Cyclic system for an algebraic theory of alternating parity automata

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Outline

1. Languages of infinite words
2. A cyclic proof system
3. Metalogical results
4. Conclusions

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- States may be $\exists$xistentially or $\forall$niversally branching.
- Each state is coloured by a natural number.
- An run is **accepting** if the least colour occurring infinitely often is even.

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Fact

$L \subseteq \mathcal{A}^\omega$ is recognised by an APA $\iff L = \bigcup_{i<n} A_i B_i^\omega$, for A_i, B_i regular and $\not\ni \varepsilon$.

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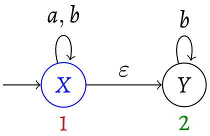
NB: there are many equivalent models of automata for ω-regular languages, but APAs enjoy elegant symmetries.

Examples of APAs

finitely many as :



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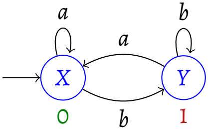

Key

: existential state
: universal state
n : even colour
n : odd colour

EXAMPLES OF APAS

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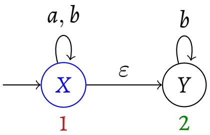

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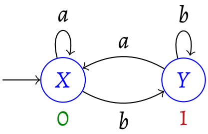

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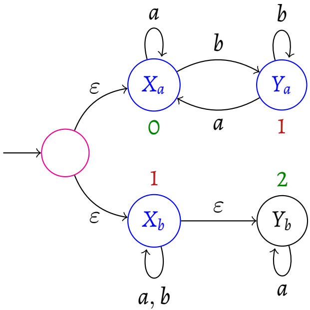
and :

finitely many bs



We can reason about APAs via expressions with fixed points.

RLL expressions:

$$
\begin{array}{rcccccccccc}
e, f, \ldots & ::= & X & \mid & ae & \mid & 0 & \mid & e + f & \mid & \mu X\, e \\
 & & & & & \mid & \top & \mid & e \cap f & \mid & \nu X\, e
\end{array}
$$

NB: symmetry of APAs is reflected by duality of syntax.

This syntax can be seen as an *alternative* to the modal logic μLTL.

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Language semantics:

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\begin{aligned}
\mathcal{L}(A) &:= A & \qquad \mathcal{L}(ae) &:= \{aw \mid w \in \mathcal{L}(e)\} \\
\mathcal{L}(0) &:= \varnothing & \mathcal{L}(\top) &:= \mathcal{A}^\omega \\
\mathcal{L}(e+f) &:= \mathcal{L}(e) \cup \mathcal{L}(f) & \mathcal{L}(e \cap f) &:= \mathcal{L}(e) \cap \mathcal{L}(f) \\
\mathcal{L}(\mu X e(X)) &:= \mathrm{LFP}[A \mapsto \mathcal{L}(e(A))] & \mathcal{L}(\nu X e(X)) &:= \mathrm{GFP}[A \mapsto \mathcal{L}(e(A))]
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NB: *Knaster-Tarski Theorem* $\implies$ $\mathcal{L}(e)$ is well-defined.

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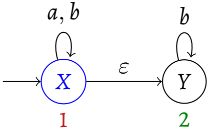

$f_a : \mu X(aX + bX + \nu Y(bY))$

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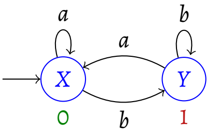

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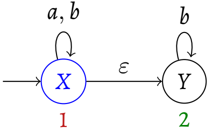

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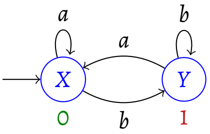

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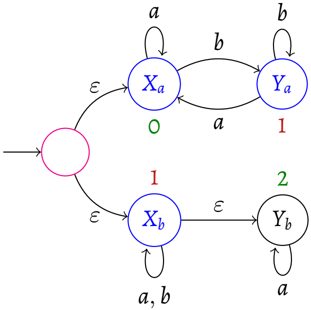

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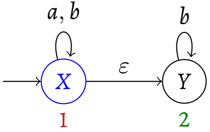

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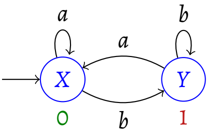

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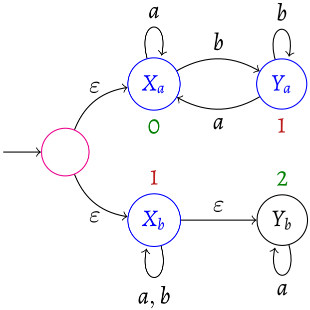

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NB: *equivalence* between an expression and associated APA is not immediate...

