Are Two Binary Operators Necessary to Finitely Axiomatise Parallel Composition?

Luca Aceto
Reykjavik University, Iceland
Gran Sasso Science Institute, Italy

Valentina Castiglioni
Reykjavik University, Iceland

Wan Fokkink
Vrije Universiteit Amsterdam, The Netherlands

Anna Ingólfsdóttir
Reykjavik University, Iceland

Bas Luttik
Eindhoven University of Technology, The Netherlands

Abstract

Bergstra and Klop have shown that *bisimilarity* has a *finite* equational axiomatisation over ACP/CCS extended with the binary *left* and *communication merge* operators. Moller proved that auxiliary operators are *necessary* to obtain a finite axiomatisation of bisimilarity over CCS, and Aceto et al. showed that this remains true when *Hennessy's merge* is added to that language. These results raise the question of whether there is *one* auxiliary *binary* operator whose addition to CCS leads to a finite axiomatisation of bisimilarity. This study provides a *negative answer* to that question based on three reasonable assumptions.

2012 ACM Subject Classification Theory of computation → Equational logic and rewriting

Keywords and phrases Equational logic, CCS, bisimulation, parallel composition.

Digital Object Identifier 10.4230/LIPIcs...

Funding This work has been supported by the project '*Open Problems in the Equational Logic of Processes*' (OPEL) of the Icelandic Research Fund (grant No. 196050-051).

1 Introduction

The purpose of this paper is to provide an answer to the following problem (see [1, Problem 8]): *Are the left merge and the communication merge operators* necessary *to obtain a finite equational axiomatisation of bisimilarity over the language CCS?* The interest in this problem is threefold, as an answer to it would: **1.** provide the first study on the finite axiomatisability of operators whose operational semantics is not determined a priori, **2.** clarify the status of the auxiliary operators *left merge* and *communication merge*, proposed in [11], in the finite axiomatisation of parallel composition, and **3.** give further insight into properties that auxiliary operators used in the finite equational characterisation of parallel composition ought to afford. We prove that, under some reasonable simplifying assumptions, whose role in our technical developments we discuss below, there is no auxiliary binary operator that can be added to CCS to yield a finite equational axiomatisation of bisimilarity. Despite falling short of solving the above-mentioned problem in full generality, our negative result is a substantial generalisation of previous non-finite-axiomatisability theorems by Moller [23, 24] and Aceto et al. [4].

In order to put our contribution in context, we first describe the history of the problem we tackle and then give a bird's eye view of our results.

The story so far In the late 1970s, Milner developed the *Calculus of Communicating Systems* (CCS) [20], a formal language based on a message-passing paradigm and aimed at describing communicating processes from an operational point of view. In detail, a *labelled transition system* (LTS) [18] was used to equip language expressions with an *operational semantics* [27] and was defined using a collection of syntax-driven rules. The analysis of process behaviour was carried out via an observational *bisimulation*-based theory [26] that defines when two states in an LTS describe the same behaviour. In particular, CCS included a *parallel composition operator* $\|$ to model the interactions among processes. Such an operator, also known as *merge* [11, 12], allows one both to *interleave* the behaviours of its argument processes (modelling concurrent computations) and to enable some form of *synchronisation* between them (modelling interactions). Later on, in collaboration with Hennessy, Milner studied the *equational theory* of (recursion free) CCS and proposed a *ground-complete axiomatisation* for it modulo bisimilarity [17]. More precisely, Hennessy and Milner presented a set $\mathcal{E}$ of *equational axioms* from which all equations over closed CCS terms (namely those with no occurrences of variables) that are *valid modulo bisimilarity* can be derived using the rules of *equational logic* [28]. Notably, the set $\mathcal{E}$ included infinitely many axioms, which were instances of the *expansion law* that was used to 'simulate equationally' the operational semantics of the parallel composition operator.

The ground-completeness result by Hennessy and Milner started the quest for a finite axiomatisation of CCS's parallel composition operator modulo bisimilarity.

Bergstra and Klop showed in [11] that a finite ground-complete axiomatisation modulo bisimilarity can be obtained by enriching CCS with two auxiliary operators, namely the *left merge* $\llfloor$ and the *communication merge* $|$, expressing respectively one step in the asymmetric pure interleaving and the synchronous behaviour of $\|$. Their result was then strengthened by Aceto et al. in [6], where it is proved that, over the fragment of CCS without recursion, restriction and relabelling, the auxiliary operators $\llfloor$ and $|$ allow for finitely axiomatising $\|$ modulo bisimilarity also when CCS terms with variables are considered. Moreover, in [9] that result is extended to the fragment of CCS with relabelling and restriction, but without communication. From those studies, we can infer that the left merge and communication merge operators are *sufficient* to finitely axiomatise parallel composition modulo bisimilarity. But is the addition of auxiliary operators *necessary* to obtain a finite equational axiomatisation, or can the use of the expansion law in the original axiomatisation of bisimilarity by Hennessy and Milner be replaced by a finite set of sound CCS equations?

To address that question, in [23, 24] Moller considered a minimal fragment of CCS, including only action prefixing, nondeterministic choice and interleaving, and proved that, even in the presence of a single action, bisimilarity does not afford a finite ground-complete axiomatisation over the closed terms in that language. This showed that auxiliary operators are indeed necessary to obtain a finite equational axiomatisation of bisimilarity. Adapting Moller's proof technique, Aceto et al. proved, in [4], that if we replace $\llfloor$ and $|$ with the so called *Hennessy's merge* $\not\llfloor$ [16], which denotes an asymmetric interleaving with communication, then the collection of equations that hold modulo bisimilarity over the recursion, restriction and relabelling free fragment of CCS enriched with $\not\llfloor$ is not finitely based (in the presence of at least two distinct complementary actions).

A natural question that arises from those *negative* results is the following:

Can one obtain a finite axiomatisation of the parallel composition operator in bisimulation semantics by adding only one binary operator *to the signature of (recursion, restriction, and relabelling free) CCS?* (P)

In this paper, we provide a partial *negative answer* to that question. (Note that, in (P), we focus on binary operators, like all the variations on parallel composition mentioned above, since using a ternary operator one can express the left and communication merge operators and, in fact, an arbitrary number of binary operators.)

Our contribution We analyse the axiomatisability of parallel composition over the language CCS_f, namely CCS enriched with a binary operator f that we use to express $\|$ as a derived operator. We prove that, under three reasonable assumptions, an auxiliary operator f alone does not allow us to obtain a finite ground-complete axiomatisation of CCS_f modulo bisimilarity.

To this end, the only knowledge we assume on the operational semantics of f is that it is formally defined by rules in the de Simone format [14] (Assumption 1) and that the behaviour of the parallel composition operator is expressed equationally by a law that is akin to the one used by Bergstra and Klop to define $\|$ in terms of $\lfloor\!\lfloor$ and $|$ (Assumption 2). We then argue that the latter assumption yields that the equation

$$x \| y \approx f(x,y) + f(y,x) \tag{A}$$

is valid modulo bisimilarity. Next we proceed by a case analysis over the possible sets of de Simone rules defining the behaviour of f, in such a way that the validity of Equation (A) modulo bisimilarity is guaranteed. To fully characterise the sets of rules that may define f, we introduce a third simplifying assumption: the target of each rule for f is either a variable or a term obtained by applying a single CCS_f operator to the variables of the rule, according to the constraints of the de Simone format (Assumption 3). Then, for each of the resulting cases, we show the desired negative result using proof-theoretic techniques that have their roots in Moller's classic results in [23, 24]. This means that we identify a (case-specific) property of terms denoted by W_n for $n \geq 0$. The idea is that, when n is *large enough*, W_n is preserved by provability from finite, sound axiom systems. Hence, whenever $\mathcal{E}$ is a finite, sound axiom system and an equation $p \approx q$ is derivable from $\mathcal{E}$, then either both terms p and q satisfy W_n, or none of them does. The negative result is then obtained by exhibiting a (case-specific) infinite family of valid equations $\{e_n \mid n \geq 0\}$ in which W_n is not preserved, that is, for each $n \geq 0$, W_n is satisfied only by one side of e_n. Due to the choice of W_n, this means that the equations in the family cannot all be derived from a finite set of valid axioms and therefore no finite, sound axiom system can be complete.

To the best of our knowledge, in this paper we propose the first non-finite axiomatisability result for a process algebra in which one of the operators, namely the auxiliary operator f, does not have a fixed semantics. However, for our technical developments, it has been necessary to restrict the search space for f by means of the aforementioned simplifying assumptions. To our mind, those assumptions are 'reasonable' because they allow us to simplify the combinatorial complexity of our analysis without excessively narrowing down the set of operators captured by our approach. There are three main reasons behind Assumption 1:

- The de Simone format is the simplest congruence format for bisimilarity. Hence we must be able to deal with this case before proceeding to any generalisation.
- The specification of parallel composition, left merge and communication merge operators (and of the vast majority of process algebraic operators) is in de Simone format. Hence, that format was a natural choice also for operator f.
- The simplicity of the de Simone rules allows us to reduce considerably the complexity of our case analysis over the sets of available rules for the operator f. However, as witnessed

by the developments in this article, even with this simplification, the proof of the desired negative result requires a large amount of delicate, technical work.

Assumptions 2 and 3 still allow us to obtain a significant generalisation of related works, such as [4], as we can see them as an attempt to identify the requirements needed to apply Moller's proof technique to Hennessy's merge like operators. We stress that the reason for adding Assumption 3 is purely technical: it plays a role in the proof of *one* of the claims in our combinatorial analysis of the rules that f may have (see Lemma 10). Although we conjecture that the assumption is not actually necessary to obtain that claim, we were unable to prove it without the assumption.

Even though the vast literature on process algebras offers a plethora of non-finite axiomatisability results for a variety of languages and semantics (see, for instance, the survey [5] from 2005), we are not aware of any previous attempt at proving a result akin to the one we present here. We have already addressed at length how our contribution fits within the study of the equational logic of processes and how it generalises previous results in that field. The proof-theoretic tools and the approach we adopt in proving our main theorem, which links equational logic with structural operational semantics and builds on a number of previous achievements (such as those in [2]), may have independent interest for researchers in logic in computer science. To our mind, achieving an answer to question (P) in full generality would be very pleasing for the concurrency-theory community, as it would finally clarify the canonical role of Bergstra and Klop's auxiliary operators in the finite axiomatisation of parallel composition modulo bisimilarity.

2 Background

In this section we introduce the basic definitions and results on which the technical developments to follow are based.

Labelled Transition Systems and Bisimilarity As semantic model we consider classic *labelled transition systems* [18].

▶ **Definition 1.** *A labelled transition system (LTS) is a triple $(S, A, \rightarrow)$, where S is a set of states, A is a set of actions, and $\rightarrow \subseteq S \times A \times S$ is a (labelled) transition relation.*

As usual, we use $t \xrightarrow{\mu} t'$ in lieu of $(t, \mu, t') \in \rightarrow$. For each $t \in S$ and $\mu \in A$, we write $t \xrightarrow{\mu}$ if $t \xrightarrow{\mu} t'$ holds for some t', and $t \stackrel{\mu}{\not\rightarrow}$ otherwise.

In this paper, we shall consider the states in a labelled transition system modulo bisimilarity [21, 26], allowing us to establish whether two processes have the same behaviour.

▶ **Definition 2.** *Let $(S, A, \rightarrow)$ be a labelled transition system. Bisimilarity, denoted by $\underline{\leftrightarrow}$, is the largest binary symmetric relation over S such that whenever $t \underline{\leftrightarrow} u$ and $t \xrightarrow{\mu} t'$, then there is a transition $u \xrightarrow{\mu} u'$ with $t' \underline{\leftrightarrow} u'$. If $t \underline{\leftrightarrow} u$, then we say that t and u are bisimilar.*

It is well-known that bisimilarity is an equivalence relation (see, e.g., [21, 26]).

The Language CCS$_f$ The language we consider in this paper is obtained by adding a single binary operator f to the recursion, restriction and relabelling free subset of Milner's CCS [21], henceforth referred to as CCS$_f$, and is given by the following grammar:

$$t ::= \mathbf{0} \mid x \mid a.t \mid \bar{a}.t \mid \tau.t \mid t + t \mid t \,\|\, t \mid f(t, t) \ ,$$

where x is a variable drawn from a countably infinite set $\mathcal{V}$, a is an action, and $\bar{a}$ is its complement. We assume that the actions a and $\bar{a}$ are distinct. Following [21], the action symbol τ will result from the synchronised occurrence of the complementary actions a and $\bar{a}$.

In order to obtain the desired negative results, it will be sufficient to consider the above language with three unary prefixing operators; so there is only one action a with its corresponding complementary action $\bar{a}$. Our results carry over unchanged to a setting with an arbitrary number of actions, and corresponding unary prefixing operators. Henceforth, we let $\mu \in \{a, \bar{a}, \tau\}$ and $\alpha \in \{a, \bar{a}\}$. As usual, we postulate that $\bar{\bar{a}} = a$. We shall use the meta-variables t, u, v, w to range over process terms, and write $var(t)$ for the collection of variables occurring in the term t. The *size* of a term is the number of operator symbols in it. A process term is *closed* if it does not contain any variables. Closed terms, or *processes*, will be typically denoted by p, q, r. Moreover, trailing $\mathbf{0}$'s will often be omitted from terms.

A *(closed) substitution* is a mapping from process variables to (closed) CCS_f terms. For every term t and substitution σ, the term obtained by replacing every occurrence of a variable x in t with the term $\sigma(x)$ will be written $\sigma(t)$. Note that $\sigma(t)$ is closed, if so is σ. We shall sometimes write $\sigma[x \mapsto p]$ to denote the substitution that maps the variable x into process p and behaves like σ on all other variables.

In the remainder of this paper, we exploit the associativity and commutativity of $+$ modulo bisimilarity and we consider process terms modulo them, namely we do not distinguish $t + u$ and $u + t$, nor $(t + u) + v$ and $t + (u + v)$. In what follows, the symbol $=$ will denote equality modulo the above identifications. We use a *summation* $\sum_{i \in \{1,\ldots,k\}} t_i$ to denote the term $t = t_1 + \cdots + t_k$, where the empty sum represents $\mathbf{0}$. We can also assume that the terms t_i, for $i \in \{1, \ldots, k\}$, do not have $+$ as head operator, and refer to them as the *summands* of t.

Henceforth, for each action μ and $m \geq 0$, we let μ^0 denote $\mathbf{0}$ and μ^{m+1} denote $\mu(\mu^m)$. For each action μ and positive integer $i \geq 0$, we also define

$$\mu^{\leq i} = \mu + \mu^2 + \cdots + \mu^i \ .$$

Equational Logic An *axiom system* $\mathcal{E}$ is a collection of (*process*) *equations* $t \approx u$ over CCS_f. An equation $t \approx u$ is *derivable* from an axiom system $\mathcal{E}$, notation $\mathcal{E} \vdash t \approx u$, if there is an *equational proof* for it from $\mathcal{E}$, namely if $t \approx u$ can be inferred from the axioms in $\mathcal{E}$ using the *rules* of *equational logic*, which are reflexivity, symmetry, transitivity, substitution and closure under CCS_f contexts. We refer the interested reader to Appendix B for a complete presentation of such rules.

We are interested in equations that are valid modulo some congruence relation $\mathcal{R}$ over closed terms. The equation $t \approx u$ is said to be *sound* modulo $\mathcal{R}$ if $\sigma(t) \,\mathcal{R}\, \sigma(u)$ for all closed substitutions σ. For simplicity, if $t \approx u$ is sound, then we write $t \,\mathcal{R}\, u$. An axiom system is *sound* modulo $\mathcal{R}$ if, and only if, all of its equations are sound modulo $\mathcal{R}$. Conversely, we say that $\mathcal{E}$ is *ground-complete* modulo $\mathcal{R}$ if $p \,\mathcal{R}\, q$ implies $\mathcal{E} \vdash p \approx q$ for all closed terms p, q. We say that $\mathcal{R}$ has a *finite*, ground-complete, axiomatisation, if there is a *finite* axiom system $\mathcal{E}$ that is sound and ground-complete for $\mathcal{R}$.

3 The simplifying assumptions

The aim of this paper is to investigate whether bisimilarity admits a finite equational axiomatisation over CCS_f, for some binary operator f. Of course, this question only makes sense if f is an operator that preserves bisimilarity. In this section we discuss two assumptions we shall make on the auxiliary operator f in order to meet such requirement and to tackle problem (P) in a simplified technical setting.

3.1 The de Simone format

One way to guarantee that f preserves bisimilarity is to postulate that the behaviour of f is described using Plotkin-style rules that fit a rule format that is known to preserve bisimilarity, see, e.g., [8] for a survey of such rule formats. The simplest format satisfying this criterion is the format proposed by de Simone in [14]. We believe that if we can't deal with operations specified in that format, then there is little hope to generalise our results. Therefore, we make the following

▶ **Assumption 1.** The behaviour of f is described by rules in de Simone format.

▶ **Definition 3.** *An SOS rule ρ for f is in de Simone format if it has the form*

$$\rho = \frac{\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}}{f(x_1, x_2) \xrightarrow{\mu} t} \tag{1}$$

where $I \subseteq \{1, 2\}$, $\mu, \mu_i \in \{a, \bar{a}, \tau\}$ $(i \in I)$, and moreover

- *the variables x_1, x_2 and y_i $(i \in I)$ are all different and are called the variables of the rule,*
- *t is a CCS_f term over variables $\{x_1, x_2, y_i \mid i \in I\}$, called the target of the rule, such that*
  - *each variable occurs at most once in t, and*
  - *if $i \in I$, then x_i does not occur in t.*

Henceforth, we shall assume, without loss of generality, that the variables x_1, x_2, y_1 and y_2 are the only ones used in operational rules. Moreover, if μ is the label of the transition in the conclusion of a de Simone rule ρ, we shall say that ρ has μ as *label*.

The SOS rules for all of the classic CCS operators, reported below, are in de Simone format, and so are those for Hennessy's $\not|$ operator from [16] and for Bergstra and Klop's left and communication merge operators [10], at least if we disregard issues related to the treatment of successful termination. Thus restricting ourselves to operators whose operational behaviour is described by de Simone rules leaves us with a good degree of generality.

$$\frac{}{\mu.x \xrightarrow{\mu} x} \qquad \frac{x \xrightarrow{\mu} x'}{x + y \xrightarrow{\mu} x'} \qquad \frac{y \xrightarrow{\mu} y'}{x + y \xrightarrow{\mu} y'} \qquad \frac{x \xrightarrow{\mu} x'}{x \parallel y \xrightarrow{\mu} x' \parallel y} \qquad \frac{y \xrightarrow{\mu} y'}{x \parallel y \xrightarrow{\mu} x \parallel y'} \qquad \frac{x \xrightarrow{\alpha} x',\ y \xrightarrow{\bar{\alpha}} y'}{x \parallel y \xrightarrow{\tau} x' \parallel y'}$$

The transition rules for the classic CCS operators above and those for the operator f give rise to transitions between CCS_f terms. The operational semantics for CCS_f is thus given by the LTS whose states are CCS_f terms, and whose transitions are those that are provable using the rules.

In what follows, we shall consider the collection of *closed CCS_f terms* modulo bisimilarity. Since the SOS rules defining the operational semantics of CCS_f are in de Simone's format, we have that bisimilarity is a congruence with respect to CCS_f operators, that is, $\mu p \underline{\leftrightarrow} \mu q$, $p + p' \underline{\leftrightarrow} q + q'$, $p \| p' \underline{\leftrightarrow} q \| q'$ and $f(p, p') \underline{\leftrightarrow} f(q, q')$ hold whenever $p \underline{\leftrightarrow} q$, $p' \underline{\leftrightarrow} q'$ and p, p', q, q' are closed CCS_f terms.

Bisimilarity is extended to arbitrary CCS_f terms thus:

▶ **Definition 4.** *Let t, u be CCS_f terms. Then $t \underline{\leftrightarrow} u$ if and only if $\sigma(t) \underline{\leftrightarrow} \sigma(u)$ for every closed substitution σ.*

3.2 Axiomatising $\|$ with f

Our second simplifying assumption concerns how the operator f can be used to axiomatise parallel composition. To this end, a fairly natural assumption on an axiom system over CCS_f is that it includes an equation of the form

$$x \| y \approx t(x,y) \tag{2}$$

where t is a CCS_f term that does not contain occurrences of $\|$ with $var(t) \subseteq \{x,y\}$. More precisely, the term will be in the general form $t(x,y) = \sum_{i \in I} t_i(x,y)$, where I is a finite index set and, for each $i \in I$, $t_i(x,y)$ does not have $+$ as head operator. Equation (2) essentially states that $\|$ is a derived operator in CCS_f modulo bisimilarity. To our mind, this is a natural, initial assumption to make in studying the problem we tackle in the paper.

We now proceed to refine the form of the term $t(x,y)$, in order to guarantee the soundness, modulo bisimilarity, of Equation (2). Intuitively, no term $t_i(x,y)$ can have prefixing as head operator. In fact, if $t(x,y)$ had a summand $\mu.t'(x,y)$, for some $\mu \in \{a, \bar{a}, \tau\}$, then one could easily show that $\mathbf{0} \| \mathbf{0} \not\mathrel{\underline{\leftrightarrow}} t(\mathbf{0}, \mathbf{0})$, since $t(\mathbf{0}, \mathbf{0})$ could perform a μ-transition, unlike $\mathbf{0} \| \mathbf{0}$. Similarly, $t(x,y)$ cannot have a variable as a summand, for otherwise we would have $a \| \tau \not\mathrel{\underline{\leftrightarrow}} t(a, \tau)$. Indeed, assume, without loss of generality, that $t(x,y)$ has a summand x. Then, $t(a,\tau) \xrightarrow{a} \mathbf{0}$, whereas $a \| \tau$ cannot terminate in one step. We can therefore assume that, for each $i \in I$, $t_i(x,y) = f(t_i^1(x,y), t_i^2(x,y))$ for some CCS_f terms $t_i^j(x,y)$, with $j \in \{1,2\}$. To further narrow down the options on the form that the subterms $t_i^j(x,y)$ might have, we would need to make some assumptions on the behaviour of the operator f. For the sake of generality, we assume that the terms $t_i^j(x,y)$ are in the simplest form, namely they are variables in $\{x,y\}$. Such an assumption is reasonable because to allow prefixing and/or nested occurrences of f-terms in the scope of the terms $t_i(x,y)$ we would need to define (at least partially) the operational semantics of f, thus making our results less general as, roughly speaking, we would need to study one possible auxiliary operator at a time (the one identified by the considered set of de Simone rules). Moreover, if we look at how parallel composition is expressed equationally as a derived operator in terms of Hennessy's merge or Bergstra and Klop's left and communication merge or as in [2], viz. via the equations

$$x \parallel y \approx (x \mathbin{|\!\!/} y) + (y \mathbin{|\!\!/} x)$$

$$x \parallel y \approx (x \mathbin{|\!\!\lfloor} y) + (y \mathbin{|\!\!\lfloor} x) + (x \mid y) \qquad x \parallel y \approx (x \mathbin{|\!\!\lfloor} y) + (x \mathbin{\rfloor\!\!|} y) + (x \mid y) \;,$$

we see the emergence of a pattern: the parallel composition operator is always expressed in terms of sums of terms built from the auxiliary operators and variables.

Therefore, from now on we'll make the following:

▶ **Assumption 2.** For some $J \subseteq \{x,y\}^2$, the equation

$$x \parallel y \approx \sum \{ f(z_1, z_2) \mid (z_1, z_2) \in J \} \tag{3}$$

holds modulo bisimilarity. We shall use t_J to denote the right-hand side of the above equation and use $t_J(p,q)$ to stand for the process $\sigma[x \mapsto p, y \mapsto q](t_J)$, for any closed substitution σ.

Using our assumptions, we further investigate the relation between operator f and parallel composition, obtaining a refined form for Equation (3) (Proposition 7 below).

▶ **Lemma 5.** *Assume that Assumptions 1 and 2 hold. Then:*

1. *The index set J on the right-hand side of (3) is non-empty.*

2. *The set of transition rules for f is non-empty.*
3. *Each transition rule for f has some premise.*
4. *The terms $f(x,x)$ and $f(y,y)$ are not summands of t_J.*

As consequence, we may infer that the index set J in the term t_J is either one of the singletons $\{(x,y)\}$ or $\{(y,x)\}$, or it is the set $\{(x,y),(y,x)\}$. Due to Moller's results to the effect that bisimilarity has no finite ground-complete axiomatisation over CCS [23, 25], the former option can be discarded, as shown in the following:

▶ **Proposition 6.** *If J is a singleton, then CCS_f admits no finite equational axiomatisation modulo bisimilarity.*

As a consequence, we can restate our Assumption 2 in the following simplified form:

▶ **Proposition 7.** *Equation* (3) *can be refined to the form:*

$$x \parallel y \approx f(x,y) + f(y,x) \ . \tag{4}$$

Moreover, in the light of Moller's results in [23, 25], we can restrict ourselves to considering only operators f such that $x \parallel y \approx f(x,y)$ does not hold modulo bisimilarity.

4 The operational semantics of f

In order to obtain the desired results, we shall, first of all, understand what rules f may and must have in order for Equation (4) to hold modulo bisimilarity (Proposition 11 below). We begin this analysis by restricting the possible forms the SOS rules for f *may* take.

▶ **Lemma 8.** *Suppose that f meets Assumption 1, and that Equation* (4) *is sound modulo bisimilarity. Let ρ be a de Simone rule for f with μ as label. Then:*

1. *If $\mu = \tau$ then the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ of ρ can only have one of the following possible forms:*
   - *$\{x_i \xrightarrow{\tau} y_i\}$ for some $i \in \{1,2\}$, or*
   - *$\{x_1 \xrightarrow{\alpha} y_1, x_2 \xrightarrow{\bar{\alpha}} y_2\}$ for some $\alpha \in \{a, \bar{a}\}$.*
2. *If $\mu = \alpha$ for some $\alpha \in \{a, \bar{a}\}$, then the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ can only have the form $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1,2\}$.*

The previous lemma limits the form of the premises that rules for f *may* have in order for Equation (4) to hold modulo bisimilarity. We now characterise the rules that f *must* have in order for it to satisfy that equation.

Firstly, we deal with *synchronisation*.

▶ **Lemma 9.** *Assume that Equation (4) holds modulo bisimilarity. Then the operator f must have a rule of the form*

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} t(y_1,y_2)} \tag{5}$$

for some $\alpha \in \{a, \bar{a}\}$ and term t. Moreover, for each rule for f of the above form the term $t(x,y)$ is bisimilar to $x \parallel y$.

Henceforth we assume, without loss of generality that the target of a rule of the form (5) is $y_1 \| y_2$. We introduce the unary predicates $S^f_{a,\bar{a}}$ and $S^f_{\bar{a},a}$ to identify which rules of type (5) are available for f. In detail, $S^f_{a,\bar{a}}$ holds if f has a rule of type (5) with premises $x_1 \xrightarrow{a} y_1$ and $x_2 \xrightarrow{\bar{a}} y_2$. $S^f_{\bar{a},a}$ holds in the symmetric case.

We consider now the *interleaving* behaviour in the rules for f. In order to properly characterise the rules for f as done in the previous Lemma 9, we consider an additional simplifying assumption on the form that the targets of the rules for f might have.

▶ **Assumption 3.** If t is the target of a rule for f, then t is either a variable or a term obtained by applying a single CCS_f operator to the variables of the rule, according to the constraints of the de Simone format.

▶ **Lemma 10.** *Let $\mu \in \{a, \bar{a}, \tau\}$. Then the operator f must have a rule of the form*

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1, x_2) \xrightarrow{\mu} t(y_1, x_2)} \tag{6}$$

or a rule of the form

$$\frac{x_2 \xrightarrow{\mu} y_2}{f(x_1, x_2) \xrightarrow{\mu} t(x_1, y_2)} \tag{7}$$

for some term t. Moreover, under Assumption 3, for each rule for f of the above forms the term $t(x, y)$ is bisimilar to $x \parallel y$.

Henceforth we assume, without loss of generality, that the target of a rule of the form (6) is $y_1 \| x_2$ and the target of a rule of the form (7) is $x_1 \| y_2$.

For each $\mu \in \{a, \bar{a}, \tau\}$, we introduce two unary predicates, L^f_μ and R^f_μ, that allow us to identify which rules with label μ are available for f. In detail,

- L^f_μ holds if f has a rule of the form (6) with label μ;
- R^f_μ holds if f has a rule of the form (7) with label μ.

