Analytic Proofs for Tense Logic

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"Iceland moss is not moss but lichen"

“A logic without cut elimination is like a car without an engine”

J.-Y. Girard

Our proposal:

“A logic without cut restriction is like a car without an engine”

Cut-restriction programme in a nutshell

Most logics do not have a cut-free sequent calculus

Adapt Gentzen's cut-elimination reductive algorithm to restrict the cut formulas when elimination is not possible

This is cut-restriction

Analytic cut-restriction: reductive arguments restricting to analytic cut instances

Sometimes we need to restrict more than the cut-rule (like in this talk):

Analytic restriction: reductive arguments restricting to analytic rule instances

Solution since 1950s: extend sequent calculus structural language

Extended proof systems regain cut-elimination but hard to tame structure

$Rxy, Rxz, x : A \Rightarrow y : B, y : C$

$\Gamma \Rightarrow \Delta // \mathcal{H}$

$\nabla\{\Gamma \Rightarrow A, \Delta^*\}$

$((A; B), C); D \Rightarrow E$

$\mathcal{H}|((A; B), C); D \Rightarrow E$

$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$

$A \Rightarrow B \mid C, D \Rightarrow E \mid \ \Rightarrow F$

$A, [B, [C, D], [E]], F$

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \Rightarrow A}{\Gamma_1' \Rightarrow A}}{\vdots}\quad \dfrac{\dfrac{\Gamma_2 \Rightarrow B}{\Gamma_2' \Rightarrow B}}{\vdots}}{}$$

$$\dfrac{\Gamma \Rightarrow C \qquad \Gamma \Rightarrow C}{\Gamma \Rightarrow C}$$

$\Gamma_1 \Rightarrow \Delta_1 || \ldots || \Gamma_n \Rightarrow \Delta_n$

$A, \overset{1}{[}B, \overset{1}{C]}, \overset{3}{[}D, \overset{2}{[}E\overset{2}{]}, \overset{1}{[}F\overset{13}{]]}$

Total$\geq$ 14 In last decade$\geq$ 5

Instead, we want SC with restricted cuts. Is it reasonable? Based on slide by Tim Lyon

Is there evidence that restricted cuts could work?

Yes!

Isolated results obtaining cut-restricted calculi: Fitting (1978); Takano (several papers); Avron Lahav 2013; Bezhanishvili Ghilardi 2014; Kowalski Ono 2017; Ono Sano 2022

Proof of concept over many substructural, intermediate, modal logics (Ciabattoni, Lang, RR 2019, 2021):

The infinitely many logics that have a cut-free hypersequent calculus have cut-restricted sequent calculi

From evidence to action. What next?

1. Applications of restricted calculi
   - Craig Interpolation (Bi-Int: Kowalski Ono 2017; Bi-Kt: Ono Sano 2022)
   - Complexity results (Ciabattoni Lang RR 2021)
   - Automated theorem provers (Haaksema Otten RR 2024)

2. Reductive argument to obtain restricted calculi
   - Takano's argument for S5 from 1992 seems to be the first
   - Sufficient conditions analytic cut-restriction (Ciabattoni Lang RR 2023)
   - New transformations (this talk)
   - What about other restriction classes? . . . open. . .

Proof of concept from cut-free hypersequent calculus

Sequentialize a lowermost hypersequent rule by adding cuts

$$
\dfrac{\begin{array}{c} P_1 \\ \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \end{array} \qquad \begin{array}{c} P_2 \\ \Gamma_2, \Delta_2 \Rightarrow \Pi_2 \end{array}}{\begin{array}{c} \Gamma_1, \Delta_2 \Rightarrow \Pi_1 | \Gamma_2, \Delta_1 \Rightarrow \Pi_2 \\ \eta \\ \Rightarrow F \end{array}} \,(com)
$$

cut-freeness implies $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2 \subseteq \mathrm{subf}(F)$

$$
\dfrac{\Delta_2 \Rightarrow \wedge\Delta_2 \qquad \dfrac{\begin{array}{c} P_1 \\ \Gamma_1, \Delta_1 \Rightarrow \Pi_1 \end{array}}{\Gamma_1, \wedge\Delta_1 \Rightarrow \Pi_1}}{\begin{array}{c} \wedge\Delta_2 \to \wedge\Delta_1, \Gamma_1, \Delta_2 \Rightarrow \Pi_1 \\ \eta' \\ \wedge\Delta_2 \to \wedge\Delta_1 \Rightarrow F \end{array}} \,(\to\mathrm{L})
$$

We obtained component of hypersequent at cost of cuts on known formulas

In this way, obtain sequent proofs of each component, then combine them

What about a reductive algorithm i.e., cut-restriction?

