A Proof-Theoretic View of Basic Intuitionistic Conditional Logic

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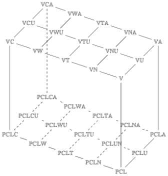
Conditional logic

- Account of conditionals *'if φ, then ψ'* different from material implication
- Reasoning with conditional sentences:

$$\begin{array}{ll} \varphi \mathrel{\Box\!\!\rightarrow} \psi & \textit{`If } \varphi \textit{ were the case, then } \psi \textit{ would be the case'} \\ \varphi \mathrel{\Diamond\!\!\rightarrow} \psi & \textit{`If } \varphi \textit{ were the case, then } \psi \textit{ might be the case'} \end{array}$$

- Tipically, extension of classical logic: CPL + would/might-principles
- Many and diverse applications: counterfactuals, conditional obligations, preferences, ceteris paribus, comparative similarity
- Several alternative semantics: selection functions, preferential models, sphere models
- Big family of logics



and many more

Basic conditional logic

- CK introduced by Chellas 1975
- Conditional logic core: few fundamental principles shared by most conditional logics
- Congruence on the left

$$\mathrm{RA}_{\Box}\ \frac{\varphi \leftrightarrow \rho}{(\varphi \mathrel{\Box\!\!\rightarrow} \psi) \leftrightarrow (\rho \mathrel{\Box\!\!\rightarrow} \psi)}$$

- Normality on the right

$$\begin{array}{lll}
\mathrm{RC}_{\Box} & \dfrac{\psi \leftrightarrow \chi}{(\varphi \mathrel{\Box\!\!\rightarrow} \psi) \leftrightarrow (\varphi \mathrel{\Box\!\!\rightarrow} \chi)} & \text{(congruence)} \\
\mathrm{CM}_{\Box} & (\varphi \mathrel{\Box\!\!\rightarrow} \psi \wedge \chi) \rightarrow (\varphi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\varphi \mathrel{\Box\!\!\rightarrow} \chi) & \text{(monotonicity)} \\
\mathrm{CC}_{\Box} & (\varphi \mathrel{\Box\!\!\rightarrow} \psi) \wedge (\varphi \mathrel{\Box\!\!\rightarrow} \chi) \rightarrow (\varphi \mathrel{\Box\!\!\rightarrow} \psi \wedge \chi) & \text{(agglomeration)} \\
\mathrm{CN}_{\Box} & \varphi \mathrel{\Box\!\!\rightarrow} \top & \text{(necessitation)}
\end{array}$$

- Smallest normal conditional logic
- CK is to conditional logics as K is to modal logics
- Basic system to be extended with further conditional principles

Intuitionistic conditional logic

- Can intuitionistic logic support reasoning on conditionals?
- Which classical systems have intuitionistic counterparts?
- Analogy with intuitionistic modal logic

Basic intuitionistic conditional logic

Initiated by Weiss 2019

- Intuitionistic variant of Chellas' CK
- Defined in the language with only $\Box\!\!\rightarrow$
- Axiomatically, IPL + $\Box\!\!\rightarrow$-axioms of CK
- Kripke semantics ('Weiss models' in our paper) combining intutionistic order and Chellas 'selection function' (in Bozic-Dosen style)
- Called $\mathsf{ConstCK}^{\Box\!\!\rightarrow}$ in our paper

Continued by Ciardelli & Liu 2020

- Alternative semantics with general frames
- Further extensions

Dufty & de Groot 2025

- Very recently, further analysis and extensions

Two problems raised by Weiss 2019

1. Extensions of CK without intuitionistic counterparts
   - Stalnaker's logic C2, classical system stronger than CK
   - Characteristic axiom: conditional excluded middle $\mathrm{CEM}_{\Box}$
     $(\varphi \mathrel{\Box\!\!\rightarrow} \psi) \vee (\varphi \mathrel{\Box\!\!\rightarrow} \neg\psi)$
   - $\mathrm{CEM}_{\Box}$ + conditional modus ponens $(\varphi \mathrel{\Box\!\!\rightarrow} \psi) \rightarrow (\varphi \rightarrow \psi)$ $\Rightarrow_{\mathrm{IPL}}$ $\varphi \vee \neg\varphi$
   - Weiss' claim: Intuitionistic variant not possible
2. Addition of the might conditional $\mathrel{\Diamond\!\!\rightarrow}$
   - $\mathrel{\Diamond\!\!\rightarrow}$ behaviour not determined by $\mathrel{\Box\!\!\rightarrow}$
   - Many different relations bewteen $\mathrel{\Box\!\!\rightarrow}$ and $\mathrel{\Diamond\!\!\rightarrow}$ are possible
   - Analogous to $\Box, \Diamond$ of intuitionistic modal logics
   - Left open by Weiss 2019

