Difference of Constrained Patterns in Logically Constrained Term Rewrite Systems

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Pattern Completeness

- Non-existence of undefined patterns
  - pattern: $f(t_1, \ldots, t_n)$ with a defined symbol f and constructors $t_1, \ldots, t_n$
- Usually checked by compilers/interpreters of programming languages
  - Guards are not taken into account, while warnings may occur
- Equivalent to quasi-reducibility of many-sorted term rewrite systems (TRS)
  - TRS $\mathcal{R}$ is quasi-reducible if all ground patterns are redexes of $\mathcal{R}$
- Usually assumed in using Rewriting Induction [Reddy, 1990]
  - Also used in proving ground confluence via RI [Aoto et al., 2017]

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Quasi-reducibility of Rewrite Systems

- Non-existence of undefined patterns $f(t_1, \ldots, t_n)$

Example (list of natural numbers)

- $\mathcal{S} = \{\, \mathit{nat},\ \mathit{list},\ \mathit{bool} \,\}$
- $\Sigma = \{\, \mathrm{nil} : \mathit{list},\ \mathrm{cons} : \mathit{nat} \times \mathit{list} \Rightarrow \mathit{list},\ 0 : \mathit{nat},\ \mathrm{s} : \mathit{nat} \Rightarrow \mathit{nat},\ \mathrm{true}, \mathrm{false} : \mathit{bool},\ \mathrm{even} : \mathit{list} \Rightarrow \mathit{bool} \,\}$
  - $\mathcal{D} = \{\, \mathrm{even} \,\}$: defined symbols $\quad \mathcal{C} = \{\, 0, \mathrm{s}, \mathrm{nil}, \mathrm{cons}, \mathrm{true}, \mathrm{false} \,\}$: constructors
- $\mathcal{R} = \left\{ \begin{array}{r} \mathrm{even}(\mathrm{nil}) \to \mathrm{true} \\ \mathrm{even}(\mathrm{cons}(x, \mathrm{cons}(y, zs))) \to \mathrm{even}(zs) \end{array} \right\}$ is not quasi-reducible

- Decidable for TRSs [Kapur et al., 1987]
- Complement algorithm for left-linear TRSs [Lazrek et al., 1990, Higashiwada and Aoto, 2019]
- Well-designed formalized algorithm in co-NP for TRSs [Thiemann and Yamada, 2024, Thiemann and Yamada, 2025]
- No result for decidability of constrained systems
  - Some sufficient conditions for Logically Constrained TRSs [Sakata et al., 2009, Kop, 2017]

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Logically Constrained Term Rewrite System (LCTRS) [Kop and Nishida, 2013]

- Computation models of functional and imperative programs [Fuhs et al., 2017]
- Calculation rules are implicitly included, e.g., $x + y \to z\ [\, z = x + y\,]$

Example (LCTRS with Integer Theory)

- $\mathcal{S}_{theory} = \{\, bool,\ int \,\}$: theory sorts
- $\mathcal{V}al = \{\, \mathsf{true}, \mathsf{false} : bool \,\} \cup \{\, \mathsf{n} : int \mid n \in \mathbb{Z} \,\}$: values
- $\Sigma_{theory} = \mathcal{V}al \cup \left\{ \begin{array}{l} +, -, \times, /, \ldots : int \times int \Rightarrow int, \\ =_{int}, \neq_{int}, <, \leq, \ldots : int \times int \Rightarrow bool, \\ \wedge, \vee, \ldots : bool \times bool \Rightarrow bool,\ \neg : bool \Rightarrow bool \end{array} \right\}$: theory symbols
- $\Sigma_{terms} = \{\, \mathsf{sum} : int \Rightarrow int \,\}$: user-defined symbols
- $\mathcal{R} = \left\{ \begin{array}{ll} \mathsf{sum}(n) \to n & [\, n \leq 0\,] \\ \mathsf{sum}(n) \to n + \mathsf{sum}(n + 1) & [\, n > 0\,] \end{array} \right\}$: user-defined rules
- $\mathsf{sum}(3)$

