Improved decision procedures for multi-modal tense logic using CEGAR-tableaux or Rational reconstruction of CEGAR-tableaux from a tableau perspective

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Modal Tableaux: CPL-tableaux with extra (K) rule

Negation normal form: rewrite implications $A \to B$ as $\neg A \vee B$ and push negations down to atoms using de Morgan dualities

Old way: cpl-tableaux plus (K) rule for modalities in nnf

$$(\mathrm{Id})\ \frac{p; \neg p; X}{} \quad (\wedge)\ \frac{A \wedge B; X}{A; B; X} \qquad (\vee)\ \frac{A \vee B; X}{A; X \qquad B; X} \qquad (\mathrm{K})\ \frac{\Diamond A; \Box X; \Diamond Y; L}{A; X}$$

where X, Y are finite sets, and comma is set partition and L is a set of atoms and negated atoms and (K) chooses A non-deterministically

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(\Box(p \to q) \to (\Box p \to \Box q))}{\Box(\neg p \vee q) \wedge \Box p \wedge \Diamond \neg q}\,\text{nnf}}{\Box(\neg p \vee q)\ ;\ \Box p\ ;\ \Diamond \neg q}\,(\wedge)x2}{\neg p \vee q\ ;\ p\ ;\ \neg q}\,(K)}{\dfrac{\neg p\ ;\ p\ ;\ \neg q}{}\,(id) \qquad \dfrac{q\ ;\ p\ ;\ \neg q}{}\,(id)}\,(\vee)$$

Closed tableau for negation of K-axiom $\Box(p \to q) \to (\Box p \to \Box q)$

Using a CPL Oracle via a Restart Rule

cpl: as an oracle sets up OR-branching but needs backtracking

$$(\mathrm{cpl})\ \dfrac{Z_0}{\dfrac{L_1; \Box X_1; \Diamond A_1; \cdots ; \Diamond A_m}{\begin{array}{c} A_1; X_1 \\ \times \end{array}}\,(\mathrm{K}) \qquad L_2; \Box X_2; \Diamond Y_2 \qquad \cdots}$$

Intuitions: don't make the same mistake twice!

Restart Z_0 by learning negation of conflict set $\varphi := \neg(\Box X_1; \Diamond A_1)$

$$(\mathrm{cpl})\ \dfrac{Z_0}{\dfrac{L_1; \Box X_1; \Diamond A_1; \cdots ; \Diamond A_m}{\begin{array}{c} A_1; X_1 \\ \times \end{array}}\,(\mathrm{K}) \qquad Z_0; \varphi}$$

Too naive: $\neg\Box B_1 \vee \cdots \vee \neg\Box B_n \vee \neg\Diamond A_1$ creates work!

Relevance: which members of conflict set are necessary for clash?

Modal KE-Tableaux Using Negation Normal Form

$$\text{(Id)}\ \frac{p; \neg p; X}{\times} \qquad (\wedge)\ \frac{A \wedge B; X}{A; B; X} \qquad \text{(KE)}\ \frac{A; \bar{A} \vee B; X}{A; B; X} \qquad \text{(K)}\ \frac{\Diamond A; \Box X; \Diamond Y; L}{A; X}$$

$$\text{(PB)}\ \frac{X}{A; X \qquad \bar{A}; X} \qquad \frac{A \vee B; X}{A; X \qquad \dfrac{\bar{A}; A \vee B; X}{\bar{A}; B; X}\ \text{(KE)}}\ \text{(PB)}$$

D'Agostino and Mondadori 1994: KE-tableaux plus (K) rule in nnf

Explicit Cut rule: (PB) rule necessary for completeness

(KE): is (ineffective) modus ponens from A and $A \to B$ infer B

$(\vee)$ rule: with mutually exclusive OR-branches is derivable

Example: Closed KE-tableau for negation of K-axiom

$$\cfrac{\neg(\Box(p \to q) \to (\Box p \to \Box q))}{\cfrac{\Box(\neg p \vee q) \wedge \Box p \wedge \Diamond\neg q}{\cfrac{\Box(\neg p \vee q)\ ;\ \Box p\ ;\ \Diamond\neg q}{\cfrac{\neg p \vee q\ ;\ p\ ;\ \neg q}{\cfrac{q\ ;\ p\ ;\ \neg q}{\times}\ (id)}\ (KE)}\ (K)}\ (\wedge)x2}\ \text{nnf}$$

Note: atomic KE rule is effective!

