Analytic calculi for logics of indicative conditionals

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Overview

We study a family of finite-valued logics that model indicative conditionals: 'if . . ., then . . .' sentences that talk about what could be true (as opposed to counterfactuals).

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- If the train is on time, we will be home by ten. (indicative conditional)
- If the train had been on time, we would have been home by ten. (counterfactual)

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In this work

Analytic calculi and finite axiomatizations for these logics in a uniform and modular way.

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We study a family of finite-valued logics that model indicative conditionals: 'if ..., then ...' sentences that talk about what could be true (as opposed to counterfactuals).

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In this talk

- Briefly present these logics and their main formal/algebraic aspects.
- Illustrate and advertise our axiomatization methods for the study of many-valued logics:

$$\begin{tikzcd}
\boxed{\mathcal{M} := \{\langle \mathbf{A}, D_i \rangle\}_{i \leq k}} \arrow[r, bend left, "\text{PNmatrix packing}"] & \boxed{\mathbb{M} := \langle \mathbf{B}, D \rangle} \arrow[r, bend left, "\text{Axiomatizer}"] & \boxed{\text{Analytic calculi}} \arrow[r, bend left, "\vee\text{-conversion}"] & \boxed{\text{Hilbert calculi}}
\end{tikzcd}$$

(Some conditions have to be met at each step.)

Outline

Logics of indicative conditionals

They try to formalize the semantic gap $\mathbf{0} \to x$ by adding a new truth-value $\mathbf{1/2}$.

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This leads to different proposals regarding the behavior of $\mathbf{1/2}$:

$\to_{\mathsf{DF}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$

$\to_{\mathsf{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$

$\to_{\mathsf{F}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
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In some cases, this also involves accounts of conjunction and disjunction different to the strong Kleene ones:

$\wedge_{\mathsf{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$

$\vee_{\mathsf{OL}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{1}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

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Let $A_3 := \{\mathbf{0}, \mathbf{1/2}, \mathbf{1}\}$. Our target logics are:

- De Finetti's logic DF, induced by $\langle \mathbf{DF_3}, \{\mathbf{1/2}, \mathbf{1}\}\rangle$, with $\mathbf{DF_3} := \langle A_3; \neg, \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{DF}}\rangle$.
- Cooper's logic OL, induced by $\langle \mathbf{OL_3}, \{\mathbf{1/2}, \mathbf{1}\}\rangle$, with $\mathbf{OL_3} := \langle A_3; \neg, \wedge_{\mathsf{OL}}, \vee_{\mathsf{OL}}, \to_{\mathsf{OL}}\rangle$.
- Farrell's logic F, induced by $\langle \mathbf{F_3}, \{\mathbf{1/2}, \mathbf{1}\}\rangle$, with $\mathbf{F_3} := \langle A_3; \neg, \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{F}}\rangle$.
- Cantwell's logic CN, induced by $\langle \mathbf{CN_3}, \{\mathbf{1/2}, \mathbf{1}\}\rangle$, with $\mathbf{CN_3} := \langle A_3; \neg, \wedge_{\mathsf{K}}, \vee_{\mathsf{K}}, \to_{\mathsf{OL}}\rangle$.
- Logics obtained from these by varying the designated elements.
- Logics of order obtained from the algebras above.

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- Logics of order obtained from the algebras above.

These logics were introduced decades ago, and recently we started a systematic exploration of their formal/algebraic properties:

- V.G., S.M. and U.R., *Axiomatizing the Logic of Ordinary Discourse.* IPMU 2024.
- U.R., *The Algebra of Ordinary Discourse.* Archive for Mathematical Logic, 2025.
- U.R. and M.M.P., *Indicative conditionals: some algebraic considerations.* WoLLIC, 2025.

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In general, axiomatizability via algebra for such logics can be quite challenging, and the resulting systems are not analytic.

Here, we axiomatize all these logics by analytic multiple-conclusion and traditional Hilbert calculi, following a uniform and modular approach.

