Generating Higher Identity Proofs in Homotopy Type Theory

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Dealing with identity types in Rocq

Example: transitivity lemma

$$x \overset{p}{=\!=} y \overset{q}{=\!=} z \quad \longrightarrow \quad x \overset{\mathrm{trans}(p,q)}{=\!=\!=\!=} z$$

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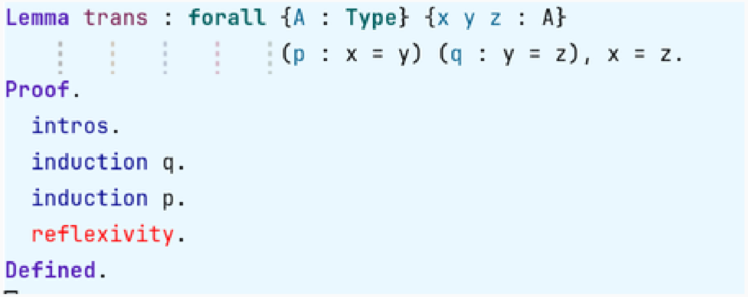
$$\begin{tikzcd} x \arrow[r, equal, "p"] & y \arrow[r, equal, "q"] & z \arrow[r] & x \arrow[r, equal, "\mathrm{trans}(p,q)"] & z \end{tikzcd}$$



```
Lemma trans : forall {A : Type} {x y z : A}
                     (p : x = y) (q : y = z), x = z.
Proof.
  intros.
  induction q.
  induction p.
  reflexivity.
Defined.
```

A direct definition of the transitivity proof term

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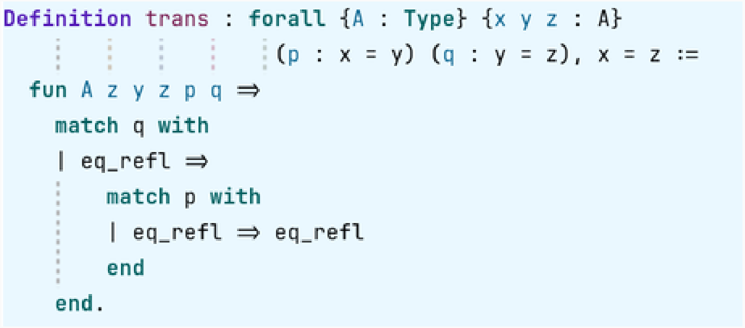

```
Definition trans : forall {A : Type} {x y z : A}
                          (p : x = y) (q : y = z), x = z :=
  fun A z y z p q ⇒
    match q with
    | eq_refl ⇒
        match p with
        | eq_refl ⇒ eq_refl
        end
    end.
```

A more involved example: “whiskering”

