On Solving String Equations via

Powers and Parikh Images

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Super Compact Summary

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We have a new String Solver!

What this Talk is About

- Background
  - What is string equation solving?
  - Nielsen transformation

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  - ground power introduction
  - generalised Parikh images
    - multi-sequence (MS) Parikh
    - multi-sequence error bounded (MSEB) Parikh
  - lazy equality decomposition
  - optimised Nielsen rules
  - integer reasoning

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For mathematicians

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For computer scientists

Finding values for string variables s.t. equations evaluate true

- String term: finite concatenation of string variables and characters
- String variable: represent finite character sequences
- Empty string: neutral element denoted by ε (aka "").
- Important reasoning problem in software/protocol verification

Example String Equation

Example

Problem:

Formally: $axaz = yby$

Java: `"a" + x + "a" + z == y + "b" + y`

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Solution:

Formally: $\mathscr{I} = \{\, x/b,\ \ y/a,\ \ z/\varepsilon \,\}$

Java: `x = "b", y = "a", z = ""`

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Formally: $\mathscr{I} = \{\, x/b,\ \ y/a,\ \ z/\varepsilon \,\}$

Java: `x = "b", y = "a", z = ""`

Check:

Formally: $aba = aba$

Java: `"a" + "b" + "a" + "" == "a" + "b" + "a"`

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String Tokens:

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- Still: SMT solver incomplete approaches (theory interplay)
  - Automatons (z3-noodler, Ostrich, Sloth, TRAU, ...)
  - SAT (Woorpje, nfa2sat)
  - Unwinding/model repair (z3, cvc5, S3, z3Str3, ...)
  - Nielsen transformation (more or less in all solvers)
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  - ...
- Termination only on "common-fragments"

  quadratic, 2-variable, straight-line, chain-free, ...

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Originally used for free-*groups* (1921)

High-level idea

For $tu = sv$ with token t and s

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Align a and y – *nothing else matters* (for now...)!

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Represented as "(destructive) unsigned tableau calculus"

Nielsen Rules

$$\frac{au = bv}{\bot} \qquad \frac{au = \varepsilon}{\bot} \qquad \frac{wu = wv}{\mathrm{u} = \mathrm{v}} \qquad \frac{xu = \varepsilon}{(xu = \varepsilon)[x/\varepsilon]}$$

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$$\frac{xu = av}{(xu = av)[x/\varepsilon] \qquad (xu = av)[x/ax']}$$

$$\frac{xu = yv}{(xu = yv)[x/y] \qquad (xu = yv)[x/yx'] \qquad (xu = yv)[y/xy']}$$

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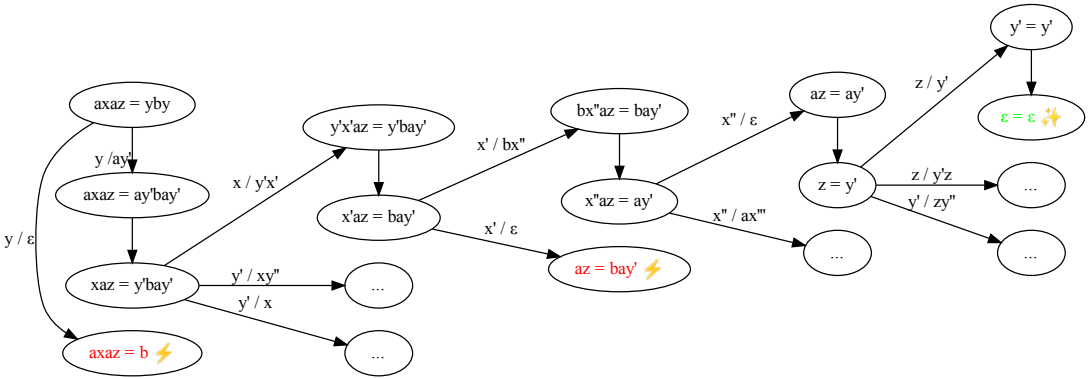
Nielsen Transformation as Graph



Thank you for attention?

