Are Two Binary Operators Necessary to Finitely Axiomatize Parallel Composition?*

(Or How Canonical are Bergstra and Klop's Merges?)

A bunch of smart people

A natural question to ask is whether there is a single *binary* operator that preserves bisimulation equivalence, and whose addition to the recursion free fragment of CCS allows for the finite equational axiomatization of parallel composition—see [1, Problem 8]. We conjecture that no such operator exists, and that the use of *two* auxiliary operators is therefore necessary to achieve a finite axiomatization of parallel composition in bisimulation semantics. This result would offer the definitive justification we seek for the canonical standing of the operators proposed by Bergstra and Klop. This note presents a collection of random thoughts on a simplified, but arguably still meaningful and hopefully interesting, version of this problem.

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1 Introduction

The purpose of this paper is to provide an answer to the following problem:

> Can one obtain a finite axiomatization of the parallel composition operator in bisimulation semantics by adding *only one binary operator* to the signature of CCS?

The interest in this problem is twofold, as an answer to it would: 1. clarify the status of the auxiliary operators *left merge* and *communication merge*, proposed in [10], in the axiomatization of parallel composition, and 2. shed further insight into properties that *good* auxiliary operators used in the finite equational characterization of parallel composition ought to afford.

In detail, we prove that, under some reasonable simplifying assumptions, the answer is *negative*.

In order to put our contribution in context, we first describe the history of the problem we tackle and then give a bird's eye view of our results and their limitations.

1.1 The story so far

In the late 1970s, Robin Milner developed the Calculus of Communicating Systems (CCS) [18], a formal algebraic language based on a message-passing paradigm and aimed at describing communicating processes from an operational point of view. In detail, a labelled transition system (LTS) [16] was used to

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equip language expressions with an *operational semantics* [25]; the LTS giving the operational semantics of each CCS expression was defined using a collection of syntax-driven rules. The analysis of process behaviour was carried out via an (observational) *bisimulation*-based theory [24] that defines when two LTSs describe the same behaviour. In particular, to model the interactions among processes a *parallel composition operator* $\|$ was introduced. Such an operator, also known as *merge* [10, 11], allows one both to *interleave* the behaviours of the composed processes (thus modelling concurrent computations) and to enable some form of *synchronisation* between them (thus modelling communications). Later on, in collaboration with Matthew Hennessy, the same author studied the *equational theory* of (recursion free) CCS and proposed a *ground-complete axiomatization* for it modulo bisimilarity [15]. More precisely, they presented a set E of *equational axioms* from which all equations over closed CCS terms (namely with no occurrences of variables) that are *valid modulo bisimilarity* can be derived using the rules of *equational logic* [27]. Notably, the set E included infinitely many axioms, mainly generated by the *expansion law* they used to deal with the parallel composition operator.

The ground-completeness result by Hennessy and Milner started the quest for a finite axiomatization modulo bisimilarity of CCS's parallel composition operator.

Jan Bergstra and Jan Willem Klop proved, in [10], that if we enrich CCS[1] with two auxiliary operators, namely the *left merge* $\llfloor$ and the *communication merge* $|$, expressing respectively the asymmetric pure interleaving and the synchronous behaviour of $\|$, then a finite ground-complete axiomatization modulo bisimilarity exists. Their result was then strengthen by Luca Aceto et al. in [5], where it is proved that over the fragment of CCS without recursion, restriction and relabelling, the two auxiliary operators $\llfloor$ and $|$ allow for finitely axiomatizing $\|$ modulo bisimilarity also when CCS terms with variables are considered.

From these results, we can infer that the left merge and communication merge operators are *sufficient* to finitely axiomatize the parallel composition operator modulo bisimilarity.

Conversely, in [21,22] Faron Moller considered a minimal fragment of CSS, namely with action prefixing, nondeterministic choice and purely interleaving parallel composition, and proved that bisimilarity does not afford a finite axiomatization over the closed terms in such fragment of CCS without the left merge operator. Following Moller's technique, Luca Aceto et al. proved, in [4], that if we replace $\llfloor$ and $|$ with the so called *Hennessy's merge* $|\!/$ [14], which is a sort of combination of the auxiliary operators as it denotes an asymmetric interleaving with communication, then the collection of equations that hold modulo bisimilarity over the recursion, restriction and relabelling free fragment of CCS enriched with $|\!/$ is not finitely based (in the presence of at least two distinct complementary actions).

A natural question that arises from these *negative* results is then the following:

> *Are the left merge and the communication merge operators* necessary *to obtain a finite aziomatization of bisimilarity over (recursion, restriction and relabelling free) CCS?*

In this paper we aim to take a first step towards answering to such a question.

1.2 Our contribution

We investigate the problem of whether it is possible to obtain a finite ground-complete axiomatization of bismilarity by adding only one auxiliary binary operator to CCS. In detail, we analyse the axiomatizability of parallel composition over the language CCS_f, namely CCS enriched with a binary operator f

[1] To be precise, [10] provides a finite axiomatization over the language ACP and not CCS. However, the existence of a finite equational axiomatization of $\|$ depends only on the use of the auxiliary operators $\llfloor$ and $|$, and not on any other operator in ACP. Thus, as long as we focus only on $\|$, the result obtained for ACP holds also for CCS.

that we use to decompose the behaviour of $\|$. We prove that, under reasonable assumptions, the auxiliary operator f alone does not allow us to obtain a finite ground-complete axiomatization.

Roughly, we assume that the operational semantics of f is in the de Simone format [26] and that the behaviour of the parallel composition operator is expressed equationally by a law that is akin to the one used by Bergstra and Klop to define $\|$ in terms of $\llbracket$ and $|$. We then argue that this assumption yields that the equation

$$x \| y \approx f(x,y) + f(y,x) \tag{A}$$

is valid modulo bisimilarity. Next we proceed by a case analysis over the possible sets of SOS rules defining the behaviour of f, in such a way that the validity of Equation (A) modulo bisimilarity is guaranteed, and, for each case, we provide the desired negative result using proof-theoretic techniques that have their roots in Moller's classic results in [21, 22]. This means that we identify a (case-specific) property of terms, called the *witness property* and denoted by W, associated with each finite set of sound axioms E and a natural number $n \geq 0$. The idea is that, when n is *large enough*, whenever the equation $p \approx q$ is derivable from E, then either both terms p and q satisfy $W(n)$, or none of them does. The negative result is then obtained by exhibiting a (case-specific) family of infinitely many valid equations $\{e(n) \mid n \geq 0\}$ in which the witness property is not preserved, that is, for each $n \geq 0$, $W(n)$ is satisfied only by one side of $e(n)$. Due to the choice of W, this means that the equations in the family cannot have been derived from a finite set of valid axioms and therefore they have to be *all* included in the axiomatization. As there are infinitely many such equations, the axiomatization cannot be finitely based.

The choice of limiting ourselves to consider a binary operator f in the de Simone format is threefold:

- The de Simone format is the simplest congruence format for bisimilarity. Hence we must be able to deal with this case before proceeding to any generalization.
- The specification of parallel composition, left merge and communication merge operators (and of the vast majority of process algebraic operators) is in de Simone format. Hence, that format was a natural choice also for operator f.
- The simplicity of the de Simone rules allows us to reduce considerably the complexity of our case analysis on the sets of available SOS rules for the operator f. Due to the unavoidably high amount of technical results that are already necessary to obtain the desired negative result, we opted for a simplification in this sense.

We hope that the results we present in this paper will entice others to study the question we address here and to generalize our contribution, leading to a *Book Proof theorem* [8].

1.3 Organization of contents

2 Preliminaries

We begin by introducing the basic definitions and results on which the technical developments to follow are based.

2.1 Labelled Transition Systems and Bisimulation Equivalence

As semantic model we consider classic *labelled transition systems* [16].

Definition 1. A *labelled transition system* is a triple $(S,A,\rightarrow)$, where S is a set of *states*, A is a set of *actions*, and $\rightarrow \subseteq S \times A \times S$ is a (*labelled*) *transition relation*.

As usual, we shall use the more suggestive notation $t \xrightarrow{\mu} t'$ in lieu of (t,μ,t'). For each term t and action μ, we shall write $t \xrightarrow{\mu}$ if $t \xrightarrow{\mu} t'$ holds for some state t', and $t \stackrel{\mu}{\not\rightarrow}$ if there is no t' such that $t \xrightarrow{\mu} t'$ holds. The *initials* of t are the actions $\mathrm{init}(t) = \{\mu \mid t \xrightarrow{\mu}\}$ that label the outgoing transitions of t. The transition relations $\xrightarrow{\mu}$ naturally compose to determine the possible effects that performing a sequence of actions may have on a state.

Definition 2. For a sequence of actions $s = \mu_1 \cdots \mu_k$ $(k \geq 0)$, and states t,t', we write $t \xrightarrow{s} t'$ iff there exists a sequence of transitions

$$t = t_0 \xrightarrow{\mu_1} t_1 \xrightarrow{\mu_2} \cdots \xrightarrow{\mu_k} t_k = t' \ .$$

If $t \xrightarrow{s} t'$ holds for some state t', then s is a *trace* of t. Moreover, we say that s is maximal if $\mathrm{init}(t') = \emptyset$.

By means of traces, we associate two classic measures with a state t: its *depth*, denoted by $depth(t)$, and its *norm*, denoted by $norm(t)$. They express, respectively, the length of a *longest* trace of t and that of its *shortest* maximal trace. Formally

$$\begin{aligned} depth(t) &= \sup\{k \mid \ t \text{ has a trace of length } k\} \\ norm(t) &= \min\{k \mid \ t \text{ has a maximal trace of length } k\} \ . \end{aligned}$$

In this paper, we shall consider the states in a labelled transition system modulo bisimulation equivalence [19, 24], allowing us to establish whether two processes have the same behaviour.

Definition 3. Let $(S,A,\rightarrow)$ be a labelled transition system. *Bisimulation equivalence*, or simply *bisimilarity*, denoted by $\underline{\leftrightarrow}$, is the largest binary symmetric relation over S such that whenever $t \ \underline{\leftrightarrow}\ u$ and $t \xrightarrow{\mu} t'$, then there is a transition $u \xrightarrow{\mu} u'$ with $t' \ \underline{\leftrightarrow}\ u'$. If $t \ \underline{\leftrightarrow}\ u$, then we say that t and u are *bisimilar*.

It is well-known that bisimilarity is indeed an equivalence relation (see, e.g., [19, 24]). Moreover, two bisimilar states have the same depth and norm.

2.2 The Language CCS$_f$

The language for processes we shall consider in this paper is obtained by adding a single binary operator f to the recursion, restriction and relabelling free subset of Milner's CCS [19], and henceforth referred to as CCS$_f$. This language is given by the following grammar:

$$t ::= \mathbf{0} \mid x \mid a.t \mid \bar{a}.t \mid \tau.t \mid t+t \mid t \,\|\, t \mid f(t,t) \ ,$$

where x is a variable drawn from a countably infinite set $\mathscr{V}$, a is an action, and $\bar{a}$ is its complement. We assume that the actions a and $\bar{a}$ are distinct. Following Milner [19], the action symbol τ will result from the synchronized occurrence of the complementary actions a and $\bar{a}$.

In order to simplify the analysis (and to obtain the desired negative results), it will be sufficient to consider the above language with three unary prefixing operators, so there is only one action a with its corresponding complementary action $\bar{a}$. The results we shall present in what follows carry over unchanged to a setting with an arbitrary number of actions, and corresponding unary prefixing operators.

Henceforth, we let $\mu \in \{a,\bar{a},\tau\}$ and $\alpha \in \{a,\bar{a}\}$. As usual, we postulate that $\bar{\bar{a}} = a$. We shall use the meta-variables t,u,v,w to range over process terms, and write $var(t)$ for the collection of variables occurring in the term t. The *size* of a term is the number of operator symbols in it. A process term is *closed* if it does not contain any variables. Closed terms, or *processes*, will be typically denoted by p,q,r. Following standard practice in the literature on CCS and related languages, trailing $\mathbf{0}$'s will often be omitted from terms.

A *(closed) substitution* is a mapping from process variables to (closed) CCS_f terms. For every term t and substitution σ, the term obtained by replacing every occurrence of a variable x in t with the term $\sigma(x)$ will be written $\sigma(t)$. Note that $\sigma(t)$ is closed, if so is σ. We shall sometimes write $\sigma[x \mapsto p]$ to denote the substitution that maps the variable x into process p and behaves like σ on all other variables.

In the remainder of this paper, process terms are considered modulo associativity and commutativity of $+$. In other words, we do not distinguish $t+u$ and $u+t$, nor $(t+u)+v$ and $t+(u+v)$. In what follows, the symbol $=$ will denote equality modulo the above identifications. We use a *summation* $\sum_{i\in\{1,\dots,k\}} t_i$ to denote the term $t = t_1 + \cdots + t_k$, where the empty sum represents $\mathbf{0}$. We can also assume that the terms t_i, for $i \in \{1,\dots,k\}$, do not have $+$ as head operator, and they are referred to as the *summands* of t.

Henceforth, for each action μ and $m \geq 0$, we let μ^0 denote $\mathbf{0}$, and μ^{m+1} denote $\mu(\mu^m)$. For each action μ and positive integer i, we also define

$$\mu^{\leq i} = \mu + \mu^2 + \cdots + \mu^i \ .$$

2.3 Equational Logic

An *axiom system* E is a collection of (*process*) *equations* $t \approx u$ over the language CCS_f. An equation $t \approx u$ is *derivable* from an axiom system E, notation $E \vdash t \approx u$, if there is an *equational proof* for it from E, namely if it can be inferred from the axioms in E using the *rules* of *equational logic*, which are reflexivity, symmetry, transitivity, substitution and closure under CCS_f contexts.

$$t \approx t \quad \frac{t \approx u}{u \approx t} \quad \frac{t \approx u \quad u \approx v}{t \approx v} \quad \frac{t \approx u}{\sigma(t) \approx \sigma(u)} \quad \frac{t \approx u}{\mu.t \approx \mu.u}$$
$$\frac{t \approx u \quad t' \approx u'}{t + t' \approx u + u'} \quad \frac{t \approx u \quad t' \approx u'}{f(t,t') \approx f(u,u')} \quad \frac{t \approx u \quad t' \approx u'}{t \parallel t' \approx u \parallel u'} \ .$$

Without loss of generality one may assume that substitutions happen first in equational proofs, i.e., that the rule

$$\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

may only be used when $(t \approx u) \in E$. In this case $\sigma(t) \approx \sigma(u)$ is called a *substitution instance* of an axiom in E. Moreover, by postulating that for each axiom in E also its symmetric counterpart is present in E, one may assume that applications of symmetry happen first in equational proofs, i.e., that the rule

$$\frac{t \approx u}{u \approx t}$$

is never used in equational proofs. In the remainder of this paper, we shall always tacitly assume that equational axiom systems are closed with respect to symmetry.

We are interested in equations that are valid modulo a chosen congruence relation $\mathscr{R}$. The equation $t \approx u$ is said to be *sound* with respect to $\mathscr{R}$ if $\sigma(t) \mathscr{R} \sigma(u)$ for all closed substitutions σ. For simplicity, if $t \approx u$ is sound, then we write $t \mathscr{R} u$. An axiom system is *sound* modulo $\mathscr{R}$ if and only if all of its equations are sound modulo $\mathscr{R}$. Conversely, we say that E is *ground-complete* modulo $\mathscr{R}$ if $p \mathscr{R} q$ implies $E \vdash p \approx q$ for all processes p,q. In the remainder of the paper, we shall say that E is a set of *valid* equations, if E is sound and ground-complete. We say that $\mathscr{R}$ affords a *finite* axiomatization, if there is a *finite* axiom system E s.t. $E \vdash t \approx u$ if and only if $t \mathscr{R} u$.

3 The Problem

Our purpose is to further investigate the axiomatizability of the parallel composition operator of CCS in bisimulation semantics. As we discussed in the Introduction, bisimilarity affords a finite equational axiomatization over CCS extended with the left and communication merge operators from [10], which allow us to derive any instance of the expansion law for the parallel composition operator [15] from a finite set of axioms. Moreover, in [21, 22] it was proved that the use of auxiliary operators in indeed necessary to obtain a finite axiomatization of bisimilarity in that setting.

The aim of this paper is to offer a contribution towards the solution of the following problem:

> Are the left merge and communication merge operators necessary to obtain a finite axiomatization of bisimulation equivalence? Can one obtain a finite axiomatization by adding only *one* binary operator to the signature of CCS?

In particular, this boils down to asking whether bisimulation equivalence admits a finite equational axiomatization over the language CCS_f, for some binary operator f. Of course, this question only makes sense if f is an operator that preserves bisimulation equivalence. In this section we discuss two assumptions we shall make on the auxiliary operator f in order to meet such requirement and to present our results in a slightly simplified technical setting.

3.1 The de Simone format

One way to guarantee that f preserves bisimilarity is to postulate that the behaviour of f is described using Plotkin-style rules that fit a rule format that is known to preserve bisimulation equivalence, see, e.g., [7] for a survey of such rule formats. The simplest format satisfying this criterion is the format proposed by de Simone in [26]. We believe that if we can't deal with operations specified in that format, then there is little hope to generalize our results. We feel therefore inclined to make the following

Simplifying Assumption 1. The behaviour of f is described by rules in de Simone format.

Definition 4 (De Simone rules, [26])**.** An SOS rule ρ for f is in de Simone format if it has the form

$$\rho = \frac{\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}}{f(x_1,x_2) \xrightarrow{\mu} t} \tag{1}$$

where

- $I \subseteq \{1,2\}$,
- the variables x_1, x_2 and y_i ($i \in I$) are all different (and are called the *variables of the rule*),
- μ and μ_i ($i \in I$) are contained in $\{a, \bar{a}, \tau\}$, and
- t (sometimes called the *target of the rule*) is a CCS_f term over variables $\{x_1, x_2, y_i \mid i \in I\}$ such that
  - each variable occurs at most once in t, and
  - if $i \in I$, then x_i does not occur in t.

Henceforth, we shall assume, without loss of generality, that the variables x_1, x_2, y_1 and y_2 are the only ones used in operational rules. The pair $(\mu_1, -)$ if $I = \{1\}$, $(-, \mu_2)$ if $I = \{2\}$, and (μ_1, μ_2) if $I = \{1,2\}$, of actions mentioned in the premises of a de Simone rule will sometimes be called the *trigger* of the rule. Moreover, if μ is the label of the transition in the conclusion of a de Simone rule ρ, we shall say that ρ has μ as *label*.

$$\frac{}{\mu x \xrightarrow{\mu} x} \qquad \frac{x \xrightarrow{\mu} x'}{x+y \xrightarrow{\mu} x'} \qquad \frac{y \xrightarrow{\mu} y'}{x+y \xrightarrow{\mu} y'}$$

$$\frac{x \xrightarrow{\mu} x'}{x \parallel y \xrightarrow{\mu} x' \parallel y} \qquad \frac{y \xrightarrow{\mu} y'}{x \parallel y \xrightarrow{\mu} x \parallel y'} \qquad \frac{x \xrightarrow{\alpha} x',\ y \xrightarrow{\bar{\alpha}} y'}{x \parallel y \xrightarrow{\tau} x' \parallel y'}$$

Table 1: SOS Rules for the CCS Operators ($\mu \in \{a, \bar{a}, \tau\}$ and $\alpha \in \{a, \bar{a}\}$)

The SOS rules for all of the classic CCS operators are in de Simone's format (see Table 1), and so are those for Hennessy's $|\!/$ operator from [14] and for Bergstra and Klop's left and communication merge operators [9], at least if we disregard issues related to the treatment of successful termination. Thus restricting ourselves to operators whose operational behaviour is described by de Simone rules already gives us a good degree of generality.

The transition rules in Table 1 and those for the operator f give rise to transitions between CCS$_f$ terms. The operational semantics for CCS$_f$ is thus given by the labelled transition system whose states are CCS$_f$ terms, and whose labelled transitions are those that are provable using the rules (see Definition 1).

In what follows, we shall consider the collection of *closed CCS$_f$ terms* modulo bisimulation equivalence (see Definition 3). Since the SOS rules defining the operational semantics of the language CCS$_f$ are in de Simone's format, we have that:

Fact 1. Bisimulation equivalence is a congruence with respect to the operations in the language CCS$_f$, that is, $\mu p \leftrightarrow \mu q$, $p + p' \leftrightarrow q + q'$, $p \| p' \leftrightarrow q \| q'$ and $f(p, p') \leftrightarrow f(q, q')$ hold whenever $p \leftrightarrow q$, $p' \leftrightarrow q'$ and p, p', q, q' are closed CCS$_f$ terms.

Bisimulation equivalence is extended to arbitrary CCS$_f$ terms thus:

Definition 5. Let t, u be CCS$_f$ terms. Then $t \leftrightarrow u$ if and only if $\sigma(t) \leftrightarrow \sigma(u)$ for every closed substitution σ.

Hence $t \leftrightarrow u$ if, and only if, the equation $t \approx u$ holds in the quotient algebra of closed CCS$_f$ terms modulo bisimilarity.

3.2 Axiomatizing $\|$ with f

In order to make some progress on the problem at hand, we would also like to consider, at least initially, a simplified view of how the operator f can be used to axiomatize parallel composition. To this end, a fairly natural assumption on an axiom system over CCS$_f$ is that it includes an equation of the form

$$x \| y \approx t(x, y) \tag{2}$$

where t is a CCS$_f$ term that does not contain occurrences of $\|$. More precisely, the term will be in the general form $t(x, y) = \sum_{i \in I} t_i(x, y)$, where $I \neq \emptyset$ is a finite set of indexes and for each $i \in I$ the CCS$_f$ term $t_i(x, y)$ does not have $+$ as head operator.

We now proceed to refine the form of the term $t(x, y)$, in order to guarantee the soundness, modulo bisimilarity, of Equation (2). Intuitively, no term $t_i(x, y)$ can have prefixing on top. In fact, if $t(x, y)$ had a summand $\mu.t'(x, y)$, for some $\mu \in \{a, \bar{a}, \tau\}$, then one could easily show that $\mathbf{0} \| \mathbf{0} \not\leftrightarrow t(\mathbf{0}, \mathbf{0})$, since $t(\mathbf{0}, \mathbf{0})$ could perform a μ-transition. Hence, Equation (2) would be unsound. Similarly, $t(x, y)$ cannot have a variable as a summand, for otherwise we would have $\mu_1 \| \mu_2 \not\leftrightarrow t(\mu_1, \mu_2)$: assume, without loss of generality, that $t(x,y)$ has a summand x. Then, $t(\mu_1, \mu_2) \xrightarrow{\mu_1} \mathbf{0}$, namely $t(\mu_1, \mu_2)$ would have norm 1, whereas $\mu_1 \| \mu_2$ has norm 2. We can therefore assume that, for each $i \in I$

$$t_i(x,y) = f(t_i^1(x,y), t_i^2(x,y))$$

for some CCS$_f$ terms $t_i^j(x,y)$, with $j \in \{1,2\}$.

To further narrow down the options on the form that the subterms $t_i^j(x,y)$ might have, we would need to make some assumptions on the behaviour of the operator f. However, to favour the generality of our results with respect to the choice of the auxiliary binary operator f, we limit ourselves to assume that the terms $t_i^j(x,y)$ are in the simplest form, namely they are variables in $\{x,y\}$. Such an assumption is reasonable because to allow prefixing and/or nested occurrences of f-terms in the scope of the terms $t_i(x,y)$ we would need to study the behaviour of f. This would entail the (at least partial) definition of the operational semantics of f, thus making our results less general as, roughly speaking, we would need to study one possible auxiliary operator at a time (the one identified by the considered set of de Simone rules). Moreover, if we look at how parallel composition is expressed equationally as a derived operator in terms of Hennessy's merge or Bergstra and Klop's left and communication merge or as in reference [2], viz. via the equations

$$\begin{aligned} x \| y &\approx (x \,|^{/}\, y) + (y \,|^{/}\, x) \\ x \| y &\approx (x \lfloor\!\lfloor y) + (y \lfloor\!\lfloor x) + (x \mid y) \\ x \| y &\approx (x \lfloor\!\lfloor y) + (x \rfloor\!\rfloor y) + (x \mid y) \;, \end{aligned}$$

we see the emergence of a pattern: the parallel composition operator is always expressed in terms of sums of terms built from the auxiliary operators and variables.

Therefore, from now on we'll make the following:

Simplifying Assumption 2. For some $J \subseteq \{x,y\}^2$, the equation

$$x \| y \approx \sum \{f(z_1, z_2) \mid (z_1, z_2) \in J\} \tag{3}$$

holds modulo bisimulation equivalence. In what follows, we shall sometimes use t_J to denote the right-hand side of the above equation and use $t_J(p,q)$ to stand for the process $\sigma[x \mapsto p, y \mapsto q](t_J)$.

Starting from this simplified setting, we now proceed to further investigate the relation between operator f and the parallel composition, obtaining a refined form for Equation (3) (Simplifying Assumption 3 below).

Lemma 1. *Assume that Assumptions 1 and 2 hold. Then:*

1. *The index set J on the right-hand side of (3) is non-empty.*
2. *The set of rules for f is non-empty.*
3. *Each rule for f has some premise.*

Proof: The proof can be found in Appendix A.1. □

Using the basic observations collected above, we can now argue that:

Lemma 2. *The terms $f(x,x)$ and $f(y,y)$ are not summands of t_J, the right-hand side of Equation (3).*

Proof: The proof can be found in Appendix A.2. □

As consequence of this result, we may infer that the index set J in the term on the right-hand side of Equation (3) is either one of the singletons $\{(x,y)\}$ or $\{(y,x)\}$, or it is the set $\{(x,y),(y,x)\}$. However, due to Moller's results to the effect that bisimilarity has no finite ground-complete axiomatization over CCS [21, 23], the former option can be discarded, as shown in the following:

Proposition 1. *If J is a singleton, then the language CCS_f admits no finite equational axiomatization modulo bisimulation equivalence.*

Proof: The proof can be found in Appendix A.3. □

As a consequence, using the commutativity of $+$ modulo bisimulation equivalence, we can restate our Assumption 2 in the following simplified form:

Simplifying Assumption 3. Equation (3) can be refined to the following form:

$$x \parallel y \approx f(x,y) + f(y,x) \quad . \tag{4}$$

Moreover, we can restrict ourselves to considering only operators f such that $x \parallel y \approx f(x,y)$ does not hold modulo bisimulation equivalence, namely that

$$x \parallel y \not\approx f(x,y) \quad . \tag{5}$$

We remark that Equation (5) is equivalent to stating that there exists a closed substitution σ such that $\sigma(x)\|\sigma(y) \not\leftrightarrow f(\sigma(x),\sigma(y))$. However, it could still be the case that, due to the available rules for f, for some closed substitution σ' it holds that $\sigma'(x)\|\sigma'(y) \leftrightarrow f(\sigma'(x),\sigma'(y))$.