Tackling the problems

1. Ciardelli & Liu 2020: Never give up
   - Intuitionistic variant of Stalnaker's C2 possible considering the classically but not intuitionistically equivalent axiom of conditional determinacy $(\varphi \boxright \psi \vee \chi) \to (\varphi \boxright \psi) \vee (\varphi \boxright \chi)$
2. Olkhovikov's 2024 proposal: IntCK
   - Intuitionistic variant of CK with both $\boxright$ and $\diamondright$
   - Fischer Servi style semantics
   - But: extension not conservative: validates $\boxright$ principles that are not valid in $\mathrm{ConstCK}^{\boxright}$
   - Analogus to intuitionistic modal logics: $\Box$-fragment of IK vs iK (Das & Marin 2023)

Our contribution

Basic intuitionistic conditional logic through the lens of proof-theory

- Make order: systematic view of the existing systems
- Conservative extension of $\mathsf{ConstCK}^{\Box\!\!\rightarrow}$ with $\Diamond\!\!\rightarrow$
- Byproduct: Aternative intuitionistic variant of $\mathsf{CEM}_{\Box}$

Why proof theory...

... instead of axioms?

- Axioms equivalent over CPL but not over IPL. How to select?
- Example of $\mathrm{CEM}_{\Box}$: Weiss 2019 vs. Ciardelli & Liu 2020 vs. alternative formulations
- Which axioms for $\Diamond\!\!\rightarrow$?
- Which interactions between $\Box\!\!\rightarrow$ and $\Diamond\!\!\rightarrow$?

... lost in the intuitionistic conditional jungle

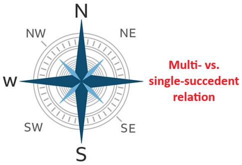
Why proof theory...

... instead of semantics?

- A possible way: follow the heredity property. But:
- Not necessarily conservative over $\mathsf{ConstCK}^{\Box\mapsto}$ (e.g. Olkhovikov's IntCK)
- Many different ways to satisfy it (think about intu. variants of modal logic K)

... still easy to get lost

Proof-theoretic compass to orientate in the intuitionistic conditional jungle

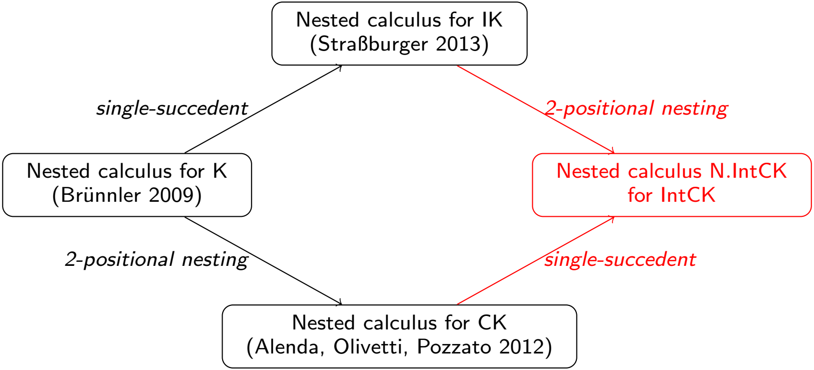


The direction we follow: Multi- vs. single-succedent sequents relation

Multi- vs. single-succedent relation

- Understand the relations bewteen CK and existing intu. counterparts of it
- Define further intuitionistic counterparts and extensions
- Known to hold between CPL and IPL since Gentzen 1935
- Applied to intuitionistic modal logics (Straßburger 2013, Das & Marin 2023, ...)