We write $L^f_\mu \wedge R^f_\mu$ to denote that f has both a rule of the form (6) and one of the form (7) with label μ. We stress that, for each action μ, the validity of predicate L^f_μ does not prevent R^f_μ from holding, and vice versa. Throughout the paper, in case *only one* of the two predicates holds, we will clearly state it.

Summing up, we have obtained that:

▶ **Proposition 11.** *If f meets Assumptions 1 and 3 and Equation (4) is sound modulo bisimilarity, then f must satisfy $S^f_{\alpha,\bar{\alpha}}$ for at least one $\alpha \in \{a, \bar{a}\}$, and, for each $\mu \in \{a, \bar{a}, \mu\}$, at least one of L^f_μ and R^f_μ.*

The next proposition states that this is enough to obtain the soundness of Equation (4).

▶ **Proposition 12.** *Assume that all of the rules for f have the form (5), (6), or (7). If $S^f_{\alpha,\bar{\alpha}}$ holds for at least one $\alpha \in \{a, \bar{a}\}$, and, for each $\mu \in \{a, \bar{a}, \tau\}$, at least one of L^f_μ and R^f_μ holds, then Equation (4) is sound modulo bisimilarity.*

When the set of actions is $\{a, \bar{a}, \tau\}$, there are 81 operators that satisfy the constraints in Propositions 11 and 12, including parallel composition and Hennessy's merge. In general, when the set of actions has $2n + 1$ elements, there are 3^{3n+1} possible operators meeting those constraints.

5 The main theorem and its proof strategy

Our order of business will now be to use the information collected so far to prove our main result, namely the following theorem:

▶ **Theorem 13.** *Assume that f satisfies Assumptions 1 and 3, and that Equation* (4) *holds modulo bisimilarity. Then bisimilarity admits no finite equational axiomatisation over CCS_f.*

In this section, we discuss the general reasoning behind the proof of Theorem 13. In light of Propositions 11 and 12, to prove Theorem 13 we will proceed by a case analysis over the possible sets of allowed SOS rules for operator f. In each case, our proof method will follow the same general schema, which has its roots in Moller's arguments to the effect that bisimilarity is not finitely based over CCS (see, e.g., [4, 23, 24, 25]), and that we present here at an informal level.

The main idea is to identify a *witness property of the negative result*. This is a specific property of CCS_f terms, say W_n for $n \geq 0$, that, when n is *large enough*, is preserved by provability from finite axiom systems. Roughly, this means that if $\mathcal{E}$ is a finite set of axioms that are sound modulo bisimilarity, the equation $p \approx q$ is provable from $\mathcal{E}$, and n is greater than the size of all the terms in the equations in $\mathcal{E}$, then either both p and q satisfy W_n, or none of them does. Then, we exhibit an infinite family of valid equations, say e_n, called accordingly *witness family of equations for the negative result*, in which W_n is not preserved, namely it is satisfied only by one side of each equation. Thus, Theorem 13 specialises to:

▶ **Theorem 14.** *Suppose that Assumptions 1–3 are met. Let $\mathcal{E}$ be a finite axiom system over CCS_f that is sound modulo bisimilarity. Then there is an infinite family e_n, $n \geq 0$, of sound equations such that $\mathcal{E}$ does not prove the equation e_n, for each n that is larger than the size of each term in the equations in $\mathcal{E}$.*

In this paper, the property W_n corresponds to having a summand that is bisimilar to a specific process. In detail:

1. We identify, for each case, a family of processes $f(\mu, p_n)$, for $n \geq 0$, and the choices of μ and p_n are tailored to the particular set of SOS rules allowed for f. Moreover, process p_n will have size at least n, for each $n \geq 0$. Sometimes, we shall refer to the processes $f(\mu, p_n)$ as the *witness processes*.
2. We prove that by choosing n *large enough*, given a finite set of valid equations $\mathcal{E}$ and processes $p, q \leftrightarrow f(\mu, p_n)$, if $\mathcal{E} \vdash p \approx q$ and p has a summand bisimilar to $f(\mu, p_n)$, then also q has a summand bisimilar to $f(\mu, p_n)$. Informally, we will choose n greater than the size of all the terms in the equations in $\mathcal{E}$, so that we are guaranteed that the behaviour of the summand bisimilar to $f(\mu, p_n)$ is due to a closed substitution instance of a variable.
3. We provide an infinite family of valid equations e_n in which one side has a summand bisimilar to $f(\mu, p_n)$, but the other side does not. In light of item 2, this implies that such a family of equations cannot be derived from any finite collection of valid equations over CCS_f, modulo bisimilarity, thus proving Theorem 14.

To narrow down the combinatorial analysis over the allowed sets of SOS rules for f we examine first the *distributivity properties*, modulo $\leftrightarrow$, of the operator f over summation.

First of all, we notice that f cannot distribute over summation in both arguments. This is a consequence of our previous analysis of the operational rules that such an operator f may and must have in order for Equation (4) to hold. However, it can also be shown in a purely algebraic manner as we do in Appendix E.1.

▶ **Lemma 15.** *A binary operator satisfying Equation (4) cannot distribute over + in both arguments.*

Hence, we can limit ourselves to considering binary operators satisfying our constraints that, modulo bisimilarity, distribute over + in one argument or in none.

We consider these two possibilities in turn.

Distributivity in one argument Due to our Assumptions 1–3, we can exploit a result from [2] to characterise the rules for an operator f that distributes over summation in one of its arguments. More specifically, [2, Lemma 4.3] gives a condition on the rules for a *smooth operator* g in a GSOS system that includes the + operator in its signature, which guarantees that g distributes over summation in one of its arguments. (The rules defining the semantics of smooth operators are a generalisation of those in de Simone format.) Here we show that, for operator f, the condition in [2, Lemma 4.3] is both necessary and sufficient for distributivity of f in one of its two arguments.

▶ **Lemma 16.** *Let $i \in \{1, 2\}$. Modulo bisimilarity, operator f distributes over summation in its i-th argument if and only if each rule for f has a premise $x_i \xrightarrow{\mu_i} y_i$, for some μ_i.*

By Proposition 11, Lemma 16 implies that, when f is distributive in one argument, either L^f_μ holds for all $\mu \in \{a, \bar{a}, \tau\}$ or R^f_μ holds for all $\mu \in \{a, \bar{a}, \tau\}$, and $S_{\alpha,\bar{\alpha}}$ holds for at least one $\alpha \in \{a, \bar{a}\}$. Notice that if L^f_μ holds for each action μ and both $S^f_{a,\bar{a}}$ and $S^f_{\bar{a},a}$ hold, then f behaves as Hennessy's merge $|\!\!/$ [16], and our Theorem 14 specialises to [4, Theorem 22]. Hence we assume, without loss of generality, that $S^f_{\alpha,\bar{\alpha}}$ holds for only one $\alpha \in \{a, \bar{a}\}$. A similar reasoning applies if R^f_μ holds for each action μ.

In Section 6 we will present the proof of Theorem 14 in the case of an operator f that distributes over summation in its first argument (see Theorem 17).

Distributivity in neither argument We now consider the case in which f does not distribute over summation in either argument.

Also in this case, we can exploit Lemma 16 to obtain a characterisation of the set of rules allowed for an operator f satisfying the desired constraints. In detail, we infer that there must be $\mu, \nu \in \{a, \bar{a}, \tau\}$, not necessarily distinct, such that L^f_μ and R^f_ν hold. Otherwise, as f must have at least one rule for each action (see Proposition 11), at least one argument would be involved in the premises of each rule, and this would entail distributivity over summation in that argument.

We will split the proof of Theorem 14 for an operator f that, modulo bisimilarity, does not distribute over summation in either argument into three main cases:

1. In Section 7, we consider the case of $L^f_\alpha \wedge R^f_\alpha$ holding, for some $\alpha \in \{a, \bar{a}\}$ (Theorem 18).
2. In Section 8, we deal with the case of f having only one rule for α, only one rule for $\bar{\alpha}$, and such rules are of different forms. As we will see, we will need to distinguish two subcases, according to which predicate $S^f_{\alpha,\bar{\alpha}}$ holds (Theorem 19 and Theorem 20).
3. Finally, in Section 9, we study the case of f having only one rule with label α, only one rule with label $\bar{\alpha}$, and such rules are of the same type (Theorem 21).

The technical development of the aforementioned results can be found in Appendices F–H.

6 Negative result: the case $L^f_a, L^f_{\bar{a}}, L^f_\tau$

In this section we discuss the nonexistence of a finite axiomatisation of CCS_f in the case of an operator f that, modulo bisimilarity, distributes over summation in one of its arguments. We expand only the case of f distributing in the first argument. (The case of distributivity in the second argument follows by a straightforward adaptation of the arguments we use in this section.) Hence, in the current setting, we can assume the following set of SOS rules for f:

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1,x_2) \xrightarrow{\mu} y_1 \| x_2} \; \forall\, \mu \in \{a, \bar{a}, \tau\} \qquad \frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} y_1 \| y_2}$$

namely, only L^f_μ holds for each action μ, and only $S_{\alpha,\bar{\alpha}}$ holds for some $\alpha \in \{a, \bar{a}\}$.

According to the proof strategy sketched in Section 5, we now introduce a particular family of equations on which we will build our negative result. We define

$$p_n = \sum_{i=0}^{n} \bar{\alpha}\alpha^{\leq i} \qquad (n \geq 0)$$

$$e_n: \quad f(\alpha, p_n) \approx \alpha p_n + \sum_{i=0}^{n} \tau\alpha^{\leq i} \qquad (n \geq 0) \;.$$

It is not difficult to check that the infinite family of equations e_n is sound modulo bisimilarity.

Our order of business is now to prove the instance of Theorem 14 considering the family of equations e_n above, showing that no finite collection of equations over CCS_f that are sound modulo bisimilarity can prove all of the equations e_n $(n \geq 0)$.

Formally, we prove the following theorem:

▶ **Theorem 17.** *Assume an operator f such that only L^f_μ holds for each action μ and only $S^f_{\alpha,\bar{\alpha}}$ holds. Let $\mathcal{E}$ be a finite axiom system over CCS_f that is sound modulo $\underline{\leftrightarrow}$, n be larger than the size of each term in the equations in $\mathcal{E}$, and p, q be closed terms such that $p, q \underline{\leftrightarrow} f(\alpha, p_n)$. If $\mathcal{E} \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.*

Then, since the left-hand side of equation e_n, viz. the term $f(\alpha, p_n)$, has a summand bisimilar to $f(\alpha, p_n)$, whilst the right-hand side, viz. the term $\alpha p_n + \sum_{i=0}^{n} \tau\alpha^{\leq i}$, does not, we can conclude that the infinite collection of equations $\{e_n \mid n \geq 0\}$ is the desired witness family. Theorem 14 is then proved for the considered class of auxiliary binary operators.

7 Negative result: the case $L^f_\alpha \wedge R^f_\alpha$

In this section we investigate the first case, out of three, related to an operator f that does not distribute, modulo bisimilarity, over summation in either of its arguments.

We choose $\alpha \in \{a, \bar{a}\}$ and we assume that the set of rules for f includes

$$\frac{x_1 \xrightarrow{\alpha} y_1}{f(x_1,x_2) \xrightarrow{\alpha} y_1 \| x_2} \qquad \frac{x_2 \xrightarrow{\alpha} y_2}{f(x_1,x_2) \xrightarrow{\alpha} x_1 \| y_2} \;,$$

namely, predicate $L^f_\alpha \wedge R^f_\alpha$ holds for f.

We stress that the validity of the negative result we prove in this section does not depend on which types of rules with labels $\bar{\alpha}$ and τ are available for f. Moreover, the case of an operator for which $L^f_{\bar{\alpha}} \wedge R^f_{\bar{\alpha}}$ holds can be easily obtained from the one we are considering, and it is therefore omitted.

We now introduce the infinite family of valid equations, modulo bisimilarity, that will allow us to obtain the negative result in the case at hand. We define

$$q_n = \sum_{i=0}^{n} \alpha\bar{\alpha}^{\leq i} \qquad (n \geq 0)$$

$$e_n: \quad f(\alpha, q_n) \approx \alpha q_n + \sum_{i=0}^{n} \alpha(\alpha \| \bar{\alpha}^{\leq i}) \qquad (n \geq 0) \ .$$

Following the proof strategy from Section 5, we aim to show that, when n is *large enough*, the witness property of having a summand bisimilar to $f(\alpha, q_n)$ is preserved by derivations from a finite, sound axiom system $\mathcal{E}$, as stated in the following theorem:

▶ **Theorem 18.** *Assume an operator f such that $L^f_\alpha \wedge R^f_\alpha$ holds. Let $\mathcal{E}$ be a finite axiom system over CCS_f that is sound modulo $\leftrightarrow$, n be larger than the size of each term in the equations in $\mathcal{E}$, and p, q be closed terms such that $p, q \leftrightarrow f(\alpha, q_n)$. If $\mathcal{E} \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, q_n)$, then so does q.*

Then, we can conclude that the infinite collection of equations $\{e_n \mid n \geq 0\}$ is the desired witness family. In fact, the left-hand side of equation e_n, viz. the term $f(\alpha, q_n)$, has a summand bisimilar to $f(\alpha, q_n)$, whilst the right-hand side, viz. the term $\alpha q_n + \sum_{i=0}^{n} \alpha(\alpha \| \bar{\alpha}^{\leq i})$, does not. This concludes the proof of Theorem 14 in this case.

8 Negative result: the case L^f_α, $R^f_{\bar{\alpha}}$

In this section we deal with the second case related to an operator f that does not distribute over summation in either argument. This time, given $\alpha \in \{a, \bar{a}\}$, we assume that operator f has only one rule with label α and only one rule with label $\bar{\alpha}$, and moreover we assume such rules to be of different types. In detail, we expand the case in which for action α only the predicate L^f_α holds, and for action $\bar{\alpha}$ only $R^f_{\bar{\alpha}}$ holds, namely f has rules:

$$\frac{x_1 \xrightarrow{\alpha} y_1}{f(x_1, x_2) \xrightarrow{\alpha} y_1 \| x_2} \qquad \frac{x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1, x_2) \xrightarrow{\bar{\alpha}} x_1 \| y_2} \ .$$

Once again, the proof for the symmetric case with $L^f_{\bar{\alpha}}$ and R^f_α holding is omitted.

To obtain the proof of the negative result, we consider the same family of witness processes $f(\alpha, p_n)$ from Section 6. However, differently from the previous case, the definition of the witness family of equations depends on which rules of type (5) are available for f. More precisely, we need to split the proof of the negative result into two cases, according to whether the rules for f allow α and p_n to synchronise or not.

Case 1: Possibility of synchronisation Assume first that $S^f_{\alpha,\bar{\alpha}}$ holds, so that the rule

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1, x_2) \xrightarrow{\tau} y_1 \| y_2}$$

allows for synchronisation between α and p_n. In this setting, the infinite family of equations

$$e_n: \quad f(\alpha, p_n) \approx \alpha p_n + \sum_{i=0}^{n} \bar{\alpha}(\alpha \| \alpha^{\leq i}) + \sum_{i=0}^{n} \tau\alpha^{\leq i} \qquad (n \geq 0)$$

is sound modulo bisimilarity and it constitutes a family of witness equations.

▶ **Theorem 19.** *Assume an operator f such that only L^f_α holds for α, only $R^f_{\bar\alpha}$ holds for $\bar\alpha$, and $S^f_{\alpha,\bar\alpha}$ holds. Let $\mathcal{E}$ be a finite axiom system over CCS_f that is sound modulo $\underline{\leftrightarrow}$, n be larger than the size of each term in the equations in $\mathcal{E}$, and p, q be closed terms such that $p, q \underline{\leftrightarrow} f(\alpha, p_n)$. If $\mathcal{E} \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.*

This proves Theorem 14 in the considered setting, as the left-hand side of equation e_n, viz. the term $f(\alpha, p_n)$, has a summand bisimilar to $f(\alpha, p_n)$, whilst the right-hand side, viz. the term $\alpha p_n + \sum_{i=0}^{n} \bar\alpha(\alpha \| \bar\alpha^{\leq i}) + \sum_{i=0}^{n} \tau\alpha^{\leq i}$, does not.

Case 2: No synchronisation Assume now that the synchronisation between α and p_n is prevented, namely only $S^f_{\bar\alpha,\alpha}$ holds. Then, the witness family of equations changes as follows:

$$e_n: \quad f(\alpha, p_n) \approx \alpha p_n + \sum_{i=0}^{n} \bar\alpha(\alpha \| \alpha^{\leq i}) \qquad (n \geq 0) \ .$$

Our order of business is then to prove the following:

▶ **Theorem 20.** *Assume an operator f such that only L^f_α holds for α, only $R^f_{\bar\alpha}$ holds for $\bar\alpha$, and only $S^f_{\bar\alpha,\alpha}$ holds. Let $\mathcal{E}$ be a finite axiom system over CCS_f that is sound modulo $\underline{\leftrightarrow}$, n be larger than the size of each term in the equations in $\mathcal{E}$, and p, q be closed terms such that $p, q \underline{\leftrightarrow} f(\alpha, p_n)$. If $\mathcal{E} \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.*

Once again, the validity of Theorem 14 follows by noticing that the left-hand side of equation e_n, viz. the term $f(\alpha, p_n)$, has a summand bisimilar to $f(\alpha, p_n)$, whilst the right-hand side, viz. the term $\alpha p_n + \sum_{i=0}^{n} \bar\alpha(\alpha \| \bar\alpha^{\leq i})$, does not.

9 Negative result: the case L^f_τ

This section considers the last case in our analysis, namely that of an operator f that does not distribute, modulo bisimilarity, over summation in either argument and that has the same rule type for actions $\alpha, \bar\alpha$. Here, we present solely the case in which L^f_τ holds, and only $R^f_\alpha, R^f_{\bar\alpha}$ hold for $\alpha, \bar\alpha$, namely f has rules:

$$\frac{x_1 \xrightarrow{\tau} y_1}{f(x_1, x_2) \xrightarrow{\tau} y_1 \| x_2} \qquad \frac{x_2 \xrightarrow{\alpha} y_2}{f(x_1, x_2) \xrightarrow{\alpha} x_1 \| y_2} \qquad \frac{x_2 \xrightarrow{\bar\alpha} y_2}{f(x_1, x_2) \xrightarrow{\bar\alpha} x_1 \| y_2} .$$

The symmetric case can be obtained from this one in a straightforward manner.

Interestingly, the validity of the negative result we consider in this section is independent of which rules of type (5) are available for f, and of the validity of the predicate R^f_τ.

Consider the family of equations defined by:

$$e_n: \quad f(\tau, q_n) \approx \tau q_n + \sum_{i=0}^{n} \alpha(\tau \| \bar\alpha^{\leq i}) \qquad (n \geq 0)$$

where the processes q_n are the same used in Section 7. Theorem 21 below proves that the collection of equations e_n, $n \geq 0$, is a witness family of equations for our negative result.

▶ **Theorem 21.** *Assume an operator f such that L^f_τ holds and only R^f_α and $R^f_{\bar\alpha}$ hold for actions α and $\bar\alpha$. Let $\mathcal{E}$ be a finite axiom system over CCS_f that is sound modulo $\underline{\leftrightarrow}$, n be larger than the size of each term in the equations in $\mathcal{E}$, and p, q be closed terms such that $p, q \underline{\leftrightarrow} f(\tau, q_n)$. If $\mathcal{E} \vdash p \approx q$ and p has a summand bisimilar to $f(\tau, q_n)$, then so does q.*

As the left-hand side of equation e_n, viz. the term $f(\tau, q_n)$, has a summand bisimilar to $f(\tau, q_n)$, whilst the right-hand side, viz. the term $\tau q_n + \sum_{i=0}^{n} \alpha(\tau \| \bar{\alpha}^{\leq i})$, does not, we can conclude that the collection of infinitely many equations e_n $(n \geq 0)$ is the desired witness family. This concludes the proof of Theorem 14 for this case and our proof of Theorem 13.

10 Conclusions

In this paper, we have shown that, under a number of reasonable assumptions, we cannot use a single binary auxiliary operator f, whose semantics is defined via inference rules in the de Simone format, to obtain a finite axiomatisation of bisimilarity over the recursion, restriction and relabelling free fragment of CCS. Our result constitutes a first step towards a definitive justification of the canonical standing of the left and communication merge operators by Bergstra and Klop. We envisage the following ways in which we might generalise the contribution presented in this study. Firstly, we will try to get rid of Assumptions 2 and 3. Next, it is natural to relax Assumption 1 by considering the GSOS format [13] in place of the de Simone format. However, as shown by the heavy amount of technical results necessary to prove our main result even in our simplified setting, we believe that this generalisation cannot be obtained in a straightforward manner and that it will require the introduction of new techniques. It would also be very interesting to explore whether some version of problem (P) can be solved using existing results from equational logic and universal algebra.

References

1. Luca Aceto. Some of my favourite results in classic process algebra. *Bulletin of the EATCS*, 81:90–108, 2003.
2. Luca Aceto, Bard Bloom, and Frits W. Vaandrager. Turning SOS rules into equations. *Inf. Comput.*, 111(1):1–52, 1994. `doi:10.1006/inco.1994.1040`.
3. Luca Aceto, Taolue Chen, Anna Ingólfsdóttir, Bas Luttik, and Jaco van de Pol. On the axiomatizability of priority II. *Theor. Comput. Sci.*, 412(28):3035–3044, 2011. `doi:10.1016/j.tcs.2011.02.033`.
4. Luca Aceto, Wan Fokkink, Anna Ingólfsdóttir, and Bas Luttik. CCS with hennessy's merge has no finite-equational axiomatization. *Theor. Comput. Sci.*, 330(3):377–405, 2005. `doi:10.1016/j.tcs.2004.10.003`.
5. Luca Aceto, Wan Fokkink, Anna Ingólfsdóttir, and Bas Luttik. Finite equational bases in process algebra: Results and open questions. In Aart Middeldorp, Vincent van Oostrom, Femke van Raamsdonk, and Roel C. de Vrijer, editors, *Processes, Terms and Cycles: Steps on the Road to Infinity, Essays Dedicated to Jan Willem Klop, on the Occasion of His 60th Birthday*, volume 3838 of *Lecture Notes in Computer Science*, pages 338–367. Springer, 2005. URL: `https://doi.org/10.1007/11601548_18`.
6. Luca Aceto, Wan Fokkink, Anna Ingólfsdóttir, and Bas Luttik. A finite equational base for CCS with left merge and communication merge. *ACM Trans. Comput. Log.*, 10(1):6:1–6:26, 2009. `doi:10.1145/1459010.1459016`.
7. Luca Aceto, Wan Fokkink, Anna Ingólfsdóttir, and Sumit Nain. Bisimilarity is not finitely based over BPA with interrupt. *Theor. Comput. Sci.*, 366(1-2):60–81, 2006. `doi:10.1016/j.tcs.2006.07.003`.
8. Luca Aceto, Wan Fokkink, and Chris Verhoef. Structural operational semantics. In *Handbook of Process Algebra*, pages 197–292. North-Holland / Elsevier, 2001. `doi:10.1016/b978-044482830-9/50021-7`.
9. Luca Aceto, Anna Ingólfsdóttir, Bas Luttik, and Paul van Tilburg. Finite equational bases for fragments of CCS with restriction and relabelling. In *Proceedings of IFIP TCS 2008*, volume 273 of *IFIP*, pages 317–332, 2008. `doi:10.1007/978-0-387-09680-3\_22`.

10 Jan A. Bergstra and Jan Willem Klop. The algebra of recursively defined processes and the algebra of regular processes. In *Proceedings of ICALP 2011*, volume 172 of *Lecture Notes in Computer Science*, pages 82–94, 1984. doi:10.1007/3-540-13345-3_7.

11 Jan A. Bergstra and Jan Willem Klop. Process algebra for synchronous communication. *Information and Control*, 60(1-3):109–137, 1984. doi:10.1016/S0019-9958(84)80025-X.

12 Jan A. Bergstra and Jan Willem Klop. Algebra of communicating processes with abstraction. *Theor. Comput. Sci.*, 37:77–121, 1985. doi:10.1016/0304-3975(85)90088-X.

13 Bard Bloom, Sorin Istrail, and Albert R. Meyer. Bisimulation can't be traced. *J. ACM*, 42(1):232–268, 1995. doi:10.1145/200836.200876.

14 Robert de Simone. Higher-level synchronising devices in meije-SCCS. *Theor. Comput. Sci.*, 37:245–267, 1985. doi:10.1016/0304-3975(85)90093-3.

15 Jan Friso Groote. A new strategy for proving omega-completeness applied to process algebra. In *Proceedings of CONCUR '90*, volume 458 of *Lecture Notes in Computer Science*, pages 314–331, 1990. doi:10.1007/BFb0039068.

16 Matthew Hennessy. Axiomatising finite concurrent processes. *SIAM J. Comput.*, 17(5):997–1017, 1988. doi:10.1137/0217063.

17 Matthew Hennessy and Robin Milner. Algebraic laws for nondeterminism and concurrency. *J. ACM*, 32(1):137–161, 1985. doi:10.1145/2455.2460.

18 Robert M. Keller. Formal verification of parallel programs. *Commun. ACM*, 19(7):371–384, 1976. doi:10.1145/360248.360251.

19 Bas Luttik and Vincent van Oostrom. Decomposition orders another generalisation of the fundamental theorem of arithmetic. *Theor. Comput. Sci.*, 335(2-3):147–186, 2005. doi:10.1016/j.tcs.2004.11.019.

20 Robin Milner. *A Calculus of Communicating Systems*, volume 92 of *Lecture Notes in Computer Science*. Springer, 1980. doi:10.1007/3-540-10235-3.

21 Robin Milner. *Communication and concurrency*. PHI Series in computer science. Prentice Hall, 1989.

22 Robin Milner and Faron Moller. Unique decomposition of processes. *Theor. Comput. Sci.*, 107(2):357–363, 1993. doi:10.1016/0304-3975(93)90176-T.

23 Faron Moller. *Axioms for Concurrency*. PhD thesis, Department of Computer Science, University of Edinburgh, July 1989. Report CST-59-89. Also published as ECS-LFCS-89-84.

24 Faron Moller. The importance of the left merge operator in process algebras. In *Proceedings of ICALP '90*, volume 443 of *Lecture Notes in Computer Science*, pages 752–764, 1990. doi:10.1007/BFb0032072.

25 Faron Moller. The nonexistence of finite axiomatisations for CCS congruences. In *Proceedings of LICS '90*, pages 142–153, 1990. doi:10.1109/LICS.1990.113741.

26 David M. R. Park. Concurrency and automata on infinite sequences. In *Proceedings of GI-Conference*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183, 1981. doi:10.1007/BFb0017309.

27 Gordon D. Plotkin. A structural approach to operational semantics. Report DAIMI FN-19, Computer Science Department, Aarhus University, 1981.

28 Walter Taylor. Equational logic. In *Contributions to Universal Algebra*, pages 465 – 501. North-Holland, 1977. doi:https://doi.org/10.1016/B978-0-7204-0725-9.50040-X.

A Depth and norm of processes

We introduce here some additional notation and notions that will be useful for the technical development of our results.

The *initials* of t are the actions that label the outgoing transitions of t, that is, $\mathrm{init}(t) = \{\mu \mid t \xrightarrow{\mu}\}$. For a sequence of actions $s = \mu_1 \cdots \mu_k$ ($k \geq 0$), and states t, t', we write $t \xrightarrow{s} t'$ iff there exists a sequence of transitions $t = t_0 \xrightarrow{\mu_1} t_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_k} t_k = t'$. If $t \xrightarrow{s} t'$ holds for some state t', then s is a *trace* of t. Moreover, we say that s is a maximal trace of t if $\mathrm{init}(t') = \emptyset$. By means of traces, we associate two classic notions with a state t: its *depth*, denoted by $depth(t)$, and its *norm*, denoted by $norm(t)$. For a state t whose set of traces is finite, they express, respectively, the length of a *longest* trace of t and that of a *shortest* maximal trace. Formally, $depth(t) = \sup\{k \mid \ t \text{ has a trace of length } k\}$ and $norm(t) = \inf\{k \mid \ t \text{ has a maximal trace of length } k\}$. Moreover, two bisimilar states have the same depth and norm.