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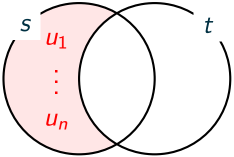
Complement Algorithm for Linear Patterns

[Lazrek et al., 1990, Higashiwada and Aoto, 2019]

- Based on difference operator $\ominus$ over linear patterns

  $s \ominus t = \{u_1, \ldots, u_n\}$: finite set of linear patterns

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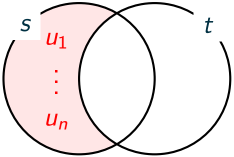

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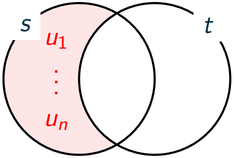

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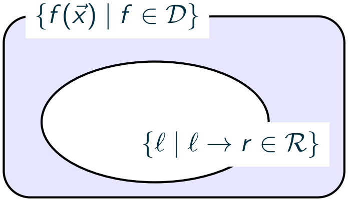

Applications to LCTRSs

- Equivalence verification via RI for LCTRSs [Fuhs et al., 2017]
  - Termination and quasi-reducibility of given LCTRSs are assumed

- Proof system for All-Path Reachability (APR) problems $P \Rightarrow^{\forall} Q$ [Ciobâcă and Lucanu, 2018]
  - Difference of constrained terms is computed: Some rule reduces $P \Rightarrow^{\forall} Q$ to $(P \setminus Q) \Rightarrow^{\forall} Q$

Example

- $\mathcal{S} = \{\, \mathit{bool},\ \mathit{int},\ \mathit{list} \,\}$
- $\mathcal{C} = \mathcal{V}\mathit{al} \cup \{\, \mathsf{nil} : \mathit{list},\ \mathsf{cons} : \mathit{int} \times \mathit{list} \Rightarrow \mathit{list} \,\}$
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- Can we decide it?

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Goal and Contributions

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Difference operator and Complement Algorithm for Logically Constrained TRSs

Contributions

- $\ominus$ over constrained patterns and constrained linear patterns
  - LHSs of $\ominus$ do not have to be linear, while RHSs are linear
- Complement Algorithm for finite sets of constrained linear patterns
- Quasi-reducibility of left-linear LCTRSs with decidable theories is decidable

LCTRSs in This Talk

- No non-value ground constructor term with a theory sort
  - Example: Declaration of $\mathsf{s} : \mathit{int} \Rightarrow \mathit{int}$ is not allowed for integer LCTRSs
  - All theory sorts are inextensible [Fuhs et al., 2025]
- Finitely many non-theory symbols

- In practical terms, these are not limitations

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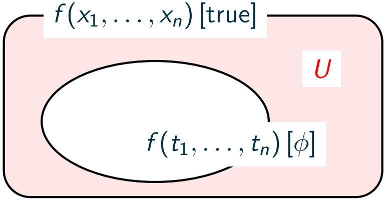
2. Complement of Patterns
3. Difference Operator over Constrained Patterns
4. Complement Algorithm for Quasi-Reducibility of LCTRSs
5. Conclusion

Constrained Patterns and Complements

- Constrained pattern $t\,[\phi]$ is a pair of pattern t and constraint ϕ
  - Pattern is a term $f(t_1,\ldots,t_n)$ s.t. $f \in \mathcal{D}$ and $t_1,\ldots,t_n \in T(\mathcal{C},\mathcal{V})$
  - $\mathcal{G}(t) := \{\, t\sigma \mid \sigma$ is a ground constructor substitution $\}$ and $\mathcal{G}(U) := \bigcup_{u \in U} \mathcal{G}(u)$
- $\mathcal{G}(t\,[\phi]) := \{\, t\sigma \mid \sigma$ is a ground constructor substitution, $\forall x \in \mathcal{V}ar(\phi).\ x\sigma \in \mathcal{V}al,\ [\![\phi\sigma]\!] = \top \,\}$
- Complement of constrained pattern $f(t_1,\ldots,t_n)\,[\phi]$ is a set U of constrained patterns s.t.