For later use, we note a useful consequence of the soundness of Equation (4) modulo bisimulation equivalence.

Lemma 3. *Assume that Equation (4) holds modulo bisimulation equivalence. Then $depth(p)$ is finite for each closed CCS_f term.*

Proof: The proof can be found in Appendix A.4. □

We can therefore refine the Assumption 2 to its version given in Equation (4) and prove the following:

Theorem 1. *Assume that f satisfies Assumption 1 and that Equation* (4) *holds modulo bisimulation equivalence. Then bisimulation equivalence admits no finite equational axiomatization over the language CCS_f.*

4 The operational semantics of f

In order to prove Theorem 1, we shall, first of all, understand what rules f may and must have in order for Equation (4) to hold modulo bisimulation equivalence (Proposition 2 below).

We begin this analysis by restricting the possible forms the SOS rules for the operator f *may* take.

Lemma 4. *Suppose that f meets Assumption 1, and that Equation* (4) *is sound modulo bisimulation equivalence. Let ρ be a de Simone rule for f of the form (1). Then the following statements hold.*

1. *If $\mu = \tau$ then the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ of ρ can only have one of the following possible forms:*
   - *$\{x_i \xrightarrow{\tau} y_i\}$ for some $i \in \{1,2\}$, or*
   - *$\{x_1 \xrightarrow{\alpha} y_1, x_2 \xrightarrow{\bar{\alpha}} y_2\}$ for some $\alpha \in \{a,\bar{a}\}$.*

2. *If $\mu = \alpha$ for some $\alpha \in \{a, \bar{a}\}$, then the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ can only have the form $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1,2\}$.*

Proof: The proof can be found in Appendix B.1. □

The previous lemma limits the form of the premises that rules for f *may* have in order for Equation (4) to hold modulo bisimulation equivalence. We now characterize the rules that f *must* have in order for it to satisfy that equation. Firstly, we deal with *synchronization*.

Lemma 5. *Assume that Equation (4) holds modulo bisimulation equivalence. Then the operator f must have a rule of the form*

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} t(y_1,y_2)} \tag{6}$$

for some $\alpha \in \{a, \bar{a}\}$ and term t. Moreover, for each rule for f of the above form the term $t(x,y)$ is bisimilar to $x \,\|\, y$.

Proof: The proof can be found in Appendix B.2. □

We shall sometimes write that the rule in (6) is of the form $S_{\alpha,\bar{\alpha}}$, to underline the fact that it allows for the synchronization of the two arguments of f by testing action α on the first argument and $\bar{\alpha}$ on the second argument.

We consider now the *interleaving* behaviour in the rules for f. In order to properly characterize the rules for f as done in the previous Lemma 5, we consider an additional simplifying assumption on the form that the targets of the rules for f might have.

Simplifying Assumption 4. If t is the target of a rule for f, then t us either a variable or a term obtained by applying a single CCS_f operator to the variables of the rule, according to the constraints of the de Simone format.

Notice that Assumption 4 does not contradict Lemma 5. Moreover, we remark that such an assumption is not too strict as, to the best of our knowledge, an example of an operator having rules whose target is a term obtained by applying more than one language operator to the variables of the rule is the *copy operator* from [12]. However, such an operator is out of the de Simone format, in that the aforementioned rule double tests the argument of the copy operator.

Lemma 6. *Let $\mu \in \{a, \bar{a}, \tau\}$. Then the operator f must have either a rule of the form*

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1,x_2) \xrightarrow{\mu} t(y_1,x_2)} \tag{7}$$

or a rule of the form

$$\frac{x_2 \xrightarrow{\mu} y_2}{f(x_1,x_2) \xrightarrow{\mu} t(x_1,y_2)} \tag{8}$$

for some term t. Moreover, under Assumption 4, for each rule for f of the above forms the term $t(x,y)$ is bisimilar to $x \,\|\, y$.

Proof: The proof can be found in Appendix B.3. □

Remark 1. Simplifying Assumption 4 is only used in the proof of the second claim in Lemma 6. We have been unable to to prove the claim without it. We do not know whether that assumption is needed either, but we conjecture that is not.

We shall sometimes write that a rule of type (7) is of the form A^l_μ to stress that it tests a μ-move by the left argument of f. Similarly, a rule of type (8) will be of the form A^r_μ. Moreover, we shall sometimes extend this notation to sets of rules in the natural way. For instance, we shall let $A^l_{\alpha,\tau}, A^r_{\bar{\alpha}}$ denote that the set of rules for f includes rules of type (7) with labels α and τ, and a rule of type (8) with label $\bar{\alpha}$.

Summing up, we have obtained that f must afford at least one rule of type (6) (where we can assume without loss of generality that the target of the rule is $y_1 \parallel y_2$), and, for each $\mu \in \{a, \bar{a}, \mu\}$, at least one rule of type (7) or (8) (where, using Assumption 4 we can assume without loss of generality that the target of rule (7) is $y_1 \parallel x_2$, and that of rule (8) is $x_1 \parallel y_2$).

The following proposition states that this is enough to obtain the soundness of Equation (4).

Proposition 2. *Assume that all of the rules for f have the form (6) with target $y_1 \parallel y_2$, (7) with target $y_1 \parallel x_2$, or (8) with target $x_1 \parallel y_2$. If the rules for f contain at least one rule of type (6), and, for each $\mu \in \{a, \bar{a}, \tau\}$, at least one rule of type (7) or (8), then Equation (4) is sound modulo bisimulation equivalence.*

Proof: The proof can be found in Appendix B.4. □

In light of the above result, we shall henceforth tacitly assume that f has a minimal amount of rules prescribed by Proposition 2. As an immediate consequence of the form of the rules for f given in Proposition 2, we have that:

Lemma 7. *Assume that all of the rules for f have the form (6) with target $y_1 \parallel y_2$, (7) with target $y_1 \parallel x_2$, or (8) with target $x_1 \parallel y_2$. Then each closed term p in the language CCS_f is finitely branching—that is, the set $\{(\mu, q) \mid p \xrightarrow{\mu} q\}$ is finite.*

Remark 2. A standard consequence of the finiteness of the depth (Lemma 3), and the finite branching, of closed terms in the language CCS_f is that each closed CCS_f term is bisimilar to a synchronization tree [18], that is, a closed term built only using the constant $\mathbf{0}$, the unary prefixing operations and the binary $+$ operation. Since bisimulation equivalence is a congruence over the language CCS_f (Fact 1), this means, in particular, that an equation $t \approx u$ over the language CCS_f is sound modulo bisimilarity if, and only if, the closed terms $\sigma(t)$ and $\sigma(u)$ are bisimilar for each substitution mapping variables to synchronization trees. Moreover, we can use the sub-language of synchronization trees, which is common to all of the languages CCS_f, to compare terms from these languages for different choices of binary operation f with respect to bisimilarity. Finally, in this setting, the notion of depth can be extended to equations by letting $depth(t \approx u) = \max\{depth(t), depth(u)\}$.

5 The proof strategy

In this section we discuss the general reasoning behind the proof of Theorem **??** and, thus, present the proof method we use to obtain the desired negative result (Theorem 2 below). In light of Proposition 2, to prove Theorem 1 we will proceed by a case analysis over the possible sets of allowed SOS rules for operator f. However, in each case, our proof method will follow the same general schema, which has its roots in Moller's arguments to the effect that bisimulation equivalence is not finitely based over the language CCS (see, e.g., [4, 21–23]), and that we present here at an informal level.

The main idea is to identify a *witness property of the negative result*. This is a specific property of terms associated with each finite axiom system E over the language CCS_f that is sound with respect to bisimulation equivalence, say $W(n)$ for $n \geq 0$, such that, when n is *large enough*, whenever the equation $p \approx q$ can be derived from E then either both p and q satisfy $W(n)$, or none of them does. Then, we exhibit an infinite family of valid equations, say e_n, called accordingly *witness family of equations* *for the negative result*, in which $W(n)$ is not preserved, namely it is satisfied only by one side of each equation. Thus, our order of business will be to prove the following result:

Theorem 2. *Let E be a finite axiom system over the language CCS_f that is sound with respect to bisimulation equivalence. Let n be larger than the size of each term in the equations in E. Then E does not prove the sound equation e_n.*

In this paper, the witness property $W(n)$ corresponds to having a summand that is bisimilar to a specific process. In detail:

1. We identify, for each case, a family of processes $f(\mu, p_n)$, where $n \in \mathbb{N}$, and the choice of μ and p_n is tailored to the particular set of SOS rules allowed for f. Moreover, process p_n will have depth at least n, for each $n \geq 0$. Sometimes, we shall refer to the processes $f(\mu, p_n)$ as the *witness processes*.

2. We prove that by choosing n *large enough*, given a finite set of valid equations E and processes $p, q \leftrightarrow f(\mu, p_n)$, if $E \vdash p \approx q$ and p has a summand bisimilar to $f(\mu, p_n)$, then also q has a summand bisimilar to $f(\mu, p_n)$. Informally, we will choose n greater than the depth of all the equations in E, so that we are guaranteed that behaviour of the summand bisimilar to $f(\mu, p_n)$ is due to a closed substitution instance of a variable.

3. We provide an infinite family of valid equations e_n in which one side has a summand bisimilar to $f(\mu, p_n)$, but the other side does not. In light of item 2, this implies that such a family of equations cannot be derived from any finite collection of valid equations over CCS_f, modulo bisimilarity, thus proving Theorem 2.

To narrow down the combinatorial analysis over the allowed sets of SOS rules for f, and to exploit some general result, we examine first the *distributivity properties*, modulo bisimilarity, of the operator f with respect to summation.

First of all, we notice that f cannot be distributive with respect to summation in both arguments. This is a consequence of our previous analysis of the operational rules that such an operator f may and must have in order for Equation (4) to hold. However, it can also be shown in a purely algebraic manner, as we do below.

Lemma 8. *A binary operator satisfying Equation (4) cannot be distributive with respect to summation in both arguments.*

Proof: The proof can be found in Appendix B.5. □

Hence, we can limit ourselves to considering binary operators satisfying our constraints that, modulo bisimilarity, are distributive with respect to summation in one argument or in none.

We consider these two possibilities in turn.

5.1 Distributivity in one argument

Due to Remark 2 and Assumption 1, we can exploit a result from [2] to characterize the rules for an operator f that distributes over summation in one of its arguments.

Lemma 9 ([2, Lemma 4.3])**.** *Suppose that i is an argument of f for which each rule for f has a (positive) premise testing it. Then, modulo bisimilarity, f distributes over summation in its i-th argument.*

We provide an informal explanation of Lemma 9 via the following example. Assume, without loss of generality, that, modulo bisimilarity, f distributes with respect to summation in the first argument. This means that the equation

$$f(x+y,z) \approx f(x,z)+f(y,z) \tag{9}$$

is sound over the language CCS_f modulo bisimulation equivalence. Consider now the term

$$p = (\bar{a}.a.\mathbf{0}+\bar{a}.a.a.\mathbf{0}) \parallel (a.\mathbf{0}+\bar{a}.\mathbf{0}+\tau.\mathbf{0}) \ .$$

It is not hard to see that p is bisimilar to

$$\begin{aligned} q = {} & \bar{a}.(a.\mathbf{0} \parallel (a.\mathbf{0}+\bar{a}.\mathbf{0}+\tau.\mathbf{0})) + \bar{a}.(a.a.\mathbf{0} \parallel (a.\mathbf{0}+\bar{a}.\mathbf{0}+\tau.\mathbf{0}))+ \\ & \tau.a.\mathbf{0}+\tau.a.a.\mathbf{0}+\sum_{\mu\in\{a,\bar{a},\tau\}} \mu.(\bar{a}.a.\mathbf{0}+\bar{a}.a.a.\mathbf{0}) \ . \end{aligned}$$

Since the Equation (4) is sound over the language CCS_f modulo bisimilarity, using Equation (9) we have that

$$\begin{aligned} q \underline{\leftrightarrow} {} & f(\bar{a}.a.\mathbf{0}, a.\mathbf{0}+\bar{a}.\mathbf{0}+\tau.\mathbf{0})+ \\ & f(\bar{a}.a.a.\mathbf{0}, a.\mathbf{0}+\bar{a}.\mathbf{0}+\tau.\mathbf{0})+f(a.\mathbf{0}+\bar{a}.\mathbf{0}+\tau.\mathbf{0}, \bar{a}.a.\mathbf{0}+\bar{a}.a.a.\mathbf{0}) \ . \end{aligned}$$

This equivalence entails that, for each $\mu \in \{a,\bar{a},\tau\}$, the operation f has no rule of type (8). In fact, if f had a rule of that type for some $\mu \in \{a,\bar{a},\tau\}$, then we could use it to prove the transition

$$f(\bar{a}.a.\mathbf{0}, a.\mathbf{0}+\bar{a}.\mathbf{0}+\tau.\mathbf{0}) \xrightarrow{\mu} \bar{a}.a.\mathbf{0} \parallel \mathbf{0} \ .$$

Therefore the term on the right-hand side of the above equivalence would afford a μ-labelled transition leading to the term

$$\bar{a}.a.\mathbf{0} \parallel \mathbf{0} \ .$$

However, it is easy to see that, for each $\mu \in \{a,\bar{a},\tau\}$, the term q has no transition leading to a term that is bisimilar to $\bar{a}.a.\mathbf{0} \parallel \mathbf{0}$.

By Proposition 2, this means that f has a rule of the form (7) for each $\mu \in \{a,\bar{a},\tau\}$, and at least one rule of type (6). Notice now that if f has both rules of type (6), then it is nothing but Hennessy's merge operator $|^{/}$ [14], and our Theorem 1 specializes to [4, Theorem 18]. We may therefore assume, without loss of generality, that f has only one rule of the form (6).

Clearly, a similar reasoning holds for an operator f that, modulo bisimilarity, distributes over summation in its second argument. If this is the case, then we would obtain that f has a rule of the form (8) for each $\mu \in \{a,\bar{a},\tau\}$, no rule of the form (7), and exactly one rule of the form (6).

In Section 8 we will formalize the proof of Theorem 1 in the case of an operator f that distributes over summation in its first argument (viz. Theorem 3). The proof in the case of distributivity in the second argument can be easily obtained from the one we present by changing the set of the rules for f, accordingly.

5.2 Distributivity in neither argument

We now consider the case in which f does not distribute with respect to summation in either argument.

Also in this case, we can exploit [2, Lemma 4.3] to obtain a characterization of the set of rules allowed for an operator f satisfying the desired constraints. In detail, we infer that there must be $\mu,\nu \in \{a,\bar{a},\tau\}$, not necessarily distinct, such that f has a rule of the form (7) with label μ and a rule of the form (8) with label ν. Otherwise, as f must have at least one rule for each action (see Proposition 2), at least one argument would be involved in the premises of each rule, and this would entail distributivity with respect to summation in that argument (see Lemma 9).

We will split the proof of Theorem 1 for an operator f that, modulo bisimilarity, does not distribute over summation in either argument into three main cases:

1. In Section 9 we consider the case of f having both a rule of type (7) and one of type (8) with label $\alpha \in \{a, \bar{a}\}$, or equivalently with label $\bar{\alpha}$ (Theorem 4).
2. In Section 10 we deal with the case of f having only one rule for α, only one rule for $\bar{\alpha}$, and such rules are of different forms. As we will see, we will need to distinguish two subcases, according to which rules of type (6) are available for f (Theorem 5 and Theorem 6).
3. Finally, in Section 11, we study the case of f having only one rule with label α, only one rule with label $\bar{\alpha}$, and such rules are of the same type (Theorem 7).

Before proceeding to the proofs of the various cases, we dedicate Section 6 and Section 7 to the introduction of some general preliminary results and observations, concerning, respectively, the *equational theory over CCS$_f$* and the *decomposition* of the semantics of terms, that will play a key role in the technical development of our proofs.

6 The equational theory of CCS$_f$

In this section we study some aspects of the equational theory of CCS$_f$ modulo bisimilarity that are useful in the proofs of our negative results. In particular, we show that, due to the relation between f and the parallel composition operator given by Equation (4), proving the negative result over CCS$_f$ is equivalent to proving it over the simpler fragment CCS$_f^-$, which does not contain occurrences of $\|$ (Proposition 3 below).

Furthermore, we discuss the relation between the available rules for f and the bisimulation equivalence of terms of the form $f(p,q)$ with $\mathbf{0}$. As we will see, in the case of an operator f that distributes with respect to summation in one argument it is possible to *saturate* the axiom systems [21] yielding a simplification in the proofs (Proposition 4 below). On the other hand, saturation is not possible for an operator f that distributes with respect to $+$ in neither of its arguments.

6.1 Simplifying equational proofs

As a first stepping stone towards the proof of Theorem 1, we now proceed to argue that it is sufficient to show that bisimulation equivalence admits no finite equational axiomatization over the language CCS$_f^-$, consisting of the CCS$_f$ terms that do not contain occurrences of the parallel composition operator. Even though this observation is not unexpected, as Equation (4) essentially states that parallel composition is a derived operator in the algebra of CCS$_f$ terms modulo bisimulation equivalence, we now argue for it in some detail for the sake of completeness.

Definition 6 (The CCS$_f^-$ fragment)**.** For each CCS$_f$ term t, we define $\hat{t}$ as follows:

$$\begin{array}{rclcrcl}
\hat{\mathbf{0}} & = & \mathbf{0} & \qquad & \widehat{t+u} & = & \hat{t}+\hat{u} \\
\hat{x} & = & x & & \widehat{f(t,u)} & = & f(\hat{t},\hat{u}) \\
\widehat{\mu t} & = & \mu\hat{t} & & \widehat{t\|u} & = & f(\hat{t},\hat{u})+f(\hat{u},\hat{t}) \ .
\end{array}$$

Then, for any axiom system E over the language CCS_f, we let $\widehat{E} = \{\hat{t} \approx \hat{u} \mid (t \approx u) \in E\}$.

We notice that, for each CCS_f term t, the term $\hat{t}$ is in the language CCS_f^-. Moreover, if t contains no occurrences of the parallel composition operator, then $\hat{t} = t$. Since Equation (4) is sound with respect to bisimilarity, which is a congruence relation, it is not hard to show that:

Fact 2. Each term t in the language CCS_f is bisimilar to $\hat{t}$. Therefore if E is an axiom system over the language CCS_f that is sound with respect to bisimilarity, then $\widehat{E}$ is an axiom system over the language CCS_f^- that is sound with respect to bisimilarity.

The following result states the promised reduction of the non-finite axiomatizability of bisimilarity over the language CCS_f to that of bisimilarity over the language CCS_f^-.

Proposition 3. *Let E be an axiom system over the language CCS_f. Then the following statements hold.*

1. *If E proves the equation $t \approx u$, then $\widehat{E}$ proves the equation $\hat{t} \approx \hat{u}$.*
2. *If E gives a complete axiomatization of bisimulation equivalence over the language CCS_f, then $\widehat{E}$ completely axiomatizes bisimulation equivalence over the language CCS_f^-.*
3. *If bisimulation equivalence admits no finite equational axiomatization over the language CCS_f^-, then it has no finite equational axiomatization over the language CCS_f either.*

Proof: The proof can be found in Appendix C.1. □

In light of this result, henceforth we shall focus on proving that bisimulation equivalence affords no finite equational axiomatization over the language CCS_f^-.

6.2 Bisimilarity with 0

For a further simplification, we can focus on the **0** *absorption properties* of CCS_f^- operators. Informally, we can restrict the axiom system to a collection of equations that do not introduce unnecessary terms that are bisimilar to **0** in the equational proofs. However the **0** absorprtion properties of f depend crucially on the allowed set of SOS rules for f. In particular, we have different results, according to the distributivity properties of f.

6.2.1 0 absorption for distributive f

We examine first the case of an operator f that, modulo bisimilarity, distributes over summation in its first argument.

An example of a collection of equations over the language CCS_f^- that are sound with respect to $\underline{\leftrightarrow}$ is given by axioms A1–A4, F0–F2 in Table 2. Interestingly, the axioms A4 and F1 in Table 2 (used from left to right) are enough to establish that each CCS_f^- term that is bisimilar to **0** is also provably equal to **0**.

Lemma 10. *Let t be a CCS_f^- term. Then $t \underline{\leftrightarrow} \mathbf{0}$ if, and only if, the equation $t \approx \mathbf{0}$ is provable using axioms A4 and F1 in Table 2 from left to right.*

Proof: The proof is sketched in Appendix C.2. □

In light of the above result, in the technical developments to follow, when dealing with an operator f that distributes over $+$ in its first argument we shall assume, without loss of generality, that each axiom system we consider includes the equations A1–A4, F0–F2 in Table 2. This assumption means, in particular, that our axiom systems will allow us to identify each term that is bisimilar to **0** with **0**.

Some common axioms

A1 $x+x \approx x$
A2 $x+y \approx y+x$
A3 $(x+y)+z \approx x+(y+z)$
A4 $x+\mathbf{0} \approx x$
F0 $f(\mathbf{0},\mathbf{0}) \approx \mathbf{0}$

Some axioms for $A^l_{\mathscr{A}}\ A^r_{\emptyset}$

F1 $f(\mathbf{0},x) \approx \mathbf{0}$
F2 $f(x,\mathbf{0}) \approx x$

Some axioms for $A^l_{\emptyset}\ A^r_{\mathscr{A}}$

F3 $f(x,\mathbf{0}) \approx \mathbf{0}$
F4 $f(\mathbf{0},x) \approx x$

Some axioms for $A^l_{\mathscr{A}}\ A^r_{\alpha}$

F5 $f(\mathbf{0},\bar{\alpha}.x+w) \approx f(\mathbf{0},w)$
F6 $f(\mathbf{0},\tau.x+w) \approx f(\mathbf{0},w)$
and F2

Some axioms for $A^l_{\mathscr{A}}\ A^r_{\bar{\alpha}}$

F7 $f(\mathbf{0},\alpha.x+w) \approx f(\mathbf{0},w)$
and F2, F6

Some axioms for $A^l_{\mathscr{A}}\ A^r_{\tau}$

F2, F5, F7

Some axioms for $A^l_{\alpha}\ A^r_{\mathscr{A}}$

F8 $f(\bar{\alpha}.x+w,\mathbf{0}) \approx f(w,\mathbf{0})$
F9 $f(\tau.x+w,\mathbf{0}) \approx f(w,\mathbf{0})$
and F4

Some axioms for $A^l_{\bar{\alpha}}\ A^r_{\mathscr{A}}$

F10 $f(\alpha.x+w,\mathbf{0}) \approx f(w,\mathbf{0})$
and F4, F9

Some axioms for $A^l_{\tau}\ A^r_{\mathscr{A}}$

F4, F8, F10

Some axioms for $A^l_{\mathscr{A}}\ A^r_{\alpha,\bar{\alpha}}$

F2, F6

Some axioms for $A^l_{\mathscr{A}}\ A^r_{\alpha,\tau}$

F2, F5

Some axioms for $A^l_{\mathscr{A}}\ A^r_{\bar{\alpha},\tau}$

F2, F7

Some axioms for $A^l_{\alpha,\bar{\alpha}}\ A^r_{\mathscr{A}}$

F4, F9

Some axioms for $A^l_{\alpha,\tau}\ A^r_{\mathscr{A}}$

F4, F8

Some axioms for $A^l_{\bar{\alpha},\tau}\ A^r_{\mathscr{A}}$

F4,F10

Some axioms for $A^l_{\alpha,\tau}\ A^r_{\bar{\alpha}}\ S_{\alpha,\bar{\alpha}}$

F11 $f(\bar{\alpha}.x+w,\tau.y) \approx f(w,\tau.y)$
F12 $f(\bar{\alpha}.x,\tau.y+w) \approx f(\bar{\alpha}.x,w)$
F13 $f(\bar{\alpha}.x+w,\alpha.y) \approx f(w,\alpha.y)$
F14 $f(\bar{\alpha}.x,\alpha.y+w) \approx f(\bar{\alpha}.x,w)$
F15 $f(\bar{\alpha}.x,\tau.y) \approx \mathbf{0}$
F16 $f(\bar{\alpha}.x,\alpha.y) \approx \mathbf{0}$
and F6, F7, F8

Some axioms for $A^l_{\alpha,\tau}\ A^r_{\bar{\alpha}}\ S_{\bar{\alpha},\alpha}$

F6, F7, F8, F11, F12, F15

Some axioms for $A^l_{\alpha,\tau}\ A^r_{\alpha,\bar{\alpha}}$

F6, F8, F11, F12, F15

Some axioms for $A^l_{\alpha,\tau}\ A^r_{\bar{\alpha},\tau}\ S_{\alpha,\bar{\alpha}}$

F7, F8, F13, F14, F16

Some axioms for $A^l_{\alpha,\tau}\ A^r_{\bar{\alpha},\tau}\ S_{\bar{\alpha},\alpha}$

F7, F8

Some axioms for $A^l_{\bar{\alpha}}\ A^r_{\alpha,\tau}\ S_{\bar{\alpha},\alpha}$

F17 $f(\alpha.x+w,\bar{\alpha}.y) \approx f(w,\bar{\alpha}.y)$
F18 $f(\alpha.x,\bar{\alpha}.y+w) \approx f(\alpha.x,w)$
F19 $f(\tau.x+w,\bar{\alpha}.y) \approx f(w,\bar{\alpha}.y)$
F20 $f(\tau.x,\bar{\alpha}.y+w) \approx f(\tau.x,w)$
F21 $f(\alpha.x,\bar{\alpha}.y) \approx \mathbf{0}$
F22 $f(\tau.x,\bar{\alpha}.y) \approx \mathbf{0}$
and F5, F9, F10

Some axioms for $A^l_{\bar{\alpha}}\ A^r_{\alpha,\tau}\ S_{\alpha,\bar{\alpha}}$

F5, F9, F10, F19, F20, F22

Some axioms for $A^l_{\bar{\alpha},\alpha}\ A^r_{\alpha,\tau}$

F9, F19, F20, F22

Some axioms for $A^l_{\bar{\alpha},\tau}\ A^r_{\alpha,\tau}\ S_{\alpha,\bar{\alpha}}$

F5, F10, F17, F18, F21

Some axioms for $A^l_{\bar{\alpha},\tau}\ A^r_{\alpha,\tau}\ S_{\bar{\alpha},\alpha}$

F5, F10

Some axioms for $A^l_{\alpha}\ A^r_{\bar{\alpha},\tau}\ S_{\alpha,\bar{\alpha}}$

F23 $f(\tau.x+w,\alpha.y) \approx f(w,\alpha.y)$
F24 $f(\tau.x,\alpha.y+w) \approx f(\tau.x,w)$
F25 $f(\tau.x,\alpha.y) \approx \mathbf{0}$
and F7, F8, F9, F13, F14, F16

Some axioms for $A^l_{\alpha}\ A^r_{\bar{\alpha},\tau}\ S_{\bar{\alpha},\alpha}$

F7, F8, F9, F23, F24, F25

Some axioms for $A^l_{\tau}\ A^r_{\alpha,\bar{\alpha}}$

F26 $f(\alpha.x+w,\tau.y) \approx f(w,\tau.y)$
F27 $f(\alpha.x,\tau.y+w) \approx f(\alpha.x,w)$
F28 $f(\alpha.x,\tau.y) \approx \mathbf{0}$
and F6, F8, F10, F11, F12, F15

Some axioms for $A^l_{\bar{\alpha},\tau}\ A^r_{\alpha}\ S_{\bar{\alpha},\alpha}$

F5, F6, F10, F17, F18, F21, F26, F27, F28

Some axioms for $A^l_{\bar{\alpha},\tau}\ A^r_{\alpha}\ S_{\alpha,\bar{\alpha}}$

F5, F6, F10, F26, F27, F28

Some axioms for $A^l_{\alpha,\bar{\alpha}}\ A^r_{\tau}$

F5, F7, F9, F19, F20, F22, F23, F24, F25

Some axioms for $A^l_{\alpha,\bar{\alpha}}\ A^r_{\bar{\alpha},\tau}$

F7, F9, F23, F24, F25

Some axioms for $A^l_{\bar{\alpha},\tau}\ A^r_{\alpha,\bar{\alpha}}$

F6, F10, F26, F27, F28

Table 2: Some sets of axioms, according to which rules are available for f.

