$$(e_1)\ t \approx t \quad (e_2)\ \frac{t \approx u}{u \approx t} \quad (e_3)\ \frac{t \approx u \quad u \approx v}{t \approx v} \quad (e_4)\ \frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

$$(e_5)\ \frac{t \approx u}{\mu.t \approx \mu.u} \quad (e_6)\ \frac{t \approx u \quad t' \approx u'}{t + t' \approx u + u'}$$

$$(e_7)\ \frac{t \approx u \quad t' \approx u'}{f(t,t') \approx f(u,u')} \quad (e_8)\ \frac{t \approx u \quad t' \approx u'}{t \parallel t' \approx u \parallel u'} \ .$$

Table 1. The rules of equational logic

A Equational logic

In Table 1 we report the rules of equational logic over CCS_f. As in operational semantics, they allow us to infer equations by proceeding inductively over the structure of terms. Let $\mathcal{E}$ be a sound set of axioms. Rules (e_1)-(e_4) are common for all process languages and they ensure that $\mathcal{E}$ is closed with respect to reflexivity, symmetry, transitivity and substitution, respectively. Rules (e_5)-(e_8) are tailored for CCS_f and they ensure the closure of $\mathcal{E}$ under CCS_f contexts. They are therefore referred to as the *congruence rules*. Briefly, rule (e_5) is the rule for prefixing, rule (e_6) deals with the nondeterministic choice operator. Rules (e_7) and (e_8) ensure, respectively, that the binary operator f and the parallel composition operator preserve the equivalence of terms.

Without loss of generality one may assume that substitutions happen first in equational proofs, i.e., that the rule

$$\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

may only be used when $(t \approx u) \in \mathcal{E}$. In this case $\sigma(t) \approx \sigma(u)$ is called a *substitution instance* of an axiom in $\mathcal{E}$. Moreover, by postulating that for each axiom in $\mathcal{E}$ also its symmetric counterpart is present in $\mathcal{E}$, one may assume that applications of symmetry happen first in equational proofs, i.e., that the rule

$$\frac{t \approx u}{u \approx t}$$

is never used in equational proofs. In the remainder of Appendix, we shall always tacitly assume that equational axiom systems are closed with respect to symmetry.

B Proofs of the results in Section 3

B.1 Proof of Lemma 3.3

Proof of Lemma 3.3. Statements 1 and 2 are trivial because the equation

$$x \| y \approx \mathbf{0}$$

is not sound modulo bisimilarity.

Let us focus now on the proof for statement 3. To this end, assume, towards a contradiction, that f has a rule of the form

$$f(x_1, x_2) \xrightarrow{\mu} t(x_1, x_2) \ ,$$

for some action μ and term t. This rule can be used to derive that

$$f(\mathbf{0}, \mathbf{0}) \xrightarrow{\mu} t(\mathbf{0}, \mathbf{0}) \ .$$

Since the set J on the right-hand side of (3) is non-empty by statement 1, the term $f(\mathbf{0}, \mathbf{0})$ occurs as a summand of $t_J(\mathbf{0}, \mathbf{0})$. It follows that

$$t_J(\mathbf{0}, \mathbf{0}) \xrightarrow{\mu} t(\mathbf{0}, \mathbf{0}) \ .$$

Therefore,

$$\mathbf{0} \parallel \mathbf{0} \leftrightarrow \mathbf{0} \not\leftrightarrow t_J(\mathbf{0}, \mathbf{0}) \ ,$$

contradicting our Assumption 2.

Finally, we deal with statement 4. Assume, towards a contradiction, that $f(x, x)$, say, is a summand of t_J. Since $a \parallel \mathbf{0} \xrightarrow{a} \mathbf{0} \parallel \mathbf{0} \leftrightarrow \mathbf{0}$ and equation (3) holds modulo bisimulation equivalence, there is a closed term p such that

$$t_J(a, \mathbf{0}) \xrightarrow{a} p \text{ and } p \leftrightarrow \mathbf{0} \ .$$

This means that there is a summand $f(z_1, z_2)$ of t_J such that

$$f(p_1, p_2) \xrightarrow{a} p \ ,$$

where, for $i \in \{1, 2\}$,

$$p_i = \begin{cases} a & \text{if } z_i = x \ , \\ \mathbf{0} & \text{if } z_i = y \ . \end{cases}$$

The transition $f(p_1, p_2) \xrightarrow{a} p$ must be provable using some de Simone rule ρ for f (see Equation (1) in Definition 3.1). Such a rule has some premise by Lemma 3.3(3), and each such premise must have the form $x_1 \xrightarrow{\mu} y_1$ or $x_2 \xrightarrow{\mu} y_2$, for some action μ. If both z_1 and z_2 are y then $p_1 = p_2 = \mathbf{0}$, and none of those premises can be met. Therefore at least one of z_1 and z_2 in the summand $f(z_1, z_2)$ is x. Moreover, if $x_i \xrightarrow{\mu} y_i$ $(i \in \{1, 2\})$ is a premise of ρ, then $z_i = x$ and $\mu = a$ (or else the premise could not be met). So the rule ρ can have one of the following three forms:

$$\frac{x_1 \xrightarrow{a} y_1}{f(x_1, x_2) \xrightarrow{a} t_1(y_1, x_2)} \qquad \frac{x_2 \xrightarrow{a} y_2}{f(x_1, x_2) \xrightarrow{a} t_2(x_1, y_2)}$$

$$\frac{x_1 \xrightarrow{a} y_1 \quad x_2 \xrightarrow{a} y_2}{f(x_1, x_2) \xrightarrow{a} t_3(y_1, y_2)}$$

for some terms t_1, t_2 and t_3. We now proceed to argue that the existence of each of these rules contradicts the soundness of Equation (3) modulo bisimulation equivalence.

If ρ has the form

$$\frac{x_1 \xrightarrow{a} y_1 \quad x_2 \xrightarrow{a} y_2}{f(x_1, x_2) \xrightarrow{a} t_3(y_1, y_2)}$$

then $z_1 = z_2 = x$ and

$$f(a, a) \xrightarrow{a} p \ .$$

Since the term $f(a, a)$ is a summand of $t_J(a, a)$, it follows that

$$t_J(a, a) \xrightarrow{a} p$$

also holds. However, this contradicts the soundness of equation (3) because, for each transition $a \parallel a \xrightarrow{a} q$, we have that $q \leftrightarrow a \not\leftrightarrow \mathbf{0} \leftrightarrow p$.

Assume now, without loss of generality, that ρ has the form

$$\frac{x_1 \xrightarrow{a} y_1}{f(x_1, x_2) \xrightarrow{a} t_1(y_1, x_2)}$$

Using this rule, we can infer that

$$f(a, a) \xrightarrow{a} t_1(\mathbf{0}, a) \ .$$

Since $f(x, x)$ is a summand of t_J by our assumption, the term $f(a, a)$ is a summand of $t_J(a, \mathbf{0})$. Hence,

$$t_J(a, \mathbf{0}) \xrightarrow{a} t_1(\mathbf{0}, a)$$

also holds. As equation (3) holds modulo bisimulation equivalence, we have that

$$a \parallel \mathbf{0} \leftrightarrow t_J(a, \mathbf{0}) \ .$$

Therefore $t_1(\mathbf{0}, a) \leftrightarrow \mathbf{0}$, because $a \parallel \mathbf{0} \xrightarrow{a} \mathbf{0} \parallel \mathbf{0}$ is the only transition afforded by the term $a \parallel \mathbf{0}$. Observe now that

$$t_J(a, a) \xrightarrow{a} t_1(\mathbf{0}, a) \leftrightarrow \mathbf{0} \ .$$

also holds. However, this contradicts the soundness of equation (3) as above because, for each transition $a \parallel a \xrightarrow{a} q$, we have that $q \leftrightarrow a \not\leftrightarrow \mathbf{0} \leftrightarrow p$.

This proves that $f(x, x)$ is not a summand of t_J, which was to be shown. □

B.2 Proof of Proposition 3.4

Proof of Proposition 3.4. If J is a singleton, then, since $\parallel$ is commutative modulo bisimulation equivalence, the equation

$$x \parallel y \approx f(x, y)$$

holds modulo bisimilarity. Therefore the result follows from the nonexistence of a finite equational axiomatisation for CCS proven by Moller in [23, 25]. □

B.3 Proof of Lemma 3.6

Proof of Lemma 3.6. By structural induction on closed terms. For all of the standard CCS operators, it is well known that the depth of closed terms can be characterized inductively thus:

$$\begin{aligned} depth(\mathbf{0}) &= 0 \\ depth(\mu p) &= 1 + depth(p) \\ depth(p + q) &= \max\{depth(p), depth(q)\} \\ depth(p \| q) &= depth(p) + depth(q) \ . \end{aligned}$$

So the depth of a closed term of the form μp, $p + q$ or $p \| q$ is finite, if so are the depths of p and q.

Consider now a closed term of the form $f(p, q)$. Since bisimilar terms have the same depth and, by the proviso of the lemma, Equation (4) holds modulo bisimulation equivalence, we have that

$$depth(f(p, q)) \leq depth(f(p, q) + f(q, p)) = depth(p \| q) \ .$$

It follows that $depth(f(p, q))$ is finite, if so are the depths of p and q. □

C Proofs of the results in Section 4

C.1 Proof of Lemma 4.1

Proof of Lemma 4.1. We only detail the proof for statement 1. (The proof for statement 2 follows similar lines, and is left to the reader.)

Assume, towards a contradiction, that $\mu = \tau$ and the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ of ρ has some form that differs from those in the statement. Then the set of premises of ρ has one of the following two forms:

- $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1, 2\}$ and $\alpha \in \{a, \bar{a}\}$, or
- $\{x_1 \xrightarrow{\mu_1} y_1, x_2 \xrightarrow{\mu_2} y_2\}$ for some $\mu_1, \mu_2 \in \{a, \bar{a}, \tau\}$ such that
  - either $\mu_1 = \tau$ or $\mu_2 = \tau$, or
  - $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$.

We now proceed to argue that the existence of either of these rules for f contradicts the soundness of Equation (4).

- Assume that the set of premises of ρ has the form $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1, 2\}$ and $\alpha \in \{a, \bar{a}\}$. In this case, we can use that rule to prove the existence of the transition
  $$f(\alpha, \mathbf{0}) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \text{ or } f(\mathbf{0}, \alpha) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ ,$$
  depending on whether $i = 1$ or $i = 2$. Therefore
  $$f(\alpha, \mathbf{0}) + f(\mathbf{0}, \alpha) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0})$$
  also holds. However, the existence of this transition immediately contradicts the soundness of Equation (4) modulo bisimulation equivalence because $\alpha \parallel \mathbf{0}$ affords no τ-transition.
- Assume that the set of premises of ρ has the form $\{x_1 \xrightarrow{\mu_1} y_1, x_2 \xrightarrow{\mu_2} y_2\}$ for some $\mu_1, \mu_2 \in \{a, \bar{a}, \tau\}$ such that
  - either $\mu_1 = \tau$ or $\mu_2 = \tau$, or
  - $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$.

  In the this case, we can use that rule to prove the existence of the transition
  $$f(\mu_1, \mu_2) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ .$$
  Therefore
  $$f(\mu_1, \mu_2) + f(\mu_2, \mu_1) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0})$$
  also holds. By the soundness of Equation (4), we have that
  $$\mu_1 \| \mu_2 \leftrightarrow f(\mu_1, \mu_2) + f(\mu_2, \mu_1) \ .$$
  Hence $\mu_1 \| \mu_2 \xrightarrow{\tau} p$ for some p such that $p \leftrightarrow t(\mathbf{0}, \mathbf{0})$. If $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$, then the above transition cannot exist, because $\alpha \parallel \alpha$ affords no τ-transition. This immediately contradicts the soundness of Equation (4) modulo bisimulation equivalence. We therefore

proceed with the proof by assuming that at least one of μ_1 and μ_2 is τ. In this case, we have that $\mu_1 \parallel \mu_2 \xrightarrow{\tau} p$ implies that $p \mathrel{\underline{\leftrightarrow}} \mu_1$ and $\mu_2 = \tau$, or $p \mathrel{\underline{\leftrightarrow}} \mu_2$ and $\mu_1 = \tau$. Assume, without loss of generality, that $\mu_1 = \tau$ and

$$t(\mathbf{0},\mathbf{0}) \mathrel{\underline{\leftrightarrow}} \mu_2 \ . \tag{8}$$

Pick now an action $\alpha \neq \mu_2$. (Such an action exists as we have three actions in our language.) The soundness of Equation (4) yields that

$$\tau \parallel (\mu_2 + \alpha) \mathrel{\underline{\leftrightarrow}} f(\tau, \mu_2 + \alpha) + f(\mu_2 + \alpha, \tau) \ .$$

Using the rule for f we assumed we had and the rules for $+$, we can prove the existence of the transition

$$f(\tau, \mu_2 + \alpha) + f(\mu_2 + \alpha, \tau) \xrightarrow{\tau} t(\mathbf{0},\mathbf{0}) \ .$$

Since the source of the above transition is bisimilar to $\tau \parallel (\mu_2 + \alpha)$, there must be a term p such that $\tau \parallel (\mu_2 + \alpha) \xrightarrow{\tau} p$ and $p \mathrel{\underline{\leftrightarrow}} t(\mathbf{0},\mathbf{0})$. By Equation (8), this term p can only be $\tau \parallel \mathbf{0}$. In fact,

$$t(\mathbf{0},\mathbf{0}) \mathrel{\underline{\leftrightarrow}} \mu_2 \not\mathrel{\underline{\leftrightarrow}} (\mu_2 + \alpha) \mathrel{\underline{\leftrightarrow}} \mathbf{0} \parallel (\mu_2 + \alpha) \ ,$$

for we chose $\alpha \in \{a, \bar{a}\}$ different from μ_2. We have therefore proven that $\mu_1 = \mu_2 = \tau$.

We are now ready to reach the promised contradiction to the soundness of Equation (4). In fact, consider the term $f(\tau + a, \tau + a)$. Using the rule for f we assumed we had, we can again prove the existence of the transition

$$f(\tau + a, \tau + a) \xrightarrow{\tau} t(\mathbf{0},\mathbf{0}) \ .$$

By Equation (8) and our observation that $\mu_2 = \tau$, the term $t(\mathbf{0},\mathbf{0})$ is bisimilar to τ. On the other hand, $(\tau + a) \parallel (\tau + a) \xrightarrow{\tau} p$ implies that $p \mathrel{\underline{\leftrightarrow}} (\tau + a) \not\mathrel{\underline{\leftrightarrow}} \tau$, contradicting the soundness of Equation (4) modulo bisimulation equivalence.

□

C.2 Proof of Lemma 4.2

Proof of Lemma 4.2. We first argue that f must have a rule of the form (5) for some $\alpha \in \{a, \bar{a}\}$ and term t. To this end, assume, towards a contradiction, that f has no such rule. Observe that the term $a \parallel \bar{a}$ affords the transition

$$a \parallel \bar{a} \xrightarrow{\tau} \mathbf{0} \parallel \mathbf{0} \ .$$

However, neither the term $f(a, \bar{a})$ nor the term $f(\bar{a}, a)$ affords a τ-transition. In fact, using our assumption that f has no rule of the form (5) and Lemma 4.1(1), each rule for f with a τ-transition as a consequent must have the form

$$\frac{x_i \xrightarrow{\tau} y_i}{f(x_1, x_2) \xrightarrow{\tau} t}$$

for some $i \in \{1, 2\}$ and term t. Such a rule cannot be used to infer a transition from $f(a, \bar{a})$ or $f(\bar{a}, a)$. It follows that

$$a \parallel \bar{a} \not\mathrel{\underline{\leftrightarrow}} f(a, \bar{a}) + f(\bar{a}, a) \ ,$$

contradicting the soundness of Equation (4). Therefore f must have a rule of the form (5).

We now proceed to argue that $t(x, y)$ is bisimilar to $x \parallel y$, for each rule of the form (5) for f. Pick a rule for f of the form (5). We shall argue that

$$p \parallel q \mathrel{\underline{\leftrightarrow}} t(p, q) \ ,$$

for all closed CCS$_f$ terms p and q. To this end, consider the terms $\alpha.p \parallel \bar{\alpha}.q$ and $f(\alpha.p, \bar{\alpha}.q) + f(\bar{\alpha}.q, \alpha.p)$. Using rule (5) and the rules for $+$, we have that

$$f(\alpha.p, \bar{\alpha}.q) + f(\bar{\alpha}.q, \alpha.p) \xrightarrow{\tau} t(p, q) \ .$$

By the soundness of Equation (4), we have that

$$\alpha.p \parallel \bar{\alpha}.q \mathrel{\underline{\leftrightarrow}} f(\alpha.p, \bar{\alpha}.q) + f(\bar{\alpha}.q, \alpha.p) \ .$$

Therefore there is a closed term r such that $\alpha.p \parallel \bar{\alpha}.q \xrightarrow{\tau} r$ and $r \mathrel{\underline{\leftrightarrow}} t(p, q)$. Note now that the only τ-transition afforded by $\alpha.p \parallel \bar{\alpha}.q$ is

$$\alpha.p \parallel \bar{\alpha}.q \xrightarrow{\tau} p \parallel q \ .$$

Therefore $r = p \parallel q \mathrel{\underline{\leftrightarrow}} t(p, q)$, which was to be shown. □

C.3 Proof of Lemma 4.3

Proof of Lemma 4.3. Let $\mu \in \{a, \bar{a}, \tau\}$. We first argue that f must have a rule of the form (6) or (7) for some term t. To this end, assume, towards a contradiction, that f has no such rules. Observe that the term $\mu \parallel \mathbf{0}$ affords the transition

$$\mu \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0} \ .$$

However, neither the term $f(\mu, \mathbf{0})$ nor the term $f(\mathbf{0}, \mu)$ affords a μ-transition. In fact, using our assumption that f has no rule of the form (6) or (7), Lemma 4.1 yields that

- either f has no rule with a μ-transition as a consequent,
- or $\mu = \tau$, and each rule for f with a τ-transition as a consequent has the form

  $$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1, x_2) \xrightarrow{\tau} t(y_1, y_2)}$$

  for some $\alpha \in \{a, \bar{a}\}$.

In the latter case, such a rule cannot be used to infer a transition from $f(\mu, \mathbf{0})$ or $f(\mathbf{0}, \mu)$. It follows that

$$\mu \parallel \mathbf{0} \not\mathrel{\underline{\leftrightarrow}} f(\mu, \mathbf{0}) + f(\mathbf{0}, \mu) \ ,$$

contradicting the soundness of equation (4). Therefore f must have a rule of the form (6) or (7) for each action μ.

To conclude the proof we need to show that for each rule of the form (6) or (7) the target term $t(x, y)$ is bisimilar to $x \parallel y$. For simplicity, we expand the proof only for the case of rules of the form (6). The proof for rules of the form (7) follows by the same reasoning.