$$\mathcal{G}(U) \;=\; \mathcal{G}(f(x_1,\ldots,x_n)\,[\mathsf{true}]) \setminus \mathcal{G}(f(t_1,\ldots,t_n)\,[\phi])$$



- Finite complements are expected

Complement Operator for Linear Constructor Terms

[Lazrek et al., 1990, Higashiwada and Aoto, 2019]

Definition ($\overline{\cdot}$ for linear constructor terms)

- $\overline{x : \iota} = \emptyset$
- $\overline{c(u_1, \ldots, u_n) : \iota} = \{d(y_1, \ldots, y_m) \mid d : \iota_1 \times \cdots \times \iota_m \Rightarrow \iota \in \mathcal{C},\ c \neq d\}$
  $\cup \{c(u_1, \ldots, u_{i-1}, u'_i, y_{i+1}, \ldots, y_n) \mid u_i \notin \mathcal{V},\ u'_i \in \overline{u_i}\}$

- $\mathcal{C}$ is assumed to be finite for finiteness of $\overline{u}$

Example

- $\mathcal{S} = \{\, nat,\ bool,\ list,\ pair \,\}$
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- $\overline{\mathrm{nil}} =$
- $\overline{\mathrm{cons}(x, \mathrm{nil})} =$

- Linearity is necessary for finite complements of patterns with infinite sorts
  - "Complement of p(x, x)" =

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- $\overline{\mathrm{nil}} =$ cons
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- Linearity is necessary for finite complements of patterns with infinite sorts
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Complement Operator for Linear Constructor Terms

[Lazrek et al., 1990, Higashiwada and Aoto, 2019]

Definition ($\overline{\cdot}$ for linear constructor terms)

- $\overline{x : \iota} = \emptyset$
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- Linearity is necessary for finite complements of patterns with infinite sorts
  - "Complement of $\mathrm{p}(x, x)$" $= \{\, \mathrm{p}(t_1, t_2) \in T(\mathcal{C}) \mid t_1, t_2 : nat,\ t_1 \neq t_2 \,\}$

Complements of Values in LCTRS Setting

- For finite results, complement operator $\overline{\cdot}$ assumes finiteness of $\mathcal{C}$ and linearity of terms
- $\mathcal{V}al$ $(= \Sigma_{theory} \cap \mathcal{C})$ may be infinite, e.g., $\mathcal{C}$ of integer LCTRSs includes all integers
- Complements of values may be infinite, e.g., $\overline{0} = \mathbb{Z} \setminus \{0\}$ is infinite
- Make s of $s\,[\phi]$ value-free, e.g., $s[0]_p\,[\phi]$ is equivalent to $s[x]_p\,[\phi \land x = 0]$
- $\mathcal{C} \setminus \mathcal{V}al$ $(\subseteq \Sigma_{terms})$ should be finite
- Logical Variables in term part can be linearized, e.g, $s[x, x]_p\,[\phi]$ is equivalent to $s[x, y]_p\,[\phi \land x = y]$

Proposition [Kop, 2017, Kojima and Nishida, 2024]

For any constrained term $s\,[\phi]$, there exists a value-free $s'\,[\phi']$ s.t. $\mathcal{G}(s\,[\phi]) = \mathcal{G}(s'\,[\phi'])$

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LCTRSs in This Talk (repeat)

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- Finitely many non-theory symbols, i.e., Σ_{terms} is finite

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Proposition [Kop, 2017, Kojima and Nishida, 2024]

For any constrained term $s\,[\phi]$, there exists a value-free LV-linear $s'\,[\phi']$ s.t. $\mathcal{G}(s\,[\phi]) = \mathcal{G}(s'\,[\phi'])$

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Contents of This Talk

1. Background
2. Complement of Patterns
3. Difference Operator over Constrained Patterns
4. Complement Algorithm for Quasi-Reducibility of LCTRSs
5. Conclusion