Nested calculus for IntCK



Nested calculus for IntCK

- From Straßburger 2013: treatment of intuitionistic base, input and output polarities
- From Alenda, Olivetti, Pozzato 2012: indexing nested components with formulas to evaluate conditional operators
- Conditional rules:

$$\Box\!\!\rightarrow^{\bullet}\ \frac{\varphi^{\bullet},\eta^{\circ} \quad \eta^{\bullet},\varphi^{\circ} \quad \Gamma\{\varphi \Box\!\!\rightarrow \psi^{\bullet}, [\eta : \psi^{\bullet}, \Delta]\}}{\Gamma\{\varphi \Box\!\!\rightarrow \psi^{\bullet}, [\eta : \Delta]\}} \qquad \Box\!\!\rightarrow^{\circ}\ \frac{\Gamma\{[\varphi : \psi^{\circ}]\}}{\Gamma\{\varphi \Box\!\!\rightarrow \psi^{\circ}\}}$$

$$\Diamond\!\!\rightarrow^{\bullet}\ \frac{\Gamma\{[\varphi : \psi^{\bullet}]\}}{\Gamma\{\varphi \Diamond\!\!\rightarrow \psi^{\bullet}\}} \qquad \Diamond\!\!\rightarrow^{\circ}\ \frac{\varphi^{\bullet},\eta^{\circ} \quad \eta^{\bullet},\varphi^{\circ} \quad \Gamma\{[\eta : \psi^{\circ}, \Delta]\}}{\Gamma\{\varphi \Diamond\!\!\rightarrow \psi^{\circ}, [\eta : \Delta]\}}$$

- Sound and complete calculus for IntCK
- Admissibility of structural rules and cut elimination

Sequent calculus for ConstCK$^{\Box\!\!\rightarrow}$

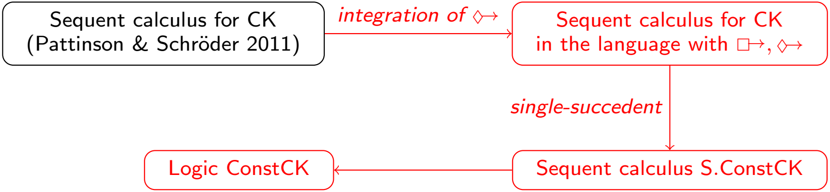
Sequent calculus for CK (Pattinson & Schröder 2011) $\xrightarrow{\textit{single-succedent}}$ Sequent calculus S.ConstCK$^{\Box\!\!\rightarrow}$ for ConstCK$^{\Box\!\!\rightarrow}$

- Conditional rule:

$$\Box\!\!\rightarrow\ \frac{\{\varphi \Leftrightarrow \rho_i\}_{i \leq n} \qquad \sigma_1, \ldots, \sigma_n \Rightarrow \psi}{\Gamma, \rho_1 \Box\!\!\rightarrow \sigma_1, \ldots, \rho_n \Box\!\!\rightarrow \sigma_n \Rightarrow \varphi \Box\!\!\rightarrow \psi}$$

- Sound and complete calculus for ConstCK$^{\Box\!\!\rightarrow}$
- Admissibility of structural rules and cut elimination

Adding $\Diamond\!\!\rightarrow$ to ConstCK$^{\Box\!\!\rightarrow}$: the logic ConstCK



Adding $\Diamond\!\!\rightarrow$ to ConstCK$^{\Box\!\!\rightarrow}$: the logic ConstCK

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{Sequent calculus for CK}\\ \text{(Pattinson \& Schröder 2011)}\end{array}} \arrow[r, "\textit{integration of } \Diamond\!\!\rightarrow"] & \boxed{\begin{array}{c}\text{Sequent calculus for CK}\\ \text{in the language with } \Box\!\!\rightarrow, \Diamond\!\!\rightarrow\end{array}} \arrow[d, "\textit{single-succedent}"'] \\
\boxed{\text{Logic ConstCK}} & \boxed{\text{Sequent calculus S.ConstCK}} \arrow[l]
\end{tikzcd}
$$