B Equational logic

In Table 1 we report the rules of equational logic over CCS_f. As in operational semantics, they allow us to infer equations by proceeding inductively over the structure of terms. Let $\mathcal{E}$ be a sound set of axioms. Rules (e_1)-(e_4) are common for all process languages and they ensure that $\mathcal{E}$ is closed with respect to reflexivity, symmetry, transitivity and substitution, respectively. Rules (e_5)-(e_8) are tailored for CCS_f and they ensure the closure of $\mathcal{E}$ under CCS_f contexts. They are therefore referred to as the *congruence rules*. Briefly, rule (e_5) is the rule for prefixing, rule (e_6) deals with the nondeterministic choice operator. Rules (e_7) and (e_8) ensure, respectively, that the binary operator f and the parallel composition operator preserve the equivalence of terms.

$$(e_1)\ t \approx t \qquad (e_2)\ \frac{t \approx u}{u \approx t} \qquad (e_3)\ \frac{t \approx u \quad u \approx v}{t \approx v} \qquad (e_4)\ \frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

$$(e_5)\ \frac{t \approx u}{\mu.t \approx \mu.u} \qquad (e_6)\ \frac{t \approx u \quad t' \approx u'}{t + t' \approx u + u'} \qquad (e_7)\ \frac{t \approx u \quad t' \approx u'}{f(t,t') \approx f(u,u')} \qquad (e_8)\ \frac{t \approx u \quad t' \approx u'}{t \parallel t' \approx u \parallel u'}\ .$$

Table 1 The rules of equational logic

Without loss of generality one may assume that substitutions happen first in equational proofs, i.e., that the rule

$$\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

may only be used when $(t \approx u) \in \mathcal{E}$. In this case $\sigma(t) \approx \sigma(u)$ is called a *substitution instance* of an axiom in $\mathcal{E}$. Moreover, by postulating that for each axiom in $\mathcal{E}$ also its symmetric counterpart is present in $\mathcal{E}$, one may assume that applications of symmetry happen first in equational proofs, i.e., that the rule

$$\frac{t \approx u}{u \approx t}$$

is never used in equational proofs. In the remainder of Appendix, we shall always tacitly assume that equational axiom systems are closed with respect to symmetry.

C Proofs of the results in Section 3

C.1 Proof of Lemma 5

▶ **Lemma 5.** *Assume that Assumptions 1 and 2 hold. Then:*

1. *The index set J on the right-hand side of (3) is non-empty.*
2. *The set of transition rules for f is non-empty.*
3. *Each transition rule for f has some premise.*
4. *The terms $f(x,x)$ and $f(y,y)$ are not summands of t_J.*

Proof of Lemma 5. Statements 1 and 2 are trivial because the equation

$$x \| y \approx \mathbf{0}$$

is not sound modulo bisimilarity.

Let us focus now on the proof for statement 3. To this end, assume, towards a contradiction, that f has a rule of the form

$$f(x_1, x_2) \xrightarrow{\mu} t(x_1, x_2) \enspace ,$$

for some action μ and term t. This rule can be used to derive that

$$f(\mathbf{0}, \mathbf{0}) \xrightarrow{\mu} t(\mathbf{0}, \mathbf{0}) \enspace .$$

Since the set J on the right-hand side of (3) is non-empty by statement 1, the term $f(\mathbf{0}, \mathbf{0})$ occurs as a summand of $t_J(\mathbf{0}, \mathbf{0})$. It follows that

$$t_J(\mathbf{0}, \mathbf{0}) \xrightarrow{\mu} t(\mathbf{0}, \mathbf{0}) \enspace .$$

Therefore,

$$\mathbf{0} \parallel \mathbf{0} \leftrightarrow \mathbf{0} \not\leftrightarrow t_J(\mathbf{0}, \mathbf{0}) \enspace ,$$

contradicting our Assumption 2.

Finally, we deal with statement 4. Assume, towards a contradiction, that $f(x,x)$, say, is a summand of t_J. Since $a \parallel \mathbf{0} \xrightarrow{a} \mathbf{0} \parallel \mathbf{0} \leftrightarrow \mathbf{0}$ and equation (3) holds modulo bisimulation equivalence, there is a closed term p such that

$$t_J(a, \mathbf{0}) \xrightarrow{a} p \text{ and } p \leftrightarrow \mathbf{0} \enspace .$$

This means that there is a summand $f(z_1, z_2)$ of t_J such that

$$f(p_1, p_2) \xrightarrow{a} p \enspace ,$$

where, for $i \in \{1, 2\}$,

$$p_i = \begin{cases} a & \text{if } z_i = x \enspace , \\ \mathbf{0} & \text{if } z_i = y \enspace . \end{cases}$$

The transition $f(p_1, p_2) \xrightarrow{a} p$ must be provable using some de Simone rule ρ for f (see Equation (1) in Definition 3). Such a rule has some premise by Lemma 5(3), and each such premise must have the form $x_1 \xrightarrow{\mu} y_1$ or $x_2 \xrightarrow{\mu} y_2$, for some action μ. If both z_1 and z_2 are y then $p_1 = p_2 = \mathbf{0}$, and none of those premises can be met. Therefore at least one of z_1 and z_2 in the summand $f(z_1, z_2)$ is x. Moreover, if $x_i \xrightarrow{\mu} y_i$ ($i \in \{1, 2\}$) is a premise of ρ, then

$z_i = x$ and $\mu = a$ (or else the premise could not be met). So the rule ρ can have one of the following three forms:

$$\frac{x_1 \xrightarrow{a} y_1}{f(x_1,x_2) \xrightarrow{a} t_1(y_1,x_2)} \qquad \frac{x_2 \xrightarrow{a} y_2}{f(x_1,x_2) \xrightarrow{a} t_2(x_1,y_2)} \qquad \frac{x_1 \xrightarrow{a} y_1 \quad x_2 \xrightarrow{a} y_2}{f(x_1,x_2) \xrightarrow{a} t_3(y_1,y_2)}$$

for some terms t_1, t_2 and t_3. We now proceed to argue that the existence of each of these rules contradicts the soundness of Equation (3) modulo bisimulation equivalence.

If ρ has the form

$$\frac{x_1 \xrightarrow{a} y_1 \quad x_2 \xrightarrow{a} y_2}{f(x_1,x_2) \xrightarrow{a} t_3(y_1,y_2)}$$

then $z_1 = z_2 = x$ and

$$f(a,a) \xrightarrow{a} p \ .$$

Since the term $f(a,a)$ is a summand of $t_J(a,a)$, it follows that

$$t_J(a,a) \xrightarrow{a} p$$

also holds. However, this contradicts the soundness of equation (3) because, for each transition $a \parallel a \xrightarrow{a} q$, we have that $q \underline{\leftrightarrow} a \not\underline{\leftrightarrow} \mathbf{0} \underline{\leftrightarrow} p$.

Assume now, without loss of generality, that ρ has the form

$$\frac{x_1 \xrightarrow{a} y_1}{f(x_1,x_2) \xrightarrow{a} t_1(y_1,x_2)}$$

Using this rule, we can infer that

$$f(a,a) \xrightarrow{a} t_1(\mathbf{0},a) \ .$$

Since $f(x,x)$ is a summand of t_J by our assumption, the term $f(a,a)$ is a summand of $t_J(a,\mathbf{0})$. Hence,

$$t_J(a,\mathbf{0}) \xrightarrow{a} t_1(\mathbf{0},a)$$

also holds. As equation (3) holds modulo bisimulation equivalence, we have that

$$a \parallel \mathbf{0} \underline{\leftrightarrow} t_J(a,\mathbf{0}) \ .$$

Therefore $t_1(\mathbf{0},a) \underline{\leftrightarrow} \mathbf{0}$, because $a \parallel \mathbf{0} \xrightarrow{a} \mathbf{0} \parallel \mathbf{0}$ is the only transition afforded by the term $a \parallel \mathbf{0}$. Observe now that

$$t_J(a,a) \xrightarrow{a} t_1(\mathbf{0},a) \underline{\leftrightarrow} \mathbf{0} \ .$$

also holds. However, this contradicts the soundness of equation (3) as above because, for each transition $a \parallel a \xrightarrow{a} q$, we have that $q \underline{\leftrightarrow} a \not\underline{\leftrightarrow} \mathbf{0} \underline{\leftrightarrow} p$.

This proves that $f(x,x)$ is not a summand of t_J, which was to be shown. ◀

C.2 Proof of Proposition 6

▶ **Proposition 6.** *If J is a singleton, then CCS_f admits no finite equational axiomatisation modulo bisimilarity.*

Proof of Proposition 6. If J is a singleton, then, since $\parallel$ is commutative modulo bisimulation equivalence, the equation

$$x \parallel y \approx f(x,y)$$

holds modulo bisimilarity. Therefore the result follows from the nonexistence of a finite equational axiomatisation for CCS proven by Moller in [23, 25]. ◀

C.3 Lemma 22

For later use, we note a useful consequence of the soundness of Equation (4) modulo bisimilarity.

▶ **Lemma 22.** *Assume that Equation (4) holds modulo $\leftrightarrow$. Then depth(p) is finite for each closed CCS_f term p.*

Proof: By structural induction on closed terms. For all of the standard CCS operators, it is well known that the depth of closed terms can be characterized inductively thus:

$$\begin{aligned} depth(\mathbf{0}) &= 0 \\ depth(\mu p) &= 1 + depth(p) \\ depth(p+q) &= \max\{depth(p), depth(q)\} \\ depth(p\|q) &= depth(p) + depth(q) \ . \end{aligned}$$

So the depth of a closed term of the form μp, $p+q$ or $p\|q$ is finite, if so are the depths of p and q.

Consider now a closed term of the form $f(p,q)$. Since bisimilar terms have the same depth and, by the proviso of the lemma, Equation (4) holds modulo bisimulation equivalence, we have that

$$depth(f(p,q)) \leq depth(f(p,q) + f(q,p)) = depth(p\|q) \ .$$

It follows that $depth(f(p,q))$ is finite, if so are the depths of p and q. □

D Proofs of the results in Section 4

D.1 Proof of Lemma 8

▶ **Lemma 8.** *Suppose that f meets Assumption 1, and that Equation (4) is sound modulo bisimilarity. Let ρ be a de Simone rule for f with μ as label. Then:*

1. *If $\mu = \tau$ then the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ of ρ can only have one of the following possible forms:*
   - *$\{x_i \xrightarrow{\tau} y_i\}$ for some $i \in \{1,2\}$, or*
   - *$\{x_1 \xrightarrow{\alpha} y_1, x_2 \xrightarrow{\bar{\alpha}} y_2\}$ for some $\alpha \in \{a, \bar{a}\}$.*
2. *If $\mu = \alpha$ for some $\alpha \in \{a, \bar{a}\}$, then the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ can only have the form $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1,2\}$.*

Proof of Lemma 8. We only detail the proof for statement 1. (The proof for statement 2 follows similar lines, and is left to the reader.)

Assume, towards a contradiction, that $\mu = \tau$ and the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ of ρ has some form that differs from those in the statement. Then the set of premises of ρ has one of the following two forms:

- $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1,2\}$ and $\alpha \in \{a, \bar{a}\}$, or
- $\{x_1 \xrightarrow{\mu_1} y_1, x_2 \xrightarrow{\mu_2} y_2\}$ for some $\mu_1, \mu_2 \in \{a, \bar{a}, \tau\}$ such that
  - either $\mu_1 = \tau$ or $\mu_2 = \tau$, or
  - $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$.

We now proceed to argue that the existence of either of these rules for f contradicts the soundness of Equation (4).

- Assume that the set of premises of ρ has the form $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1,2\}$ and $\alpha \in \{a, \bar{a}\}$. In this case, we can use that rule to prove the existence of the transition

$$f(\alpha, \mathbf{0}) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \text{ or } f(\mathbf{0}, \alpha) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ ,$$

depending on whether $i = 1$ or $i = 2$. Therefore

$$f(\alpha, \mathbf{0}) + f(\mathbf{0}, \alpha) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0})$$

also holds. However, the existence of this transition immediately contradicts the soundness of Equation (4) modulo bisimulation equivalence because $\alpha \parallel \mathbf{0}$ affords no τ-transition.

- Assume that the set of premises of ρ has the form $\{x_1 \xrightarrow{\mu_1} y_1, x_2 \xrightarrow{\mu_2} y_2\}$ for some $\mu_1, \mu_2 \in \{a, \bar{a}, \tau\}$ such that
  - either $\mu_1 = \tau$ or $\mu_2 = \tau$, or
  - $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$.

  In the this case, we can use that rule to prove the existence of the transition

$$f(\mu_1, \mu_2) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ .$$

Therefore

$$f(\mu_1, \mu_2) + f(\mu_2, \mu_1) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0})$$

also holds. By the soundness of Equation (4), we have that

$$\mu_1 \| \mu_2 \leftrightarrow f(\mu_1, \mu_2) + f(\mu_2, \mu_1) \ .$$

Hence $\mu_1 \| \mu_2 \xrightarrow{\tau} p$ for some p such that $p \leftrightarrow t(\mathbf{0}, \mathbf{0})$. If $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$, then the above transition cannot exist, because $\alpha \parallel \alpha$ affords no τ-transition. This immediately contradicts the soundness of Equation (4) modulo bisimulation equivalence. We therefore proceed with the proof by assuming that at least one of μ_1 and μ_2 is τ. In this case, we have that $\mu_1 \parallel \mu_2 \xrightarrow{\tau} p$ implies that $p \leftrightarrow \mu_1$ and $\mu_2 = \tau$, or $p \leftrightarrow \mu_2$ and $\mu_1 = \tau$. Assume, without loss of generality, that $\mu_1 = \tau$ and

$$t(\mathbf{0}, \mathbf{0}) \leftrightarrow \mu_2 \ . \tag{8}$$

Pick now an action $\alpha \neq \mu_2$. (Such an action exists as we have three actions in our language.) The soundness of Equation (4) yields that

$$\tau \parallel (\mu_2 + \alpha) \leftrightarrow f(\tau, \mu_2 + \alpha) + f(\mu_2 + \alpha, \tau) \ .$$

Using the rule for f we assumed we had and the rules for $+$, we can prove the existence of the transition

$$f(\tau, \mu_2 + \alpha) + f(\mu_2 + \alpha, \tau) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ .$$

Since the source of the above transition is bisimilar to $\tau \parallel (\mu_2 + \alpha)$, there must be a term p such that $\tau \parallel (\mu_2 + \alpha) \xrightarrow{\tau} p$ and $p \leftrightarrow t(\mathbf{0}, \mathbf{0})$. By Equation (8), this term p can only be $\tau \parallel \mathbf{0}$. In fact,

$$t(\mathbf{0}, \mathbf{0}) \leftrightarrow \mu_2 \not\leftrightarrow (\mu_2 + \alpha) \leftrightarrow \mathbf{0} \parallel (\mu_2 + \alpha) \ ,$$

for we chose $\alpha \in \{a, \bar{a}\}$ different from μ_2. We have therefore proven that $\mu_1 = \mu_2 = \tau$.

We are now ready to reach the promised contradiction to the soundness of Equation (4). In fact, consider the term $f(\tau + a, \tau + a)$. Using the rule for f we assumed we had, we can again prove the existence of the transition

$$f(\tau + a, \tau + a) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ .$$

By Equation (8) and our observation that $\mu_2 = \tau$, the term $t(\mathbf{0}, \mathbf{0})$ is bisimilar to τ. On the other hand, $(\tau + a) \parallel (\tau + a) \xrightarrow{\tau} p$ implies that $p \leftrightarroweq (\tau + a) \not\leftrightarroweq \tau$, contradicting the soundness of Equation (4) modulo bisimulation equivalence.

◀

D.2 Proof of Lemma 9

▶ **Lemma 9.** *Assume that Equation (4) holds modulo bisimilarity. Then the operator f must have a rule of the form*

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1, x_2) \xrightarrow{\tau} t(y_1, y_2)} \tag{5}$$

for some $\alpha \in \{a, \bar{a}\}$ and term t. Moreover, for each rule for f of the above form the term $t(x, y)$ is bisimilar to $x \parallel y$.

Proof of Lemma 9. We first argue that f must have a rule of the form (5) for some $\alpha \in \{a, \bar{a}\}$ and term t. To this end, assume, towards a contradiction, that f has no such rule. Observe that the term $a \parallel \bar{a}$ affords the transition

$$a \parallel \bar{a} \xrightarrow{\tau} \mathbf{0} \parallel \mathbf{0} \ .$$

However, neither the term $f(a, \bar{a})$ nor the term $f(\bar{a}, a)$ affords a τ-transition. In fact, using our assumption that f has no rule of the form (5) and Lemma 8(1), each rule for f with a τ-transition as a consequent must have the form

$$\frac{x_i \xrightarrow{\tau} y_i}{f(x_1, x_2) \xrightarrow{\tau} t}$$

for some $i \in \{1, 2\}$ and term t. Such a rule cannot be used to infer a transition from $f(a, \bar{a})$ or $f(\bar{a}, a)$. It follows that

$$a \parallel \bar{a} \not\leftrightarroweq f(a, \bar{a}) + f(\bar{a}, a) \ ,$$

contradicting the soundness of Equation (4). Therefore f must have a rule of the form (5).

We now proceed to argue that $t(x, y)$ is bisimilar to $x \parallel y$, for each rule of the form (5) for f. Pick a rule for f of the form (5). We shall argue that

$$p \parallel q \leftrightarroweq t(p, q) \ ,$$

for all closed CCS_f terms p and q. To this end, consider the terms $\alpha.p \parallel \bar{\alpha}.q$ and $f(\alpha.p, \bar{\alpha}.q) + f(\bar{\alpha}.q, \alpha.p)$. Using rule (5) and the rules for $+$, we have that

$$f(\alpha.p, \bar{\alpha}.q) + f(\bar{\alpha}.q, \alpha.p) \xrightarrow{\tau} t(p, q) \ .$$

By the soundness of Equation (4), we have that

$$\alpha.p \parallel \bar{\alpha}.q \leftrightarroweq f(\alpha.p, \bar{\alpha}.q) + f(\bar{\alpha}.q, \alpha.p) \ .$$

Therefore there is a closed term r such that $\alpha.p \parallel \bar{\alpha}.q \xrightarrow{\tau} r$ and $r \leftrightarrow t(p,q)$. Note now that the only τ-transition afforded by $\alpha.p \parallel \bar{\alpha}.q$ is

$$\alpha.p \parallel \bar{\alpha}.q \xrightarrow{\tau} p \parallel q \ .$$

Therefore $r = p \parallel q \leftrightarrow t(p,q)$, which was to be shown. ◀

D.3 Proof of Lemma 10

▶ **Lemma 10.** *Let $\mu \in \{a, \bar{a}, \tau\}$. Then the operator f must have a rule of the form*

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1,x_2) \xrightarrow{\mu} t(y_1,x_2)} \tag{6}$$

or a rule of the form

$$\frac{x_2 \xrightarrow{\mu} y_2}{f(x_1,x_2) \xrightarrow{\mu} t(x_1,y_2)} \tag{7}$$

for some term t. Moreover, under Assumption 3, for each rule for f of the above forms the term $t(x,y)$ is bisimilar to $x \parallel y$.

Proof of Lemma 10. Let $\mu \in \{a, \bar{a}, \tau\}$. We first argue that f must have a rule of the form (6) or (7) for some term t. To this end, assume, towards a contradiction, that f has no such rules. Observe that the term $\mu \parallel \mathbf{0}$ affords the transition

$$\mu \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0} \ .$$

However, neither the term $f(\mu, \mathbf{0})$ nor the term $f(\mathbf{0}, \mu)$ affords a μ-transition. In fact, using our assumption that f has no rule of the form (6) or (7), Lemma 8 yields that

- either f has no rule with a μ-transition as a consequent,
- or $\mu = \tau$, and each rule for f with a τ-transition as a consequent has the form

  $$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} t(y_1,y_2)}$$

  for some $\alpha \in \{a, \bar{a}\}$.

In the latter case, such a rule cannot be used to infer a transition from $f(\mu, \mathbf{0})$ or $f(\mathbf{0}, \mu)$. It follows that

$$\mu \parallel \mathbf{0} \not\leftrightarrow f(\mu, \mathbf{0}) + f(\mathbf{0}, \mu) \ ,$$

contradicting the soundness of equation (4). Therefore f must have a rule of the form (6) or (7) for each action μ.

To conclude the proof we need to show that for each rule of the form (6) or (7) the target term $t(x,y)$ is bisimilar to $x \parallel y$. For simplicity, we expand the proof only for the case of rules of the form (6). The proof for rules of the form (7) follows by the same reasoning.

We proceed by a case analysis over the structure of $t(y_1,x_2)$, which, we recall, under assumption 3 can be either a variable in $\{y_1,x_2\}$ or a term of the form $g(y_1,x_2)$ for some CCS_f operator g. Our aim is to show that the only possibility is to have $t(y_1,x_2) = y_1 \parallel x_2$, as any other process term would invalidate one of our simplifying assumptions.

- Case t is a variable in $\{y_1, x_2\}$. We can distinguish two cases, according to which variable is considered:
  - $t = y_1$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (6) we can infer that $f(p,p) \xrightarrow{\mu} \mathbf{0}$, and thus $f(p,p) + f(p,p) \xrightarrow{\mu} \mathbf{0}$. However, there is no μ-transition from $p \parallel p$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel p \xrightarrow{\mu} q$, then q is a process that will always be able to perform a second μ-transition. Hence, we would have $p \parallel p \not\leftrightarrow f(p,p) + f(p,p)$, thus contradicting the soundness of Equation (4).
  - $t = x_2$. Consider process $p = \mu.\mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mu.\mathbf{0}$, from an application of rule (6) we can infer that $f(p,\mathbf{0}) \xrightarrow{\mu} \mathbf{0}$ and thus $f(p,\mathbf{0}) + f(\mathbf{0},p) \xrightarrow{\mu} \mathbf{0}$. However, there is no μ-transition from $p \parallel \mathbf{0}$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel \mathbf{0} \xrightarrow{\mu} q$, then q is a process that will always be able to perform a second μ-transition. Hence we would have $p \parallel \mathbf{0} \not\leftrightarrow f(p,\mathbf{0}) + f(\mathbf{0},p)$, thus contradicting the soundness of Equation (4).
- Case t is a term of the form $g(y_1, x_2)$ for some CCS$_f$ operator g. We can distinguish three cases, according to which operator is used:
  - g is the prefix operator. We can distinguish two cases, according to which variable of the rule occurs in t:
    * $t = \nu.y_1$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (6) we can infer that $f(p,\mathbf{0}) \xrightarrow{\mu} \nu.\mathbf{0} \xrightarrow{\nu} \mathbf{0}$, and thus $f(p,\mathbf{0}) + f(\mathbf{0},p) \xrightarrow{\mu}\xrightarrow{\nu} \mathbf{0}$. However, $p \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0} \stackrel{\nu}{\not\rightarrow}$. Hence, we would have that $p \parallel \mathbf{0} \not\leftrightarrow f(p,\mathbf{0}) + f(\mathbf{0},p)$, thus contradicting the soundness of Equation (4).
    * $t = \nu.x_2$. This case is analogous to the previous one.
  - g is the nondeterministic choice operator and thus $t = y_1 + x_2$. Consider processes $p = \mu.\mu.\mathbf{0}$ and $q = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} q$, from an application of rule (6) we can infer that $f(p,q) \xrightarrow{\mu} q + q \xrightarrow{\mu} \mathbf{0}$, and thus $f(p,q) + f(q,p) \xrightarrow{\mu}\xrightarrow{\mu} \mathbf{0}$. However, there is no process p' such that $p \parallel q \xrightarrow{\mu}\xrightarrow{\mu} p'$ and $p' \leftrightarrow \mathbf{0}$, since p' can always perform an additional μ-transition. Hence, we would have $p \parallel q \not\leftrightarrow f(p,q) + f(q,p)$, which contradicts the soundness of Equation (4).
  - $g = f$. First of all, we notice that in this case we can infer that f cannot have both types of rules of the form (5), and both types of rules, (6) and (7), for all actions. In fact, if this was the case, due to Lemmas 9 and 10, the set of rules defining the behaviour of $f(x_1, x_2)$ would be

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1,x_2) \xrightarrow{\mu} f(y_1,x_2)} \qquad \frac{x_2 \xrightarrow{\mu} y_2}{f(x_1,x_2) \xrightarrow{\mu} f(x_1,y_2)} \qquad \frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} y_1 \parallel y_2} \qquad \frac{x_1 \xrightarrow{\bar{\alpha}} y_1 \quad x_2 \xrightarrow{\alpha} y_2}{f(x_1,x_2) \xrightarrow{\tau} y_1 \parallel y_2}$$

    with $\mu \in \{a, \bar{a}, \tau\}$ and $\alpha \in \{a, \bar{a}\}$. Clearly, operator f would then be a mere renaming of the parallel composition operator. In particular, as a one-to-one correspondence between the rules for f and those for $\parallel$ could be established, we have that $f(x,y)$ would be *bisimilar under formal hypothesis* to $x \parallel y$ (see [14, Definition 1.10]) and therefore, by [14, Theorem 1.12], we could directly conclude that $f(x,y) \approx x \parallel y$ for all x, y. However, this would contradict the fact that $x \parallel y \not\approx f(x,y)$. Let us now consider the case of an operator f having both types of rules, (6) and (7), and only one type of rules of the form (5), say the rule

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} y_1 \parallel y_2}.$$

    We proceed towards contradiction and distinguish two subcases, according to whether the order of the arguments is preserved or not by the rules of type (6) with label α. Similar arguments would allow us to deal with rules of type (7).

- The target of the rule of type (6) with label α is $f(y_1, x_2)$. Then $f(\alpha.\bar{\alpha}, \alpha) \xrightarrow{\alpha} f(\bar{\alpha}, \alpha) \leftrightarrow \bar{\alpha}+\alpha$. However, there is no α-transition from $\alpha.\bar{\alpha}\|\alpha$ to a process bisimilar to $\bar{\alpha} + \alpha$, thus contradicting the soundness of Equation (4).
- The target of the rule of type (6) with label α is $f(x_2, y_1)$. Then $f(\alpha.\alpha, \bar{\alpha}.\alpha) \xrightarrow{\alpha} f(\bar{\alpha}.\alpha, \alpha) \stackrel{\tau}{\not\rightarrow}$. However, whenever $\alpha.\alpha\|\bar{\alpha}.\alpha$ performs an α-transition, it always reaches a process that can perform a τ-move. This contradicts the soundness of Equation (4).

Finally, let us deal with the case in which there is at least one action $\mu \in \{a, \bar{a}, \tau\}$ for which only one rule among (6) and (7) is available. According to our current simplifying assumptions, let (6) be the available rule for f with label μ. We can distinguish two cases, according to the occurrences of the variables of the rule in t:

- $t = f(y_1, x_2)$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (6) we can infer that $f(p,p) \xrightarrow{\mu} f(\mathbf{0}, a)$, and thus $f(p,p) + f(p,p) \xrightarrow{\mu} f(\mathbf{0}, p)$, with $f(\mathbf{0}, p) \leftrightarrow \mathbf{0}$, since only rules of the form (6) are available with respect to action μ. However, there is no μ-transition from $p \parallel p$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel p \xrightarrow{\mu} q$ then q is a process that will always be able to perform a second μ-transition. Hence, we would have $p \parallel p \not\leftrightarrow f(p,p) + f(p,p)$, thus contradicting the soundness of Equation (4).
- $t = f(x_2, y_1)$. Consider process $p = \mu.\mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mu.\mathbf{0}$, and only rules of the from (6) are available with respect to action μ, we can infer that $f(p, \mathbf{0}) \xrightarrow{\mu} f(\mathbf{0}, \mu.\mathbf{0}) \stackrel{\mu}{\not\rightarrow}$ and $f(\mathbf{0}, p) \not\rightarrow$, which means that $f(p, \mathbf{0}) + f(\mathbf{0}, p)$ cannot perform two μ-transitions in a row. However, we have that $p \parallel \mathbf{0} \xrightarrow{\mu} \mu.\mathbf{0} \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0}$. Hence, we would have $p \parallel \mathbf{0} \not\leftrightarrow f(p, \mathbf{0}) + f(\mathbf{0}, p)$, thus contradicting the soundness of Equation (4).

◀

D.4 Proof of Proposition 12

▶ **Proposition 12.** *Assume that all of the rules for f have the form (5), (6), or (7). If $S^f_{\alpha,\bar{\alpha}}$ holds for at least one $\alpha \in \{a, \bar{a}\}$, and, for each $\mu \in \{a, \bar{a}, \tau\}$, at least one of L^f_μ and R^f_μ holds, then Equation (4) is sound modulo bisimilarity.*

Proof of Proposition 12. We argue that the relation

$$\mathcal{B} = \{(p \parallel q, f(p,q) + f(q,p)) \mid p, q \text{ closed terms}\} \cup \leftrightarrow$$

is a bisimulation. To this end, pick closed terms p, q. Now show, using the information on the rules for f given in the proviso of the proposition, that, for each action μ and closed term r,

- whenever $p \parallel q \xrightarrow{\mu} r$, there is a term r' that is equal to r up to commutativity of $\parallel$ such that $f(p,q) + f(q,p) \xrightarrow{\mu} r'$, and
- whenever $f(p,q) + f(q,p) \xrightarrow{\mu} r$, there is a term r' that is equal to r up to commutativity of $\parallel$ such that $p \parallel q \xrightarrow{\mu} r'$.