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Players: *Eloise* ($\exists$) and *Abelard* ($\forall$).

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By now standard techniques we can show:

Theorem (Adequacy)

$w \in \mathcal{L}(e) \iff$ *$\exists$ has a (positional) winning strategy from* (w, e).

$\rightsquigarrow$ an RLL expression and its associated APA compute the same ω-language.

Algebraic vs modal perspectives:

There is already a syntax for APAs: the modal logic $\mu\mathsf{LTL}$!

- RLL expressions are based in the language of lattices while $\mu\mathsf{LTL}$ is based in the language of Boolean algebras. (*no complements!*)
- Resulting theories are subtly but formally different: there are *Right-linear Lattices* that are not Boolean algebras, and not complete lattices [Kai95, DD25].
- Nonetheless, we are heavily inspired by previous work for μ-calculi, in particular [DHL06]: we are using now standard techniques, cf. [NW96].

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The algebraic context:

Our work is building on the *algebraic tradition*:

- *Kleene algebras*, for regular expressions, recast as *Right-linear algebras* (RLAs) [DD24b]. (*no multiplication!*) Both may be construed a theory of NFAs.
- RLLs strictly extend RLAs, unlike ω*-regular algebras*.
  Theory of APAs vs theory of Büchi automata.

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This work is an application of cyclic proofs to algebras for regularity.

Outline

1. $(0, \top, +, \cap)$ forms a *bounded distributive lattice*.
2. Each $a \in \mathcal{A}$ is a (lower) semibounded lattice homomorphism:

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4. $\mu X e(X)$ is a least prefixed point of $X \mapsto e(X)$:

$$\begin{aligned} &e(\mu X e(X)) \leqslant \mu X e(X) \\ &e(f) \leqslant f \implies \mu X e(X) \leqslant f \end{aligned}$$

5. $\nu X e(X)$ is a greatest postfixed point of $X \mapsto e(X)$:

$$\begin{aligned} &\nu X e(X) \leqslant e(\nu X e(X)) \\ &f \leqslant e(f) \implies f \leqslant \nu X e(X) \end{aligned}$$

Sequents: $\Gamma \to \Delta$, where Γ, Δ are *sets* of formulas.

Starting point: a standard system for bounded distributive lattices.

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Definition

The system $\widehat{\mathsf{RLL}}$ has the following additional rules:

$\mathcal{A}$ **rules:**

$$\mathsf{p}\text{-}l\,\frac{}{ae, bf \to}\; a \neq b \qquad \mathsf{h}_a\,\frac{\Gamma \to \Delta}{a\Gamma \to a\Delta}\;\Gamma \neq \varnothing \qquad \mathsf{p}\text{-}r\,\frac{\{\to \Gamma_a\}_{a\in\mathcal{A}}}{\to \{a\Gamma_a\}_{a\in\mathcal{A}}}$$

Fixed point rules:

$$\mu\text{-}l\,\frac{\Gamma, e(\mu X e(X)) \to \Delta}{\Gamma, \mu X e(X) \to \Delta} \qquad \mu\text{-}r\,\frac{\Gamma \to \Delta, e(\mu X e(X))}{\Gamma \to \Delta, \mu X e(X)}$$

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NB: these rules do not guarantee extremality of fixed points!

Definition

- **Preproofs** are generated coinductively from the inference rules.
- A preproof is **cyclic/regular** if it has only finitely many distinct sub-preproofs.
- A **proof** is a preproof where each infinite branch has a 'good formula trace'.

'Good formula traces' given by $\exists$-winning plays on RHS, or $\forall$-winning plays on LHS.

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Example

a^ω has infinitely many as

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\nu X(aX) \to i_a}\;{\scriptstyle \nu\text{-}l,\nu\text{-}r}\ \bullet}{a\nu X(aX) \to ai_a}\;{\scriptstyle \mathsf{h}_a}}{a\nu X(aX) \to ai_a + bi'_a}\;{\scriptstyle +\text{-}r,\mathsf{wk}\text{-}r}}{a\nu X(aX) \to i'_a}\;{\scriptstyle \mu\text{-}r}}{\nu X(aX) \to i_a}\;{\scriptstyle \nu\text{-}l,\nu\text{-}r}\ \bullet$$

a^ω does not have finitely many as

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{f_a, \nu Y(aY) \to}\ \circ}{af_a, a\nu Y(aY) \to}\;{\scriptstyle \mathsf{h}_a}\quad \text{(easy)}}{af_a + bf_a + \nu Y(bY), a\nu Y(aY) \to}\;{\scriptstyle +\text{-}l}}{f_a, a\nu Y(aY) \to}\;{\scriptstyle \mu\text{-}l}}{f_a, \nu Y(aY) \to}\;{\scriptstyle \nu\text{-}l}\ \circ}{f_a \cap \nu Y(aY) \to}\;{\scriptstyle \cap\text{-}l}$$