Definition 7. We say that a term t has a $\mathbf{0}$ *factor* if it contains a subterm of the form $f(t',t'')$, where either t' or t'' is bisimilar to $\mathbf{0}$.

For example, the term $f(x, f(\mathbf{0},y))$ has a $\mathbf{0}$ factor. The notion of $\mathbf{0}$ factor will play a central role in allowing us to identify those terms containing some occurrences of variables that can never contribute to the behaviour of their closed instantiations (cf. Proposition 6 and Lemma 20 in Section 7).

It is easy to see that, modulo equations A1–A4, F0–F2 in Table 2, every CCS_f^- term t has the form $\sum_{i\in I} t_i$, for some finite index set I, and terms t_i ($i \in I$) that are not $\mathbf{0}$ and do not have themselves the form $t'+t''$, for some terms t' and t''. The terms t_i ($i \in I$) will be referred to as the *summands* of t. Moreover, again modulo the considered equations in Table 2, each of the t_i can be assumed to have no $\mathbf{0}$ factors. (Recall that this means that, whenever a term of the form $f(t',t'')$ is a subterm of t_i, then $t' \not\underline{\leftrightarrow} \mathbf{0}$ and $t'' \not\underline{\leftrightarrow} \mathbf{0}$.) For example, a term of the form $f(a\mathbf{0}+\bar{a}\mathbf{0},\mathbf{0})$ will *not* be considered a summand in what follows because, using equation F2 in Table 2, that term can be proven equal to $a\mathbf{0}+\bar{a}\mathbf{0}$.

It is well-known (cf., e.g., Sect. 2 in [13]) that if an equation relating two closed terms can be proven from an axiom system E, then there is a closed proof for it. We shall now argue that if E satisfies a further closure property in addition to those mentioned earlier, and that closed equation relates two terms containing no occurrences of $\mathbf{0}$ as a summand or factor, then there is a closed proof for it in which all of the terms have no occurrences of $\mathbf{0}$ as a summand or factor—see [21, Proposition 5.1.5].

Definition 8. For each CCS_f^- term t, we define $t/\mathbf{0}$ thus:

$$\mathbf{0}/\mathbf{0} = \mathbf{0} \qquad x/\mathbf{0} = x \qquad \mu t/\mathbf{0} = \mu(t/\mathbf{0})$$

$$(t+u)/\mathbf{0} = \begin{cases} u/\mathbf{0} & \text{if } t \underline{\leftrightarrow} \mathbf{0} \\ t/\mathbf{0} & \text{if } u \underline{\leftrightarrow} \mathbf{0} \\ (t/\mathbf{0})+(u/\mathbf{0}) & \text{otherwise} \end{cases}$$

$$f(t,u)/\mathbf{0} = \begin{cases} \mathbf{0} & \text{if } t \underline{\leftrightarrow} \mathbf{0} \\ t/\mathbf{0} & \text{if } u \underline{\leftrightarrow} \mathbf{0} \\ f(t/\mathbf{0},u/\mathbf{0}) & \text{otherwise} \end{cases}$$

Intuitively, $t/\mathbf{0}$ is the term that results by removing *all* occurrences of $\mathbf{0}$ as a summand or factor from t.

The following lemma, whose simple proof by structural induction on terms is omitted, collects the basic properties of the above construction.

Lemma 11. *For each CCS_f^- term t, the following statements hold:*

1. *the equation $t \approx t/\mathbf{0}$ can be proven using the equations A1–A4, F0–F2 in Table 2, and therefore $t \underline{\leftrightarrow} t/\mathbf{0}$;*
2. *the term $t/\mathbf{0}$ has no occurrence of $\mathbf{0}$ as a summand or factor;*
3. *$t/\mathbf{0} = t$, if t has no occurrence of $\mathbf{0}$ as a summand or factor;*
4. *$\sigma(t/\mathbf{0})/\mathbf{0} = \sigma(t)/\mathbf{0}$, for each substitution σ.*

Definition 9. We say that a substitution σ is a $\mathbf{0}$-*substitution* iff $\sigma(x) \neq x$ implies that $\sigma(x) = \mathbf{0}$, for each variable x.

Definition 10. Let E be an axiom system. We define the axiom system $\mathrm{cl}(E)$ thus:

$$\mathrm{cl}(E) = E \cup \{\sigma(t)/\mathbf{0} \approx \sigma(u)/\mathbf{0} \mid (t \approx u) \in E,\ \sigma \text{ a } \mathbf{0}\text{-substitution}\} \ .$$

An axiom system E is *saturated* if $E = \mathrm{cl}(E)$.

The following lemma collects some basic sanity properties of the closure operator $\mathrm{cl}(\cdot)$. (Note, in particular, that the application of $\mathrm{cl}(\cdot)$ to an axiom system preserves closure with respect to symmetry.)

Lemma 12. *Let E be an axiom system. Then the following statements hold.*

1. $\mathrm{cl}(E) = \mathrm{cl}(\mathrm{cl}(E))$.
2. $\mathrm{cl}(E)$ *is finite, if so is E.*
3. $\mathrm{cl}(E)$ *is sound, if so is E.*
4. $\mathrm{cl}(E)$ *is closed with respect to symmetry, if so is E.*
5. $\mathrm{cl}(E)$ *and E prove the same equations, if E contains the equations A1–A4, F0–F2 in Table 2.*

Proof: The proof is sketched in Appendix C.3. □

We are now ready to state our counterpart of [21, Proposition 5.1.5].

Proposition 4. *Assume that E is a saturated axiom system. Suppose furthermore that we have a closed proof from E of the closed equation $p \approx q$. Then replacing each term r in that proof with $r/\mathbf{0}$ yields a closed proof of the equation $p/\mathbf{0} \approx q/\mathbf{0}$. In particular, the proof from E of an equation $p \approx q$, where p and q are terms not containing occurrences of $\mathbf{0}$ as a summand or factor, need not use terms containing occurrences of $\mathbf{0}$ as a summand or factor.*

Proof: The proof follows the lines of that of [21, Proposition 5.1.5], and is therefore omitted. □

In light of this result, since the saturation of a finite axiom system that includes the equations A1–A4, F0–F2 in Table 2 results in an equivalent, finite collection of equations (Lemma 12(2) and (5)), henceforth, when dealing with an operator f that distributes modulo $\underline{\leftrightarrow}$ over summation in one of its arguments, we shall limit ourselves to considering saturated axiom systems.

The use of saturated axiom systems will play an important role in the proof of our Theorem 3.

6.2.2 0 absorption for a non distributive f

In Section 5.2, we argued that the set of allowed rules for an operator f that does not distribute over summation in either argument has to include at least a rule of type (7) *and* at least one of type (8). We also notice that for an operator f having both types of rules for all actions we can distinguish two cases, according to which rules of type (6) are available:

- If f has both rules of type (6), then it would be a mere rewriting of the parallel composition operator (cf. Appendix B.3, proof of Lemma 6).
- If f has only one rule of type (6), then one can easily observe that Moller's argument could be directly applied to f, immediately yielding the desired negative result.

Hence, we can assume that there is an action $\mu \in \{a, \bar{a}, \tau\}$ such that f has only one rule, of type either (7) or (8), with μ as label.

As shown by the axioms in Table 2, this asymmetry in the set of rules for f can cause some CCS_f^- term to behave as $\mathbf{0}$ when occurring in the scope of f, despite not being bisimilar to $\mathbf{0}$ at all.

Example **1.** Consider the term $t = f(\alpha + \bar{\alpha}.u, \tau)$, for some term u, and assume that the set of rules for f is of the form $A^l_{\alpha,\tau}, A^r_{\bar{\alpha}}$. One can easily check that, since the initial execution of the τ-move in the second argument is prevented by the rules for f, then the subterm $\bar{\alpha}.u$ can never contribute to the behaviour of t. Thus, $t \underline{\leftrightarrow} \alpha.\tau$, even though $\bar{\alpha}.u \not\underline{\leftrightarrow} \mathbf{0}$ for each term u. ◀

From a technical point of view, this implies that Lemmas 10 and 11.1 no longer hold. In fact, one can always construct a term t of the form $t = f(\sum_{i=1}^{n} \mu.x_i, \sum_{j=1}^{m} \nu.y_j)$ for some $n,m \geq 0$, with μ, ν chosen according to the available set of rules for f (notice that one of the two arguments could be bisimilar to $\mathbf{0}$, cf. Table 2), such that $t \,\underline{\leftrightarrow}\, \mathbf{0}$. However, we conjecture that *since we are considering an operator f that does not distribute over summation in either of its arguments, the valid equation, modulo bisimilarity, $t \approx \mathbf{0}$ cannot be proved by means of any finite (sub)set of axioms.* Roughly speaking, this is due to the fact that no valid axiom can be established for a term of the form $f(\mu.x+z, \nu.y+w)$ in that the behaviour of the terms substituted for the variables z and w is crucial to determine that of a closed instantiation of the term.

Summarizing, this would imply that, in the case at end, we cannot use saturation to simplify the axiom systems and, moreover, the family of equations

$$f(\sum_{i=1}^{n} \mu.p_i, \sum_{j=1}^{m} \nu.q_j) \approx \mathbf{0} \qquad n,m \geq 0$$

for some processes p_i, q_j, could play the role of witness family of equations for our desired negative result. Unfortunately, the presence of two summations would force us to introduce a number of additional technical results that would make the proof of the negative results even heavier than it already is. Moreover, those supplementary results are not necessary to treat the case of the witness families that we are going to introduce in Sections 9–11 to obtain the proof of Theorem 2.

Nevertheless, this gives us to opportunity to prove an even stronger result: we can in fact prove that *even if we would be able to provide a finite axiom system E allowing us to infer that a term that is bisimilar to $\mathbf{0}$ is also provably equal to $\mathbf{0}$, then E could not be finitely extended in order to obtain a finite axiomatization of bisimilarity over CCS_f^-.*

7 Preliminary results

As briefly outlined in Section 4, to obtain the desired negative results we will proceed by a case analysis on the actual set of rules that are available for operator f. However, there are a few preliminary results that hold for all the allowed behaviours of f and that will be useful in the upcoming proofs. We dedicate this section to presenting such results and notions.

7.1 Unique prime decomposition

In the proof of our main results, we shall often make use of some notions from [20, 21]. These we now proceed to introduce for the sake of completeness and readability.

Definition 11 (Prime process)**.** A closed term p is *irreducible* if $p \,\underline{\leftrightarrow}\, q \| r$ implies $q \,\underline{\leftrightarrow}\, \mathbf{0}$ or $r \,\underline{\leftrightarrow}\, \mathbf{0}$, for all closed terms q, r. We say that p is *prime* if it is irreducible and is not bisimilar to $\mathbf{0}$.

For example, each term p of depth (respectively, norm) 1 is prime because every term of the form $q \| r$ that does not involve $\mathbf{0}$ factors has depth (resp., norm) at least 2, and thus cannot be bisimilar to p.

The following lemma states the primality of two families of closed terms that will play a key role in the proof of our main result.

Lemma 13.

1. *The term $\mu^{\leq m}$ is prime, for each $m \geq 1$.*

2. *Let $\nu \in \{a,\bar{a}\}$, $\mu \in \{a,\bar{a},\tau\}$, $\nu \neq \mu$, $m \geq 1$ and $1 \leq i_1 < \ldots < i_m$. Then the term $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m}$ is prime.*

Proof: The proof can be found in Appendix D.1. □

In [20] the notion of *unique prime decomposition* of a process p was introduced, as the unique multiset $\{| q_1,\ldots,q_n |\}$ of primes for which $p \underline{\leftrightarrow} q_1 \| \ldots \| q_n$. Later on, [17] proposed an algebraic characterization of prime decompositions. Interestingly, CCS_f modulo $\underline{\leftrightarrow}$ is a *commutative monoid* with respect to $\|$, having $\mathbf{0}$ as unit, and the transition relation defines a *decomposition preorder* over closed terms. Then, by [17, Theorem 32], we can conclude that the following result holds:

Proposition 5. *Any CCS_f term can be expressed uniquely, up to $\underline{\leftrightarrow}$, as a parallel composition of primes.*

As we will see, this property will play a crucial role in some of the upcoming proofs.

7.2 Decomposing the semantics of terms

In the proofs to follow, we shall sometimes need to establish a correspondence between the behaviour of open terms and the semantics of their closed instances, with a special focus on the role of variables. In detail, we need to consider the possible origins of a transition of the form $\sigma(t) \xrightarrow{\alpha} p$, for some action $\alpha \in \{a,\bar{a}\}$, closed substitution σ, CCS_f term t and closed term p. In fact, the equational theory is defined over process terms, whereas the semantic properties can be verified only on their closed instances. We dedicate the remainder of this section to provide the notions and theoretical results necessary to establish the desired behavioural correspondence.

Naturally enough, we expect that $\sigma(t) \xrightarrow{\alpha} \sigma(t')$ whenever $t \xrightarrow{\alpha} t'$.

Lemma 14. *Let t,t' be process terms, let $\mu \in \{a,\bar{a},\tau\}$. Then for all substitutions σ it holds that if $t \xrightarrow{\mu} t'$ then $\sigma(t) \xrightarrow{\mu} \sigma(t')$.*

However, a transition $t \xrightarrow{\alpha} t'$ may also derive from the initial behaviour of some closed term $\sigma(x)$, provided that the collection of initial moves of $\sigma(t)$ depends, in some formal sense, on that of the closed term substituted for the variable x. Roughly speaking, our aim is now to provide the conditions under which we could infer $\sigma(t) \xrightarrow{\mu} p$ from $\sigma(x) \xrightarrow{\nu} q$, for some $\mu,\nu \in \{a,\bar{a},\tau\}$ and processes p,q. As one can expect, in our setting the provability of transitions needs to be parametric with respect to the allowed sets of rules for f.

Example **2.** Consider the CCS_f^- term $t = f(x,\tau)$. Firstly, we notice that if f has a rule of type (8) with label τ, then we can infer that $\sigma(t) \xrightarrow{\tau} \sigma(x)\|\mathbf{0}$ for all closed substitutions σ (cf. Lemma 14 above).

Assume now that $\sigma(x) = a$. Clearly, we can derive $\sigma(t) \xrightarrow{a} \mathbf{0}\|\tau$ only if f has a rule of type (7) with label a. ◀

To fully describe this situation, for each action $\mu \in \{a,\bar{a},\tau\}$, we introduce two unary predicates, L^f_μ and R^f_μ, that allow us to identify which rules with label μ are available for f. In detail,

- L^f_μ holds if f has a rule of the form (7) with label μ;
- R^f_μ holds if f has a rule of the form (8) with label μ.

We write $L^f_\mu \wedge R^f_\mu$ to denote that f has both a rule of the form (7) and one of the form (8) with label μ. We stress that, for each action μ, the validity of predicate L^f_μ does not prevent R^f_μ to hold, and vice versa. Throughout the paper, in case that *only one* of the two predicates holds, we will clearly state it.

$$(a_1)\ \frac{L^f_\mu}{x \xrightarrow{x_\mathrm{l}}_\mu x_\mathrm{d}} \qquad (a_2)\ \frac{R^f_\mu}{x \xrightarrow{x_\mathrm{r}}_\mu x_\mathrm{d}} \qquad (a_3)\ \frac{L^f_\mu \wedge R^f_\mu}{x \xrightarrow{x_\mathrm{b}}_\mu x_\mathrm{d}}$$

$$(a_4)\ \frac{t_1 \xrightarrow{x_\mathrm{w}}_\mu c}{t_1 + t_2 \xrightarrow{x_\mathrm{w}}_\mu c}\ \mathrm{w} \in \{\mathrm{l},\mathrm{r},\mathrm{b}\} \qquad (a_5)\ \frac{t_2 \xrightarrow{x_\mathrm{w}}_\mu c}{t_1 + t_2 \xrightarrow{x_\mathrm{w}}_\mu c}\ \mathrm{w} \in \{\mathrm{l},\mathrm{r},\mathrm{b}\}$$

$$(a_6)\ \frac{t_1 \xrightarrow{x_\mathrm{l}}_\mu c}{f(t_1,t_2) \xrightarrow{x_\mathrm{l}}_\mu c \| t_2} \qquad (a_7)\ \frac{t_2 \xrightarrow{x_\mathrm{r}}_\mu c}{f(t_1,t_2) \xrightarrow{x_\mathrm{r}}_\mu t_1 \| c} \qquad (a_8)\ \frac{t_1 \xrightarrow{x_\mathrm{b}}_\mu c}{f(t_1,t_2) \xrightarrow{x_\mathrm{b}}_\mu c \| t_2} \qquad (a_9)\ \frac{t_2 \xrightarrow{x_\mathrm{b}}_\mu c}{f(t_1,t_2) \xrightarrow{x_\mathrm{b}}_\mu t_1 \| c}$$

Table 3: Operational semantics of open terms with respect to action μ.

Then, for each action $\mu \in \{a, \bar{a}, \tau\}$, we introduce the auxiliary transition relation over open terms $\rightarrow_\mu$ defined via the inference rules in Table 3. To this end, we present the notion of *configuration* over CCS_f terms, which stems from [6]. Configurations are terms defined over a set of variables $\mathscr{V}_\mathrm{d} = \{x_d \mid x \in \mathscr{V}\}$, disjoint from $\mathscr{V}$, and CCS_f terms. Intuitively, the symbol x_d (read "during x") will be used to denote that the closed term substituted for an occurrence of variable x has begun its execution.

Definition 12. The collection of *CCS_f configurations* is given by the following grammar:

$$c ::= t \mid x_d \mid c \,\|\, t \mid t \,\|\, c \ ,$$

where t is a CCS_f term, and $x_d \in \mathscr{V}_d$.

For example, the configuration $x_d \,\|\, f(a,x)$ is meant to describe a state of the computation of some term in which the (closed term substituted for the) occurrence of variable x on the left-hand side of the $\|$ operator has begun its execution, but the one on the right-hand side has not.

We introduce also special labels for the auxiliary transitions $\rightarrow_\mu$, to keep track of which rules for f are available, and thus which one triggered the move by the closed instance of x. In detail, we let x_l denote that the closed instance of x is responsible for the transition when L^f_μ holds. In case R^f_μ holds, we use x_r. Finally, x_b is used when $L^f_\mu \wedge R^f_\mu$ holds. The reason behind these distinctions is that they are useful in the derivation of a proper semantics for open terms having f as head operator.

Example **3.** Consider the term $t = f(x,\tau)$ from Example 2. Assume, for instance, that f has a rule of type (7) with label a. Thus, the predicate L^f_a holds, yielding the transition $x \xrightarrow{x_\mathrm{l}}_a x_\mathrm{d}$, due to rule (a_1) in Table 3. Then, an application of rule (a_6) in the same table would give $f(x,\tau) \xrightarrow{x_\mathrm{l}}_a x_\mathrm{d} \| \tau$ with the following meaning: since the rules for f allow for testing a-moves of the first argument, then an a-transition by (an instance of) variable x occurring in the first argument of f will induce an a-move of $f(x,\tau)$.

Conversely, assume that f has only a rule of type (8) with label a, namely only R^f_a holds. Then, by applying rule (a_2) in Table 3 we obtain that $x \xrightarrow{x_\mathrm{r}}_a x_\mathrm{d}$ and, from the rules in the same table, it is not possible to derive any $\rightarrow_a$ transition of $f(x,\tau)$ from that of x, modelling the fact that the rules for f allow us to test a only on the second argument of f. ◀

Lemmas 15 and 16 below formalize the decomposition of the semantics of CCS^-_f terms. We remark that, in light of Proposition 2, we have that at least one of L^f_μ, R^f_μ and $L^f_\mu \wedge R^f_\mu$ holds for each action μ.

Lemma 15. *Let $\mu \in \{a,\bar{a},\tau\}$, t be a CCS^-_f term, x be a variable,* $\mathrm{w} \in \{\mathrm{l},\mathrm{r},\mathrm{b}\}$ *and σ be a closed substitution. Then $\sigma(x) \xrightarrow{\mu} p$ for some process p, and $t \xrightarrow{x_\mathrm{w}}_\mu c$ for some configuration c, then $\sigma(t) \xrightarrow{\mu} \sigma[x_\mathrm{d} \mapsto p](c)$.*

Proof: The proof follows by as easy induction on the structure of t and the derivation of the auxiliary transition $t \xrightarrow{x_{\mathrm{w}}}_{\mu} c$. □

Lemma 16. *Let $\alpha \in \{a, \bar{a}\}$, t be a CCS$_f^-$ term, σ be a closed substitution and p be a closed term. Whenever $\sigma(t) \xrightarrow{\alpha} p$, then one of the following holds:*

1. *There is a CCS$_f$ term t' such that $t \xrightarrow{\alpha} t'$ and $\sigma(t') = p$.*
2. *There are a variable x, a process q and a configuration c such that:*
   (a) *either only L^f_μ holds, $\sigma(x) \xrightarrow{\alpha} q$, $t \xrightarrow{x_{\mathrm{l}}}_{\alpha} c$ and $\sigma[x_{\mathrm{d}} \mapsto q](c) = p$;*
   (b) *or only R^f_μ holds, $\sigma(x) \xrightarrow{\alpha} q$, $t \xrightarrow{x_{\mathrm{r}}}_{\alpha} c$ and $\sigma[x_{\mathrm{d}} \mapsto q](c) = p$;*
   (c) *or $L^f_\mu \wedge R^f_\mu$ holds, $\sigma(x) \xrightarrow{\alpha} q$, $t \xrightarrow{x_{\mathrm{b}}}_{\alpha} c$ and $\sigma[x_{\mathrm{d}} \mapsto q](c) = p$.*

Proof: The proof can be found in Appendix D.2. □

Interestingly, we can also give a syntactic characterization of the occurrences in a term of the variables that can contribute to the behaviour of closed instances of that term. Formally, to infer the behaviour of a term t from that of (a closed instance of) a variable x, the latter must occur *unguarded* in t, namely x cannot occur in the scope of a prefixing operator in t. Inspired by [3], for $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$, we introduce a relation $\triangleleft^{\mu}_{\mathrm{w}}$ between a variable x and a term t. Intuitively, the role of the label w is the same as in the auxiliary transitions, namely, to identify which predicates hold (and thus which rules for f are available) for f with respect to action μ. Then, $x \triangleleft^{\mu}_{\mathrm{w}} t$ holds if x occurs ungarded in t, the predicate associated with w holds for f, and whenever t has a subterm of the form $f(t_1, t_2)$ and x occurs in t_i, with $i = 1$ if $\mathrm{w} \in \{\mathrm{l}, \mathrm{b}\}$ and $i = 2$ if $\mathrm{w} \in \{\mathrm{r}, \mathrm{b}\}$, then the occurrence of x can contribute to an initial μ-transition of $\sigma(t)$ when $\sigma(x) \xrightarrow{\mu}$.