The claim follows because $\parallel$ is commutative modulo $\leftrightarrow$. ◀

As an immediate consequence of the form of the rules for f given in Proposition 12, we have the following lemma:

▶ **Lemma 23.** *Assume that all of the rules for f have the form (5), (6), or (7). Then each closed term p in CCS_f is finitely branching, that is, the set $\{(\mu, q) \mid p \xrightarrow{\mu} q\}$ is finite.*

▶ Remark 24. A standard consequence of the finiteness of the depth (Lemma 22) and the finite branching of closed terms in CCS_f is that each closed CCS_f term is bisimilar to a synchronisation tree [20], that is, a closed term built only using the constant $\mathbf{0}$, the unary prefixing operations and the binary $+$ operation. Since bisimilarity is a congruence over CCS_f, this means, in particular, that an equation $t \approx u$ over CCS_f is sound modulo bisimilarity if, and only if, the closed terms $\sigma(t)$ and $\sigma(u)$ are bisimilar for each substitution mapping variables to synchronisation trees. Moreover, we can use the sub-language of synchronisation trees, which is common to all of the languages CCS_f, to compare terms from these languages for different choices of binary operation f with respect to bisimilarity.

E Proofs of results in Section 5

E.1 Proof of Lemma 15

▶ **Lemma 15.** *A binary operator satisfying Equation (4) cannot distribute over $+$ in both arguments.*

Proof of Lemma 15. Assume, towards a contradiction, that f is distributive in both arguments with respect to summation. Then, using Equation (4), we have that:

$$\begin{aligned}(x+y) \parallel z &\approx f(x+y,z) + f(z,x+y)\\ &\approx f(x,z)+f(y,z)+f(z,x)+f(z,y)\\ &\approx (x \parallel z) + (y \parallel z)\ .\end{aligned}$$

However, this is a contradiction because, as is well known, the equation

$$(x+y) \parallel z \approx (x \parallel z) + (y \parallel z)$$

is not sound in bisimulation semantics. For example, our readers can easily verify that

$$(a+\tau) \parallel a \not\leftrightarrow (a \parallel a) + (\tau \parallel a)\ .$$

◀

E.2 Proof of Lemma 16

▶ **Lemma 16.** *Let $i \in \{1,2\}$. Modulo bisimilarity, operator f distributes over summation in its i-th argument if and only if each rule for f has a premise $x_i \xrightarrow{\mu_i} y_i$, for some μ_i.*

Proof of Lemma 16. We prove the two implications separately.

($\Leftarrow$) This case follows by similar arguments to those used in the proof of [2, Lemma 4.3] and it is therefore omitted.

($\Rightarrow$) Assume that f distributes with respect to $+$ in some argument. We recall that by Lemmas 9 and 10 for each action μ at least one between L^f_μ and R^f_μ must hold. We aim to prove that either L^f_μ holds for all actions μ and none of the R^f_μ does, or vice versa. Indeed, suppose towards a contradiction that there are rules satisfying L^f_μ and R^f_ν for some actions μ and ν. Then

- $f(\tau+\tau^2,\nu)$ is not bisimilar to $f(\tau,\nu)+f(\tau^2,\nu)$, because the validity of R^f_ν allows us to prove that $f(\tau+\tau^2,\nu) \xrightarrow{\nu} (\tau+\tau^2)\|\mathbf{0}$ and $f(\tau,\nu)+f(\tau^2,\nu)$ cannot match that transition up to bisimilarity.

- $f(\mu, \tau + \tau^2)$ is not bisimilar to $f(\mu, \tau) + f(\mu, \tau^2)$, because the validity of L^f_μ allows us to prove that $f(\mu, \tau + \tau^2) \xrightarrow{\mu} \mathbf{0} \| (\tau + \tau^2)$ and $f(\mu, \tau) + f(\mu, \tau^2)$ cannot match that transition up to bisimilarity.

◀

F The equational theory of CCS$_f$

In this section we study some aspects of the equational theory of CCS$_f$ modulo bisimilarity that are useful in the proofs of our negative results. In particular, we show that, due to Equation (4), proving the negative result over CCS$_f$ is equivalent to proving it over its reduct CCS^-_f, whose signature does not contain occurrences of $\|$ (Proposition 26 below).

Furthermore, we discuss the relation between the available rules for f and the bisimilarity of terms of the form $f(p, q)$ with $\mathbf{0}$. As we will see, in the case of an operator f that distributes with respect to summation in one argument, it is possible to *saturate* the axiom systems [23] yielding a simplification in the proofs (Proposition **??** below). On the other hand, we cannot rely on saturation for an operator f that distributes with respect to $+$ in neither of its arguments.

F.1 Simplifying equational proofs

We show that it is sufficient to prove that bisimilarity admits no finite equational axiomatisation over CCS^-_f, consisting of the CCS$_f$ terms that do not contain occurrences of $\|$.

▶ **Definition 25.** *For each CCS$_f$ term t, we define $\hat{t}$ as follows:*

$$\begin{array}{ll} \hat{\mathbf{0}} = \mathbf{0} & \widehat{t+u} = \hat{t} + \hat{u} \\ \hat{x} = x & \widehat{f(t,u)} = f(\hat{t}, \hat{u}) \\ \widehat{\mu t} = \mu \hat{t} & \widehat{t \| u} = f(\hat{t}, \hat{u}) + f(\hat{u}, \hat{t}) \ . \end{array}$$

Then, for any axiom system $\mathcal{E}$ over CCS$_f$, we let $\widehat{\mathcal{E}} = \{\hat{t} \approx \hat{u} \mid (t \approx u) \in \mathcal{E}\}$.

We notice that, for each CCS$_f$ term t, the term $\hat{t}$ is in CCS^-_f. Moreover, if t contains no occurrences of the parallel composition operator, then $\hat{t} = t$. Since Equation (4) is sound with respect to bisimilarity, which is a congruence relation, it is not hard to show that each term t in CCS$_f$ is bisimilar to $\hat{t}$. Therefore if $\mathcal{E}$ is an axiom system over CCS$_f$ that is sound with respect to bisimilarity, then $\widehat{\mathcal{E}}$ is an axiom system over CCS^-_f that is sound with respect to bisimilarity.

The following result states the reduction of the non-finite axiomatisability of $\leftrightarrow$ over CCS$_f$ to that of $\leftrightarrow$ over CCS^-_f.

▶ **Proposition 26.** *Let $\mathcal{E}$ be an axiom system over CCS$_f$. Then:*

1. *If $\mathcal{E} \vdash t \approx u$, then $\widehat{\mathcal{E}} \vdash \hat{t} \approx \hat{u}$.*
2. *If $\mathcal{E}$ is a complete axiomatisation of $\leftrightarrow$ over CCS$_f$, then $\widehat{\mathcal{E}}$ completely axiomatises $\leftrightarrow$ over CCS^-_f.*
3. *If bisimilarity is not finitely axiomatisable over CCS^-_f, then it is not finitely axiomatisable over CCS$_f$ either.*

Proof: We prove the three statements separately.

- Proof of Statement 1. Assume that $\mathcal{E} \vdash t \approx u$. We shall argue that $\widehat{\mathcal{E}}$ proves the equation $\hat{t} \approx \hat{u}$ by induction on the depth of the proof of $t \approx u$ from $\mathcal{E}$. We proceed by a case analysis on the last rule used in the proof. Below we only consider the two most interesting cases in this analysis.
  - Case $\mathcal{E} \vdash t \approx u$, because $\sigma(t') = t$ and $\sigma(u') = u$ for some equation $(t' \approx u') \in \mathcal{E}$. Note, first of all, that, by the definition of $\widehat{\mathcal{E}}$, the equation $\widehat{t'} \approx \widehat{u'}$ is contained in $\widehat{\mathcal{E}}$. Observe now that

$$\hat{t} = \hat{\sigma}(\widehat{t'}) \text{ and } \hat{u} = \hat{\sigma}(\widehat{u'}) \ ,$$

    where $\hat{\sigma}$ is the substitution mapping each variable x to the term $\widehat{\sigma(x)}$. It follows that the equation $\hat{t} \approx \hat{u}$ can be proven from the axiom system $\widehat{\mathcal{E}}$ by instantiating the equation $\widehat{t'} \approx \widehat{u'}$ with the substitution $\hat{\sigma}$, and we are done.
  - Case $\mathcal{E} \vdash t \approx u$, because $t = t_1 \| t_2$ and $u = u_1 \| u_2$ for some t_i, u_i $(i = 1, 2)$ such that $\mathcal{E} \vdash t_i \approx u_i$ $(i = 1, 2)$. Using the inductive hypothesis twice, we have that $\widehat{\mathcal{E}} \vdash \widehat{t_i} \approx \widehat{u_i}$ $(i = 1, 2)$. Therefore, using substitutivity, $\widehat{\mathcal{E}}$ proves that

$$\hat{t} = f(\widehat{t_1}, \widehat{t_2}) + f(\widehat{t_2}, \widehat{t_1}) \approx f(\widehat{u_1}, \widehat{u_2}) + f(\widehat{u_2}, \widehat{u_1}) = \hat{u} \ ,$$

    which was to be shown.

  The remaining cases are simpler, and we leave the details to the reader.
- Proof of Statement 2. Assume that t and u are two bisimilar terms in the language CCS_f^-. We shall argue that $\widehat{\mathcal{E}}$ proves the equation $t \approx u$. To this end, we begin by noting that the equation $t \approx u$ also holds in the algebra of CCS_f terms modulo bisimulation. In fact, for each term v in the language CCS_f and closed substitution σ mapping variables to CCS_f terms, we have that

$$\sigma(v) \leftrightarrow \hat{\sigma}(v) \ ,$$

  where the substitution $\hat{\sigma}$ is defined as above.
  Since $\mathcal{E}$ is complete for bisimilarity over CCS_f by our assumptions, it follows that $\mathcal{E}$ proves the equation $t \approx u$. Therefore, by statement 1 of the proposition, we have that $\widehat{\mathcal{E}}$ proves the equation $\hat{t} \approx \hat{u}$. The claim now follows because $\hat{t} = t$ and $\hat{u} = u$.
- Proof of Statement 3. This is an immediate consequence of statement 2 because $\widehat{\mathcal{E}}$ has the same cardinality of $\mathcal{E}$, and is therefore finite, if so is $\mathcal{E}$.

□

In light of this result, henceforth we shall focus on proving that $\leftrightarrow$ affords no finite equational axiomatisation over CCS_f^-.

F.2 Bisimilarity with $\mathbf{0}$

As a further simplification, we can focus on the $\mathbf{0}$ *absorption properties* of CCS_f^- operators. Informally, we can restrict the axiom system to a collection of equations that do not introduce unnecessary terms that are bisimilar to $\mathbf{0}$ in the equational proofs, namely $\mathbf{0}$ summands and $\mathbf{0}$ *factors*.

▶ **Definition 27.** *We say that a CCS_f^- term t has a $\mathbf{0}$ factor if it contains a subterm of the form $f(t', t'')$, where either t' or t'' is bisimilar to $\mathbf{0}$.*

The $\mathbf{0}$ absorption properties of f depend crucially on the allowed set of SOS rules for f. Notably, we have different results, according to the distributivity properties of f.

F.3 0 absorption for f that distributes in one argument

We examine first the case of an operator f that, modulo bisimilarity, distributes over summation in its first argument.

In this case, an example of a collection of equations over CCS_f^- that are sound with respect to $\leftrightarrow$ is given by axioms A0–A3, F0–F1:

$$\begin{array}{llll} \mathrm{A0} & x + \mathbf{0} \approx x & \mathrm{F0} & f(\mathbf{0}, x) \approx \mathbf{0} \\ \mathrm{A1} & x + y \approx y + x & \mathrm{F1} & f(x, \mathbf{0}) \approx x \ . \\ \mathrm{A2} & (x + y) + z \approx x + (y + z) & & \\ \mathrm{A3} & x + x \approx x & & \end{array}$$

Axioms A0 and F0 are enough to establish that each CCS_f^- term that is bisimilar to $\mathbf{0}$ is also provably equal to $\mathbf{0}$.

▶ **Lemma 28.** *Let t be a CCS_f^- term. Then $t \leftrightarrow \mathbf{0}$ if, and only if, the equation $t \approx \mathbf{0}$ is provable using axioms A0 and F0 from left to right.*

Before proceeding to the technical proof, we observe the following:

▶ Remark 29. Whenever a process term t has neither $\mathbf{0}$ summands nor factors then we can assume that, for some finite non-empty index set I, $t = \sum_{i \in I} t_i$ for some terms t_i such that none of them has $+$ as head operator and moreover, none of them has $\mathbf{0}$ summands nor factors.

Proof of Lemma 28. The "if" implication is an immediate consequence of the soundness of the equations A4 and F1 with respect to $\leftrightarrow$. To prove the "only if" implication, define, first of all, the collection NIL of CCS_f^- terms as the set of terms generated by the following grammar:

$$t ::= \mathbf{0} \mid t + t \mid f(t, u) \ ,$$

where u is an arbitrary CCS_f^- term. We claim that:

▷ Claim 30. Each CCS_f^- term t is bisimilar to $\mathbf{0}$ if, and only if, $t \in \mathrm{NIL}$.

Using this claim and structural induction on $t \in \mathrm{NIL}$, it is a simple matter to show that if $t \leftrightarrow \mathbf{0}$, then $t \approx \mathbf{0}$ is provable using axioms A0 and F0 from left to right, which was to be shown.

To complete the proof, it therefore suffices to show the above claim. To establish the "if" implication in the statement of the claim, one proves, using structural induction on t and the congruence properties of bisimilarity, that if $t \in \mathrm{NIL}$, then $\sigma(t) \leftrightarrow \mathbf{0}$ for every closed substitution σ. To show the "only if" implication, we establish the contrapositive statement, viz. that if $t \notin \mathrm{NIL}$, then $\sigma(t) \not\leftrightarrow \mathbf{0}$ for some closed substitution σ. To this end, it suffices only to show, using structural induction on t, that if $t \notin \mathrm{NIL}$, then $\sigma_a(t) \xrightarrow{\mu}$ for some action $\mu \in \{a, \bar{a}, \tau\}$, where σ_a is the closed substitution mapping each variable to the closed term $a\mathbf{0}$. The details of this argument are not hard, and are therefore left to the reader. ◀

In light of the above result, in the technical developments to follow, when dealing with an operator f that distributes over $+$ in its first argument we shall assume, without loss of generality, that each axiom system we consider includes the equations A0–A3, F0–F1. This assumption means, in particular, that our axiom systems will allow us to identify each term that is bisimilar to $\mathbf{0}$ with $\mathbf{0}$.

It is well-known (see, e.g., Sect. 2 in [15]) that if an equation relating two closed terms can be proved from an axiom system $\mathcal{E}$, then there is a closed proof for it. Moreover, if $\mathcal{E}$

satisfies a further closure property, called *saturation*, in addition to those mentioned earlier, and that closed equation relates two terms containing no occurrences of $\mathbf{0}$ as a summand or factor, then there is a closed proof for it in which all of the terms have no occurrences of $\mathbf{0}$ as a summand or factor.

▶ **Definition 31.** *For each CCS_f^- term t, we define $t/\mathbf{0}$ thus:*

$$\mathbf{0}/\mathbf{0} = \mathbf{0} \qquad x/\mathbf{0} = x \qquad \mu t/\mathbf{0} = \mu(t/\mathbf{0})$$

$$(t+u)/\mathbf{0} = \begin{cases} u/\mathbf{0} & \text{if } t \underline{\leftrightarrow} \mathbf{0} \\ t/\mathbf{0} & \text{if } u \underline{\leftrightarrow} \mathbf{0} \\ (t/\mathbf{0}) + (u/\mathbf{0}) & \text{otherwise} \end{cases}$$

$$f(t,u)/\mathbf{0} = \begin{cases} \mathbf{0} & \text{if } t \underline{\leftrightarrow} \mathbf{0} \\ t/\mathbf{0} & \text{if } u \underline{\leftrightarrow} \mathbf{0} \\ f(t/\mathbf{0}, u/\mathbf{0}) & \text{otherwise} \end{cases}$$

Intuitively, $t/\mathbf{0}$ is the term that results by removing *all* occurrences of $\mathbf{0}$ as a summand or factor from t.

The following lemma, whose simple proof by structural induction on terms is omitted, collects the basic properties of the above construction.

▶ **Lemma 32.** *For each CCS_f^- term t, the following statements hold:*

1. *the equation $t \approx t/\mathbf{0}$ can be proven using the equations A0–A3, F0–F1, and therefore $t \underline{\leftrightarrow} t/\mathbf{0}$;*
2. *the term $t/\mathbf{0}$ has no occurrence of $\mathbf{0}$ as a summand or factor;*
3. *$t/\mathbf{0} = t$, if t has no occurrence of $\mathbf{0}$ as a summand or factor;*
4. *$\sigma(t/\mathbf{0})/\mathbf{0} = \sigma(t)/\mathbf{0}$, for each substitution σ.*

▶ **Definition 33.** *We say that a substitution σ is a $\mathbf{0}$-substitution iff $\sigma(x) \neq x$ implies that $\sigma(x) = \mathbf{0}$, for each variable x.*

▶ **Definition 34.** *Let $\mathcal{E}$ be an axiom system. We define the axiom system $\mathrm{cl}(\mathcal{E})$ thus:*

$$\mathrm{cl}(\mathcal{E}) = \mathcal{E} \cup \{\sigma(t)/\mathbf{0} \approx \sigma(u)/\mathbf{0} \mid (t \approx u) \in \mathcal{E},\ \sigma \text{ a } \mathbf{0}\text{-substitution}\} \ .$$

An axiom system $\mathcal{E}$ is saturated if $\mathcal{E} = \mathrm{cl}(\mathcal{E})$.

The following lemma collects some basic sanity properties of the closure operator $\mathrm{cl}(\cdot)$. (Note, in particular, that the application of $\mathrm{cl}(\cdot)$ to an axiom system preserves closure with respect to symmetry.)

▶ **Lemma 35.** *Let $\mathcal{E}$ be an axiom system. Then the following statements hold.*

1. $\mathrm{cl}(\mathcal{E}) = \mathrm{cl}(\mathrm{cl}(\mathcal{E}))$.
2. $\mathrm{cl}(\mathcal{E})$ *is finite, if so is* $\mathcal{E}$.
3. $\mathrm{cl}(\mathcal{E})$ *is sound, if so is* $\mathcal{E}$.
4. $\mathrm{cl}(\mathcal{E})$ *is closed with respect to symmetry, if so is* $\mathcal{E}$.
5. $\mathrm{cl}(\mathcal{E})$ *and* $\mathcal{E}$ *prove the same equations, if* $\mathcal{E}$ *contains the equations A0–A3, F0–F1.*

Proof: We limit ourselves to sketching the proofs of statements 1 and 5 in the lemma.

In the proof of statement 1, the only non-trivial thing to check is that the equation

$$\sigma(\sigma'(t)/\mathbf{0}))/\mathbf{0} \approx \sigma(\sigma'(u)/\mathbf{0}))/\mathbf{0}$$

is contained in $\mathrm{cl}(\mathcal{E})$, whenever $(t \approx u) \in \mathcal{E}$ and σ, σ' are $\mathbf{0}$-substitutions. This follows from Lemma 32(4) because the collection of $\mathbf{0}$-substitutions is closed under composition.

To show statement 5, it suffices only to argue that each equation $t \approx u$ that is provable from $\mathrm{cl}(\mathcal{E})$ is also provable from $\mathcal{E}$, if $\mathcal{E}$ contains the equations A0–A3, F0–F1. This can be done by induction on the depth of the proof of the equation $t \approx u$ from $\mathrm{cl}(\mathcal{E})$, using Lemma 32(1) for the case in which $t \approx u$ is a substitution instance of an axiom in $\mathrm{cl}(\mathcal{E})$. ◻

Notice that, in light of this result, the saturation of a finite axiom system that includes the equations A0–A3, F0–F1 results in an equivalent, finite collection of equations (Lemma 35(2) and (5)).

We are now ready to state our counterpart of [23, Proposition 5.1.5].

▶ **Proposition 36.** *Assume that $\mathcal{E}$ is a saturated axiom system. Suppose furthermore that we have a closed proof from $\mathcal{E}$ of the closed equation $p \approx q$. Then replacing each term r in that proof with $r/\mathbf{0}$ yields a closed proof of the equation $p/\mathbf{0} \approx q/\mathbf{0}$. In particular, the proof from $\mathcal{E}$ of an equation $p \approx q$, where p and q are terms not containing occurrences of $\mathbf{0}$ as a summand or factor, need not use terms containing occurrences of $\mathbf{0}$ as a summand or factor.*

Proof: The proof follows the lines of that of [23, Proposition 5.1.5], and is therefore omitted. ◻

In light of Proposition 36, henceforth, when dealing with an operator f that distributes with respect to $+$ in one of its arguments, we shall limit ourselves to considering saturated axiom systems.

F.4 0 absorption for a non distributive f

In Section 5, we argued that the set of allowed rules for an operator f that does not distribute over summation in either argument has to include at least a rule of type (6) *and* at least one of type (7). We also notice that for an operator f having both types of rules for all actions we can distinguish two cases, according to which rules of type (5) are available: (i) If f has both rules of type (5), then it would be a mere rewriting of the parallel composition operator (see Appendix D.3, proof of Lemma 10). (ii) If f has only one rule of type (5), then one can observe that Moller's argument to the effect that bisimilarity is not finitely based over the fragment of CCS with action prefixing, nondeterministic choice and purely interleaving parallel composition, could be applied to f, yielding the desired negative result.

Hence, we can assume that there is an action $\mu \in \{a, \bar{a}, \tau\}$ such that f has only one rule, of type either (6) or (7), with μ as label. This asymmetry in the set of rules for f can cause some CCS_f^- term to behave as $\mathbf{0}$ when occurring in the scope of f, despite not being bisimilar to $\mathbf{0}$ at all.

▶ **Example 37.** Consider the term $t = f(a + \bar{a}.u, \tau)$, for some term u, and assume that f has only rules of type (6) with labels a and τ and only a rule of type (7) with label $\bar{a}$. One can easily check that, since the initial execution of the τ-move in the second argument is prevented by the rules for f, then the subterm $\bar{a}.u$ can never contribute to the behaviour of t. Thus, $t \leftrightarrow a.\tau$, even though $\bar{a}.u \not\leftrightarrow \mathbf{0}$ for each term u.

From a technical point of view, this implies that Lemmas 28 and 32.1 no longer hold. In fact, one can always construct a term t of the form $t = f(\sum_{i=1}^{n} \mu.x_i, \sum_{j=1}^{m} \nu.y_j)$ for some $n, m \geq 0$, with μ, ν chosen according to the available set of rules for f, such that $t \leftrightarrow \mathbf{0}$. We conjecture that *since we are considering an operator f that does not distribute over summation in either of its arguments, the valid equations, modulo bisimilarity, of the form $t \approx \mathbf{0}$ cannot be proved by means of any finite, sound set of axioms.* Roughly speaking, this is due to the fact that no valid axiom can be established for a term of the form $f(\mu.x+z, \nu.y+w)$ in that the behaviour of the terms substituted for the variables z and w is crucial to determine that of a closed instantiation of the term.

Summarizing, this would imply that, in the case at hand, we cannot assume that we can use saturation to simplify the axiom systems and, moreover, the family of equations

$$f(\sum_{i=1}^{n} \mu.p_i, \sum_{j=1}^{m} \nu.q_j) \approx \mathbf{0} \qquad n, m \geq 0$$

for some processes p_i, q_j, could play the role of witness family of equations for our desired negative result. Unfortunately, the presence of two summations would force us to introduce a number of additional technical results that would make the proof of the negative results even heavier than it already is. Moreover, those supplementary results are not necessary to treat the case of the witness families that we are going to introduce in Sections 7–9 to obtain the proof of Theorem 14.

G Unique prime decomposition

In the proof of our main results, we shall often make use of some notions from [22, 23]. These we now proceed to introduce for the sake of completeness and readability.

▶ **Definition 38.** *A closed term p is irreducible if $p \leftrightarrow q \| r$ implies $q \leftrightarrow \mathbf{0}$ or $r \leftrightarrow \mathbf{0}$, for all closed terms q, r. We say that p is prime if it is irreducible and is not bisimilar to $\mathbf{0}$.*

For example, each term p of depth (respectively, norm) 1 is prime because every term of the form $q \| r$ that does not involve $\mathbf{0}$ factors has depth (resp., norm) at least 2, and thus cannot be bisimilar to p.

The following lemma states the primality of two families of closed terms that will play a key role in the proof of our main result.

▶ **Lemma 39.** **1.** *The term $\mu^{\leq m}$ is prime, for each $m \geq 1$.*
2. *Let $\nu \in \{a, \bar{a}\}$, $\mu \in \{a, \bar{a}, \tau\}$, $\nu \neq \mu$, $m \geq 1$ and $1 \leq i_1 < \ldots < i_m$. Then the term $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m}$ is prime.*

Proof: The first claim is immediate because the norm of $\mu^{\leq m}$ is one, for each $m \geq 1$.

For the second claim, assume by contradiction that there are process terms p, q such that $p, q \not\leftrightarrow \mathbf{0}$ and $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m} \leftrightarrow p \| q$. Clearly, this would imply the existence of process terms p', q' such that $p \xrightarrow{\nu} p'$ and $q \xrightarrow{\nu} q'$ so that $p \| q \xrightarrow{\nu} p' \| q$ and $p \| q \xrightarrow{\nu} p \| q'$. However, these transitions would in turn imply that $p \| q \xrightarrow{\nu} p' \| q \xrightarrow{\nu} p' \| q'$, namely $p \| q$ could perform two ν-moves in a row, whereas $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m}$ cannot perform such a sequence of actions, thus contradicting $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m} \leftrightarrow p \| q$. □

In [22] the notion of *unique prime decomposition* of a process p was introduced, as the unique multiset $\{\!| q_1, \ldots, q_n |\!\}$ of primes s.t. $p \leftrightarrow q_1 \| \ldots \| q_n$. Inspired by the unique prime decomposition result of [22], the authors of [19] proposed the notion of *decomposition order for commutative monoids*, and proved that the existence of a decomposition order on a commutative monoid implies that the monoid has the unique prime decomposition property. CCS_f modulo $\underline{\leftrightarrow}$ is a *commutative monoid* with respect to $\|$, having $\mathbf{0}$ as unit, and the transition relation defines a *decomposition order* over bisimilarity equivalence classes of closed terms. Then, by [19, Theorem 32], the following result holds:

▶ **Proposition 40.** *Any CCS_f term can be expressed uniquely, up to $\underline{\leftrightarrow}$, as a parallel composition of primes.*

As we will see, this property will play a crucial role in some of the upcoming proofs.

H Decomposing the semantics of terms

As outlined in Section 5, to obtain the desired negative results we will proceed by a case analysis on the operational rules for operator f. However, there are a few preliminary results that hold for all cases and that will be useful in the upcoming proofs. We dedicate this section to presenting these results and some auxiliary notions.

In the proofs to follow, we shall sometimes need to establish a correspondence between the behaviour of open terms and the semantics of their closed instances, with a special focus on the role of variables. In detail, we need to consider the possible origins of a transition of the form $\sigma(t) \xrightarrow{\alpha} p$, for some action $\alpha \in \{a, \bar{a}\}$, closed substitution σ, CCS_f^- term t and closed term p. In fact, the equational theory is defined over process terms, whereas the semantic properties can be verified only on their closed instances.

▶ **Lemma 41.** *Let $\mu \in \{a, \bar{a}, \tau\}$. Then for all t, t' and substitutions σ it holds that if $t \xrightarrow{\mu} t'$ then $\sigma(t) \xrightarrow{\mu} \sigma(t')$.*

However, a transition $\sigma(t) \xrightarrow{\mu} p$ may also derive from the initial behaviour of some closed term $\sigma(x)$, provided that the collection of initial moves of $\sigma(t)$ depends, in some formal sense, on that of the closed term substituted for the variable x. Roughly speaking, our aim is now to provide the conditions under which $\sigma(t) \xrightarrow{\mu} p$ can be inferred from $\sigma(x) \xrightarrow{\nu} q$, for some $\mu, \nu \in \{a, \bar{a}, \tau\}$ and processes p, q. As one might expect, in our setting the provability of transitions needs to be parametric with respect to the rules for f.

