f} & [tmp_1] & \mathsf{f}_2(\mathsf{Imply}(l,r), tmp_2) & \to \mathfrak{t} & [tmp_2] \\
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$\Rightarrow$ easy to prove terminating

Experiments with Stainless and AProVE

Benchmarks from work on autograding by equivalence proving
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Name	LOC	#	Inference	Transfer	**LCTRS**	Total Proven
formula	59	37	0	27	**24**	28
sigma	10	704	0	678	**0**	678
prime	21	22	0	5	**14**	14
gcd	9	41	0	22	**15**	27

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- Benchmark selection cannot be handled by measure inference
- *sigma* benchmark has higher-order arguments, outside of the scope of AProVE's Integer TRSs
- Measure transfer and termination proofs via constrained rewriting complement each other well

Limitations / future work

- Translation represents bounded Scala integers as unbounded. But:

```
def overflow_fun(i: Int, n: Int): Int =
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$\Rightarrow$ optimise via pattern differences [Nishida, Kojima, Nakamura, *FroCoS '25*]

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Conclusion: from Scala to LCTRSs for termination analysis

- Integration of translation to LCTRSs into the Stainless analysis pipeline
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Proving Termination of Scala Programs by Constrained Term Rewriting

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Carsten Fuhs
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Take Home Messages

- **Static analysis workflow:**

Program P in language L $\longrightarrow$ Intermediate representation $\mathcal{R}_P$ $\longrightarrow$ Result for $\mathcal{R}_P$ + proof

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Thanks a lot for your attention!