Skolemization Beyond Intuitionistic Logic: The Role of Quantifier Shifts

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Outline

Skolemization

A general practice: Suppose in the process of proving a theorem, you get

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- (proof-theoretic view): replacing universal quantifiers by new function symbols (then, a validity-equivalent formula is obtained).

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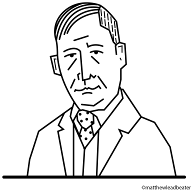
As we are considering Skolemization for any logic in general, we will choose the proof theoretic view.

Skolemization in Classical logic

Skolemization

Skolemization (proof-theoretic view): a method to remove *strong* quantifiers (i.e., positive occurrences of the universal quantifier and negative occurrences of the existential quantifier) from a first-order formula φ, and replace them with *fresh* function symbols.



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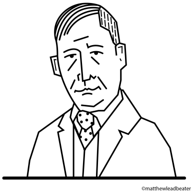

The result is called Skolemization of φ and is denoted by φ^S.

Example

$$(\forall x \exists y \forall z \varphi(x, y, z))^S = \exists y \varphi(c, y, f(y)).$$

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- First impression: Skolemization may not be done for intuitionistic predicate logic, IQC, because we know that quantifiers and connectives do not commute freely.
- Indeed! In the context of intuitionistic logic, as well as many intermediate logics, Skolemization is a non-trivial affair.

Related work

Definition

Skolemization is *sound* and *complete* for a logic L, ($\Rightarrow$ and $\Leftarrow$, resp.) when for any formula φ we have

$$L \vdash \varphi \;\Leftrightarrow\; L \vdash \varphi^S.$$

Sometimes we also say L *admits* Skolemization.

Related work

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- (Mints '66) For prenex formulas Skolemization is sound and complete in the setting of IQC.
- (Baaz, Iemhoff '10, '16, '21) Alternative methods of Skolemization in intermediate logics (in certain conservative extensions of IQC).
- (Baaz, Metcalfe, Cintula '08, '15) Skolemization for substructural and fuzzy logics.

A natural question

What if we strengthen IQC in such a way that quantifiers and connectives commute freely? (and as a consequence for each formula there is a provably equivalent formula in *prenex normal form*)
Does the new logic have Skolemization? If not, for which class of formulas does the Skolemization hold?

The motivation of this research!

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Consider the commuting of each quantifier $\forall, \exists$ with each connective $\wedge, \vee, \rightarrow$. The following principles, that we call the *quantifier shifts*, are the only ones not valid in IQC:

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Denote the logic $\mathrm{IQC} + \{\mathrm{CD}, \mathrm{SW}, \mathrm{ED}\}$ by QFS, which we call the *logic of quantifier shifts*.

Main Result

We provide a characterization of all intermediate logics that satisfy the soundness and completeness of Skolemization.

Theorem

An intermediate logic admits Skolemization if and only if it contains all quantifier shift principles.

Kripke frames and models

Definition

- A *Kripke frame* for IQC is a triple $(W, \preccurlyeq, D)$, where $W \neq \emptyset$ is a set of worlds, $\preccurlyeq$ is a binary reflexive and transitive relation over W, and D is a function assigning to each $w \in W$ a non-empty set $D(w)$, called the *domain* of w, such that if $w \preccurlyeq w'$ then $D(w) \subseteq D(w')$.
- A *Kripke model* for IQC is a quadruple $(W, \preccurlyeq, D, V)$ where $(W, \preccurlyeq, D)$ is a Kripke frame and V is a valuation function in its usual sense.