Definition 13 (Relation $\triangleleft$)**.** Let $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. The relation $\triangleleft^{\mu}_{\mathrm{w}}$ between variables and terms is defined inductively as follows:

$$1.\ x \triangleleft^{\mu}_{\mathrm{l}} x \text{ if } L^f_\mu \text{ holds} \qquad 2.\ x \triangleleft^{\mu}_{\mathrm{r}} x \text{ if } R^f_\mu \text{ holds} \qquad 3.\ x \triangleleft^{\mu}_{\mathrm{b}} x \text{ if } L^f_\mu \wedge R^f_\mu \text{ holds}$$

$$4.\ x \triangleleft^{\mu}_{\mathrm{w}} t \ \Rightarrow\ x \triangleleft^{\mu}_{\mathrm{w}} t + u \ \wedge\ x \triangleleft^{\mu}_{\mathrm{w}} u + t$$

$$5.\ x \triangleleft^{\mu}_{\mathrm{l}} t \ \Rightarrow\ x \triangleleft^{\mu}_{\mathrm{l}} f(t, u) \qquad 6.\ x \triangleleft^{\mu}_{\mathrm{r}} t \ \Rightarrow\ x \triangleleft^{\mu}_{\mathrm{r}} f(u, t)$$

$$7.\ x \triangleleft^{\mu}_{\mathrm{b}} t \ \Rightarrow\ x \triangleleft^{\mu}_{\mathrm{b}} f(t, u) \ \wedge\ x \triangleleft^{\mu}_{\mathrm{b}} f(u, t).$$

Example **4.** Assume, for instance, that L^f_a, $R^f_{\bar{a}}$ and $L^f_\tau \wedge R^f_\tau$ are the only predicates holding. Then, for $t = f(x, \tau)$ we have that $x \triangleleft^{a}_{\mathrm{l}} t$, $x \triangleleft^{\tau}_{\mathrm{l}} t$ and $x \triangleleft^{\tau}_{\mathrm{b}} t$. ◀

Interestingly, there is a close relation between unguarded occurrences of variables in terms and the auxiliary transitions in Table 3, as stated in the following:

Lemma 17. *Let $\mu \in \{a, \bar{a}, \tau\}$ and $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. Then $x \triangleleft^{\mu}_{\mathrm{w}} t$ if and only if $t \xrightarrow{x_{\mathrm{w}}}_{\mu} c$ for a configuration $c \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ for some CCS$_f$ term t'.*

Proof: The proof can be found in Appendix D.3. □

Next, we proceed to a more detailed analysis of the contribution of variables to the behaviour of closed instantiations of terms in which they occur.

Lemma 18. *Let t be a term in the language CCS_f^-, σ be a closed substitution and $\alpha \in \{a, \bar{a}\}$. Assume that*

$$\sigma(t) \underline{\leftrightarrow} \sum_{i=1}^{n} \alpha.p_i + q$$

for some n greater than the size of t and closed terms p_i, q with $p_i \not\underline{\leftrightarrow} p_j$ whenever $i \neq j$. Then t has a summand x, for some variable x, such that

$$\sigma(x) \underline{\leftrightarrow} \sum_{j \in J} \alpha.p_j + q'$$

for some $J \subset \{1, \ldots, n\}$, with $|J| \geq 2$, and some closed term q'.

Proof: The proof can be found in Appendix D.4. □

The following result shows a particular case of previous Lemma 18, in which we can infer that, provided the term t has neither $\mathbf{0}$ summands nor factors, not only a variable is responsible for the additional behaviour of t, but that t coincides with such a variable.

Lemma 19. *Let t be a term in the language CCS_f^- that does not have $+$ as head operator, and let σ be a closed substitution. Let $\alpha \in \{a, \bar{a}\}$ and $\mu \in \{a, \bar{a}, \tau\}$ with $\alpha \neq \mu$. Assume that $\sigma(t)$ has neither $\mathbf{0}$ summands nor factors, and that*

$$\sigma(t) \underline{\leftrightarrow} \alpha.\mu^{\leq i_1} + \cdots + \alpha.\mu^{\leq i_m} \quad ,$$

for some $m > 1$ and $1 \leq i_1 < \ldots < i_m$. Then $t = x$, for some variable x.

Proof: The proof can be found in Appendix D.5. □

Remark 3. Whenever a process term t has neither $\mathbf{0}$ summands nor factors then we can assume that, for some finite non-empty index set I,

$$t = \sum_{i \in I} t_i \tag{10}$$

for some terms t_i such that none of them has $+$ as head operator and moreover, none of them has $\mathbf{0}$ summands nor factors.

We can now establish whether the behaviour of two bisimilar terms is determined by the same variable.

Proposition 6. *Let $\alpha \in \{a, \bar{a}\}$, x be a variable and t, u be CCS_f^- with $t \underline{\leftrightarrow} u$ and such that neither t nor u has $\mathbf{0}$ summands or factors. If $x \lhd_{\mathrm{w}}^{\alpha} t$ for some $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$, then $x \lhd_{\mathrm{w}}^{\alpha} u$. In particular, if $x \lhd_{\mathrm{w}}^{\alpha} t$ because t has a summand x, then so does u.*

Proof: The proof can be found in Appendix D.6. □

The following lemma gives the necessary conditions to relate the depth of closed instances of a term to the depth of the closed instances of the variables occurring in it.

Lemma 20. *Let t be a CCS_f^- term and σ be a closed substitution. If t has no $\mathbf{0}$ summands or factors and $x \lhd_{\mathrm{w}}^{\mu} t$ for some $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$ and $\mu \in \{a, \bar{a}, \tau\}$ with $\mathrm{init}(\sigma(x)) \subseteq \{\mu \mid x \lhd_{\mathrm{w}}^{\mu} t\}$, then $depth(\sigma(t)) \geq depth(\sigma(x))$.*

Proof: The proof can be found in Appendix D.7. □

Example **5.** We remark that, due to the potential asymmetry of the rules for f, the requirement on the set of initials of $\sigma(x)$ cannot be relaxed in any trivial way. Consider, for instance, the term $t = f(x,\tau)$ from our running example and assume that the only predicates holding are L^f_α, L^f_τ and $R^f_{\bar{\alpha}}$. Notice that $x \lhd^\alpha_1 t$ and $x \lhd^\tau_1 t$. Consider the closed substitution σ with $\sigma(x) = \alpha + \tau + \bar{\alpha}.\alpha^n$, for some $n \geq 2$, so that $\{\alpha,\tau\} \subset \mathrm{init}(\sigma(x)) = \{\alpha,\tau,\bar{\alpha}\}$. As $L^f_{\bar{\alpha}}$ and R^f_τ do not hold, the only inferable initial transitions for $\sigma(t)$ are those resulting from the α-move and the τ-move by $\sigma(x)$. Thus, we get that $depth(\sigma(t)) = 2$, whereas $depth(\sigma(x)) \geq 3$. This is due to the fact that the computation of $\sigma(x)$ starting with a $\bar{\alpha}$-move is *blocked* by the rules for f and, thus, it cannot contribute to the behaviour of t. ◀

8 Negative result for f that distributes in one argument

In this section we discuss the existence of a finite axiomatization of CCS^-_f in the case of an operator f that, modulo bisimilarity, distributes over summation in one of its arguments. As briefly outlined in Section 5.1, we will expand only the case of f distributing in the first argument. The case of distributivity in the second argument follows by a straightforward adaptation of the arguments we will use in this section.

Hence, in the current setting, we can assume that the set of allowed SOS-rules for f is the following:

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1,x_2) \xrightarrow{\mu} y_1 \| x_2}\ \forall \mu \in \{a,\bar{a},\tau\} \qquad\qquad \frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} y_1 \| y_2}$$

namely, one rule of type (7) for each action and exactly one rule of type (6).

According to the proof strategy sketched in Section 5, we now introduce a particular family of f-terms on which we will build our negative result.

For each $n \geq 0$, we define the term p_n as:

$$p_n = \sum_{i=0}^{n} \bar{\alpha}\alpha^{\leq i} \quad (n \geq 0) \ .$$

It is not difficult to check that the following family of (infinitely many) equations

$$e_n : \quad f(\alpha, p_n) \approx \alpha p_n + \sum_{i=0}^{n} \tau\alpha^{\leq i} \quad (n \geq 0)$$

is sound with respect to bisimilarity.

Remark 4. We have defined the processes p_n in a such a way that an initial synchronization, in the scope of operator f, with the process α is always possible. This choice will allow us to slightly simplify the reasoning in the proof of the upcoming Proposition 7 and thus of the negative result (cf., for instance, with the proof of Proposition 8 in Section 9). Clearly, the possibility of synchronization is directly related to which rules of type (6) are available for f. However, since f has a rule of type (7) for all actions, it is then always possible to identify a pair μ, p_n such that $f(\mu, p_n) \xrightarrow{\tau}$ due to an application of the rule of type (6) allowed for f.

Our order of business is now to prove the instance of Theorem 2 (see Section 5) considering the family of equations e_n above, so that no finite collection of equations over CCS^-_f that are sound modulo bisimilarity can prove all of the equations e_n ($n \geq 0$).

To this end, following the reasoning sketched in Section 5,

- we need to identify a witness property that is preserved by provability from finite axiom systems,
- but, for suitably large values of n, the left-hand side of equality e_n, viz. the term $f(\alpha, p_n)$, affords it, whilst the right-hand side, viz. the term $\alpha p_n + \sum_{i=0}^{n} \tau\alpha^{\leq i}$, does not.

Formally, we aim at proving the following theorem:

Theorem 3. *Assume an operator f that, modulo bisimilarity, distributes over summation in its first argument and that has one rule of type (7) for each action and exactly one rule of type (6).*

Let E be a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence. Let n be larger than the size of each term in the equations in E. Assume that p and q are closed terms that are bisimilar to $f(\alpha, p_n)$, and contain no occurrences of $\mathbf{0}$ as a summand or factor. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

The remainder of this section is entirely devoted to a proof of the above statement.

8.1 Case specific properties of $f(\alpha, p_n)$

We start by studying some properties of the processes $f(\alpha, p_n)$, which also depend on the particular configuration of rules for f that we are considering.

Lemma 21. *The term $f(\alpha, p_n)$ is prime, for each $n \geq 0$.*

Proof: The proof can be found in Appendix E.1. □

The following lemma states a useful decomposition property of $f(\alpha, p_n)$.

Lemma 22. *Let $n \geq 1$. Assume that $f(p,q) \leftrightarrow f(\alpha, p_n)$, where q is a closed term that is not bisimilar to $\mathbf{0}$. Then $p \leftrightarrow \alpha$ and $q \leftrightarrow p_n$.*

Proof: The proof can be found in Appendix E.2. □

Finally we present a technical lemma stating that, under the considered set of rules for f, if a closed term $\sigma(t)$ is not bisimilar to $\mathbf{0}$, then by instantiating the variables in t with a process which is not bisimilar to $\mathbf{0}$ we cannot obtain an closed instance of t which is bisimilar to $\mathbf{0}$.

Lemma 23. *Let t be a term in the language CCS_f^-, and let σ be a substitution with $\sigma(t) \not\leftrightarrow \mathbf{0}$. Assume that u is a term in the language CCS_f^- that is not bisimilar to $\mathbf{0}$. Then $\sigma[x \mapsto u](t) \not\leftrightarrow \mathbf{0}$ for each variable x.*

Proof: By induction on the structure t. □

8.2 Proving Theorem 3

We now proceed to present a detailed proof of Theorem 3. The following result, stating that the property mentioned in the statement of that theorem holds for all closed instantiations of axioms in E, will be the crux in such a proof.

Proposition 7. *Assume an operator f that, modulo bisimilarity, distributes over summation in its first argument and that has one rule of type (7) for each action and exactly one rule of type (6).*

Let $t \approx u$ be an equation over the language CCS_f^- that is sound with respect to bisimulation equivalence. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q are bisimilar to $f(\alpha, p_n)$ for some n larger than the size of t, and have neither $\mathbf{0}$ summands or factors. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: Observe, first of all, that since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Hence, by Remark 3, we have that for some finite non-empty index sets I, J,

$$t = \sum_{i \in I} t_i \quad \text{and} \quad u = \sum_{j \in J} u_j \enspace ,$$

where none of the t_i ($i \in I$) and u_j ($j \in J$) is $\mathbf{0}$, has $+$ as its head operator, has $\mathbf{0}$ summands and factors.

Since $p = \sigma(t)$ has a summand bisimilar to $f(\alpha, p_n)$, there is an index $i \in I$ such that $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, p_n)$.

Our aim is now to show that there is an index $j \in J$ such that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$, proving that $q = \sigma(u)$ also has a summand bisimilar to $f(\alpha, p_n)$.

We proceed by a case analysis on the form t_i may have.

1. CASE $t_i = x$ FOR SOME VARIABLE x. In this case, we have $\sigma(x) \underline{\leftrightarrow} f(\alpha, p_n)$, and t has x as a summand. As $t \approx u$ is sound with respect to bisimilarity and neither t nor u have $\mathbf{0}$ summands or factors, it follows that u also has x as a summand (Proposition 6). Thus there is an index $j \in J$ such that $u_j = x$, and, modulo bisimulation, $\sigma(u)$ has $f(\alpha, p_n)$ as a summand, which was to be shown.

2. CASE $t_i = \mu t'$ FOR SOME TERM t'. This case is vacuous because, since $\mu\sigma(t') \xrightarrow{\mu} \sigma(t')$ is the only transition afforded by $\sigma(t_i)$, this term cannot be bisimilar to $f(\alpha, p_n)$. Indeed $f(\alpha, p_n)$ can perform both, an α-labelled transition triggered by the first argument, and the τ-move due to the synchronization between α and p_n.

3. CASE $t_i = f(t', t'')$ FOR SOME TERMS t', t''. In this case, we have $f(\sigma(t'), \sigma(t'')) \underline{\leftrightarrow} f(\alpha, p_n)$. As $\sigma(t_i)$ has no $\mathbf{0}$ factors, it follows that $\sigma(t') \not\underline{\leftrightarrow} \mathbf{0}$ and $\sigma(t'') \not\underline{\leftrightarrow} \mathbf{0}$. Thus $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\sigma(t'') \underline{\leftrightarrow} p_n$ (Lemma 22). Now, t'' can be written as $t'' = v_1 + \cdots + v_\ell, (\ell > 0)$, where none of the summands v_i is $\mathbf{0}$ or a sum. Observe that, since n is larger than the size of t, we have that $\ell < n$. Hence, since $\sigma(t'') \underline{\leftrightarrow} p_n = \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i}$, there must be some $h \in \{1, \ldots, \ell\}$ such that $\sigma(v_h) \underline{\leftrightarrow} \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m}$ for some $m > 1$ and $1 \leq i_1 < \ldots < i_m \leq n$. The term $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors—or else, so would $\sigma(t'')$, and thus $p = \sigma(t)$. By Lemma 19, it follows that v_h can only be a variable x and thus that
$$\sigma(x) \underline{\leftrightarrow} \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} \enspace . \tag{11}$$
Observe, for later use, that, since t' has no $\mathbf{0}$ factors, the above equation yields that $x \notin var(t')$—or else $\sigma(t') \not\underline{\leftrightarrow} \alpha$ (Lemma 20). So, modulo bisimilarity, t_i has the form $f(t', (x + t'''))$, for some term t''', with $x \notin var(t')$ and $\sigma(t') \underline{\leftrightarrow} \alpha$.
Our order of business will now be to use the information collected so far in this case of the proof to argue that $\sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$. To this end, consider the substitution
$$\sigma' = \sigma[x \mapsto \bar{\alpha} f(\alpha, p_n)] \enspace .$$
We have that
$$\begin{aligned} \sigma'(t_i) &= f(\sigma'(t'), \sigma'(t'')) \\ &= f(\sigma(t'), \sigma'(t'')) && \text{(As } x \notin var(t')\text{)} \\ &\underline{\leftrightarrow} f(\alpha, (\bar{\alpha} f(\alpha, p_n) + \sigma'(t'''))) && \text{(As } t'' = x + t'''\text{).} \end{aligned}$$
Thus, $\sigma'(t_i) \xrightarrow{\tau} p' \underline{\leftrightarrow} f(\alpha, p_n)$ for some p', so that
$$\sigma'(t) \xrightarrow{\tau} p' \underline{\leftrightarrow} f(\alpha, p_n)$$

also holds. Since $t \approx u$ is sound with respect to $\underline{\leftrightarrow}$, it follows that

$$\sigma'(t) \underline{\leftrightarrow} \sigma'(u) \ .$$

Hence, we can infer that there are a $j \in J$ and a q' such that

$$\sigma'(u_j) \xrightarrow{\tau} q' \underline{\leftrightarrow} f(\alpha, p_n) \ . \tag{12}$$

Recall that, by one of the assumptions of the proposition, $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$, and thus $\sigma(u)$ has depth $n+2$. On the other hand, by (12),

$$depth(\sigma'(u_j)) \geq n+3 \ .$$

Since σ and σ' differ only in the closed term they map variable x to, it follows that

$$x \in var(u_j) \ . \tag{13}$$

We now proceed to show that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$ by a further case analysis on the form a term u_j satisfying (12) and (13) may have.

(a) CASE $u_j = x$. This case is vacuous because $\sigma'(x) = \bar{\alpha} f(\alpha, p_n) \stackrel{\tau}{\nrightarrow}$, and thus this possible form for u_j does not meet (12).

(b) CASE $u_j = \mu u'$ FOR SOME TERM u'. In light of (12), we have that $\mu = \tau$ and $q' = \sigma'(u') \underline{\leftrightarrow} f(\alpha, p_n)$. Using (13) and the fact that u' has no $\mathbf{0}$ factors, we have that $depth(\sigma'(u')) \geq n+3$ (Lemma 20). Since $f(\alpha, p_n)$ has depth $n+2$, this contradicts $\sigma'(u') \underline{\leftrightarrow} f(\alpha, p_n)$.

(c) CASE $u_j = f(u', u'')$ FOR SOME TERMS u', u''. Our assumption that $\sigma(u)$ has no $\mathbf{0}$ factors yields that none of the terms $u', u'', \sigma(u')$ and $\sigma(u'')$ is bisimilar to $\mathbf{0}$. Moreover, by (13), either $x \in var(u')$ or $x \in var(u'')$.
Since $\sigma'(u_j) = f(\sigma'(u'), \sigma'(u''))$ affords transition (12), we have that $q' = q_1 \| q_2$ for some q_1, q_2. As $f(\alpha, p_n)$ is prime (Lemma 21), it follows that either $q_1 \underline{\leftrightarrow} \mathbf{0}$ or $q_2 \underline{\leftrightarrow} \mathbf{0}$. Hence, we can distinguish two cases, according to the possible origins for transition (12):

i. $\sigma'(u') \xrightarrow{\tau} q_1$ and $q_2 = \sigma'(u'')$. Assume that $\sigma'(u') \xrightarrow{\tau} q_1$ and $q_2 = \sigma'(u'')$. We now proceed to argue that this case produces a contradiction.
To this end, note first of all that $\sigma'(u'') \not\underline{\leftrightarrow} \mathbf{0}$, because $\sigma(u'') \not\underline{\leftrightarrow} \mathbf{0}$ (Lemma 23). Thus it must be the case that $q_1 \underline{\leftrightarrow} \mathbf{0}$ and $q_2 = \sigma'(u'') \underline{\leftrightarrow} f(\alpha, p_n)$. In light of the definition of σ', it follows that x occurs in u', but not in u'' (Lemma 20). Therefore, since σ and σ' only differ at the variable x,

$$\sigma(u'') = \sigma'(u'') \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

Since $\underline{\leftrightarrow}$ is a congruence, we derive that

$$\sigma(u_j) = f(\sigma(u'), \sigma(u'')) \underline{\leftrightarrow} f(\sigma(u'), f(\alpha, p_n)) \ . \tag{14}$$

Since $\sigma(u') \not\underline{\leftrightarrow} \mathbf{0}$ because $q = \sigma(u)$ has no $\mathbf{0}$-factors, we may infer that

$$\begin{aligned}
n+2 &= depth(f(\alpha, p_n)) \\
&= depth(\sigma(u)) && (\text{As } \sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)) \\
&\geq depth(\sigma(u_j)) \\
&= depth(\sigma(u')) + n + 2 && (\text{By (14)}) \\
&> n+2 && (\text{As } depth(\sigma(u')) > 0),
\end{aligned}$$

which is the desired contradiction.

ii. $\sigma'(u') \xrightarrow{\alpha} q_1$ and $\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2$. Assume now that $\sigma'(u') \xrightarrow{\alpha} q_1$ and $\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2$. Recall that exactly one of q_1, q_2 is bisimilar to $\mathbf{0}$. We proceed with the proof by considering these two possible cases in turn.

- CASE $q_1 \underline{\leftrightarrow} \mathbf{0}$. Our order of business will be to argue that, in this case, $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$, and thus that $q = \sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$.
  To this end, observe, first of all, that $q_2 \underline{\leftrightarrow} f(\alpha, p_n)$ by (12). It follows that $x \in var(u'')$, for otherwise we could derive a contradiction thus:

$$\begin{aligned}
depth(f(\alpha,p_n)) &= depth(\sigma(u)) && (\text{As } \sigma(u) \underline{\leftrightarrow} f(\alpha,p_n))\\
&\geq depth(\sigma(u_j)) &&\\
&> depth(\sigma(u'')) && (\text{As } depth(\sigma(u')) > 0)\\
&= depth(\sigma'(u'')) && (\text{As } x \notin var(u''))\\
&> depth(f(\alpha,p_n)) && (\text{As } \sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \underline{\leftrightarrow} f(\alpha,p_n)).
\end{aligned}$$

  Moreover, we claim that $x \notin var(u')$. Indeed, if x also occurred in u', then, since u' has no $\mathbf{0}$ factors, the term $\sigma(x)$ would contribute to the behaviour of $\sigma(u_j)$. Therefore, by (11), the term $\sigma(u_j)$ would afford a sequence of actions containing two occurrences of $\bar{\alpha}$, contradicting our assumption that $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$.
  Observe now that, as $\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \underline{\leftrightarrow} f(\alpha, p_n)$, it must be the case that u'' has a summand x. To see that this does hold, we examine the other possible forms a summand w of u'' responsible for the transition

$$\sigma'(u'') \xrightarrow{\bar{\alpha}} q_2 \underline{\leftrightarrow} f(\alpha,p_n)$$

  may have, and argue that each of them leads to a contradiction.
  A. CASE $w = \bar{\alpha}w'$, FOR SOME TERM w'. In this case, $q_2 = \sigma'(w')$. However, the depth of such a q_2 is either smaller than $n+2$ (if $x \notin var(w')$), or larger than $n+2$ (if $x \in var(w')$). More precisely, in the former case $x \notin var(w')$ implies $\sigma(w) = \sigma'(w)$ and thus $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ gives $n+2 = depth(\sigma(u)) \geq depth(\sigma(w)) = 1 + depth(\sigma(w'))$, giving $depth(\sigma'(w')) \leq n+1$. In the latter case, as $x \in var(w')$ and w' does not have $\mathbf{0}$ factors (or otherwise u'' would have $\mathbf{0}$ factors), by Lemma 20, we would have $depth(\sigma'(w')) \geq depth(\sigma'(x)) = n+3$. Both cases then contradict the fact that q_2 is bisimilar to $f(\alpha, p_n)$, because the latter term has depth $n+2$.
  B. CASE $w = f(w_1, w_2)$, FOR SOME TERMS w_1 AND w_2. Observe, first of all, that $\sigma(w_1)$ and $\sigma(w_2)$ are not bisimilar to $\mathbf{0}$, because $\sigma(u)$ has no $\mathbf{0}$ factors. It follows that $\sigma'(w_1)$ and $\sigma'(w_2)$ are not bisimilar to $\mathbf{0}$ either (Lemma 23).
  Now, since

$$\sigma'(w) = f(\sigma'(w_1), \sigma'(w_2)) \xrightarrow{\bar{\alpha}} q_2 \ ,$$

  there is a closed term q_3 such that $\sigma'(w_1) \xrightarrow{\bar{\alpha}} q_3$ and

$$q_2 = q_3 \| \sigma'(w_2) \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

  As the term $f(\alpha, p_n)$ is prime, and $\sigma'(w_2)$ is not bisimilar to $\mathbf{0}$, we may infer that $q_3 \underline{\leftrightarrow} \mathbf{0}$ and

$$\sigma'(w_2) \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

It follows that $x \notin var(w_2)$, or else the depth of $\sigma'(w_2)$ would be at least $n+3$, and therefore that

$$\sigma'(w_2) = \sigma(w_2) \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

However, this contradicts our assumption that

$$q = \sigma(u) \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

Summing up, we have argued that u'' has a summand x. Therefore, by (11),

$$\sigma(u'') \underline{\leftrightarrow} \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} + r'' \ ,$$

for some closed term r''. We have already noted that

$$\sigma(u') = \sigma'(u') \xrightarrow{\alpha} q_1 \underline{\leftrightarrow} \mathbf{0} \ .$$

Therefore, we have that

$$\sigma(u') \underline{\leftrightarrow} \alpha + r' \ ,$$

for some closed term r'. Using the congruence properties of bisimulation equivalence, we may infer that

$$\begin{aligned}
\sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\
&\underline{\leftrightarrow} f((\alpha + r'), (\bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} + r'')) \ .
\end{aligned}$$

In light of this equivalence, we have that

$$\sigma(u_j) \xrightarrow{\alpha} r \underline{\leftrightarrow} \bar{\alpha}.\alpha^{\leq i_1} + \cdots + \bar{\alpha}.\alpha^{\leq i_m} + r'' \underline{\leftrightarrow} \sigma(u'') \ ,$$

for some closed term r, and thus

$$q = \sigma(u) \xrightarrow{\alpha} r \ .$$

Since $q = \sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ by our assumption, it must be the case that $r \underline{\leftrightarrow} \sigma(u'') \underline{\leftrightarrow} p_n$. So, again using the congruence properties of $\underline{\leftrightarrow}$, we have that

$$\sigma(u_j) = f(\sigma(u'), \sigma(u'')) \underline{\leftrightarrow} f((\alpha + r'), p_n) \ .$$

As $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$, using Lemma 22 it is now a simple matter to infer that

$$\sigma(u') \underline{\leftrightarrow} \alpha \ .$$

Hence $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$. Note that $\sigma(u_j)$ is a summand of $q = \sigma(u)$. Therefore q has a summand bisimilar to $f(\alpha, p_n)$, which was to be shown.