Difference Operator $\ominus$ over Unconstrained Linear Patterns

[Lazrek et al., 1990, Higashiwada and Aoto, 2019]

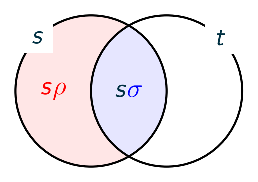
- Assume w.l.o.g. that s, t of $s \ominus t$ have no shared variables: $\mathcal{V}ar(s) \cap \mathcal{V}ar(t) = \emptyset$

Definition

$$s \ominus t \;=\; \begin{cases} \{\, s\rho \mid \rho \in \overline{\sigma|_{\mathcal{V}ar(s)}} \,\} & \text{if } s, t \text{ are unifiable with mgu } \sigma \\ \{\, s \,\} & \text{o/w} \end{cases}$$

where

$$\overline{\sigma} = \{\, \rho \mid \mathcal{D}om(\rho) = \mathcal{D}om(\sigma),\ \rho \neq \sigma,\ \forall x \in \mathcal{D}om(\sigma).\ x\rho \in \overline{x\sigma} \cup \{x\sigma\} \,\}$$



- $s \ominus t$ is a finite set of patterns s.t. $\mathcal{G}(s \ominus t) = \mathcal{G}(s) \setminus \mathcal{G}(t)$

Example (cont'd)

- $\mathsf{even}(\mathsf{cons}(x, \mathsf{nil})) \ominus \mathsf{even}(\mathsf{cons}(0, ys)) =$
  - $\sigma = \{\, x \mapsto 0,\ ys \mapsto \mathsf{nil} \,\}$

- Linearity of s and t ensures linearity of $x\sigma$, but s does not have to be linear [new]

Proposition If t is linear, then $x\sigma$ is linear for any $x \in \mathcal{V}ar(s)$ [new]

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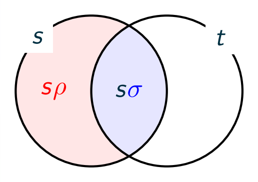

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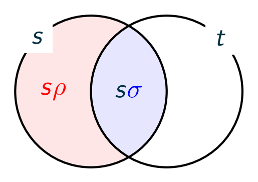

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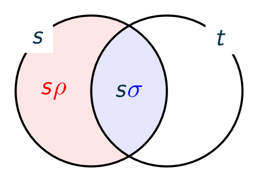

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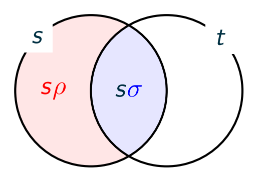
$$s \ominus t = \begin{cases} \{\, s\rho \mid \rho \in \overline{\sigma|_{\mathcal{V}ar(s)}} \,\} & \text{if } s, t \text{ are unifiable with mgu } \sigma \\ \{\, s \,\} & \text{o/w} \end{cases}$$

where

$$\overline{\sigma} = \{\, \rho \mid \mathcal{D}om(\rho) = \mathcal{D}om(\sigma),\ \rho \neq \sigma,\ \forall x \in \mathcal{D}om(\sigma).\ x\rho \in \overline{x\sigma} \cup \{x\sigma\} \,\}$$



- $s \ominus t$ is a finite set of patterns s.t. $\mathcal{G}(s \ominus t) = \mathcal{G}(s) \setminus \mathcal{G}(t)$

Example (cont'd)

- $\mathsf{even}(\mathsf{cons}(x, \mathsf{nil})) \ominus \mathsf{even}(\mathsf{cons}(0, ys)) = \{\, \mathsf{even}(\mathsf{cons}(\mathsf{s}(y), \mathsf{nil})) \,\}$
  - $\sigma = \{\, x \mapsto 0,\ ys \mapsto \mathsf{nil} \,\}$ and thus $\overline{\sigma|_{\{x\}}} = \{\, x \mapsto \mathsf{s}(y) \,\}$

- Linearity of s and t ensures linearity of $x\sigma$, but s does not have to be linear [new]

Proposition If t is linear, then $x\sigma$ is linear for any $x \in \mathcal{V}ar(s)$ [new]