Outline

Partial non-deterministic matrices (PNmatrices)

A Σ-PNmatrix $\mathbb{M}$ is a structure $\langle \mathbf{A}, D \rangle$ such that:

- $\mathbf{A} := \langle A, \cdot_{\mathbf{A}} \rangle$ is a Σ-multialgebra, that is,
  - $A \neq \varnothing$ (the set of truth-values);
  - $©_{\mathbf{A}} : A^k \to \mathcal{P}(A)$, for each k-ary $© \in \Sigma$. (generalization of truth tables)
- $D \subseteq A$ is the set of designated truth-values.
- $\Phi \rhd_{\mathbb{M}} \Psi$ iff there is no valuation v on $\mathbb{M}$ such that $v[\Phi] \subseteq D$ and $v[\Psi] \subseteq \overline{D}$.

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Example $\Sigma = \{\star, \neg\}, \mathbb{E} = \langle \mathbf{E}, \{\mathbf{1}\}\rangle$

$\star_{\mathbf{E}}$	$\mathbf{0}$	$1/2$	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$1/2$	$1/2, \mathbf{1}$
$1/2$	$\mathbf{0}, 1/2$	$\mathbf{0}, 1/2$	$\varnothing$
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	$\neg_{\mathbf{E}}$
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	$\neg_{\mathbf{E}}$
$\mathbf{0}$	$\mathbf{1}$
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Notes

- A matrix is a (total deterministic) PNmatrix.
- For a PNmatrix $\mathbb{M}$, $\Phi \vdash_{\mathbb{M}} \varphi$ iff $\Phi \rhd_{\mathbb{M}} \{\varphi\}$.
- $\rhd_{\{\mathbb{M}_i\}_i} := \bigcap_i \rhd_{\mathbb{M}_i}$.
- $\vdash_{\{\mathbb{M}_i\}_i} := \bigcap_i \vdash_{\mathbb{M}_i}$.

Multiple-conclusion calculi

Definition by example

R_{CL}: classical logic over $\Sigma_{\neg\vee\wedge\to}$

$$\frac{p,\, q}{p \wedge q}\, r_1 \qquad \frac{p \wedge q}{p}\, r_2 \qquad \frac{p \wedge q}{q}\, r_3$$

$$\frac{p}{p \vee q}\, r_4 \qquad \frac{q}{p \vee q}\, r_5 \qquad \frac{p \vee q}{p,\, q}\, r_6$$

$$\frac{p,\, p \to q}{q}\, r_7 \qquad \frac{q}{p \to q}\, r_8 \qquad \frac{\varnothing}{p,\, p \to q}\, r_9$$

$$\frac{p,\, \neg p}{\varnothing}\, r_{10} \qquad \frac{\varnothing}{p,\, \neg p}\, r_{11}$$

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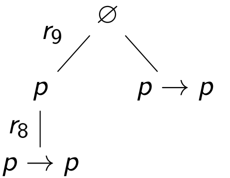
$\varnothing \rhd_{\mathsf{R}_{CL}} p \to p$



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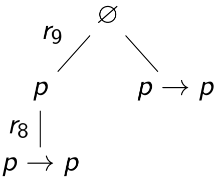
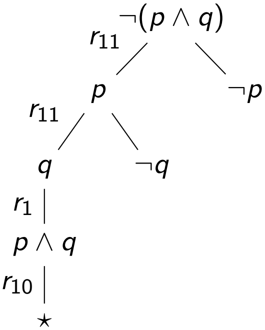


$\neg(p \wedge q) \rhd_{\mathsf{R}_{CL}} \neg p, \neg q$



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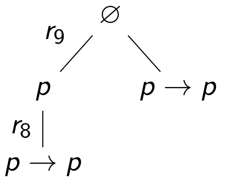

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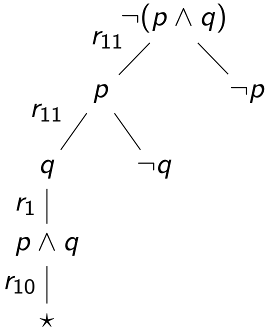

R_{CL} is analytic and induces a decision procedure

Only subformulas of the statement of interest need to be considered in proof search.

What are these $\rhd_{\mathbb{M}}$ and $\rhd_{\mathrm{R}}$?