- Conditional rules:

$$
\Box\!\!\rightarrow \; \frac{\{\varphi \Leftrightarrow \rho_i\}_{i \leq n} \qquad \sigma_1, \ldots, \sigma_n \Rightarrow \psi}{\Gamma, \rho_1 \Box\!\!\rightarrow \sigma_1, \ldots, \rho_n \Box\!\!\rightarrow \sigma_n \Rightarrow \varphi \Box\!\!\rightarrow \psi}
$$

$$
\Diamond\!\!\rightarrow \; \frac{\{\varphi \Leftrightarrow \rho_i\}_{i \leq n} \qquad \varphi \Leftrightarrow \eta \qquad \sigma_1, \ldots, \sigma_n, \psi \Rightarrow \vartheta}{\Gamma, \rho_1 \Box\!\!\rightarrow \sigma_1, \ldots, \rho_n \Box\!\!\rightarrow \sigma_n, \varphi \Diamond\!\!\rightarrow \psi \Rightarrow \eta \Diamond\!\!\rightarrow \vartheta}
$$

$$
\Diamond\!\!\rightarrow_\bot \; \frac{\{\varphi \Leftrightarrow \rho_i\}_{i \leq n} \qquad \sigma_1, \ldots, \sigma_n, \psi \Rightarrow}{\Gamma, \rho_1 \Box\!\!\rightarrow \sigma_1, \ldots, \rho_n \Box\!\!\rightarrow \sigma_n, \varphi \Diamond\!\!\rightarrow \psi \Rightarrow \Delta}
$$

Adding $\Diamond\!\!\rightarrow$ to $\mathsf{ConstCK}^{\Box\!\!\rightarrow}$: the logic ConstCK

Properties of S.ConstCK and ConstCK

- Admissibility of structural rules and cut elimination
- Equivalent axiomatic system: ConstCK:

$$\begin{array}{ll} \multicolumn{2}{l}{\mathsf{IPL} + \text{congruence rules of left and right} +} \\ \mathsf{CM}_\Box & (\varphi \Box\!\!\rightarrow \psi \wedge \chi) \to (\varphi \Box\!\!\rightarrow \psi) \wedge (\varphi \Box\!\!\rightarrow \chi) \\ \mathsf{CC}_\Box & (\varphi \Box\!\!\rightarrow \psi) \wedge (\varphi \Box\!\!\rightarrow \chi) \to (\varphi \Box\!\!\rightarrow \psi \wedge \chi) \\ \mathsf{CN}_\Box & \varphi \Box\!\!\rightarrow \top \\ \mathsf{CN}_\Diamond & \neg(\varphi \Diamond\!\!\rightarrow \bot) \\ \mathsf{CK}_\Diamond & (\varphi \Box\!\!\rightarrow (\psi \to \chi)) \to ((\varphi \Diamond\!\!\rightarrow \psi) \to (\varphi \Diamond\!\!\rightarrow \chi)) \end{array}$$

- Soundness and completeness w.r.t. *Constructive Chellas models*
- $\Diamond\!\!\rightarrow$-free fragment of ConstCK amounts to $\mathsf{ConstCK}^{\Box\!\!\rightarrow}$:

$$\text{For } \varphi \in \mathcal{L}^{\Box\!\!\rightarrow},\ \mathsf{ConstCK} \vdash \varphi \text{ iff } \mathsf{ConstCK}^{\Box\!\!\rightarrow} \vdash \varphi.$$

Why these names for intuitionistic conditional logics

Int/const modal logics embeddable into int conditional logics via the translation

$$(\Box\varphi)^t := \top \Box\!\!\rightarrow \varphi^t \qquad (\Diamond\varphi)^t := \top \Diamond\!\!\rightarrow \varphi^t$$

In particular:

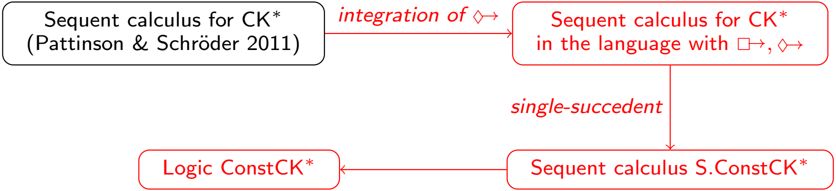
$$\mathsf{iK} \rightsquigarrow \mathsf{ConstCK}^{\Box\!\!\rightarrow} \qquad \mathsf{IK} \rightsquigarrow \mathsf{IntCK} \qquad \mathsf{WK} \rightsquigarrow \mathsf{ConstCK}$$

