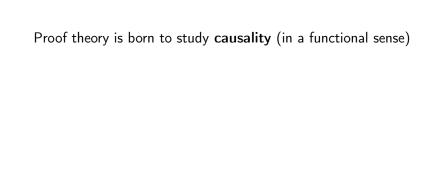
Intuitionistic BV (& Friends)

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- Motivations
- 2 IMLL and BV
- IBV
- 4 INML
- 6 Conclusion

Motivations



Proof theory is born to study **causality** (in a functional sense)

 $A \Rightarrow B$

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Theorem (Deduction Theorem [in any 'reasonable' logic])

If I can prove B from A, then I can prove $A \Rightarrow B$.

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Theorem (Deduction Theorem [in any 'reasonable' logic])

If I can prove B from A, then I can prove $A \Rightarrow B$.

... but what if we want to consider **sequentiality**?

$$a \mid \bar{a}.b \mid \bar{b} \mid c \mid \bar{c}$$

Imperative programming

$$x = 5$$

$$y = x + 2$$

$$x = 2$$

- Sequential algorithms

$$a \mid \bar{a}.b \mid \bar{b} \mid c \mid \bar{c}$$

Imperative programming

- Sequential algorithms
- . . .

How to model these things logically?

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Imperative programming

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How to model these things logically?

WHY?

$$a \mid \bar{a}.b \mid \bar{b} \mid c \mid \bar{c}$$

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- Sequential algorithms
- . . .

How to model these things logically?

WHY?

Type systems for imperative programming, (better) type systems for process calculi, logical models of sequential algorithms, ...

IMLL and BV



IMLL

IMLL

Trivia about IMLL:

- LJ = IMLL + structural rules
- IMLL type system for linear λ-calculus
- categorical model: symmetric monoidal closed category
- Cut-elimination (β-reduction)







$$A, B := a \mid \bar{a} \mid \mathbb{I} \mid A \otimes B \mid A \otimes B$$

Rules

$$\equiv \frac{A}{B} \ \dagger \qquad \mathrm{s} \frac{A \otimes (B \otimes C)}{(A \otimes B) \otimes C} \qquad \qquad \mathrm{ai} \downarrow \frac{\mathbb{I}}{a \otimes \bar{a}}$$

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$$ai\downarrow \frac{1}{a \otimes \bar{a}}$$

$$\dot{}$$
 = $$\otimes$ is associative and commutative} I is a unit for \otimes , and $\otimes$$

⊗ is associative and commutative

$$ai\uparrow \frac{a\otimes \bar{a}}{\mathbb{I}}$$

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$$\begin{tabular}{ll} $ \otimes$ is associative and commutative \\ $ \otimes$ is associative and commutative \\ $ \mathbb{I}$ is a unit for \otimes, and \otimes \\ \end{tabular}$$

$$a \otimes \bar{a}$$
 $a \otimes \bar{a}$

Derivations (Deep Inference)

$$\mathcal{D}, \mathcal{D}' \coloneqq A \mid \mathcal{D} \otimes \mathcal{D}' \mid \mathcal{D} \triangleleft \mathcal{D}' \mid \mathcal{D} \otimes \mathcal{D}' \mid r \frac{\mathcal{D}}{\mathcal{D}'} \mid r \frac{\mathcal{D}}{\mathcal{D}'} \mid r \frac{\mathcal{D}}{\mathcal{D}'}$$

$$A, B := a \mid \bar{a} \mid \mathbb{I} \mid A \otimes B \mid A \otimes B \mid A \triangleleft B$$

Rules

$$\equiv \frac{A}{B} \ \dagger \qquad \mathsf{s} \frac{A \otimes (B \otimes C)}{(A \otimes B) \otimes C} \qquad \qquad \mathsf{ai} \downarrow \frac{\mathbb{I}}{a \otimes \bar{a}} \qquad \mathsf{q} \downarrow \frac{(A \otimes B) \triangleleft (C \otimes D)}{(A \triangleleft C) \otimes (B \triangleleft D)}$$

$$ai\downarrow \frac{1}{a\otimes \bar{a}}$$

$$\operatorname{ql} \frac{(A \otimes B) \triangleleft (C \otimes D)}{(A \triangleleft C) \otimes (B \triangleleft D)}$$

$$\label{eq:definition} \dot{\uparrow} = \begin{bmatrix} \otimes \text{ is associative and commutative} \\ & \text{ is associative} \\ & \otimes \text{ is associative and commutative} \\ & \mathbb{I} \text{ is a unit for } \otimes, \ & \text{ } \text{ } \text{, } \text{ and } \otimes \\ \end{bmatrix}$$

$$\boxed{ \text{ai} \uparrow \frac{a \otimes \bar{a}}{\mathbb{I}} \qquad \text{q} \uparrow \frac{(A \triangleleft C) \otimes (B \triangleleft D)}{(A \otimes B) \triangleleft (C \otimes D)} }$$

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Trivia about BV:

• it extends $MLL \cup \{mix\}$ with a non-commutative connective \triangleleft ;

- it extends MLL ∪ {mix} with a non-commutative connective <;
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Theorem

The rule $\operatorname{cut} \frac{A \otimes \bar{A}}{\mathbb{T}}$ is admissible in BV.