Difference Operator $\ominus$ over Unconstrained Linear Patterns

[Lazrek et al., 1990, Higashiwada and Aoto, 2019]

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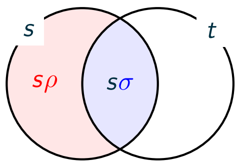

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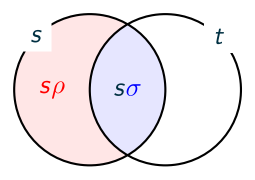

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Difference Operator $\ominus$ over Value-free LV-linear Constrained Terms

Definition (repeat)

$$s \ominus t \;=\; \begin{cases} \{\, s\rho \mid \rho \in \overline{\sigma|_{\mathcal{V}ar(s)}} \,\} & \text{if } s, t \text{ are unifiable with mgu } \sigma \\ \{\, s \,\} & \text{o/w} \end{cases}$$

where $\overline{\sigma} = \{\, \rho \mid \mathcal{D}om(\rho) = \mathcal{D}om(\sigma),\ \rho \neq \sigma,\ \forall x \in \mathcal{D}om(\sigma).\ x\rho \in \overline{x\sigma} \cup \{x\sigma\} \,\}$

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- $\mathcal{G}(s\,[\phi]) = \mathcal{G}(s\rho\,[\phi\sigma]) \uplus \underline{\mathcal{G}(s\sigma\,[\phi\sigma])}$

Definition [new]

$$s\,[\phi] \ominus t\,[\psi] \;=\; \begin{cases} \{\, s\rho\,[\phi\sigma] \mid \rho \in \overline{\sigma|_{\mathcal{V}ar(s)}} \,\} & \text{if } s, t \text{ are unifiable with mgu } \sigma \\ \cup\, \{\, s\sigma\,[\phi\sigma \wedge \neg\psi\sigma] \mid \phi\sigma \wedge \neg\psi\sigma \text{ is SAT} \,\} & \quad \text{and } \phi\sigma \wedge \psi\sigma \text{ is SAT} \\ \{\, s\,[\phi] \,\} & \text{o/w} \end{cases}$$

Difference Operator $\ominus$ over Value-free LV-linear Constrained Terms

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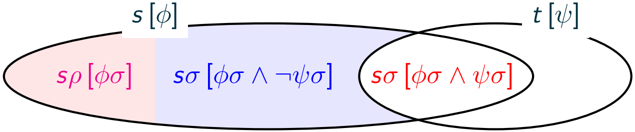

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  $||$

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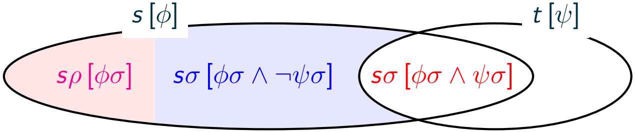

Definition

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Example: Difference of Constrained Patterns

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Example (cont'd)

- $\mathsf{f}(xs, x)\,[\mathsf{true}] \ominus \mathsf{f}(\mathsf{nil}, y_1)\,[y_1 \leq 0] = \Big\{ \qquad \Big\}$
  - $\sigma = \{\, xs \mapsto \mathsf{nil},\ y_1 \mapsto x \,\}$
  - $\rho = \{\, xs \mapsto \mathsf{cons}(v, vs) \,\}$
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Example: Difference of Constrained Patterns

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Example: Difference of Constrained Patterns

Definition (repeat) [new]

$$s\,[\phi] \ominus t\,[\psi] \;=\; \begin{cases} \{\, s\rho\,[\phi\sigma] \mid \rho \in \overline{\sigma|_{\mathcal{V}ar(s)}}\,\} & \text{if } s,\ t \text{ are unifiable with mgu } \sigma \\ \cup\, \{\, s\sigma\,[\phi\sigma \wedge \neg\psi\sigma] \mid \phi\sigma \wedge \neg\psi\sigma \text{ is SAT}\,\} & \quad \text{and } \phi\sigma \wedge \psi\sigma \text{ is SAT} \\ \{\, s\,[\phi]\,\} & \text{o/w} \end{cases}$$