Extensions of ConstCK

We study intuitionistic counterparts of the extensions of CK with:

- Identity: $\mathsf{ID}_{\Box}$ $\varphi \Box\!\!\rightarrow \varphi$
- Conditional modus ponens: $\mathsf{MP}_{\Box}$ $(\varphi \Box\!\!\rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$
- Conditional excluded middle: $\mathsf{CEM}_{\Box}$ $(\varphi \Box\!\!\rightarrow \psi) \vee (\varphi \Box\!\!\rightarrow \neg\psi)$

That is, the conditional logics covered in Pattinson & Schröder 2011



Extensions of ConstCK

- Conditional rules:

$$\mathrel{\Box\!\!\rightarrow}^{\mathsf{id}}\ \frac{\{\varphi \Leftrightarrow \rho_i\}_{i\leq n} \qquad \sigma_1, \dots, \sigma_n, \varphi \Rightarrow \psi}{\Gamma, \rho_1 \mathrel{\Box\!\!\rightarrow} \sigma_1, \dots, \rho_n \mathrel{\Box\!\!\rightarrow} \sigma_n \Rightarrow \varphi \mathrel{\Box\!\!\rightarrow} \psi}$$

$$\mathrel{\Diamond\!\!\rightarrow}^{\mathsf{id}}\ \frac{\{\varphi \Leftrightarrow \rho_i\}_{i\leq n} \qquad \varphi \Leftrightarrow \eta \qquad \sigma_1, \dots, \sigma_n, , \varphi, \psi \Rightarrow \vartheta}{\Gamma, \rho_1 \mathrel{\Box\!\!\rightarrow} \sigma_1, \dots, \rho_n \mathrel{\Box\!\!\rightarrow} \sigma_n, \varphi \mathrel{\Diamond\!\!\rightarrow} \psi \Rightarrow \eta \mathrel{\Diamond\!\!\rightarrow} \vartheta}$$

$$\mathrel{\Diamond\!\!\rightarrow}^{\mathsf{id}}_{\bot}\ \frac{\{\varphi \Leftrightarrow \rho_i\}_{i\leq n} \qquad \sigma_1, \dots, \sigma_n, , \varphi, \psi \Rightarrow}{\Gamma, \rho_1 \mathrel{\Box\!\!\rightarrow} \sigma_1, \dots, \rho_n \mathrel{\Box\!\!\rightarrow} \sigma_n, \varphi \mathrel{\Diamond\!\!\rightarrow} \psi \Rightarrow \Delta}$$

$$\mathsf{mp}_{\Box}\ \frac{\Gamma, \varphi \mathrel{\Box\!\!\rightarrow} \psi \Rightarrow \varphi \qquad \Gamma, \varphi \mathrel{\Box\!\!\rightarrow} \psi, \psi \Rightarrow \Delta}{\Gamma, \varphi \mathrel{\Box\!\!\rightarrow} \psi \Rightarrow \Delta} \qquad \mathsf{mp}_{\Diamond}\ \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \mathrel{\Diamond\!\!\rightarrow} \psi}$$

$$\mathrel{\Diamond\!\!\rightarrow}^{\mathsf{cem}}\ \frac{\{\varphi \Leftrightarrow \rho_i\}_{i\leq n} \qquad \{\varphi \Leftrightarrow \xi_j\}_{j\leq k} \qquad \varphi \Leftrightarrow \eta \qquad \sigma_1, \dots, \sigma_n, \chi_1, \dots, \chi_k, \psi \Rightarrow \vartheta}{\Gamma, \rho_1 \mathrel{\Box\!\!\rightarrow} \sigma_1, \dots, \rho_n \mathrel{\Box\!\!\rightarrow} \sigma_n, \xi_1 \mathrel{\Diamond\!\!\rightarrow} \chi_1, \dots, \xi_k \mathrel{\Diamond\!\!\rightarrow} \chi_k, \varphi \mathrel{\Diamond\!\!\rightarrow} \psi \Rightarrow \eta \mathrel{\Diamond\!\!\rightarrow} \vartheta}$$