▶ **Example 42.** Consider the CCS_f^- term $t = f(x, \tau)$. Firstly, we notice that if R^f_τ holds then we can infer that $\sigma(t) \xrightarrow{\tau} \sigma(x)\|\mathbf{0}$ for all closed substitutions σ. Assume now that $\sigma(x) = a$. Clearly, we can derive $\sigma(t) \xrightarrow{a} \mathbf{0}\|\tau$ only if L^f_a holds.

To fully describe this situation, for each $\mu \in \{a, \bar{a}, \tau\}$, we introduce the auxiliary transition relation $\rightarrow_\mu$ over open terms. To this end, we present the notion of *configuration* over CCS_f^- terms, which stems from [7]. Configurations are terms defined over a set of variables $\mathcal{V}_\mathrm{d} = \{x_d \mid x \in \mathcal{V}\}$, disjoint from $\mathcal{V}$, and CCS_f^- terms. Intuitively, the symbol x_d (read "during x") will be used to denote that the closed term substituted for an occurrence of variable x has begun its execution.

▶ **Definition 43.** *The collection of CCS_f^- configurations is given by the following grammar:*

$$c ::= t \mid x_d \mid c \| t \mid t \| c \ ,$$

where t is a CCS_f^- term, and $x_d \in \mathcal{V}_d$.

For example, the configuration $x_d \parallel f(a, x)$ is meant to describe a state of the computation of some term in which the (closed term substituted for the) occurrence of variable x on the left-hand side of the $\parallel$ operator has begun its execution, but the one on the right-hand side has not.

We introduce also special labels for the auxiliary transitions $\rightarrow_\mu$, to keep track of which rules for f are available, and thus which one triggered the move by the closed instance of x. In detail, we let x_{l} denote that the closed instance of x is responsible for the transition when L^f_μ holds. In case R^f_μ holds, we use x_{r}. Finally, x_{b} is used when $L^f_\mu \wedge R^f_\mu$ holds.

The auxiliary transitions of the form $\rightarrow_\mu$ are then formally defined via the inference rules below

$$(a_1)\ \frac{L^f_\mu}{x \xrightarrow{x_{\mathrm{l}}}_\mu x_{\mathrm{d}}} \qquad (a_2)\ \frac{R^f_\mu}{x \xrightarrow{x_{\mathrm{r}}}_\mu x_{\mathrm{d}}} \qquad (a_3)\ \frac{L^f_\mu \wedge R^f_\mu}{x \xrightarrow{x_{\mathrm{b}}}_\mu x_{\mathrm{d}}}$$

$$(a_4)\ \frac{t_1 \xrightarrow{x_{\mathrm{w}}}_\mu c}{t_1 + t_2 \xrightarrow{x_{\mathrm{w}}}_\mu c}\ \mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\} \qquad (a_5)\ \frac{t_2 \xrightarrow{x_{\mathrm{w}}}_\mu c}{t_1 + t_2 \xrightarrow{x_{\mathrm{w}}}_\mu c}\ \mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$$

$$(a_6)\ \frac{t_1 \xrightarrow{x_{\mathrm{l}}}_\mu c}{f(t_1, t_2) \xrightarrow{x_{\mathrm{l}}}_\mu c \| t_2} \qquad (a_7)\ \frac{t_2 \xrightarrow{x_{\mathrm{r}}}_\mu c}{f(t_1, t_2) \xrightarrow{x_{\mathrm{r}}}_\mu t_1 \| c}$$

$$(a_8)\ \frac{t_1 \xrightarrow{x_{\mathrm{b}}}_\mu c}{f(t_1, t_2) \xrightarrow{x_{\mathrm{b}}}_\mu c \| t_2} \qquad (a_9)\ \frac{t_2 \xrightarrow{x_{\mathrm{b}}}_\mu c}{f(t_1, t_2) \xrightarrow{x_{\mathrm{b}}}_\mu t_1 \| c}$$

▶ **Example 44.** Consider the term $t = f(x, \tau)$ from Example 42. Assume, for instance, that L^f_a holds, yielding the transition $x \xrightarrow{x_{\mathrm{l}}}_a x_{\mathrm{d}}$, due to rule (a_1). Then, an application of rule (a_6) would give $f(x, \tau) \xrightarrow{x_{\mathrm{l}}}_a x_{\mathrm{d}} \| \tau$ with the following meaning: since the rules for f allow a-moves of the first argument to yield a-moves of terms of the form $f(p, q)$, then an a-transition by (an instance of) variable x occurring in the first argument of f will induce an a-move of $f(x, \tau)$.

Conversely, assume that only R^f_a holds. Then, by applying rule (a_2) we obtain that $x \xrightarrow{x_{\mathrm{r}}}_a x_{\mathrm{d}}$ and, from the rules, it is not possible to derive any $\rightarrow_a$ transition of $f(x, \tau)$ from that of x, modelling the fact that the rules for f prevent the execution of a-moves from the first argument.

Lemmas 45 and 46 formalise the decomposition of the semantics of CCS^-_f terms. We remark that, due to Lemma 10, at least one between L^f_μ and R^f_μ holds for each μ.

▶ **Lemma 45.** *Let $\mu \in \{a, \bar{a}, \tau\}$, t be a CCS^-_f term, x be a variable, $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$ and σ be a closed substitution. If $\sigma(x) \xrightarrow{\mu} p$ for some process p, and $t \xrightarrow{x_{\mathrm{w}}}_\mu c$ for some configuration c, then $\sigma(t) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto p](c)$.*

Proof: The proof follows by induction on the structure of t and the derivation of the auxiliary transition $t \xrightarrow{x_{\mathrm{w}}}_\mu c$. □

▶ **Lemma 46.** *Let $\alpha \in \{a, \bar{a}\}$, t be a CCS^-_f term, σ be a closed substitution and p be a closed term. Whenever $\sigma(t) \xrightarrow{\alpha} p$, then one of the following holds:*

1. *There is term t' such that $t \xrightarrow{\alpha} t'$ and $\sigma(t') = p$.*
2. *There are a variable x, a process q and a configuration c such that:*
   a. *only L^f_α holds, $\sigma(x) \xrightarrow{\alpha} q$, $t \xrightarrow{x_{\mathrm{l}}}_\alpha c$ and $\sigma[x_{\mathrm{d}} \mapsto q](c) = p$;*
   b. *only R^f_α holds, $\sigma(x) \xrightarrow{\alpha} q$, $t \xrightarrow{x_{\mathrm{r}}}_\alpha c$ and $\sigma[x_{\mathrm{d}} \mapsto q](c) = p$; or*
   c. *$L^f_\alpha \wedge R^f_\alpha$ holds, $\sigma(x) \xrightarrow{\alpha} q$, $t \xrightarrow{x_{\mathrm{b}}}_\alpha c$ and $\sigma[x_{\mathrm{d}} \mapsto q](c) = p$.*

Proof: The proof is by induction on the structure of t. The only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$, which we expand below. According to which rules are available for f with respect to α, we can distinguish three cases:

1. Case only L^f_α holds. Then, $f(\sigma(t_1), \sigma(t_2)) \xrightarrow{\alpha} p$ can be inferred only from a transition of the form $\sigma(t_1) \xrightarrow{\alpha} p'$ for some closed term p' with $p = p' \| \sigma(t_2)$. By induction over the derivation of $\sigma(t_1) \xrightarrow{\alpha} p'$, and considering that only L^f_α holds, we can then distinguish two cases:
   - There is a term t' such that $t_1 \xrightarrow{\alpha} t'$ and $\sigma(t'_1) = p'$. As f has the rule of the form (6) for α we can immediately infer that $t \xrightarrow{\alpha} t' \| t_2$. Hence, by letting $t' = t'_1 \| t_2$, we obtain $t \xrightarrow{\alpha} t'$ and $\sigma(t') = p$.
   - There are a variable x, a closed term q and a configuration c_1 such that $\sigma(x) \xrightarrow{\alpha} q$, $t_1 \xrightarrow{x_1}_\alpha c_1$ with $\sigma[x_{\mathrm{d}} \mapsto q](c_1) = p'$. Hence, by applying the auxiliary rule (a_6) we can infer that $f(t_1, t_2) \xrightarrow{x_1}_\alpha c_1 \| t_2$ and moreover, since x_{d} may occur only in c_1, we have $p = p' \| \sigma(t_2) = \sigma[x_{\mathrm{d}} \mapsto q](c_1 \| t_2)$.
2. Case only R^f_α holds. This case is analogous to the previous one (it is enough to switch the roles of t_1 and t_2 and consider x_{r} in place of x_{l}) and therefore omitted.
3. Case $L^f_\alpha \wedge R^f_\alpha$ holds. This case follows by noticing that $t \xrightarrow{x_{\mathrm{b}}}_\alpha$ can be inferred from both $t_1 \xrightarrow{x_{\mathrm{b}}}_\alpha$ and $t_2 \xrightarrow{x_{\mathrm{b}}}_\alpha$, and therefore the follows from the structure of the previous two cases, using rules (a_8) and (a_9).

$\square$

Next, we proceed to a more detailed analysis of the contribution of variables to the behaviour of closed instantiations of terms in which they occur.

▶ **Lemma 47.** *Let t be a term in CCS^-_f, σ be a closed substitution and $\alpha \in \{a, \bar{a}\}$. Assume that $\sigma(t) \leftrightarrow \sum_{i=1}^{n} \alpha.p_i + q$ for some n greater than the size of t and closed terms p_i, q with $p_i \not\leftrightarrow p_j$ whenever $i \neq j$. Then t has a summand x, for some variable x, such that $\sigma(x) \leftrightarrow \sum_{j \in J} \alpha.p_j + q'$ for some $J \subseteq \{1, \ldots, n\}$, with $|J| \geq 2$, and some closed term q'.*

Proof: For simplicity of notation let $I = \{1, \ldots, n\}$. Since there is a transition $\sum_{i \in I} \alpha.p_i + q \xrightarrow{\alpha} p_i$ for each $i \in I$, from $\sigma(t) \leftrightarrow \sum_{i \in I} \alpha.p_i + q$ we get that $\sigma(t) \xrightarrow{\alpha} r_i$ with $r_i \leftrightarrow p_i$, for all $i \in I$. Since n is greater than the size of t, we infer that Lemma 46.1 can be applied only to m such transitions, for some $m < n$, so that there are an index set $H \subset I$ (possibly empty) and CCS$_f$ terms t_h, for $h \in H$ such that $|H| = m$, $t \xrightarrow{\alpha} t_h$ and $\sigma(t_h) \leftrightarrow p_h$. Notice that since $p_i \not\leftrightarrow p_j$ for $i \neq j$ we get that the t_h are pairwise distinct. Let $J = I \setminus H$. For the remaining α-transitions $\sigma(t) \xrightarrow{\alpha} r_j$ for $j \in J$ we have that one among cases 2a–2c of Lemma 46 applies, according to which rules are available for f with respect to action α. Hence, we have that, for each $j \in J$ there are a variable x_j, a closed term q_j and a configuration c_j such that $\sigma(x_j) \xrightarrow{\alpha} q_j$, $t \xrightarrow{x_{j,\mathrm{w}}}_\alpha c_j$ and $\sigma[x_{j,\mathrm{d}} \mapsto q_j] = r_j$, where $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$ depends on the rules for f. Once again, since n is greater than the size of t there cannot be more than $|J| - 1$ distinct variables x_j occurring in t and causing such α-moves. Hence, there is at least one variable $x \in var(t)$ such that $\sigma(x) \leftrightarrow \alpha.q_{j_1} + \alpha.q_{j_2} + q'$ for some $j_1 \neq j_2 \in J$ and closed term q'. $\square$

The next result shows a particular case of Lemma 47, in which we can infer that, provided the term t has only one summand and has neither **0** summands nor factors, not only is a variable x responsible for the additional behaviour of t, but that t coincides with x.

▶ **Lemma 48.** *Let t be a term in CCS^-_f that does not have $+$ as head operator, and let σ be a closed substitution. Let $\alpha \in \{a, \bar{a}\}$ and $\mu \in \{a, \bar{a}, \tau\}$ with $\alpha \neq \mu$. Assume that $\sigma(t)$ has* *neither* **0** *summands nor factors, and that* $\sigma(t) \leftrightarrow \alpha.\mu^{\leq i_1} + \cdots + \alpha.\mu^{\leq i_m}$, *for some* $m > 1$ *and* $1 \leq i_1 < \ldots < i_m$. *Then* $t = x$, *for some variable* x.

Proof: Assume, towards a contradiction, that t is not a variable. We proceed by a case analysis on the possible form this term may have.

1. Case $t = \nu.t'$ for some term t'. Then $\nu = \alpha$ and $\mu^{\leq i_1} \leftrightarrow \sigma(t') \leftrightarrow \mu^{\leq i_m}$. However, this is a contradiction because, since $i_1 < i_m$, the terms $\mu^{\leq i_1}$ and $\mu^{\leq i_m}$ have different depths, and are therefore not bisimilar.
2. Case $t = f(t', t'')$ for some terms t', t''. Since $\sigma(t)$ has no **0** factors, we have that $\sigma(t') \not\leftrightarrow \mathbf{0}$ and $\sigma(t'') \not\leftrightarrow \mathbf{0}$.
   Observe now that $\alpha.\mu^{\leq i_1} + \alpha.\mu^{\leq i_m} \xrightarrow{\alpha} \mu^{\leq i_m}$. Thus, as
   $$\sigma(t) = f(\sigma(t'), \sigma(t'')) \leftrightarrow \alpha.\mu^{\leq i_1} + \cdots + \alpha.\mu^{\leq i_m} \quad ,$$
   according to which rules are available for f with respect to ν, we can distinguish the following two cases:
   - L^f_α holds and there is a term p' such that
     $$\sigma(t') \xrightarrow{\alpha} p' \text{ and } p' \| \sigma(t'') \leftrightarrow \mu^{\leq i_m} \quad .$$
     As $\sigma(t'') \not\leftrightarrow \mathbf{0}$ and $\mu^{\leq i_m}$ is prime (Lemma 39(1)), this implies that $p' \leftrightarrow \mathbf{0}$ and
     $$\sigma(t'') \leftrightarrow \mu^{\leq i_m} \quad .$$
     Since $\alpha.\mu^{\leq i_1} + \cdots + \alpha.\mu^{\leq i_m} \xrightarrow{\alpha} \mu^{\leq i_1}$, a similar reasoning allows us to conclude that
     $$\sigma(t'') \leftrightarrow \mu^{\leq i_1}$$
     also holds. However, this is a contradiction because by the proviso of the lemma $m > 1$ and $1 \leq i_1 < \ldots < i_m$, and therefore $\mu^{\leq i_1}$ and $\mu^{\leq i_m}$ are not bisimilar.
   - R^f_α holds and there is a term p'' such that
     $$\sigma(t'') \xrightarrow{\alpha} p'' \text{ and } \sigma(t') \| p'' \leftrightarrow \mu^{\leq i_m} \quad .$$
     This case is analogous to the previous one and leads as well to a contradiction.

We may therefore conclude that t must be a variable, which was to be shown. ◻

We can now establish whether some of the initial behaviour of two bisimilar terms is determined by the same variable (Proposition 54).

We start by arguing that we can also give a syntactic characterization of the occurrences in a term of the variables that can contribute to the behaviour of closed instances of that term. Formally, to infer the behaviour of a term t from that of (a closed instance of) a variable x, the latter must occur *unguarded* in t, namely x cannot occur in the scope of a prefixing operator in t. Inspired by [3], for $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$, we introduce a relation $\triangleleft^\mu_\mathrm{w}$ between a variable x and a term t. Intuitively, the role of the label w is the same as in the auxiliary transitions, namely, to identify which predicates hold (and thus which rules for f are available) for f with respect to action μ. Then $x \triangleleft^\mu_\mathrm{w} t$ holds if the predicate associated with w holds for f and whenever t has a subterm of the form $f(t_1, t_2)$ and x occurs in t_i, with $i = 1$ if $\mathrm{w} \in \{\mathrm{l}, \mathrm{b}\}$ and $i = 2$ if $\mathrm{w} \in \{\mathrm{r}, \mathrm{b}\}$, then the occurrence of x is unguarded and can contribute to an initial μ-transition of $\sigma(t)$ when $\sigma(x) \xrightarrow{\mu}$.

▶ **Definition 49** (Relation $\lhd$). *Let $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. The relation $\lhd^\mu_\mathrm{w}$ between variables and terms is defined inductively as follows:*

$$
\begin{array}{l}
\textit{1. } x \lhd^\mu_\mathrm{l} x \textit{ if } L^f_\mu \quad \textit{2. } x \lhd^\mu_\mathrm{r} x \textit{ if } R^f_\mu \quad \textit{3. } x \lhd^\mu_\mathrm{b} x \textit{ if } L^f_\mu \wedge R^f_\mu \\
\textit{4. } x \lhd^\mu_\mathrm{w} t \;\Rightarrow\; x \lhd^\mu_\mathrm{w} t + u \;\wedge\; x \lhd^\mu_\mathrm{w} u + t \\
\textit{5. } x \lhd^\mu_\mathrm{l} t \;\Rightarrow\; x \lhd^\mu_\mathrm{l} f(t,u) \qquad \textit{6. } x \lhd^\mu_\mathrm{r} t \;\Rightarrow\; x \lhd^\mu_\mathrm{r} f(u,t) \\
\textit{7. } x \lhd^\mu_\mathrm{b} t \;\Rightarrow\; x \lhd^\mu_\mathrm{b} f(t,u) \;\wedge\; x \lhd^\mu_\mathrm{b} f(u,t).
\end{array}
$$

▶ **Example 50.** Assume, for instance, that L^f_a, $R^f_{\bar{a}}$ and $L^f_\tau \wedge R^f_\tau$ are the only predicates holding. Then, for $t = f(x,\tau)$ we have that $x \lhd^a_\mathrm{l} t$, $x \lhd^\tau_\mathrm{l} t$ and $x \lhd^\tau_\mathrm{b} t$.

There is a close relation between unguarded occurrences of variables in terms and the auxiliary transitions, as stated in the following:

▶ **Lemma 51.** *Let $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. Then $x \lhd^\mu_\mathrm{w} t$ if and only if $t \xrightarrow{x_\mathrm{w}}_\mu c$ for a configuration $c \leftrightarrow x_\mathrm{d} \| t'$ for some CCS^-_f term t'.*

Proof: We prove the two implications separately.

($\Rightarrow$) We proceed by induction over the structure of t. The only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$ which we expand below, by distinguishing three cases, according to which rules for f are available:

- $x \lhd^\mu_\mathrm{l} f(t_1,t_2)$. This can only be due to $x \lhd^\mu_\mathrm{l} t_1$. By the induction hypothesis for t_1, this implies that $t_1 \xrightarrow{x_\mathrm{l}}_\mu c_1$ with $c_1 \leftrightarrow x_\mathrm{d} \| t'_1$ for some t'_1. By applying the auxiliary rule (a_6), we infer $f(t_1,t_2) \xrightarrow{x_\mathrm{l}}_\mu c$ with $c = c_1 \| t_2$ and, since $\leftrightarrow$ is a congruence with respect to $\|$ and $\|$ is associative with respect to $\leftrightarrow$, we get $c \leftrightarrow (x_\mathrm{d} \| t'_1) \| t_2 \leftrightarrow x_\mathrm{d} \| t'$ with $t' \leftrightarrow t'_1 \| t_2$.
- $x \lhd^\mu_\mathrm{r} f(t_1,t_2)$. This can only be due to $x \lhd^\mu_\mathrm{r} t_2$. Thus, we can proceed as in the previous case, by applying the auxiliary rule (a_7) in place of rule (a_6) and using the commutativity of $\|$ with respect to $\leftrightarrow$.
- $x \lhd^\mu_\mathrm{b} f(t_1,t_2)$. This can be due to either $x \lhd^\mu_\mathrm{b} t_1$ or $x \lhd^\mu_\mathrm{b} t_2$. For both, we can proceed as in the previous cases, by applying the auxiliary rules (a_8) or, respectively, (a_9) in place of rules (a_6) and (a_7).

($\Leftarrow$) We proceed by induction over the derivation of the open transition $t \xrightarrow{x_\mathrm{w}}_\mu c$. Again, the only interesting case is the inductive step corresponding to $t = f(t_1,t_2)$, which we expand below by considering three cases, according to which rules are available for f:

- $f(t_1,t_2) \xrightarrow{x_\mathrm{l}}_\mu c$ with $c \leftrightarrow x_\mathrm{d} \| t'$ for some t'. According to the auxiliary operational semantics, it must be the case that $t_1 \xrightarrow{x_\mathrm{l}}_\mu c_1$ for some c_1 such that $c = c_1 \| t_2$. Notice that since x_d can occur only in c_1, from $c = c_1 \| t_2$ and $c \leftrightarrow x_\mathrm{d} \| t'$, we infer $c_1 \leftrightarrow x_\mathrm{d} \| t''$ for some t'' such that $t'' \| t_2 \leftrightarrow t'$. Hence, we can apply the induction hypothesis to the transition from t_1 and obtain $x \lhd^\mu_\mathrm{l} t_1$. Since $t = f(t_1,t_2)$ we can immediately conclude that $x \lhd^\mu_\mathrm{l} t$.
- $f(t_1,t_2) \xrightarrow{x_\mathrm{r}}_\mu c$. It follows by a similar reasoning.
- $f(t_1,t_2) \xrightarrow{x_\mathrm{b}}_\mu c$. It follows by a similar reasoning.

□

We now discuss the necessary conditions to relate the depth of closed instances of a term to the depth of the closed instances of the variables occurring in it.

▶ **Lemma 52.** *Let t be a CCS_f^- term and σ be a closed substitution. If t has no* **0** *summands or factors and $x \triangleleft_{\mathrm{w}}^{\mu} t$ for some $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$ and $\mu \in \{a, \bar{a}, \tau\}$ with $\mathrm{init}(\sigma(x)) \subseteq \{\mu \mid x \triangleleft_{\mathrm{w}}^{\mu} t\}$, then $depth(\sigma(t)) \geq depth(\sigma(x))$.*

Proof: The proof proceeds by structural induction over t and a case analysis over $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. The only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$ which we expand below for the case of $\mathrm{w} = \mathrm{l}$. The other cases can be obtained by applying a similar reasoning.

Moreover, always for sake of simplicity, assume that there is only one action μ such that $x \triangleleft_{\mathrm{l}}^{\mu} t$, so that $\mathrm{init}(\sigma(x)) = \{\mu\}$. Once again, the general case can be easily derived from this one. Notice that this implies the existence of a closed term q such that $\sigma(x) \xrightarrow{\mu} q$ and $depth(\sigma(x)) = depth(q) + 1$. We have that $x \triangleleft_{\mathrm{l}}^{\mu} f(t_1, t_2)$ can be derived only by $x \triangleleft_{\mathrm{l}}^{\mu} t_1$. Hence, structural induction over t_1 gives $depth(\sigma(t_1)) \geq depth(\sigma(x))$. Moreover, by Lemma 51 we obtain that $t_1 \xrightarrow{x_1}_{\mu} c_1$ for some $c_1 \leftrightarrow x_{\mathrm{d}} \| t'$ for some term t'. Furthermore, $\sigma(x) \xrightarrow{\mu} q$ together with Lemma 45 gives $\sigma(t_1) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto q](c_1)$. Then we can infer that $\sigma(t) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto q](c_1) \| \sigma(t_2) \leftrightarrow q \| (\sigma(t') \| \sigma(t_2))$. We have therefore obtained

$$\begin{aligned} depth(\sigma(t)) &\geq 1 + depth(q \| (\sigma(t') \| \sigma(t_2))) \\ &= 1 + depth(q) + depth(\sigma(t') \| \sigma(t_2)) \\ &\geq 1 + depth(q) \\ &= depth(\sigma(x)). \end{aligned}$$

∎

▶ **Example 53.** We remark that, due to the potential asymmetry of the rules for f, the requirement on the set of initials of $\sigma(x)$ cannot be relaxed in any trivial way. Consider, for instance, the term $t = f(x, \tau)$ from our running example and assume that the only predicates holding are L_{α}^{f}, L_{τ}^{f} and $R_{\bar{\alpha}}^{f}$. Notice that $x \triangleleft_{\mathrm{l}}^{\alpha} t$ and $x \triangleleft_{\mathrm{l}}^{\tau} t$. Consider the closed substitution σ with $\sigma(x) = \alpha + \tau + \bar{\alpha}.\alpha^n$, for some $n \geq 2$, so that $\{\alpha, \tau\} \subset \mathrm{init}(\sigma(x)) = \{\alpha, \tau, \bar{\alpha}\}$. As $L_{\bar{\alpha}}^{f}$ and R_{τ}^{f} do not hold, the only inferable initial transitions for $\sigma(t)$ are those resulting from the α-move and the τ-move by $\sigma(x)$. Thus, we get that $depth(\sigma(t)) = 2$, whereas $depth(\sigma(x)) \geq 3$. This is due to the fact that the computation of $\sigma(x)$ starting with a $\bar{\alpha}$-move is *blocked* by the rules for f and, thus, it cannot contribute to the behaviour of t.

We can now proceed to prove the following:

▶ **Proposition 54.** *Let $\alpha \in \{a, \bar{a}\}$, x be a variable and t, u be CCS_f^- with $t \leftrightarrow u$ and such that neither t nor u has* **0** *summands or factors. If $x \triangleleft_{\mathrm{w}}^{\alpha} t$ for some $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$, then $x \triangleleft_{\mathrm{w}}^{\alpha} u$. In particular, if $x \triangleleft_{\mathrm{w}}^{\alpha} t$ because t has a summand x, then so does u.*

Proof: Observe, first of all, that since t and u have no **0** summands or factors, by Remark 29 we can assume that $t = \sum_{i \in I} t_i$ and $u = \sum_{j \in J} u_j$ for some finite non-empty index sets I, J, where none of the t_i ($i \in I$) and u_j ($j \in J$) has $+$ as its head operator, and none of the t_i ($i \in I$) and u_j ($j \in J$) have **0** summands or factors. Therefore, $x \triangleleft_{\mathrm{w}}^{\alpha} t$ implies that there is some index $i \in I$ such that $x \triangleleft_{\mathrm{w}}^{\alpha} t_i$. We then proceed by a case analysis on the rules available for f. Actually we expand only the case in which only L_{α}^{f} holds, as the other two cases, in which respectively only R_{α}^{f} holds, or $L_{\alpha}^{f} \wedge R_{\alpha}^{f}$ holds, can be obtained analogously.