Any ω-word over $\{a, b\}$ has finitely many as or infinitely many as

$$
+\text{-}r\,\dfrac{\mu\text{-}r,\nu\text{-}r\,\dfrac{+\text{-}r\,\dfrac{\nu\text{-}l,\nu\text{-}r,\mu\text{-}r\,\dfrac{+\text{-}r\,\dfrac{+_l\,\dfrac{\mathsf{h}_a\,\dfrac{\dfrac{\vdots}{\top \to f_a, i_a}\;\bullet}{a\top \to af_a, ai_a} \qquad \mathsf{h}_a\,\dfrac{\mu\text{-}r,+\text{-}r\,\dfrac{\dfrac{\vdots}{\top \to af_a, bf_a, b^\omega, i'_a}\;\circ}{\top \to f_a, b^\omega, i'_a}}{b\top \to bf_a, bb^\omega, bi'_a}}{a\top + b\top \to af_a, bf_a, bb^\omega, ai_a, bi'_a}}{a\top + b\top \to af_a, bf_a, bb^\omega, ai_a + bi'_a}}{\top \to af_a, bf_a, b^\omega, i'_a}\;\circ}{\top \to af_a + bf_a + b^\omega, i'_a}}{\top \to f_a, i_a}\;\bullet}{\top \to f_a + i_a}
$$

Key

$$b^\omega := \nu Y(bY)$$
$$i'_a := \mu Y(ai_a + bY)$$

Correctness

$-\circ^\omega$: orange trace good

$-\bullet^\omega$: green trace good

$(- \bullet - \circ)^\omega$: green/blue trace good

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If $\widehat{\mathsf{RLL}}$ *has a proof of* $e \to f$ *then* $\mathcal{L}(e) \subseteq \mathcal{L}(f)$.

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Proof idea.
Let P be a $\widehat{\mathsf{RLL}}$ preproof of $e \to f$, but $w \in \mathcal{L}(e) \setminus \mathcal{L}(f)$.

- Let $\mathfrak{e}/\mathfrak{a}$ be $\exists/\forall$ winning strategies from $(w, e)/(w, f)$, by adequacy.
- $\mathfrak{e}$, $\mathfrak{a}$ together determine an infinite branch $B_{\mathfrak{e},\mathfrak{a}}$ of P:
  - $\mathfrak{e}$ decides the direction at a left branching rule;
  - $\mathfrak{a}$ decides the direction at a right branching rule.
- By construction, any LHS / RHS trace of $B_{\mathfrak{e},\mathfrak{a}}$ is a play of $\mathfrak{e}/\mathfrak{a}$.
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Corollary
$\mathsf{CRLL} \vdash e \to f \implies \mathcal{L}(e) \subseteq \mathcal{L}(f)$.

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The proof search game for $\widehat{\mathsf{RLL}}$ is finite memory determined.

Theorem (Completeness)

$\mathcal{L}(e) \subseteq \mathcal{L}(f) \implies \mathsf{CRLL} \vdash e \to f$, *for* e, f *guarded*.

For completeness, we need further combinatorial game theoretic machinery.

Construe proof search as a 2-player game between `Prover` and `Refuter`.

By ω-regularity of the correctness condition for $\widehat{\mathsf{RLL}}$:

Lemma (cf. Büchi-Landweber)
The proof search game for $\widehat{\mathsf{RLL}}$ is finite memory determined.

Theorem (Completeness)
$\mathcal{L}(e) \subseteq \mathcal{L}(f) \implies \mathsf{CRLL} \vdash e \to f$, *for* e, f *guarded.*

Proof idea.
Assume there is a `Refuter` winning strategy $\mathfrak{R}$ from $e \to f$.

- Play $\mathfrak{R}$ against a canonical `Prover` strategy to obtain a branch B.
- By guardedness, B determines an infinite word $w_B \in \mathcal{A}^\omega$.
- By construction, the LHSs of B determine an $\exists$ strategy winning from (w_B, e).
- Dually, the RHSs of B determine an $\forall$ strategy winning from (w_B, f).

Thus by adequacy we have $w_B \in \mathcal{L}(e) \setminus \mathcal{L}(f)$. $\square$

Summary:

- We proposed an algebraic syntax for APAs and developed a cyclic proof system.
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THANK YOU.

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