Generalized consequence relations (grc)

A gcr $\rhd$ is a binary relation on sets of formulas satisfying **overlap**, **dilution** and **cut**.
$\rhd$ may additionally satisfy **substitution-invariance** and **finitariness**.

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When R axiomatizes $\rhd$, R is also a proof system for $\vdash_{\rhd}$.

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Given $x, y \in A$, a formula $\mathrm{S}(p)$ is a separator for x and y in $\mathbb{M}$ if $\mathrm{S}_{\mathbf{A}}(x) \subseteq D$ and $\mathrm{S}_{\mathbf{A}}(y) \subseteq \overline{D}$, or vice-versa.

Example: $\neg p$ is a separator for $\mathbf{1}$ and $1/2$ in $\mathbb{E}$ (but p is not!).

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A set $\mathscr{D}^x \subseteq L_\Sigma(\{p\})$ isolates x in $\mathbb{M}$ if
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A discriminator for $\mathbb{M}$ is a family $\{(\mathscr{D}^x_+, \mathscr{D}^x_-)\}_{x \in A}$ such that:

- $\mathscr{D}^x_+ \cup \mathscr{D}^x_-$ isolates x
- $\mathrm{S}_{\mathbf{A}}(x) \subseteq D$, if $\mathrm{S} \in \mathscr{D}^x_+$
- $\mathrm{S}_{\mathbf{A}}(x) \subseteq \overline{D}$, if $\mathrm{S} \in \mathscr{D}^x_-$

We call $\mathbb{M}$ monadic when there is a discriminator for it.

A discriminator for $\mathbb{E}$

x	$\mathscr{D}^x_+$	$\mathscr{D}^x_-$
$\mathbf{0}$	$\neg p$	p
$\mathbf{1/2}$	$\varnothing$	$p, \neg p$
$\mathbf{1}$	p	$\varnothing$

Analytic multiple-conclusion calculi for logics of monadic PNmatrices

Θ-analyticity, for $\Theta(p) \subseteq L_\Sigma(\{p\})$

- A proof of (Φ, Ψ) in R is Θ-analytic when it only contains formulas in

$$\mathsf{sub}(\Phi \cup \Psi) \cup \bigcup_{\delta \in \mathsf{sub}(\Phi \cup \Psi)} \Theta(p \mapsto \delta).$$

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Theorem

(Shoesmith and Smiley, 1978; Caleiro and Marcelino, 2019)

Let $\mathbb{M}$ be a monadic PNmatrix with discriminator $\mathscr{D}$ and Θ be the set of separators in $\mathscr{D}$. Then $\rhd_{\mathbb{M}}$ is axiomatized by a finite Θ-analytic calculus effectively generated from $\mathbb{M}$ and $\mathscr{D}$.

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Let $\mathbb{M}$ be a monadic PNmatrix with discriminator $\mathscr{D}$ and Θ be the set of separators in $\mathscr{D}$. Then $\rhd_{\mathbb{M}}$ is axiomatized by a finite Θ-analytic calculus effectively generated from $\mathbb{M}$ and $\mathscr{D}$.

The axiomatization algorithm is modular in the signature:

- adding new connectives preserves monadicity, and rules of other connectives are preserved;
- good to axiomatize fragments and expansions.

From multiple to single-conclusion

A finite multiple-conclusion calculus cannot always be converted into a finite Hilbert calculus.

However, if the language is expressive enough, a conversion is possible.

A derived connective Ⓒ is

- a disjunction in $\vdash$ whenever $\Gamma, \varphi \,Ⓒ\, \psi \vdash \gamma$ iff $\Gamma, \varphi \vdash \gamma$ and $\Gamma, \psi \vdash \gamma$
- an implication in $\vdash$ whenever $\Gamma \vdash \varphi \,Ⓒ\, \psi$ iff $\Gamma, \varphi \vdash \psi$

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Theorem

If $\vdash_{\rhd}$ has a disjunction or an implication, then a m.c. axiomatization for $\rhd$ is convertible to a s.c. axiomatization for $\vdash_{\rhd}$, preserving finiteness.