Want to investigate a proof theory that stays in the sequent calculus...

Also motivated by the wild success of cut-elimination...

Cut-elimination is a robust algorithm that is semantics-independent

CE is uniform over many logics and proof formalisms

How can we adapt the cut-elimination algorithm to restrict cuts?

restrict what you can't eliminate

Cut-elimination is the boundary case of cut-restriction...

... Analytic (cut-)restriction—cut formula is a subformula of the conclusion of the cut rule—is the next step towards generality

Some success: Takano's S5 (1992); sufficient conditions for analytic cut-restriction (Ciabattoni Lang RR 2023)... but this is not broad enough... still uncovering new transformations (as seen in this talk)

Let us recall Gentzen's reductive argument to illustrate our goal...

Gentzen's cut-elimination argument: parametric & principal reductions

Choose a topmost cut

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ X' \Rightarrow A\end{matrix}}{X \Rightarrow {\color{red}A}}\ \text{some rule } r \qquad \begin{matrix}\vdots\\ {\color{red}A}, Y \Rightarrow C\end{matrix}}{X, Y \Rightarrow C}\ \text{cut}$$

Permute the cut upwards (parametric reduction) to obtain smaller cut:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ X' \Rightarrow {\color{red}A}\end{matrix} \qquad \begin{matrix}\vdots\\ {\color{red}A}, Y \Rightarrow C\end{matrix}}{X', Y \Rightarrow C}\ \text{cut}}{X, Y \Rightarrow C}\ \text{same rule } r\text{, slightly different instance}$$

(the sum of the heights of the premises of new cut are smaller)

When the cut is principal in both premises...

$$\dfrac{\dfrac{\vdots \atop X \Rightarrow A}{X \Rightarrow {\color{red}A \vee B}}\,\vee R \qquad \dfrac{\vdots \atop A, Y \Rightarrow C \qquad \vdots \atop B, Y \Rightarrow C}{{\color{red}A \vee B}, Y \Rightarrow C}\,\vee L}{X, Y \Rightarrow C}\,\text{cut}$$

Now apply cut on smaller subformulas (principal reduction)

$$\dfrac{\vdots \atop X \Rightarrow {\color{red}A} \qquad \vdots \atop {\color{red}A}, Y \Rightarrow C}{X, Y \Rightarrow C}\,\text{cut}$$

(the cut formula in new cut has smaller size)

In this way we can repeatedly replace a topmost cut with strictly smaller cuts and finally eliminate it

This talk: reductive argument to establish restricted calculus for tense logic

Basic normal tense logic Kt

Extends K with modal operator $\blacksquare$ and necessitation rule for it

Also add the normal axiom and converse axioms

$$\blacksquare(A \to B) \to (\blacksquare A \to \blacksquare B) \qquad A \to \Box\neg\blacksquare\neg A \qquad A \to \blacksquare\neg\Box\neg A$$

Kripke semantics via frames (W, R) with

$\Box A$ as 'A holds at every point in the future'

$\blacksquare A$ as 'A holds at every point in the past'

Sequent calculus: extend LK with following rules (Nishimura 1980)

$$\frac{\Gamma \Rightarrow A, \blacksquare\Delta}{\Box\Gamma \Rightarrow \Box A, \Delta}\ (\Box) \qquad \frac{\Gamma \Rightarrow A, \Box\Delta}{\blacksquare\Gamma \Rightarrow \blacksquare A, \Delta}\ (\blacksquare)$$