- CASE $q_2 \underline{\leftrightarrow} \mathbf{0}$. We now proceed to argue that this case produces a contradiction. To this end, observe, first of all, that $q_1 \underline{\leftrightarrow} f(\alpha, p_n)$. Reasoning as in the analysis of the previous case, we may infer that x occurs in u', but x does not occur in u''. Moreover, since $\sigma'(u') \xrightarrow{\alpha} q_1 \underline{\leftrightarrow} f(\alpha, p_n)$, it must be the case that $u' \xrightarrow{\alpha} u'''$ for some u''' such that

$$\sigma'(u''') = q_1 \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

(For, otherwise, using Lemma 16.2a, we would have that $\sigma'(u') \xrightarrow{\alpha} q_1$ because $u' \xrightarrow{y} c$, $\sigma(y) \xrightarrow{\alpha} q'_1$ and $q_1 = \sigma'[y_d \mapsto q'_1](c)$, for some variable y, configuration c and closed term q'_1. Then we would necessarily have that $y \neq x$. In fact, if $y = x$, then we would have that $\alpha = \bar{\alpha}$ by the definition of σ', contradicting the distinctness of these two complementary actions. Observe now that, again in light of the definition of σ', the variable x cannot occur in c, or else the depth of

$$q_1 = \sigma'[y_d \mapsto q'_1](c)$$

would be at least $n+3$, contradicting our assumption that

$$q_1 \leftrightarrow f(\alpha, p_n) \ .$$

Hence, since the variable y is different from x, it is not hard to see that $\sigma(u') \xrightarrow{\alpha} q_1$ also holds, and thus that

$$depth(q_1) < depth(\sigma(u)) = n+2 \ ,$$

contradicting our assumption that $q_1 \leftrightarrow f(\alpha, p_n)$.) Since u contains no $\mathbf{0}$ factors, in light of the definition of σ', this u''' cannot contain occurrences of the variable x. (For, otherwise, Lemma 20 would yield that

$$depth(\sigma'(u''')) = depth(q_1) \geq n+3 \ ,$$

contradicting our assumption that $q_1 \leftrightarrow f(\alpha, p_n)$.) So

$$\sigma(u''') = q_1 \leftrightarrow f(\alpha, p_n)$$

also holds. Thus

$$\begin{aligned} n+2 &= depth(f(\alpha, p_n)) \\ &= depth(\sigma(u)) && (\text{As } \sigma(u) \leftrightarrow f(\alpha, p_n)) \\ &\geq depth(\sigma(u_j)) \\ &= depth(f(\sigma(u'), \sigma(u''))) \\ &> depth(\sigma(u''')) + depth(\sigma(u'')) && (\text{As } \sigma(u') \xrightarrow{\alpha} \sigma(u''')) \\ &> n+2 \end{aligned}$$

where the last inequality follows by $depth(\sigma(u'')) > 0$ and $depth(\sigma(u''')) = n+2$, and gives the desired contradiction.

This completes the proof for the case $u_j = f(u', u'')$ for some terms u', u''.

The proof of Proposition 7 is now complete. □

We are now ready to prove Theorem 3, thus completing the proof of Theorem 2 in the considered case of an operator f that, modulo bisimilarity, distributes over summation in its first argument.

Proof of Theorem 3. Assume that E is a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in E:

1. $E \vdash p \approx q$,
2. $p \leftrightarroweq q \leftrightarroweq f(\alpha, p_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, p_n)$.

We prove that q also has a summand bisimilar to $f(\alpha, p_n)$ by induction on the depth of the closed proof of the equation $p \approx q$ from E. Recall that, without loss of generality, we may assume that the closed terms involved in the proof of the equation$p \approx q$ have no $\mathbf{0}$ summands or factors (by Proposition 4, as E may be assumed to be saturated), and that applications of symmetry happen first in equational proofs (that is, E is closed with respect to symmetry).

We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from E. The case of reflexivity is trivial, and that of transitivity follows immediately by using the inductive hypothesis twice. Below we only consider the other possibilities.

- CASE $E \vdash p \approx q$, BECAUSE $\sigma(t) = p$ AND $\sigma(u) = q$ FOR SOME EQUATION $(t \approx u) \in E$ AND CLOSED SUBSTITUTION σ. Since $\sigma(t) = p$ and $\sigma(u) = q$ have no $\mathbf{0}$ summands or factors, and n is larger than the size of each term mentioned in equations in E, the claim follows by Proposition 7.
- CASE $E \vdash p \approx q$, BECAUSE $p = \mu p'$ AND $q = \mu q'$ FOR SOME p', q' SUCH THAT $E \vdash p' \approx q'$. This case is vacuous because $p = \mu p' \not\leftrightarroweq f(\alpha, p_n)$, and thus p does not have a summand bisimilar to $f(\alpha, p_n)$.
- CASE $E \vdash p \approx q$, BECAUSE $p = p' + p''$ AND $q = q' + q''$ FOR SOME p', q', p'', q'' SUCH THAT $E \vdash p' \approx q'$ AND $E \vdash p'' \approx q''$. Since p has a summand bisimilar to $f(\alpha, p_n)$, we have that so does either p' or p''. Assume, without loss of generality, that p' has a summand bisimilar to $f(\alpha, p_n)$. Since p is bisimilar to $f(\alpha, p_n)$, so is p'. Using the soundness of E modulo bisimulation, it follows that $q' \leftrightarroweq f(\alpha, p_n)$. The inductive hypothesis now yields that q' has a summand bisimilar to $f(\alpha, p_n)$. Hence, q has a summand bisimilar to $f(\alpha, p_n)$, which was to be shown.
- CASE $E \vdash p \approx q$, BECAUSE $p = f(p', p'')$ AND $q = f(q', q'')$ FOR SOME p', q', p'', q'' SUCH THAT $E \vdash p' \approx q'$ AND $E \vdash p'' \approx q''$. Since the proof involves no uses of $\mathbf{0}$ as a summand or a factor, we have that $p', p'' \not\leftrightarroweq \mathbf{0}$ and $q', q'' \not\leftrightarroweq \mathbf{0}$. It follows that q is a summand of itself. By our assumptions,

  $$f(\alpha, p_n) \leftrightarroweq q \ .$$

  Therefore we have that q has a summand bisimilar to $f(\alpha, p_n)$, and we are done.

This completes the proof of Theorem 3 and thus of Theorem 2 in the case of an operator f that, modulo bisimilarity distributes over summation in its first argument. □

9 Negative result for f that does not distribute: the case $L^f_\alpha \wedge R^f_\alpha$

In this section we investigate the first case, out of three, related to an operator f that does no distribute, modulo bisimilarity, over summation in either of its arguments.

We choose $\alpha \in \{a, \bar{a}\}$ and we assume that the allowed set of rules for f includes both a rule of type (7) and one of type (8) with label α, namely:

$$\frac{x_1 \xrightarrow{\alpha} y_1}{f(x_1, x_2) \xrightarrow{\alpha} y_1 \| x_2} \qquad \frac{x_2 \xrightarrow{\alpha} y_2}{f(x_1, x_2) \xrightarrow{\alpha} x_1 \| y_2} \ .$$

Notice that this is equivalent to say that the predicate $L^f_\alpha \wedge R^f_\alpha$ holds for f.

We stress that the validity of the negative result does not depend on which types of rules with labels $\bar{\alpha}$ and τ are available for f. Moreover, the arguments we are going to apply would still hold if we exchange α with $\bar{\alpha}$. Hence, the case of an operator for which $L^f_{\bar{\alpha}} \wedge R^f_{\bar{\alpha}}$ holds can be easily obtained from the one we are considering, and it is therefore omitted.

We now introduce the family of (infinitely many) valid equations, modulo bisimilarity, that will allow us to obtain the negative result in the case at hand.

$$q_n = \sum_{i=0}^{n} \alpha\bar{\alpha}^{\leq i} \qquad (n \geq 0)$$

$$e_n : f(\alpha, q_n) \approx \alpha q_n + \sum_{i=0}^{n} \alpha(\alpha \| \bar{\alpha}^{\leq i}) \qquad (n \geq 0).$$

Following the proof strategy from Section 5, we aim to show that, when n is *large enough*, the witness property of having a summand bisimilar to $f(\alpha, q_n)$ is preserved by a finite axiom system E, as stated in the following theorem:

Theorem 4. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments and that has both a rule of type (7) and one rule of type (8) with label α.*

Let E be a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence. Let n be larger than the size of each term in the equations in E. Assume p and q are closed terms bisimilar to $f(\alpha, q_n)$ and contain no $\mathbf{0}$ summands or factors. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, q_n)$, then so does q.

Then, since the left-hand side of equality e_n, viz. the term $f(\alpha, q_n)$, enjoy such a property, whilst the right-hand side, viz. the term $\alpha q_n + \sum_{i=0}^{n} \alpha(\alpha \| \bar{\alpha}^{\leq i})$, does not, we can conclude that the collection of infinitely many equations e_n ($n \geq 0$) is the desired witness family.

9.1 Case specific properties of $f(\alpha, q_n)$

Before proceeding to the proof of Theorem 4, we discuss a few useful properties of the processes $f(\alpha, q_n)$. Such properties are stated in Lemmas 24 and 25 and they are the updated versions of, respectively, Lemmas 21 and 22 from Section 8 with respect to the current set of SOS rules that are allowed for f.

Firstly, due to the form of processes q_n and the validity of $L^f_\alpha \wedge R^f_\alpha$, we have that $f(\alpha, q_n)$ is bisimilar to the parallel composition of α with q_n.

Lemma 24. *For each $n \geq 0$ it holds that $f(\alpha, q_n) \leftrightarrow \alpha \| q_n$.*

Secondly, we study the decomposition properties of $f(\alpha, q_n)$.

Lemma 25. *Let $n \geq 1$. Assume that $p, q \not\leftrightarrow \mathbf{0}$ and $f(p, q) \leftrightarrow f(\alpha, q_n)$. Then*

- *either $p \leftrightarrow \alpha$ and $q \leftrightarrow q_n$,*
- *or $q \leftrightarrow \alpha$ and $p \leftrightarrow q_n$.*

Proof: The proof can be found in Appendix F.1. □

9.2 Proving Theorem 4

The negative result stated in Theorem 4 is strongly based on the following proposition, which ensures that the property of having a summand bisimilar to $f(\alpha, q_n)$ is preserved by the closure under substitution of equations in a finite sound axiom system.

Proposition 8. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments and that has both a rule of type (7) and one rule of type (8) with label α.*

Let $t \approx u$ be an equation over the language CCS_f^- that is sound with respect bisimulation equivalence. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q are bisimilar to $f(\alpha, q_n)$ for some n larger than the size of t, and have neither $\mathbf{0}$ summands nor factors. If p has a summand bisimilar to $f(\alpha, q_n)$, then so does q.

Proof: First of all we notice that since $\sigma(t)$ and $\sigma(u)$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Therefore by Remark 3 we get that

$$t = \sum_{i \in I} t_i \qquad \text{and} \qquad u = \sum_{j \in J} u_j$$

for some finite non-empty index sets I, J with all the t_i and u_j not having $+$ as head operator, $\mathbf{0}$ summands nor factors. By the hypothesis, there is some $i \in I$ with $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, q_n)$. We proceed by a case analysis over the structure of t_i to show that there is a u_j such that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, q_n)$.

1. CASE $t_i = x$ FOR SOME VARIABLE x SUCH THAT $\sigma(x) \underline{\leftrightarrow} f(\alpha, q_n)$. By Proposition 6, t having a summand x implies that u has a summand x as well. Thus, we can immediately conclude that $\sigma(u)$ has a summand bisimilar to $f(\alpha, q_n)$ as required.

2. CASE $t_i = \mu.t'$ FOR SOME TERM t'. This case is vacuous, as it contradicts our assumption $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, q_n)$. Indeed, if $\mu = \alpha$ then $\sigma(t')$ cannot be bisimilar to both q_n and $\alpha \| \bar{\alpha}^{\leq i}$, for any $i \in \{1, \dots, n\}$.

3. CASE $t_i = f(t', t'')$ FOR SOME TERMS t', t''. Since $\sigma(t)$ has no $\mathbf{0}$ factors, we are guaranteed that $\sigma(t'), \sigma(t'') \not\underline{\leftrightarrow} \mathbf{0}$. Hence, from $f(\sigma(t'), \sigma(t'')) \underline{\leftrightarrow} f(\alpha, q_n)$ and Lemma 25 we can distinguish two cases: (a) either $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\sigma(t'') \underline{\leftrightarrow} q_n$, (b) or $\sigma(t') \underline{\leftrightarrow} q_n$ and $\sigma(t'') \underline{\leftrightarrow} \alpha$. We expand only the former case, as the latter follows from an identical (symmetrical) reasoning. By Remark 3, from $\sigma(t'') \underline{\leftrightarrow} q_n$ we infer that $t'' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator and have no $\mathbf{0}$-summands or factors. Since n is larger that the size of t, we have that $|H| < n$ and thus there is some $h \in H$ such that $\sigma(v_h) \underline{\leftrightarrow} \sum_{k=1}^{m} \alpha \bar{\alpha}^{\leq i_k}$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. Since $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors, from Lemma 19 we infer that v_h can only be a variable x with
$$\sigma(x) \underline{\leftrightarrow} \sum_{k=1}^{m} \alpha \bar{\alpha}^{\leq i_k}. \tag{15}$$
Therefore, $t_i = f(t', x + t''')$ for some t''' such that $\sigma(x + t''') \underline{\leftrightarrow} q_n$. We also notice that since $\sigma(t') \underline{\leftrightarrow} \alpha$ and $\sigma(t')$ has no $\mathbf{0}$ summands or factors, then it cannot be the case that $x \in var(t')$.

   To prove that u has a summand bisimilar to $f(\alpha, q_n)$, consider the closed substitution
$$\sigma' = \sigma[x \mapsto \alpha q_n].$$
   Since R_α^f and Lemma 24 hold, we have
$$\sigma'(t_i) \xrightarrow{\alpha} p' \underline{\leftrightarrow} \alpha \| q_n \underline{\leftrightarrow} f(\alpha, q_n).$$

As $t \approx u$ implies $\sigma'(t) \leftrightarrow \sigma'(u)$, we infer that there must be a summand u_j such that $\sigma'(u_j) \xrightarrow{\alpha} r$ for some $r \leftrightarrow f(\alpha,q_n)$. Notice that, since $\sigma(u) \leftrightarrow f(\alpha,q_n)$ and $\sigma(u_j) = \sigma'(u_j)$ if $x \notin var(u_j)$, then it must be the case that $x \in var(u_j)$, or otherwise we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha,q_n)$, as $\sigma(u_j) = \sigma'(u_j) \xrightarrow{\alpha} r$ would give $\sigma(u) \xrightarrow{\alpha} r \leftrightarrow f(\alpha,q_n)$. However, there is no r' such that $f(\alpha,q_n) \xrightarrow{\alpha} r'$ and $r' \leftrightarrow f(\alpha,q_n)$. By Lemma 16, as $L^f_\alpha \wedge R^f_\alpha$ holds, we can distinguish two cases:

(a) There is a term u' such that $u_j \xrightarrow{\alpha} u'$ and $\sigma'(u') \leftrightarrow f(\alpha,q_n)$. Then, since $f(\alpha,q_n) \leftrightarrow \alpha \parallel q_n$ (Lemma 24) we can apply the expansion law, obtaining $\sigma'(u') \leftrightarrow \sum_{i=1}^{n} \alpha(\alpha\|\bar{\alpha}^{\leq i}) + \alpha q_n$. As n is greater than the size of u, and thus of those of u_j and u', by Lemma 18 we get that u' has a summand y, for some variable y, such that $\sigma'(y) \leftrightarrow \sum_{k=1}^{m'} \alpha(\alpha\|\bar{\alpha}^{\leq i'_k}) + r'$, for some $m' > 1$, $1 \leq i'_1 < \cdots < i'_{m'} \leq n$ and closed term r'. Notice that we can infer that $y \neq x$, as $\sigma'(x) \not\leftrightarrow \sigma'(y)$ for any closed term r'. Thus we have $\sigma'(y) = \sigma(y)$ and we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha,q_n)$ in that $\sigma(u_j)$ would be able to perform three α-moves in a row. In fact

$$
\begin{aligned}
\sigma(u_j) &\xrightarrow{\alpha} \sigma(u') && (u' \text{ has a summand } y)\\
&\xrightarrow{\alpha} \alpha\|\bar{\alpha}^{\leq i'_k} && \text{for some } k \in \{1,\dots,m'\}\\
&\xrightarrow{\alpha} \bar{\alpha}^{\leq i'_k},
\end{aligned}
$$

whereas $\sigma(u) \leftrightarrow f(\alpha,q_n)$ can perform only two such transitions.

(b) There are a variable y, a closed term r' and a configuration c such that $\sigma'(y) \xrightarrow{\alpha} r'$, $u_j \xrightarrow{y_b}_\alpha c$ and $\sigma'[y_d \mapsto r'](c) \leftrightarrow f(\alpha,q_n)$. We claim that it must be the case that $y = x$. To see this, assume towards a contradiction that $y \neq x$. We proceed by a case analysis on the possible occurrences of x in c.

- $x \notin var(c)$ or $x \in var(c)$ but its occurrence is in a guarded context that prevents the execution of its closed instances. In this case we get $r = \sigma[y_d \mapsto r'](c) \leftrightarrow \sigma'[y_d \mapsto r'](c) \leftrightarrow f(\alpha,q_n)$. This contradicts $\sigma(u) \leftrightarrow f(\alpha,q_n)$ since we would have $\sigma(u) \xrightarrow{\alpha} r \leftrightarrow f(\alpha,q_n)$, and such a transition cannot be mimicked by $f(\alpha,q_n)$.
- $x \in var(c)$ and its execution is not prevented. We can distinguish two sub-cases, according to whether the occurrence of x is guarded or not.
  - Assume that x occurs guarded in c. In this case we get a contradiction with $r \leftrightarrow f(\alpha,q_n)$ in that

$$
\begin{aligned}
n+2 &= depth(f(\alpha,q_n))\\
&= depth(r)\\
&\geq 1 + depth(\sigma'(x)) && (x \text{ is guarded})\\
&= n+3.
\end{aligned}
$$

  - Assume now that $x \lhd^\alpha_b c$. We proceed by a case analysis on the structure of c.
    * $c \leftrightarrow y_d\|(x+u_1)\|u_2$. Notice that in this case we have $r = r'\|\sigma'(x)+\sigma'(u_1)\|\sigma'(u_2)$. Then, the only transition available for $\sigma'(x)$ is $\sigma'(x) \xrightarrow{\alpha} q_n$, which gives $r \xrightarrow{\alpha} r'\|q_n\|\sigma'(u_2)$. As $r \leftrightarrow f(\alpha,q_n)$, then it must be the case that $f(\alpha,q_n) \xrightarrow{\alpha} r''$ for some $r'' \leftrightarrow r'\|q_n\|\sigma'(u_2)$. Since q_n is prime, we can infer that $r'' \leftrightarrow q_n$ and thus that $r' \leftrightarrow \mathbf{0} \leftrightarrow \sigma'(u_2)$. Hence, we have that $r \leftrightarrow \sigma'(x)+\sigma'(u_1)$. As the one we wrote is the only transition available for $\sigma'(x)$, we can infer that, for all $i \in \{1,\dots,n\}$, the transitions $r \xrightarrow{\alpha} \alpha\|\bar{\alpha}^{\leq i}$ cannot be derived from $\sigma'(x)$, but

only from $\sigma'(u_1)$. Moreover, notice that $y \neq x$ gives $\sigma'(y) = \sigma(y)$, and from $\text{init}(\sigma'(x)) = \text{init}(\sigma(x)) = \{\alpha\}$ and the fact that $L^f_\alpha \wedge R^f_\alpha$ holds, we can infer that $\sigma(u_2) \underline{\leftrightarrow} \sigma'(u_2) \underline{\leftrightarrow} \mathbf{0}$. Therefore, we get a contradiction with $\sigma(u) \underline{\leftrightarrow} f(\alpha, q_n)$, since $\sigma(u) \xrightarrow{\alpha} r' \| \sigma(x) + \sigma(u_1) \| \sigma(u_2) \underline{\leftrightarrow} \sigma(x) + \sigma(u_1) \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq i}$, for any $i \in \{1, \ldots, n\}$. Process $f(\alpha, q_n)$, in turn, by performing two α-moves can only reach processes bisimilar to $\bar{\alpha}^{\leq i}$, for $i \in \{1, \ldots, n\}$.

* c has a subterm u_3 of the form $u_3 \underline{\leftrightarrow} f(x + u_2, u_1)$ or $u_3 \underline{\leftrightarrow} f(u_1, x + u_2)$. In both cases, we get that $\sigma'(x) \xrightarrow{\alpha} q_n$ implies $\sigma'(u_3) \xrightarrow{\alpha} q_n \| \sigma'(u_1)$. However, $f(\alpha, q_n) \xrightarrow{\alpha} \mathbf{0} \| q_n \underline{\leftrightarrow} q_n$ and q_n prime give $\sigma'(u_1) \underline{\leftrightarrow} \mathbf{0}$. One can then argue that, as $\text{init}(\sigma'(x)) = \{\alpha\}$, either x does not occur in u_1, or it does it in a guarded context that prevents its execution. Hence, we infer $\sigma(u_1) \underline{\leftrightarrow} \sigma'(u_1) \underline{\leftrightarrow} \mathbf{0}$, thus contradicting $\sigma(u)$ not having $\mathbf{0}$ factors.

Therefore, we can conclude that it must be the case that $y = x$ and $r' = q_n$. In particular, notice that $x \triangleleft^\alpha_\flat u_j$. We now proceed by a case analysis on the structure of u_j to show that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, q_n)$.

i. $u_j = x$. This case is vacuous, as $\sigma'(x) \xrightarrow{\alpha} q_n \not\underline{\leftrightarrow} f(\alpha, q_n)$.

ii. $u_j = f(u', u'')$ for some u', u''. Notice that $x \triangleleft^\alpha_\flat u_j$ can be due either to $x \triangleleft^\alpha_\flat u'$ or $x \triangleleft^\alpha_\flat u''$. As both $\sigma'(u')$ and $\sigma'(u'')$ can be responsible for the α-move by $\sigma'(u_j)$, we distinguish two cases:

A. $\sigma'(u') \xrightarrow{\alpha} r_1$ and $r_1 \| \sigma'(u'') \underline{\leftrightarrow} f(\alpha, q_n)$. As $f(\alpha, q_n) \underline{\leftrightarrow} \alpha \| q_n$ and both α and q_n are prime, by the existence of a unique prime decomposition, we distinguish two cases:

- $r_1 \underline{\leftrightarrow} \alpha$ and $\sigma'(u'') \underline{\leftrightarrow} q_n$. Since $x \triangleleft^\alpha_\flat u''$ is in contradiction with $\sigma'(u'') \underline{\leftrightarrow} q_n$, we infer that $x \triangleleft^\alpha_\flat u'$. Moreover $\text{init}(\sigma(x)) = \text{init}(\sigma'(x)) = \{\alpha\}$, $L^f_\alpha \wedge R^f_\alpha$, $\sigma'(u') \underline{\leftrightarrow} q_n$ and the fact that $\sigma(u)$ has no $\mathbf{0}$ factors we get that either $x \notin var(u'')$ or x occurs in u'' but its execution is prevented by the rules for f. Therefore $\sigma'(u'') \underline{\leftrightarrow} \sigma(u'') \underline{\leftrightarrow} q_n$. However, $depth(\sigma(x)) \geq 3$, and $x \triangleleft^\alpha_\flat u'$ with $\text{init}(\sigma(x)) = \{\alpha\}$ give us, by Lemma 20, that $depth(\sigma(u')) \geq depth(\sigma(x))$. Therefore we get a contradiction, in that

$$\begin{aligned}
n + 2 &= depth(f(\alpha, q_n)) \\
&= depth(\sigma(u)) \\
&\geq depth(\sigma(u_j)) \\
&= depth(f(\sigma(u'), \sigma(u''))) \\
&\geq depth(\sigma(x)) + depth(\sigma(u'')) \\
&\geq 3 + n + 1 \\
&= n + 4.
\end{aligned}$$

- $r_1 \underline{\leftrightarrow} q_n$ and $\sigma'(u'') \underline{\leftrightarrow} \alpha$. By reasoning as above, we can infer that either $x \notin var(u'')$ or its execution is blocked by the rules for f, so that $\sigma'(u'') \underline{\leftrightarrow} \sigma(u'')$. Moreover, we get that $x \triangleleft^\alpha_\flat u'$. We aim at showing that u' has a summand x. We proceed by showing that the only other possibility, namely $u' = f(w_1, w_2)$ for some w_1, w_2, leads to a contradiction. As $u' = f(w_1, w_2)$ we have that either $x \triangleleft^\alpha_\flat w_1$ or $x \triangleleft^\alpha_\flat w_2$. However, $\sigma'(u') \xrightarrow{\alpha} r_1 \underline{\leftrightarrow} q_n$ gives two possibilities:
  - $\sigma'(w_1) \xrightarrow{\alpha} r'_1$ and $r'_1 \| \sigma'(w_2) \underline{\leftrightarrow} q_n$. Since q_n is prime, then either $r'_1 \underline{\leftrightarrow} \mathbf{0}$ and $\sigma'(w_2) \underline{\leftrightarrow} q_n$, or $r'_1 \underline{\leftrightarrow} q_n$ and $\sigma'(w_2) \underline{\leftrightarrow} \mathbf{0}$. In both cases we infer that either

$x \notin var(w_2)$ or its execution in it is always prevented, so that $\sigma(w_2) \leftrightarrow \sigma'(w_2)$. Therefore, the former case, combined with $\sigma(u'') \leftrightarrow \alpha$, contradicts $\sigma(u) \leftrightarrow f(\alpha, q_n)$. The latter case contradicts $\sigma(u)$ not having $\mathbf{0}$ factors.

– $\sigma'(w_2) \xrightarrow{\alpha} r_2'$ and $\sigma'(w_1) \| r_2' \leftrightarrow q_n$. The same reasoning as in the previous case allows us to conclude that this case gives a contradiction.