We proceed by a case analysis over the structure of $t(y_1, x_2)$, which, we recall, under assumption 3 can be either a variable in $\{y_1, x_2\}$ or a term of the form $g(y_1, x_2)$ for some CCS$_f$ operator g. Our aim is to show that the only possibility is to have $t(y_1, x_2) = y_1 \parallel x_2$, as any other process term would invalidate one of our simplifying assumptions.

- Case t is a variable in $\{y_1, x_2\}$. We can distinguish two cases, according to which variable is considered:
  - $t = y_1$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (6) we can infer that $f(p, p) \xrightarrow{\mu} \mathbf{0}$, and thus $f(p, p) + f(p, p) \xrightarrow{\mu} \mathbf{0}$. However, there is no μ-transition from $p \parallel p$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel p \xrightarrow{\mu} q$, then q is a process that will always be able to perform a second μ-transition. Hence, we would have $p \parallel p \,\underline{\not\leftrightarrow}\, f(p, p) + f(p, p)$, thus contradicting the soundness of Equation (4).
  - $t = x_2$. Consider process $p = \mu.\mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mu.\mathbf{0}$, from an application of rule (6) we can infer that $f(p, \mathbf{0}) \xrightarrow{\mu} \mathbf{0}$ and thus $f(p, \mathbf{0}) + f(\mathbf{0}, p) \xrightarrow{\mu} \mathbf{0}$. However, there is no μ-transition from $p \parallel \mathbf{0}$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel \mathbf{0} \xrightarrow{\mu} q$, then q is a process that will always be able to perform a second μ-transition. Hence we would have $p \parallel \mathbf{0} \,\underline{\not\leftrightarrow}\, f(p, \mathbf{0}) + f(\mathbf{0}, p)$, thus contradicting the soundness of Equation (4).
- Case t is a term of the form $g(y_1, x_2)$ for some CCS$_f$ operator g. We can distinguish three cases, according to which operator is used:
  - g is the prefix operator. We can distinguish two cases, according to which variable of the rule occurs in t:
    * $t = \nu.y_1$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (6) we can infer that $f(p, \mathbf{0}) \xrightarrow{\mu} \nu.\mathbf{0} \xrightarrow{\nu} \mathbf{0}$, and thus $f(p, \mathbf{0}) + f(\mathbf{0}, p) \xrightarrow{\mu}\xrightarrow{\nu} \mathbf{0}$. However, $p \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0} \stackrel{\nu}{\not\rightarrow}$. Hence, we would have that $p \parallel \mathbf{0} \,\underline{\not\leftrightarrow}\, f(p, \mathbf{0}) + f(\mathbf{0}, p)$, thus contradicting the soundness of Equation (4).
    * $t = \nu.x_2$. This case is analogous to the previous one.
  - g is the nondeterministic choice operator and thus $t = y_1 + x_2$. Consider processes $p = \mu.\mu.\mathbf{0}$ and $q = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} q$, from an application of rule (6) we can infer that $f(p, q) \xrightarrow{\mu} q + q \xrightarrow{\mu} \mathbf{0}$, and thus $f(p, q) + f(q, p) \xrightarrow{\mu}\xrightarrow{\mu} \mathbf{0}$. However, there is no process p' such that $p \parallel q \xrightarrow{\mu}\xrightarrow{\mu} p'$ and $p' \,\underline{\leftrightarrow}\, \mathbf{0}$, since p' can always perform an additional μ-transition. Hence, we would have $p \parallel q \,\underline{\not\leftrightarrow}\, f(p, q) + f(q, p)$, which contradicts the soundness of Equation (4).
  - $g = f$. First of all, we notice that in this case we can infer that f cannot have both types of rules of the form (5), and both types of rules, (6) and (7), for all actions. In fact, if this was the case, due to Lemmas 4.2 and 4.3, the set of rules defining the behaviour of $f(x_1, x_2)$ would be

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1, x_2) \xrightarrow{\mu} f(y_1, x_2)} \qquad \frac{x_2 \xrightarrow{\mu} y_2}{f(x_1, x_2) \xrightarrow{\mu} f(x_1, y_2)}$$

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1, x_2) \xrightarrow{\tau} y_1 \parallel y_2} \qquad \frac{x_1 \xrightarrow{\bar{\alpha}} y_1 \quad x_2 \xrightarrow{\alpha} y_2}{f(x_1, x_2) \xrightarrow{\tau} y_1 \parallel y_2}$$

with $\mu \in \{a, \bar{a}, \tau\}$ and $\alpha \in \{a, \bar{a}\}$. Clearly, operator f would then be a mere renaming of the parallel composition operator. In particular, as a one-to-one correspondence between the rules for f and those for $\parallel$ could be established, we have that $f(x, y)$ would be *bisimilar under formal hypothesis* to $x \parallel y$ (see [14, Definition 1.10]) and therefore, by [14, Theorem 1.12], we could directly conclude that $f(x, y) \approx x \parallel y$ for all x, y. However, this would contradict the fact that $x \parallel y \not\approx f(x, y)$. Let us now consider the case of an operator f having both types of rules, (6) and (7), and only one type of rules of the form (5), say the rule

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1, x_2) \xrightarrow{\tau} y_1 \parallel y_2}.$$

We proceed towards contradiction and distinguish two subcases, according to whether the order of the arguments is preserved or not by the rules of type (6) with label α. Similar arguments would allow us to deal with rules of type (7).

* The target of the rule of type (6) with label α is $f(y_1, x_2)$. Then $f(\alpha.\bar{\alpha}, \alpha) \xrightarrow{\alpha} f(\bar{\alpha}, \alpha) \,\underline{\leftrightarrow}\, \bar{\alpha} + \alpha$. However, there is no α-transition from $\alpha.\bar{\alpha} \| \alpha$ to a process bisimilar to $\bar{\alpha} + \alpha$, thus contradicting the soundness of Equation (4).
* The target of the rule of type (6) with label α is $f(x_2, y_1)$. Then $f(\alpha.\alpha, \bar{\alpha}.\alpha) \xrightarrow{\alpha} f(\bar{\alpha}.\alpha, \alpha) \stackrel{\tau}{\not\rightarrow}$. However, whenever $\alpha.\alpha \| \bar{\alpha}.\alpha$ performs an α-transition, it always reaches a process that can perform a τ-move. This contradicts the soundness of Equation (4).

Finally, let us deal with the case in which there is at least one action $\mu \in \{a, \bar{a}, \tau\}$ for which only one rule among (6) and (7) is available. According to our current simplifying assumptions, let (6) be the available rule for f with label μ. We can distinguish two cases, according to the occurrences of the variables of the rule in t:

* $t = f(y_1, x_2)$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (6) we can infer that $f(p, p) \xrightarrow{\mu} f(\mathbf{0}, a)$, and thus $f(p, p) + f(p, p) \xrightarrow{\mu} f(\mathbf{0}, p)$, with $f(\mathbf{0}, p) \,\underline{\leftrightarrow}\, \mathbf{0}$, since only rules of the form (6) are available with respect to action μ. However, there is no μ-transition from $p \parallel p$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel$

$p \xrightarrow{\mu} q$ then q is a process that will always be able to perform a second μ-transition. Hence, we would have $p \parallel p \not\underline{\leftrightarrow} f(p, p) + f(p, p)$, thus contradicting the soundness of Equation (4).

∗ $t = f(x_2, y_1)$. Consider process $p = \mu.\mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mu.\mathbf{0}$, and only rules of the from (6) are available with respect to action μ, we can infer that $f(p, \mathbf{0}) \xrightarrow{\mu} f(\mathbf{0}, \mu.\mathbf{0}) \stackrel{\mu}{\not\rightarrow}$ and $f(\mathbf{0}, p) \not\rightarrow$, which means that $f(p, \mathbf{0}) + f(\mathbf{0}, p)$ cannot perform two μ-transitions in a row. However, we have that $p \parallel \mathbf{0} \xrightarrow{\mu} \mu.\mathbf{0} \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0}$. Hence, we would have $p \parallel \mathbf{0} \not\underline{\leftrightarrow} f(p, \mathbf{0}) + f(\mathbf{0}, p)$, thus contradicting the soundness of Equation (4).

□

C.4 Proof of Proposition 4.5

Proof of Proposition 4.5. We argue that the relation

$$\mathcal{B} = \{(p \parallel q, f(p, q) + f(q, p)) \mid p, q \text{ closed terms}\} \cup \underline{\leftrightarrow}$$

is a bisimulation. To this end, pick closed terms p, q. Now show, using the information on the rules for f given in the proviso of the proposition, that, for each action μ and closed term r,

- whenever $p \parallel q \xrightarrow{\mu} r$, there is a term r' that is equal to r up to commutativity of $\parallel$ such that $f(p, q) + f(q, p) \xrightarrow{\mu} r'$, and
- whenever $f(p, q) + f(q, p) \xrightarrow{\mu} r$, there is a term r' that is equal to r up to commutativity of $\parallel$ such that $p \parallel q \xrightarrow{\mu} r'$.

The claim follows because $\parallel$ is commutative modulo $\underline{\leftrightarrow}$. □

D Proofs of results in Section 5

D.1 Proof of Lemma 5.2

Proof of Lemma 5.2. Assume, towards a contradiction, that f is distributive in both arguments with respect to summation. Then, using Equation (4), we have that:

$$\begin{aligned} (x + y) \parallel z &\approx f(x + y, z) + f(z, x + y) \\ &\approx f(x, z) + f(y, z) + f(z, x) + f(z, y) \\ &\approx (x \parallel z) + (y \parallel z) \ . \end{aligned}$$

However, this is a contradiction because, as is well known, the equation

$$(x + y) \parallel z \approx (x \parallel z) + (y \parallel z)$$

is not sound in bisimulation semantics. For example, our readers can easily verify that

$$(a + \tau) \parallel a \not\underline{\leftrightarrow} (a \parallel a) + (\tau \parallel a) \ .$$

□

D.2 Proof of Lemma 5.3

Proof of Lemma 5.3. We prove the two implications separately.

($\Leftarrow$) This case follows by similar arguments to those used in the proof of [2, Lemma 4.3] and it is therefore omitted.

($\Rightarrow$) Assume that f distributes with respect to $+$ in some argument. We recall that by Lemmas 4.2 and 4.3 for each action μ at least one between L^f_μ and R^f_μ must hold. We aim to prove that either L^f_μ holds for all actions μ and none of the R^f_μ does, or vice versa. Indeed, suppose towards a contradiction that there are rules satisfying L^f_μ and R^f_ν for some actions μ and ν. Then

- $f(\tau + \tau^2, \nu)$ is not bisimilar to $f(\tau, \nu) + f(\tau^2, \nu)$, because the validity of R^f_ν allows us to prove that $f(\tau + \tau^2, \nu) \xrightarrow{\nu} (\tau + \tau^2) \| \mathbf{0}$ and $f(\tau, \nu) + f(\tau^2, \nu)$ cannot match that transition up to bisimilarity.
- $f(\mu, \tau + \tau^2)$ is not bisimilar to $f(\mu, \tau) + f(\mu, \tau^2)$, because the validity of L^f_μ allows us to prove that $f(\mu, \tau + \tau^2) \xrightarrow{\mu} \mathbf{0} \| (\tau + \tau^2)$ and $f(\mu, \tau) + f(\mu, \tau^2)$ cannot match that transition up to bisimilarity.

□

E Proofs of results in Section 6

E.1 Proof of Proposition 6.2

Proof of Proposition 6.2. We prove the three statements separately.

- Proof of Statement 1. Assume that $\mathcal{E} \vdash t \approx u$. We shall argue that $\widehat{\mathcal{E}}$ proves the equation $\hat{t} \approx \hat{u}$ by induction on the depth of the proof of $t \approx u$ from $\mathcal{E}$. We proceed by a case analysis on the last rule used in the proof. Below we only consider the two most interesting cases in this analysis.
  - Case $\mathcal{E} \vdash t \approx u$, because $\sigma(t') = t$ and $\sigma(u') = u$ for some equation $(t' \approx u') \in \mathcal{E}$. Note, first of all, that, by the definition of $\widehat{\mathcal{E}}$, the equation $\widehat{t'} \approx \widehat{u'}$ is contained in $\widehat{\mathcal{E}}$. Observe now that
  $$\hat{t} = \hat{\sigma}(\widehat{t'}) \text{ and } \hat{u} = \hat{\sigma}(\widehat{u'}) \ ,$$
  where $\hat{\sigma}$ is the substitution mapping each variable x to the term $\widehat{\sigma(x)}$. It follows that the equation $\hat{t} \approx \hat{u}$ can be proven from the axiom system $\widehat{\mathcal{E}}$ by instantiating the equation $\widehat{t'} \approx \widehat{u'}$ with the substitution $\hat{\sigma}$, and we are done.
  - Case $\mathcal{E} \vdash t \approx u$, because $t = t_1 \| t_2$ and $u = u_1 \| u_2$ for some t_i, u_i $(i = 1, 2)$ such that $\mathcal{E} \vdash t_i \approx u_i$ $(i = 1, 2)$. Using the inductive hypothesis twice, we have that $\widehat{\mathcal{E}} \vdash \widehat{t_i} \approx \widehat{u_i}$ $(i = 1, 2)$. Therefore, using substitutivity, $\widehat{\mathcal{E}}$ proves that

$$\hat{t} = f(\widehat{t_1}, \widehat{t_2}) + f(\widehat{t_2}, \widehat{t_1}) \approx f(\widehat{u_1}, \widehat{u_2}) + f(\widehat{u_2}, \widehat{u_1}) = \hat{u} \ ,$$

which was to be shown.

The remaining cases are simpler, and we leave the details to the reader.

- PROOF OF STATEMENT 2. Assume that t and u are two bisimilar terms in the language CCS_f^-. We shall argue that $\widehat{\mathcal{E}}$ proves the equation $t \approx u$. To this end, we begin by noting that the equation $t \approx u$ also holds in the algebra of CCS_f terms modulo bisimulation. In fact, for each term v in the language CCS_f and closed substitution σ mapping variables to CCS_f terms, we have that
$$\sigma(v) \leftrightarrow \hat{\sigma}(v) \ ,$$
where the substitution $\hat{\sigma}$ is defined as above.
Since $\mathcal{E}$ is complete for bisimilarity over CCS_f by our assumptions, it follows that $\mathcal{E}$ proves the equation $t \approx u$. Therefore, by statement 1 of the proposition, we have that $\widehat{\mathcal{E}}$ proves the equation $\hat{t} \approx \hat{u}$. The claim now follows because $\hat{t} = t$ and $\hat{u} = u$.
- PROOF OF STATEMENT 3. This is an immediate consequence of statement 2 because $\widehat{\mathcal{E}}$ has the same cardinality of $\mathcal{E}$, and is therefore finite, if so is $\mathcal{E}$.

□

E.2 Proof of Lemma 6.4

Before proceeding to the technical proof, we observe the following:

Remark 3. *Whenever a process term t has neither $\mathbf{0}$ summands nor factors then we can assume that, for some finite non-empty index set I, $t = \sum_{i \in I} t_i$ for some terms t_i such that none of them has $+$ as head operator and moreover, none of them has $\mathbf{0}$ summands nor factors.*

Proof of Lemma 6.4. The "if" implication is an immediate consequence of the soundness of the equations A4 and F1 with respect to $\leftrightarrow$. To prove the "only if" implication, define, first of all, the collection NIL of CCS_f^- terms as the set of terms generated by the following grammar:
$$t ::= \mathbf{0} \mid t + t \mid f(t, u) \ ,$$
where u is an arbitrary CCS_f^- term. We claim that:

Claim 1. *Each CCS_f^- term t is bisimilar to $\mathbf{0}$ if, and only if, $t \in NIL$.*

Using this claim and structural induction on $t \in \mathrm{NIL}$, it is a simple matter to show that if $t \leftrightarrow \mathbf{0}$, then $t \approx \mathbf{0}$ is provable using axioms A0 and F0 from left to right, which was to be shown.

To complete the proof, it therefore suffices to show the above claim. To establish the "if" implication in the statement of the claim, one proves, using structural induction on t and the congruence properties of bisimilarity, that if $t \in \mathrm{NIL}$, then $\sigma(t) \leftrightarrow \mathbf{0}$ for every closed substitution σ. To show the "only if" implication, we establish the contrapositive statement, viz. that if $t \notin \mathrm{NIL}$, then $\sigma(t) \not\leftrightarrow \mathbf{0}$ for some closed substitution σ. To this end, it suffices only to show, using structural induction on t, that if $t \notin \mathrm{NIL}$, then $\sigma_a(t) \xrightarrow{\mu}$ for some action $\mu \in \{a, \bar{a}, \tau\}$, where σ_a is the closed substitution mapping each variable to the closed term $a\mathbf{0}$. The details of this argument are not hard, and are therefore left to the reader. □

E.3 Proof of Proposition 6.5

We present here the definitions and preliminary results necessary to prove (the full version of) Proposition 6.5 (see Proposition E.6 below).

Definition E.1. For each CCS_f^- term t, we define $t/\mathbf{0}$ thus:
$$\mathbf{0}/\mathbf{0} \;=\; \mathbf{0} \quad x/\mathbf{0} \;=\; x \quad \mu t/\mathbf{0} \;=\; \mu(t/\mathbf{0})$$
$$(t+u)/\mathbf{0} \;=\; \begin{cases} u/\mathbf{0} & \text{if } t \leftrightarrow \mathbf{0} \\ t/\mathbf{0} & \text{if } u \leftrightarrow \mathbf{0} \\ (t/\mathbf{0}) + (u/\mathbf{0}) & \text{otherwise} \end{cases}$$
$$f(t,u)/\mathbf{0} \;=\; \begin{cases} \mathbf{0} & \text{if } t \leftrightarrow \mathbf{0} \\ t/\mathbf{0} & \text{if } u \leftrightarrow \mathbf{0} \\ f(t/\mathbf{0}, u/\mathbf{0}) & \text{otherwise} \end{cases}$$

Intuitively, $t/\mathbf{0}$ is the term that results by removing *all* occurrences of $\mathbf{0}$ as a summand or factor from t.

The following lemma, whose simple proof by structural induction on terms is omitted, collects the basic properties of the above construction.

Lemma E.2. *For each CCS_f^- term t, the following statements hold:*

1. *the equation $t \approx t/\mathbf{0}$ can be proven using the equations A0–A3, F0–F1, and therefore $t \leftrightarrow t/\mathbf{0}$;*
2. *the term $t/\mathbf{0}$ has no occurrence of $\mathbf{0}$ as a summand or factor;*
3. *$t/\mathbf{0} = t$, if t has no occurrence of $\mathbf{0}$ as a summand or factor;*
4. *$\sigma(t/\mathbf{0})/\mathbf{0} = \sigma(t)/\mathbf{0}$, for each substitution σ.*

Definition E.3. We say that a substitution σ is a $\mathbf{0}$-*substitution* iff $\sigma(x) \neq x$ implies that $\sigma(x) = \mathbf{0}$, for each variable x.