'Extends K *modal rule with converse box going the other way'*

Note: cut-free calculi for Kt have been presented via various formalisms (labelled/hypersequent/display). Our focus here is a reductive argument to obtain a restricted sequent calculus

Cut-elimination does not hold (Nishimura 1980)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\neg p \Rightarrow \neg p}}{\blacksquare\neg p \Rightarrow \blacksquare\neg p}}{\Rightarrow \neg\blacksquare\neg p, \blacksquare\neg p}}{\Rightarrow \Box\neg\blacksquare\neg p, \neg p} \qquad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}}{p \Rightarrow \Box\neg\blacksquare\neg p}\ (cut)$$

Only rule instances of the following might cause non-analyticity

$$\dfrac{\Gamma \Rightarrow A, \blacksquare\Delta}{\Box\Gamma \Rightarrow \Box A, \Delta}\ (\Box) \qquad \dfrac{\Gamma \Rightarrow A, \Box\Delta}{\blacksquare\Gamma \Rightarrow \blacksquare A, \Delta}\ (\blacksquare)$$

$$\dfrac{\Gamma \Rightarrow A, \Delta \qquad \Sigma, A \Rightarrow \Pi}{\Sigma, \Gamma \Rightarrow \Delta, \Pi}\ (cut)$$

Definition (grade of non-analytic rule instance): the maximum of the grades (i.e., size) of all the non-analytic formulas in it

Our approach to analytic restriction: repeatedly eliminate topmost non-analytic rules. Whether modal or cut-rule, the crucial lemma is

Proposition

Let α be GKt*-proof whose only non-analytic rule is modal of grade $k > 0$.*
$\exists$GKt*-proof of the same sequent whose non-analytic rule instances grade $< k$.*

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$$\boxed{\beta_i}$$

$$\dfrac{\Sigma^i \Rightarrow B^i, \Box\Psi^i}{\blacksquare\Sigma^i \Rightarrow \blacksquare B^i, \Psi^i}\,(\blacksquare)^i \quad i \in I$$

$$\boxed{core(\alpha_0)}$$

$$\dfrac{\Gamma \Rightarrow A, \blacksquare\Delta, \blacksquare\Pi}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi}\,(\Box)$$

A formula in any Σ^i's or $\Box\Psi^i$ is called a critical formula.

Let Ξ be the set of all critical formulas.

$$\dfrac{\dfrac{\dfrac{\cdots}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_1, C_2} \quad \dfrac{\cdots}{C_2, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_1}}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_1}\,(cut) \quad \dfrac{\dfrac{\cdots}{C_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, C_2} \quad \dfrac{\cdots}{C_1, C_2, \Box\Gamma \Rightarrow \Box A, \Delta,}}{C_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi}}{\Box\Gamma \Rightarrow \Box A, \Delta, \Pi}\,(cut)$$

Critical formulas are $\{\Sigma^i \mid i \in I\} \cup \{\Box\Psi^i \mid i \in I\}$. Cuts are binary tree with $2^{|\Xi|}$-many leaves, each of following form for some partition $\Xi_1 \cup \Xi_2 = \Xi$:

$$\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2$$

$\Xi_1^{-\Box} := \{C \mid \Box C \in \Xi_1\}$

and $\Xi_2^* := \Xi_2 \cap (\bigcup_i \Sigma^i)$

$$\boxed{\beta_i}$$

$$\dfrac{\Sigma^i \Rightarrow B^i, \Box\Psi^i}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2}\ (w),\ (c)$$

$(\Xi_1; \Xi_2)$ aligned

$$\boxed{\text{axiom expansion}}$$

$$\Xi_1^{-\Box}, \blacksquare\Sigma^i \Rightarrow \Psi^i, \blacksquare\Xi_2^*$$

$$\boxed{core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}}$$

$$\dfrac{\dfrac{\Xi_1^{-\Box}, \Gamma \Rightarrow A, \blacksquare\Pi, \blacksquare\Xi_2^*}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Pi, \Xi_2^*}\ (\Box)}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2}\ (w)$$