Modal Clausal Normal Form

MCNF: convert given $\neg\varphi_0$ into modal clausal normal form via fresh atomic formulae (names) for subformulae of $\neg\varphi_0$ (Mints)

cpl-clauses: $l_1 \wedge \cdots \wedge l_n \rightarrow r_1 \vee \cdots \vee r_m$ $\quad (\mathcal{C}^{cpl})$

box-clauses: $a \rightarrow \Box b$ $\quad (\mathcal{C}^{box})$

dia-clauses: $c \rightarrow \Diamond d$ $\quad (\mathcal{C}^{dia})$

Lemma: A formula of modal depth κ can be put into a modal clausal normal form below which is equi-satisfiable

$$mcnf(nnf(\neg\varphi_0)) = \Box^0\mathcal{C}_0 \; ; \; \Box^1\mathcal{C}_1 \; ; \; \cdots \; ; \; \Box^\kappa\mathcal{C}_\kappa$$

where every $\mathcal{C}_i$ is a set of modal clauses (recursive)

Example: consider the negation of the K axiom

$$\text{(mcnf)}\ \frac{\text{(nnf)}\ \dfrac{\neg(\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q))}{\Box(\neg p \vee q) \; ; \; \Box p \; ; \; \Diamond\neg q}}{a_1; a_2; c_1; a_1 \rightarrow \Box b_1; a_2 \rightarrow \Box p; c_1 \rightarrow \Diamond\neg q; \Box(b_1 \rightarrow (\neg p \vee q))}$$

Modal KE-tableaux with clause learning in MCNF?

Example: what we can learn from a jump that closes?

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{C}^{cpl} \; ; \; \mathcal{C}^{box} \; ; \; \mathcal{C}^{dia} \; ; \; \Box\mathcal{MC}}{a_1; a_2; c_1 \; ; \; a_1 \to \Box b_1; a_2 \to \Box p \; ; \; c_1 \to \Diamond\neg q \; ; \; \Box(b_1 \to \neg p \vee q)}}{a_1; a_2; c_1; \Box b_1; \Box p; \Diamond\neg q; \Box(b_1 \to \neg p \vee q)}\,(\text{KE})\text{x}3}{b_1; p; \neg q; b_1 \to (\neg p \vee q)}\,(\text{K})}{\dfrac{b_1; p; \neg q; q}{\times}\,(\text{id})}\,(KE)x2$$

Fact: each of $\{b_1, p, \neg q\}$ is essential for clash

Fact: so $(\Box b_1; \Box p; \Diamond\neg q)$ is K-unsatisfiable

Could learn $\neg(\Box b_1; \Box p; \Diamond\neg q) = (\neg\Box b_1 \vee \neg\Box p \vee \neg\Diamond\neg q)$

But: the culprits are actually their names $(a_1; a_2; c_1)$

Instead Learn $\varphi := (\neg a_1 \vee \neg a_2 \vee \neg c_1)$ and restart

$$\dfrac{\dfrac{\mathcal{C}^{cpl} \; ; \; \mathcal{C}^{box} \; ; \; \mathcal{C}^{dia} \; ; \; \Box\mathcal{MC}}{a_1; a_2; c_1; \neg a_1 \vee \neg a_2 \vee \neg c_1 \; ; \; \mathcal{C}^{box} \; ; \; \mathcal{C}^{dia} \; ; \; \Box\mathcal{MC}}\,(\text{restart } (\mathcal{C}^{cpl}; \varphi))}{\dfrac{a_1; a_2; c_1; \neg c_1}{\times}\,(\text{id})}\,(\text{KE})\text{x}3$$

CEGAR-tableaux Rules (Downwards)

Consider: $\mathcal{C}^{cpl}$; $\mathcal{C}^{box}$; $\mathcal{C}^{dia}$; $\Box\mathcal{MC}$

cpl-clauses: $l_1 \wedge \cdots \wedge l_n \to r_1 \vee \cdots \vee r_m$, $\cdots$ $\quad(\mathcal{C}^{cpl})$

box-clauses: $a \to \Box b$, $\cdots$ $\quad(\mathcal{C}^{box})$

dia-clauses: $c \to \Diamond d$, $\cdots$ $\quad(\mathcal{C}^{dia})$

modal context: $\Box\mathcal{C}_1; \Box^2\mathcal{C}_2; \cdots ; \Box^\kappa\mathcal{C}_\kappa$ $\quad(\Box\mathcal{MC})$