Definition E.4. Let $\mathcal{E}$ be an axiom system. We define the axiom system $\mathrm{cl}(\mathcal{E})$ thus:
$$\mathrm{cl}(\mathcal{E}) = \mathcal{E} \cup \{\sigma(t)/\mathbf{0} \approx \sigma(u)/\mathbf{0} \mid (t \approx u) \in \mathcal{E},\ \sigma \text{ a } \mathbf{0}\text{-substitution}\}$$
An axiom system $\mathcal{E}$ is *saturated* if $\mathcal{E} = \mathrm{cl}(\mathcal{E})$.

The following lemma collects some basic sanity properties of the closure operator $\mathrm{cl}(\cdot)$. (Note, in particular, that the application of $\mathrm{cl}(\cdot)$ to an axiom system preserves closure with respect to symmetry.)

Lemma E.5. *Let $\mathcal{E}$ be an axiom system. Then the following statements hold.*

1. $\mathrm{cl}(\mathcal{E}) = \mathrm{cl}(\mathrm{cl}(\mathcal{E}))$.
2. $\mathrm{cl}(\mathcal{E})$ *is finite, if so is* $\mathcal{E}$.
3. $\mathrm{cl}(\mathcal{E})$ *is sound, if so is* $\mathcal{E}$.
4. $\mathrm{cl}(\mathcal{E})$ *is closed with respect to symmetry, if so is* $\mathcal{E}$.
5. $\mathrm{cl}(\mathcal{E})$ *and* $\mathcal{E}$ *prove the same equations, if* $\mathcal{E}$ *contains the equations A0–A3, F0–F1.*

Proof: We limit ourselves to sketching the proofs of statements 1 and 5 in the lemma.

In the proof of statement 1, the only non-trivial thing to check is that the equation

$$\sigma(\sigma'(t)/\mathbf{0}))/\mathbf{0} \approx \sigma(\sigma'(u)/\mathbf{0}))/\mathbf{0}$$

is contained in $\mathrm{cl}(\mathcal{E})$, whenever $(t \approx u) \in \mathcal{E}$ and σ, σ' are $\mathbf{0}$-substitutions. This follows from Lemma E.2(4) because the collection of $\mathbf{0}$-substitutions is closed under composition.

To show statement 5, it suffices only to argue that each equation $t \approx u$ that is provable from $\mathrm{cl}(\mathcal{E})$ is also provable from $\mathcal{E}$, if $\mathcal{E}$ contains the equations A0–A3, F0–F1. This can be done by induction on the depth of the proof of the equation $t \approx u$ from $\mathrm{cl}(\mathcal{E})$, using Lemma E.2(1) for the case in which $t \approx u$ is a substitution instance of an axiom in $\mathrm{cl}(\mathcal{E})$. □

Notice that, in light of this result, the saturation of a finite axiom system that includes the equations A0–A3, F0–F1 results in an equivalent, finite collection of equations (Lemma E.5(2) and (5)).

We are now ready to state our counterpart of [23, Proposition 5.1.5].

Proposition E.6. *Assume that* $\mathcal{E}$ *is a saturated axiom system. Suppose furthermore that we have a closed proof from* $\mathcal{E}$ *of the closed equation* $p \approx q$. *Then replacing each term* r *in that proof with* $r/\mathbf{0}$ *yields a closed proof of the equation* $p/\mathbf{0} \approx q/\mathbf{0}$. *In particular, the proof from* $\mathcal{E}$ *of an equation* $p \approx q$, *where* p *and* q *are terms not containing occurrences of* $\mathbf{0}$ *as a summand or factor, need not use terms containing occurrences of* $\mathbf{0}$ *as a summand or factor.*

Proof: The proof follows the lines of that of [23, Proposition 5.1.5], and is therefore omitted. □

E.4 0 absorption for a non distributive f

In Section 5.2, we argued that the set of allowed rules for an operator f that does not distribute over summation in either argument has to include at least a rule of type (6) *and* at least one of type (7). We also notice that for an operator f having both types of rules for all actions we can distinguish two cases, according to which rules of type (5) are available: (i) If f has both rules of type (5), then it would be a mere rewriting of the parallel composition operator (see Appendix C.3, proof of Lemma 4.3). (ii) If f has only one rule of type (5), then one can observe that Moller's argument to the effect that bisimilarity is not finitely based over the fragment of CCS with action prefixing, nondeterministic choice and purely interleaving parallel composition, could be applied to f, yielding the desired negative result.

Hence, we can assume that there is an action $\mu \in \{a, \bar{a}, \tau\}$ such that f has only one rule, of type either (6) or (7), with μ as label. This asymmetry in the set of rules for f can cause some CCS_f^- term to behave as $\mathbf{0}$ when occurring in the scope of f, despite not being bisimilar to $\mathbf{0}$ at all.

Example E.7. Consider the term $t = f(a + \bar{a}.u, \tau)$, for some term u, and assume that f has only rules of type (6) with labels a and τ and only a rule of type (7) with label $\bar{a}$. One can easily check that, since the initial execution of the τ-move in the second argument is prevented by the rules for f, then the subterm $\bar{a}.u$ can never contribute to the behaviour of t. Thus, $t \leftrightarrow a.\tau$, even though $\bar{a}.u \not\leftrightarrow \mathbf{0}$ for each term u.

From a technical point of view, this implies that Lemmas 6.4 and E.2.1 no longer hold. In fact, one can always construct a term t of the form $t = f(\sum_{i=1}^{n} \mu.x_i, \sum_{j=1}^{m} \nu.y_j)$ for some $n, m \geq 0$, with μ, ν chosen according to the available set of rules for f, such that $t \leftrightarrow \mathbf{0}$. We conjecture that *since we are considering an operator* f *that does not distribute over summation in either of its arguments, the valid equations, modulo bisimilarity, of the form* $t \approx \mathbf{0}$ *cannot be proved by means of any finite, sound set of axioms.* Roughly speaking, this is due to the fact that no valid axiom can be established for a term of the form $f(\mu.x + z, \nu.y + w)$ in that the behaviour of the terms substituted for the variables z and w is crucial to determine that of a closed instantiation of the term.

Summarizing, this would imply that, in the case at hand, we cannot assume that we can use saturation to simplify the axiom systems and, moreover, the family of equations

$$f(\sum_{i=1}^{n} \mu.p_i, \sum_{j=1}^{m} \nu.q_j) \approx \mathbf{0} \qquad n, m \geq 0$$

for some processes p_i, q_j, could play the role of witness family of equations for our desired negative result. Unfortunately, the presence of two summations would force us to introduce a number of additional technical results that would make the proof of the negative results even heavier than it already is. Moreover, those supplementary results are not necessary to treat the case of the witness families that we are going to introduce in Sections 9–11 to obtain the proof of Theorem 5.1.

F Proofs of results in Section 7

F.1 Proof of Lemma 7.6

Proof of Lemma 7.6. The proof is by induction on the structure of t. The only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$, which we expand below. According to which rules are available for f with respect to α, we can distinguish three cases:

1. CASE ONLY L_α^f HOLDS. Then, $f(\sigma(t_1), \sigma(t_2)) \xrightarrow{\alpha} p$ can be inferred only from a transition of the form $\sigma(t_1) \xrightarrow{\alpha} p'$ for some closed term p' with $p = p' \| \sigma(t_2)$. By induction over the derivation of $\sigma(t_1) \xrightarrow{\alpha} p'$, and

considering that only L^f_α holds, we can then distinguish two cases:

- There is a term t' such that $t_1 \xrightarrow{\alpha} t'$ and $\sigma(t'_1) = p'$. As f has the rule of the form (6) for α we can immediately infer that $t \xrightarrow{\alpha} t' \| t_2$. Hence, by letting $t' = t'_1 \| t_2$, we obtain $t \xrightarrow{\alpha} t'$ and $\sigma(t') = p$.
- There are a variable x, a closed term q and a configuration c_1 such that $\sigma(x) \xrightarrow{\alpha} q$, $t_1 \xrightarrow{x_\mathrm{l}}_\alpha c_1$ with $\sigma[x_\mathrm{d} \mapsto q](c_1) = p'$. Hence, by applying the auxiliary rule (a_6) we can infer that $f(t_1, t_2) \xrightarrow{x_\mathrm{l}}_\alpha c_1 \| t_2$ and moreover, since x_d may occur only in c_1, we have $p = p' \| \sigma(t_2) = \sigma[x_\mathrm{d} \mapsto q](c_1 \| t_2)$.

2. Case only R^f_α holds. This case is analogous to the previous one (it is enough to switch the roles of t_1 and t_2 and consider x_r in place of x_l) and therefore omitted.
3. Case $L^f_\alpha \wedge R^f_\alpha$ holds. This case follows by noticing that $t \xrightarrow{x_\mathrm{b}}_\alpha$ can be inferred from both $t_1 \xrightarrow{x_\mathrm{b}}_\alpha$ and $t_2 \xrightarrow{x_\mathrm{b}}_\alpha$, and therefore the follows from the structure of the previous two cases, using rules (a_8) and (a_9).

$\square$

F.2 Proof of Lemma 7.7

Proof of Lemma 7.7. For simplicity of notation let $I = \{1, \dots, n\}$. Since there is a transition $\sum_{i \in I} \alpha.p_i + q \xrightarrow{\alpha} p_i$ for each $i \in I$, from $\sigma(t) \underline{\leftrightarrow} \sum_{i \in I} \alpha.p_i + q$ we get that $\sigma(t) \xrightarrow{\alpha} r_i$ with $r_i \underline{\leftrightarrow} p_i$, for all $i \in I$. Since n is greater than the size of t, we infer that Lemma 7.6.1 can be applied only to m such transitions, for some $m < n$, so that there are an index set $H \subset I$ (possibly empty) and CCS$_f$ terms t_h, for $h \in H$ such that $|H| = m$, $t \xrightarrow{\alpha} t_h$ and $\sigma(t_h) \underline{\leftrightarrow} p_h$. Notice that since $p_i \not\underline{\leftrightarrow} p_j$ for $i \neq j$ we get that the t_h are pairwise distinct. Let $J = I \setminus H$. For the remaining α-transitions $\sigma(t) \xrightarrow{\alpha} r_j$ for $j \in J$ we have that one among cases 2a–2c of Lemma 7.6 applies, according to which rules are available for f with respect to action α. Hence, we have that, for each $j \in J$ there are a variable x_j, a closed term q_j and a configuration c_j such that $\sigma(x_j) \xrightarrow{\alpha} q_j$, $t \xrightarrow{x_{j,\mathrm{w}}}_\alpha c_j$ and $\sigma[x_{j,\mathrm{d}} \mapsto q_j] = r_j$, where $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$ depends on the rules for f. Once again, since n is greater than the size of t there cannot be more than $|J| - 1$ distinct variables x_j occurring in t and causing such α-moves. Hence, there is at least one variable $x \in var(t)$ such that $\sigma(x) \underline{\leftrightarrow} \alpha.q_{j_1} + \alpha.q_{j_2} + q'$ for some $j_1 \neq j_2 \in J$ and closed term q'. $\square$

F.3 Unique prime decomposition

In the proof of our main results, we shall often make use of some notions from [22, 23]. These we now proceed to introduce for the sake of completeness and readability.

Definition F.1. A closed term p is *irreducible* if $p \underline{\leftrightarrow} q \| r$ implies $q \underline{\leftrightarrow} \mathbf{0}$ or $r \underline{\leftrightarrow} \mathbf{0}$, for all closed terms q, r. We say that p is *prime* if it is irreducible and is not bisimilar to $\mathbf{0}$.

For example, each term p of depth (respectively, norm) 1 is prime because every term of the form $q \| r$ that does not involve $\mathbf{0}$ factors has depth (resp., norm) at least 2, and thus cannot be bisimilar to p.

The following lemma states the primality of two families of closed terms that will play a key role in the proof of our main result.

Lemma F.2. *1. The term $\mu^{\leq m}$ is prime, for each $m \geq 1$.*
2. Let $\nu \in \{a, \bar{a}\}$, $\mu \in \{a, \bar{a}, \tau\}$, $\nu \neq \mu$, $m \geq 1$ and $1 \leq i_1 < \dots < i_m$. Then the term $\nu.\mu^{\leq i_1} + \dots + \nu.\mu^{\leq i_m}$ is prime.

Proof: The first claim is immediate because the norm of $\mu^{\leq m}$ is one, for each $m \geq 1$.

For the second claim, assume by contradiction that there are process terms p, q such that $p, q \not\underline{\leftrightarrow} \mathbf{0}$ and $\nu.\mu^{\leq i_1} + \dots + \nu.\mu^{\leq i_m} \underline{\leftrightarrow} p \| q$. Clearly, this would imply the existence of process terms p', q' such that $p \xrightarrow{\nu} p'$ and $q \xrightarrow{\nu} q'$ so that $p \| q \xrightarrow{\nu} p' \| q$ and $p \| q \xrightarrow{\nu} p \| q'$. However, these transitions would in turn imply that $p \| q \xrightarrow{\nu} p' \| q \xrightarrow{\nu} p' \| q'$, namely $p \| q$ could perform two ν-moves in a row, whereas $\nu.\mu^{\leq i_1} + \dots + \nu.\mu^{\leq i_m}$ cannot perform such a sequence of actions, thus contradicting $\nu.\mu^{\leq i_1} + \dots + \nu.\mu^{\leq i_m} \underline{\leftrightarrow} p \| q$. $\square$

In [22] the notion of *unique prime decomposition* of a process p was introduced, as the unique multiset $\{\!| q_1, \dots, q_n |\!\}$ of primes s.t. $p \underline{\leftrightarrow} q_1 \| \dots \| q_n$. Inspired by the unique prime decomposition result of [22], the authors of [19] proposed the notion of *decomposition order for commutative monoids*, and proved that the existence of a decomposition order on a commutative monoid implies that the monoid has the unique prime decomposition property. CCS$_f$ modulo $\underline{\leftrightarrow}$ is a *commutative monoid* with respect to $\|$, having $\mathbf{0}$ as unit, and the transition relation defines a *decomposition order* over bisimilarity equivalence classes of closed terms. Then, by [19, Theorem 32], the following result holds:

Proposition F.3. *Any CCS$_f$ term can be expressed uniquely, up to $\underline{\leftrightarrow}$, as a parallel composition of primes.*

As we will see, this property will play a crucial role in some of the upcoming proofs.

F.4 Proof of Lemma 7.8

Proof of Lemma 7.8. Assume, towards a contradiction, that t is not a variable. We proceed by a case analysis on the possible form this term may have.

1. Case $t = \nu.t'$ for some term t'. Then $\nu = \alpha$ and $\mu^{\leq i_1} \underline{\leftrightarrow} \sigma(t') \underline{\leftrightarrow} \mu^{\leq i_m}$. However, this is a contradiction because, since $i_1 < i_m$, the terms $\mu^{\leq i_1}$ and $\mu^{\leq i_m}$ have different depths, and are therefore not bisimilar.
2. Case $t = f(t', t'')$ for some terms t', t''. Since $\sigma(t)$ has no $\mathbf{0}$ factors, we have that $\sigma(t') \not\underline{\leftrightarrow} \mathbf{0}$ and $\sigma(t'') \not\underline{\leftrightarrow} \mathbf{0}$. Observe now that $\alpha.\mu^{\leq i_1} + \alpha.\mu^{\leq i_m} \xrightarrow{\alpha} \mu^{\leq i_m}$. Thus, as

$$\sigma(t) = f(\sigma(t'), \sigma(t'')) \underline{\leftrightarrow} \alpha.\mu^{\leq i_1} + \dots + \alpha.\mu^{\leq i_m} \ ,$$

according to which rules are available for f with respect to v, we can distinguish the following two cases:

- L^f_α holds and there is a term p' such that

  $$\sigma(t') \xrightarrow{\alpha} p' \text{ and } p' \| \sigma(t'') \underline{\leftrightarrow} \mu^{\leq i_m} \ .$$

  As $\sigma(t'') \not\underline{\leftrightarrow} \mathbf{0}$ and $\mu^{\leq i_m}$ is prime (Lemma F.2(1)), this implies that $p' \underline{\leftrightarrow} \mathbf{0}$ and

  $$\sigma(t'') \underline{\leftrightarrow} \mu^{\leq i_m} \ .$$

  Since $\alpha.\mu^{\leq i_1} + \cdots + \alpha.\mu^{\leq i_m} \xrightarrow{\alpha} \mu^{\leq i_1}$, a similar reasoning allows us to conclude that

  $$\sigma(t'') \underline{\leftrightarrow} \mu^{\leq i_1}$$

  also holds. However, this is a contradiction because by the proviso of the lemma $m > 1$ and $1 \leq i_1 < \ldots < i_m$, and therefore $\mu^{\leq i_1}$ and $\mu^{\leq i_m}$ are not bisimilar.
- R^f_α holds and there is a term p'' such that

  $$\sigma(t'') \xrightarrow{\alpha} p'' \text{ and } \sigma(t') \| p'' \underline{\leftrightarrow} \mu^{\leq i_m} \ .$$

  This case is analogous to the previous one and leads as well to a contradiction.

We may therefore conclude that t must be a variable, which was to be shown. □

F.5 Proof of Proposition 7.9

Before giving the proof, we argue that we can also give a syntactic characterization of the occurrences in a term of the variables that can contribute to the behaviour of closed instances of that term. Formally, to infer the behaviour of a term t from that of (a closed instance of) a variable x, the latter must occur *unguarded* in t, namely x cannot occur in the scope of a prefixing operator in t. Inspired by [3], for $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$, we introduce a relation $\triangleleft^\mu_{\mathrm{w}}$ between a variable x and a term t. Intuitively, the role of the label w is the same as in the auxiliary transitions, namely, to identify which predicates hold (and thus which rules for f are available) for f with respect to action μ. Then $x \triangleleft^\mu_{\mathrm{w}} t$ holds if the predicate associated with w holds for f and whenever t has a subterm of the form $f(t_1, t_2)$ and x occurs in t_i, with $i = 1$ if $\mathrm{w} \in \{\mathrm{l}, \mathrm{b}\}$ and $i = 2$ if $\mathrm{w} \in \{\mathrm{r}, \mathrm{b}\}$, then the occurrence of x is unguarded and can contribute to an initial μ-transition of $\sigma(t)$ when $\sigma(x) \xrightarrow{\mu}$.

Definition F.4 (Relation $\triangleleft$)**.** Let $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. The relation $\triangleleft^\mu_{\mathrm{w}}$ between variables and terms is defined inductively as follows:

1. $x \triangleleft^\mu_{\mathrm{l}} x$ if L^f_μ 2. $x \triangleleft^\mu_{\mathrm{r}} x$ if R^f_μ 3. $x \triangleleft^\mu_{\mathrm{b}} x$ if $L^f_\mu \wedge R^f_\mu$
4. $x \triangleleft^\mu_{\mathrm{w}} t \ \Rightarrow\ x \triangleleft^\mu_{\mathrm{w}} t + u \ \wedge\ x \triangleleft^\mu_{\mathrm{w}} u + t$
5. $x \triangleleft^\mu_{\mathrm{l}} t \ \Rightarrow\ x \triangleleft^\mu_{\mathrm{l}} f(t, u)$ 6. $x \triangleleft^\mu_{\mathrm{r}} t \ \Rightarrow\ x \triangleleft^\mu_{\mathrm{r}} f(u, t)$
7. $x \triangleleft^\mu_{\mathrm{b}} t \ \Rightarrow\ x \triangleleft^\mu_{\mathrm{b}} f(t, u) \ \wedge\ x \triangleleft^\mu_{\mathrm{b}} f(u, t)$.