$$\mathrel{\Diamond\!\!\rightarrow}^{\mathsf{cem}}_{\bot}\ \frac{\{\varphi \Leftrightarrow \rho_i\}_{i\leq n} \qquad \{\varphi \Leftrightarrow \xi_j\}_{j\leq k} \qquad \sigma_1, \dots, \sigma_n, \chi_1, \dots, \chi_k, \psi \Rightarrow}{\Gamma, \rho_1 \mathrel{\Box\!\!\rightarrow} \sigma_1, \dots, \rho_n \mathrel{\Box\!\!\rightarrow} \sigma_n, \xi_1 \mathrel{\Diamond\!\!\rightarrow} \chi_1, \dots, \xi_k \mathrel{\Diamond\!\!\rightarrow} \chi_k, \varphi \mathrel{\Diamond\!\!\rightarrow} \psi \Rightarrow \Delta}$$

Extensions of ConstCK

- Equivalent axiomatics systems characterised by axioms

$\mathsf{ID}_\Box$	$\varphi \Box\!\!\rightarrow \varphi$
$\mathsf{MP}_\Box$	$(\varphi \Box\!\!\rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$
$\mathsf{MP}_\Diamond$	$\varphi \wedge \psi \rightarrow (\varphi \Diamond\!\!\rightarrow \psi)$
$\mathsf{CEM}_\Diamond$	$(\varphi \Diamond\!\!\rightarrow \psi) \wedge (\varphi \Diamond\!\!\rightarrow \chi) \rightarrow (\varphi \Diamond\!\!\rightarrow \psi \wedge \chi)$

- Constructive Chellas models for S.ConstCKID, S.ConstCKMP, S.ConstCKMPID: soundness and completeness
- $\Diamond\!\!\rightarrow$-free fragments of S.ConstCKID, S.ConstCKMP, S.ConstCKMPID coincide with the logics by Weiss 2019
- A different counterpart for logics with CEM: no $\Box\!\!\rightarrow$ axiom, but $\mathsf{CEM}_\Diamond$

Let's stop here for today 

Where we arrived using the proof-theoretic compass

- Clear proof-theoretic relations between CK and its intuitionistic counterparts
- New intuitionistic counterpart of CK conservative over ConstCK$^{\Box\to}$
- Definition of some extensions
- Semantical characterisation of most of them

A cardinal direction is not the same as a GPS track

There still are details to be checked and choices to be made

- Schröder, Pattinson, Hausmann 2010: calculi for logics with cautious monotonicity. Is the same strategy still applicable?
- Different ways of being single-succedent: e.g. modal logics CK vs. WK
- Extensions of nested calculi

Just an example of a possible fork

- CEM rule by Alenda, Olivetti, Pozato 2012:

$$\text{cem}\ \frac{\varphi^\bullet, \eta^\circ \quad \eta^\bullet, \varphi^\circ \quad \Gamma\{[\varphi : \Sigma, \Delta], [\eta : \Delta]\}}{\Gamma\{[\varphi : \Sigma], [\eta : \Delta]\}}$$

- Single-succedent intuitionistic base (Straßburger 2013 style):
  - CEM$_\Diamond$ $(\varphi \Diamond\!\!\rightarrow \psi) \wedge (\varphi \Diamond\!\!\rightarrow \chi) \rightarrow (\varphi \Diamond\!\!\rightarrow \psi \wedge \chi)$ derivable
  - CD $(\varphi \Box\!\!\rightarrow \psi \vee \chi) \rightarrow (\varphi \Box\!\!\rightarrow \psi) \vee (\varphi \Box\!\!\rightarrow \chi)$ not derivable
- Multi-succedent intuitionistic base (Kuznets & Straßburger 2019 style):
  - CEM$_\Diamond$ $(\varphi \Diamond\!\!\rightarrow \psi) \wedge (\varphi \Diamond\!\!\rightarrow \chi) \rightarrow (\varphi \Diamond\!\!\rightarrow \psi \wedge \chi)$ derivable
  - CD $(\varphi \Box\!\!\rightarrow \psi \vee \chi) \rightarrow (\varphi \Box\!\!\rightarrow \psi) \vee (\varphi \Box\!\!\rightarrow \chi)$ derivable
- Different intuitionistionizations give rise to different counterparts
- Many questions open
  - Cut admissibility?
  - Semantics?
  - Relations with classical counterparts?
  - ...

The exploration of the intuitionistic conditional jungle is only just beginning

Thank you!