Example (cont'd)

- $\mathsf{f}(xs, x)\,[\mathsf{true}] \ominus \mathsf{f}(\mathsf{nil}, y_1)\,[y_1 \leq 0] = \left\{ \begin{array}{rl} \mathsf{f}(\mathsf{cons}(v, vs), x) & [\mathsf{true}] \\ \mathsf{f}(\mathsf{nil}, x) & [\mathsf{true} \wedge \neg(x \leq 0)] \end{array} \right\}$
  - $\sigma = \{\, xs \mapsto \mathsf{nil},\ y_1 \mapsto x\,\}$
  - $\rho = \{\, xs \mapsto \mathsf{cons}(v, vs)\,\}$
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  - $\phi\sigma \wedge \neg\psi\sigma$ is $x \leq 0 \wedge \neg\mathsf{true}$, which is UNSAT

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Contents of This Talk

Extension to Finite Sets of Unconstrained Linear Patterns

[Lazrek et al., 1990, Higashiwada and Aoto, 2019]

Definition

$$P \oslash Q = \begin{cases} ((P \setminus \{s\}) \cup (s \ominus t)) \oslash ((Q \setminus \{t\}) \cup (t \ominus s)) & \text{if } \exists s \in P, t \in Q.\ s \ominus t \neq \{s\} \\ P & \text{o/w} \end{cases}$$

- Both P and Q are assumed to be sets of linear patterns
  - Not all patterns have to be linear
  - Linearity of s is required for linearity of $t \ominus s$
- Patterns in P are w.l.o.g. assumed to be pairwise disjoint
  - s, t are disjoint if $\mathcal{G}(s) \cap \mathcal{G}(t) = \emptyset$ (i.e., s, t are not unifiable)
  - If s and t are unifiable with mgu σ, then we replace $\{s, t\}$ by $(s \ominus t) \uplus \{s\sigma\} \uplus (t \ominus s)$
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Left-linear LCTRS $\mathcal{R}$ is quasi-reducible iff $\{f(\vec{x})\,[\mathsf{true}] \mid f \in \mathcal{D}\} \oslash \{\ell\,[\phi] \mid \ell \to r\,[\phi] \in \mathcal{R}\} = \emptyset$

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Example: Quasi-Reducibility of LCTRSs

Example (cont'd)

$$\left\{\begin{array}{lrl} (1) & \mathsf{f}(\mathsf{nil}, y_1) \to 0 & [\, y_1 \leq 0 \,] \\ (2) & \mathsf{f}(\mathsf{cons}(x_2, xs_2), y_2) \to \mathsf{f}(xs_2, y_2 - 1) & [\, x_2 \leq 0 \wedge y_2 > 0 \,] \\ (3) & \mathsf{f}(\mathsf{cons}(x_3, \mathsf{cons}(z_3, zs_3)), y_3) \to x_3 + \mathsf{f}(zs_3, y_3 - 2) & [\, x_3 > 0 \wedge y_3 > 1 \,] \end{array}\right\}$$ is not quasi-reducible as

$$\{\, \mathsf{f}(xs, y)\, [\mathsf{true}] \,\} \oslash \left\{\begin{array}{lrl} (1) & \mathsf{f}(\mathsf{nil}, y_1) & [\, y_1 \leq 0 \,] \\ (2) & \mathsf{f}(\mathsf{cons}(x_2, xs_2), y_2) & [\, x_2 \leq 0 \wedge y_2 > 0 \,] \\ (3) & \mathsf{f}(\mathsf{cons}(x_3, \mathsf{cons}(z_3, zs_3)), y_3) & [\, x_3 > 0 \wedge y_3 > 1 \,] \end{array}\right\}$$