Nielsen Transformation – Properties

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- semi-complete
  - If there is a model, we will find it
- not terminating in general

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Nielsen Transformation can Diverge

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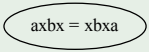

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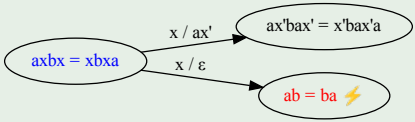

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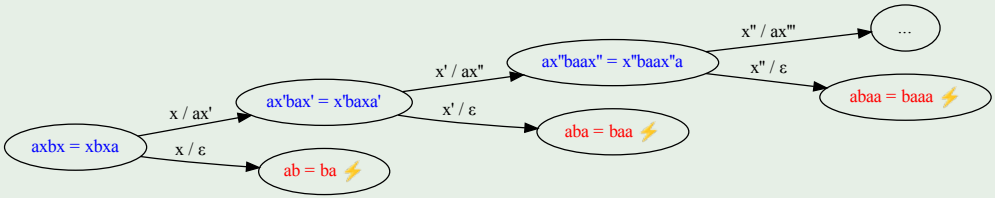

. . . but some Cases can be Fixed

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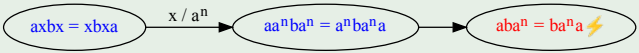

Ground Power Introduction

(Ground) Powers in Detail

We define u^n as

$$u^0 := \varepsilon \qquad u^{n+1} := uu^n$$

Example $(((ab)^2c)^3)$

Represents: *ababc ababc ababc*

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$$wxu = xv \models \exists n \in \mathbb{N} : \exists w' \in prefix(w) : x = w^n w'$$

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- w var-free $\rightsquigarrow$ $prefix(w)$ finite, var-free, and computable $\rightsquigarrow$ finite splits

Examples Power Introduction (1/3)

Example ($abx = xba$)

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Examples Power Introduction (2/3)

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Ground power introduction not applicable ¯_(ツ)_/¯

Examples Power Introduction (3/3)

Example $((ab)^n cx = xbac)$

Examples Power Introduction (3/3)

Example $((ab)^n cx = xbac)$

Three cases

- $x = ((ab)^n c)^m (ab)^l \wedge 0 \leq l < m$
- $x = ((ab)^n c)^m (ab)^l a \wedge 0 \leq l < m$
- $x = ((ab)^n c)^m (ab)^n$

Some Power Rules

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$$\frac{xu = wxv,\ w \text{ var-free}}{(xu = wxv)[x/w^n \text{prefix}(w)]} \qquad \frac{xu = w^n v}{(xu = w^n v)[x/w^n x'] \quad (xu = w^n v)[x/\text{prefix}(w^n)]}$$

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Combination with integer reasoning

One Second to Recap

- String equation solving: Aligning tokens
- Algorithmically/Calculus: Nielsen transformation
  - Incomplete for UNSAT
  - Extended by now with explicit powers
    - Performance improvement
    - More termination

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Length and Parikh Abstraction

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Clearly,

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Example ($xayy = ybxx$)

$$xayy = ybxx \rightsquigarrow |xayy| = |yaxx| \rightsquigarrow |x| = |y|$$

Clearly,

$$u = v \;\models\; |u| = |v|$$

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Thus, $x = y \;\rightsquigarrow\; xaxx = xbxx \;\rightsquigarrow\; a = b$ ↯

Parikh Image

Length abstraction often too coarse:

Example ($xay = ybx$)

Length abstraction gives $0 = 0$ ¯_(ツ)_/¯

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Parikh Image $\alpha_a(u)$

Number of a in u.

$$u = v \models \alpha_a(u) = \alpha_a(v)$$

$$\alpha_a(uv) \rightsquigarrow \alpha_a(u) + \alpha_a(v)$$

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Example ($xay = ybx$ – ctd.)

$$\alpha_a(xay) = \alpha_a(ybx)$$

$$\alpha_a(x) + 1 + \alpha_a(y) = \alpha_a(y) + \alpha_a(x) \lightning$$

Generalizing Parikh Image

Example ($xabcy = ybacx$)

unsatisfiable . . . but presented approaches will diverge

Parikh abstraction: powerful, but often still too weak.

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- $xabcy = ybacx \rightsquigarrow$ multi-sequence (MS) Parikh
- $xaxaaabbby = xxababababy \rightsquigarrow$ multi-sequence error bounded (MSEB) Parikh

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Multi-Sequence (MS) Parikh

MS Parikh – $xabcy = ybacx$

We realize that the number of bc are different on both sides, so

$$\alpha_{bc}(xabcy) = \alpha_{bc}(x) + 1 + \alpha_{bc}(y)$$

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Question

Why not search for *ab*?