Definition

A formula A is defined to be *valid* in a frame F, denoted by $F \vDash A$, and valid in a model M, denoted by $M \vDash A$, as usual:

- $M, v \Vdash \top$, $M, v \nVdash \bot$
- $M, v \Vdash p$ iff $\quad v \in V(p)$, for a propositional variable p,
- $M, v \Vdash A \wedge B$ iff $\quad M, v \Vdash A$ and $M, v \Vdash B$
- $M, v \Vdash A \vee B$ iff $\quad M, v \Vdash A$ or $M, v \Vdash B$
- $M, v \Vdash A \to B$ iff $\quad \forall w \succcurlyeq v$ if $M, w \Vdash A$ then $M, w \Vdash B$
- $M, v \Vdash \exists x A(x)$ iff $\exists d \in D(v)$ such that $M, v \Vdash A(d)$
- $M, v \Vdash \forall x A(x)$ iff $\forall w \succcurlyeq v$ and $\forall d \in D(w)$ we have $M, w \Vdash A(d)$

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For any $w \in W$ define $\preccurlyeq [w] := \{v \in W \mid v \succcurlyeq w\}$.

Example of failure of Skolemization in IQC

Example

The axiom $\forall x(A(x) \vee B) \to \forall x A(x) \vee B$ is not provable in IQC (as illustrated below) but its Skolemization $\forall x(A(x) \vee B) \to A(c) \vee B$ is.

$$\begin{array}{c} D_v = \{c, d\} \; v \Vdash A(c), B \; v \nVdash A(d) \\ \uparrow \\ D_w = \{c\} \; w \Vdash A(c) \; w \nVdash B \end{array}$$

Because

$$w \Vdash \forall x(A(x) \vee B) \quad \text{but} \quad w \nVdash \forall x A(x) \vee B$$

Class of frames

As the first step of studying the logic of quantifier shifts, let us investigate the semantics.

Definition

Let $F = (W, \preccurlyeq, D)$ be a frame. Define:
(WF) $\forall w \in W$ the set $\preccurlyeq [w]$ is well-founded.
(cWF) $\forall w \in W$ the set $\preccurlyeq [w]$ is conversely well-founded.

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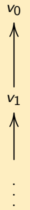
Remark

A frame F satisfying WF does not imply that $\preccurlyeq$ is well-founded on W:



Class of frames

We present rich classes of frames for QFS and its fragments.

Definition

Define $\mathcal{F}$ as the class of the following Kripke frames F closed under the disjoint union: F is constant domain with domain $\mathcal{D}$, satisfying one of the following conditions:

1. $|\mathcal{D}| = 1$,
2. $|\mathcal{D}| > 1$, $\mathcal{D}$ is finite, and F is linear,
3. $\mathcal{D}$ is infinite, F is linear, and satisfies both WF and cWF.

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Define $\mathcal{F}_{\mathrm{ED}}$ (resp. $\mathcal{F}_{\mathrm{SW}}$) as the class of Kripke frames containing $\mathcal{F}$ and also frames of the form

- constant domain, linear, infinite domains, satisfying WF (resp. cWF).

Frame characterization

$$\begin{aligned}\mathrm{CD}: &\ \forall x(A(x) \vee B) \to \forall x A(x) \vee B \\ \mathrm{SW}: &\ (\forall x A(x) \to B) \to \exists x(A(x) \to B) \\ \mathrm{ED}: &\ (B \to \exists x A(x)) \to \exists x(B \to A(x))\end{aligned}$$

Theorem (Frame characterization)

Let F be a frame.

- $F \models \mathrm{CD}$ *if and only if* F *is constant domain.*
- $F \models \mathrm{ED}$ *if and only if* $F \in \mathcal{F}_{\mathrm{ED}}$.
- $F \models \mathrm{SW}$ *if and only if* $F \in \mathcal{F}_{\mathrm{SW}}$.
- $F \models \mathrm{QFS}$ *if and only if* $F \in \mathcal{F}$.

Example

$$
\begin{tikzcd}
\vdots & \mathcal{D} \quad k_1 \\
\mathcal{D} \quad w_3 \arrow[u] & \mathcal{D} \quad k_2 \arrow[u] \\
\mathcal{D} \quad w_2 \arrow[u] & \mathcal{D} \quad k_3 \arrow[u] \\
\mathcal{D} \quad w_1 \arrow[u] & \vdots \arrow[u]
\end{tikzcd}
$$

- The the left frame is a frame for ED:

$$w_1 \Vdash B \to \exists x A(x) \quad \text{and} \quad w_1 \Vdash \exists x (B \to A(x))$$

- The right one is a frame for QFS.