Proof.

• Splitting lemma $(\odot \in \{\triangleleft, \otimes\})$:

$$\vdash_{\mathsf{BV}} (A \odot B) \otimes K \quad \Rightarrow \quad \begin{matrix} K_A \bar{\odot} K_B \\ \parallel \\ K \end{matrix} \quad \text{and} \quad \begin{matrix} \parallel \\ K_A \otimes A \end{matrix} \quad \text{and} \quad \begin{matrix} \parallel \\ K_B \otimes B \end{matrix} \; .$$

Context reduction:

$$\vdash_{\mathsf{BV}} C[A] \quad \Rightarrow \quad \begin{array}{c} K \otimes X \\ \mathcal{P}_X \parallel \\ P[X] \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{P}_A \parallel \\ K \otimes A \end{array} \quad \text{for any formula } X.$$

up-rules elimination:

$$\vdash_{\mathsf{BV}} C \left[r_{\uparrow} \frac{A}{B} \right] \Rightarrow \left[\begin{matrix} \mathsf{BV} \\ C[B] \end{matrix} \right]$$

П



Where to start?

Take MLL

$$\mathsf{ax} \frac{}{\vdash \mathsf{a}, \bar{\mathsf{a}}} \quad \otimes \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \otimes B} \quad \otimes \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

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Polarize formulas

positive:
$$A^{\circ}, B^{\circ} := a \mid A^{\bullet} \otimes B^{\circ} \mid A^{\circ} \otimes B^{\circ}$$

negative: $A^{\bullet}, B^{\bullet} := \bar{a} \mid A^{\circ} \otimes B^{\bullet} \mid A^{\bullet} \otimes B^{\bullet}$

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IMLL = positive MLL

Where to start?

Take MLL

$$\mathsf{ax}\frac{}{\vdash a,\bar{a}} \quad \otimes \frac{\vdash \Gamma,A,B}{\vdash \Gamma,A\otimes B} \quad \otimes \frac{\vdash \Gamma,A \quad \vdash \Delta,B}{\vdash \Gamma,\Delta,A\otimes B} \quad \min \frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma,\Delta}$$

Polarize formulas

positive:
$$A^{\circ}, B^{\circ} := a \mid A^{\bullet} \otimes B^{\circ} \mid A^{\circ} \otimes B^{\circ}$$

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 $IMLL = positive MLL_{mix}$

BV from IBV

$$\equiv \frac{A}{B} \qquad s \frac{A \otimes (B \otimes C)}{(A \otimes B) \otimes C} \qquad \text{ai} \downarrow \frac{\mathbb{I}}{a \otimes \bar{a}} \qquad \text{q} \downarrow \frac{(A \otimes B) \triangleleft (C \otimes D)}{(A \triangleleft C) \otimes (B \triangleleft D)}$$

BV from IBV

Take BV

$$\equiv \frac{A}{B} \qquad s \frac{A \otimes (B \otimes C)}{(A \otimes B) \otimes C} \qquad \text{ai} \downarrow \frac{\mathbb{I}}{a \otimes \bar{a}} \qquad \text{q} \downarrow \frac{(A \otimes B) \triangleleft (C \otimes D)}{(A \triangleleft C) \otimes (B \triangleleft D)}$$

Polarize formulas

positive:
$$A^{\circ}, B^{\circ} := \mathbb{I} | a | A^{\bullet} \otimes B^{\circ} | A^{\circ} \otimes B^{\circ} | A^{\circ} \triangleleft B^{\circ}$$

negative: $A^{\bullet}, B^{\bullet} := \mathbb{I} | \bar{a} | A^{\circ} \otimes B^{\bullet} | A^{\bullet} \otimes B^{\bullet} | A^{\bullet} \triangleleft B^{\bullet}$

IBV

... fine tune units (otherwise ⊗ and ⊲ collapse)

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IBV (Categorically)

IMLL	BV
$\langle \otimes, \otimes, \mathbb{I} angle$ symmetric monoidal closed category	$ \langle \otimes, \otimes, \mathbb{I} \rangle $ isomix category $ + $ degenerate linear functor $ ((A \triangleleft C) \otimes (B \triangleleft D)) \Rightarrow ((A \otimes B) \triangleleft (C \otimes D)) $

IBV (Categorically)

IMLL	BV
$\langle \otimes, \otimes, \mathbb{I} angle$ symmetric monoidal closed category	$\langle \otimes, \otimes, \mathbb{I} \rangle$ isomix category $+$ degenerate linear functor \triangleleft $\left((A \triangleleft C) \otimes (B \triangleleft D) \right) \Rightarrow \left((A \otimes B) \triangleleft (C \otimes D) \right)$

$$\langle \otimes, \neg \circ, \mathbb{I} \rangle$$
 + degenerate linear functor-ish $\triangleleft + \begin{pmatrix} A \multimap (A \triangleleft \mathbb{I}) \\ A \multimap (\mathbb{I} \triangleleft A) \end{pmatrix}$

Theorem

The rule $\operatorname{cut} \frac{A \multimap \bar{A}}{\mathbb{T}}$ is admissible in IBV.