Fact

All of the considered logics of indicative conditionals have a disjunction.

Outline

Single-matrix logics, the case $D = \{\mathbf{1/2}, \mathbf{1}\}$

Since we already have a single matrix, the PNmatrix packing is not needed.

All these logics have $\neg$, which is enough to provide separators.

Logic: $\neg$-fragment

$$\mathsf{R}_{\neg}^{\{\mathbf{1},\mathbf{1/2}\}} \qquad \frac{p}{\neg\neg p} \qquad \frac{\neg\neg p}{p} \qquad \frac{}{p, \neg p}$$

This calculus is $\{p, \neg p\}$-analytic.

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$$\mathsf{R}_{\wedge_{\mathsf{K}}}^{\{\mathbf{1}, 1/2\}} \quad \frac{p \wedge q}{p} \quad \frac{p \wedge q}{q} \quad \frac{p, q}{p \wedge q} \quad \frac{\neg(p \wedge q)}{\neg p, \neg q} \quad \frac{\neg p}{\neg(p \wedge q)} \quad \frac{\neg q}{\neg(p \wedge q)}$$

$$\mathsf{R}_{\to_{\mathsf{F}}}^{\{\mathbf{1}, 1/2\}} \quad \frac{}{p \to q, p} \quad \frac{}{\neg(p \to q), p} \quad \frac{\neg(p \to q)}{\neg p, \neg q} \quad \frac{\neg p}{\neg(p \to q)} \quad \frac{\neg q}{\neg(p \to q)} \quad \frac{q}{p \to q} \quad \frac{p, p \to q}{q}$$

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Single-matrix logics, the case $D = \{1/2, \mathbf{1}\}$

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Logic: OL

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$$\mathsf{R}_{\wedge_{\mathsf{OL}}}^{\{\mathbf{1},1/2\}} \quad \frac{p \wedge q}{p} \quad \frac{p \wedge q}{q} \quad \frac{p, q}{p \wedge q} \quad \frac{p \wedge q, \neg(p \wedge q)}{\neg p} \quad \frac{p \wedge q, \neg(p \wedge q)}{\neg q} \quad \frac{\neg p, \neg q}{\neg(p \wedge q)}$$

$$\mathsf{R}_{\to_{\mathsf{OL}}}^{\{\mathbf{1},1/2\}} \quad \frac{}{p \to q, p} \quad \frac{}{\neg(p \to q), p} \quad \frac{\neg q}{\neg(p \to q)} \quad \frac{q}{p \to q} \quad \frac{p, p \to q}{q} \quad \frac{\neg(p \to q), p}{\neg q}$$

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This calculus is $\{p, \neg p\}$-analytic.

Similar for $D = \{\mathbf{1}\}$. The case $D = \{1/2\}$ demands $\{p, p \to p\}$ as separators.

Outline

Logics that preserve degrees of truth

Let $\mathbf{A}$ be an algebra where values are ordered by $\leq$.

We define the order-preserving logic of $\mathbf{A}$, denoted $\vdash^{\leq}_{\mathbf{A}}$, by

$$\Gamma \vdash^{\leq}_{\mathbf{A}} \varphi \text{ iff for all } \mathbf{A}\text{-valuations } v \text{ and all } a \in A, \text{ if } v(\Gamma) \geq a, \text{ then } v(\varphi) \geq a.$$

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All the mentioned algebras of indicative conditionals are ordered (some of them have more than one definable ordering). How to axiomatize them?

Packing matrices in a single PNmatrix

Consider the family of matrices $\mathcal{M} := \{\langle \mathbf{O}_3, \{\mathbf{1/2}, \mathbf{1}\}\rangle, \langle \mathbf{O}_3, \{\mathbf{1/2}\}\rangle\}$.

$\vdash_{\mathcal{M}}$ is the logic that preserves degrees of truth of $\mathbf{O}_3$ according to the order induced by $\wedge_{\mathsf{OL}}$.

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In order to apply the axiomatization algorithm, we first use partiality to pack these matrices into one PNmatrix.