Frame incompleteness

Definition

The logic L is *sound and complete* w.r.t. the class $\mathcal{C}$ of Kripke frames when for any formula φ

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Theorem

The following logics are frame-incomplete:

$$\mathsf{QFS} \qquad \mathsf{IQC} + \{\mathsf{CD}, \mathsf{SW}\} \qquad \mathsf{IQC} + \{\mathsf{CD}, \mathsf{ED}\}$$

Proof sketch for QFS

Proof.

To show that QFS is frame-incomplete, we have to prove that for any class $\mathcal{C}$ of Kripke frames for QFS, there exists a formula φ such that $\mathcal{C} \vDash \varphi$ but QFS $\nvDash \varphi$.

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To show that QFS is frame-incomplete, we have to prove that for any class $\mathcal{C}$ of Kripke frames for QFS, there exists a formula φ such that $\mathcal{C} \models \varphi$ but QFS $\not\models \varphi$. We claim taking an instance of $\varphi = \mathsf{Lin} \vee \mathsf{OEP}$ works, where

$$\mathsf{Lin} := (C \to D) \vee (D \to C) \quad \text{and} \quad \mathsf{OEP} := \exists x A(x) \to \forall x A(x)$$

are the Linearity and One Element Principle schemata.

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1. If a frame F validates QFS, then it is constant domain. Moreover, F is either linear or its domain is just a singleton. This means that F validates all instances of the axiom schema $\mathsf{Lin} \vee \mathsf{OEP}$.

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2. There is an instance of Lin $\vee$ OEP such that QFS $\not\models$ Lin $\vee$ OEP.

These two points together prove that QFS is frame-incomplete. □

The following model is a model of QFS but $w \nVdash$ Lin $\vee$ OEP. Let $\mathcal{D}_w = \mathcal{D}_{v_1} = \mathcal{D}_{v_2} = \mathcal{D} = \{a, b\}$.

$$
\begin{tikzcd}
v_1 \vDash R(a), R(b), P \quad v_1 \nVdash Q & & v_2 \vDash R(a), R(b), Q \quad v_2 \nVdash P \\
& w \Vdash R(a) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Separation of fragments

What do we know about fragments of QFS? Are they distinct?

Theorem

The following hold:

1. $\mathsf{CD}, \mathsf{SW} \nvdash \mathsf{ED}$ $\quad$ (*hence:* $\mathsf{CD} \nvdash \mathsf{ED}$ *and* $\mathsf{SW} \nvdash \mathsf{ED}$)
2. $\mathsf{CD}, \mathsf{ED} \nvdash \mathsf{SW}$ $\quad$ (*hence:* $\mathsf{CD} \nvdash \mathsf{SW}$ *and* $\mathsf{ED} \nvdash \mathsf{SW}$)

Therefore, QFS, $\mathsf{IQC} + \{\mathsf{CD}, \mathsf{SW}\}$, and $\mathsf{IQC} + \{\mathsf{CD}, \mathsf{ED}\}$ are all distinct.

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Theorem

$\mathsf{SW} \vdash \mathsf{CD}$ $\qquad$ *and* $\qquad$ $\mathsf{ED} \nvdash \mathsf{CD}$

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Revisiting Skolemization

A thought:

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Corollary

QFS admits Skolemization.

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Corollary

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Unfortunately, this is not the case.

Prenex fragments of QFS and IQC

Example

QFS $\vdash$ CD

QFS $\vdash \forall x(A(x) \vee B) \to \forall y A(y) \vee B$

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Prenex fragments of QFS and IQC

Example

QFS $\vdash$ CD

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However, IQC $\nvdash \exists x \forall y((A(x) \vee B) \to (A(y) \vee B))$. Take:

$$\begin{tikzcd}
\{1,2\}\ w_2 \Vdash A(1) \\
\{1\}\ w_1 \arrow[u]
\end{tikzcd}$$

where $\mathcal{D}_1 = \{1\}$ and $\mathcal{D}_2 = \{1,2\}$. Then $w_2 \Vdash A(1) \vee B$ but $w_2 \nVdash A(2) \vee B$. Hence, $w_1 \nVdash \exists x \forall y((A(x) \vee B) \to (A(y) \vee B))$.