$(\Xi_1; \Xi_2)$ unaligned

$$\boxed{\text{cuts on critical formulas}}$$

$$\Box\Gamma \Rightarrow \Box A, \Delta, \Pi$$

$core(\alpha_0)$ to $core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$: replace $\blacksquare\Delta$-ancestors with $\Xi_1^{-\Box}, \ldots \Rightarrow \ldots, \blacksquare\Xi_2^*$

Final step: argue that $core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$ is analytic

1) $core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$ is obtained from $core(\alpha_0)$ by replacing $\blacksquare\Delta$-ancestors

$$\ldots \Rightarrow \ldots, \blacksquare B, \ldots \quad \text{is replaced with} \quad \Xi_1^{-\Box}, \ldots \Rightarrow \ldots, \blacksquare\Xi_2^*$$

2) Here $\Xi_1^{-\Box} := \{C \mid \Box C \in \Xi_1\}$ and $\Xi_2^* := \Xi_2 \cap (\bigcup_i \Sigma^i)$.

Non-analyticity may be due to adding too much, or deleting a witness

$$\Xi_1^{-\Box}, \blacksquare\Sigma^i \Rightarrow \Psi^i, \blacksquare\Xi_2^*$$

$$\boxed{core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}}$$

$$\dfrac{\dfrac{\Xi_1^{-\Box}, \Gamma \Rightarrow A, \blacksquare\Pi, \blacksquare\Xi_2^*}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Pi, \Xi_2^*}\,(\Box)}{\Xi_1, \Box\Gamma \Rightarrow \Box A, \Delta, \Pi, \Xi_2}\,(\mathrm{w})$$

If analytic cut-rule on D in $core(\alpha_0) \rightsquigarrow$ non-analytic in $core(\alpha_0)^{\Xi_1^{-\Box}, \blacksquare\Xi_2^*}$, then in $core(\alpha_0)$: D is subformula of larger D' in $\blacksquare\Delta$ (so grade$(D) < k$)

If $(\Box)$ is non-analytic, then consider $D \in \blacksquare\Xi_2^*$. As $D \in \blacksquare\Sigma^i$, it occurs in $core(\alpha_0)$ from leaf $\blacksquare\Sigma^i \Rightarrow \blacksquare B^i, \Psi^i$. Then, D appears in root of $core(\alpha_0)$ (so outside $\blacksquare\Delta$), or D is analytic cut-formula in $core(\alpha_0)$ (argue as above)

Potential interpretation of algorithm/case split in countermodels

Consider the proof. Key: non-analytic formula critical formula

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p \vee q}}{\blacksquare p \Rightarrow \blacksquare(p \vee q)}\ (\blacksquare)}{\Rightarrow \neg\blacksquare p, \blacksquare(p \vee q)}}{\Rightarrow \square\neg\blacksquare p, p \vee q}\ (\square)$$

In countermodel construction: <u>antecedent as true</u> and <u>succedent as false</u> and attempt to build model (e.g., falsify a $\square$ formula at a future world)

Here is the result of our algorithm. It begins with cut on every critical formula:

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\blacksquare p \Rightarrow \blacksquare p}\ (\blacksquare)}{\Rightarrow \neg\blacksquare p, \blacksquare p}}{\Rightarrow \square\neg\blacksquare p, p}\ (\square) \qquad \dfrac{p \Rightarrow p}{p \Rightarrow p \vee q}}{\Rightarrow p \vee q, \square\neg\blacksquare p}\ (cut)$$

We might read this as a distinct approach when faced with having to introduce an non-analytic formulas: do analytic cuts at current world instead

Conclusion

no cuts required

only analytic rules required

mild violations of analyticity (e.g. Takano on $K45$)

$\iddots \quad \ddots$

cuts are sets of subformulas

cuts are multisets of subformulas

$\vdots$

arbitrary cuts required

What next?

- Extending the reductive algorithm to larger families of restrictions
- Using restricted proof systems for proving logical properties
- Complexity issues (size of cut-restricted proof wrt extended cut-free proof)? Computational interpretations of cut-restriction?
- Modular automated theorem provers using restricted cuts, and heuristics
- Rocq/Isabelle formalisation of analytic restriction

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