Summing up, we have argued that u' has a summand x. Therefore, by Equation (15),

$$\sigma(u') \leftrightarrow \sum_{k=1}^{m} \alpha.\bar{\alpha}^{\leq i_k} + r'' \quad ,$$

for some closed term r''. We have already noted that

$$\sigma(u'') \leftrightarrow \sigma'(u'') \leftrightarrow \alpha \quad .$$

Therefore, using the congruence properties of bisimulation equivalence, we may infer that

$$\begin{aligned} \sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\ &\leftrightarrow f(\sum_{k=1}^{m} \alpha\bar{\alpha}^{\leq i_k} + r'', \alpha) \quad . \end{aligned}$$

In light of this equivalence, we have $\sigma(u_j) \xrightarrow{\alpha} r' \leftrightarrow \sigma(u')$ and thus $\sigma(u) \xrightarrow{\alpha} r'$. Since $\sigma(u) \leftrightarrow f(\alpha, q_n)$ we have that either $r' \leftrightarrow q_n$, or $r' \leftrightarrow \alpha \| \alpha^{\leq i}$ for some $i \in \{1, \ldots, n\}$. However, the latter case is in contradiction with $r' \leftrightarrow \sigma(u')$, and thus it must be the case that $r' \leftrightarrow q_n$. Therefore, we can conclude that $\sigma(u_j) \leftrightarrow f(q_n, \alpha)$. It is easy to check that $f(\alpha, q_n) \leftrightarrow f(q_n, \alpha)$. Hence, $\sigma(u)$ has the desired summand.

B. $\sigma'(u'') \xrightarrow{\alpha} r_2$ and $\sigma'(u') \| r_2 \leftrightarrow f(\alpha, q_n)$. This case follows as the previous one and allows us to conclude as well that $\sigma(u)$ has the desired summand.

The proof of Proposition 8 is now complete. □

We have now all the necessary ingredients for the proof of Theorem 4, which we present below.

Proof of Theorem 4. Assume that E is a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in E:

1. $E \vdash p \approx q$,
2. $p \leftrightarrow q \leftrightarrow f(\alpha, q_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, q_n)$.

We proceed by induction on the depth of the closed proof of the equation $p \approx q$ from E, to prove that also q has a summand bisimilar to $f(\alpha, q_n)$. Recall that, without loss of generality, we may assume that E is closed with respect to symmetry, and thus applications of symmetry happen first in equational proofs. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from E. The case of reflexivity is trivial, and that of transitivity follows by applying twice the inductive hypothesis. We proceed now to a detailed analysis of the remaining cases:

1. CASE $E \vdash p \approx q$ BECAUSE $\sigma(t) = p$ AND $\sigma(u) = q$ FOR SOME TERMS t,u WITH $E \vdash t \approx u$ AND CLOSED SUBSTITUTION σ. The proof of this case follows by Proposition 8.

2. CASE $E \vdash p \approx q$ BECAUSE $p = \mu.p'$ AND $q = \mu.q'$ FOR SOME p',q' WITH $E \vdash p' \approx q'$. This case is vacuous in that $p = \mu.p' \not\leftrightarrow f(\alpha,q_n)$ and thus p does not have a summand bisimilar to $f(\alpha,q_n)$.

3. CASE $E \vdash p \approx q$ BECAUSE $p = r_1 + r_2$ AND $q = s_1 + s_2$ FOR SOME r_i,s_i WITH $E \vdash r_i \approx s_i$, FOR $i \in \{1,2\}$. Since p has a summand bisimilar to $f(\alpha,q_n)$ then so does either r_1 or r_2. Assume without loss of generality that r_1 has such a summand. As $p \leftrightarrow f(\alpha,q_n)$ then $r_1 \leftrightarrow f(\alpha,q_n)$ holds as well. Then, from $E \vdash r_1 \approx s_1$ we infer $s_1 \leftrightarrow f(\alpha,q_n)$. Thus, by the inductive hypothesis we obtain that s_1 has a summand bisimilar to $f(\alpha,q_n)$ and, consequently, so does q.

4. CASE $E \vdash p \approx q$ BECAUSE $p = f(r_1,r_2)$ AND $q = f(s_1,s_2)$ FOR SOME r_i,s_i WITH $E \vdash r_i \approx s_i$, FOR $i \in \{1,2\}$. By the proviso of the theorem p,q have neither $\mathbf{0}$ summands nor factors, thus implying $r_i,s_i \not\leftrightarrow \mathbf{0}$. Hence, from $p \leftrightarrow f(\alpha,q_n)$ and $p = f(r_1,r_2)$ and Lemma 25 we obtain $r_i \leftrightarrow \alpha$ and $r_{3-i} \leftrightarrow q_n$, thus implying, by the soundness of the equations in E, that $s_i \leftrightarrow \alpha$ and $s_{3-i} \leftrightarrow q_n$, so that either $q = f(\alpha,q_n)$ or $q = f(q_n,\alpha)$. In both cases, we can infer that q has itself as the desired summand.

This completes the proof of Theorem 4 and thus of Theorem 2 in the case of an operator f that does not distribute over summation in either argument. □

10 Negative result for f that does not distribute: the case L^f_α, $R^f_{\bar{\alpha}}$

In this section we deal with the second case related to an operator f that does not distribute over summation in either arguments. This time, given $\alpha \in \{a,\bar{a}\}$, we assume that operator f has only one rule with label α and only one rule with label $\bar{\alpha}$ (for, otherwise, the result treated in Section 9 would apply) and moreover we assume such rules to be distinct. In detail, we expand the case in which the only rule for f with label α is of type (7) and the only one with label $\bar{\alpha}$ is of type (8), namely

$$\frac{x_1 \xrightarrow{\alpha} y_1}{f(x_1,x_2) \xrightarrow{\alpha} y_1 \| x_2} \qquad \frac{x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\bar{\alpha}} x_1 \| y_2}$$

so that, for action α only the predicate L^f_α holds, and for action $\bar{\alpha}$ only $R^f_{\bar{\alpha}}$ holds.

Once again, switching the roles of α and $\bar{\alpha}$ would not affect the technical development of the negative result and, thus, the symmetric case with $L^f_{\bar{\alpha}}$ and R^f_α holding is omitted.

To obtain the proof of the negative result, we consider the same family of witness processes from Section 8, namely $f(\alpha,p_n)$ with

$$p_n = \sum_{i=0}^{n} \bar{\alpha}\alpha^{\leq i} \qquad (n \geq 0) \ .$$

However, differently from the previous case, the definition of the witness family of equations depends on which rules of type (6) are allowed for f. More precisely, we need to split the proof of the negative result into two cases, according to whether the configuration of the rules for f allow α and p_n to synchronize or not.

10.1 Case 1: Possibility of synchronization

Assume first that the set of rules for f allows for synchronization between α and p_n, namely that it includes the rule

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} y_1 \| y_2}\ .$$

In the current setting, the family of (infinitely many) equations

$$e_n : f(\alpha, p_n) \approx \alpha p_n + \sum_{i=0}^{n} \bar{\alpha}(\alpha \| \alpha^{\leq i}) + \sum_{i=0}^{n} \tau \alpha^{\leq i} \quad (n \geq 0),$$

can be proved to be sound modulo bisimilarity. The remainder of this (sub)section is devoted to showing that the equations e_n, for $n \geq 0$, constitute a family of witness equations, as stated in the following theorem:

Theorem 5. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments, for action α has only one rule of type (7), and for action $\bar{\alpha}$ has only one rule of type (8). Moreover, assume that f has a rule of the form $S_{\alpha,\bar{\alpha}}$ of type (6).*

Let E be a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence. Let n be larger than the size of each term in the equations in E. Assume p and q are closed terms bisimilar to $f(\alpha, p_n)$ and contain no $\mathbf{0}$ summands or factors. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Before proceeding to the proof, we remark that the processes $f(\alpha, p_n)$ enjoy the following properties, according to the current set of allowed rules for operator f:

Lemma 26. *For each $n \geq 0$ it holds that $f(\alpha, p_n) \underline{\leftrightarrow} \alpha \| p_n$.*

Lemma 27. *Let $n \geq 1$. If $p, q \not\underline{\leftrightarrow} \mathbf{0}$ are such that $f(p,q) \underline{\leftrightarrow} f(\alpha, p_n)$ then $p \underline{\leftrightarrow} \alpha$ and $q \underline{\leftrightarrow} p_n$.*

Proof: The proof is analogous to that of Lemma 22 and therefore omitted. □

10.1.1 Proving Theorem 5

The crucial point in the proof of the negative result is (also in this case) the preservation of the witness property when instantiating an equation from a finite, sound axiom system. We expand this case in the following proposition:

Proposition 9. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments, for action α has only one rule of type (7), and for action $\bar{\alpha}$ has only one rule of type (8). Moreover, assume that f has a rule of the form $S_{\alpha,\bar{\alpha}}$ of type (6).*

Let $t \approx u$ be an equation over the language CCS_f^- that is sound with respect bisimulation equivalence. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q are bisimilar to $f(\alpha, p_n)$ for some n larger than the size of t, and have neither $\mathbf{0}$ summands nor factors. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: First of all we notice that since $\sigma(t)$ and $\sigma(u)$ have no $\mathbf{0}$ summands or factors, then neither do t and u. Therefore by Remark 3 we get that

$$t = \sum_{i \in I} t_i \qquad \text{and} \qquad u = \sum_{j \in J} u_j$$

for some finite non-empty index sets I, J with all the t_i and u_j not having $+$ as head operator, $\mathbf{0}$ summands nor factors. By the hypothesis, there is some $i \in I$ with $\sigma(t_i) \underline{\leftrightarrow} f(\alpha, p_n)$. We proceed by a case analysis on the structure of t_i to show that there is a u_j such that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$, establishing our claim.

1. Case $t_i = x$ for some variable x such that $\sigma(x) \leftrightarrow f(\alpha, p_n)$. By Proposition 6, t having a summand x implies that u has a summand x as well. Thus, we can immediately conclude that $\sigma(u)$ has a summand bisimilar to $f(\alpha, p_n)$ as required.

2. Case $t_i = \mu.t'$ for some term t'. This case is vacuous, as it contradicts $\sigma(t_i) \leftrightarrow f(\alpha, p_n)$.

3. Case $t_i = f(t', t'')$ for some terms t', t''. Since $\sigma(t)$ has no $\mathbf{0}$ factors, we are guaranteed that $\sigma(t'), \sigma(t'') \not\leftrightarrow \mathbf{0}$. Hence, from $f(\sigma(t'), \sigma(t'')) \leftrightarrow f(\alpha, p_n)$ and Lemma 27 we obtain $\sigma(t') \leftrightarrow \alpha$ and $\sigma(t'') \leftrightarrow p_n$. By Remark 3 we infer that $t'' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator and have no $\mathbf{0}$-summands or factors. Since n is larger that the size of t, we have that $|H| < n$ and thus there is some $h \in H$ such that $\sigma(v_h) \leftrightarrow \sum_{k=1}^{m} \bar{\alpha}\alpha^{\leq i_k}$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. Since $\sigma(v_h)$ has no $\mathbf{0}$ summands or factors, from Lemma 19 we infer that v_h can only be a variable x with
$$\sigma(x) \leftrightarrow \sum_{k=1}^{m} \bar{\alpha}\alpha^{\leq i_k}. \tag{16}$$
Therefore, $t_i = f(t', x + t''')$ for some t''' such that $\sigma(x + t''') \leftrightarrow p_n$. We also notice that since $\sigma(t') \leftrightarrow \alpha$ and $\mathrm{init}(\sigma(x)) = \{\bar{\alpha}\}$, we can infer that $x \triangleleft_{\mathrm{r}}^{\bar{\alpha}} t'$ does not hold (otherwise, $\sigma'(t)$ would afford an initial $\bar{\alpha}$-transition and would not be bisimilar to α).

   To prove that u has a summand bisimilar to $f(\alpha, p_n)$, consider the closed substitution
$$\sigma' = \sigma[x \mapsto \bar{\alpha}p_n].$$
Notice that, since $\sigma(t') \leftrightarrow \alpha$, $\sigma(t')$ has no $\mathbf{0}$ summands or factors, $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\bar{\alpha}\}$ and x is the only variable which is affected when changing σ into σ', then we can infer that either $x \notin var(t')$ or its execution is always prevented. In both cases we get $\sigma(t') \leftrightarrow \sigma'(t') \leftrightarrow \alpha$. Then, using Lemma 26 and $t_i = f(t', x + t'')$, we have
$$\sigma'(t_i) \xrightarrow{\bar{\alpha}} p' \leftrightarrow \alpha \| p_n \leftrightarrow f(\alpha, p_n).$$
As $t \approx u$ implies $\sigma'(t) \leftrightarrow \sigma'(u)$, we infer that there must be a summand u_j such that $\sigma'(u_j) \xrightarrow{\bar{\alpha}} r$ for some $r \leftrightarrow f(\alpha, p_n)$. Notice that, since $\sigma(u) \leftrightarrow f(\alpha, p_n)$ and $\sigma(u_j) = \sigma'(u_j)$ if $x \notin var(u_j)$, then it must be the case that $x \in var(u_j)$, or otherwise we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, p_n)$. By Lemma 16, as only $R_{\bar{\alpha}}^{f}$ holds, we can distinguish two cases:

   (a) There is a term u' such that $u_j \xrightarrow{\bar{\alpha}} u'$ and $\sigma'(u') \leftrightarrow f(\alpha, p_n)$. Then, since $f(\alpha, p_n) \leftrightarrow \alpha \parallel p_n$ (Lemma 26) we can apply the expansion law, obtaining $\sigma'(u') \leftrightarrow \alpha p_n + \sum_{i=1}^{n} \bar{\alpha}(\alpha \| \alpha^{\leq i}) + \sum_{i=1}^{n} \tau\alpha^{\leq i}$. As n is greater than the size of u, and thus of those of u_j and u', by Lemma 18 we get that u' has a summand y, for some variable y, such that $\sigma'(y) \leftrightarrow \sum_{k=1}^{m'} \bar{\alpha}(\alpha \| \alpha^{\leq i'_k}) + r'$, for some $m' > 1$, $1 \leq i'_1 < \cdots < i'_{m'} \leq n$ and closed term r'. Notice that we can infer that $y \neq x$, as $\sigma'(x) \not\leftrightarrow \sigma'(y)$ for any closed term r'. Thus we have $\sigma'(y) = \sigma(y)$ and we get a contradiction with $\sigma(u) \leftrightarrow f(\alpha, p_n)$ in that $\sigma(u_j)$ would be able to perform two $\bar{\alpha}$-moves in a row unlike $f(\alpha, p_n)$.

   (b) There are a variable y, a closed term r' and a configuration c such that $\sigma'(y) \xrightarrow{\bar{\alpha}} r'$, $u_j \xrightarrow{y_{\mathrm{r}}}_{\bar{\alpha}} c$ and $\sigma'[y_{\mathrm{d}} \mapsto r'](c) \leftrightarrow f(\alpha, p_n)$. We claim that it must be the case that $y = x$. To see this claim, assume towards a contradiction that $y \neq x$. We proceed by a case analysis on the possible occurrences of x in c.

- $x \notin var(c)$ or $x \in var(c)$ but its occurrence is in a guarded context that prevents the execution of its closed instances. In this case we get $\sigma[y_{\mathrm{d}} \mapsto r'](c) \leftrightarrow \sigma'[y_{\mathrm{d}} \mapsto r'](c) \leftrightarrow f(\alpha, p_n)$. This contradicts $\sigma(u) \leftrightarrow f(\alpha, p_n)$ since we would have $\sigma(u) \xrightarrow{\bar{\alpha}} r \leftrightarrow f(\alpha, p_n)$, and such a transitions cannot be mimicked by $f(\alpha, p_n)$.
- $x \in var(c)$ and its execution is not prevented. We can distinguish two sub-cases, according to whether the occurrence of x is guarded or not.
  - Assume that x occurs guarded in c. In this case we get a contradiction with $r \leftrightarrow f(\alpha, p_n)$ in that

$$
\begin{aligned}
n+2 &= depth(f(\alpha, p_n)) \\
&= depth(r) \\
&\geq 1 + depth(\sigma'(x)) && (x \text{ is guarded}) \\
&= n+3.
\end{aligned}
$$

  - Assume now that $x \lhd_{\mathrm{b}}^{\alpha} c$. This case contradicts our assumption that $\sigma(u) \leftrightarrow f(\alpha, p_n)$ since we would have $\sigma(u) \xrightarrow{\bar{\alpha}} \sigma[y_{\mathrm{d}} \mapsto r'](c) \xrightarrow{\bar{\alpha}}$, due to Lemmas 17 and 15, whereas $f(\alpha, p_n)$ cannot perform two $\bar{\alpha}$-moves in a row.

Therefore, we can conclude that it must be the case that $y = x$ and $r' = p_n$. In particular, notice that $x \lhd_{\mathrm{r}}^{\bar{\alpha}} u_j$. We now proceed by a case analysis on the structure of u_j to show that $\sigma(u_j) \leftrightarrow f(\alpha, p_n)$.

i. $u_j = x$. This case is vacuous, as $\sigma'(x) \xrightarrow{\bar{\alpha}} p_n \not\leftrightarrow f(\alpha, p_n)$.

ii. $u_j = f(u', u'')$ for some u', u''. Notice that $x \lhd_{\mathrm{r}}^{\bar{\alpha}} u_j$ can be due only to $x \lhd_{\mathrm{r}}^{\bar{\alpha}} u''$. We have $\sigma'(u'') \xrightarrow{\bar{\alpha}} r_1$ and $\sigma'(u_j) \xrightarrow{\bar{\alpha}} \sigma'(u') \| r_1 \leftrightarrow f(\alpha, p_n)$. As $f(\alpha, p_n) \leftrightarrow \alpha \| p_n$ and both α and p_n are prime, by the existence of a unique prime decomposition, we distinguish two cases:
   - Case $\sigma'(u') \leftrightarrow \alpha$ and $r_1 \leftrightarrow p_n$. As above, $\mathrm{init}(\sigma(x)) = \mathrm{init}(\sigma'(x)) = \{\bar{\alpha}\}$, $R^f_{\bar{\alpha}}$, $\sigma'(u') \leftrightarrow \alpha$ and the fact that $\sigma(u)$ has no $\mathbf{0}$ factors we get that either $x \notin var(u')$ or x occurs in u' but its execution is prevented by the rules for f. Therefore $\sigma'(u') \leftrightarrow \sigma(u') \leftrightarrow \alpha$. We aim at showing that u'' has a summand x. We proceed by proving that the only other possibility, namely $u'' = f(w_1, w_2)$ for some w_1, w_2 with $x \lhd_{\mathrm{r}}^{\bar{\alpha}} w_2$, leads to a contradiction.
     As $\sigma'(u'') \xrightarrow{\bar{\alpha}} r_1 \leftrightarrow p_n$, we have $\sigma'(w_2) \xrightarrow{\bar{\alpha}} r_2$ and $\sigma'(w_1) \| r_2 \leftrightarrow p_n$. Since, p_n is prime, we have that either $\sigma'(w_1) \leftrightarrow \mathbf{0}$ and $r_2 \leftrightarrow p_n$, or $\sigma'(w_1) \leftrightarrow p_n$ and $r_2 \leftrightarrow \mathbf{0}$. In both cases, as $\sigma'(x) \not\leftrightarrow \sigma'(w_1)$ and the previous considerations, we infer $\sigma(w_1) \leftrightarrow \sigma'(w_1)$. Hence, the former case contradicts $\sigma(u)$ not having $\mathbf{0}$ factors. The latter case contradicts $\sigma(u) \leftrightarrow f(\alpha, p_n)$ as, considering that $x \lhd_{\mathrm{r}}^{\bar{\alpha}} w_2$, the transition $\sigma'(w_2) \xrightarrow{\bar{\alpha}} r_2 \leftrightarrow \mathbf{0}$ cannot be due to $\sigma'(x)$ and therefore it would be available also to $\sigma(w_2)$ thus implying $\sigma(u_j) \xrightarrow{\bar{\alpha}} r''$ with $r'' \leftrightarrow f(\alpha, p_n)$.
     Summing up, we have argued that u'' has a summand x. Therefore, by Equation (16),

$$\sigma(u'') \leftrightarrow \sum_{k=1}^{m} \bar{\alpha}.\alpha^{\leq i_k} + r'' \quad ,$$

     for some closed term r''. We have already noted that

$$\sigma(u') \leftrightarrow \sigma'(u') \leftrightarrow \alpha \quad .$$

Thus, using the congruence properties of bisimulation equivalence, we may infer that

$$\begin{aligned} \sigma(u_j) &= f(\sigma(u'), \sigma(u'')) \\ &\underline{\leftrightarrow} f(\alpha, \sum_{k=1}^{m} \bar{\alpha}\alpha^{\leq i_k} + r'') \ . \end{aligned}$$

In light of this equivalence, we have $\sigma(u_j) \xrightarrow{\alpha} r' \underline{\leftrightarrow} \sigma(u'')$ and thus $\sigma(u) \xrightarrow{\alpha} r'$. Since $\sigma(u) \underline{\leftrightarrow} f(\alpha, p_n)$ then it must be the case that $r' \underline{\leftrightarrow} p_n$. Therefore, we can conclude that $\sigma(u_j) \underline{\leftrightarrow} f(\alpha, p_n)$. Hence, $\sigma(u)$ has the desired summand.

- Case $\sigma'(u') \underline{\leftrightarrow} p_n$ and $r_1 \underline{\leftrightarrow} \alpha$. By reasoning as above, we can infer that either $x \notin var(u')$ or it is blocked by the rules for f, so that $\sigma'(u') \underline{\leftrightarrow} \sigma(u') \underline{\leftrightarrow} p_n$. However, $depth(\sigma(x)) \geq 3$, and $x \triangleleft_{\mathrm{r}}^{\bar{\alpha}} u''$ with $\mathrm{init}(\sigma(x)) = \{\bar{\alpha}\}$ give us, by Lemma 20, that $depth(\sigma(u'')) \geq depth(\sigma(x))$. Therefore we get a contradiction, in that

$$\begin{aligned} n+2 &= depth(f(\alpha, p_n)) \\ &= depth(\sigma(u)) \\ &\geq depth(\sigma(u_j)) \\ &= depth(f(\sigma(u'), \sigma(u''))) \\ &\geq depth(\sigma(u')) + depth(\sigma(u'')) \\ &\geq depth(\sigma(u')) + depth(\sigma(x)) \\ &\geq n+1+3 \\ &= n+4. \end{aligned}$$

The proof of Proposition 9 is now complete. □

We can now formalize the proof of Theorem 5.

Proof of Theorem 5. Assume that E is a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence, and that the following hold, for some closed terms p and q and positive integer n larger than the size of each term in the equations in E:

1. $E \vdash p \approx q$,
2. $p \underline{\leftrightarrow} q \underline{\leftrightarrow} f(\alpha, p_n)$,
3. p and q contain no occurrences of $\mathbf{0}$ as a summand or factor, and
4. p has a summand bisimilar to $f(\alpha, p_n)$.

We proceed by induction on the depth of the closed proof of the equation $p \approx q$ from E, to prove that q has a summand bisimilar to $f(\alpha, p_n)$ as well. Recall that, without loss of generality, we may assume that E is closed with respect to symmetry, and thus applications of symmetry happen first in equational proofs. We proceed by a case analysis on the last rule used in the proof of $p \approx q$ from E. The case of reflexivity is trivial, and that of transitivity follows by applying twice the inductive hypothesis. We proceed now to a detailed analysis of the remaining cases:

1. CASE $E \vdash p \approx q$ BECAUSE $\sigma(t) = p$ AND $\sigma(u) = q$ FOR SOME TERMS t, u WITH $E \vdash t \approx u$ AND CLOSED SUBSTITUTION σ. The proof of this case follows by Proposition 9.

2. CASE $E \vdash p \approx q$ BECAUSE $p = \mu.p'$ AND $q = \mu.q'$ FOR SOME p', q' WITH $E \vdash p' \approx q'$. This case is vacuous in that $p = \mu.p' \not\leftrightarrow f(\alpha, p_n)$ and thus p does not have a summand bisimilar to $f(\alpha, p_n)$.
3. $E \vdash p \approx q$ because $p = p_1 + p_2$ and $q = q_1 + q_2$ for some p_i, q_i with $E \vdash p_i \approx q_i$, for $i \in \{1,2\}$. Since p has a summand bisimilar to $f(\alpha, p_n)$ then so does either p_1 or p_2. Assume without loss of generality that p_1 has such a summand. As $p \leftrightarrow f(\alpha, p_n)$ then $p_1 \leftrightarrow f(\alpha, p_n)$ holds as well. Then, from $E \vdash p_1 \approx q_1$ we infer $q_1 \leftrightarrow f(\alpha, p_n)$. Thus, by the inductive hypothesis we obtain that q_1 has a summand bisimilar to $f(\alpha, p_n)$ and, consequently, so does q.
4. $E \vdash p \approx q$ because $p = f(p_1, p_2)$ and $q = f(q_1, q_2)$ for some p_i, q_i with $E \vdash p_i \approx q_i$, for $i \in \{1,2\}$. By the proviso of the theorem p, q have neither $\mathbf{0}$ summands nor factors, thus implying $p_i, q_i \not\leftrightarrow \mathbf{0}$. Hence, from $p \leftrightarrow f(\alpha, p_n)$ and $p = f(p_1, p_2)$ and Lemma 27 we obtain $p_1 \leftrightarrow \alpha$ and $p_2 \leftrightarrow p_n$, thus implying, by the soundness of the equations in E, that $q_1 \leftrightarrow \alpha$ and $q_2 \leftrightarrow p_n$, so that $q = f(\alpha, p_n)$. In both cases, we can infer that q has itself as the desired summand.

This completes the proof of Theorem 5 and thus of Theorem 2 in the case of an operator f that does not distribute over summation in either argument. □

10.2 Case 2: No synchronization

Assume now that the rules for f do not allow $f(\alpha, p_n)$ to perform a τ-move due to a synchronization between α and p_n. This is equivalent to saying that f has only the following rule of type (6)

$$\frac{x_1 \xrightarrow{\bar{\alpha}} y_1 \quad x_2 \xrightarrow{\alpha} y_2}{f(x_1, x_2) \xrightarrow{\tau} y_1 \| y_2} \ .$$

Thus, the witness family of equations changes as follows:

$$e_n : f(\alpha, p_n) \approx \alpha p_n + \sum_{i=0}^{n} \bar{\alpha}(\alpha \| \alpha^{\leq i}) \quad (n \geq 0),$$

and our order of business is then be to prove the following result:

Theorem 6. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments, for action α has only one rule of type (7), and for action $\bar{\alpha}$ has only one rule of type (8). Moreover, assume that f has a rule of the form $S_{\bar{\alpha},\alpha}$ of type (6).*

Let E be a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence. Let n be larger than the size of each term in the equations in E. Assume p and q are closed terms bisimilar to $f(\alpha, p_n)$ and contain no $\mathbf{0}$ summands or factors. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

10.2.1 Proving Theorem 6

The proof of Theorem 6 follows that of Theorem 5 in a step by step manner, by exploiting Proposition 10 below in place of Proposition 9. The only difference in the two results is that, in the case at hand, Lemma 26 does not hold anymore. (In fact one could prove, as done for Lemma 21, that $f(\alpha, p_n)$ is prime for all $n \geq 0$.)