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- We will use this sequent calculus.

Definitions

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- The **characteristic variable** of an inference is a, if the inference yields a strongly quantified formula $QxA(x)$ from $A(a)$, where a is a free variable.
- Let π be a derivation. We say b is a **side variable** of a in π ($a <_\pi b$) if π contains a strong-quantifier inference of one of the forms:
$$\frac{\Gamma \Rightarrow A(a, b, \vec{c})}{\Gamma \Rightarrow \forall x A(x, b, \vec{c})} \qquad \frac{A(a, b, \vec{c}), \Gamma \Rightarrow \Delta}{\exists x A(x, b, \vec{c}), \Gamma \Rightarrow \Delta}$$

Sequent calculi **LK** and **LJ**

First-order **LK** is the extension of the usual propositional **LK** for classical logic obtained by adding quantifier inferences:

$$\frac{\Gamma, \varphi(t) \Rightarrow \Delta}{\Gamma, \forall x \varphi(x) \Rightarrow \Delta}\ \forall L \qquad \frac{\Gamma \Rightarrow \varphi(y), \Delta}{\Gamma \Rightarrow \forall x \varphi(x), \Delta}\ \forall R$$

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The sequent calculus **QFS** is defined as **LJ** $+ \{\mathrm{CD}, \mathrm{ED}, \mathrm{SW}\}$.

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For instance, each quantifier inference of **LJ** as in the previous slide is suitable for every regular **LJ**-proof.

Justification of a suitable quantifier

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Note that in $*_1$ and $*_2$ we have $b <_\pi a$ and $a <_\pi b$, respectively.

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$\mathbf{LK}^{++}$ and $\mathbf{LJ}^{++}$

Definition (Aguilera, Baaz '19)

The calculus $\mathbf{LK}^{++}$ (resp. $\mathbf{LJ}^{++}$) is defined like **LK** (resp. **LJ**), except that the constraint on $\exists L$ and $\forall R$ is removed and one adds the restriction that a proof may only contain quantifier inferences that are *suitable* for it.

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An interesting fact: all the quantifier shift axioms are provable in $\mathbf{LJ}^{++}$.

Theorem (Aguilera, Baaz '19)

Let S be a sequent. We have $\mathbf{LJ}^{++} \vdash S$ *iff* $\mathbf{QFS} \vdash S$

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If the Skolemization of $\Gamma \Rightarrow \Delta$ *is cut-free* $\mathbf{LJ}^{++}$*-derivable, then* $\Gamma \Rightarrow \Delta$ *is cut-free* $\mathbf{LJ}^{++}$*-derivable.*

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Corollary

QFS admits Skolemization.

Proof of Corollary

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- As **LJ** $\vdash \Rightarrow \varphi \to \varphi^S$ for any φ and **LJ** $\vdash \Rightarrow (B_1 \wedge \cdots \wedge B_n) \to A^S$, we have **LJ** $\vdash \Rightarrow ((B_1 \wedge \cdots \wedge B_n) \to A^S)^S$.

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- Hence, there is a cut-free proof π of $\Rightarrow ((B_1 \wedge \cdots \wedge B_n) \to A^S)^S$ in **LJ**.
- Since π is also an $\mathbf{LJ}^{++}$-proof, by Theorem we obtain $\mathbf{LJ}^{++} \vdash \Rightarrow (B_1 \wedge \cdots \wedge B_n) \to A$, which yields $\mathbf{QFS} \vdash \Rightarrow A$ hence QFS $\vdash A$.

$\square$

Concluding remarks

- We introduced a logic QFS by adding the quantifier shifts to IQC.
- QFS is Kripke frame-incomplete.
- **Main result:** An intermediate logic admits Skolemization if and only if it contains all quantifier shift principles.

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Future work:

- Does the logic QFS have the disjunction property (DP)? (We know that it doesn not have the existence property (EP).)
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Thank you for your attention.