Since only L_{α}^{f} holds, then it must be the case that $x \triangleleft_{\mathrm{l}}^{\alpha} t_i$. By Lemma 51 we get that $t_i \xrightarrow{x_1}_{\alpha} c$ for some configuration c with $c \leftrightarrow x_{\mathrm{d}} \| t'$ for some t'. Let n be greater than the size of t and consider the substitution σ such that

$$\sigma(y) = \begin{cases} \alpha \sum_{i=1}^{n} \bar{\alpha} \alpha^{\leq i} & \text{if } y = x \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

For simplicity of notation, let $p_n = \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i}$. Clearly $\sigma(x) \xrightarrow{\alpha} p_n$. By Lemma 45 we obtain that $\sigma(t_i) \xrightarrow{\alpha} p$ with $p = \sigma[x_{\rm d} \mapsto p_n](c)$ and, thus, $p \leftrightarrow p_n \| \sigma(t')$. As $t \leftrightarrow u$ implies $\sigma(t) \leftrightarrow \sigma(u)$, we get that there is an index $j \in J$ such that $\sigma(u_j) \xrightarrow{\alpha} q$ for some $q \leftrightarrow p_n \| \sigma(t')$. As only L^f_α holds, by Lemma 46 we can distinguish two cases:

- There are a variable y, a closed term q' and a configuration c' such that $\sigma(y) \xrightarrow{\alpha} q'$, $u_j \xrightarrow{y_1}_\alpha c'$ and $q = \sigma[y_{\rm d} \mapsto q'](c')$. Since σ maps all variables but x to $\mathbf{0}$, we can directly infer that $y = x$, $q' = p_n$. Moreover, as p_n is prime and there is a unique prime decomposition of processes, we also infer that $c' \leftrightarrow x_{\rm d} \| u'$ for some u' with $\sigma(u') \leftrightarrow \sigma(t')$. Consequently, by Lemma 51 we can conclude that $x \lhd^\alpha_{\rm l} u_j$ and thus $x \lhd^\alpha_{\rm l} u$ as required.
- There is a term u' such that $u_j \xrightarrow{\alpha} u'$ and $\sigma(u') \leftrightarrow p_n \| \sigma(t')$. We proceed to show that this case leads to a contradiction. We distinguish two cases:
  - $\sigma(t') \leftrightarrow \mathbf{0}$. Thus $\sigma(u') \leftrightarrow p_n$ and we can rewrite $u' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator. Moreover, since u not having $\mathbf{0}$ summands nor factors implies that neither u_j no u' have some, the same holds for all the v_h. Since n is larger than the size of u, and thus than that of u', by Lemma 48 $\sigma(u') \leftrightarrow p_n$ implies that there is one index $h \in H$ such that $v_h = y$ for some variable y and $\sigma(y) \leftrightarrow \bar{\alpha}\alpha^{\leq i_1} + \dots + \bar{\alpha}\alpha^{\leq i_m}$ for some $m > 1$ and $1 \leq i_1 < \dots < i_m \leq n$. However, by the choice of σ, all variables but x are mapped to $\mathbf{0}$, and moreover $\sigma(x) \not\leftrightarrow \bar{\alpha}\alpha^{\leq i_1} + \dots + \bar{\alpha}\alpha^{\leq i_m}$ thus contradicting $\sigma(u') \leftrightarrow p_n$.
  - $\sigma(t') \not\leftrightarrow \mathbf{0}$. Consequently, $\sigma(t') \leftrightarrow \sum_{h \in H} \mu_h q_h$ for some actions $\mu_h \in \{a, \bar{a}, \tau\}$ and closed terms q_h. We can therefore apply the expansion law for parallel composition obtaining

$$
\begin{aligned}
\sigma(u') &\leftrightarrow p_n \| \sigma(t') \\
&\leftrightarrow \sum_{i=1}^{n} \bar{\alpha}(\alpha^{\leq i} \| \sigma(t')) + \sum_{h \in H} \mu_h (p_n \| q_h) + \\
&\quad + \sum_{\substack{i=1,\dots,n \\ h \in H \text{ s.t. } \mu_h = \alpha}} \tau(\alpha^{\leq i} \| q_h).
\end{aligned}
$$

    We notice that the first term in the expansion has size at least $n+1$ and therefore greater than the size of u and in particular of u'. Moreover $\alpha^{\leq i} \| \sigma(t') \not\leftrightarrow \alpha^{\leq j} \| \sigma(t')$ whenever $i \neq j$. Therefore, by Lemma 47 there is a variable $y \in \mathit{var}(u')$ such that $\sigma(y) \leftrightarrow \bar{\alpha}(\alpha^{\leq i_1} \| \sigma(t')) + \dots + \bar{\alpha}(\alpha^{\leq i_m} \| \sigma(t')) + r$ for some $m > 1$ and $1 \leq i_1 < \dots < i_m$ and closed term r. However, $\sigma(y) = \mathbf{0}$ whenever $y \neq x$ and $\sigma(x) \not\leftrightarrow \bar{\alpha}(\alpha^{\leq i_1} \| \sigma(t')) + \dots + \bar{\alpha}(\alpha^{\leq i_m} \| \sigma(t')) + r$, for any closed term r, thus contradicting $\sigma(u') \leftrightarrow p_n \| \sigma(t')$.

We have therefore obtained that whenever $x \lhd^\alpha_{\rm l} t$ then also $x \lhd^\alpha_{\rm l} u$.

Assume now that t has a summand x. We aim to show that u has a summand x as well. Since $x \lhd^\alpha_{\rm l} x$ gives $x \lhd^\alpha_{\rm l} t$, by the first part of the Proposition we get $x \lhd^\alpha_{\rm l} u$ and thus there is an index $j \in J$ such that $x \lhd^\alpha_{\rm l} u_j$. We now treat the cases of an operator f that distributes over $+$ in its first argument and of an operator f that does not distribute in either argument separately.

Case of an operator f that distributes over $+$ in its first argument. Consider the substitution $\sigma_{\mathbf{0}}$ mapping each variable to $\mathbf{0}$. Pick an integer m larger than the depth of $\sigma_{\mathbf{0}}(t)$ and of $\sigma_{\mathbf{0}}(u)$. Let σ be the substitution mapping x to the term a^{m+1} and agreeing with $\sigma_{\mathbf{0}}$ on all the other variables.

As $t \approx u$ is sound with respect to bisimulation equivalence, we have that

$$\sigma(t) \leftrightarrow \sigma(u) \ .$$

Moreover, the term $\sigma(t)$ affords the transition $\sigma(t) \xrightarrow{a} a^m$, for $t_i = x$ and $\sigma(x) = a^{m+1} \xrightarrow{a} a^m$. Hence, for some closed term p,

$$\sigma(u) = \sum_{j \in J} \sigma(u_j) \xrightarrow{a} p \leftrightarrow a^m \ .$$

This means that there is a $j \in J$ such that $\sigma(u_j) \xrightarrow{a} p$. We claim that this u_j can only be the variable x. To see that this claim holds, observe, first of all, that $x \in \mathit{var}(u_j)$. In fact, if x did not occur in u_j, then we would reach a contradiction thus:

$$\begin{aligned} m = \mathit{depth}(p) &< \mathit{depth}(\sigma(u_j)) \\ = \mathit{depth}(\sigma_{\mathbf{0}}(u_j)) &\leq \mathit{depth}(\sigma_{\mathbf{0}}(u)) < m \ . \end{aligned}$$

Using this observation and Lemma 52, it is not hard to show that, for each of the other possible forms u_j may have, $\sigma(u_j)$ does not afford an a-labelled transition leading to a term of depth m. We may therefore conclude that $u_j = x$, which was to be shown.

Case of an operator f that does not distribute over $+$ in either argument. Notice that in the case at hand, there must be at least one action $\mu \in \{a, \bar{a}, \tau\}$ such that R^f_μ holds. Assume such an action μ. Again, let n be greater than the size of t and consider the substitution

$$\sigma_1(y) = \begin{cases} \alpha\alpha^{\leq n} & \text{if } y = x \\ \alpha + \mu & \text{otherwise.} \end{cases}$$

Thus $\sigma_1(x) \xrightarrow{\alpha} \alpha^{\leq n}$ and consequently $\sigma_1(t) \xrightarrow{\alpha} \alpha^{\leq n}$. Since $\sigma_1(t) \leftrightarrow \sigma_1(u)$ it must hold that $\sigma_1(u) \xrightarrow{\alpha} q$ for some $q \leftrightarrow \alpha^{\leq n}$. As n is greater than the size of u, one can infer that u can have a summand given by at most $\lfloor \frac{n-2}{2} \rfloor$ nested occurrences of f (which is a binary operator of size at least 3). Since, moreover, all variables but x are mapped into a term of depth 1, we can infer that the only term that can be responsible for the α-move to q is a summand u_j such that $x \lhd^\alpha_1 u_j$. To show $u_j = x$ we show that the only other possible case, namely $u_j = f(u', u'')$ with $x \lhd^\alpha_1 u'$ leads to a contradiction. Recall that by the proviso of the Proposition u has no $\mathbf{0}$ factors, which implies that $u', u'' \not\leftrightarrow \mathbf{0}$. Since moreover, $x \lhd^\alpha_1 u'$, by Lemma 51 and Lemma 46 we get $u' \xrightarrow{x_1}_\alpha c$ and thus $u_j \xrightarrow{x_1}_\alpha c \| u''$ for some configuration $c \leftrightarrow x_{\mathrm{d}} \| u'''$ for some term u''', so that $\sigma_1(u_j) \xrightarrow{\alpha} \sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\leq n}](c) \| \sigma_1(u'') = q$. However, $u'' \not\leftrightarrow \mathbf{0}$ implies that either there is a term v such that $u'' \xrightarrow{\nu} v$, for some action ν, or in u'' at least one variable occurs unguarded. Hence, by the choice of σ_1, as both L^f_α and R^f_μ hold, we can infer that $\mathit{depth}(\sigma_1(u'')) \geq 1$ which gives

$$\begin{aligned} n &= \mathit{depth}(\alpha^{\leq n}) \\ &= \mathit{depth}(q) \\ &= \mathit{depth}(\sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\leq n}](c) \| \sigma_1(u'')) \\ &= \mathit{depth}(\sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\leq n}](c)) + \mathit{depth}(\sigma_1(u'')) \\ &\geq \mathit{depth}(\alpha^{\leq n}) + \mathit{depth}(\sigma_1(u'')) \\ &\geq n + 1 \end{aligned}$$

thus contradicting $q \leftrightarrow \alpha^{\leq n}$. $\square$

I Proof of Theorem 17

Before proceeding to the proof, we present a technical lemma stating that, under the considered set of rules for f, if a closed term $\sigma(t)$ is not bisimilar to $\mathbf{0}$, then by instantiating the variables in t with a process which is not bisimilar to $\mathbf{0}$ we cannot obtain a closed instance of t which is bisimilar to $\mathbf{0}$.

▶ **Lemma 55.** *Let t be a CCS_f^- term and let σ be a substitution with $\sigma(t) \not\leftrightarrow \mathbf{0}$. Assume that u is a CCS_f^- term that is not bisimilar to $\mathbf{0}$. Then $\sigma[x \mapsto u](t) \not\leftrightarrow \mathbf{0}$ for each variable x.*

Proof: By induction on the structure t. □

▶ Remark 56. We have defined the processes p_n in a such a way that an initial synchronization, in the scope of operator f, with the process α is always possible. This choice will allow us to slightly simplify the reasoning in the proof of the upcoming Proposition 59 and thus of the negative result (cf., for instance, with the proof of Proposition 63 in Section 7). Clearly, the possibility of synchronization is directly related to which rules of type (5) are available for f. However, since f has a rule of type (6) for all actions, it is then always possible to identify a pair μ, p_n such that $f(\mu, p_n) \xrightarrow{\tau}$ due to an application of the rule of type (5) allowed for f.

Finally, we study some properties of the processes $f(\alpha, p_n)$, which also depend on the particular configuration of rules for f that we are considering.

▶ **Lemma 57.** *The term $f(\alpha, p_n)$ is prime, for each $n \geq 0$.*

Proof: Since $f(\alpha, p_n)$ is not bisimilar to $\mathbf{0}$, to prove the statement it suffices only to show that $f(\alpha, p_n)$ is irreducible for $n \geq 0$.

If $n = 0$ then $f(\alpha, p_n) = f(\alpha, \mathbf{0})$ is a term of depth 1, and is therefore irreducible as claimed.

Consider now $n \geq 1$. Assume, towards a contradiction, that $f(\alpha, p_n) \leftrightarrow p \| q$ for two closed terms p and q with $p \not\leftrightarrow \mathbf{0}$ and $q \not\leftrightarrow \mathbf{0}$, that is, $f(\alpha, p_n)$ is *not* irreducible. We have that

$$f(\alpha, p_n) \xrightarrow{\alpha} \mathbf{0} \| p_n \leftrightarrow p_n \ .$$

As $f(\alpha, p_n) \leftrightarrow p \| q$, there is a transition $p \| q \xrightarrow{\alpha} r$ for some $r \leftrightarrow p_n$. Without loss of generality, we may assume that $p \xrightarrow{\alpha} p'$ and $r = p' \| q$. Since we have assumed that $n \geq 1$, by statement 2 and our assumption that $q \not\leftrightarrow \mathbf{0}$, we have that $p' \leftrightarrow \mathbf{0}$ and $q \leftrightarrow p_n$. Again using that $n \geq 1$, it follows that $q \xrightarrow{\bar{\alpha}} q'$ for some q'. This means that $p \| q \xrightarrow{\bar{\alpha}}$, contradicting the assumption that $f(\alpha, p_n) \leftrightarrow p \| q$. Thus $f(\alpha, p_n)$ is irreducible, which was to be shown. □

▶ **Lemma 58.** *Let $n \geq 1$. Assume that $f(p, q) \leftrightarrow f(\alpha, p_n)$, where $q \not\leftrightarrow \mathbf{0}$. Then $p \leftrightarrow \alpha$ and $q \leftrightarrow p_n$.*

Proof: Since $f(p, q) \leftrightarrow f(\alpha, p_n)$ and $f(\alpha, p_n) \xrightarrow{\alpha} \mathbf{0} \| p_n \leftrightarrow p_n$, there is a p' such that $p \xrightarrow{\alpha} p'$ and $p' \| q \leftrightarrow p_n$. It follows that $q \leftrightarrow p_n$ and $p' \leftrightarrow \mathbf{0}$, because p_n is prime (Lemma 39(2)) and $q \not\leftrightarrow \mathbf{0}$. We are therefore left to prove that p is bisimilar to α. To this end, note, first of all, that, as $\leftrightarrow$ is a congruence over the language CCS_f, we have that

$$f(p, p_n) \leftrightarrow f(\alpha, p_n) \ .$$

Assume now that $p \xrightarrow{\mu} p''$ for some action μ and closed term p''. In light of the above equivalence, one of the following two cases may arise:

1. $\mu = \alpha$ and $p''\|p_n \leftrightarrow p_n$ or
2. $\mu = \tau$ and $p''\|p_n \leftrightarrow \alpha^{\leq i}$, for some $i \in \{1, \dots, n\}$.

In the former case, p'' must have depth 0 and is thus bisimilar to $\mathbf{0}$. The latter case is impossible, because the depth of $p''\|p_n$ is at least $n+1$.

We may therefore conclude that every transition of p is of the form $p \xrightarrow{\alpha} p''$, for some $p'' \leftrightarrow \mathbf{0}$. Since we have already seen that p affords an α-labelled transition leading to $\mathbf{0}$, modulo bisimulation equivalence, it follows that $p \leftrightarrow \alpha$, which was to be shown. ◻

The following result, stating that the property mentioned in the statement of that theorem holds for all closed instantiations of axioms in $\mathcal{E}$, will be the crux in the proof of Theorem 17.

▶ **Proposition 59.** *Assume an operator f that, modulo $\leftrightarrow$, distributes over $+$ in its first argument and such that only L^f_μ holds for each action μ, and only $S^f_{\alpha,\bar{\alpha}}$ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\leftrightarrow$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands or factors and $p, q \leftrightarrow f(\alpha, p_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: Observe, first of all, that since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Hence, by Remark 29, we have that for some finite non-empty index sets I, J,

$$t = \sum_{i \in I} t_i \quad \text{and} \quad u = \sum_{j \in J} u_j \ ,$$

where none of the t_i ($i \in I$) and u_j ($j \in J$) is $\mathbf{0}$, has $+$ as its head operator, has $\mathbf{0}$ summands and factors.

Since $p = \sigma(t)$ has a summand bisimilar to $f(\alpha, p_n)$, there is an index $i \in I$ such that $\sigma(t_i) \leftrightarrow f(\alpha, p_n)$.

Our aim is now to show that there is an index $j \in J$ such that $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$, proving that $q = \sigma(u)$ also has a summand bisimilar to $f(\alpha, p_n)$.

We proceed by a case analysis on the form t_i may have.

1. Case $t_i = x$ for some variable x. In this case, we have $\sigma(x) \leftrightarrow f(\alpha, p_n)$, and t has x as a summand. As $t \approx u$ is sound with respect to bisimilarity and neither t nor u have $\mathbf{0}$ summands or factors, it follows that u also has x as a summand (Proposition 54). Thus there is an index $j \in J$ such that $u_j = x$, and, modulo bisimulation, $\sigma(u)$ has $f(\alpha, p_n)$ as a summand, which was to be shown.
2. Case $t_i = \mu t'$ for some term t'. This case is vacuous because, since $\mu\sigma(t') \xrightarrow{\mu} \sigma(t')$ is the only transition afforded by $\sigma(t_i)$, this term cannot be bisimilar to $f(\alpha, p_n)$. Indeed $f(\alpha, p_n)$ can perform both, an α-labelled transition triggered by the first argument, and the τ-move due to the synchronization between α and p_n.
3. Case $t_i = f(t', t'')$ for some terms t', t''. In this case, we have $f(\sigma(t'), \sigma(t'')) \leftrightarrow f(\alpha, p_n)$. As $\sigma(t_i)$ has no $\mathbf{0}$ factors, it follows that $\sigma(t') \not\leftrightarrow \mathbf{0}$ and $\sigma(t'') \not\leftrightarrow \mathbf{0}$. Thus $\sigma(t') \leftrightarrow \alpha$ and $\sigma(t'') \leftrightarrow p_n$ (Lemma 58). Now, t'' can be written as $t'' = v_1 + \cdots + v_\ell, (\ell > 0)$, where none of the summands v_i is $\mathbf{0}$ or a sum. Observe that, since n is larger than the size of t, we have that $\ell < n$. Hence, since $\sigma(t'') \leftrightarrow p_n = \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i}$, there must be some $h \in \{1, \dots, \ell\}$ such that $\sigma(v_h) \leftrightarrow \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m}$ for some $m > 1$ and $1 \leq i_1 < \ldots < i_m \leq n$. The term $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors—or else, so would

$\sigma(t'')$, and thus $p = \sigma(t)$. By Lemma 48, it follows that v_h can only be a variable x and thus that

$$\sigma(x) \leftrightarrow \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} \ . \tag{9}$$

Observe, for later use, that, since t' has no $\mathbf{0}$ factors, the above equation yields that $x \notin \mathit{var}(t')$—or else $\sigma(t') \not\leftrightarrow \alpha$ (Lemma 52). So, modulo bisimilarity, t_i has the form $f(t', (x + t'''))$, for some term t''', with $x \notin \mathit{var}(t')$ and $\sigma(t') \leftrightarrow \alpha$.
Our order of business will now be to use the information collected so far in this case of the proof to argue that $\sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$. To this end, consider the substitution

$$\sigma' = \sigma[x \mapsto \bar{\alpha} f(\alpha, p_n)] \ .$$

We have that

$$\begin{aligned} \sigma'(t_i) &= f(\sigma'(t'), \sigma'(t'')) \\ &= f(\sigma(t'), \sigma'(t'')) && (\text{As } x \notin \mathit{var}(t')) \\ &\leftrightarrow f(\alpha, (\bar{\alpha} f(\alpha, p_n) + \sigma'(t'''))) && (\text{As } t'' = x + t'''). \end{aligned}$$

Thus, $\sigma'(t_i) \xrightarrow{\tau} p' \leftrightarrow f(\alpha, p_n)$ for some p', so that

$$\sigma'(t) \xrightarrow{\tau} p' \leftrightarrow f(\alpha, p_n)$$

also holds. Since $t \approx u$ is sound with respect to $\leftrightarrow$, it follows that

$$\sigma'(t) \leftrightarrow \sigma'(u) \ .$$

Hence, we can infer that there are a $j \in J$ and a q' such that

$$\sigma'(u_j) \xrightarrow{\tau} q' \leftrightarrow f(\alpha, p_n) \ . \tag{10}$$

Recall that, by one of the assumptions of the proposition, $\sigma(u) \leftrightarrow f(\alpha, p_n)$, and thus $\sigma(u)$ has depth $n + 2$. On the other hand, by (10),

$$\mathit{depth}(\sigma'(u_j)) \geq n + 3 \ .$$

Since σ and σ' differ only in the closed term they map variable x to, it follows that

$$x \in \mathit{var}(u_j) \ . \tag{11}$$

We now proceed to show that $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$ by a further case analysis on the form a term u_j satisfying (10) and (11) may have.

a. CASE $u_j = x$. This case is vacuous because $\sigma'(x) = \bar{\alpha} f(\alpha, p_n) \stackrel{\tau}{\not\rightarrow}$, and thus this possible form for u_j does not meet (10).

b. CASE $u_j = \mu u'$ FOR SOME TERM u'. In light of (10), we have that $\mu = \tau$ and $q' = \sigma'(u') \leftrightarrow f(\alpha, p_n)$. Using (11) and the fact that u' has no $\mathbf{0}$ factors, we have that $\mathit{depth}(\sigma'(u')) \geq n + 3$ (Lemma 52). Since $f(\alpha, p_n)$ has depth $n + 2$, this contradicts $q' \leftrightarrow f(\alpha, p_n)$.

c. CASE $u_j = f(u', u'')$ FOR SOME TERMS u', u''. Our assumption that $\sigma(u)$ has no $\mathbf{0}$ factors yields that none of the terms $u', u'', \sigma(u')$ and $\sigma(u'')$ is bisimilar to $\mathbf{0}$. Moreover, by (11), either $x \in \mathit{var}(u')$ or $x \in \mathit{var}(u'')$.
Since $\sigma'(u_j) = f(\sigma'(u'), \sigma'(u''))$ affords transition (10), we have that $q' = q_1 \| q_2$ for some q_1, q_2. As $f(\alpha, p_n)$ is prime (Lemma 57), it follows that either $q_1 \leftrightarrow \mathbf{0}$ or $q_2 \leftrightarrow \mathbf{0}$. Hence, we can distinguish two cases, according to the possible origins for transition (10):

i. $\sigma'(u') \xrightarrow{\tau} q_1$ and $q_2 = \sigma'(u'')$. We now proceed to argue that this case produces a contradiction.

To this end, note first of all that $\sigma'(u'') \not\leftrightarrow \mathbf{0}$, because $\sigma(u'') \not\leftrightarrow \mathbf{0}$ (Lemma 55). Thus it must be the case that $q_1 \leftrightarrow \mathbf{0}$ and $q_2 = \sigma'(u'') \leftrightarrow f(\alpha, p_n)$. In light of the definition of σ', it follows that x occurs in u', but not in u'' (Lemma 52). Therefore, since σ and σ' only differ at the variable x,

$$\sigma(u'') = \sigma'(u'') \leftrightarrow f(\alpha, p_n) \ .$$

Since $\leftrightarrow$ is a congruence, we derive that

$$\sigma(u_j) = f(\sigma(u'), \sigma(u'')) \leftrightarrow f(\sigma(u'), f(\alpha, p_n)). \tag{12}$$

Since $\sigma(u') \not\leftrightarrow \mathbf{0}$ because $q = \sigma(u)$ has no $\mathbf{0}$-factors, we may infer that

$$\begin{aligned}
& n+2 \\
={}& \mathit{depth}(f(\alpha, p_n)) \\
={}& \mathit{depth}(\sigma(u)) && (\text{As } \sigma(u) \leftrightarrow f(\alpha, p_n)) \\
\geq{}& \mathit{depth}(\sigma(u_j)) \\
={}& \mathit{depth}(\sigma(u')) + n + 2 && (\text{By (12)}) \\
>{}& n+2 && (\text{As } \mathit{depth}(\sigma(u')) > 0),
\end{aligned}$$

which is the desired contradiction.

ii. $\sigma'(u') \xrightarrow{\alpha} q_1$ and $\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2$. Recall that exactly one of q_1, q_2 is bisimilar to $\mathbf{0}$. We proceed with the proof by considering these two possible cases in turn.

- Case $q_1 \leftrightarrow \mathbf{0}$. Our order of business will be to argue that, in this case, $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$, and thus that $q = \sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$. To this end, observe, first of all, that $q_2 \leftrightarrow f(\alpha, p_n)$ by (10). It follows that $x \in \mathit{var}(u'')$, for otherwise we could derive a contradiction thus:

$$\begin{aligned}
& \mathit{depth}(f(\alpha, p_n)) \\
={}& \mathit{depth}(\sigma(u)) && (\text{As } \sigma(u) \leftrightarrow f(\alpha, p_n)) \\
\geq{}& \mathit{depth}(\sigma(u_j)) \\
>{}& \mathit{depth}(\sigma(u'')) && (\text{As } \mathit{depth}(\sigma(u')) > 0) \\
={}& \mathit{depth}(\sigma'(u'')) && (\text{As } x \notin \mathit{var}(u'')) \\
>{}& \mathit{depth}(f(\alpha, p_n)) && (\text{As } \sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \leftrightarrow f(\alpha, p_n)).
\end{aligned}$$

  Moreover, we claim that $x \notin \mathit{var}(u')$. Indeed, if x also occurred in u', then, since u' has no $\mathbf{0}$ factors, the term $\sigma(x)$ would contribute to the behaviour of $\sigma(u_j)$. Therefore, by (9), the term $\sigma(u_j)$ would afford a sequence of actions containing two occurrences of $\bar{\alpha}$, contradicting our assumption that $\sigma(u) \leftrightarrow f(\alpha, p_n)$.

  Observe now that, as $\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \leftrightarrow f(\alpha, p_n)$, it must be the case that u'' has a summand x. To see that this does hold, we examine the other possible forms a summand w of u'' responsible for the transition

$$\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \leftrightarrow f(\alpha, p_n)$$

  may have, and argue that each of them leads to a contradiction.

A. Case $w = \bar{\alpha}w'$, for some term w'. In this case, $q_2 = \sigma'(w')$. However, the depth of such a q_2 is either smaller than $n+2$ (if $x \notin \mathit{var}(w')$), or larger than $n+2$ (if $x \in \mathit{var}(w')$). More precisely, in the former case $x \notin \mathit{var}(w')$ implies $\sigma(w) = \sigma'(w)$ and thus $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ gives $n+2 = \mathit{depth}(\sigma(u)) \geq \mathit{depth}(\sigma(w)) = 1 + \mathit{depth}(\sigma(w'))$, giving $\mathit{depth}(\sigma'(w')) \leq n+1$. In the latter case, as $x \in \mathit{var}(w')$ and w' does not have $\mathbf{0}$ factors (or otherwise u'' would have $\mathbf{0}$ factors), by Lemma 52, we would have $\mathit{depth}(\sigma'(w')) \geq \mathit{depth}(\sigma'(x)) = n+3$. Both cases then contradict the fact that q_2 is bisimilar to $f(\alpha, p_n)$, because the latter term has depth $n+2$.

B. Case $w = f(w_1, w_2)$, for some terms w_1 and w_2. Observe, first of all, that $\sigma(w_1)$ and $\sigma(w_2)$ are not bisimilar to $\mathbf{0}$, because $\sigma(u)$ has no $\mathbf{0}$ factors. It follows that $\sigma'(w_1)$ and $\sigma'(w_2)$ are not bisimilar to $\mathbf{0}$ either (Lemma 55). Now, since

$$\sigma'(w) = f(\sigma'(w_1), \sigma'(w_2)) \xrightarrow{\bar{\alpha}} q_2 \ ,$$

there is a closed term q_3 such that $\sigma'(w_1) \xrightarrow{\bar{\alpha}} q_3$ and

$$q_2 = q_3 \| \sigma'(w_2) \ \underline{\leftrightarrow}\ f(\alpha, p_n) \ .$$

As the term $f(\alpha, p_n)$ is prime, and $\sigma'(w_2)$ is not bisimilar to $\mathbf{0}$, we may infer that $q_3 \underline{\leftrightarrow} \mathbf{0}$ and

$$\sigma'(w_2) \ \underline{\leftrightarrow}\ f(\alpha, p_n) \ .$$

It follows that $x \notin \mathit{var}(w_2)$, or else the depth of $\sigma'(w_2)$ would be at least $n+3$, and therefore that

$$\sigma'(w_2) = \sigma(w_2) \ \underline{\leftrightarrow}\ f(\alpha, p_n) \ .$$

However, this contradicts our assumption that

$$q = \sigma(u) \ \underline{\leftrightarrow}\ f(\alpha, p_n) \ .$$

Summing up, we have argued that u'' has a summand x. Therefore, by (9),

$$\sigma(u'') \ \underline{\leftrightarrow}\ \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} + r'' \ ,$$

for some closed term r''. We have already noted that

$$\sigma(u') = \sigma'(u') \xrightarrow{\alpha} q_1 \ \underline{\leftrightarrow}\ \mathbf{0} \ .$$

Therefore, we have that

$$\sigma(u') \ \underline{\leftrightarrow}\ \alpha + r' \ ,$$

for some closed term r'. Using the congruence properties of bisimulation equivalence, we may infer that

$$\begin{aligned} \sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\ &\underline{\leftrightarrow}\ f((\alpha + r'), (\sum_{j=1}^{m} \bar{\alpha}.\alpha^{\leq i_j} + r'')). \end{aligned}$$

In light of this equivalence, we have that

$$\sigma(u_j) \xrightarrow{\alpha} r \leftrightarrow \sum_{j=1}^{m} \bar{\alpha}.\alpha^{\leq i_j} + r'' \leftrightarrow \sigma(u''),$$

for some closed term r, and thus

$$q = \sigma(u) \xrightarrow{\alpha} r \ .$$

Since $q = \sigma(u) \leftrightarrow f(\alpha, p_n)$ by our assumption, it must be the case that $r \leftrightarrow \sigma(u'') \leftrightarrow p_n$. So, again using the congruence properties of $\leftrightarrow$, we have that

$$\sigma(u_j) = f(\sigma(u'), \sigma(u'')) \leftrightarrow f((\alpha + r'), p_n).$$

As $\sigma(u) \leftrightarrow f(\alpha, p_n)$, using Lemma 58 it is now a simple matter to infer that

$$\sigma(u') \leftrightarrow \alpha \ .$$

Hence $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$. Note that $\sigma(u_j)$ is a summand of $q = \sigma(u)$. Therefore q has a summand bisimilar to $f(\alpha, p_n)$, which was to be shown.