Define $\mathbf{A} := \langle \{\mathbf{0}, \mathbf{1/2}, \mathbf{1}^-, \mathbf{1}^+\}, \cdot_{\mathbf{A}}\rangle$ and $\mathbb{M} := \langle \mathbf{A}, \{\mathbf{1/2}, \mathbf{1}^+\}\rangle$ such that

$\wedge_{\mathbf{A}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}^-$	$\mathbf{1}^+$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}^-$	$\mathbf{1}^+$
$\mathbf{1}^-$	$\mathbf{0}$	$\mathbf{1}^-$	$\mathbf{1}^-$	$\varnothing$
$\mathbf{1}^+$	$\mathbf{0}$	$\mathbf{1}^+$	$\varnothing$	$\mathbf{1}^+$

$\to_{\mathbf{A}}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}^-$	$\mathbf{1}^+$
$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$	$\mathbf{1/2}$
$\mathbf{1/2}$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}^-$	$\mathbf{1}^+$
$\mathbf{1}^-$	$\mathbf{0}$	$\mathbf{1/2}$	$\mathbf{1}^-$	$\varnothing$
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$\mathbf{0}$	$\mathbf{1}^-, \mathbf{1}^+$
$\mathbf{1/2}$	$\mathbf{1/2}$
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$\mathbf{1}^+$	$\mathbf{0}$

Then $\rhd_{\mathcal{M}} = \rhd_{\mathbb{M}}$, and $\mathbb{M}$ is monadic, thus we can proceed with the axiomatization.

Packing matrices in a single PNmatrix

How does this work?

Consider the same family of matrices $\mathcal{M} := \{\langle \mathbf{O}_3, \{\mathbf{1/2}, \mathbf{1}\}\rangle, \langle \mathbf{O}_3, \{\mathbf{1/2}\}\rangle\}$.

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$\mathbf{1}^-$	$\mathbf{0}$	$\mathbf{1}^-$	$\mathbf{1}^-$	$\varnothing$
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$\mathbf{1}^-$	$\mathbf{0}$
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The submatrices in blue and red are isomorphic to the matrices in $\mathcal{M}$, and are the maximal total components of $\mathbb{M}$, which determine $\rhd_{\mathbb{M}}$.

Outline

Final considerations

- This work: systematic axiomatization of logics of indicative conditionals, by analytic m.c. calculi and Hilbert calculi, with the help of PNmatrices.

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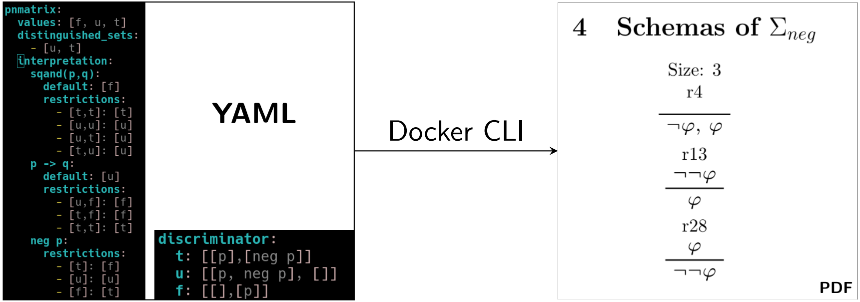
- The axiomatizer has a prototype implementation at

https://github.com/greati/logicantsy



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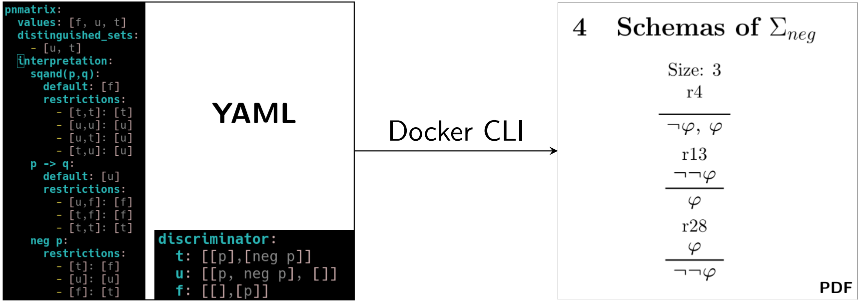

- What about non-monadic cases?