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$$= \left\{\begin{array}{lll} (8) & \mathsf{f}(\mathsf{cons}(x, \mathsf{nil}), y_1) & [\, y_1 \leq 0 \land \neg(x \leq 0 \land y_1 > 0) \,] \\ (9) & \mathsf{f}(\mathsf{cons}(x, \mathsf{cons}(z, zs)), y_1) & [\, y_1 \leq 0 \land \neg(x \leq 0 \land y_1 > 0) \land \neg(x > 0 \land y_1 > 1) \,] \\ (5) & \dots & \end{array}\right\} \oslash \left\{\begin{array}{ll} (7) & \dots \\ (3c) & \dots \end{array}\right\}$$

$$= \{\, (8),\ (9),\ (5) \,\} \neq \emptyset$$

Example: Quasi-Reducibility of LCTRSs

Example (cont'd)

$$\left\{\begin{array}{lrl} (1) & \mathsf{f}(\mathsf{nil}, y_1) \to 0 & [\, y_1 \leq 0\,] \\ (2) & \mathsf{f}(\mathsf{cons}(x_2, xs_2), y_2) \to \mathsf{f}(xs_2, y_2 - 1) & [\, x_2 \leq 0 \wedge y_2 > 0\,] \\ (3) & \mathsf{f}(\mathsf{cons}(x_3, \mathsf{cons}(z_3, zs_3)), y_3) \to x_3 + \mathsf{f}(zs_3, y_3 - 2) & [\, x_3 > 0 \wedge y_3 > 1\,] \end{array}\right\}$$ is not quasi-reducible as

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Contents of This Talk

Conclusion

Summary

- $\ominus$ over constrained patterns and constrained linear patterns
  - LHSs of $\ominus$ do not have to be linear, while RHSs are linear
- Complement Algorithm for finite sets of constrained linear patterns
- Quasi-reducibility of left-linear LCTRSs with decidable theories is decidable

Future Work

- Extension of co-NP Algorithm [Thiemann and Yamada, 2024] to LCTRSs
- Implementation

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Appendix

Subsumption Rules of All-Path Reachability Proof Systems

Subsumption Rule for ARSs

[Ciobâcă and Lucanu, 2018]

$$\textsc{Subs}\ \frac{(P \setminus Q) \Rightarrow^{\forall} Q \qquad P \cap Q \neq \emptyset}{P \Rightarrow^{\forall} Q}$$

Subsumption Rule for LCTRSs with $\to_\varepsilon$

[Ciobâcă and Lucanu, 2018]

$$\textsc{Subs}\ \frac{s\,[\phi \wedge \neg(\exists \vec{x}.\ (s = t \wedge \phi \wedge \psi))] \Rightarrow^{\forall} t\,[\psi] \qquad \exists \vec{x}.\ (s = t \wedge \phi \wedge \psi) \text{ is SAT}}{s\,[\phi] \Rightarrow^{\forall} t\,[\psi]}$$

where $\{\vec{x}\} = \mathcal{V}ar(t, \psi) \setminus \mathcal{V}ar(s, \phi)$

- "$s\,[\phi \wedge \neg(\exists \vec{x}.\ (s = t \wedge \phi \wedge \psi))]$" $=$ "$\mathcal{G}(s\,[\phi]) \setminus \mathcal{G}(t\,[\psi])$"
- "$\exists \vec{x}.\ (s = t \wedge \phi \wedge \psi)$ is SAT" $=$ "$\mathcal{G}(s\,[\phi]) \cap \mathcal{G}(t\,[\psi]) \neq \emptyset$"

Subsumption Rule for LCTRSs with $\to_\varepsilon$

$$\textsc{Subs}\ \frac{u_1\,[\phi_1] \Rightarrow^{\forall} t\,[\psi] \quad \ldots \quad u_n\,[\phi_n] \Rightarrow^{\forall} t\,[\psi] \qquad s\,[\phi],\ t\,[\psi] \text{ are unifiable}}{s\,[\phi] \Rightarrow^{\forall} t\,[\psi]}$$

where $s\,[\phi] \ominus t\,[\psi] = \{\, u_1\,[\phi_1], \ldots, u_n\,[\phi_n] \,\}$

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