Example F.5. Assume, for instance, that L^f_a, $R^f_{\bar{a}}$ and $L^f_\tau \wedge R^f_\tau$ are the only predicates holding. Then, for $t = f(x, \tau)$ we have that $x \triangleleft^a_{\mathrm{l}} t$, $x \triangleleft^\tau_{\mathrm{l}} t$ and $x \triangleleft^\tau_{\mathrm{b}} t$.

There is a close relation between unguarded occurrences of variables in terms and the auxiliary transitions, as stated in the following:

Lemma F.6. *Let $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. Then $x \triangleleft^\mu_{\mathrm{w}} t$ if and only if $t \xrightarrow{x_{\mathrm{w}}}_\mu c$ for a configuration $c \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ for some CCS^-_f term t'.*

Proof: We prove the two implications separately.

($\Rightarrow$) We proceed by induction over the structure of t. The only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$ which we expand below, by distinguishing three cases, according to which rules for f are available:

- $x \triangleleft^\mu_{\mathrm{l}} f(t_1, t_2)$. This can only be due to $x \triangleleft^\mu_{\mathrm{l}} t_1$. By the induction hypothesis for t_1, this implies that $t_1 \xrightarrow{x_{\mathrm{l}}}_\mu c_1$ with $c_1 \underline{\leftrightarrow} x_{\mathrm{d}} \| t_1'$ for some t_1'. By applying the auxiliary rule (a_6), we infer $f(t_1, t_2) \xrightarrow{x_{\mathrm{l}}}_\mu c$ with $c = c_1 \| t_2$ and, since $\underline{\leftrightarrow}$ is a congruence with respect to $\|$ and $\|$ is associative with respect to $\underline{\leftrightarrow}$, we get $c \underline{\leftrightarrow} (x_{\mathrm{d}} \| t_1') \| t_2 \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ with $t' \underline{\leftrightarrow} t_1' \| t_2$.
- $x \triangleleft^\mu_{\mathrm{r}} f(t_1, t_2)$. This can only be due to $x \triangleleft^\mu_{\mathrm{r}} t_2$. Thus, we can proceed as in the previous case, by applying the auxiliary rule (a_7) in place of rule (a_6) and using the commutativity of $\|$ with respect to $\underline{\leftrightarrow}$.
- $x \triangleleft^\mu_{\mathrm{b}} f(t_1, t_2)$. This can be due to either $x \triangleleft^\mu_{\mathrm{b}} t_1$ or $x \triangleleft^\mu_{\mathrm{b}} t_2$. For both, we can proceed as in the previous cases, by applying the auxiliary rules (a_8) or, respectively, (a_9) in place of rules (a_6) and (a_7).

($\Leftarrow$) We proceed by induction over the derivation of the open transition $t \xrightarrow{x_{\mathrm{w}}}_\mu c$. Again, the only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$, which we expand below by considering three cases, according to which rules are available for f:

- $f(t_1, t_2) \xrightarrow{x_{\mathrm{l}}}_\mu c$ with $c \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ for some t'. According to the auxiliary operational semantics, it must be the case that $t_1 \xrightarrow{x_{\mathrm{l}}}_\mu c_1$ for some c_1 such that $c = c_1 \| t_2$. Notice that since x_{d} can occur only in c_1, from $c = c_1 \| t_2$ and $c \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$, we infer $c_1 \underline{\leftrightarrow} x_{\mathrm{d}} \| t''$ for some t'' such that $t'' \| t_2 \underline{\leftrightarrow} t'$. Hence, we can apply the induction hypothesis to the transition from t_1 and obtain $x \triangleleft^\mu_{\mathrm{l}} t_1$. Since $t = f(t_1, t_2)$ we can immediately conclude that $x \triangleleft^\mu_{\mathrm{l}} t$.
- $f(t_1, t_2) \xrightarrow{x_{\mathrm{r}}}_\mu c$. It follows by a similar reasoning.
- $f(t_1, t_2) \xrightarrow{x_{\mathrm{b}}}_\mu c$. It follows by a similar reasoning.

□

We now discuss the necessary conditions to relate the depth of closed instances of a term to the depth of the closed instances of the variables occurring in it.

Lemma F.7. *Let t be a CCS_f^- term and σ be a closed substitution. If t has no $\mathbf{0}$ summands or factors and $x \triangleleft^{\mu}_{\mathrm{w}} t$ for some $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$ and $\mu \in \{a, \bar{a}, \tau\}$ with $\mathrm{init}(\sigma(x)) \subseteq \{\mu \mid x \triangleleft^{\mu}_{\mathrm{w}} t\}$, then $depth(\sigma(t)) \geq depth(\sigma(x))$.*

Proof: The proof proceeds by structural induction over t and a case analysis over $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. The only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$ which we expand below for the case of $\mathrm{w} = \mathrm{l}$. The other cases can be obtained by applying a similar reasoning.

Moreover, always for sake of simplicity, assume that there is only one action μ such that $x \triangleleft^{\mu}_{\mathrm{l}} t$, so that $\mathrm{init}(\sigma(x)) = \{\mu\}$. Once again, the general case can be easily derived from this one. Notice that this implies the existence of a closed term q such that $\sigma(x) \xrightarrow{\mu} q$ and $depth(\sigma(x)) = depth(q)+1$. We have that $x \triangleleft^{\mu}_{\mathrm{l}} f(t_1, t_2)$ can be derived only by $x \triangleleft^{\mu}_{\mathrm{l}} t_1$. Hence, structural induction over t_1 gives $depth(\sigma(t_1)) \geq depth(\sigma(x))$. Moreover, by Lemma F.6 we obtain that $t_1 \xrightarrow{x_{\mathrm{l}}}_{\mu} c_1$ for some $c_1 \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ for some term t'. Furthermore, $\sigma(x) \xrightarrow{\mu} q$ together with Lemma 7.5 gives $\sigma(t_1) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto q](c_1)$. Then we can infer that $\sigma(t) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto q](c_1) \| \sigma(t_2) \ \underline{\leftrightarrow}\ q \| (\sigma(t') \| \sigma(t_2))$. We have therefore obtained

$$\begin{aligned} depth(\sigma(t)) &\geq 1 + depth(q \| (\sigma(t') \| \sigma(t_2))) \\ &= 1 + depth(q) + depth(\sigma(t') \| \sigma(t_2)) \\ &\geq 1 + depth(q) \\ &= depth(\sigma(x)). \end{aligned}$$

□

Example F.8. We remark that, due to the potential asymmetry of the rules for f, the requirement on the set of initials of $\sigma(x)$ cannot be relaxed in any trivial way. Consider, for instance, the term $t = f(x, \tau)$ from our running example and assume that the only predicates holding are L^f_α, L^f_τ and $R^f_{\bar{\alpha}}$. Notice that $x \triangleleft^{\alpha}_{\mathrm{l}} t$ and $x \triangleleft^{\tau}_{\mathrm{l}} t$. Consider the closed substitution σ with $\sigma(x) = \alpha + \tau + \bar{\alpha}.\alpha^n$, for some $n \geq 2$, so that $\{\alpha, \tau\} \subset \mathrm{init}(\sigma(x)) = \{\alpha, \tau, \bar{\alpha}\}$. As $L^f_{\bar{\alpha}}$ and R^f_τ do not hold, the only inferable initial transitions for $\sigma(t)$ are those resulting from the α-move and the τ-move by $\sigma(x)$. Thus, we get that $depth(\sigma(t)) = 2$, whereas $depth(\sigma(x)) \geq 3$. This is due to the fact that the computation of $\sigma(x)$ starting with a $\bar{\alpha}$-move is *blocked* by the rules for f and, thus, it cannot contribute to the behaviour of t.

We can now proceed to prove a more general version of Proposition 7.9, namely the following:

Proposition F.9. *Let $\alpha \in \{a, \bar{a}\}$, x be a variable and t, u be CCS_f^- with $t \underline{\leftrightarrow} u$ and such that neither t nor u has $\mathbf{0}$ summands or factors. If $x \triangleleft^{\alpha}_{\mathrm{w}} t$ for some $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$, then $x \triangleleft^{\alpha}_{\mathrm{w}} u$. In particular, if $x \triangleleft^{\alpha}_{\mathrm{w}} t$ because t has a summand x, then so does u.*

Proof: Observe, first of all, that since t and u have no $\mathbf{0}$ summands or factors, by Remark 3 we can assume that $t = \sum_{i \in I} t_i$ and $u = \sum_{j \in J} u_j$ for some finite non-empty index sets I, J, where none of the t_i ($i \in I$) and u_j ($j \in J$) has $+$ as its head operator, and none of the t_i ($i \in I$) and u_j ($j \in J$) have $\mathbf{0}$ summands or factors. Therefore, $x \triangleleft^{\alpha}_{\mathrm{w}} t$ implies that there is some index $i \in I$ such that $x \triangleleft^{\alpha}_{\mathrm{w}} t_i$. We then proceed by a case analysis on the rules available for f. Actually we expand only the case in which only L^f_α holds, as the other two cases, in which respectively only R^f_α holds, or $L^f_\alpha \wedge R^f_\alpha$ holds, can be obtained analogously.

Since only L^f_α holds, then it must be the case that $x \triangleleft^{\alpha}_{\mathrm{l}} t_i$. By Lemma F.6 we get that $t_i \xrightarrow{x_{\mathrm{l}}}_{\alpha} c$ for some configuration c with $c \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ for some t'. Let n be greater than the size of t and consider the substitution σ such that

$$\sigma(y) = \begin{cases} \alpha \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i} & \text{if } y = x \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

For simplicity of notation, let $p_n = \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i}$. Clearly $\sigma(x) \xrightarrow{\alpha} p_n$. By Lemma 7.5 we obtain that $\sigma(t_i) \xrightarrow{\alpha} p$ with $p = \sigma[x_{\mathrm{d}} \mapsto p_n](c)$ and, thus, $p \underline{\leftrightarrow} p_n \| \sigma(t')$. As $t \underline{\leftrightarrow} u$ implies $\sigma(t) \underline{\leftrightarrow} \sigma(u)$, we get that there is an index $j \in J$ such that $\sigma(u_j) \xrightarrow{\alpha} q$ for some $q \underline{\leftrightarrow} p_n \| \sigma(t')$. As only L^f_α holds, by Lemma 7.6 we can distinguish two cases:

- There are a variable y, a closed term q' and a configuration c' such that $\sigma(y) \xrightarrow{\alpha} q'$, $u_j \xrightarrow{y_{\mathrm{l}}}_{\alpha} c'$ and $q = \sigma[y_{\mathrm{d}} \mapsto q'](c')$. Since σ maps all variables but x to $\mathbf{0}$, we can directly infer that $y = x$, $q' = p_n$. Moreover, as p_n is prime and there is a unique prime decomposition of processes, we also infer that $c' \underline{\leftrightarrow} x_{\mathrm{d}} \| u'$ for some u' with $\sigma(u') \underline{\leftrightarrow} \sigma(t')$. Consequently, by Lemma F.6 we can conclude that $x \triangleleft^{\alpha}_{\mathrm{l}} u_j$ and thus $x \triangleleft^{\alpha}_{\mathrm{l}} u$ as required.
- There is a term u' such that $u_j \xrightarrow{\alpha} u'$ and $\sigma(u') \underline{\leftrightarrow} p_n \| \sigma(t')$. We proceed to show that this case leads to a contradiction. We distinguish two cases:
  - $\sigma(t') \underline{\leftrightarrow} \mathbf{0}$. Thus $\sigma(u') \underline{\leftrightarrow} p_n$ and we can rewrite $u' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator. Moreover, since u not having $\mathbf{0}$ summands nor factors implies that neither u_j no u' have some, the same holds for all the v_h. Since n is larger than the size of u, and thus than that of u', by Lemma 7.8 $\sigma(u') \underline{\leftrightarrow} p_n$ implies that there is one index $h \in H$ such that $v_h = y$ for some variable y and $\sigma(y) \underline{\leftrightarrow} \bar{\alpha}\alpha^{\leq i_1} + \cdots + \bar{\alpha}\alpha^{\leq i_m}$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. However, by the choice of σ, all variables but x are mapped to $\mathbf{0}$, and moreover $\sigma(x) \not\underline{\leftrightarrow} \bar{\alpha}\alpha^{\leq i_1} + \cdots + \bar{\alpha}\alpha^{\leq i_m}$ thus contradicting $\sigma(u') \underline{\leftrightarrow} p_n$.
  - $\sigma(t') \not\underline{\leftrightarrow} \mathbf{0}$. Consequently, $\sigma(t') \underline{\leftrightarrow} \sum_{h \in H} \mu_h q_h$ for some actions $\mu_h \in \{a, \bar{a}, \tau\}$ and closed terms q_h. We can therefore apply the expansion law for parallel

composition obtaining

$$\sigma(u') \leftrightarrow p_n \| \sigma(t') \leftrightarrow \sum_{i=1}^{n} \bar{\alpha}(\alpha^{\le i} \| \sigma(t')) + \sum_{h \in H} \mu_h (p_n \| q_h) + \sum_{\substack{i=1,\dots,n \\ h \in H \text{ s.t. } \mu_h = \alpha}} \tau(\alpha^{\le i} \| q_h).$$

We notice that the first term in the expansion has size at least $n+1$ and therefore greater than the size of u and in particular of u'. Moreover $\alpha^{\le i} \| \sigma(t') \not\leftrightarrow \alpha^{\le j} \| \sigma(t')$ whenever $i \neq j$. Therefore, by Lemma 7.7 there is a variable $y \in var(u')$ such that $\sigma(y) \leftrightarrow \bar{\alpha}(\alpha^{\le i_1} \| \sigma(t')) + \cdots + \bar{\alpha}(\alpha^{\le i_m} \| \sigma(t')) + r$ for some $m > 1$ and $1 \le i_1 < \cdots < i_m$ and closed term r. However, $\sigma(y) = \mathbf{0}$ whenever $y \neq x$ and $\sigma(x) \not\leftrightarrow \bar{\alpha}(\alpha^{\le i_1} \| \sigma(t')) + \cdots + \bar{\alpha}(\alpha^{\le i_m} \| \sigma(t')) + r$, for any closed term r, thus contradicting $\sigma(u') \leftrightarrow p_n \| \sigma(t')$.

We have therefore obtained that whenever $x \triangleleft_1^{\alpha} t$ then also $x \triangleleft_1^{\alpha} u$.

Assume now that t has a summand x. We aim to show that u has a summand x as well. Since $x \triangleleft_1^{\alpha} x$ gives $x \triangleleft_1^{\alpha} t$, by the first part of the Proposition we get $x \triangleleft_1^{\alpha} u$ and thus there is an index $j \in J$ such that $x \triangleleft_1^{\alpha} u_j$. We now treat the cases of an operator f that distributes over + in its first argument and of an operator f that does not distribute in either argument separately.

Case of an operator f that distributes over + in its first argument. Consider the substitution σ_0 mapping each variable to $\mathbf{0}$. Pick an integer m larger than the depth of $\sigma_0(t)$ and of $\sigma_0(u)$. Let σ be the substitution mapping x to the term a^{m+1} and agreeing with σ_0 on all the other variables.

As $t \approx u$ is sound with respect to bisimulation equivalence, we have that

$$\sigma(t) \leftrightarrow \sigma(u) \quad .$$

Moreover, the term $\sigma(t)$ affords the transition $\sigma(t) \xrightarrow{a} a^m$, for $t_i = x$ and $\sigma(x) = a^{m+1} \xrightarrow{a} a^m$. Hence, for some closed term p,

$$\sigma(u) = \sum_{j \in J} \sigma(u_j) \xrightarrow{a} p \leftrightarrow a^m \quad .$$

This means that there is a $j \in J$ such that $\sigma(u_j) \xrightarrow{a} p$. We claim that this u_j can only be the variable x. To see that this claim holds, observe, first of all, that $x \in var(u_j)$. In fact, if x did not occur in u_j, then we would reach a contradiction thus:

$$m = depth(p) < depth(\sigma(u_j)) = depth(\sigma_0(u_j)) \le depth(\sigma_0(u)) < m \quad .$$

Using this observation and Lemma F.7, it is not hard to show that, for each of the other possible forms u_j may have, $\sigma(u_j)$ does not afford an a-labelled transition leading to a term of depth m. We may therefore conclude that $u_j = x$, which was to be shown.

Case of an operator f that does not distribute over + in either argument. Notice that in the case at hand, there must be at least one action $\mu \in \{a, \bar{a}, \tau\}$ such that R_{μ}^{f} holds. Assume such an action μ. Again, let n be greater than the size of t and consider the substitution

$$\sigma_1(y) = \begin{cases} \alpha\alpha^{\le n} & \text{if } y = x \\ \alpha + \mu & \text{otherwise.} \end{cases}$$

Thus $\sigma_1(x) \xrightarrow{\alpha} \alpha^{\le n}$ and consequently $\sigma_1(t) \xrightarrow{\alpha} \alpha^{\le n}$. Since $\sigma_1(t) \leftrightarrow \sigma_1(u)$ it must hold that $\sigma_1(u) \xrightarrow{\alpha} q$ for some $q \leftrightarrow \alpha^{\le n}$. As n is greater than the size of u, one can infer that u can have a summand given by at most $\lfloor \frac{n-2}{2} \rfloor$ nested occurrences of f (which is a binary operator of size at least 3). Since, moreover, all variables but x are mapped into a term of depth 1, we can infer that the only term that can be responsible for the α-move to q is a summand u_j such that $x \triangleleft_1^{\alpha} u_j$. To show $u_j = x$ we show that the only other possible case, namely $u_j = f(u', u'')$ with $x \triangleleft_1^{\alpha} u'$ leads to a contradiction. Recall that by the proviso of the Proposition u has no $\mathbf{0}$ factors, which implies that $u', u'' \not\leftrightarrow \mathbf{0}$. Since moreover, $x \triangleleft_1^{\alpha} u'$, by Lemma F.6 and Lemma 7.6 we get $u' \xrightarrow{x_1}_{\alpha} c$ and thus $u_j \xrightarrow{x_1}_{\alpha} c \| u''$ for some configuration $c \leftrightarrow x_{\mathrm{d}} \| u'''$ for some term u''', so that $\sigma_1(u_j) \xrightarrow{\alpha} \sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\le n}](c) \| \sigma_1(u'') = q$. However, $u'' \not\leftrightarrow \mathbf{0}$ implies that either there is a term v such that $u'' \xrightarrow{\nu} v$, for some action ν, or in u'' at least one variable occurs unguarded. Hence, by the choice of σ_1, as both L_{α}^{f} and R_{μ}^{f} hold, we can infer that $depth(\sigma_1(u'')) \ge 1$ which gives

$$\begin{aligned} n &= depth(\alpha^{\le n}) \\ &= depth(q) \\ &= depth(\sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\le n}](c) \| \sigma_1(u'')) \\ &= depth(\sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\le n}](c)) + depth(\sigma_1(u'')) \\ &\ge depth(\alpha^{\le n}) + depth(\sigma_1(u'')) \\ &\ge n + 1 \end{aligned}$$

thus contradicting $q \leftrightarrow \alpha^{\le n}$. □

G Proof of Theorem 8.1

Before proceeding to the proof, we present a technical lemma stating that, under the considered set of rules for f, if a closed term $\sigma(t)$ is not bisimilar to $\mathbf{0}$, then by instantiating the variables in t with a process which is not bisimilar to $\mathbf{0}$ we cannot obtain a closed instance of t which is bisimilar to $\mathbf{0}$.