$$\begin{tikzcd}
x \arrow[r, bend left, equal, "p"] \arrow[r, bend right, equal, "q"'] \arrow[r, phantom, "\| a"] & y \arrow[r, equal, "r"] & z \arrow[r] & x \arrow[r, bend left, equal, "\mathrm{trans}(p{,}r)"] \arrow[r, bend right, equal, "\mathrm{trans}(q{,}r)"'] \arrow[r, phantom, "\|"] & z
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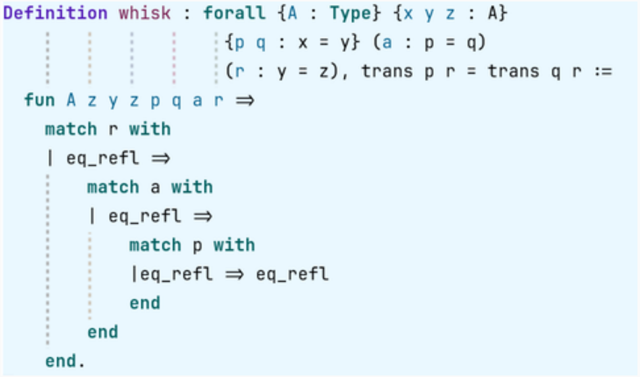

```
Definition whisk : forall {A : Type} {x y z : A}
                          {p q : x = y} (a : p = q)
                          (r : y = z), trans p r = trans q r :=
  fun A z y z p q a r ⇒
    match r with
    | eq_refl ⇒
        match a with
        | eq_refl ⇒
            match p with
            |eq_refl ⇒ eq_refl
            end
        end
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- Yet, they are not trivial
  See the Eckmann-Hilton example at the end of the talk

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- Yet, they are not trivial
  See the Eckmann-Hilton example at the end of the talk
- Aim of the talk: streamline as much of these arguments as possible

Identity types, weak ω-groupoids and Catt

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- Weak ω-groupoids are the algebraic structure that describe identity types.
  They describe exaclty the arguments we want to streamline
- They are a flavour of higher-dimensional categories.
- In this talk, we present the language Catt, a DSL to work with weak ω-categories.
  We will not introduce formally the theory of weak ω-groupoids
- Weak ω-groupoids are a particular case of weak ω-categories.
  We use a language for the latter, for historical reasons, but we will ignore the difference in this talk

Pasting diagrams

Pasting diagrams are equality schemes that can be completely pattern-matched away against `eq_refl`[1]

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$$\begin{tikzcd} x \arrow[r, equal, bend left, "p"{name=P}] \arrow[r, equal, bend right, "q"'{name=Q}] \arrow[from=P, to=Q, phantom, "\| a"] & y \arrow[r, equal, "r"] & z \end{tikzcd} \quad \rightsquigarrow \quad (x(p(a)q)y(r)z)$$

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$$\begin{tikzcd} x \arrow[r, bend left, equal, "p"] \arrow[r, bend right, equal, "q"'] & y \end{tikzcd}$$

Pattern-matching q onto `eq_refl` creates loop

The language Catt[2]

- types : $\star$, and $a \sim b$, where a, b are terms of the same type
  Intuition:$a \sim b$ are abstract equalities

[2] terms and conditions: groupoids vs categories

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- types : $\star$, and $a \sim b$, where a, b are terms of the same type
  Intuition:$a \sim b$ are abstract equalities
- terms : generated by variables and $\mathrm{coh}(\Gamma, A)$ where Γ is a pasting diagram and A is a type in Γ
  Intuition: In a pasting diagram, every pair of terms of the same type are equal, by pattern-matching the entire diagram away

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  Intuition: In a pasting diagram, every pair of terms of the same type are equal, by pattern-matching the entire diagram away
- In the implemented language, additionnal conditions are put on the type A in a coherence, to model categories.
  I hope to implement the version presented here soon

[2]terms and conditions: groupoids vs categories

Live demo

Generating identity proof from Catt terms

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Principle

- Use terms in Catt to generate identity proofs in Rocq
- Boilerplate inductive mechanism for the structure of the theory
- Base case: $\mathsf{coh}(\Gamma, A)$ is handled by pattern-matching away all equalities described by Γ and the proof is then `eq_refl`.
- Implicit arguments are managed automatically

Live demo

Building complex proofs

Proof automation in Catt

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  Suspension, opposites, functoriality, naturality...

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  Contrarily to Rocq, which is wide and general purpose
- We can algorithmically manipulate Catt as a language to generate proof-terms. Several meta-operations are already implemented
  Suspension, opposites, functoriality, naturality...
- Our export to Rocq allows to leverage these to build proofs on the structure of identity

The Eckmann-Hilton argument

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  It is tightly connected to homotopy theory, and in Rocq, is proven by purely structural manipulation on identity types

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- The Eckmann-Hilton argument is an important argument in topology
  It is tightly connected to homotopy theory, and in Rocq, is proven by purely structural manipulation on identity types
- It also provides a refutation axiom K
  It allows one to construct an equality between a term and itself, which in some models is not the reflexivity

The Eckmann-Hilton argument: live demo

Thank you!