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Theorem

These rules are admissible in IBV:

$$ai_{\uparrow}^{\bullet} \xrightarrow{A \circ a} u_{\uparrow}^{\bullet} \xrightarrow{A \circ A} u_{\uparrow}^{\bullet} \xrightarrow{A \circ I} u_{\uparrow}^{\bullet} \xrightarrow{A \circ I} u_{\uparrow}^{\bullet} \xrightarrow{A \circ I} u_{\uparrow}^{\bullet} \xrightarrow{A \circ A} u_{\downarrow}^{\bullet} \xrightarrow{A \circ A} u_{\downarrow}^{\bullet}$$

Lemma (splitting)

• If $\vdash_{\mathsf{IBV}} K \multimap (A \otimes B)$, then there are formulas K_A and K_B such that

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Lemma (Atomic Splitting)

- $\textcircled{1} \ \textit{if} \vdash_{\mathsf{IBV}} \mathsf{K} \multimap \textit{a, then there is a negative derivation}$
- 2 if $\vdash_{\mathsf{IBV}} \mathsf{a} \multimap \mathsf{K}$, then there is a positive derivation

Lemma (Context Reduction)

1 If $\vdash_{\mathsf{IBV}} P[A]$ with $P[\cdot]$ a positive context

$$\begin{array}{c|c} K \multimap X \\ \mathcal{P}_X \parallel & \text{and} & \begin{array}{c} \mathcal{P}_A \parallel \\ K \multimap A \end{array} \text{ for any formula } X.$$

2 If $\vdash_{\mathsf{IBV}} N[A]$ with $N[\cdot]$ a negative context

$$X \multimap K$$
 $\mathcal{P}_X \parallel$ and $\mathcal{P}_A \parallel$ for any formula X .
 $N[X]$



$$\mathsf{INML} = \mathsf{IMLL} \cup \left\{ \neg \frac{\Gamma, A_1, \dots, A_n \vdash A \quad \Delta, B_1, \dots, B_n \vdash B}{\Gamma, \Delta, A_1 \triangleleft B_1, \dots, A_n \triangleleft B_n \vdash A \triangleleft B} \ n \ge 0 \right\}$$

$$\mathsf{INML} = \mathsf{IMLL} \cup \left\{ \sqrt[4]{\frac{\Gamma, A_1, \dots, A_n \vdash A \quad \Delta, B_1, \dots, B_n \vdash B}{\Gamma, \Delta, A_1 \triangleleft B_1, \dots, A_n \triangleleft B_n \vdash A \triangleleft B}} \ n \ge 0 \right\}$$

Theorem (Cut Elimination)

The cut-rule is admissible in INML.

$$\mathsf{INML} = \mathsf{IMLL} \cup \left\{ \neg \frac{\Gamma, A_1, \dots, A_n \vdash A \quad \Delta, B_1, \dots, B_n \vdash B}{\Gamma, \Delta, A_1 \triangleleft B_1, \dots, A_n \triangleleft B_n \vdash A \triangleleft B} \ n \ge 0 \right\}$$

Theorem (Cut Elimination)

The cut-rule is admissible in INML.

Theorem

INML is a conservative extension of IMLL.

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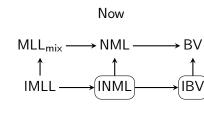
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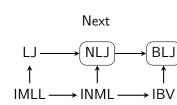
$$\mathsf{IBV} = \mathsf{INML} \cup \left\{ \begin{array}{l} \Gamma \vdash (A \triangleleft B) \triangleleft C \quad A \triangleleft (B \triangleleft C), \Delta \vdash D \\ \hline \Gamma, \Delta \vdash D \\ \mathsf{a-cut}_R \frac{\Gamma \vdash A \triangleleft (B \triangleleft C) \quad (A \triangleleft B) \triangleleft C, \Delta \vdash D}{\Gamma, \Delta \vdash D} \end{array} \right\}$$

Conclusion

Now

 $\begin{array}{c} \mathsf{MLL_{mix}} \longrightarrow \mathsf{NML} \longrightarrow \mathsf{B^{N}} \\ \uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow \\ \mathsf{IMLL} \longrightarrow \mathsf{INML} \longrightarrow \mathsf{IB} \end{array}$





Thanks

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Questions?