cpl: (id)-like rule

$$(\text{cpl})\ \frac{\mathcal{C}^{cpl}\ ;\ \mathcal{C}^{box}\ ;\ \mathcal{C}^{dia}\ ;\ \Box\mathcal{MC}}{\times}\ cpl(\mathcal{C}^{cpl}) = unsat$$

jump: (KE+K) rule which builds in multiple modus ponens steps

$$(\text{jump})\ \frac{\vartheta\ ;\ \mathcal{C}^{dia}\ ;\ \mathcal{C}^{box}\ ;\ \Box\mathcal{MC}}{d\ ;\ B\ ;\ \mathcal{MC}}\ cpl(\mathcal{C}^{cpl}) = \vartheta$$

where $(c \to \Diamond d) \in \mathcal{C}^{dia}$ & $c \in \vartheta$ via (KE)

and $B := \{b \mid (a \to \Box b) \in \mathcal{C}^{box}$ & $a \in \vartheta\}$ via (KE)+(K)

CEGAR-tableaux Rules (both ways)

SAT-solver: a cpl oracle which either returns a cpl-valuation ϑ or-else returns a "minimal unsatisfiable core" UC

(id): rule to close a branch

$$(\mathrm{id})\ \frac{\mathcal{C}^{cpl}\ ;\ \mathcal{C}^{box}\ ;\ \mathcal{C}^{dia}\ ;\ \Box\mathcal{MC}}{\times}\ sat(\mathcal{C}^{cpl}) = (unsat, UC)$$

(jump/restart): (KE+K) rule which builds in multiple modus ponens steps and relevant clause-learning and restart

$$\frac{w_0 := \mathcal{C}^{cpl}\ ;\ \mathcal{C}^{dia}\ ;\ \mathcal{C}^{box}\ ;\ \Box\mathcal{MC}}{d\ ;\ B\ ;\ \mathcal{MC} \qquad w_0; \varphi}\ sat(\mathcal{C}^{cpl}) = (sat, \vartheta)$$

(jump): exists $(c \to \Diamond d) \in \mathcal{C}^{dia}$ & $c \in \vartheta$ and $B := \{b \mid (a \to \Box b) \in \mathcal{C}^{box}$ & $a \in \vartheta\}$

(restart): if $d; B; \mathcal{MC}$ returns $(unsat, UC)$ then

conflict set: $CS(w_0) = \{c\}; \{a \mid (a \to \Box b) \in \mathcal{C}^{box}$ & $b \in UC\}$

clause learning: restart with $\varphi := \neg CS(w_0) = (\neg l_1 \vee \cdots \vee \neg l_n)$

Three valued logic: a "false" is not the same as $\Box b$ "false"

CEGAR-tableaux Rules (both ways) Example

Example: consider negation of the K axiom

$$(\text{jump/restart})\ \dfrac{\mathcal{C}^{cpl}\ ;\ \mathcal{C}^{box}\ ;\ \mathcal{C}^{dia}\ ;\ \Box\mathcal{MC} \atop a_1; a_2; c_1; a_1 \to \Box b_1; a_2 \to \Box p; c_1 \to \Diamond\neg q; \Box(b_1 \to \neg p \vee q)}{(\text{id})\ \dfrac{\neg q; b_1; p; \atop b_1 \to \neg p \vee q}{(unsat, \{\neg q, b_1, p\})} \qquad \dfrac{(\neg a_1 \vee \neg a_2 \vee \neg c_1); a_1; a_2; c_1; \atop \mathcal{C}^{box}\ ;\ \mathcal{C}^{dia}\ ;\ \Box\mathcal{MC}}{(unsat, \{a_1, a_2, c_1\})}\ (\text{id})}$$

SAT-solver: $sat(\mathcal{C}^{cpl}) = (sat, \vartheta)$ where $\vartheta = \{a_1, a_2, c_1\}$

pre-jump: exists $(c_1 \to \Diamond\neg q) \in \mathcal{C}^{dia}$ such that $c_1 \in \vartheta$

pre-jump: let $B = \{b \mid (a \to \Box b) \in \mathcal{C}^{box}\ \&\ a \in \vartheta\} = \{b_1, p\}$

jump: to node $(d; B; \mathcal{MC}) = (\neg q\ ;\ b_1\ ;\ p\ ;\ b_1 \to \neg p \vee q)$

SAT-solver: $sat(\mathcal{C}^{cpl}) = (unsat, UC)$ where $UC = \{\neg q, b_1, p\}$