Proposition 10. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments, for action α has only one rule of type (7), and for action $\bar{\alpha}$ has only one rule of type (8). Moreover, assume that f has a rule of the form $S_{\bar{\alpha},\alpha}$ of type (6).*

Let $t \approx u$ be an equation over the language CCS_f^- that is sound with respect bisimulation equivalence. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q are bisimilar to $f(\alpha, p_n)$ for some n larger than the size of t, and have neither $\mathbf{0}$ summands nor factors. If p has a summand bisimilar to $f(\alpha, p_n)$, then so does q.

Proof: The proof follows exactly as the proof of Proposition 9, with the only difference that when we consider the derived transition

$$\sigma'(t_1) \xrightarrow{\bar{\alpha}} p'$$

we have that $p' \leftrightarrow \alpha \| p_n \not\leftrightarrow f(\alpha, p_n)$. However, by substituting $f(\alpha, p_n)$ with $\alpha \| p_n$ in the remaining of the proof, the same arguments hold. □

Proof of Theorem 6. The proof is identical to the proof of Theorem 5, by exploiting Proposition 10 in place of Proposition 9. □

11 Negative result for f that does not distribute: the case L_τ^f

This section considers the last case in our analysis, namely that of an operator f that does not distribute, modulo bisimilarity, over summation in either argument and that has the same rule type for actions $\alpha, \bar{\alpha}$. Here, we present solely the case in which f has only rules of type (8) with labels $\alpha, \bar{\alpha}$ and a rule of type (7) with label τ, namely

$$\frac{x_1 \xrightarrow{\tau} y_1}{f(x_1, x_2) \xrightarrow{\tau} y_1 \| x_2} \qquad \frac{x_2 \xrightarrow{\alpha} y_2}{f(x_1, x_2) \xrightarrow{\alpha} x_1 \| y_2} \qquad \frac{x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1, x_2) \xrightarrow{\bar{\alpha}} x_1 \| y_2} \ .$$

Hence, notice that L_τ^f holds and only R_α^f and $R_{\bar{\alpha}}^f$ hold for actions α and $\bar{\alpha}$. The symmetric case of only rules of type (7) for $\alpha, \bar{\alpha}$ and a rule of type (8) for τ can be obtained from this one in a straightforward manner.

Interestingly, the validity of the negative result is independent of which rules of type (6) are available for f, and of the validity of the predicate R_τ^f. (We recall that asking for f having a rule of type (7) with label τ does not exclude the possibility of having also a rule of type (8) with the same label).

Consider the family of (infinitely many) equations defined by:

$$q_n = \sum_{i=0}^{n} \alpha \bar{\alpha}^{\leq i} \qquad (n \geq 0)$$

$$e_n : f(\tau, q_n) \approx \tau q_n + \sum_{i=0}^{n} \alpha(\tau \| \bar{\alpha}^{\leq i}) \qquad (n \geq 0)$$

where the processes q_n are the same used in Section 9. Then, Theorem 7 below proves that the collection of equations e_n, $n \geq 0$, is a witness family of equations for our negative result.

Theorem 7. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments, has only rules of type (8) with labels $\alpha, \bar{\alpha}$, and has a rule of type (7) with label τ.*

Let E be a finite axiom system over the language CCS_f^- that is sound with respect to bisimulation equivalence. Let n be larger than the size of each term in the equations in E. Assume p and q are closed terms bisimilar to $f(\tau, q_n)$ and contain no $\mathbf{0}$ summands or factors. If $E \vdash p \approx q$ and p has a summand bisimilar to $f(\tau, q_n)$, then so does q.

The witness processes $f(\tau, q_n)$ enjoy the properties formalized in Lemmas 28 and 29 below.

Lemma 28. *For each $n \geq 0$ it holds that $f(\tau, q_n) \underline{\leftrightarrow} \tau \| q_n$.*

Lemma 29. *Let $n \geq 1$. If $p, q \not\underline{\leftrightarrow} \mathbf{0}$ are such that $f(p,q) \underline{\leftrightarrow} f(\tau, q_n)$ then $p \underline{\leftrightarrow} \tau$ and $q \underline{\leftrightarrow} q_n$.*

Proof: The proof can be found in Appendix G.1. □

The proof of Theorem 7 strongly depends on the following result.

Proposition 11. *Assume an operator f that, modulo bisimilarity, does not distribute over summation in either of its arguments, has only rules of type (8) with labels $\alpha, \bar{\alpha}$, and has a rule of type (7) with label τ.*

Let $t \approx u$ be an equation over the language CCS_f^- that is sound with respect bisimulation equivalence. Let σ be a closed substitution with $p = \sigma(t)$ and $q = \sigma(u)$. Suppose that p and q are bisimilar to $f(\tau, q_n)$ for some n larger than the size of t, and have neither $\mathbf{0}$ summands nor factors. If p has a summand bisimilar to $f(\tau, q_n)$, then so does q.

Proof: The claim follows by the same arguments used in the proof of Proposition 9 and by considering the substitution

$$\sigma' = \sigma[x \mapsto \alpha q_n].$$

□

The negative result of our last case can then be formalized as in the previous sections.

Proof of Theorem 7. The same reasoning used in the proof of Theorem 5 applies, by exploiting Proposition 11 in place of Proposition 9. □

12 Conclusions

Please ignore this section for the moment. It is still "under construction". My apologizes for the inconvenience.

With this paper we have taken a first step towards the problem of finitely based axiomatizations of bisimulation over languages including the CCS's parallel composition operator. We have shown that, under reasonable simplifying assumptions, we cannot use a single binary auxiliary operator f, whose semantics is defined via inference rules in the de Simone format, to obtain a finite axiomatization of bisimilarity.

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A Proofs of the results in Section 3

A.1 Proof of Lemma 1

Proof of Lemma 1. Statements 1 and 2 are trivial because the equation

$$x \| y \approx \mathbf{0}$$

is not sound modulo bisimilarity. We therefore limit ourselves to presenting a proof for statement 3. To this end, assume, towards a contradiction, that f has a rule of the form

$$f(x_1,x_2) \xrightarrow{\mu} t(x_1,x_2) \ ,$$

for some action μ and term t. This rule can be used to derive that

$$f(\mathbf{0},\mathbf{0}) \xrightarrow{\mu} t(\mathbf{0},\mathbf{0}) \ .$$

Since the set J on the right-hand side of (3) is non-empty by statement 1, the term $f(\mathbf{0},\mathbf{0})$ occurs as a summand of $t_J(\mathbf{0},\mathbf{0})$. It follows that

$$t_J(\mathbf{0},\mathbf{0}) \xrightarrow{\mu} t(\mathbf{0},\mathbf{0}) \ .$$

Therefore,

$$\mathbf{0} \| \mathbf{0} \underline{\leftrightarrow} \mathbf{0} \not\underline{\leftrightarrow} t_J(\mathbf{0},\mathbf{0}) \ ,$$

contradicting our Assumption 2. □

A.2 Proof of Lemma 2

Proof of Lemma 2. Assume, towards a contradiction, that $f(x,x)$, say, is a summand of t_J. Since $a \| \mathbf{0} \xrightarrow{a} \mathbf{0} \| \mathbf{0} \underline{\leftrightarrow} \mathbf{0}$ and Equation (3) holds modulo bisimulation equivalence, there is a closed term p such that

$$t_J(a,\mathbf{0}) \xrightarrow{a} p \text{ and } p \underline{\leftrightarrow} \mathbf{0} \ .$$

This means that there is a summand $f(z_1,z_2)$ of t_J such that

$$f(p_1,p_2) \xrightarrow{a} p \ ,$$

where, for $i \in \{1,2\}$,

$$p_i = \begin{cases} a & \text{if } z_i = x \ , \\ \mathbf{0} & \text{if } z_i = y \ . \end{cases}$$

The transition $f(p_1,p_2) \xrightarrow{a} p$ must be provable using some rule ρ for f of the form (1). Such a rule has some premise by Lemma 1(3), and each such premise must have the form $x_1 \xrightarrow{\mu} y_1$ or $x_2 \xrightarrow{\mu} y_2$, for some action μ. If both z_1 and z_2 are y then $p_1 = p_2 = \mathbf{0}$, and none of those premises can be met. Therefore at least one of z_1 and z_2 in the summand $f(z_1,z_2)$ is x. Moreover, if $x_i \xrightarrow{\mu} y_i$ ($i \in \{1,2\}$) is a premise of ρ, then $z_i = x$ and $\mu = a$ (or else the premise could not be met). So the rule ρ can have one of the following three forms:

$$\frac{x_1 \xrightarrow{a} y_1}{f(x_1,x_2) \xrightarrow{a} t_1(y_1,x_2)} \qquad \frac{x_2 \xrightarrow{a} y_2}{f(x_1,x_2) \xrightarrow{a} t_2(x_1,y_2)}$$

$$\frac{x_1 \xrightarrow{a} y_1 \quad x_2 \xrightarrow{a} y_2}{f(x_1,x_2) \xrightarrow{a} t_3(y_1,y_2)}$$

for some terms t_1, t_2 and t_3. We now proceed to argue that the existence of each of these rules contradicts the soundness of Equation (3) modulo bisimulation equivalence.

If ρ has the form

$$\frac{x_1 \xrightarrow{a} y_1 \quad x_2 \xrightarrow{a} y_2}{f(x_1,x_2) \xrightarrow{a} t_3(y_1,y_2)}$$

then $z_1 = z_2 = x$ and

$$f(a,a) \xrightarrow{a} p \ .$$

Since the term $f(a,a)$ is a summand of $t_J(a,a)$, it follows that

$$t_J(a,a) \xrightarrow{a} p$$

also holds. However, this contradicts the soundness of Equation (3) because, for each transition $a \| a \xrightarrow{a} q$, we have that $q \mathrel{\underline{\leftrightarrow}} a \mathrel{\underline{\not\leftrightarrow}} \mathbf{0} \mathrel{\underline{\leftrightarrow}} p$.

Assume now, without loss of generality, that ρ has the form

$$\frac{x_1 \xrightarrow{a} y_1}{f(x_1,x_2) \xrightarrow{a} t_1(y_1,x_2)}$$

Using this rule, we can infer that

$$f(a,a) \xrightarrow{a} t_1(\mathbf{0},a) \ .$$

Since $f(x,x)$ is a summand of t_J by our assumption, the term $f(a,a)$ is a summand of $t_J(a,\mathbf{0})$. Hence,

$$t_J(a,\mathbf{0}) \xrightarrow{a} t_1(\mathbf{0},a)$$

also holds. As Equation (3) holds modulo bisimulation equivalence, we have that

$$a \| \mathbf{0} \mathrel{\underline{\leftrightarrow}} t_J(a,\mathbf{0}) \ .$$

Therefore $t_1(\mathbf{0},a) \mathrel{\underline{\leftrightarrow}} \mathbf{0}$, because $a \| \mathbf{0} \xrightarrow{a} \mathbf{0} \| \mathbf{0}$ is the only transition afforded by the term $a \| \mathbf{0}$. Observe now that

$$t_J(a,a) \xrightarrow{a} t_1(\mathbf{0},a) \mathrel{\underline{\leftrightarrow}} \mathbf{0} \ .$$

also holds. However, this contradicts the soundness of Equation (3) as above because, for each transition $a \| a \xrightarrow{a} q$, we have that $q \mathrel{\underline{\leftrightarrow}} a \mathrel{\underline{\not\leftrightarrow}} \mathbf{0} \mathrel{\underline{\leftrightarrow}} p$.

This proves that $f(x,x)$ is not a summand of t_J, which was to be shown. □

A.3 Proof of Proposition 1

Proof of Proposition 1. If J is a singleton, then, since $\|$ is commutative modulo bisimulation equivalence, the equation

$$x \| y \approx f(x,y)$$

holds modulo bisimilarity. Therefore the result follows from the nonexistence of a finite equational axiomatization for CCS proven by Moller in [21, 23]. □

A.4 Proof of Lemma 3

Proof of Lemma 3. By structural induction on closed terms. For all of the standard CCS operators, it is well known that the depth of closed terms can be characterized inductively thus:

$$\begin{array}{rcl} \mathit{depth}(\mathbf{0}) & = & 0 \\ \mathit{depth}(\mu p) & = & 1+\mathit{depth}(p) \\ \mathit{depth}(p+q) & = & \max\{\mathit{depth}(p),\mathit{depth}(q)\} \\ \mathit{depth}(p\|q) & = & \mathit{depth}(p)+\mathit{depth}(q) \ . \end{array}$$

So the depth of a closed term of the form μp, $p+q$ or $p\|q$ is finite, if so are the depths of p and q.

Consider now a closed term of the form $f(p,q)$. Since bisimilar terms have the same depth and, by the proviso of the lemma, Equation (4) holds modulo bisimulation equivalence, we have that

$$\mathit{depth}(f(p,q)) \leq \mathit{depth}(f(p,q)+f(q,p)) = \mathit{depth}(p\|q) \ .$$

It follows that $\mathit{depth}(f(p,q))$ is finite, if so are the depths of p and q. $\square$

B Proofs of the results in Section 4

B.1 Proof of Lemma 4

Proof of Lemma 4. We only detail the proof for statement 1. (The proof for statement 2 follows similar lines, and is left to the reader.)

Assume, towards a contradiction, that $\mu = \tau$ and the set of premises $\{x_i \xrightarrow{\mu_i} y_i \mid i \in I\}$ of ρ has some form that differs from those in the statement. Then the set of premises of ρ has one of the following two forms:

- $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1,2\}$ and $\alpha \in \{a, \bar{a}\}$, or
- $\{x_1 \xrightarrow{\mu_1} y_1, x_2 \xrightarrow{\mu_2} y_2\}$ for some $\mu_1, \mu_2 \in \{a, \bar{a}, \tau\}$ such that
  - either $\mu_1 = \tau$ or $\mu_2 = \tau$, or
  - $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$.

We now proceed to argue that the existence of either of these rules for f contradicts the soundness of Equation (4).

- Assume that the set of premises of ρ has the form $\{x_i \xrightarrow{\alpha} y_i\}$ for some $i \in \{1,2\}$ and $\alpha \in \{a, \bar{a}\}$. In this case, we can use that rule to prove the existence of the transition

  $$f(\alpha, \mathbf{0}) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \text{ or } f(\mathbf{0}, \alpha) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ ,$$

  depending on whether $i = 1$ or $i = 2$. Therefore

  $$f(\alpha, \mathbf{0}) + f(\mathbf{0}, \alpha) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0})$$

  also holds. However, the existence of this transition immediately contradicts the soundness of Equation (4) modulo bisimulation equivalence because $\alpha \parallel \mathbf{0}$ affords no τ-transition.

- Assume that the set of premises of ρ has the form $\{x_1 \xrightarrow{\mu_1} y_1, x_2 \xrightarrow{\mu_2} y_2\}$ for some $\mu_1, \mu_2 \in \{a, \bar{a}, \tau\}$ such that
  - either $\mu_1 = \tau$ or $\mu_2 = \tau$, or
  - $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$.

  In the this case, we can use that rule to prove the existence of the transition

  $$f(\mu_1, \mu_2) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0}) \ .$$

  Therefore

  $$f(\mu_1, \mu_2) + f(\mu_2, \mu_1) \xrightarrow{\tau} t(\mathbf{0}, \mathbf{0})$$

  also holds. By the soundness of Equation (4), we have that

  $$\mu_1 \| \mu_2 \underline{\leftrightarrow} f(\mu_1, \mu_2) + f(\mu_2, \mu_1) \ .$$

  Hence $\mu_1 \| \mu_2 \xrightarrow{\tau} p$ for some p such that $p \underline{\leftrightarrow} t(\mathbf{0}, \mathbf{0})$. If $\mu_1 = \mu_2 = \alpha$ for some $\alpha \in \{a, \bar{a}\}$, then the above transition cannot exist, because $\alpha \parallel \alpha$ affords no τ-transition. This immediately contradicts the soundness of Equation (4) modulo bisimulation equivalence. We therefore proceed with the proof by assuming that at least one of μ_1 and μ_2 is τ. In this case, we have that $\mu_1 \parallel \mu_2 \xrightarrow{\tau} p$ implies that $p \leftrightarrow \mu_1$ and $\mu_2 = \tau$, or $p \leftrightarrow \mu_2$ and $\mu_1 = \tau$. Assume, without loss of generality, that $\mu_1 = \tau$ and

$$t(\mathbf{0},\mathbf{0}) \leftrightarrow \mu_2 \ . \tag{17}$$

Pick now an action $\alpha \neq \mu_2$. (Such an action exists as we have three actions in our language.) The soundness of Equation (4) yields that

$$\tau \parallel (\mu_2 + \alpha) \leftrightarrow f(\tau, \mu_2 + \alpha) + f(\mu_2 + \alpha, \tau) \ .$$

Using the rule for f we assumed we had and the rules for $+$, we can prove the existence of the transition

$$f(\tau, \mu_2 + \alpha) + f(\mu_2 + \alpha, \tau) \xrightarrow{\tau} t(\mathbf{0},\mathbf{0}) \ .$$

Since the source of the above transition is bisimilar to $\tau \parallel (\mu_2 + \alpha)$, there must be a term p such that $\tau \parallel (\mu_2 + \alpha) \xrightarrow{\tau} p$ and $p \leftrightarrow t(\mathbf{0},\mathbf{0})$. By Equation (17), this term p can only be $\tau \parallel \mathbf{0}$. In fact,

$$t(\mathbf{0},\mathbf{0}) \leftrightarrow \mu_2 \not\leftrightarrow (\mu_2 + \alpha) \leftrightarrow \mathbf{0} \parallel (\mu_2 + \alpha) \ ,$$

for we chose $\alpha \in \{a, \bar{a}\}$ different from μ_2. We have therefore proven that $\mu_1 = \mu_2 = \tau$.

We are now ready to reach the promised contradiction to the soundness of Equation (4). In fact, consider the term $f(\tau + a, \tau + a)$. Using the rule for f we assumed we had, we can again prove the existence of the transition

$$f(\tau + a, \tau + a) \xrightarrow{\tau} t(\mathbf{0},\mathbf{0}) \ .$$

By Equation (17) and our observation that $\mu_2 = \tau$, the term $t(\mathbf{0},\mathbf{0})$ is bisimilar to τ. On the other hand, $(\tau + a) \parallel (\tau + a) \xrightarrow{\tau} p$ implies that $p \leftrightarrow (\tau + a) \not\leftrightarrow \tau$, contradicting the soundness of Equation (4) modulo bisimulation equivalence.

$\square$

B.2 Proof of Lemma 5

Proof of Lemma 5. We first argue that f must have a rule of the form (6) for some $\alpha \in \{a, \bar{a}\}$ and term t. To this end, assume, towards a contradiction, that f has no such rule. Observe that the term $a \parallel \bar{a}$ affords the transition

$$a \parallel \bar{a} \xrightarrow{\tau} \mathbf{0} \parallel \mathbf{0} \ .$$

However, neither the term $f(a, \bar{a})$ nor the term $f(\bar{a}, a)$ affords a τ-transition. In fact, using our assumption that f has no rule of the form (6) and Lemma 4(1), each rule for f with a τ-transition as a consequent must have the form

$$\frac{x_i \xrightarrow{\tau} y_i}{f(x_1, x_2) \xrightarrow{\tau} t}$$

for some $i \in \{1, 2\}$ and term t. Such a rule cannot be used to infer a transition from $f(a, \bar{a})$ or $f(\bar{a}, a)$. It follows that

$$a \parallel \bar{a} \not\leftrightarrow f(a, \bar{a}) + f(\bar{a}, a) \ ,$$

contradicting the soundness of Equation (4). Therefore f must have a rule of the form (6).

We now proceed to argue that $t(x, y)$ is bisimilar to $x \parallel y$, for each rule of the form (6) for f. Pick a rule for f of the form (6). We shall argue that

$$p \parallel q \leftrightarrow t(p, q) \ ,$$

for all closed CCS$_f$ terms p and q. To this end, consider the terms $\alpha.p \parallel \bar{\alpha}.q$ and $f(\alpha.p,\bar{\alpha}.q)+f(\bar{\alpha}.q,\alpha.p)$. Using rule (6) and the rules for $+$, we have that

$$f(\alpha.p,\bar{\alpha}.q)+f(\bar{\alpha}.q,\alpha.p) \xrightarrow{\tau} t(p,q) \ .$$

By the soundness of Equation (4), we have that

$$\alpha.p \parallel \bar{\alpha}.q \underline{\leftrightarrow} f(\alpha.p,\bar{\alpha}.q)+f(\bar{\alpha}.q,\alpha.p) \ .$$

Therefore there is a closed term r such that $\alpha.p \parallel \bar{\alpha}.q \xrightarrow{\tau} r$ and $r \underline{\leftrightarrow} t(p,q)$. Note now that the only τ-transition afforded by $\alpha.p \parallel \bar{\alpha}.q$ is

$$\alpha.p \parallel \bar{\alpha}.q \xrightarrow{\tau} p \parallel q \ .$$

Therefore $r = p \parallel q \underline{\leftrightarrow} t(p,q)$, which was to be shown. □

B.3 Proof of Lemma 6

Proof of Lemma 6. Let $\mu \in \{a,\bar{a},\tau\}$. We first argue that f must have a rule of the form (7) or (8) for some term t. To this end, assume, towards a contradiction, that f has no such rules. Observe that the term $\mu \parallel \mathbf{0}$ affords the transition

$$\mu \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0} \ .$$

However, neither the term $f(\mu,\mathbf{0})$ nor the term $f(\mathbf{0},\mu)$ affords a μ-transition. In fact, using our assumption that f has no rule of the form (7) or (8), Lemma 4 yields that

- either f has no rule with a μ-transition as a consequent,
- or $\mu = \tau$, and each rule for f with a τ-transition as a consequent has the form

  $$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} t(y_1,y_2)}$$

  for some $\alpha \in \{a,\bar{a}\}$.

In the latter case, such a rule cannot be used to infer a transition from $f(\mu,\mathbf{0})$ or $f(\mathbf{0},\mu)$. It follows that

$$\mu \parallel \mathbf{0} \underline{\not\leftrightarrow} f(\mu,\mathbf{0})+f(\mathbf{0},\mu) \ ,$$

contradicting the soundness of Equation (4). Therefore f must have a rule of the form (7) or (8) for each action μ.

To conclude the proof we need to show that for each rule of the form (7) or (8) the target term $t(x,y)$ is bisimilar to $x \parallel y$. For simplicity, we expand the proof only for the case of rules of the form (7). The proof for rules of the form (8) follows by the same reasoning.

We proceed by a case analysis over the structure of $t(y_1,x_2)$, which, we recall, under assumption 4 can be either a variable in $\{y_1,x_2\}$ or a term of the form $g(y_1,x_2)$ for some CCS$_f$ operator g. Our aim is to show that the only possibility is to have $t(y_1,x_2) = y_1 \parallel x_2$, as any other process term would invalidate one of our simplifying assumptions.

- CASE t IS A VARIABLE IN $\{y_1,x_2\}$. We can distinguish two cases, according to which variable is considered:

- $t = y_1$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (7) we can infer that $f(p,p) \xrightarrow{\mu} \mathbf{0}$, and thus $f(p,p)+f(p,p) \xrightarrow{\mu} \mathbf{0}$. However, there is no μ-transition form $p \parallel p$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel p \xrightarrow{\mu} q$, then q is a process that will always be able to perform a second μ-transition. Hence, we would have $p \parallel p \not\underline{\leftrightarrow} f(p,p)+f(p,p)$, thus contradicting the soundness of Equation (4).
- $t = x_2$. Consider process $p = \mu.\mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mu.\mathbf{0}$, from an application of rule (7) we can infer that $f(p,\mathbf{0}) \xrightarrow{\mu} \mathbf{0}$ and thus $f(p,\mathbf{0})+f(\mathbf{0},p) \xrightarrow{\mu} \mathbf{0}$. However, there is no μ-transition form $p \parallel \mathbf{0}$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel \mathbf{0} \xrightarrow{\mu} q$, then q is a process that will always be able to perform a second μ-transition. Hence we would have $p \parallel \mathbf{0} \not\underline{\leftrightarrow} f(p,\mathbf{0})+f(\mathbf{0},p)$, thus contradicting the soundness of Equation (4).