Lemma G.1. *Let t be a CCS_f^- term and let σ be a substitution with $\sigma(t) \not\leftrightarrow \mathbf{0}$. Assume that u is a CCS_f^- term that is not bisimilar to $\mathbf{0}$. Then $\sigma[x \mapsto u](t) \not\leftrightarrow \mathbf{0}$ for each variable x.*

Proof: By induction on the structure t. □

Remark 4. *We have defined the processes p_n in a such a way that an initial synchronization, in the scope of operator f, with the process α is always possible. This choice will allow us to slightly simplify the reasoning in the proof of the upcoming Proposition G.4 and thus of the negative result (cf., for instance, with the proof of Proposition H.3 in Section 9). Clearly, the possibility of synchronization is directly related to which rules of type (5) are available for f. However, since f has a rule of type (6) for all actions, it is then always possible to identify a pair μ, p_n such that $f(\mu, p_n) \xrightarrow{\tau}$ due to an application of the rule of type (5) allowed for f.*

Finally, we study some properties of the processes $f(\alpha, p_n)$, which also depend on the particular configuration of rules for f that we are considering.

Lemma G.2. *The term $f(\alpha, p_n)$ is prime, for each $n \geq 0$.*

Proof: Since $f(\alpha, p_n)$ is not bisimilar to $\mathbf{0}$, to prove the statement it suffices only to show that $f(\alpha, p_n)$ is irreducible for $n \geq 0$.

If $n = 0$ then $f(\alpha, p_n) = f(\alpha, \mathbf{0})$ is a term of depth 1, and is therefore irreducible as claimed.

Consider now $n \geq 1$. Assume, towards a contradiction, that $f(\alpha, p_n) \leftrightarrow p \| q$ for two closed terms p and q with $p \not\leftrightarrow \mathbf{0}$ and $q \not\leftrightarrow \mathbf{0}$, that is, $f(\alpha, p_n)$ is *not* irreducible. We have that

$$f(\alpha, p_n) \xrightarrow{\alpha} \mathbf{0} \| p_n \leftrightarrow p_n \ .$$

As $f(\alpha, p_n) \leftrightarrow p \| q$, there is a transition $p \| q \xrightarrow{\alpha} r$ for some $r \leftrightarrow p_n$. Without loss of generality, we may assume that $p \xrightarrow{\alpha} p'$ and $r = p' \| q$. Since we have assumed that $n \geq 1$, by statement 2 and our assumption that $q \not\leftrightarrow \mathbf{0}$, we have that $p' \leftrightarrow \mathbf{0}$ and $q \leftrightarrow p_n$. Again using that $n \geq 1$, it follows that $q \xrightarrow{\bar{\alpha}} q'$ for some q'. This means that $p \| q \xrightarrow{\bar{\alpha}}$, contradicting the assumption that $f(\alpha, p_n) \leftrightarrow p \| q$. Thus $f(\alpha, p_n)$ is irreducible, which was to be shown. □

Lemma G.3. *Let $n \geq 1$. Assume that $f(p, q) \leftrightarrow f(\alpha, p_n)$, where $q \not\leftrightarrow \mathbf{0}$. Then $p \leftrightarrow \alpha$ and $q \leftrightarrow p_n$.*

Proof: Since $f(p, q) \leftrightarrow f(\alpha, p_n)$ and $f(\alpha, p_n) \xrightarrow{\alpha} \mathbf{0} \| p_n \leftrightarrow p_n$, there is a p' such that $p \xrightarrow{\alpha} p'$ and $p' \| q \leftrightarrow p_n$. It follows that $q \leftrightarrow p_n$ and $p' \leftrightarrow \mathbf{0}$, because p_n is prime (Lemma F.2(2)) and $q \not\leftrightarrow \mathbf{0}$. We are therefore left to prove that p is bisimilar to α. To this end, note, first of all, that, as $\leftrightarrow$ is a congruence over the language CCS_f, we have that

$$f(p, p_n) \leftrightarrow f(\alpha, p_n) \ .$$

Assume now that $p \xrightarrow{\mu} p''$ for some action μ and closed term p''. In light of the above equivalence, one of the following two cases may arise:

1. $\mu = \alpha$ and $p'' \| p_n \leftrightarrow p_n$ or
2. $\mu = \tau$ and $p'' \| p_n \leftrightarrow \alpha^{\leq i}$, for some $i \in \{1, \ldots, n\}$.

In the former case, p'' must have depth 0 and is thus bisimilar to $\mathbf{0}$. The latter case is impossible, because the depth of $p'' \| p_n$ is at least $n + 1$.

We may therefore conclude that every transition of p is of the form $p \xrightarrow{\alpha} p''$, for some $p'' \leftrightarrow \mathbf{0}$. Since we have already seen that p affords an α-labelled transition leading to $\mathbf{0}$, modulo bisimulation equivalence, it follows that $p \leftrightarrow \alpha$, which was to be shown. □

The following result, stating that the property mentioned in the statement of that theorem holds for all closed instantiations of axioms in $\mathcal{E}$, will be the crux in the proof of Theorem 8.1.

Proposition G.4. *Assume an operator f that, modulo $\leftrightarrow$, distributes over $+$ in its first argument and such that only L^f_μ holds for each action μ, and only $S^f_{\alpha,\bar{\alpha}}$ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\leftrightarrow$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands or factors and $p, q \leftrightarrow f(\alpha, p_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: Observe, first of all, that since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Hence, by Remark 3, we have that for some finite non-empty index sets I, J,

$$t = \sum_{i \in I} t_i \quad \text{and} \quad u = \sum_{j \in J} u_j \ ,$$

where none of the t_i ($i \in I$) and u_j ($j \in J$) is $\mathbf{0}$, has $+$ as its head operator, has $\mathbf{0}$ summands and factors.

Since $p = \sigma(t)$ has a summand bisimilar to $f(\alpha, p_n)$, there is an index $i \in I$ such that $\sigma(t_i) \leftrightarrow f(\alpha, p_n)$.

Our aim is now to show that there is an index $j \in J$ such that $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$, proving that $q = \sigma(u)$ also has a summand bisimilar to $f(\alpha, p_n)$.

We proceed by a case analysis on the form t_i may have.

1. Case $t_i = x$ for some variable x. In this case, we have $\sigma(x) \leftrightarrow f(\alpha, p_n)$, and t has x as a summand. As $t \approx u$ is sound with respect to bisimilarity and neither t nor u have $\mathbf{0}$ summands or factors, it follows that u also has x as a summand (Proposition 7.9). Thus there is an index $j \in J$ such that $u_j = x$, and, modulo bisimulation, $\sigma(u)$ has $f(\alpha, p_n)$ as a summand, which was to be shown.
2. Case $t_i = \mu t'$ for some term t'. This case is vacuous because, since $\mu\sigma(t') \xrightarrow{\mu} \sigma(t')$ is the only transition afforded by $\sigma(t_i)$, this term cannot be bisimilar to $f(\alpha, p_n)$. Indeed $f(\alpha, p_n)$ can perform both, an α-labelled transition triggered by the first argument, and the τ-move due to the synchronization between α and p_n.
3. Case $t_i = f(t', t'')$ for some terms t', t''. In this case, we have $f(\sigma(t'), \sigma(t'')) \leftrightarrow f(\alpha, p_n)$. As $\sigma(t_i)$ has

no $\mathbf{0}$ factors, it follows that $\sigma(t') \not\underline{\leftrightarrow} \mathbf{0}$ and $\sigma(t'') \not\underline{\leftrightarrow} \mathbf{0}$. Thus $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\sigma(t'') \underline{\leftrightarrow} p_n$ (Lemma G.3). Now, t'' can be written as $t'' = v_1 + \cdots + v_\ell$, $(\ell > 0)$, where none of the summands v_i is $\mathbf{0}$ or a sum. Observe that, since n is larger than the size of t, we have that $\ell < n$. Hence, since $\sigma(t'') \underline{\leftrightarrow} p_n = \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i}$, there must be some $h \in \{1, \ldots, \ell\}$ such that $\sigma(v_h) \underline{\leftrightarrow} \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m}$ for some $m > 1$ and $1 \leq i_1 < \ldots < i_m \leq n$. The term $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors—or else, so would $\sigma(t'')$, and thus $p = \sigma(t)$. By Lemma 7.8, it follows that v_h can only be a variable x and thus that

$$\sigma(x) \underline{\leftrightarrow} \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} \ . \tag{9}$$

Observe, for later use, that, since t' has no $\mathbf{0}$ factors, the above equation yields that $x \notin var(t')$—or else $\sigma(t') \not\underline{\leftrightarrow} \alpha$ (Lemma F.7). So, modulo bisimilarity, t_i has the form $f(t', (x + t'''))$, for some term t''', with $x \notin var(t')$ and $\sigma(t') \underline{\leftrightarrow} \alpha$.

Our order of business will now be to use the information collected so far in this case of the proof to argue that $\sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$. To this end, consider the substitution

$$\sigma' = \sigma[x \mapsto \bar{\alpha} f(\alpha, p_n)] \ .$$

We have that

$$\begin{aligned}
\sigma'(t_i) &= f(\sigma'(t'), \sigma'(t'')) \\
&= f(\sigma(t'), \sigma'(t'')) && (\text{As } x \notin var(t')) \\
&\underline{\leftrightarrow} f(\alpha, (\bar{\alpha} f(\alpha, p_n) + \sigma'(t'''))) && (\text{As } t'' = x + t''').
\end{aligned}$$

Thus, $\sigma'(t_i) \xrightarrow{\tau} p' \underline{\leftrightarrow} f(\alpha, p_n)$ for some p', so that

$$\sigma'(t) \xrightarrow{\tau} p' \underline{\leftrightarrow} f(\alpha, p_n)$$

also holds. Since $t \approx u$ is sound with respect to $\underline{\leftrightarrow}$, it follows that

$$\sigma'(t) \underline{\leftrightarrow} \sigma'(u) \ .$$

Hence, we can infer that there are a $j \in J$ and a q' such that

$$\sigma'(u_j) \xrightarrow{\tau} q' \underline{\leftrightarrow} f(\alpha, p_n) \ . \tag{10}$$

Recall that, by one of the assumptions of the proposition, $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$, and thus $\sigma(u)$ has depth $n + 2$. On the other hand, by (10),

$$depth(\sigma'(u_j)) \geq n + 3 \ .$$

Since σ and σ' differ only in the closed term they map variable x to, it follows that

$$x \in var(u_j) \ . \tag{11}$$

We now proceed to show that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$ by a further case analysis on the form a term u_j satisfying (10) and (11) may have.

a. CASE $u_j = x$. This case is vacuous because $\sigma'(x) = \bar{\alpha} f(\alpha, p_n) \stackrel{\tau}{\nrightarrow}$, and thus this possible form for u_j does not meet (10).

b. CASE $u_j = \mu u'$ FOR SOME TERM u'. In light of (10), we have that $\mu = \tau$ and $q' = \sigma'(u') \underline{\leftrightarrow} f(\alpha, p_n)$. Using (11) and the fact that u' has no $\mathbf{0}$ factors, we have that $depth(\sigma'(u')) \geq n + 3$ (Lemma F.7). Since $f(\alpha, p_n)$ has depth n+2, this contradicts $q' \underline{\leftrightarrow} f(\alpha, p_n)$.

c. CASE $u_j = f(u', u'')$ FOR SOME TERMS u', u''. Our assumption that $\sigma(u)$ has no $\mathbf{0}$ factors yields that none of the terms $u', u'', \sigma(u')$ and $\sigma(u'')$ is bisimilar to $\mathbf{0}$. Moreover, by (11), either $x \in var(u')$ or $x \in var(u'')$.
Since $\sigma'(u_j) = f(\sigma'(u'), \sigma'(u''))$ affords transition (10), we have that $q' = q_1 \| q_2$ for some q_1, q_2. As $f(\alpha, p_n)$ is prime (Lemma G.2), it follows that either $q_1 \underline{\leftrightarrow} \mathbf{0}$ or $q_2 \underline{\leftrightarrow} \mathbf{0}$. Hence, we can distinguish two cases, according to the possible origins for transition (10):

i. $\sigma'(u') \xrightarrow{\tau} q_1$ and $q_2 = \sigma'(u'')$. We now proceed to argue that this case produces a contradiction.
To this end, note first of all that $\sigma'(u'') \not\underline{\leftrightarrow} \mathbf{0}$, because $\sigma(u'') \not\underline{\leftrightarrow} \mathbf{0}$ (Lemma G.1). Thus it must be the case that $q_1 \underline{\leftrightarrow} \mathbf{0}$ and $q_2 = \sigma'(u'') \underline{\leftrightarrow} f(\alpha, p_n)$. In light of the definition of σ', it follows that x occurs in u', but not in u'' (Lemma F.7). Therefore, since σ and σ' only differ at the variable x,

$$\sigma(u'') = \sigma'(u'') \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

Since $\underline{\leftrightarrow}$ is a congruence, we derive that

$$\sigma(u_j) = f(\sigma(u'), \sigma(u'')) \underline{\leftrightarrow} f(\sigma(u'), f(\alpha, p_n)). \tag{12}$$

Since $\sigma(u') \not\underline{\leftrightarrow} \mathbf{0}$ because $q = \sigma(u)$ has no $\mathbf{0}$-factors, we may infer that

$$\begin{aligned}
& n + 2 \\
&= depth(f(\alpha, p_n)) \\
&= depth(\sigma(u)) && (\text{As } \sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)) \\
&\geq depth(\sigma(u_j)) \\
&= depth(\sigma(u')) + n + 2 && (\text{By (12)}) \\
&> n + 2 && (\text{As } depth(\sigma(u')) > 0),
\end{aligned}$$

which is the desired contradiction.

ii. $\sigma'(u') \xrightarrow{\alpha} q_1$ and $\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2$. Recall that exactly one of q_1, q_2 is bisimilar to $\mathbf{0}$. We proceed with the proof by considering these two possible cases in turn.

- CASE $q_1 \underline{\leftrightarrow} \mathbf{0}$. Our order of business will be to argue that, in this case, $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$, and thus that $q = \sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$.
To this end, observe, first of all, that $q_2 \underline{\leftrightarrow} f(\alpha, p_n)$ by (10). It follows that $x \in var(u'')$, for otherwise we could derive a contradiction thus:

$$\begin{aligned}
& depth(f(\alpha, p_n)) \\
&= depth(\sigma(u)) && (\text{As } \sigma(u) \underline{\leftrightarrow} f(\alpha, p_n))
\end{aligned}$$

$$\geq depth(\sigma(u_j))$$
$$> depth(\sigma(u'')) \qquad (\text{As } depth(\sigma(u')) > 0)$$
$$= depth(\sigma'(u'')) \qquad (\text{As } x \notin var(u''))$$
$$> depth(f(\alpha, p_n)) \qquad (\text{As } \sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \leftrightarrow f(\alpha, p_n)).$$

Moreover, we claim that $x \notin var(u')$. Indeed, if x also occurred in u', then, since u' has no $\mathbf{0}$ factors, the term $\sigma(x)$ would contribute to the behaviour of $\sigma(u_j)$. Therefore, by (9), the term $\sigma(u_j)$ would afford a sequence of actions containing two occurrences of $\bar{\alpha}$, contradicting our assumption that $\sigma(u) \leftrightarrow f(\alpha, p_n)$.

Observe now that, as $\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \leftrightarrow f(\alpha, p_n)$, it must be the case that u'' has a summand x. To see that this does hold, we examine the other possible forms a summand w of u'' responsible for the transition

$$\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \leftrightarrow f(\alpha, p_n)$$

may have, and argue that each of them leads to a contradiction.

A. Case $w = \bar{\alpha}w'$, for some term w'. In this case, $q_2 = \sigma'(w')$. However, the depth of such a q_2 is either smaller than $n+2$ (if $x \notin var(w')$), or larger than $n + 2$ (if $x \in var(w')$). More precisely, in the former case $x \notin var(w')$ implies $\sigma(w) = \sigma'(w)$ and thus $\sigma(u) \leftrightarrow f(\alpha, p_n)$ gives $n + 2 = depth(\sigma(u)) \geq depth(\sigma(w)) = 1+depth(\sigma(w'))$, giving $depth(\sigma'(w')) \leq n+1$. In the latter case, as $x \in var(w')$ and w' does not have $\mathbf{0}$ factors (or otherwise u'' would have $\mathbf{0}$ factors), by Lemma F.7, we would have $depth(\sigma'(w')) \geq depth(\sigma'(x)) = n + 3$. Both cases then contradict the fact that q_2 is bisimilar to $f(\alpha, p_n)$, because the latter term has depth $n + 2$.

B. Case $w = f(w_1, w_2)$, for some terms w_1 and w_2. Observe, first of all, that $\sigma(w_1)$ and $\sigma(w_2)$ are not bisimilar to $\mathbf{0}$, because $\sigma(u)$ has no $\mathbf{0}$ factors. It follows that $\sigma'(w_1)$ and $\sigma'(w_2)$ are not bisimilar to $\mathbf{0}$ either (Lemma G.1). Now, since

$$\sigma'(w) = f(\sigma'(w_1), \sigma'(w_2)) \xrightarrow{\bar{\alpha}} q_2 \ ,$$

there is a closed term q_3 such that $\sigma'(w_1) \xrightarrow{\bar{\alpha}} q_3$ and

$$q_2 = q_3 \| \sigma'(w_2) \leftrightarrow f(\alpha, p_n) \ .$$

As the term $f(\alpha, p_n)$ is prime, and $\sigma'(w_2)$ is not bisimilar to $\mathbf{0}$, we may infer that $q_3 \leftrightarrow \mathbf{0}$ and

$$\sigma'(w_2) \leftrightarrow f(\alpha, p_n) \ .$$

It follows that $x \notin var(w_2)$, or else the depth of $\sigma'(w_2)$ would be at least $n+3$, and therefore that

$$\sigma'(w_2) = \sigma(w_2) \leftrightarrow f(\alpha, p_n) \ .$$

However, this contradicts our assumption that

$$q = \sigma(u) \leftrightarrow f(\alpha, p_n) \ .$$

Summing up, we have argued that u'' has a summand x. Therefore, by (9),

$$\sigma(u'') \leftrightarrow \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} + r'' \ ,$$

for some closed term r''. We have already noted that

$$\sigma(u') = \sigma'(u') \xrightarrow{\alpha} q_1 \leftrightarrow \mathbf{0} \ .$$

Therefore, we have that

$$\sigma(u') \leftrightarrow \alpha + r' \ ,$$

for some closed term r'. Using the congruence properties of bisimulation equivalence, we may infer that

$$\sigma(u_j) = f(\sigma(u'), \sigma(u''))$$
$$\leftrightarrow f((\alpha + r'), (\sum_{j=1}^{m} \bar{\alpha}.\alpha^{\leq i_j} + r'')).$$

In light of this equivalence, we have that

$$\sigma(u_j) \xrightarrow{\alpha} r \leftrightarrow \sum_{j=1}^{m} \bar{\alpha}.\alpha^{\leq i_j} + r'' \leftrightarrow \sigma(u''),$$

for some closed term r, and thus

$$q = \sigma(u) \xrightarrow{\alpha} r \ .$$

Since $q = \sigma(u) \leftrightarrow f(\alpha, p_n)$ by our assumption, it must be the case that $r \leftrightarrow \sigma(u'') \leftrightarrow p_n$. So, again using the congruence properties of $\leftrightarrow$, we have that

$$\sigma(u_j) = f(\sigma(u'), \sigma(u'')) \leftrightarrow f((\alpha + r'), p_n).$$

As $\sigma(u) \leftrightarrow f(\alpha, p_n)$, using Lemma G.3 it is now a simple matter to infer that

$$\sigma(u') \leftrightarrow \alpha \ .$$

Hence $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$. Note that $\sigma(u_j)$ is a summand of $q = \sigma(u)$. Therefore q has a summand bisimilar to $f(\alpha, p_n)$, which was to be shown.