Conflict set: $CS(w_0) = \{a_1, a_2, c_1\}$

Restart: with $\neg CS(w_0) = (\neg a_1 \vee \neg a_2 \vee \neg c_1)$

CEGAR-tableaux = Modal KE-Tableaux in MCNF

Example: consider negation of the K axiom

$$\mathcal{C}^{cpl} \setminus \mathcal{A} \;;\; \mathcal{C}^{box} \;;\; \mathcal{C}^{dia} \;;\; \Box\mathcal{MC}$$

$$\text{(PB)}\ \dfrac{a_1 \to \Box b_1; a_2 \to \Box p; c_1 \to \Diamond\neg q; \Box(b_1 \to \neg p \vee q)}{\text{(KE+K)}\ \dfrac{a_1; a_2; c_1}{\text{(KE+id)}\ \dfrac{\begin{array}{c}\neg q; b_1; p;\\ b_1 \to \neg p \vee q\end{array}}{\times}} \qquad \dfrac{\begin{array}{c}(\neg a_1 \vee \neg a_2 \vee \neg c_1); a_1; a_2; c_1;\\ \mathcal{C}^{box} \;;\; \mathcal{C}^{dia} \;;\; \Box\mathcal{MC}\end{array}}{\times}\ \text{(KE+id)}}$$

SAT-solver with assumptions: $sat(\mathcal{A}, \mathcal{C}^{cpl} \setminus \mathcal{A}) = (sat, \vartheta \supseteq \mathcal{A})$
correctly guesses the conflict set $CS(w_0) = \{a_1, a_2, c_1\} \subseteq \vartheta$

Restart: with $\neg CS(w_0) = (\neg a_1 \vee \neg a_2 \vee \neg c_1)$ is just (PB)

Conclusion: CEGAR-tableaux are modal KE-tableaux!

Residuation in Classical Propositional Tense Logic

Syntax: given some set *Atm* of atomic formulae

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A \mid \blacksquare A \mid \blacklozenge A$$

Reading of $\Box A$: "always in the future A"

Reading of $\blacksquare A$: "always in the past A"

Reading of $\Diamond A := \neg\Box\neg A$: "sometime in the future A"

Reading of $\blacklozenge A := \neg\blacksquare\neg A$: "sometime in the past A"

$\blacklozenge\Box p \to p$: if sometime in the past always in the future p then p

$$\begin{tikzcd} & p & & p \arrow[dr] & \\ \Box p \arrow[ur] \arrow[r] \arrow[dr] & \blacklozenge\Box p, p & & \Diamond\blacksquare p, p \arrow[r] & \blacksquare p \\ & p & & p \arrow[ur] & \end{tikzcd}$$

$\Diamond\blacksquare p \to p$: if sometime in the future always in the past p then p

Extensions For Handling Converse

Problem: traditional modal tableaux cannot go "back" in time

$$(K)\ \frac{\Diamond A; \Box X; \Diamond Y; L}{A; X} \qquad \Diamond A; \Box X; \Diamond Y; L \longrightarrow A; X; \blacksquare B$$

Various Solutions: labels, display calculi, (linear) nesting, hypersequents, restricted cut rule, restart rule, ...

Restart Rule: intuitions only (i.e read paper)

$$(\text{restart})\ \frac{\Diamond A; \Box X; \Diamond Y; L \longrightarrow A; X; \blacksquare B}{\Diamond A; \Box X; \Diamond Y; L; B}$$

Example: residuation axiom $\neg(\Diamond\blacksquare p \to p) = (\Diamond\blacksquare p; \neg p)$

$$\begin{array}{rc} (K) & \dfrac{\Diamond\blacksquare p; \neg p}{\Diamond\blacksquare p; \neg p \longrightarrow \blacksquare p} \\ (\text{restart}) & \dfrac{}{\Diamond\blacksquare p; \neg p; p} \\ (\text{id}) & \dfrac{}{\times} \end{array}$$

Concrete Paper: Rajeev Goré, Björn Lellmann: Syntactic Cut-Elimination and Backward Proof-Search for Tense Logic via Linear Nested Sequents. TABLEAUX 2019: 185-202