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- ab might be crossing in yb!
- bc was definitely not crossing

So Restriction to Definitely Non-Crossing?

No – we lose interesting cases!

Example ($xxabcyy = yybacxx$)

Any u with $|u| > 1$ can crossing in xx or yy!

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$\rightsquigarrow$ bc is crossing if $x/cx'b$

$\hookrightarrow$ error bounded Parikh

Parikh Grouping

1. Given some equation $u = v$ with $\mathrm{Vars}(u) = \mathrm{Vars}(v)$
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      - ↪ e.g., $w := abc$ is permitted; $w := ab$ is not

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   - ↪ e.g., using $w := abc$ gives

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3. Group u and v such there cannot be crossings with pattern w
4. Cancel out common subsequences
5. Replace variable-free groups (w) by 1 and others by 0
   - ↪ e.g., $|\{\ (abc)\ \}^{abc}| = |\{\ (bac)\ \}^{abc}| \rightsquigarrow 1 = 0 \rightsquigarrow \bot$

Another Example

For equation $xaxaaabbby = yxababab x$ and pattern $w := ab$ we get

$$\{\ (x),\ (ax),\ (aa),\ (ab),\ (bb),\ (y)\ \}^{ab} = \{\ (yx),\ (ab),\ (ab),\ (ab),\ (x)\ \}^{ab}$$

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$$\{\ (ax),\ (y)\ \}^{ab} = 2 + \{\ (yx)\ \}^{ab}$$

No conflict! Are we done?

Another Example – Error-Bounded Parikh Image

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We over-/under-approximate!

- Overapprox. LHS: $\{\ (ax),\ (y)\ \}^{ab} \leq 1 + \{\ (x),\ (y)\ \}^{ab}$
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Thus, we have

$$\{\ (ax),\ (y)\ \}^{ab} \leq\ 1 + \{\ (x),\ (y)\ \}^{ab} \lneq 2 + \{\ (y),\ (x)\ \}^{ab} \leq\ 2 + \{\ (yx)\ \}^{ab}$$

Experiments

- Prototype implementation: ZIPT
- Evaluation on "word equations-only" SMT-LIB2 benchmarks

	ZIPT	Z3	cvc5	OSTRICH	Z3-Noodler	Z3str3	Total
`track 01`	**200**	198	191	198	**200**	**200**	*200*
`track 02`	**9**	4	1	4	6	6	*9*
`track 03`	**195**	176	164	127	190	190	*200*
`track 04`	**200**	198	**200**	198	199	**200**	*200*

Recap

- String Equation Solving: Aligning tokens
- Algorithmically: Nielsen transformation
  - Extended by explicit powers
  - Extended by generalised Parikh image


thank you for
x = y +
your attention
Questions?

Appendix

Future Work

- Improve implementation
  - ↪ currently often very inefficient

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- Improve implementation
  - ↪ currently often very inefficient
- Detect Parikh cases efficiently
  - ↪ currently we try out all cases
- Major problem: regular expression membership
  - ↪ call Z3-noodler?

Nielsen Transformation can be Slow

Example

$xaxbybz = axyybzzbaa$

- has the unique model

$$x_1/aaaaaaaa, \quad x_2/aaaa, \quad x_3/aa$$

- double exponential time

Why Power Normalization?

$$\frac{u[w'(w''w')^n] = v}{u[(w'w'')^n w'] = v}$$

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$n = 0$: $(ab)^0 cx = a(ba)^m d \rightsquigarrow cx = a(ba)^m d$ ↯

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2 more cases ...