- Case $q_2 \leftrightarrow \mathbf{0}$. We now proceed to argue that this case produces a contradiction. To this end, observe, first of all, that $q_1 \leftrightarrow f(\alpha, p_n)$. Reasoning as in the analysis of the previous case, we may infer that x occurs in u', but x does not occur in u''. Moreover, since $\sigma'(u') \xrightarrow{\alpha} q_1 \leftrightarrow f(\alpha, p_n)$, it must be the case that $u' \xrightarrow{\alpha} u'''$ for some u''' such that

$$\sigma'(u''') = q_1 \leftrightarrow f(\alpha, p_n) \ .$$

(For, otherwise, using Lemma 7.6.2a, we would have that $\sigma'(u') \xrightarrow{\alpha} q_1$ because $u' \xrightarrow{y} c, \sigma(y) \xrightarrow{\alpha} q_1'$ and $q_1 = \sigma'[y_d \mapsto q_1'](c)$, for some variable y, configuration c and closed term q_1'. Then we would necessarily have that $y \neq x$. In fact, if $y = x$, then we would have that $\alpha = \bar{\alpha}$ by the definition of σ', contradicting the distinctness of these two complementary actions. Observe now that, again in light of the definition of σ', the variable x cannot occur in c, or else the depth of

$$q_1 = \sigma'[y_d \mapsto q_1'](c)$$

would be at least $n+3$, contradicting our assumption that

$$q_1 \leftrightarrow f(\alpha, p_n) \ .$$

Hence, since the variable y is different from x, it is not hard to see that $\sigma(u') \xrightarrow{\alpha} q_1$ also holds, and thus that

$$depth(q_1) < depth(\sigma(u)) = n + 2 \ ,$$

contradicting our assumption that $q_1 \leftrightarrow f(\alpha, p_n)$.) Since u contains no $\mathbf{0}$ factors, in light of the definition of σ', this u''' cannot contain occurrences of the variable x. (For, otherwise, Lemma F.7 would yield that

$$depth(\sigma'(u''')) = depth(q_1) \geq n + 3 \ ,$$

contradicting our assumption that $q_1 \leftrightarrow f(\alpha, p_n)$.) So

$$\sigma(u''') = q_1 \leftrightarrow f(\alpha, p_n)$$

also holds. Thus

$$\begin{aligned}
& n + 2 \\
= \ & depth(f(\alpha, p_n)) \\
= \ & depth(\sigma(u)) && (\text{As } \sigma(u) \leftrightarrow f(\alpha, p_n)) \\
\geq \ & depth(\sigma(u_j)) \\
= \ & depth(f(\sigma(u'), \sigma(u''))) \\
> \ & depth(\sigma(u''')) + depth(\sigma(u'')) && (\text{As } \sigma(u') \xrightarrow{\alpha} \sigma(u''')) \\
> \ & n + 2
\end{aligned}$$

where the last inequality follows by the fact that $depth(\sigma(u'')) > 0$ and $depth(\sigma(u''')) = n + 2$, and gives the desired contradiction.

This completes the proof for the case $u_j = f(u', u'')$ for some terms u', u''.

The proof of Proposition G.4 is now complete. □

G.1 Formal proof of Theorem 8.1

Proof of Theorem 8.1. Assume that $\mathcal{E}$ is a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in $\mathcal{E}$:

1. $E \vdash p \approx q$,
2. $p \leftrightarrow q \leftrightarrow f(\alpha, p_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, p_n)$.

We prove that q also has a summand bisimilar to $f(\alpha, p_n)$ by induction on the depth of the closed proof of the equation $p \approx q$ from $\mathcal{E}$. Recall that, without loss of generality, we may assume that the closed terms involved in the proof of the equation$p \approx q$ have no $\mathbf{0}$ summands or factors (by Proposition E.6, as $\mathcal{E}$ may be assumed to be saturated), and that applications of symmetry happen first in equational proofs (that is, $\mathcal{E}$ is closed with respect to symmetry).

We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from $\mathcal{E}$. The case of reflexivity is trivial, and that of transitivity follows immediately by using the inductive hypothesis twice. Below we only consider the other possibilities.

- Case $E \vdash p \approx q$, because $\sigma(t) = p$ and $\sigma(u) = q$ for some equation $(t \approx u) \in E$ and closed substitution σ. Since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, and n is larger than the size of each term mentioned in equations in $\mathcal{E}$, the claim follows by Proposition G.4.
- Case $E \vdash p \approx q$, because $p = \mu p'$ and $q = \mu q'$ for some p', q' such that $E \vdash p' \approx q'$. This case is vacuous because $p = \mu p' \not\leftrightarrow f(\alpha, p_n)$, and thus p does not have a summand bisimilar to $f(\alpha, p_n)$.
- Case $E \vdash p \approx q$, because $p = p' + p''$ and $q = q' + q''$ for some p', q', p'', q'' such that $E \vdash p' \approx q'$ and $E \vdash p'' \approx q''$. Since p has a summand bisimilar to $f(\alpha, p_n)$, we have that so does either p' or p''. Assume, without loss of generality, that p' has a summand bisimilar to $f(\alpha, p_n)$. Since p is bisimilar to $f(\alpha, p_n)$, so is p'. Using the soundness of $\mathcal{E}$ modulo bisimulation, it follows that $q' \leftrightarrow f(\alpha, p_n)$. The inductive hypothesis now yields that q' has a summand bisimilar to $f(\alpha, p_n)$. Hence, q has a summand bisimilar to $f(\alpha, p_n)$, which was to be shown.
- Case $E \vdash p \approx q$, because $p = f(p', p'')$ and $q = f(q', q'')$ for some p', q', p'', q'' such that $E \vdash p' \approx q'$ and $E \vdash p'' \approx q''$. Since the proof involves no uses of $\mathbf{0}$ as a summand or a factor, we have that $p', p'' \not\leftrightarrow \mathbf{0}$ and $q', q'' \not\leftrightarrow \mathbf{0}$. It follows that q is a summand of itself. By our assumptions,

$$f(\alpha, p_n) \leftrightarrow q \ .$$

Therefore we have that q has a summand bisimilar to $f(\alpha, p_n)$, and we are done.

This completes the proof of Theorem 8.1 and thus of Theorem 5.1 in the case of an operator f that, modulo bisimilarity distributes over summation in its first argument. □

H Proof of Theorem 9.1

Before proceeding to the proof of Theorem 9.1, we discuss a few useful properties of the processes $f(\alpha, q_n)$. Such properties are stated in Lemmas H.1 and H.2 and they are the updated versions of, respectively, Lemmas G.2 and G.3 with respect to the current set of SOS rules that are allowed for f.

Lemma H.1. *For each $n \geq 0$ it holds that $f(\alpha, q_n) \leftrightarrow \alpha \| q_n$.*

Lemma H.2. *Let $n \geq 1$. Assume that $f(p, q) \leftrightarrow f(\alpha, q_n)$ for $p, q \not\leftrightarrow \mathbf{0}$. Then (i) either $p \leftrightarrow \alpha$ and $q \leftrightarrow q_n$, (ii) or $q \leftrightarrow \alpha$ and $p \leftrightarrow q_n$.*

Proof: Since $f(p, q) \leftrightarrow f(\alpha, q_n)$ and $f(\alpha, q_n) \xrightarrow{\alpha} \mathbf{0} \| q_n \leftrightarrow q_n$, we can distinguish the following two cases depending on whether a matching transition from $f(p, q)$ stems from p or q:

- There is a p' such that $p \xrightarrow{\alpha} p'$ and $p' \| q \leftrightarrow q_n$. It follows that $q \leftrightarrow q_n$ and $p' \leftrightarrow \mathbf{0}$, because q_n is prime (Lemma F.2(2)) and $q \not\leftrightarrow \mathbf{0}$. We are therefore left to prove that p is bisimilar to α. To this end, note, first of all, that, as $\leftrightarrow$ is a congruence over the language CCS_f, we have that
$$f(p, q_n) \leftrightarrow f(\alpha, q_n) \ .$$
First of all, notice that the equivalence above implies that $depth(p) = 1$. We proceed to prove that $p \leftrightarrow \alpha$. Assume towards a contradiction that $p \not\leftrightarrow \alpha$ and thus that $p \xrightarrow{\mu} \mathbf{0}$ for some $\mu \neq \alpha$. We can distinguish two cases, according to whether the predicate L^f_μ holds or not.
  - Assume first that L^f_μ holds. Then we would have $\mathrm{init}(f(p, q_n)) = \{\alpha, \mu\}$ and $\mathrm{init}(f(\alpha, q_n)) = \{\alpha\}$, thus contradicting $f(p, q_n) \leftrightarrow f(\alpha, q_n)$.
  - Assume now that L^f_μ does not hold. Then, in light of the above equivalence, from $f(\alpha, q_n) \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq n}$ and the fact that $q_n \not\leftrightarrow \bar{\alpha}^{\leq n}$, we can infer that $f(p, q_n) \xrightarrow{\alpha} p \| \bar{\alpha}^{\leq n}$ and $p \| \bar{\alpha}^{\leq n} \leftrightarrow \alpha \| \bar{\alpha}^{\leq n}$.
    Now, if $\mu = \tau$, then $p \| \bar{\alpha}^{\leq n} \xrightarrow{\tau} \mathbf{0} \| \bar{\alpha}^{\leq n} \leftrightarrow \bar{\alpha}^{\leq n}$. However, $\alpha \| \bar{\alpha}^{\leq n}$ can perform a τ-move only due to a synchronization between α and one of the $\bar{\alpha}$, thus implying that $\alpha \| \bar{\alpha}^{\leq n} \xrightarrow{\tau} \mathbf{0} \| \bar{\alpha}^i \leftrightarrow \bar{\alpha}^i$ for some $i \in \{0, \dots, n-1\}$. Since there is no such index i such that $\bar{\alpha}^{\leq n} \leftrightarrow \bar{\alpha}^i$, this contradicts $f(p, q_n) \leftrightarrow f(\alpha, q_n)$.
    Similarly, if $\mu = \bar{\alpha}$, then $p \| \bar{\alpha}^{\leq n}$ could perform a sequence of $n+1$ transitions all with label $\bar{\alpha}$, whereas $\alpha \| \bar{\alpha}^{\leq n}$ can perform at most n $\bar{\alpha}$-moves in a row. Therefore, also this case is in contradiction with $f(p, q_n) \leftrightarrow f(\alpha, q_n)$.

  We may therefore conclude that every transition of p is of the form $p \xrightarrow{\alpha} p''$, for some $p'' \leftrightarrow \mathbf{0}$. Since we have already seen that p affords an α-labelled transition leading to $\mathbf{0}$, modulo bisimulation equivalence, it follows that $p \leftrightarrow \alpha$, which was to be shown.
- There is a q' such that $q \xrightarrow{\alpha} q'$ and $p \| q' \leftrightarrow q_n$. This case can be treated similarly to the previous case and allows us to conclude that $q \leftrightarrow \alpha$ and $p \leftrightarrow q_n$.

□

The negative result stated in Theorem 9.1 is strongly based on the following proposition, which ensures that the property of having a summand bisimilar to $f(\alpha, q_n)$ is preserved by the closure under substitution of equations in a finite sound axiom system.

Proposition H.3. *Assume an operator f such that $L^f_\alpha \wedge R^f_\alpha$ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\leftrightarrow$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor factors, and $p, q \leftrightarrow f(\alpha, q_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, q_n)$, then so does q.

Proof: First of all we notice that since $\sigma(t)$ and $\sigma(u)$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Therefore by Remark 3 we get that
$$t = \sum_{i \in I} t_i \qquad \text{and} \qquad u = \sum_{j \in J} u_j$$
for some finite non-empty index sets I, J with all the t_i and u_j not having $+$ as head operator, $\mathbf{0}$ summands nor factors. By the hypothesis, there is some $i \in I$ with $\sigma(t_i) \leftrightarrow f(\alpha, q_n)$. We proceed by a case analysis over the structure of t_i to show that there is a u_j such that $\sigma(u_j) \leftrightarrow f(\alpha, q_n)$.

1. Case $t_i = x$ for some variable x such that $\sigma(x) \leftrightarrow f(\alpha, q_n)$. By Proposition 7.9, t having a summand x implies that u has a summand x as well. Thus, we can immediately conclude that $\sigma(u)$ has a summand bisimilar to $f(\alpha, q_n)$ as required.
2. Case $t_i = \mu.t'$ for some term t'. This case is vacuous, as it contradicts our assumption $\sigma(t_i) \leftrightarrow f(\alpha, q_n)$. Indeed, if $\mu = \alpha$ then $\sigma(t')$ cannot be bisimilar to both q_n and $\alpha \| \bar{\alpha}^{\leq i}$, for any $i \in \{1, \dots, n\}$.
3. Case $t_i = f(t', t'')$ for some terms t', t''. As $\sigma(t)$ has no $\mathbf{0}$ factors, we have that $\sigma(t'), \sigma(t'') \not\leftrightarrow \mathbf{0}$. Hence, from $f(\sigma(t'), \sigma(t'')) \leftrightarrow f(\alpha, q_n)$ and Lemma H.2 we can distinguish two cases: a. either $\sigma(t') \leftrightarrow \alpha$ and $\sigma(t'') \leftrightarrow q_n$, b. or $\sigma(t') \leftrightarrow q_n$ and $\sigma(t'') \leftrightarrow \alpha$. We expand only the former case, as the latter follows from an identical (symmetrical) reasoning. By Remark 3, from $\sigma(t'') \leftrightarrow q_n$ we infer that $t'' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator and have

no $\mathbf{0}$-summands or factors. Since n is larger that the size of t, we have that $|H| < n$ and thus there is some $h \in H$ such that $\sigma(v_h) \leftrightarrow \sum_{k=1}^{m} \alpha\bar{\alpha}^{\leq i_k}$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. Since $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors, from Lemma 7.8 we infer that v_h can only be a variable x with

$$\sigma(x) \leftrightarrow \sum_{k=1}^{m} \alpha\bar{\alpha}^{\leq i_k}. \tag{13}$$

Therefore, $t_i = f(t', x + t''')$ for some t''' such that $\sigma(x + t''') \leftrightarrow q_n$. We also notice that since $\sigma(t') \leftrightarrow \alpha$ and $\sigma(t')$ has no $\mathbf{0}$ summands or factors, then it cannot be the case that $x \in var(t')$.

To prove that u has a summand bisimilar to $f(\alpha, q_n)$, consider the closed substitution

$$\sigma' = \sigma[x \mapsto \alpha q_n].$$

Since R_α^f and Lemma H.1 hold, we have

$$\sigma'(t_i) \xrightarrow{\alpha} p' \leftrightarrow \alpha \| q_n \leftrightarrow f(\alpha, q_n).$$

As $t \approx u$ implies $\sigma'(t) \leftrightarrow \sigma'(u)$, we infer that there must be a summand u_j such that $\sigma'(u_j) \xrightarrow{\alpha} r$ for some $r \leftrightarrow f(\alpha, q_n)$. Notice that, since $\sigma(u) \leftrightarrow f(\alpha, q_n)$ and $\sigma(u_j) = \sigma'(u_j)$ if $x \notin var(u_j)$, then it must be the case that $x \in var(u_j)$, or otherwise we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, q_n)$, as $\sigma(u_j) = \sigma'(u_j) \xrightarrow{\alpha} r$ would give $\sigma(u) \xrightarrow{\alpha} r \leftrightarrow f(\alpha, q_n)$. However, there is no r' such that $f(\alpha, q_n) \xrightarrow{\alpha} r'$ and $r' \leftrightarrow f(\alpha, q_n)$. By Lemma 7.6, as $L_\alpha^f \wedge R_\alpha^f$ holds, we can distinguish two cases:

a. There is a term u' s.t. $u_j \xrightarrow{\alpha} u'$ and $\sigma'(u') \leftrightarrow f(\alpha, q_n)$. Then, since $f(\alpha, q_n) \leftrightarrow \alpha \parallel q_n$ (Lemma H.1) we can apply the expansion law, obtaining

$$\sigma'(u') \leftrightarrow \sum_{i=1}^{n} \alpha(\alpha \| \bar{\alpha}^{\leq i}) + \alpha q_n.$$

As n is greater than the size of u, and thus of those of u_j and u', by Lemma 7.7 we get that u' has a summand y, for some variable y, such that

$$\sigma'(y) \leftrightarrow \sum_{k=1}^{m'} \alpha(\alpha \| \bar{\alpha}^{\leq i'_k}) + r',$$

for some $m' > 1$, $1 \leq i'_1 < \cdots < i'_{m'} \leq n$ and closed term r'. Notice that we can infer that $y \neq x$, as $\sigma'(x) \not\leftrightarrow \sigma'(y)$ for any closed term r'. Thus we have $\sigma'(y) = \sigma(y)$ and we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, q_n)$ in that $\sigma(u_j)$ would be able to perform three α-moves in a row. In fact

$$\begin{aligned} \sigma(u_j) &\xrightarrow{\alpha} \sigma(u') && (u' \text{ has a summand } y) \\ &\xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq i'_k} && \text{for some } k \in \{1, \ldots, m'\} \\ &\xrightarrow{\alpha} \bar{\alpha}^{\leq i'_k}, \end{aligned}$$

whereas $\sigma(u) \leftrightarrow f(\alpha, q_n)$ can perform only two such transitions.

b. There are a variable y, a closed term r' and a configuration c s.t. $\sigma'(y) \xrightarrow{\alpha} r'$, $u_j \xrightarrow{y_d}_\alpha c$ and $\sigma'[y_d \mapsto r'](c) \leftrightarrow f(\alpha, q_n)$. We claim that it must be the case that $y = x$. To see this, assume towards a contradiction that $y \neq x$. We proceed by a case analysis on the possible occurrences of x in c.