Residuation Again But Locally

Residuation $\Diamond\blacksquare A \to A$ and $\blacklozenge\Box A \to A$

Claim: $\Box(A \to \blacksquare B) \to (\Diamond A \to B)$ and $\blacksquare(A \to \Box B) \to (\blacklozenge A \to B)$

$$\text{(id)}\;\dfrac{\text{(restart)}\;\dfrac{\text{(MP)}\;\dfrac{\text{(K)}\;\dfrac{\text{(not-inwards)}\;\dfrac{\neg(\Box(A \to \blacksquare B) \to (\Diamond A \to B))}{\Box(A \to \blacksquare B); \Diamond A; \neg B}}{\Box(A \to \blacksquare B); \Diamond A; \neg B \longrightarrow A; A \to \blacksquare B}}{\Box(A \to \blacksquare B); \Diamond A; \neg B \longrightarrow A; A \to \blacksquare B; \blacksquare B}}{\Box(A \to \blacksquare B); \Diamond A; \neg B; B}}{\times}$$

Preprocessing rules:

$$\dfrac{X; \Box(A \to \blacksquare B)}{X; \Box(A \to \blacksquare B); \Diamond A \to B} \qquad \dfrac{X; \blacksquare(A \to \Box B)}{X; \blacksquare(A \to \Box B); \blacklozenge A \to B}$$

Problem: how to "see" $\Box(A \to \blacksquare B)$ "here" in these rules

Modal Clausal Normal Form For Tense Logic

Lemma: A formula of modal depth κ can be put into a modal clausal normal form below which is equi-satisfiable

$$\square^0 \mathcal{C}_0 \ ; \ \square^1 \mathcal{C}_1 \ ; \ \blacksquare^1 \mathcal{D}_1 \ ; \ \square^1 \blacksquare^1 \mathcal{E}_1 \ ; \ \blacksquare^1 \square^1 \mathcal{F}_1 \ ; \ \cdots \ ; \ \square^\kappa \mathcal{C}_\kappa \ ; \ \blacksquare^\kappa \mathcal{D}_\kappa$$

where every $\mathcal{C}_i, \mathcal{D}_i, \mathcal{E}_i, \mathcal{F}_i$ is a set of modal clauses

Problem: how to "see" $\square(a \to \blacksquare b)$ "here"

$$\frac{X; \square(a \to \blacksquare b)}{X; \square(a \to \blacksquare b); \lozenge a \to b} \qquad \frac{X; \blacksquare(a \to \square b)}{X; \blacksquare(a \to \square b); \blacklozenge a \to b}$$

Example: consider the negation of one of the residuation axioms

$$\dfrac{\dfrac{(\text{nnf})\ \dfrac{\neg(\blacklozenge \square p \to p)}{\blacklozenge \square p; \neg p}}{c_1; \neg p; c_1 \to \blacklozenge d_1; \blacksquare(d_1 \to \square p)}\ (\text{mcnf})}{w_0 := c_1; \neg p; c_1 \to \blacklozenge d_1; \blacksquare(d_1 \to \square p); \neg p \to \blacksquare \neg d_1}\ (\text{residuation})$$

$$(\text{jump/restart})\ \dfrac{w_0 := c_1; \neg p; c_1 \to \blacklozenge d_1; \blacksquare(d_1 \to \square p); \neg p \to \blacksquare \neg d_1}{\dfrac{d_1; \neg d_1; d_1 \to \square p}{(unsat, d_1; \neg d_1)} \qquad \dfrac{w_0; \neg c_1 \vee p}{(unsat, c_1; \neg p)}}$$