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Example $((ab)^n cx = a(ba)^m d$ – With normalisation)

$$(ab)^n cx = a(ba)^m d \rightsquigarrow (ab)^n cx = (ab)^m ad$$

$n \leq m$: $cx = (ab)^{m-n} ad \rightsquigarrow \ldots$ ↯

$m < n$: $(ab)^{n-m} cx = ad \rightsquigarrow \ldots$ ↯

Equality Decomposition

Two big hammers: Power Introduction and Parikh Image

- Restricted applicability
  - Classical Nielsen: beginning/ending of string term relevant
  - Powers: $ux \ldots = x$
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  - ~~Cry?~~

  ↪ Equality Decomposition
  two equations = two beginnings/endings ☺

Equality Decomposition

Example ($xayybx = ybxxay$)

Looks hard, but is trivial, as $|xay| = |ybx|$. We split on

$$xay^{\dagger}\, ybx = ybx^{\dagger}\, xay$$

and thus

$$xay = ybx \wedge ybx = xay$$

Lazy Equality Decomposition

Example ($xayybx = yabxxy$)

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Example ($xayybx = yabxxy$)

There is no position to put †.

	y	a	b	x	x	y
x	x / y	x / y + 1	x / y + 2	0 / y + 2	0 / x + y + 2	0 / x + 2y + 2
a	x + 1 / y	x / y	x / y + 1	0 / y + 1	0 / x + y + 1	0 / x + 2y + 1
y	x + 1 / 0	x / 0	x / 1	0 / 1	0 / x + 1	0 / x + y + 1
y	x + y + 1 / 0	x + y / 0	x + y / 1	y / 1	y / x + 1	0 / x + 1
b	x + y + 2 / 0	x + y + 1 / 0	x + y / 0	y / 0	y / x	0 / x
y	x + 2y + 2 / 0	x + 2y + 1 / 0	x + 2y / 0	2y / 0	2y / x	y / x

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x	x / y	x / y + 1	x / y + 2	0 / y + 2	0 / x + y + 2	0 / x + 2y + 2
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y	x + 1 / 0	x / 0	x / 1	0 / 1	0 / x + 1	0 / x + y + 1
y	x + y + 1 / 0	x + y / 0	x + y / 1	y / 1	y / x + 1	0 / x + 1
b	x + y + 2 / 0	x + y + 1 / 0	x + y / 0	y / 0	y / x	0 / x
y	x + 2y + 2 / 0	x + 2y + 1 / 0	x + 2y / 0	2y / 0	2y / x	y / x

They only differ by 1 at $xay^{\dagger}ybx = yabx^{\dagger}xy$.

$$\hookrightarrow y/\varepsilon \ \vee \ y/oy'$$

with o being *some unknown* character

Lazy Equality Decomposition

Example ($xayybx = yabxxy$)

$$\hookrightarrow y/\varepsilon \ \vee \ y/oy'$$

with o being *some unknown* character
If y/oy':

$$xaoy'o \ ^{\dagger} \ y'bx = oy'abx \ ^{\dagger} \ xoy'$$

As $|xaoy'o| = |oy'abx|$ we split into

$$\rightsquigarrow xaoy'o = oy'abx \wedge y'bx = xoy'$$

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$$\frac{\begin{array}{c} x_1 u_1 = w_1 x_m v_1 \\ \dots \\ x_m u_m = w_m x_{m-1} v_m \\ \text{all } w_i \text{ var-free} \end{array}}{x_i/(w_i w_{i-1} \dots w_1 w_m \dots w_{i+1})^n \,\mathrm{prefix}(w_i w_{i-1} \dots w_1 w_m \dots w_{i+1})}$$

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$$\frac{w^m u = w^n v, \quad \vdash_{\mathbb{Z}} \; m < n}{\cancel{w^m u = w^n v} \qquad u = w^{n-m} v}$$

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$$\frac{w^n u = \varepsilon}{n = 0}$$

Power Normalisation Rules

$$\frac{u[(w^n)^m] = v}{u[w^{mn}] = v} \qquad \frac{u[w^n] = v, \ \vdash_{\mathbb{Z}} \ n = 0}{u[\varepsilon] = v} \qquad \frac{u[w^n] = v, \ \vdash_{\mathbb{Z}} \ n = 1}{u[w] = v}$$

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$$\frac{u[ww] = v}{u[w^2] = v} \qquad \frac{u[w^n w] = v}{u[w^{n+1}] = v} \qquad \frac{u[ww^n] = v}{u[w^{n+1}] = v} \qquad \frac{u[w'(w''w')^n] = v}{u[(w'w'')^n w'] = v}$$

Required for termination in certain cases!

Power Introduction $\stackrel{?}{=}$ Saviour

Power Introduction $\neq$ Saviour!

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Example ($xaby = yabx \wedge xaby = byax$)

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$\ldots$

Actually, $x = b^n$ **or** $y = b^n$