- Case $q_2 \leftrightarrow \mathbf{0}$. We now proceed to argue that this case produces a contradiction. To this end, observe, first of all, that $q_1 \leftrightarrow f(\alpha, p_n)$. Reasoning as in the analysis of the previous case, we may infer that x occurs in u', but x does not occur in u''. Moreover, since $\sigma'(u') \xrightarrow{\alpha} q_1 \leftrightarrow f(\alpha, p_n)$, it must be the case that $u' \xrightarrow{\alpha} u'''$ for some u''' such that

$$\sigma'(u''') = q_1 \leftrightarrow f(\alpha, p_n) \ .$$

(For, otherwise, using Lemma 46.2a, we would have that $\sigma'(u') \xrightarrow{\alpha} q_1$ because $u' \xrightarrow{y} c$, $\sigma(y) \xrightarrow{\alpha} q_1'$ and $q_1 = \sigma'[y_d \mapsto q_1'](c)$, for some variable y, configuration c and closed term q_1'. Then we would necessarily have that $y \neq x$. In fact, if $y = x$, then we would have that $\alpha = \bar{\alpha}$ by the definition of σ', contradicting the distinctness of these two complementary actions. Observe now that, again in light of the definition of σ', the variable x cannot occur in c, or else the depth of

$$q_1 = \sigma'[y_d \mapsto q_1'](c)$$

would be at least $n + 3$, contradicting our assumption that

$$q_1 \leftrightarrow f(\alpha, p_n) \ .$$

Hence, since the variable y is different from x, it is not hard to see that $\sigma(u') \xrightarrow{\alpha} q_1$ also holds, and thus that

$$\mathit{depth}(q_1) < \mathit{depth}(\sigma(u)) = n + 2 \ ,$$

contradicting our assumption that $q_1 \leftrightarrow f(\alpha, p_n)$.) Since u contains no $\mathbf{0}$ factors, in light of the definition of σ', this u''' cannot contain occurrences of the variable x. (For, otherwise, Lemma 52 would yield that

$$\mathit{depth}(\sigma'(u''')) = \mathit{depth}(q_1) \geq n + 3 \ ,$$

contradicting our assumption that $q_1 \leftrightarrow f(\alpha, p_n)$.) So

$$\sigma(u''') = q_1 \leftrightarrow f(\alpha, p_n)$$

also holds. Thus

$$
\begin{aligned}
& n+2 \\
= \; & depth(f(\alpha, p_n)) \\
= \; & depth(\sigma(u)) && (\text{As } \sigma(u) \leftrightarrow f(\alpha, p_n)) \\
\geq \; & depth(\sigma(u_j)) \\
= \; & depth(f(\sigma(u'), \sigma(u''))) \\
> \; & depth(\sigma(u''')) + depth(\sigma(u'')) && (\text{As } \sigma(u') \xrightarrow{\alpha} \sigma(u''')) \\
> \; & n+2
\end{aligned}
$$

where the last inequality follows by the fact that $depth(\sigma(u'')) > 0$ and $depth(\sigma(u''')) = n+2$, and gives the desired contradiction.

This completes the proof for the case $u_j = f(u', u'')$ for some terms u', u''.

The proof of Proposition 59 is now complete. ◻

I.1 Formal proof of Theorem 17

By exploiting the properties discussed in Appendix F, Theorem 17 is equivalent to the following:

▶ **Theorem 60.** *Assume an operator f such that only L^f_μ holds for each action μ and only $S^f_{\alpha,\bar{\alpha}}$ holds. Let $\mathcal{E}$ be a finite axiom system over the language CCS^-_f that is sound with respect to bisimulation equivalence. Let n be larger than the size of each term in the equations in $\mathcal{E}$. Assume that p and q are closed terms that are bisimilar to $f(\alpha, p_n)$, and contain no occurrences of $\mathbf{0}$ as a summand or factor. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.*

Proof: Assume that $\mathcal{E}$ is a finite axiom system over the language CCS^-_f that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in $\mathcal{E}$:

1. $E \vdash p \approx q$,
2. $p \leftrightarrow q \leftrightarrow f(\alpha, p_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, p_n)$.

We prove that q also has a summand bisimilar to $f(\alpha, p_n)$ by induction on the depth of the closed proof of the equation $p \approx q$ from $\mathcal{E}$. Recall that, without loss of generality, we may assume that the closed terms involved in the proof of the equation$p \approx q$ have no $\mathbf{0}$ summands or factors (by Proposition 36, as $\mathcal{E}$ may be assumed to be saturated), and that applications of symmetry happen first in equational proofs (that is, $\mathcal{E}$ is closed with respect to symmetry).

We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from $\mathcal{E}$. The case of reflexivity is trivial, and that of transitivity follows immediately by using the inductive hypothesis twice. Below we only consider the other possibilities.

- Case $E \vdash p \approx q$, because $\sigma(t) = p$ and $\sigma(u) = q$ for some equation $(t \approx u) \in E$ and closed substitution σ. Since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, and n is larger than the size of each term mentioned in equations in $\mathcal{E}$, the claim follows by Proposition 59.

- Case $E \vdash p \approx q$, because $p = \mu p'$ and $q = \mu q'$ for some p', q' such that $E \vdash p' \approx q'$. This case is vacuous because $p = \mu p' \not\leftrightarrow f(\alpha, p_n)$, and thus p does not have a summand bisimilar to $f(\alpha, p_n)$.
- Case $E \vdash p \approx q$, because $p = p' + p''$ and $q = q' + q''$ for some p', q', p'', q'' such that $E \vdash p' \approx q'$ and $E \vdash p'' \approx q''$. Since p has a summand bisimilar to $f(\alpha, p_n)$, we have that so does either p' or p''. Assume, without loss of generality, that p' has a summand bisimilar to $f(\alpha, p_n)$. Since p is bisimilar to $f(\alpha, p_n)$, so is p'. Using the soundness of $\mathcal{E}$ modulo bisimulation, it follows that $q' \leftrightarrow f(\alpha, p_n)$. The inductive hypothesis now yields that q' has a summand bisimilar to $f(\alpha, p_n)$. Hence, q has a summand bisimilar to $f(\alpha, p_n)$, which was to be shown.
- Case $E \vdash p \approx q$, because $p = f(p', p'')$ and $q = f(q', q'')$ for some p', q', p'', q'' such that $E \vdash p' \approx q'$ and $E \vdash p'' \approx q''$. Since the proof involves no uses of $\mathbf{0}$ as a summand or a factor, we have that $p', p'' \not\leftrightarrow \mathbf{0}$ and $q', q'' \not\leftrightarrow \mathbf{0}$. It follows that q is a summand of itself. By our assumptions,

$$f(\alpha, p_n) \leftrightarrow q \ .$$

  Therefore we have that q has a summand bisimilar to $f(\alpha, p_n)$, and we are done.

This completes the proof of Theorem 17 and thus of Theorem 14 in the case of an operator f that, modulo bisimilarity distributes over summation in its first argument. $\square$

J Proof of Theorem 18

Before proceeding to the proof of Theorem 18, we discuss a few useful properties of the processes $f(\alpha, q_n)$. Such properties are stated in Lemmas 61 and 62 and they are the updated versions of, respectively, Lemmas 57 and 58 with respect to the current set of SOS rules that are allowed for f.

▶ **Lemma 61.** *For each $n \geq 0$ it holds that $f(\alpha, q_n) \leftrightarrow \alpha \| q_n$.*

▶ **Lemma 62.** *Let $n \geq 1$. Assume that $f(p, q) \leftrightarrow f(\alpha, q_n)$ for $p, q \not\leftrightarrow \mathbf{0}$. Then (i) either $p \leftrightarrow \alpha$ and $q \leftrightarrow q_n$, (ii) or $q \leftrightarrow \alpha$ and $p \leftrightarrow q_n$.*

Proof: Since $f(p, q) \leftrightarrow f(\alpha, q_n)$ and $f(\alpha, q_n) \xrightarrow{\alpha} \mathbf{0} \| q_n \leftrightarrow q_n$, we can distinguish the following two cases depending on whether a matching transition from $f(p, q)$ stems from p or q:

- There is a p' such that $p \xrightarrow{\alpha} p'$ and $p' \| q \leftrightarrow q_n$. It follows that $q \leftrightarrow q_n$ and $p' \leftrightarrow \mathbf{0}$, because q_n is prime (Lemma 39(2)) and $q \not\leftrightarrow \mathbf{0}$. We are therefore left to prove that p is bisimilar to α. To this end, note, first of all, that, as $\leftrightarrow$ is a congruence over the language CCS_f, we have that

$$f(p, q_n) \leftrightarrow f(\alpha, q_n) \ .$$

  First of all, notice that the equivalence above implies that $depth(p) = 1$. We proceed to prove that $p \leftrightarrow \alpha$. Assume towards a contradiction that $p \not\leftrightarrow \alpha$ and thus that $p \xrightarrow{\mu} \mathbf{0}$ for some $\mu \neq \alpha$. We can distinguish two cases, according to whether the predicate L^f_μ holds or not.

  - Assume first that L^f_μ holds. Then we would have $\mathrm{init}(f(p, q_n)) = \{\alpha, \mu\}$ and $\mathrm{init}(f(\alpha, q_n)) = \{\alpha\}$, thus contradicting $f(p, q_n) \leftrightarrow f(\alpha, q_n)$.

- Assume now that L^f_μ does not hold. Then, in light of the above equivalence, from $f(\alpha, q_n) \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq n}$ and the fact that $q_n \not\underline{\leftrightarrow} \bar{\alpha}^{\leq n}$, we can infer that $f(p, q_n) \xrightarrow{\alpha} p \| \bar{\alpha}^{\leq n}$ and $p \| \bar{\alpha}^{\leq n} \underline{\leftrightarrow} \alpha \| \bar{\alpha}^{\leq n}$.
  Now, if $\mu = \tau$, then $p \| \bar{\alpha}^{\leq n} \xrightarrow{\tau} \mathbf{0} \| \bar{\alpha}^{\leq n} \underline{\leftrightarrow} \bar{\alpha}^{\leq n}$. However, $\alpha \| \bar{\alpha}^{\leq n}$ can perform a τ-move only due to a synchronization between α and one of the $\bar{\alpha}$, thus implying that $\alpha \| \bar{\alpha}^{\leq n} \xrightarrow{\tau} \mathbf{0} \| \bar{\alpha}^{i} \underline{\leftrightarrow} \bar{\alpha}^{i}$ for some $i \in \{0, \dots, n-1\}$. Since there is no such index i such that $\bar{\alpha}^{\leq n} \underline{\leftrightarrow} \bar{\alpha}^{i}$, this contradicts $f(p, q_n) \underline{\leftrightarrow} f(\alpha, q_n)$.
  Similarly, if $\mu = \bar{\alpha}$, then $p \| \bar{\alpha}^{\leq n}$ could perform a sequence of $n+1$ transitions all with label $\bar{\alpha}$, whereas $\alpha \| \bar{\alpha}^{\leq n}$ can perform at most n $\bar{\alpha}$-moves in a row. Therefore, also this case is in contradiction with $f(p, q_n) \underline{\leftrightarrow} f(\alpha, q_n)$.

  We may therefore conclude that every transition of p is of the form $p \xrightarrow{\alpha} p''$, for some $p'' \underline{\leftrightarrow} \mathbf{0}$. Since we have already seen that p affords an α-labelled transition leading to $\mathbf{0}$, modulo bisimulation equivalence, it follows that $p \underline{\leftrightarrow} \alpha$, which was to be shown.
- There is a q' such that $q \xrightarrow{\alpha} q'$ and $p \| q' \underline{\leftrightarrow} q_n$. This case can be treated similarly to the previous case and allows us to conclude that $q \underline{\leftrightarrow} \alpha$ and $p \underline{\leftrightarrow} q_n$.

$\square$

The negative result stated in Theorem 18 is strongly based on the following proposition, which ensures that the property of having a summand bisimilar to $f(\alpha, q_n)$ is preserved by the closure under substitution of equations in a finite sound axiom system.

▶ **Proposition 63.** *Assume an operator f such that $L^f_\alpha \wedge R^f_\alpha$ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\underline{\leftrightarrow}$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor factors, and $p, q \underline{\leftrightarrow} f(\alpha, q_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, q_n)$, then so does q.

Proof: First of all we notice that since $\sigma(t)$ and $\sigma(u)$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Therefore by Remark 29 we get that

$$t = \sum_{i \in I} t_i \qquad \text{and} \qquad u = \sum_{j \in J} u_j$$

for some finite non-empty index sets I, J with all the t_i and u_j not having $+$ as head operator, $\mathbf{0}$ summands nor factors. By the hypothesis, there is some $i \in I$ with $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, q_n)$. We proceed by a case analysis over the structure of t_i to show that there is a u_j such that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, q_n)$.

1. Case $t_i = x$ for some variable x such that $\sigma(x) \underline{\leftrightarrow} f(\alpha, q_n)$. By Proposition 54, t having a summand x implies that u has a summand x as well. Thus, we can immediately conclude that $\sigma(u)$ has a summand bisimilar to $f(\alpha, q_n)$ as required.
2. Case $t_i = \mu.t'$ for some term t'. This case is vacuous, as it contradicts our assumption $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, q_n)$. Indeed, if $\mu = \alpha$ then $\sigma(t')$ cannot be bisimilar to both q_n and $\alpha \| \bar{\alpha}^{\leq i}$, for any $i \in \{1, \dots, n\}$.
3. Case $t_i = f(t', t'')$ for some terms t', t''. As $\sigma(t)$ has no $\mathbf{0}$ factors, we have that $\sigma(t'), \sigma(t'') \not\underline{\leftrightarrow} \mathbf{0}$. Hence, from $f(\sigma(t'), \sigma(t'')) \underline{\leftrightarrow} f(\alpha, q_n)$ and Lemma 62 we can distinguish two cases: **a.** either $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\sigma(t'') \underline{\leftrightarrow} q_n$, **b.** or $\sigma(t') \underline{\leftrightarrow} q_n$ and $\sigma(t'') \underline{\leftrightarrow} \alpha$. We expand only the former case, as the latter follows from an identical (symmetrical) reasoning. By Remark 29, from $\sigma(t'') \underline{\leftrightarrow} q_n$ we infer that $t'' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator and have no $\mathbf{0}$-summands or factors. Since n is larger that the size

of t, we have that $|H| < n$ and thus there is some $h \in H$ such that $\sigma(v_h) \leftrightarrow \sum_{k=1}^{m} \alpha\bar{\alpha}^{\leq i_k}$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. Since $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors, from Lemma 48 we infer that v_h can only be a variable x with

$$\sigma(x) \leftrightarrow \sum_{k=1}^{m} \alpha\bar{\alpha}^{\leq i_k}. \tag{13}$$

Therefore, $t_i = f(t', x + t''')$ for some t''' such that $\sigma(x + t''') \leftrightarrow q_n$. We also notice that since $\sigma(t') \leftrightarrow \alpha$ and $\sigma(t')$ has no $\mathbf{0}$ summands or factors, then it cannot be the case that $x \in \mathit{var}(t')$.

To prove that u has a summand bisimilar to $f(\alpha, q_n)$, consider the closed substitution

$$\sigma' = \sigma[x \mapsto \alpha q_n].$$

Since R^f_α and Lemma 61 hold, we have

$$\sigma'(t_i) \xrightarrow{\alpha} p' \leftrightarrow \alpha \| q_n \leftrightarrow f(\alpha, q_n).$$

As $t \approx u$ implies $\sigma'(t) \leftrightarrow \sigma'(u)$, we infer that there must be a summand u_j such that $\sigma'(u_j) \xrightarrow{\alpha} r$ for some $r \leftrightarrow f(\alpha, q_n)$. Notice that, since $\sigma(u) \leftrightarrow f(\alpha, q_n)$ and $\sigma(u_j) = \sigma'(u_j)$ if $x \notin \mathit{var}(u_j)$, then it must be the case that $x \in \mathit{var}(u_j)$, or otherwise we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, q_n)$, as $\sigma(u_j) = \sigma'(u_j) \xrightarrow{\alpha} r$ would give $\sigma(u) \xrightarrow{\alpha} r \leftrightarrow f(\alpha, q_n)$. However, there is no r' such that $f(\alpha, q_n) \xrightarrow{\alpha} r'$ and $r' \leftrightarrow f(\alpha, q_n)$. By Lemma 46, as $L^f_\alpha \wedge R^f_\alpha$ holds, we can distinguish two cases:

a. There is a term u' s.t. $u_j \xrightarrow{\alpha} u'$ and $\sigma'(u') \leftrightarrow f(\alpha, q_n)$. Then, since $f(\alpha, q_n) \leftrightarrow \alpha \parallel q_n$ (Lemma 61) we can apply the expansion law, obtaining

$$\sigma'(u') \leftrightarrow \sum_{i=1}^{n} \alpha(\alpha \| \bar{\alpha}^{\leq i}) + \alpha q_n.$$

As n is greater than the size of u, and thus of those of u_j and u', by Lemma 47 we get that u' has a summand y, for some variable y, such that

$$\sigma'(y) \leftrightarrow \sum_{k=1}^{m'} \alpha(\alpha \| \bar{\alpha}^{\leq i'_k}) + r',$$

for some $m' > 1$, $1 \leq i'_1 < \cdots < i'_{m'} \leq n$ and closed term r'. Notice that we can infer that $y \neq x$, as $\sigma'(x) \not\leftrightarrow \sigma'(y)$ for any closed term r'. Thus we have $\sigma'(y) = \sigma(y)$ and we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, q_n)$ in that $\sigma(u_j)$ would be able to perform three α-moves in a row. In fact

$$\begin{aligned}
\sigma(u_j) &\xrightarrow{\alpha} \sigma(u') && (u' \text{ has a summand } y) \\
&\xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq i'_k} && \text{for some } k \in \{1, \ldots, m'\} \\
&\xrightarrow{\alpha} \bar{\alpha}^{\leq i'_k},
\end{aligned}$$

whereas $\sigma(u) \leftrightarrow f(\alpha, q_n)$ can perform only two such transitions.

b. There are a variable y, a closed term r' and a configuration c s.t. $\sigma'(y) \xrightarrow{\alpha} r'$, $u_j \xrightarrow{y_b}_\alpha c$ and $\sigma'[y_d \mapsto r'](c) \leftrightarrow f(\alpha, q_n)$. We claim that it must be the case that $y = x$. To see this, assume towards a contradiction that $y \neq x$. We proceed by a case analysis on the possible occurrences of x in c.

- $x \notin var(c)$ or $x \in var(c)$ but its occurrence is in a guarded context that prevents the execution of its closed instances. In this case we get $r = \sigma[y_{\mathrm{d}} \mapsto r'](c) \leftrightarrow \sigma'[y_{\mathrm{d}} \mapsto r'](c) \leftrightarrow f(\alpha, q_n)$. This contradicts $\sigma(u) \leftrightarrow f(\alpha, q_n)$ since we would have $\sigma(u) \xrightarrow{\alpha} r \leftrightarrow f(\alpha, q_n)$, and such a transition cannot be mimicked by $f(\alpha, q_n)$.
- $x \in var(c)$ and its execution is not prevented. We can distinguish two sub-cases, according to whether the occurrence of x is guarded or not.
  - Assume that x occurs guarded in c. In this case we get a contradiction with $r \leftrightarrow f(\alpha, q_n)$ in that

$$\begin{aligned} n+2 &= depth(f(\alpha, q_n)) \\ &= depth(r) \\ &\geq 1 + depth(\sigma'(x)) && (x \text{ is guarded}) \\ &= n+3. \end{aligned}$$

  - Assume now that $x \lhd_{\mathrm{b}}^{\alpha} c$. We proceed by a case analysis on the structure of c.
    * $c \leftrightarrow y_{\mathrm{d}} \| (x + u_1) \| u_2$. Notice that in this case we have $r = r' \| \sigma'(x) + \sigma'(u_1) \| \sigma'(u_2)$. Then, the only transition available for $\sigma'(x)$ is $\sigma'(x) \xrightarrow{\alpha} q_n$, which gives $r \xrightarrow{\alpha} r' \| q_n \| \sigma'(u_2)$. Since $r \leftrightarrow f(\alpha, q_n)$, then it must be the case that $f(\alpha, q_n) \xrightarrow{\alpha} r''$ for some $r'' \leftrightarrow r' \| q_n \| \sigma'(u_2)$. Since q_n is prime, we can infer that $r'' \leftrightarrow q_n$ and thus that $r' \leftrightarrow \mathbf{0} \leftrightarrow \sigma'(u_2)$. Hence, we have that $r \leftrightarrow \sigma'(x) + \sigma'(u_1)$. As the one we wrote is the only transition available for $\sigma'(x)$, we can infer that, for all $i \in \{1, \dots, n\}$, the transitions $r \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq i}$ cannot be derived from $\sigma'(x)$, but only from $\sigma'(u_1)$. Moreover, notice that $y \neq x$ gives $\sigma'(y) = \sigma(y)$, and from $\mathrm{init}(\sigma'(x)) = \mathrm{init}(\sigma(x)) = \{\alpha\}$ and the fact that $L_\alpha^f \wedge R_\alpha^f$ holds, we can infer that $\sigma(u_2) \leftrightarrow \sigma'(u_2) \leftrightarrow \mathbf{0}$. Therefore, this contradicts $\sigma(u) \leftrightarrow f(\alpha, q_n)$, since $\sigma(u) \xrightarrow{\alpha} r' \| \sigma(x) + \sigma(u_1) \| \sigma(u_2) \leftrightarrow \sigma(x) + \sigma(u_1) \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq i}$, for any $i \in \{1, \dots, n\}$. Process $f(\alpha, q_n)$, in turn, by performing two α-moves can only reach processes bisimilar to $\bar{\alpha}^{\leq i}$, for $i \in \{1, \dots, n\}$.
    * c has a subterm u_3 of the form $u_3 \leftrightarrow f(x + u_2, u_1)$ or $u_3 \leftrightarrow f(u_1, x + u_2)$. In both cases, we get that $\sigma'(x) \xrightarrow{\alpha} q_n$ implies $\sigma'(u_3) \xrightarrow{\alpha} q_n \| \sigma'(u_1)$. However, $f(\alpha, q_n) \xrightarrow{\alpha} \mathbf{0} \| q_n \leftrightarrow q_n$ and q_n prime give $\sigma'(u_1) \leftrightarrow \mathbf{0}$. One can then argue that, as $\mathrm{init}(\sigma'(x)) = \{\alpha\}$, either x does not occur in u_1, or it does it in a guarded context that prevents its execution. Hence, we infer $\sigma(u_1) \leftrightarrow \sigma'(u_1) \leftrightarrow \mathbf{0}$, thus contradicting $\sigma(u)$ not having $\mathbf{0}$ factors.

Therefore, we can conclude that it must be the case that $y = x$ and $r' = q_n$. In particular, notice that $x \lhd_{\mathrm{b}}^{\alpha} u_j$. We now proceed by a case analysis on the structure of u_j to show that $\sigma(u_j) \leftrightarrow f(\alpha, q_n)$.

i. $u_j = x$. This case is vacuous, as $\sigma'(x) \xrightarrow{\alpha} q_n$ and $q_n \not\leftrightarrow f(\alpha, q_n)$.

ii. $u_j = f(u', u'')$ for some u', u''. Notice that $x \lhd_{\mathrm{b}}^{\alpha} u_j$ can be due either to $x \lhd_{\mathrm{b}}^{\alpha} u'$ or $x \lhd_{\mathrm{b}}^{\alpha} u''$. As both $\sigma'(u')$ and $\sigma'(u'')$ can be responsible for the α-move by $\sigma'(u_j)$, we distinguish two cases:

A. $\sigma'(u') \xrightarrow{\alpha} r_1$ and $r_1 \| \sigma'(u'') \leftrightarrow f(\alpha, q_n)$. As $f(\alpha, q_n) \leftrightarrow \alpha \| q_n$ and both α and q_n are prime, by the existence of a unique prime decomposition, we distinguish two cases:

- $r_1 \leftrightarrow \alpha$ and $\sigma'(u'') \leftrightarrow q_n$. Since $x \lhd_{\mathrm{b}}^{\alpha} u''$ is in contradiction with $\sigma'(u'') \leftrightarrow q_n$, we infer that $x \lhd_{\mathrm{b}}^{\alpha} u'$. Moreover $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\alpha\}$, $L_\alpha^f \wedge R_\alpha^f$, $\sigma'(u') \leftrightarrow q_n$ and the fact that $\sigma(u)$ has no $\mathbf{0}$ factors we get that either

$x \notin var(u'')$ or x occurs in u'' but its execution is prevented by the rules for f. Therefore

$$\sigma'(u'') \leftrightarrow \sigma(u'') \leftrightarrow q_n.$$

However, $depth(\sigma(x)) \geq 3$, and $x \lhd_{\mathrm{b}}^{\alpha} u'$ with $\mathrm{init}(\sigma(x)) = \{\alpha\}$ give us, by Lemma 52, that $depth(\sigma(u')) \geq depth(\sigma(x))$. Therefore we get a contradiction, in that

$$\begin{aligned} n+2 &= depth(f(\alpha, q_n)) \\ &= depth(\sigma(u)) \\ &\geq depth(\sigma(u_j)) \\ &= depth(f(\sigma(u'), \sigma(u''))) \\ &\geq depth(\sigma(x)) + depth(\sigma(u'')) \\ &\geq 3 + n + 1 \\ &= n+4. \end{aligned}$$

- $r_1 \leftrightarrow q_n$ and $\sigma'(u'') \leftrightarrow \alpha$. By reasoning as above, we can infer that either $x \notin var(u'')$ or its execution is blocked by the rules for f, so that $\sigma'(u'') \leftrightarrow \sigma(u'')$. Moreover, we get that $x \lhd_{\mathrm{b}}^{\alpha} u'$. We aim at showing that u' has a summand x. We proceed by showing that the only other possibility, namely $u' = f(w_1, w_2)$ for some w_1, w_2, leads to a contradiction. As $u' = f(w_1, w_2)$ we have that either $x \lhd_{\mathrm{b}}^{\alpha} w_1$ or $x \lhd_{\mathrm{b}}^{\alpha} w_2$. However, $\sigma'(u') \xrightarrow{\alpha} r_1 \leftrightarrow q_n$ gives two possibilities:
  - $\sigma'(w_1) \xrightarrow{\alpha} r_1'$ and $r_1' \| \sigma'(w_2) \leftrightarrow q_n$. Since q_n is prime, then either $r_1' \leftrightarrow \mathbf{0}$ and $\sigma'(w_2) \leftrightarrow q_n$, or $r_1' \leftrightarrow q_n$ and $\sigma'(w_2) \leftrightarrow \mathbf{0}$. In both cases we infer that either $x \notin var(w_2)$ or its execution in it is always prevented, so that $\sigma(w_2) \leftrightarrow \sigma'(w_2)$. Therefore, the former case, combined with $\sigma(u'') \leftrightarrow \alpha$, gives a contradiction with $\sigma(u) \leftrightarrow f(\alpha, q_n)$. The latter case contradicts $\sigma(u)$ not having $\mathbf{0}$ factors.
  - $\sigma'(w_2) \xrightarrow{\alpha} r_2'$ and $\sigma'(w_1) \| r_2' \leftrightarrow q_n$. The same reasoning as in the previous case allows us to conclude that this case gives a contradiction.

  Summing up, we have argued that u' has a summand x. Therefore, by Equation (13),

$$\sigma(u') \leftrightarrow \sum_{k=1}^{m} \alpha.\bar{\alpha}^{\leq i_k} + r'' \ ,$$

  for some closed term r''. We have already noted that

$$\sigma(u'') \leftrightarrow \sigma'(u'') \leftrightarrow \alpha \ .$$

  Therefore, using the congruence properties of bisimulation equivalence, we may infer that

$$\begin{aligned} \sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\ &\leftrightarrow f(\sum_{k=1}^{m} \alpha\bar{\alpha}^{\leq i_k} + r'', \alpha) \ . \end{aligned}$$

  In light of this equivalence, we have $\sigma(u_j) \xrightarrow{\alpha} r' \leftrightarrow \sigma(u')$ and thus $\sigma(u) \xrightarrow{\alpha} r'$. Since by hypothesis $\sigma(u) \leftrightarrow f(\alpha, q_n)$ we have that either $r' \leftrightarrow q_n$, or $r' \leftrightarrow \alpha \| \alpha^{\leq i}$

for some $i \in \{1, \ldots, n\}$. However, the latter case is in contradiction with $r' \leftrightarrow \sigma(u')$, and thus it must be the case that $r' \leftrightarrow q_n$. Therefore, we can conclude that $\sigma(u_j) \leftrightarrow f(q_n, \alpha)$. It is easy to check that $f(\alpha, q_n) \leftrightarrow f(q_n, \alpha)$. Hence, $\sigma(u)$ has the desired summand.