Results for Tense Logic

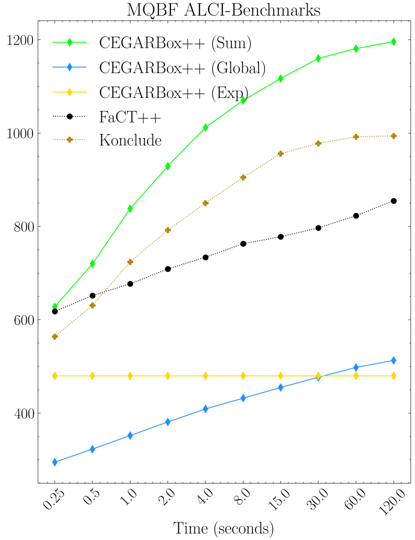
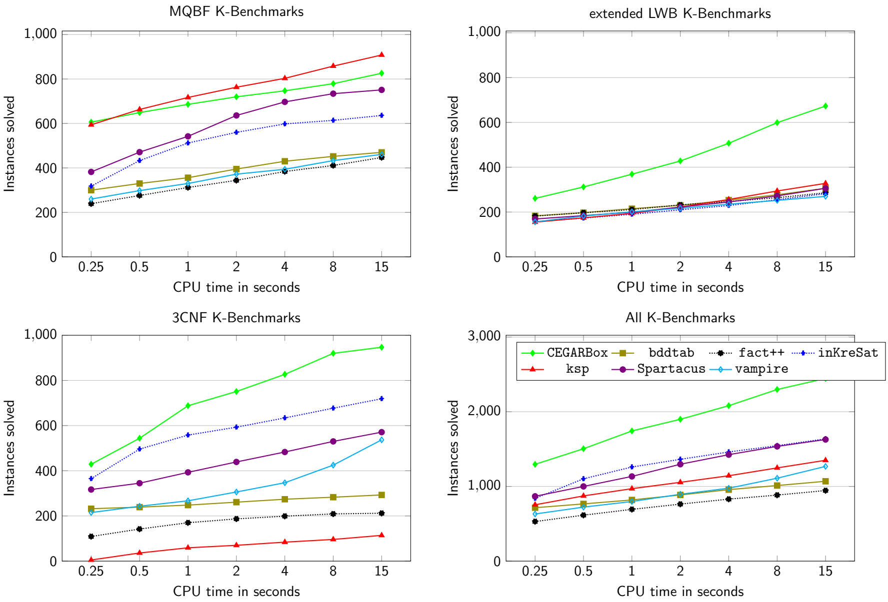
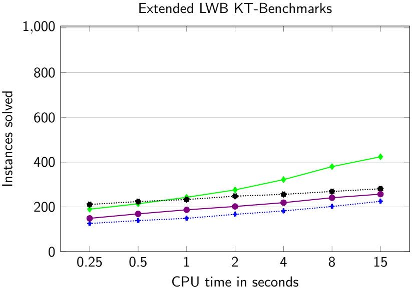


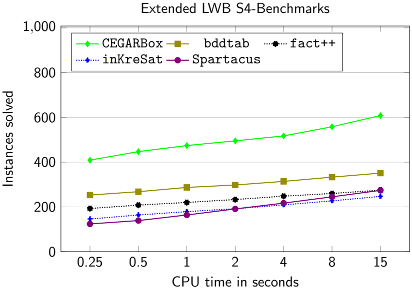
Figure: `CEGARBox++` versus `fact++` and `Konklude`

Experimental Results for K



KT (reflexive) and S4 (reflexive and transitive)





Rajeev Goré, Cormac Kikkert: CEGAR-Tableaux: Improved Modal Satisfiability via Modal Clause-Learning and SAT. TABLEAUX 2021: 74-91

Results for five basic extensions

Five Modal Extension Results for 250s							
Logic	Status	`ksp` `Local`	GMR	`CEGARBox++` `Local`	`CEGARBox++` `Global`	`CEGARBox++` `Bespoke`	`CEGARBox++` `Single`
K	sat	**100**	85	**100**	NA	NA	**100**
K	unsat	78	76	**90**	NA	NA	**90**
KD	sat	**100**	85	**100**	NA	NA	**100**
KD	unsat	75	76	**90**	NA	NA	**90**
KT	sat	69	79	97	NA	**100**	95
KT	unsat	75	77	**90**	NA	**90**	**90**
KB	sat	64	57	**100**	NA	**100**	97
KB	unsat	52	78	87	NA	89	**90**
K4	sat	58	85	78	82	84	**98**
K4	unsat	32	51	45	**69**	**69**	**69**
K5	sat	37	60	90	90	92	**97**
K5	unsat	32	45	60	42	49	**62**
NA means "prover does not exist"							

Table: Number of problems solved (out of 100 for each category) for five extensions of K using benchmarks from Nalon et al 2022.

Some version of `CEGARBox++` is best

What about the modal cube?

Simple Extensions: of K by five basic axioms of D, T, 4, 5, B

Modal Cube: 15 different logics obtained by mixing and matching

Nalon et al: give reductions of all 15 logics into K

Cláudia Nalon, Ullrich Hustadt, Fabio Papacchini, Clare Dixon: Buy One Get 14 Free: Evaluating Local Reductions for Modal Logic. CADE 2023: 382-400

Thm: for every formula φ, for every logic L in the modal cube, φ is L-satisfiable iff $\tau(\varphi)$ is K-satisfiable (Nalon et al)

Nalon et al: "the combination of the definitional reduction with CEGARBox is currently the best performing approach on our collection of benchmark formulae for the modal cube"

Questions?

Original Haskell Version: by Cormac Kikkert

`https://github.com/cormackikkert/CEGARBox-1`

New C++ Version: by Robert McArthur and Cormack Kikkert

`https://github.com/cormackikkert/CEBARBoxCPP`