- $x \notin var(c)$ or $x \in var(c)$ but its occurrence is in a guarded context that prevents the execution of its closed instances. In this case we get $r = \sigma[y_d \mapsto r'](c) \leftrightarrow \sigma'[y_d \mapsto r'](c) \leftrightarrow f(\alpha, q_n)$. This contradicts $\sigma(u) \leftrightarrow f(\alpha, q_n)$ since we would have $\sigma(u) \xrightarrow{\alpha} r \leftrightarrow f(\alpha, q_n)$, and such a transition cannot be mimicked by $f(\alpha, q_n)$.
- $x \in var(c)$ and its execution is not prevented. We can distinguish two sub-cases, according to whether the occurrence of x is guarded or not.
  - Assume that x occurs guarded in c. In this case we get a contradiction with $r \leftrightarrow f(\alpha, q_n)$ in that

$$\begin{aligned} n + 2 &= depth(f(\alpha, q_n)) \\ &= depth(r) \\ &\geq 1 + depth(\sigma'(x)) && (x \text{ is guarded}) \\ &= n + 3. \end{aligned}$$

  - Assume now that $x \lhd_b^\alpha c$. We proceed by a case analysis on the structure of c.
    * $c \leftrightarrow y_d \| (x + u_1) \| u_2$. Notice that in this case we have $r = r' \| \sigma'(x) + \sigma'(u_1) \| \sigma'(u_2)$. Then, the only transition available for $\sigma'(x)$ is $\sigma'(x) \xrightarrow{\alpha} q_n$, which gives $r \xrightarrow{\alpha} r' \| q_n \| \sigma'(u_2)$. Since $r \leftrightarrow f(\alpha, q_n)$, then it must be the case that $f(\alpha, q_n) \xrightarrow{\alpha} r''$ for some $r'' \leftrightarrow r' \| q_n \| \sigma'(u_2)$. Since q_n is prime, we can infer that $r'' \leftrightarrow q_n$ and thus that $r' \leftrightarrow \mathbf{0} \leftrightarrow \sigma'(u_2)$. Hence, we have that $r \leftrightarrow \sigma'(x) + \sigma'(u_1)$. As the one we wrote is the only transition available for $\sigma'(x)$, we can infer that, for all $i \in \{1, \ldots, n\}$, the transitions $r \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq i}$ cannot be derived from $\sigma'(x)$, but only from $\sigma'(u_1)$. Moreover, notice that $y \neq x$ gives $\sigma'(y) = \sigma(y)$, and from $\mathrm{init}(\sigma'(x)) = \mathrm{init}(\sigma(x)) = \{\alpha\}$ and the fact that $L_\alpha^f \wedge R_\alpha^f$ holds, we can infer that $\sigma(u_2) \leftrightarrow \sigma'(u_2) \leftrightarrow \mathbf{0}$. Therefore, this contradicts $\sigma(u) \leftrightarrow f(\alpha, q_n)$, since $\sigma(u) \xrightarrow{\alpha} r' \| \sigma(x) + \sigma(u_1) \| \sigma(u_2) \leftrightarrow \sigma(x) + \sigma(u_1) \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq i}$, for any $i \in \{1, \ldots, n\}$. Process $f(\alpha, q_n)$, in turn, by performing two α-moves can only reach processes bisimilar to $\bar{\alpha}^{\leq i}$, for $i \in \{1, \ldots, n\}$.
    * c has a subterm u_3 of the form $u_3 \leftrightarrow f(x + u_2, u_1)$ or $u_3 \leftrightarrow f(u_1, x + u_2)$. In both cases,

we get that $\sigma'(x) \xrightarrow{\alpha} q_n$ implies $\sigma'(u_3) \xrightarrow{\alpha} q_n \| \sigma'(u_1)$. However, $f(\alpha, q_n) \xrightarrow{\alpha} \mathbf{0} \| q_n \underline{\leftrightarrow} q_n$ and q_n prime give $\sigma'(u_1) \underline{\leftrightarrow} \mathbf{0}$. One can then argue that, as $\mathrm{init}(\sigma'(x)) = \{\alpha\}$, either x does not occur in u_1, or it does it in a guarded context that prevents its execution. Hence, we infer $\sigma(u_1) \underline{\leftrightarrow} \sigma'(u_1) \underline{\leftrightarrow} \mathbf{0}$, thus contradicting $\sigma(u)$ not having $\mathbf{0}$ factors.

Therefore, we can conclude that it must be the case that $y = x$ and $r' = q_n$. In particular, notice that $x \triangleleft^{\alpha}_{\mathrm{b}} u_j$. We now proceed by a case analysis on the structure of u_j to show that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, q_n)$.

i. $u_j = x$. This case is vacuous, as $\sigma'(x) \xrightarrow{\alpha} q_n$ and $q_n \not\underline{\leftrightarrow} f(\alpha, q_n)$.

ii. $u_j = f(u', u'')$ for some u', u''. Notice that $x \triangleleft^{\alpha}_{\mathrm{b}} u_j$ can be due either to $x \triangleleft^{\alpha}_{\mathrm{b}} u'$ or $x \triangleleft^{\alpha}_{\mathrm{b}} u''$. As both $\sigma'(u')$ and $\sigma'(u'')$ can be responsible for the α-move by $\sigma'(u_j)$, we distinguish two cases:

A. $\sigma'(u') \xrightarrow{\alpha} r_1$ and $r_1 \| \sigma'(u'') \underline{\leftrightarrow} f(\alpha, q_n)$. As $f(\alpha, q_n) \underline{\leftrightarrow} \alpha \| q_n$ and both α and q_n are prime, by the existence of a unique prime decomposition, we distinguish two cases:

- $r_1 \underline{\leftrightarrow} \alpha$ and $\sigma'(u'') \underline{\leftrightarrow} q_n$. Since $x \triangleleft^{\alpha}_{\mathrm{b}} u''$ is in contradiction with $\sigma'(u'') \underline{\leftrightarrow} q_n$, we infer that $x \triangleleft^{\alpha}_{\mathrm{b}} u'$. Moreover $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\alpha\}$, $L^f_\alpha \wedge R^f_\alpha$, $\sigma'(u') \underline{\leftrightarrow} q_n$ and the fact that $\sigma(u)$ has no $\mathbf{0}$ factors we get that either $x \notin var(u'')$ or x occurs in u'' but its execution is prevented by the rules for f. Therefore
$$\sigma'(u'') \underline{\leftrightarrow} \sigma(u'') \underline{\leftrightarrow} q_n.$$
However, $depth(\sigma(x)) \geq 3$, and $x \triangleleft^{\alpha}_{\mathrm{b}} u'$ with $\mathrm{init}(\sigma(x)) = \{\alpha\}$ give us, by Lemma F.7, that $depth(\sigma(u')) \geq depth(\sigma(x))$. Therefore we get a contradiction, in that
$$\begin{aligned} n + 2 &= depth(f(\alpha, q_n)) \\ &= depth(\sigma(u)) \\ &\geq depth(\sigma(u_j)) \\ &= depth(f(\sigma(u'), \sigma(u''))) \\ &\geq depth(\sigma(x)) + depth(\sigma(u'')) \\ &\geq 3 + n + 1 \\ &= n + 4. \end{aligned}$$
- $r_1 \underline{\leftrightarrow} q_n$ and $\sigma'(u'') \underline{\leftrightarrow} \alpha$. By reasoning as above, we can infer that either $x \notin var(u'')$ or its execution is blocked by the rules for f, so that $\sigma'(u'') \underline{\leftrightarrow} \sigma(u'')$. Moreover, we get that $x \triangleleft^{\alpha}_{\mathrm{b}} u'$. We aim at showing that u' has a summand x. We proceed by showing that the only other possibility, namely $u' = f(w_1, w_2)$ for some w_1, w_2, leads to a contradiction. As $u' = f(w_1, w_2)$ we have that either $x \triangleleft^{\alpha}_{\mathrm{b}} w_1$ or $x \triangleleft^{\alpha}_{\mathrm{b}} w_2$. However, $\sigma'(u') \xrightarrow{\alpha} r_1 \underline{\leftrightarrow} q_n$ gives two possibilities:
  - $\sigma'(w_1) \xrightarrow{\alpha} r_1'$ and $r_1' \| \sigma'(w_2) \underline{\leftrightarrow} q_n$. Since q_n is prime, then either $r_1' \underline{\leftrightarrow} \mathbf{0}$ and $\sigma'(w_2) \underline{\leftrightarrow} q_n$, or $r_1' \underline{\leftrightarrow} q_n$ and $\sigma'(w_2) \underline{\leftrightarrow} \mathbf{0}$. In both cases we infer that either $x \notin var(w_2)$ or its execution in it is always prevented, so that $\sigma(w_2) \underline{\leftrightarrow} \sigma'(w_2)$. Therefore, the former case, combined with $\sigma(u'') \underline{\leftrightarrow} \alpha$, gives a contradiction with $\sigma(u) \underline{\leftrightarrow} f(\alpha, q_n)$. The latter case contradicts $\sigma(u)$ not having $\mathbf{0}$ factors.
  - $\sigma'(w_2) \xrightarrow{\alpha} r_2'$ and $\sigma'(w_1) \| r_2' \underline{\leftrightarrow} q_n$. The same reasoning as in the previous case allows us to conclude that this case gives a contradiction.

  Summing up, we have argued that u' has a summand x. Therefore, by Equation (13),
$$\sigma(u') \underline{\leftrightarrow} \sum_{k=1}^{m} \alpha.\bar{\alpha}^{\leq i_k} + r'' \ ,$$
  for some closed term r''. We have already noted that
$$\sigma(u'') \underline{\leftrightarrow} \sigma'(u'') \underline{\leftrightarrow} \alpha \ .$$
  Therefore, using the congruence properties of bisimulation equivalence, we may infer that
$$\begin{aligned} \sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\ &\underline{\leftrightarrow} f(\sum_{k=1}^{m} \alpha\bar{\alpha}^{\leq i_k} + r'', \alpha) \ . \end{aligned}$$
  In light of this equivalence, we have $\sigma(u_j) \xrightarrow{\alpha} r' \underline{\leftrightarrow} \sigma(u')$ and thus $\sigma(u) \xrightarrow{\alpha} r'$. Since by hypothesis $\sigma(u) \underline{\leftrightarrow} f(\alpha, q_n)$ we have that either $r' \underline{\leftrightarrow} q_n$, or $r' \underline{\leftrightarrow} \alpha \| \alpha^{\leq i}$ for some $i \in \{1, \ldots, n\}$. However, the latter case is in contradiction with $r' \underline{\leftrightarrow} \sigma(u')$, and thus it must be the case that $r' \underline{\leftrightarrow} q_n$. Therefore, we can conclude that $\sigma(u_j) \underline{\leftrightarrow} f(q_n, \alpha)$. It is easy to check that $f(\alpha, q_n) \underline{\leftrightarrow} f(q_n, \alpha)$. Hence, $\sigma(u)$ has the desired summand.

B. $\sigma'(u'') \xrightarrow{\alpha} r_2$ and $\sigma'(u') \| r_2 \underline{\leftrightarrow} f(\alpha, q_n)$. This case follows as the previous one and allows us to conclude as well that $\sigma(u)$ has the desired summand.

The proof of Proposition H.3 is now complete. □

H.1 Formal proof of Theorem 9.1

Proof of Theorem 9.1. Assume that $\mathcal{E}$ is a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in $\mathcal{E}$:

1. $E \vdash p \approx q$,
2. $p \underline{\leftrightarrow} q \underline{\leftrightarrow} f(\alpha, q_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, q_n)$.

We proceed by induction on the depth of the closed proof of the equation $p \approx q$ from $\mathcal{E}$, to prove that also q has a summand bisimilar to $f(\alpha, q_n)$. Recall that, without loss of generality, we may assume that $\mathcal{E}$ is closed with respect to symmetry, and thus applications of symmetry happen first in equational proofs. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from $\mathcal{E}$. The case of reflexivity is trivial, and that of transitivity follows by applying twice the inductive hypothesis. We proceed now to a detailed analysis of the remaining cases:

1. Case $E \vdash p \approx q$ because $\sigma(t) = p$ and $\sigma(u) = q$ for some terms t, u with $E \vdash t \approx u$ and closed substitution σ. The proof of this case follows by Proposition H.3.
2. Case $E \vdash p \approx q$ because $p = \mu.p'$ and $q = \mu.q'$ for some p', q' with $E \vdash p' \approx q'$. This case is vacuous in that $p = \mu.p' \not\underline{\leftrightarrow} f(\alpha, q_n)$ and thus p does not have a summand bisimilar to $f(\alpha, q_n)$.
3. Case $E \vdash p \approx q$ because $p = r_1 + r_2$ and $q = s_1 + s_2$ for some r_i, s_i with $E \vdash r_i \approx s_i$, for $i \in \{1, 2\}$. Since p has a summand bisimilar to $f(\alpha, q_n)$ then so does either r_1 or r_2. Assume without loss of generality that r_1 has such a summand. As $p \underline{\leftrightarrow} f(\alpha, q_n)$ then $r_1 \underline{\leftrightarrow} f(\alpha, q_n)$ holds as well. Then, from $E \vdash r_1 \approx s_1$ we infer $s_1 \underline{\leftrightarrow} f(\alpha, q_n)$. Thus, by the inductive hypothesis we obtain that s_1 has a summand bisimilar to $f(\alpha, q_n)$ and, consequently, so does q.
4. Case $E \vdash p \approx q$ because $p = f(r_1, r_2)$ and $q = f(s_1, s_2)$ for some r_i, s_i with $E \vdash r_i \approx s_i$, for $i \in \{1, 2\}$. By the proviso of the theorem p, q have neither $\mathbf{0}$ summands nor factors, thus implying $r_i, s_i \not\underline{\leftrightarrow} \mathbf{0}$. Hence, from $p \underline{\leftrightarrow} f(\alpha, q_n)$ and $p = f(r_1, r_2)$ and Lemma H.2 we obtain $r_i \underline{\leftrightarrow} \alpha$ and $r_{3-i} \underline{\leftrightarrow} q_n$, thus implying, by the soundness of the equations in $\mathcal{E}$, that $s_i \underline{\leftrightarrow} \alpha$ and $s_{3-i} \underline{\leftrightarrow} q_n$, so that either $q = f(\alpha, q_n)$ or $q = f(q_n, \alpha)$. In both cases, we can infer that q has itself as the desired summand.

This completes the proof of Theorem 9.1 and thus of Theorem 5.1 in the case of an operator f that does not distribute over summation in either argument, case $L^f_\alpha \wedge R^f_\alpha$. □

I Proof of Theorem 10.1

Before proceeding to the proof, we remark that the processes $f(\alpha, p_n)$ enjoy the following properties, according to the current set of allowed rules for operator f:

Lemma I.1. *For each $n \geq 0$ it holds that $f(\alpha, p_n) \underline{\leftrightarrow} \alpha \| p_n$.*

Lemma I.2. *Let $n \geq 1$. Assume that $f(p, q) \underline{\leftrightarrow} f(\alpha, p_n)$ for $p, q \not\underline{\leftrightarrow} \mathbf{0}$. Then $p \underline{\leftrightarrow} \alpha$ and $q \underline{\leftrightarrow} p_n$.*

Proof: The proof is analogous to that of Lemma G.3 and therefore omitted. □

The crucial point in the proof of the negative result is (also in this case) the preservation of the witness property when instantiating an equation from a finite, sound axiom system. We expand this case in the following proposition:

Proposition I.3. *Assume an operator f such that only L^f_α holds for α, only $R^f_{\bar{\alpha}}$ holds for $\bar{\alpha}$, and $S_{\alpha,\bar{\alpha}}$ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\underline{\leftrightarrow}$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor factors, and $p, q \underline{\leftrightarrow} f(\alpha, p_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: First of all we notice that since $\sigma(t)$ and $\sigma(u)$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Therefore by Remark 3 we get that

$$t = \sum_{i \in I} t_i \qquad \text{and} \qquad u = \sum_{j \in J} u_j$$

for some finite non-empty index sets I, J with all the t_i and u_j not having $+$ as head operator, $\mathbf{0}$ summands nor factors. By the hypothesis, there is some $i \in I$ with $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, p_n)$. We proceed by a case analysis on the structure of t_i to show that there is a u_j such that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$, establishing our claim.