B. $\sigma'(u'') \xrightarrow{\alpha} r_2$ and $\sigma'(u')\|r_2 \leftrightarrow f(\alpha, q_n)$. This case follows as the previous one and allows us to conclude as well that $\sigma(u)$ has the desired summand.

The proof of Proposition 63 is now complete. □

J.1 Formal proof of Theorem 18

By exploiting the properties discussed in Appendix F, Theorem 18 is equivalent to the following:

▶ **Theorem 64.** *Assume an operator f such that $L^f_\alpha \wedge R^f_\alpha$ holds. Let $\mathcal{E}$ be a finite axiom system over the language CCS^-_f that is sound modulo bisimilarity. Let n be larger than the size of each term in the equations in $\mathcal{E}$. Assume p and q are closed terms bisimilar to $f(\alpha, q_n)$ and contain no $\mathbf{0}$ summands or factors. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, q_n)$, then so does q.*

Proof of Theorem 18. Assume that $\mathcal{E}$ is a finite axiom system over the language CCS^-_f that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in $\mathcal{E}$:

1. $E \vdash p \approx q$,
2. $p \leftrightarrow q \leftrightarrow f(\alpha, q_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, q_n)$.

We proceed by induction on the depth of the closed proof of the equation $p \approx q$ from $\mathcal{E}$, to prove that also q has a summand bisimilar to $f(\alpha, q_n)$. Recall that, without loss of generality, we may assume that $\mathcal{E}$ is closed with respect to symmetry, and thus applications of symmetry happen first in equational proofs. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from $\mathcal{E}$. The case of reflexivity is trivial, and that of transitivity follows by applying twice the inductive hypothesis. We proceed now to a detailed analysis of the remaining cases:

1. Case $E \vdash p \approx q$ because $\sigma(t) = p$ and $\sigma(u) = q$ for some terms t, u with $E \vdash t \approx u$ and closed substitution σ. The proof of this case follows by Proposition 63.
2. Case $E \vdash p \approx q$ because $p = \mu.p'$ and $q = \mu.q'$ for some p', q' with $E \vdash p' \approx q'$. This case is vacuous in that $p = \mu.p' \not\leftrightarrow f(\alpha, q_n)$ and thus p does not have a summand bisimilar to $f(\alpha, q_n)$.
3. Case $E \vdash p \approx q$ because $p = r_1 + r_2$ and $q = s_1 + s_2$ for some r_i, s_i with $E \vdash r_i \approx s_i$, for $i \in \{1, 2\}$. Since p has a summand bisimilar to $f(\alpha, q_n)$ then so does either r_1 or r_2. Assume without loss of generality that r_1 has such a summand. As $p \leftrightarrow f(\alpha, q_n)$ then $r_1 \leftrightarrow f(\alpha, q_n)$ holds as well. Then, from $E \vdash r_1 \approx s_1$ we infer $s_1 \leftrightarrow f(\alpha, q_n)$. Thus, by the inductive hypothesis we obtain that s_1 has a summand bisimilar to $f(\alpha, q_n)$ and, consequently, so does q.
4. Case $E \vdash p \approx q$ because $p = f(r_1, r_2)$ and $q = f(s_1, s_2)$ for some r_i, s_i with $E \vdash r_i \approx s_i$, for $i \in \{1, 2\}$. By the proviso of the theorem p, q have neither $\mathbf{0}$ summands nor factors, thus implying $r_i, s_i \not\leftrightarrow \mathbf{0}$. Hence, from $p \leftrightarrow f(\alpha, q_n)$ and $p = f(r_1, r_2)$ and

Lemma 62 we obtain $r_i \leftrightarrow \alpha$ and $r_{3-i} \leftrightarrow q_n$, thus implying, by the soundness of the equations in $\mathcal{E}$, that $s_i \leftrightarrow \alpha$ and $s_{3-i} \leftrightarrow q_n$, so that either $q = f(\alpha, q_n)$ or $q = f(q_n, \alpha)$. In both cases, we can infer that q has itself as the desired summand.

This completes the proof of Theorem 18 and thus of Theorem 14 in the case of an operator f that does not distribute over summation in either argument, case $L^f_\alpha \wedge R^f_\alpha$. ◀

K Proof of Theorem 19

Before proceeding to the proof, we remark that the processes $f(\alpha, p_n)$ enjoy the following properties, according to the current set of allowed rules for operator f:

▶ **Lemma 65.** *For each $n \geq 0$ it holds that $f(\alpha, p_n) \leftrightarrow \alpha \| p_n$.*

▶ **Lemma 66.** *Let $n \geq 1$. Assume that $f(p, q) \leftrightarrow f(\alpha, p_n)$ for $p, q \not\leftrightarrow \mathbf{0}$. Then $p \leftrightarrow \alpha$ and $q \leftrightarrow p_n$.*

Proof: The proof is analogous to that of Lemma 58 and therefore omitted. ◻

The crucial point in the proof of the negative result is (also in this case) the preservation of the witness property when instantiating an equation from a finite, sound axiom system. We expand this case in the following proposition:

▶ **Proposition 67.** *Assume an operator f such that only L^f_α holds for α, only $R^f_{\bar{\alpha}}$ holds for $\bar{\alpha}$, and $S_{\alpha,\bar{\alpha}}$ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\leftrightarrow$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor factors, and $p, q \leftrightarrow f(\alpha, p_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: First of all we notice that since $\sigma(t)$ and $\sigma(u)$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Therefore by Remark 29 we get that

$$t = \sum_{i \in I} t_i \qquad \text{and} \qquad u = \sum_{j \in J} u_j$$

for some finite non-empty index sets I, J with all the t_i and u_j not having $+$ as head operator, $\mathbf{0}$ summands nor factors. By the hypothesis, there is some $i \in I$ with $\sigma(t_i) \leftrightarrow f(\alpha, p_n)$. We proceed by a case analysis on the structure of t_i to show that there is a u_j such that $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$, establishing our claim.

1. CASE $t_i = x$ FOR SOME VARIABLE x SUCH THAT $\sigma(x) \leftrightarrow f(\alpha, p_n)$. By Proposition 54, t having a summand x implies that u has a summand x as well. Thus, we can immediately conclude that $\sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$ as required.
2. CASE $t_i = \mu.t'$ FOR SOME TERM t'. This case is vacuous, as it contradicts $\sigma(t_i) \leftrightarrow f(\alpha, p_n)$.
3. CASE $t_i = f(t', t'')$ FOR SOME TERMS t', t''. Since $\sigma(t)$ has no $\mathbf{0}$ factors, we have that $\sigma(t'), \sigma(t'') \not\leftrightarrow \mathbf{0}$. Hence, from $f(\sigma(t'), \sigma(t'')) \leftrightarrow f(\alpha, p_n)$ and Lemma 66 we obtain $\sigma(t') \leftrightarrow \alpha$ and $\sigma(t'') \leftrightarrow p_n$. By Remark 29 we infer that $t'' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator and have no $\mathbf{0}$-summands or factors. Since n is larger that the size of t, we have that $|H| < n$ and thus there is some $h \in H$ such that $\sigma(v_h) \leftrightarrow \sum_{k=1}^{m} \bar{\alpha}\alpha^{\leq i_k}$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. Since $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors, from Lemma 48 we infer that v_h can only be a variable x with

$$\sigma(x) \leftrightarrow \sum_{k=1}^{m} \bar{\alpha}\alpha^{\leq i_k}. \tag{14}$$

Therefore, $t_i = f(t', x + t''')$ for some t''' such that $\sigma(x + t''') \underline{\leftrightarrow} p_n$. We also notice that since $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\mathrm{init}(\sigma(x)) = \{\bar{\alpha}\}$, we can infer that $x \triangleleft_{\mathrm{r}}^{\bar{\alpha}} t'$ does not hold (otherwise, $\sigma'(t)$ would afford an initial $\bar{\alpha}$-transition and would not be bisimilar to α).

To prove that u has a summand bisimilar to $f(\alpha, p_n)$, consider the closed substitution

$$\sigma' = \sigma[x \mapsto \bar{\alpha}p_n].$$

Notice that, since $\sigma(t') \underline{\leftrightarrow} \alpha$, $\sigma(t')$ has no $\mathbf{0}$ summands or factors, $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\bar{\alpha}\}$ and x is the only variable which is affected when changing σ into σ', then we can infer that either $x \notin var(t')$ or its execution is always prevented. In both cases we get $\sigma(t') \underline{\leftrightarrow} \sigma'(t') \underline{\leftrightarrow} \alpha$. Then, using Lemma 65 and $t_i = f(t', x + t'')$, we have

$$\sigma'(t_i) \xrightarrow{\bar{\alpha}} p' \underline{\leftrightarrow} \alpha \| p_n \underline{\leftrightarrow} f(\alpha, p_n).$$

As $t \approx u$ implies $\sigma'(t) \underline{\leftrightarrow} \sigma'(u)$, we infer that there must be a summand u_j such that $\sigma'(u_j) \xrightarrow{\bar{\alpha}} r$ for some $r \underline{\leftrightarrow} f(\alpha, p_n)$. Notice that, since $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ and $\sigma(u_j) = \sigma'(u_j)$ if $x \notin var(u_j)$, then it must be the case that $x \in var(u_j)$, or otherwise we get a contradiction with $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$. By Lemma 46, as only $R^f_{\bar{\alpha}}$ holds, we can distinguish two cases:

a. There is a term u' s.t. $u_j \xrightarrow{\bar{\alpha}} u'$ and $\sigma'(u') \underline{\leftrightarrow} f(\alpha, p_n)$. Then, since $f(\alpha, p_n) \underline{\leftrightarrow} \alpha \parallel p_n$ (Lemma 65) we can apply the expansion law, obtaining $\sigma'(u') \underline{\leftrightarrow} \alpha p_n + \sum_{i=1}^{n} \bar{\alpha}(\alpha \| \alpha^{\leq i}) + \sum_{i=1}^{n} \tau \alpha^{\leq i}$. As n is greater than the size of u, and thus of those u_j and u', by Lemma 47 we get that u' has a summand y, for some variable y, such that $\sigma'(y) \underline{\leftrightarrow} \sum_{k=1}^{m'} \bar{\alpha}(\alpha \| \alpha^{\leq i'_k}) + r'$, for some $m' > 1$, $1 \leq i'_1 < \cdots < i'_{m'} \leq n$ and closed term r'. Notice that we can infer that $y \neq x$, as $\sigma'(x) \not\underline{\leftrightarrow} \sigma'(y)$ for any closed term r'. Thus we have $\sigma'(y) = \sigma(y)$ and we get a contradiction with $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ in that $\sigma(u_j)$ would be able to perform two $\bar{\alpha}$-moves in a row unlike $f(\alpha, p_n)$.

b. There are a variable y, a closed term r' and a configuration c s.t. $\sigma'(y) \xrightarrow{\bar{\alpha}} r'$, $u_j \xrightarrow{y_{\mathrm{r}}}_{\bar{\alpha}} c$ and $\sigma'[y_{\mathrm{d}} \mapsto r'](c) \underline{\leftrightarrow} f(\alpha, p_n)$. We claim that it must be the case that $y = x$. To see this claim, assume towards a contradiction that $y \neq x$. We proceed by a case analysis on the possible occurrences of x in c.

- $x \notin var(c)$ or $x \in var(c)$ but its occurrence is in a guarded context that prevents the execution of its closed instances. In this case we get $\sigma[y_{\mathrm{d}} \mapsto r'](c) \underline{\leftrightarrow} \sigma'[y_{\mathrm{d}} \mapsto r'](c) \underline{\leftrightarrow} f(\alpha, p_n)$. This contradicts $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ since we would have $\sigma(u) \xrightarrow{\bar{\alpha}} r \underline{\leftrightarrow} f(\alpha, p_n)$, and such a transitions cannot be mimicked by $f(\alpha, p_n)$.
- $x \in var(c)$ and its execution is not prevented. We can distinguish two sub-cases, according to whether the occurrence of x is guarded or not.
  - Assume that x occurs guarded in c. In this case we get a contradiction with $r \underline{\leftrightarrow} f(\alpha, p_n)$ in that

$$\begin{aligned} n + 2 &= depth(f(\alpha, p_n)) \\ &= depth(r) \\ &\geq 1 + depth(\sigma'(x)) && (x \text{ is guarded}) \\ &= n + 3. \end{aligned}$$

  - Assume now that $x \triangleleft_{\mathrm{b}}^{\alpha} c$. This case contradicts our assumption that $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ since we would have $\sigma(u) \xrightarrow{\bar{\alpha}} \sigma[y_{\mathrm{d}} \mapsto r'](c) \xrightarrow{\bar{\alpha}}$, due to Lemmas 51 and 45, whereas $f(\alpha, p_n)$ cannot perform two $\bar{\alpha}$-moves in a row.

Therefore, we can conclude that it must be the case that $y = x$ and $r' = p_n$. In particular, notice that $x \triangleleft_{\mathrm{r}}^{\bar\alpha} u_j$. We now proceed by a case analysis on the structure of u_j to show that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$.

i. $u_j = x$. This case is vacuous, as $\sigma'(x) \xrightarrow{\bar\alpha} p_n$ and $p_n \not\underline{\leftrightarrow} f(\alpha, p_n)$.

ii. $u_j = f(u', u'')$ for some u', u''. Notice that $x \triangleleft_{\mathrm{r}}^{\bar\alpha} u_j$ can be due only to $x \triangleleft_{\mathrm{r}}^{\bar\alpha} u''$. We have $\sigma'(u'') \xrightarrow{\bar\alpha} r_1$ and $\sigma'(u_j) \xrightarrow{\bar\alpha} \sigma'(u') \| r_1 \underline{\leftrightarrow} f(\alpha, p_n)$. Since $f(\alpha, p_n) \underline{\leftrightarrow} \alpha \| p_n$ and both α and p_n are prime, by the existence of a unique prime decomposition, we distinguish two cases:

- Case $\sigma'(u') \underline{\leftrightarrow} \alpha$ and $r_1 \underline{\leftrightarrow} p_n$. As $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\bar\alpha\}$, $R^f_{\bar\alpha}$, $\sigma'(u') \underline{\leftrightarrow} \alpha$ and $\sigma(u)$ has no $\mathbf{0}$ factors, we get that either $x \notin var(u')$ or x occurs in u' but its execution is prevented by the rules for f. Therefore $\sigma'(u') \underline{\leftrightarrow} \sigma(u') \underline{\leftrightarrow} \alpha$. We aim at showing that u'' has a summand x. We proceed by proving that the only other possibility, namely $u'' = f(w_1, w_2)$ for some w_1, w_2 with $x \triangleleft_{\mathrm{r}}^{\bar\alpha} w_2$, leads to a contradiction.
  As $\sigma'(u'') \xrightarrow{\bar\alpha} r_1 \underline{\leftrightarrow} p_n$, we have $\sigma'(w_2) \xrightarrow{\bar\alpha} r_2$ and $\sigma'(w_1) \| r_2 \underline{\leftrightarrow} p_n$. Since, p_n is prime, we have that either $\sigma'(w_1) \underline{\leftrightarrow} \mathbf{0}$ and $r_2 \underline{\leftrightarrow} p_n$, or $\sigma'(w_1) \underline{\leftrightarrow} p_n$ and $r_2 \underline{\leftrightarrow} \mathbf{0}$. In both cases, as $\sigma'(x) \not\underline{\leftrightarrow} \sigma'(w_1)$ and the previous considerations, we infer $\sigma(w_1) \underline{\leftrightarrow} \sigma'(w_1)$. Hence, the former case contradicts $\sigma(u)$ not having $\mathbf{0}$ factors. The latter case contradicts $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ as, considering that $x \triangleleft_{\mathrm{r}}^{\bar\alpha} w_2$, the transition $\sigma'(w_2) \xrightarrow{\bar\alpha} r_2 \underline{\leftrightarrow} \mathbf{0}$ cannot be due to $\sigma'(x)$ and therefore it would be available also to $\sigma(w_2)$ thus implying $\sigma(u_j) \xrightarrow{\bar\alpha} r''$ with $r'' \underline{\leftrightarrow} f(\alpha, p_n)$.
  Summing up, we have argued that u'' has a summand x. Therefore, by Equation (14),
  $$\sigma(u'') \underline{\leftrightarrow} \sum_{k=1}^{m} \bar\alpha.\alpha^{\le i_k} + r'' \enspace ,$$
  for some closed term r''. We have already noted that
  $$\sigma(u') \underline{\leftrightarrow} \sigma'(u') \underline{\leftrightarrow} \alpha \enspace .$$
  Thus, using the congruence properties of bisimulation equivalence, we may infer that
  $$\begin{aligned}\sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\ &\underline{\leftrightarrow} f(\alpha, \sum_{k=1}^{m} \bar\alpha\alpha^{\le i_k} + r'') \enspace .\end{aligned}$$
  In light of this equivalence, we have $\sigma(u_j) \xrightarrow{\alpha} r' \underline{\leftrightarrow} \sigma(u'')$ and thus $\sigma(u) \xrightarrow{\alpha} r'$. Since, by hypothesis, $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ then it must be the case that $r' \underline{\leftrightarrow} p_n$. Therefore, we can conclude that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$. Hence, $\sigma(u)$ has the desired summand.
- Case $\sigma'(u') \underline{\leftrightarrow} p_n$ and $r_1 \underline{\leftrightarrow} \alpha$. By reasoning as above, we can infer that either $x \notin var(u')$ or it is blocked by the rules for f, so that
  $$\sigma'(u') \underline{\leftrightarrow} \sigma(u') \underline{\leftrightarrow} p_n.$$
  However, $depth(\sigma(x)) \ge 3$, and $x \triangleleft_{\mathrm{r}}^{\bar\alpha} u''$ with $\mathrm{init}(\sigma(x)) = \{\bar\alpha\}$ give us, by Lemma 52, that $depth(\sigma(u'')) \ge depth(\sigma(x))$. Therefore we get a contradiction, in that
  $$n + 2 = depth(f(\alpha, p_n))$$

$$
\begin{aligned}
&= depth(\sigma(u)) \\
&\geq depth(\sigma(u_j)) \\
&= depth(f(\sigma(u'), \sigma(u''))) \\
&\geq depth(\sigma(u')) + depth(\sigma(u'')) \\
&\geq depth(\sigma(u')) + depth(\sigma(x)) \\
&\geq n + 1 + 3 \\
&= n + 4.
\end{aligned}
$$

The proof of Proposition 67 is now complete. ◻

K.1 Formal proof of Theorem 19

By exploiting the properties discussed in Appendix F, Theorem 19 is equivalent to the following:

▶ **Theorem 68.** *Assume an operator f such that only L^f_α holds for α, only $R^f_{\bar{\alpha}}$ holds for $\bar{\alpha}$, and $S^f_{\alpha,\bar{\alpha}}$ holds. Let $\mathcal{E}$ be a finite axiom system over the language CCS^-_f that is sound modulo bisimilarity. Let n be larger than the size of each term in the equations in $\mathcal{E}$. Assume p and q are closed terms bisimilar to $f(\alpha, p_n)$ and contain no $\mathbf{0}$ summands or factors. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.*

Proof of Theorem 19. Assume that $\mathcal{E}$ is a finite axiom system over the language CCS^-_f that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in $\mathcal{E}$:

1. $E \vdash p \approx q$,
2. $p \leftrightarrow q \leftrightarrow f(\alpha, p_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, p_n)$.

We proceed by induction on the depth of the closed proof of the equation $p \approx q$ from $\mathcal{E}$, to prove that q has a summand bisimilar to $f(\alpha, p_n)$ as well. Recall that, without loss of generality, we may assume that $\mathcal{E}$ is closed with respect to symmetry, and thus applications of symmetry happen first in equational proofs. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from $\mathcal{E}$. The case of reflexivity is trivial, and that of transitivity follows by applying twice the inductive hypothesis. We proceed now to a detailed analysis of the remaining cases:

1. CASE $E \vdash p \approx q$ BECAUSE $\sigma(t) = p$ AND $\sigma(u) = q$ FOR SOME TERMS t, u WITH $E \vdash t \approx u$ AND CLOSED SUBSTITUTION σ. The proof of this case follows by Proposition 67.
2. CASE $E \vdash p \approx q$ BECAUSE $p = \mu.p'$ AND $q = \mu.q'$ FOR SOME p', q' WITH $E \vdash p' \approx q'$. This case is vacuous in that $p = \mu.p' \not\leftrightarrow f(\alpha, p_n)$ and thus p does not have a summand bisimilar to $f(\alpha, p_n)$.
3. $E \vdash p \approx q$ because $p = p_1 + p_2$ and $q = q_1 + q_2$ for some p_i, q_i with $E \vdash p_i \approx q_i$, for $i \in \{1, 2\}$. Since p has a summand bisimilar to $f(\alpha, p_n)$ then so does either p_1 or p_2. Assume without loss of generality that p_1 has such a summand. As $p \leftrightarrow f(\alpha, p_n)$ then $p_1 \leftrightarrow f(\alpha, p_n)$ holds as well. Then, from $E \vdash p_1 \approx q_1$ we infer $q_1 \leftrightarrow f(\alpha, p_n)$. Thus, by the inductive hypothesis we obtain that q_1 has a summand bisimilar to $f(\alpha, p_n)$ and, consequently, so does q.

4. $E \vdash p \approx q$ because $p = f(p_1, p_2)$ and $q = f(q_1, q_2)$ for some p_i, q_i with $E \vdash p_i \approx q_i$, for $i \in \{1, 2\}$. By the proviso of the theorem p, q have neither **0** summands nor factors, thus implying $p_i, q_i \not\leftrightarrow \mathbf{0}$. Hence, from $p \leftrightarrow f(\alpha, p_n)$ and $p = f(p_1, p_2)$ and Lemma 66 we obtain $p_1 \leftrightarrow \alpha$ and $p_2 \leftrightarrow p_n$, thus implying, by the soundness of the equations in $\mathcal{E}$, that $q_1 \leftrightarrow \alpha$ and $q_2 \leftrightarrow p_n$, so that $q = f(\alpha, p_n)$. In both cases, we can infer that q has itself as the desired summand.

This completes the proof of Theorem 19 and thus of Theorem 14 in the case of an operator f that does not distribute over summation in either argument, case $L^f_\alpha, R^f_{\bar{\alpha}}, S^f_{\alpha,\bar{\alpha}}$. ◀

L Proof of Theorem 20

The proof of Theorem 20 follows that of Theorem 19 in a step by step manner, by exploiting Proposition 69 below in place of Proposition 67. The only difference with the proof of Proposition 67 is that, in the case at hand, Lemma 65 does not hold anymore. (In fact one could prove, as done for Lemma 57, that $f(\alpha, p_n)$ is prime for all $n \geq 0$.)

▶ **Proposition 69.** *Assume an operator f such that only L^f_α holds for α, only $R^f_{\bar{\alpha}}$ holds for $\bar{\alpha}$, and only $S_{\bar{\alpha},\alpha}$ holds.*

Let $t \approx u$ be an equation over CCS^-_f that is sound modulo $\leftrightarrow$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither **0** *summands nor factors, and $p, q \leftrightarrow f(\alpha, p_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.*

Proof: The proof follows exactly as the proof of Proposition 67, with the only difference that when we consider the derived transition

$$\sigma'(t_1) \xrightarrow{\bar{\alpha}} p'$$

we have that $p' \leftrightarrow \alpha \| p_n \not\leftrightarrow f(\alpha, p_n)$. However, by substituting $f(\alpha, p_n)$ with $\alpha \| p_n$ in the remaining of the proof, the same arguments hold. □

M Proof of Theorem 21

First of all, we remark that the witness processes $f(\tau, q_n)$ enjoy the properties formalized in Lemmas 70 and 71 below.

▶ **Lemma 70.** *For each $n \geq 0$ it holds that $f(\tau, q_n) \leftrightarrow \tau \| q_n$.*

▶ **Lemma 71.** *Let $n \geq 1$. Assume that $f(p, q) \leftrightarrow f(\tau, q_n)$ for $p, q \not\leftrightarrow$* **0**. *Then $p \leftrightarrow \tau$ and $q \leftrightarrow q_n$.*

Proof: The proof is analogous to that of Lemma 58. We remark that the τ-transition by $f(\tau, q_n)$ can be mimicked only by a τ-move by p. To see this, we show that any other case would lead to a contradiction with the proviso of the lemma $f(p, q) \leftrightarrow f(\tau, q_n)$. In particular, we distinguish three cases, according to which rule of type (5) is available for f and whether the predicates R^f_τ holds or not.

- Assume $p \xrightarrow{\alpha} p'$ and $q \xrightarrow{\bar{\alpha}} q'$ with $p' \| q' \leftrightarrow q_n$. This would contradict $f(\tau, q_n) \leftrightarrow f(p, q)$ since $f(p, q) \xrightarrow{\bar{\alpha}} p \| q'$, whereas $f(\tau, q_n) \overset{\bar{\alpha}}{\not\rightarrow}$.

- Assume $p \xrightarrow{\bar{\alpha}} p'$ and $q \xrightarrow{\alpha} q'$ with $p' \| q' \underline{\leftrightarrow} q_n$. Notice that since q_n is prime, then we have that either $p' \underline{\leftrightarrow} \mathbf{0}$ and $q' \underline{\leftrightarrow} q_n$, or $p' \underline{\leftrightarrow} q_n$ and $q' \underline{\leftrightarrow} \mathbf{0}$. The latter case contradicts $f(p,q) \underline{\leftrightarrow} f(\tau, q_n)$ since the transition $f(p,q) \xrightarrow{\alpha} p \| q' \underline{\leftrightarrow} p \| q_n$ cannot be mimicked by $f(\tau, q_n)$. The former case also contradicts the proviso of the lemma, since we would have $f(p,q) \xrightarrow{\alpha} p \| q' \underline{\leftrightarrow} p \xrightarrow{\bar{\alpha}} p' \underline{\leftrightarrow} q_n$, whereas $f(\tau, q_n) \xrightarrow{\alpha} \tau \| \bar{\alpha}^{\leq i}$, for some $i \in \{1, \dots, n\}$, and there is no r such that $\tau \| \bar{\alpha}^{\leq i} \xrightarrow{\bar{\alpha}} r$ and $r \underline{\leftrightarrow} q_n$, for any $i \in \{1, \dots, n\}$.
- Finally, assume that the predicate R^f_μ holds, and thus that f has a rule of type (7) with label τ. Hence, assume $q \xrightarrow{\tau} q'$, for some q', so that $f(p,q) \xrightarrow{\tau} p \| q' \underline{\leftrightarrow} q_n$. Since q_n is prime and $p \not\underline{\leftrightarrow} \mathbf{0}$, we have that $p \underline{\leftrightarrow} q_n$ and $q' \underline{\leftrightarrow} \mathbf{0}$. So, by congruence closure, we get

$$f(p,q) \underline{\leftrightarrow} f(q_n, q) \underline{\leftrightarrow} f(\tau, q_n).$$

  Since $f(\tau, q_n) \xrightarrow{\alpha} \tau \| \bar{\alpha}^{\leq n}$ and only R^f_α holds, we have that $q \xrightarrow{\alpha} q_1$ for some q_1 such that $q_n \| q_1 \underline{\leftrightarrow} \tau \| \bar{\alpha}^{\leq n}$, which is a contradiction as $q_n \xrightarrow{\alpha}$ implies $q_n \| q_1 \xrightarrow{\alpha}$, whereas $\tau \| \bar{\alpha}^{\leq n} \not\xrightarrow{\alpha}$.

□

The same reasoning used in the proof of Theorem 19 allows us to prove Theorem 21, by exploiting Proposition 72 in place of Proposition 67.

▶ **Proposition 72.** *Assume an operator f such that only R^f_α and $R^f_{\bar{\alpha}}$ hold for $\alpha, \bar{\alpha}$, and L^f_τ holds.*

Let $t \approx u$ be an equation over CCS_f^- that is sound modulo $\underline{\leftrightarrow}$. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q have neither $\mathbf{0}$ summands nor factors, and $p, q \underline{\leftrightarrow} f(\tau, q_n)$ for some n larger than the size of t. If p has a summand bisimilar to $f(\tau, q_n)$, then so does q.

Proof: The claim follows by the same arguments used in the proof of Proposition 67 and by considering the substitution

$$\sigma' = \sigma[x \mapsto \alpha q_n].$$

□