

# Justification Logic for Intuitionistic Modal Logic

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# Introduction to justification logic

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- ▶  $\Box A \rightsquigarrow t:A$ , where  $t$  is a **proof** of  $A$ , or explicit knowledge of  $A$ .
- ▶ Read this as  $t$  **justifies**  $A$ .

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$$k : \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \rightsquigarrow jk : s:(A \rightarrow B) \rightarrow (t:A \rightarrow s \cdot t:B)$$



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- ▶ Allow **monotonicity** of proofs

$$s_i:A \rightarrow s_1 + s_2:A$$

where  $i \in \{1, 2\}$ .

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- ▶ Other justification logics include:
  - ▶ Logics for S5 cube [Artemov et al., Brezhnev, Brünnler et al.]
  - ▶ Geach/Scott-Lemmon logics [Fitting]
  - ▶ Gödel-Löb logic [Shamkanov, Fitting]
  - ▶ Non-normal modal logics [Rohani and Studer]

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- ▶ Base axiomatisation extends CPL with
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- ▶ Add other proof operators and axioms if required.

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This mirrors the **external** modal view:

$$k \frac{A_1, \dots, A_n \Rightarrow B}{\Box A_1, \dots, \Box A_n \Rightarrow \Box B}$$

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- ▶ As a corollary, justification logic can emulate **necessitation**

$$\text{nec} \frac{\vdash A}{\vdash \Box A}$$

## Corollary

*If  $\text{JL} \vdash A$ , then there exists a proof term  $t$  such that*

$$\text{LP} \vdash t:A$$

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- ▶  $\Box$ s are **realised** into proof terms through a **realisation** function  $r$ .
- ▶ Realisation can be done proof-theoretically or semantically.

[Artemov, Fitting]

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- ▶ Method was expanded to hypersequent and nested sequent calculi.

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- ▶ Method was expanded to hypersequent and nested sequent calculi. [Artemov et. al, Brünnler et. al]
- ▶ Briefly outline the method for constructing a realisation with nested sequents. [Goetschi and Kuznets]

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- ▶ A **context** is a nested sequent with one or several holes  $\{ \}$  which can take the place of a formula in the sequent.
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- ▶ E.g.  $\Gamma \{ \} = A, [B, [\{ \}, [D]]]$ . Then

$$\Gamma \{ C \} = A, [B, [C, [D]]]$$

## Nested system nK

$$\begin{array}{c} \text{id} \frac{}{\Gamma\{a, \bar{a}\}} \quad \wedge \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}} \quad \vee \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}} \quad \Box \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}} \\[1em] \text{K} \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}} \end{array}$$

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### Theorem (Soundness and Completeness)

$$K \vdash A \iff \text{nK} \vdash A$$

(Proved using a cut-elimination argument)

$$\text{cut} \frac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}}$$



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- ▶ Proceed by induction on the height of the proof of  $A$ .

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- ▶ Using these theorems, we construct a realisation  $r$  on  $\Gamma$ .
- ▶ "Replicate" the soundness of this rule in justification logic.
- ▶ Difficulty here is dealing with nesting where rules are applied within brackets.



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- Consider the following:

$$\wedge \frac{p, [q_1] \quad p, [q_2]}{p, [q_1 \wedge q_2]}$$

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where  $t$  is a proof of  $A \rightarrow A$ .

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- ▶ Smallest intuitionistic modal logic with  $\Diamond$  is *constructive* modal logic CK.

[Bellin et al.]

$$k_1 : \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

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# Intuitionistic diamonds

- ▶ What about  $\Diamond$ ?
- ▶ Intuitionistic  $\Box$  and  $\Diamond$  are not De Morgan dual.

$$\Box A \not\leftrightarrow \neg \Diamond \neg A$$

- ▶ Language  $\mathcal{L}_{\Box}$ :

$$A ::= \perp \mid p \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid \Box A \mid \Diamond A$$

- ▶ Smallest intuitionistic modal logic with  $\Diamond$  is *constructive* modal logic CK. [Bellin et al.]

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- ▶ CK (and some extensions) have Gentzen-style proof theory

[Bierman and de Paiva]

# Intuitionistic diamonds

- ▶ Idea: Make diamond explicit with **satisfiers**.

$$\Diamond A \rightsquigarrow \mu:A$$

- ▶  $\mu:A$  read as  $\mu$  **satisfies**  $A$ , or  $\mu$  is a **model** of  $A$ .

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- Method of realisation is established similarly.

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## Establishing correspondence: JIK to IK

- ▶ Each proof term replaced with a  $\Box$ .
- ▶ Each satisfier term replaced with a  $\Diamond$ .
- ▶ Achieved through the following:

### Definition (Forgetful projection)

The forgetful projection is a map

$$(\cdot)^f : \mathcal{L}_J \rightarrow \mathcal{L}_\Box$$

inductively defined as follows:

$$\begin{array}{ll} \perp^f & := \perp \\ p^f & := p \\ (A * B)^f & := (A^f * B^f) \text{ where } * \in \{\wedge, \vee, \rightarrow\} \end{array} \qquad \begin{array}{ll} (t:A)^f & := \Box A^f \\ (\mu:A)^f & := \Diamond A^f \end{array}$$

# Establishing correspondence: JIK to IK

Result is formally stated as follows:

## Theorem

$\text{JIK} \vdash A \Rightarrow \text{IK} \vdash A^f.$

## Proof.

Apply forgetful projection on axioms of JIK to see we get theorems of IK.

For example:

$$\text{jk}_4 : ((\mu:A \rightarrow t:B) \rightarrow \mu \triangleright t:(A \rightarrow B))^f = (\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$$



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- ▶ Systematically replace each  $\Box$  with a proof term, and each  $\Diamond$  with a satisfier term.
- ▶ The objective is to produce a **realisation** function:

## Definition (Realisation)

A realisation is a map  $(\cdot)^r : \mathcal{L}_{\Box} \rightarrow \mathcal{L}_J$  such that  $(A^r)^f = A$  for each  $A \in \mathcal{L}_{\Box}$ .

## Establishing correspondence: IK to JIK

Correspondence is formally stated as:

Theorem (Marin and P.)

*There exists a realisation function  $r$  such that*

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- ▶ Adapt the method in the classical case.

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- ▶ Provides a treatment to make  $\Diamond$  explicit for most intuitionistic modal logics.
- ▶ Method for realisation can be applied to other nested sequent systems for intuitionistic modal logics.

[Arisaka et al., Kuznets and Straßburger]

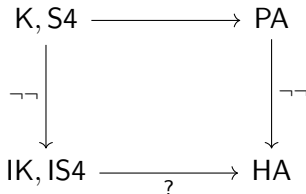
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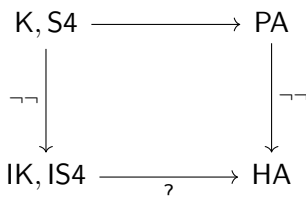
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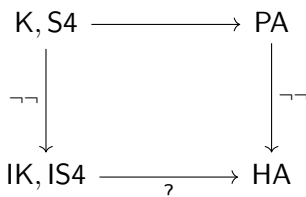
- Understanding  $\diamond$  and satisfiers in classical logic.

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- ▶ Exploring connections to arithmetic.



- ▶ Understanding  $\diamond$  and satisfiers in classical logic.

$$IK + LEM = K \quad \rightsquigarrow \quad JIK + LEM \stackrel{?}{=} JK$$

- ▶ Semantics for  $JIK$ .
- ▶ Proof theory of  $JIK$