1. Case $t_i = x$ for some variable x such that $\sigma(x) \underline{\leftrightarrow} f(\alpha, p_n)$. By Proposition 7.9, t having a summand x implies that u has a summand x as well. Thus, we can immediately conclude that $\sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$ as required.
2. Case $t_i = \mu.t'$ for some term t'. This case is vacuous, as it contradicts $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, p_n)$.
3. Case $t_i = f(t', t'')$ for some terms t', t''. Since $\sigma(t)$ has no $\mathbf{0}$ factors, we have that $\sigma(t'), \sigma(t'') \not\underline{\leftrightarrow} \mathbf{0}$. Hence, from $f(\sigma(t'), \sigma(t'')) \underline{\leftrightarrow} f(\alpha, p_n)$ and Lemma I.2 we obtain $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\sigma(t'') \underline{\leftrightarrow} p_n$. By Remark 3 we infer that $t'' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator and have no $\mathbf{0}$-summands or factors. Since n is larger that the size of t, we have that $|H| < n$ and thus there is some $h \in H$ such that $\sigma(v_h) \underline{\leftrightarrow} \sum_{k=1}^{m} \bar{\alpha}\alpha^{\leq i_k}$ for some $m > 1$ and $1 \leq i_1 < \dots < i_m \leq n$. Since $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors, from Lemma 7.8 we infer that v_h can only be a variable x with
$$\sigma(x) \underline{\leftrightarrow} \sum_{k=1}^{m} \bar{\alpha}\alpha^{\leq i_k}. \tag{14}$$
Therefore, $t_i = f(t', x + t''')$ for some t''' such that $\sigma(x + t''') \underline{\leftrightarrow} p_n$. We also notice that since $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\mathrm{init}(\sigma(x)) = \{\bar{\alpha}\}$, we can infer that $x \lessdot^{\bar{\alpha}}_{\mathrm{r}} t'$ does

not hold (otherwise, $\sigma'(t)$ would afford an initial $\bar{\alpha}$-transition and would not be bisimilar to α).
To prove that u has a summand bisimilar to $f(\alpha, p_n)$, consider the closed substitution

$$\sigma' = \sigma[x \mapsto \bar{\alpha} p_n].$$

Notice that, since $\sigma(t') \leftrightarrow \alpha$, $\sigma(t')$ has no $\mathbf{0}$ summands or factors, $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\bar{\alpha}\}$ and x is the only variable which is affected when changing σ into σ', then we can infer that either $x \notin \mathit{var}(t')$ or its execution is always prevented. In both cases we get $\sigma(t') \leftrightarrow \sigma'(t') \leftrightarrow \alpha$. Then, using Lemma I.1 and $t_i = f(t', x + t'')$, we have

$$\sigma'(t_i) \xrightarrow{\bar{\alpha}} p' \leftrightarrow \alpha \| p_n \leftrightarrow f(\alpha, p_n).$$

As $t \approx u$ implies $\sigma'(t) \leftrightarrow \sigma'(u)$, we infer that there must be a summand u_j such that $\sigma'(u_j) \xrightarrow{\bar{\alpha}} r$ for some $r \leftrightarrow f(\alpha, p_n)$. Notice that, since $\sigma(u) \leftrightarrow f(\alpha, p_n)$ and $\sigma(u_j) = \sigma'(u_j)$ if $x \notin \mathit{var}(u_j)$, then it must be the case that $x \in \mathit{var}(u_j)$, or otherwise we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, p_n)$. By Lemma 7.6, as only $R^f_{\bar{\alpha}}$ holds, we can distinguish two cases:

a. There is a term u' s.t. $u_j \xrightarrow{\bar{\alpha}} u'$ and $\sigma'(u') \leftrightarrow f(\alpha, p_n)$. Then, since $f(\alpha, p_n) \leftrightarrow \alpha \parallel p_n$ (Lemma I.1) we can apply the expansion law, obtaining $\sigma'(u') \leftrightarrow \alpha p_n + \sum_{i=1}^{n} \bar{\alpha}(\alpha \| \alpha^{\leq i}) + \sum_{i=1}^{n} \tau \alpha^{\leq i}$. As n is greater than the size of u, and thus of those of u_j and u', by Lemma 7.7 we get that u' has a summand y, for some variable y, such that $\sigma'(y) \leftrightarrow \sum_{k=1}^{m'} \bar{\alpha}(\alpha \| \alpha^{\leq i'_k}) + r'$, for some $m' > 1$, $1 \leq i'_1 < \cdots < i'_{m'} \leq n$ and closed term r'. Notice that we can infer that $y \neq x$, as $\sigma'(x) \not\leftrightarrow \sigma'(y)$ for any closed term r'. Thus we have $\sigma'(y) = \sigma(y)$ and we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, p_n)$ in that $\sigma(u_j)$ would be able to perform two $\bar{\alpha}$-moves in a row unlike $f(\alpha, p_n)$.

b. There are a variable y, a closed term r' and a configuration c s.t. $\sigma'(y) \xrightarrow{\bar{\alpha}} r'$, $u_j \xrightarrow{y_r}_{\bar{\alpha}} c$ and $\sigma'[y_d \mapsto r'](c) \leftrightarrow f(\alpha, p_n)$. We claim that it must be the case that $y = x$. To see this claim, assume towards a contradiction that $y \neq x$. We proceed by a case analysis on the possible occurrences of x in c.
   - $x \notin \mathit{var}(c)$ or $x \in \mathit{var}(c)$ but its occurrence is in a guarded context that prevents the execution of its closed instances. In this case we get $\sigma[y_d \mapsto r'](c) \leftrightarrow \sigma'[y_d \mapsto r'](c) \leftrightarrow f(\alpha, p_n)$. This contradicts $\sigma(u) \leftrightarrow f(\alpha, p_n)$ since we would have $\sigma(u) \xrightarrow{\bar{\alpha}} r \leftrightarrow f(\alpha, p_n)$, and such a transitions cannot be mimicked by $f(\alpha, p_n)$.
   - $x \in \mathit{var}(c)$ and its execution is not prevented. We can distinguish two sub-cases, according to whether the occurrence of x is guarded or not.
     - Assume that x occurs guarded in c. In this case we get a contradiction with $r \leftrightarrow f(\alpha, p_n)$ in that

$$\begin{aligned} n + 2 &= \mathit{depth}(f(\alpha, p_n)) \\ &= \mathit{depth}(r) \\ &\geq 1 + \mathit{depth}(\sigma'(x)) \qquad (x \text{ is guarded}) \\ &= n + 3. \end{aligned}$$

     - Assume now that $x \lhd^{\alpha}_{b} c$. This case contradicts our assumption that $\sigma(u) \leftrightarrow f(\alpha, p_n)$ since we would have $\sigma(u) \xrightarrow{\bar{\alpha}} \sigma[y_d \mapsto r'](c) \xrightarrow{\bar{\alpha}}$, due to Lemmas F.6 and 7.5, whereas $f(\alpha, p_n)$ cannot perform two $\bar{\alpha}$-moves in a row.

   Therefore, we can conclude that it must be the case that $y = x$ and $r' = p_n$. In particular, notice that $x \lhd^{\bar{\alpha}}_{r} u_j$. We now proceed by a case analysis on the structure of u_j to show that $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$.

   i. $u_j = x$. This case is vacuous, as $\sigma'(x) \xrightarrow{\bar{\alpha}} p_n$ and $p_n \not\leftrightarrow f(\alpha, p_n)$.

   ii. $u_j = f(u', u'')$ for some u', u''. Notice that $x \lhd^{\bar{\alpha}}_{r} u_j$ can be due only to $x \lhd^{\bar{\alpha}}_{r} u''$. We have $\sigma'(u'') \xrightarrow{\bar{\alpha}} r_1$ and $\sigma'(u_j) \xrightarrow{\bar{\alpha}} \sigma'(u') \| r_1 \leftrightarrow f(\alpha, p_n)$. Since $f(\alpha, p_n) \leftrightarrow \alpha \| p_n$ and both α and p_n are prime, by the existence of a unique prime decomposition, we distinguish two cases:
      - Case $\sigma'(u') \leftrightarrow \alpha$ and $r_1 \leftrightarrow p_n$. As $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\bar{\alpha}\}$, $R^f_{\bar{\alpha}}$, $\sigma'(u') \leftrightarrow \alpha$ and $\sigma(u)$ has no $\mathbf{0}$ factors, we get that either $x \notin \mathit{var}(u')$ or x occurs in u' but its execution is prevented by the rules for f. Therefore $\sigma'(u') \leftrightarrow \sigma(u') \leftrightarrow \alpha$. We aim at showing that u'' has a summand x. We proceed by proving that the only other possibility, namely $u'' = f(w_1, w_2)$ for some w_1, w_2 with $x \lhd^{\bar{\alpha}}_{r} w_2$, leads to a contradiction.
        As $\sigma'(u'') \xrightarrow{\bar{\alpha}} r_1 \leftrightarrow p_n$, we have $\sigma'(w_2) \xrightarrow{\bar{\alpha}} r_2$ and $\sigma'(w_1) \| r_2 \leftrightarrow p_n$. Since, p_n is prime, we have that either $\sigma'(w_1) \leftrightarrow \mathbf{0}$ and $r_2 \leftrightarrow p_n$, or $\sigma'(w_1) \leftrightarrow p_n$ and $r_2 \leftrightarrow \mathbf{0}$. In both cases, as $\sigma'(x) \not\leftrightarrow \sigma'(w_1)$ and the previous considerations, we infer $\sigma(w_1) \leftrightarrow \sigma'(w_1)$. Hence, the former case contradicts $\sigma(u)$ not having $\mathbf{0}$ factors. The latter case contradicts $\sigma(u) \leftrightarrow f(\alpha, p_n)$ as, considering that $x \lhd^{\bar{\alpha}}_{r} w_2$, the transition $\sigma'(w_2) \xrightarrow{\bar{\alpha}} r_2 \leftrightarrow \mathbf{0}$ cannot be due to $\sigma'(x)$ and therefore it would be available also to $\sigma(w_2)$ thus implying $\sigma(u_j) \xrightarrow{\bar{\alpha}} r''$ with $r'' \leftrightarrow f(\alpha, p_n)$.
        Summing up, we have argued that u'' has a summand x. Therefore, by Equation (14),

$$\sigma(u'') \leftrightarrow \sum_{k=1}^{m} \bar{\alpha}.\alpha^{\leq i_k} + r'' \ ,$$

for some closed term r''. We have already noted that

$$\sigma(u') \leftrightarrow \sigma'(u') \leftrightarrow \alpha \ .$$

Thus, using the congruence properties of bisimulation equivalence, we may infer that

$$\begin{aligned}\sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\ &\leftrightarrow f(\alpha, \sum_{k=1}^{m} \bar\alpha\alpha^{\le i_k} + r'') \ .\end{aligned}$$

In light of this equivalence, we have $\sigma(u_j) \xrightarrow{\alpha} r' \leftrightarrow \sigma(u'')$ and thus $\sigma(u) \xrightarrow{\alpha} r'$. Since, by hypothesis, $\sigma(u) \leftrightarrow f(\alpha, p_n)$ then it must be the case that $r' \leftrightarrow p_n$. Therefore, we can conclude that $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$. Hence, $\sigma(u)$ has the desired summand.

- Case $\sigma'(u') \leftrightarrow p_n$ and $r_1 \leftrightarrow \alpha$. By reasoning as above, we can infer that either $x \notin var(u')$ or it is blocked by the rules for f, so that

  $$\sigma'(u') \leftrightarrow \sigma(u') \leftrightarrow p_n.$$

  However, $depth(\sigma(x)) \geq 3$, and $x \lessdot^{\bar\alpha}_{\mathrm{r}} u''$ with $\mathrm{init}(\sigma(x)) = \{\bar\alpha\}$ give us, by Lemma F.7, that $depth(\sigma(u'')) \geq depth(\sigma(x))$. Therefore we get a contradiction, in that

$$\begin{aligned}n + 2 &= depth(f(\alpha, p_n)) \\ &= depth(\sigma(u)) \\ &\geq depth(\sigma(u_j)) \\ &= depth(f(\sigma(u'), \sigma(u''))) \\ &\geq depth(\sigma(u')) + depth(\sigma(u'')) \\ &\geq depth(\sigma(u')) + depth(\sigma(x)) \\ &\geq n + 1 + 3 \\ &= n + 4.\end{aligned}$$

The proof of Proposition I.3 is now complete. □

I.1 Formal proof of Theorem 10.1

Proof of Theorem 10.1. Assume that $\mathcal{E}$ is a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in $\mathcal{E}$:

1. $E \vdash p \approx q$,
2. $p \leftrightarrow q \leftrightarrow f(\alpha, p_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, p_n)$.

We proceed by induction on the depth of the closed proof of the equation $p \approx q$ from $\mathcal{E}$, to prove that q has a summand bisimilar to $f(\alpha, p_n)$ as well. Recall that, without loss of generality, we may assume that $\mathcal{E}$ is closed with respect to symmetry, and thus applications of symmetry happen first in equational proofs. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from $\mathcal{E}$. The case of reflexivity is trivial, and that of transitivity follows by applying twice the inductive hypothesis. We proceed now to a detailed analysis of the remaining cases:

1. Case $E \vdash p \approx q$ because $\sigma(t) = p$ and $\sigma(u) = q$ for some terms t, u with $E \vdash t \approx u$ and closed substitution σ. The proof of this case follows by Proposition I.3.
2. Case $E \vdash p \approx q$ because $p = \mu.p'$ and $q = \mu.q'$ for some p', q' with $E \vdash p' \approx q'$. This case is vacuous in that $p = \mu.p' \not\leftrightarrow f(\alpha, p_n)$ and thus p does not have a summand bisimilar to $f(\alpha, p_n)$.
3. $E \vdash p \approx q$ because $p = p_1 + p_2$ and $q = q_1 + q_2$ for some p_i, q_i with $E \vdash p_i \approx q_i$, for $i \in \{1, 2\}$. Since p has a summand bisimilar to $f(\alpha, p_n)$ then so does either p_1 or p_2. Assume without loss of generality that p_1 has such a summand. As $p \leftrightarrow f(\alpha, p_n)$ then $p_1 \leftrightarrow f(\alpha, p_n)$ holds as well. Then, from $E \vdash p_1 \approx q_1$ we infer $q_1 \leftrightarrow f(\alpha, p_n)$. Thus, by the inductive hypothesis we obtain that q_1 has a summand bisimilar to $f(\alpha, p_n)$ and, consequently, so does q.
4. $E \vdash p \approx q$ because $p = f(p_1, p_2)$ and $q = f(q_1, q_2)$ for some p_i, q_i with $E \vdash p_i \approx q_i$, for $i \in \{1, 2\}$. By the proviso of the theorem p, q have neither $\mathbf{0}$ summands nor factors, thus implying $p_i, q_i \not\leftrightarrow \mathbf{0}$. Hence, from $p \leftrightarrow f(\alpha, p_n)$ and $p = f(p_1, p_2)$ and Lemma I.2 we obtain $p_1 \leftrightarrow \alpha$ and $p_2 \leftrightarrow p_n$, thus implying, by the soundness of the equations in $\mathcal{E}$, that $q_1 \leftrightarrow \alpha$ and $q_2 \leftrightarrow p_n$, so that $q = f(\alpha, p_n)$. In both cases, we can infer that q has itself as the desired summand.

This completes the proof of Theorem 10.1 and thus of Theorem 5.1 in the case of an operator f that does not distribute over summation in either argument, case $L^f_\alpha, R^f_{\bar\alpha}, S^f_{\alpha,\bar\alpha}$. □

J Proof of Theorem 10.2

The proof of Theorem 10.2 follows that of Theorem 10.1 in a step by step manner, by exploiting Proposition J.1 below in place of Proposition I.3. The only difference with the proof of Proposition I.3 is that, in the case at hand, Lemma I.1 does not hold anymore. (In fact one could prove, as done for Lemma G.2, that $f(\alpha, p_n)$ is prime for all $n \geq 0$.)

Proposition J.1. *Assume an operator f such that only L^f_α holds for α, only $R^f_{\bar\alpha}$ holds for $\bar\alpha$, and only $S_{\bar\alpha,\alpha}$ holds.*

Let $t \approx u$ be an equation over CCS_f^- that is sound modulo $\leftrightarrow$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor factors, and $p, q \leftrightarrow f(\alpha, p_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: The proof follows exactly as the proof of Proposition I.3, with the only difference that when we consider the derived transition

$$\sigma'(t_1) \xrightarrow{\bar{\alpha}} p'$$

we have that $p' \underline{\leftrightarrow} \alpha \| p_n \not\underline{\leftrightarrow} f(\alpha, p_n)$. However, by substituting $f(\alpha, p_n)$ with $\alpha \| p_n$ in the remaining of the proof, the same arguments hold. $\square$

K Proof of Theorem 11.1

First of all, we remark that the witness processes $f(\tau, q_n)$ enjoy the properties formalized in Lemmas K.1 and K.2 below.

Lemma K.1. *For each $n \geq 0$ it holds that $f(\tau, q_n) \underline{\leftrightarrow} \tau \| q_n$.*

Lemma K.2. *Let $n \geq 1$. Assume that $f(p, q) \underline{\leftrightarrow} f(\tau, q_n)$ for $p, q \not\underline{\leftrightarrow} \mathbf{0}$. Then $p \underline{\leftrightarrow} \tau$ and $q \underline{\leftrightarrow} q_n$.*

Proof: The proof is analogous to that of Lemma G.3. We remark that the τ-transition by $f(\tau, q_n)$ can be mimicked only by a τ-move by p. To see this, we show that any other case would lead to a contradiction with the proviso of the lemma $f(p, q) \underline{\leftrightarrow} f(\tau, q_n)$. In particular, we distinguish three cases, according to which rule of type (5) is available for f and whether the predicates R^f_τ holds or not.

- Assume $p \xrightarrow{\alpha} p'$ and $q \xrightarrow{\bar{\alpha}} q'$ with $p' \| q' \underline{\leftrightarrow} q_n$. This would contradict $f(\tau, q_n) \underline{\leftrightarrow} f(p, q)$ since $f(p, q) \xrightarrow{\bar{\alpha}} p \| q'$, whereas $f(\tau, q_n) \stackrel{\bar{\alpha}}{\not\longrightarrow}$.
- Assume $p \xrightarrow{\bar{\alpha}} p'$ and $q \xrightarrow{\alpha} q'$ with $p' \| q' \underline{\leftrightarrow} q_n$. Notice that since q_n is prime, then we have that either $p' \underline{\leftrightarrow} \mathbf{0}$ and $q' \underline{\leftrightarrow} q_n$, or $p' \underline{\leftrightarrow} q_n$ and $q' \underline{\leftrightarrow} \mathbf{0}$. The latter case contradicts $f(p, q) \underline{\leftrightarrow} f(\tau, q_n)$ since the transition $f(p, q) \xrightarrow{\alpha} p \| q' \underline{\leftrightarrow} p \| q_n$ cannot be mimicked by $f(\tau, q_n)$. The former case also contradicts the proviso of the lemma, since we would have $f(p, q) \xrightarrow{\alpha} p \| q' \underline{\leftrightarrow} p \xrightarrow{\bar{\alpha}} p' \underline{\leftrightarrow} q_n$, whereas $f(\tau, q_n) \xrightarrow{\alpha} \tau \| \bar{\alpha}^{\leq i}$, for some $i \in \{1, \ldots, n\}$, and there is no r such that $\tau \| \bar{\alpha}^{\leq i} \xrightarrow{\bar{\alpha}} r$ and $r \underline{\leftrightarrow} q_n$, for any $i \in \{1, \ldots, n\}$.
- Finally, assume that the predicate R^f_μ holds, and thus that f has a rule of type (7) with label τ. Hence, assume $q \xrightarrow{\tau} q'$, for some q', so that $f(p, q) \xrightarrow{\tau} p \| q' \underline{\leftrightarrow} q_n$. Since q_n is prime and $p \not\underline{\leftrightarrow} \mathbf{0}$, we have that $p \underline{\leftrightarrow} q_n$ and $q' \underline{\leftrightarrow} \mathbf{0}$. So, by congruence closure, we get

  $$f(p, q) \underline{\leftrightarrow} f(q_n, q) \underline{\leftrightarrow} f(\tau, q_n).$$

  Since $f(\tau, q_n) \xrightarrow{\alpha} \tau \| \bar{\alpha}^{\leq n}$ and only R^f_α holds, we have that $q \xrightarrow{\alpha} q_1$ for some q_1 such that $q_n \| q_1 \underline{\leftrightarrow} \tau \| \bar{\alpha}^{\leq n}$, which is a contradiction as $q_n \xrightarrow{\alpha}$ implies $q_n \| q_1 \xrightarrow{\alpha}$, whereas $\tau \| \bar{\alpha}^{\leq n} \stackrel{\alpha}{\not\longrightarrow}$.

$\square$

The same reasoning used in the proof of Theorem 10.1 allows us to prove Theorem 11.1, by exploiting Proposition K.3 in place of Proposition I.3.

Proposition K.3. *Assume an operator f such that only R^f_α and $R^f_{\bar{\alpha}}$ hold for $\alpha, \bar{\alpha}$, and L^f_τ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\underline{\leftrightarrow}$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor factors, and $p, q \underline{\leftrightarrow} f(\tau, q_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\tau, q_n)$, then so does q.

Proof: The claim follows by the same arguments used in the proof of Proposition I.3 and by considering the substitution

$$\sigma' = \sigma[x \mapsto \alpha q_n].$$

$\square$