• CASE t IS A TERM OF THE FORM $g(y_1,x_2)$ for some CCS$_f$ operator g. We can distinguish three cases, according to which operator is used:
  - g IS THE PREFIX OPERATOR. We can distinguish two cases, according to which variable of the rule occurs in t:
    * $t = \nu.y_1$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (7) we can infer that $f(p,\mathbf{0}) \xrightarrow{\mu} \nu.\mathbf{0} \xrightarrow{\nu} \mathbf{0}$, and thus $f(p,\mathbf{0})+f(\mathbf{0},p) \xrightarrow{\mu}\xrightarrow{\nu} \mathbf{0}$. However, $p \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0} \stackrel{\nu}{\not\rightarrow}$. Hence, we would have that $p \parallel \mathbf{0} \not\underline{\leftrightarrow} f(p,\mathbf{0})+f(\mathbf{0},p)$, thus contradicting the soundness of Equation (4).
    * $t = \nu.x_2$. This case is analogous to the previous one.
  - g IS THE NONDETERMINISTIC CHOICE OPERATOR and thus $t = y_1 + x_2$. Consider processes $p = \mu.\mu.\mathbf{0}$ and $q = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} q$, from an application of rule (7) we can infer that $f(p,q) \xrightarrow{\mu} q+q \xrightarrow{\mu} \mathbf{0}$, and thus $f(p,q)+f(q,p) \xrightarrow{\mu}\xrightarrow{\mu} \mathbf{0}$. However, there is no process p' such that $p \parallel q \xrightarrow{\mu}\xrightarrow{\mu} p'$ and $p' \underline{\leftrightarrow} \mathbf{0}$, since p' can always perform an additional μ-transition. Hence, we would have $p \parallel q \not\underline{\leftrightarrow} f(p,q)+f(q,p)$, which contradicts the soundness of Equation (4).
  - $g = f$. First of all, we notice that in this case we can infer that f cannot have both types of rules, (7) and (8), for all actions. In fact, if this was the case, due to Lemmas 5 and 6, the set of rules defining the behaviour of $f(x_1,x_2)$ would be

$$\frac{x_1 \xrightarrow{\mu} y_1}{f(x_1,x_2) \xrightarrow{\mu} f(y_1,x_2)} \qquad \frac{x_2 \xrightarrow{\mu} y_2}{f(x_1,x_2) \xrightarrow{\mu} f(x_1,y_2)}$$

$$\frac{x_1 \xrightarrow{\alpha} y_1 \quad x_2 \xrightarrow{\bar{\alpha}} y_2}{f(x_1,x_2) \xrightarrow{\tau} y_1 \parallel y_2}$$

with $\mu \in \{a,\bar{a},\tau\}$ and $\alpha \in \{a,\bar{a}\}$. Clearly, operator f would then be a mere renaming of the parallel composition operator. In particular, as a one-to-one correspondence between the rules for f and those for $\parallel$ could be established, we have that $f(x,y)$ would be *bisimilar under formal hypothesis* to $x \parallel y$ (see [26, Definition 1.10]) and therefore, by [26, Theorem 1.12], we could directly conclude that $f(x,y) \approx x \parallel y$ for all x,y. However, this would contradict Equation (5). Therefore, we can infer that there is at least one action $\mu \in \{a,\bar{a},\tau\}$ for which only one rule among (7) and (8) is available. According to our current simplifying assumptions, let (7) be the available rule for f with label μ. We can distinguish two cases, according to the occurrences of the variables of the rule in t:

* $t = f(y_1, x_2)$. Consider process $p = \mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mathbf{0}$, from an application of rule (7) we can infer that $f(p,p) \xrightarrow{\mu} f(\mathbf{0},a)$, and thus $f(p,p)+f(p,p) \xrightarrow{\mu} f(\mathbf{0},a)$, with $f(\mathbf{0},a) \leftrightarrow \mathbf{0}$, since only rules of the form (7) are available with respect to action μ. However, there is no μ-transition form $p \parallel p$ to a process bisimilar to $\mathbf{0}$, as whenever $p \parallel p \xrightarrow{\mu} q$ then q is a process that will always be able to perform a second μ-transition. Hence, we would have $p \parallel p \not\leftrightarrow f(p,p)+f(p,p)$, thus contradicting the soundness of Equation (4).
* $t = f(x_2, y_1)$. Consider process $p = \mu.\mu.\mathbf{0}$. Since $p \xrightarrow{\mu} \mu.\mathbf{0}$, and only rules of the from (7) are available with respect to action μ, we can infer that $f(p,\mathbf{0}) \xrightarrow{\mu} f(\mathbf{0},\mu.\mathbf{0}) \not\xrightarrow{\mu}$ and $f(\mathbf{0},p) \not\rightarrow$, which means that $f(p,\mathbf{0})+f(\mathbf{0},p)$ cannot perform two μ-transitions in a row. However, we have that $p \parallel \mathbf{0} \xrightarrow{\mu} \mu.\mathbf{0} \parallel \mathbf{0} \xrightarrow{\mu} \mathbf{0} \parallel \mathbf{0}$. Hence, we would have $p \parallel \mathbf{0} \not\leftrightarrow f(p,\mathbf{0})+f(\mathbf{0},p)$, thus contradicting the soundness of Equation (4).

□

B.4 Proof of Proposition 2

Proof of Proposition 2. We argue that the relation

$$\mathscr{B} = \{(p \parallel q, f(p,q)+f(q,p)) \mid p,q \text{ closed terms in the language } \mathrm{CCS}_f\} \cup \leftrightarrow$$

is a bisimulation. To this end, pick closed terms p,q. Now show, using the information on the rules for f given in the proviso of the lemma, that, for each action μ and closed term r,

- whenever $p \parallel q \xrightarrow{\mu} r$, there is a term r' that is equal to r up to commutativity of $\parallel$ such that $f(p,q)+f(q,p) \xrightarrow{\mu} r'$, and
- whenever $f(p,q)+f(q,p) \xrightarrow{\mu} r$, there is a term r' that is equal to r up to commutativity of $\parallel$ such that $p \parallel q \xrightarrow{\mu} r'$.

The claim follows because $\parallel$ is commutative modulo $\leftrightarrow$. □

B.5 Proof of Lemma 8

Proof of Lemma 8. Assume, towards a contradiction, that f is distributive in both arguments with respect to summation. Then, using Equation (4), we have that:

$$\begin{array}{rcl}(x+y) \parallel z & \approx & f(x+y,z)+f(z,x+y) \\ & \approx & f(x,z)+f(y,z)+f(z,x)+f(z,y) \\ & \approx & (x \parallel z)+(y \parallel z) \ .\end{array}$$

However, this is a contradiction because, as is well known, the equation

$$(x+y) \parallel z \approx (x \parallel z)+(y \parallel z)$$

is not sound in bisimulation semantics. For example, our readers can easily verify that

$$(a+\tau) \parallel a \not\leftrightarrow (a \parallel a)+(\tau \parallel a) \ .$$

□

C Proofs of results in Section 6

C.1 Proof of Proposition 3

Proof of Proposition 3. We prove the three statements separately.

- PROOF OF STATEMENT 1. Assume that $E \vdash t \approx u$. We shall argue that $\widehat{E}$ proves the equation $\hat{t} \approx \hat{u}$ by induction on the depth of the proof of $t \approx u$ from E. We proceed by a case analysis on the last rule used in the proof. Below we only consider the two most interesting cases in this analysis.
  - CASE $E \vdash t \approx u$, BECAUSE $\sigma(t') = t$ AND $\sigma(u') = u$ FOR SOME EQUATION $(t' \approx u') \in E$. Note, first of all, that, by the definition of $\widehat{E}$, the equation $\widehat{t'} \approx \widehat{u'}$ is contained in $\widehat{E}$. Observe now that
  $$\hat{t} = \hat{\sigma}(\widehat{t'}) \text{ and } \hat{u} = \hat{\sigma}(\widehat{u'}) \enspace ,$$
  where $\hat{\sigma}$ is the substitution mapping each variable x to the term $\widehat{\sigma(x)}$. It follows that the equation $\hat{t} \approx \hat{u}$ can be proven from the axiom system $\widehat{E}$ by instantiating the equation $\widehat{t'} \approx \widehat{u'}$ with the substitution $\hat{\sigma}$, and we are done.
  - CASE $E \vdash t \approx u$, BECAUSE $t = t_1 \| t_2$ AND $u = u_1 \| u_2$ FOR SOME t_i, u_i $(i = 1,2)$ SUCH THAT $E \vdash t_i \approx u_i$ $(i = 1,2)$. Using the inductive hypothesis twice, we have that $\widehat{E} \vdash \widehat{t_i} \approx \widehat{u_i}$ $(i = 1,2)$. Therefore, using substitutivity, $\widehat{E}$ proves that
  $$\hat{t} = f(\widehat{t_1}, \widehat{t_2}) + f(\widehat{t_2}, \widehat{t_1}) \approx f(\widehat{u_1}, \widehat{u_2}) + f(\widehat{u_2}, \widehat{u_1}) = \hat{u} \enspace ,$$
  which was to be shown.

  The remaining cases are simpler, and we leave the details to the reader.
- PROOF OF STATEMENT 2. Assume that t and u are two bisimilar terms in the language CCS_f^-. We shall argue that $\widehat{E}$ proves the equation $t \approx u$. To this end, we begin by noting that the equation $t \approx u$ also holds in the algebra of CCS_f terms modulo bisimulation. In fact, for each term v in the language CCS_f and closed substitution σ mapping variables to CCS_f terms, we have that
  $$\sigma(v) \underline{\leftrightarrow} \hat{\sigma}(v) \enspace ,$$
  where the substitution $\hat{\sigma}$ is defined as above.

  Since E is complete for bisimilarity over CCS_f by our assumptions, it follows that E proves the equation $t \approx u$. Therefore, by statement 1 of the proposition, we have that $\widehat{E}$ proves the equation $\hat{t} \approx \hat{u}$. The claim now follows because $\hat{t} = t$ and $\hat{u} = u$.
- PROOF OF STATEMENT 3. This is an immediate consequence of statement 2 because $\widehat{E}$ has the same cardinality of E, and is therefore finite, if so is E.

□

C.2 Proof of Lemma 10

Proof of Lemma 10. The "if" implication is an immediate consequence of the soundness of the equations A4 and F1 with respect to $\underline{\leftrightarrow}$. To prove the "only if" implication, define, first of all, the collection NIL of CCS_f^- terms as the set of terms generated by the following grammar:

$$t ::= \mathbf{0} \mid t + t \mid f(t, u) \enspace ,$$

where u is an arbitrary CCS_f^- term. We claim that:

Claim 1. Each CCS_f^- term t is bisimilar to $\mathbf{0}$ if, and only if, $t \in \text{NIL}$.

Using this claim and structural induction on $t \in \text{NIL}$, it is a simple matter to show that if $t \underline{\leftrightarrow} \mathbf{0}$, then $t \approx \mathbf{0}$ is provable using axioms A4 and F1 from left to right, which was to be shown.

To complete the proof, it therefore suffices to show the above claim. To establish the "if" implication in the statement of the claim, one proves, using structural induction on t and the congruence properties of bisimilarity (Fact 1), that if $t \in \text{NIL}$, then $\sigma(t) \underline{\leftrightarrow} \mathbf{0}$ for every closed substitution σ. To show the "only if" implication, we establish the contrapositive statement, viz. that if $t \notin \text{NIL}$, then $\sigma(t) \underline{\not\leftrightarrow} \mathbf{0}$ for some closed substitution σ. To this end, it suffices only to show, using structural induction on t, that if $t \notin \text{NIL}$, then $\sigma_a(t) \xrightarrow{\mu}$ for some action $\mu \in \{a, \bar{a}, \tau\}$, where σ_a is the closed substitution mapping each variable to the closed term $a\mathbf{0}$. The details of this argument are not hard, and are therefore left to the reader. □

C.3 Proof of Lemma 12

Proof of Lemma 12. We limit ourselves to sketching the proofs of statements 1 and 5 in the lemma.

In the proof of statement 1, the only non-trivial thing to check is that the equation

$$\sigma(\sigma'(t)/\mathbf{0}))/\mathbf{0} \approx \sigma(\sigma'(u)/\mathbf{0}))/\mathbf{0}$$

is contained in $\text{cl}(E)$, whenever $(t \approx u) \in E$ and σ, σ' are $\mathbf{0}$-substitutions. This follows from Lemma 11(4) because the collection of $\mathbf{0}$-substitutions is closed under composition.

To show statement 5, it suffices only to argue that each equation $t \approx u$ that is provable from $\text{cl}(E)$ is also provable from E, if E contains the equations A1–A4, F0–F2 in Table 2. This can be done by induction on the depth of the proof of the equation $t \approx u$ from $\text{cl}(E)$, using Lemma 11(1) for the case in which $t \approx u$ is a substitution instance of an axiom in $\text{cl}(E)$. □

D Proofs of results in Section 7

D.1 Proof of Lemma 13

Proof of Lemma13. The first claim is immediate because the norm of $\mu^{\leq m}$ is one, for each $m \geq 1$.

For the second claim, assume by contradiction that there are process terms p,q such that $p,q \not\underline{\leftrightarrow} \mathbf{0}$ and $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m} \underline{\leftrightarrow} p\|q$. Clearly, this would imply the existence of process terms p',q' such that $p \xrightarrow{\nu} p'$ and $q \xrightarrow{\nu} q'$ so that $p\|q \xrightarrow{\nu} p'\|q$ and $p\|q \xrightarrow{\nu} p\|q'$. However, these transitions would in turn imply that $p\|q \xrightarrow{\nu} p'\|q \xrightarrow{\nu} p'\|q'$, namely $p\|q$ could perform two ν-moves in a row, whereas $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m}$ cannot perform such a sequence of actions, thus contradicting $\nu.\mu^{\leq i_1} + \cdots + \nu.\mu^{\leq i_m} \underline{\leftrightarrow} p\|q$. □

D.2 Proof of Lemma 16

Proof of Lemma 16. The proof is by induction on the structure of t. The only interesting case is the inductive step corresponding to $t = f(t_1,t_2)$, which we expand below. According to which rules are available for f with respect to α, we can distinguish three cases:

1. CASE f HAS ONLY A RULE OF THE FORM (7) FOR α. Notice that, consequently, only L^f_α holds. Then, $f(\sigma(t_1),\sigma(t_2)) \xrightarrow{\alpha} p$ can be inferred only by a transition of the form $\sigma(t_1) \xrightarrow{\alpha} p'$ for some closed term p' with $p = p'\|\sigma(t_2)$. By induction over the derivation of $\sigma(t_1) \xrightarrow{\alpha} p'$, and considering that only L^f_α holds, we can then distinguish two cases:
   - There is a term t' such that $t_1 \xrightarrow{\alpha} t'$ and $\sigma(t'_1) = p'$. As f has the rule of the form (7) for α we can immediately infer that $t \xrightarrow{\alpha} t'\|t_2$. Hence, by letting $t' = t'_1\|t_2$, we obtain $t \xrightarrow{\alpha} t'$ and $\sigma(t') = p$.
   - There are a variable x, a closed term q and a configuration c_1 such that $\sigma(x) \xrightarrow{\alpha} q$, $t_1 \xrightarrow{x_\mathrm{l}}_\alpha c_1$ with $\sigma[x_\mathrm{d} \mapsto q](c_1) = p'$. Hence, by applying the auxiliary rule (a_6) in Table 3 we can infer that $f(t_1,t_2) \xrightarrow{x_\mathrm{l}}_\alpha c_1\|t_2$ and moreover, since x_d may occur only in c_1, we have $p = p'\|\sigma(t_2) = \sigma[x_\mathrm{d} \mapsto q](c_1\|t_2)$.
2. CASE f HAS ONLY A RULE OF THE FORM (8) FOR α. Then, in this case, only R^f_α holds. This case is analogous to the previous one (it is enough to switch the roles of t_1 and t_2 and consider x_r in place of x_l) and therefore omitted.
3. CASE f HAS BOTH A RULE OF THE FORM (7) AND ONE OF THE FORM (8) FOR α. Thus $L^f_\alpha \wedge R^f_\alpha$ holds. This case follows by noticing that $t \xrightarrow{x_\mathrm{b}}_\alpha$ can be inferred from both $t_1 \xrightarrow{x_\mathrm{b}}_\alpha$ and $t_2 \xrightarrow{x_\mathrm{b}}_\alpha$, and therefore the follows from the structure of the previous two cases, using rules (a_8) and (a_9).

□

D.3 Proof of Lemma 17

Proof of Lemma 17. We prove the two implications separately.

($\Rightarrow$) We proceed by induction over the structure of t. The only interesting case is the inductive step corresponding to $t = f(t_1,t_2)$ which we expand below, by distinguishing three cases, according to which rules for f are available:

- $x \triangleleft_{\mathrm{l}}^{\mu} f(t_1,t_2)$. This can only be due to $x \triangleleft_{\mathrm{l}}^{\mu} t_1$. By structural induction over t_1, this implies that $t_1 \xrightarrow{x_{\mathrm{l}}}_{\mu} c_1$ with $c_1 \underline{\leftrightarrow} x_{\mathrm{d}} \| t_1'$ for some t_1'. By applying the auxiliary rule (a_6) in Table 3, we infer $f(t_1,t_2) \xrightarrow{x_{\mathrm{l}}}_{\mu} c$ with $c = c_1 \| t_2$ and, by congruence closure and the associativity of $\|$ with respect to $\underline{\leftrightarrow}$, we get $c \underline{\leftrightarrow} (x_{\mathrm{d}} \| t_1') \| t_2 \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ with $t' \underline{\leftrightarrow} t_1' \| t_2$.

- $x \triangleleft_{\mathrm{r}}^{\mu} f(t_1,t_2)$. This can only be due to $x \triangleleft_{\mathrm{r}}^{\mu} t_2$. Thus, we can proceed as in the previous case, by applying the auxiliary rule (a_7) in Table 3 in place of rule (a_6) and using the commutativity of $\|$ with respect to $\underline{\leftrightarrow}$.

- $x \triangleleft_{\mathrm{b}}^{\mu} f(t_1,t_2)$. This can be due to either $x \triangleleft_{\mathrm{b}}^{\mu} t_1$ or $x \triangleleft_{\mathrm{b}}^{\mu} t_2$. For both, we can proceed as in the previous cases, by applying the auxiliary rules (a_8) or, respectively, (a_9) in Table 3 in place of rules (a_6) and (a_7).

($\Leftarrow$) We proceed by induction over the derivation of the open transition $t \xrightarrow{x_{\mathrm{w}}}_{\mu} c$. Again, the only interesting case is the inductive step corresponding to $t = f(t_1,t_2)$, which we expand below by considering three cases, according to which rules are available for f:

- $f(t_1,t_2) \xrightarrow{x_{\mathrm{l}}}_{\mu} c$ with $c \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$ for some t'. According to the open operational semantics in Table 3, it must be the case that $t_1 \xrightarrow{x_{\mathrm{l}}}_{\mu} c_1$ for some c_1 such that $c = c_1 \| t_2$. Notice that since x_{d} can occur only in c_1, from $c = c_1 \| t_2$ and $c \underline{\leftrightarrow} x_{\mathrm{d}} \| t'$, we infer $c_1 \underline{\leftrightarrow} x_{\mathrm{d}} \| t''$ for some t'' such that $t'' \| t_2 \underline{\leftrightarrow} t'$. Hence, we can apply structural induction to the transition by t_1 and obtain $x \triangleleft_{\mathrm{l}}^{\mu} t_1$. Since $t = f(t_1,t_2)$ we can immediately conclude that $x \triangleleft_{\mathrm{l}}^{\mu} t$.

- $f(t_1,t_2) \xrightarrow{x_{\mathrm{r}}}_{\mu} c$. It follows by a similar reasoning.

- $f(t_1,t_2) \xrightarrow{x_{\mathrm{b}}}_{\mu} c$. It follows by a similar reasoning.

$\square$

D.4 Proof of Lemma 18

Proof of Lemma 18. For simplicity of notation let $I = \{1,\dots,n\}$. Since there is a transition $\sum_{i\in I} \alpha.p_i + q \xrightarrow{\alpha} p_i$ for each $i \in I$, from $\sigma(t) \underline{\leftrightarrow} \sum_{i\in I} \nu.p_i + q$ we get that $\sigma(t) \xrightarrow{\alpha} r_i$ with $r_i \underline{\leftrightarrow} p_i$, for all $i \in I$. Since n is greater than the size of t, we infer that Lemma 16.1 can be applied only to m such transitions, for some $m < n$, so that there are an index set $H \subset I$ and CCS$_f$ terms t_h, for $h \in H$ such that $|H| = m$, $t \xrightarrow{\alpha} t_h$ and $\sigma(t_h) \underline{\leftrightarrow} p_h$. Notice that since $p_i \not\underline{\leftrightarrow} p_j$ for $i \neq j$ we get the t_h are pairwise distinct. Let $J = I \setminus H$. For the remaining ν-transitions $\sigma(t) \xrightarrow{\alpha} r_j$ for $j \in J$ we have that one among cases 2a–2c of Lemma 16 applies, according to which rules are available for f with respect to action α. Hence, we have that, for each $j \in J$ there are a variable x_j, a closed term q_j and a configuration c_j such that $\sigma(x_j) \xrightarrow{\alpha} q_j$, $t \xrightarrow{x_{j,\mathrm{w}}}_{\alpha} c_j$ and $\sigma[x_{j,\mathrm{d}} \mapsto q_j] = r_j$, where $\mathrm{w} \in \{\mathrm{l},\mathrm{r},\mathrm{b}\}$ depends on the rules for f. Once again, since n is greater than the size of t there cannot be more than $|J| - 1$ distinct variables x_j occurring in t and causing such α-moves. Hence, there is at least one variable $x \in var(t)$ such that $\sigma(x) \underline{\leftrightarrow} \alpha.q_{j_1} + \alpha.q_{j_2} + q'$ for some $j_1 \neq j_2 \in J$ and closed term q'. $\square$

D.5 Proof of Lemma 19

Proof of Lemma19. Assume, towards a contradiction, that t is not a variable. We proceed by a case analysis on the possible form this term may have.

1. CASE $t = \nu.t'$ FOR SOME TERM t'. Then $\nu = \alpha$ and $\mu^{\leq i_1} \leftrightarrow \sigma(t') \leftrightarrow \mu^{\leq i_m}$. However, this is a contradiction because, since $i_1 < i_m$, the terms $\mu^{\leq i_1}$ and $\mu^{\leq i_m}$ have different depths, and are therefore not bisimilar.
2. CASE $t = f(t',t'')$ FOR SOME TERMS t',t''. Since $\sigma(t)$ has no $\mathbf{0}$ factors, we have that $\sigma(t') \not\leftrightarrow \mathbf{0}$ and $\sigma(t'') \not\leftrightarrow \mathbf{0}$.

   Observe now that $\alpha.\mu^{\leq i_1} + \alpha.\mu^{\leq i_m} \xrightarrow{\alpha} \mu^{\leq i_m}$. Thus, as

   $$\sigma(t) = f(\sigma(t'),\sigma(t'')) \leftrightarrow \alpha.\mu^{\leq i_1} + \cdots + \alpha.\mu^{\leq i_m} \ ,$$

   according to which rules are available for f with respect to ν, we can distinguish the following two cases:

   - L^f_α holds and there is a term p' such that

     $$\sigma(t') \xrightarrow{\alpha} p' \text{ and } p' \| \sigma(t'') \leftrightarrow \mu^{\leq i_m} \ .$$

     As $\sigma(t'') \not\leftrightarrow \mathbf{0}$ and $\mu^{\leq i_m}$ is prime (Lemma 13(1)), this implies that $p' \leftrightarrow \mathbf{0}$ and

     $$\sigma(t'') \leftrightarrow \mu^{\leq i_m} \ .$$

     Since $\alpha.\mu^{\leq i_1} + \cdots + \alpha.\mu^{\leq i_m} \xrightarrow{\alpha} \mu^{\leq i_1}$, a similar reasoning allows us to conclude that

     $$\sigma(t'') \leftrightarrow \mu^{\leq i_1}$$

     also holds. However, this is a contradiction because by the proviso of the lemma $m > 1$ and $1 \leq i_1 < \ldots < i_m$, and therefore $\mu^{\leq i_1}$ and $\mu^{\leq i_m}$ are not bisimilar.
   - R^f_α holds and there is a term p'' such that

     $$\sigma(t'') \xrightarrow{\alpha} p'' \text{ and } \sigma(t') \| p'' \leftrightarrow \mu^{\leq i_m} \ .$$

     This case is analogous to the previous one and leads as well to a contradiction.

We may therefore conclude that t must be a variable, which was to be shown. □

D.6 Proof of Proposition 6

Proof of Proposition 6. Observe, first of all, that since t and u have no $\mathbf{0}$ summands or factors, by Remark 3 we can assume that $t = \sum_{i\in I} t_i$ and $u = \sum_{j\in J} u_j$ for some finite non-empty index sets I,J, where none of the t_i ($i \in I$) and u_j ($j \in J$) has $+$ as its head operator, and none of the t_i ($i \in I$) and u_j ($j \in J$) have $\mathbf{0}$ summands or factors. Therefore, $x \lhd^\alpha_{\mathrm{w}} t$ implies that there is some index $i \in I$ such that $x \lhd^\alpha_{\mathrm{w}} t_i$. We then proceed by a case analysis on the rules available for f. Actually we expand only the case in which only L^f_α holds, as the other cases two cases, in which respectively only R^f_α holds, and $L^f_\alpha \wedge R^f_\alpha$ holds, can be obtained analogously.

Since only L^f_α holds, then it must be the case that $x \lhd^\alpha_1 t_i$. By Lemma 17 we get that $t_i \xrightarrow{x_1}_\alpha c$ for some configuration c with $c \leftrightarrow x_{\mathrm{d}} \| t'$ for some t'. Let n be greater than the size of t and consider the substitution σ such that

$$\sigma(y) = \begin{cases} \alpha \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i} & \text{if } y = x \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

For simplicity of notation, let $p_n = \sum_{i=1}^{n} \bar{\alpha}\alpha^{\leq i}$. Clearly $\sigma(x) \xrightarrow{\alpha} p_n$. By Lemma 15 we obtain that $\sigma(t_i) \xrightarrow{\alpha} p$ with $p = \sigma[x_{\mathrm{d}} \mapsto p_n](c)$ and, thus, $p \leftrightarrow p_n \| \sigma(t')$. As $t \leftrightarrow u$ implies $\sigma(t) \leftrightarrow \sigma(u)$, we get that there is an index $j \in J$ such that $\sigma(u_j) \xrightarrow{\alpha} q$ for some $q \leftrightarrow p_n \| \sigma(t')$. As only L^f_α holds, by Lemma 16 we can distinguish two cases:

- There are a variable y, a closed term q' and a configuration c' such that $\sigma(y) \xrightarrow{\alpha} q'$, $u_j \xrightarrow{y_1}_{\alpha} c'$ and $q = \sigma[y_\mathrm{d} \mapsto q'](c')$. Since σ maps all variables but x to $\mathbf{0}$, we can directly infer that $y = x$, $q' = p_n$. Moreover, as p_n is prime and there is a unique prime decomposition of processes, we also infer that $c' \leftrightarrow x_\mathrm{d} \| u'$ for some u' with $\sigma(u') \leftrightarrow \sigma(t')$. Consequently, by Lemma 17 we can conclude that $x \lhd_1^\alpha u_j$ and thus $x \lhd_1^\alpha u$ as required.
- There is a term u' such that $u_j \xrightarrow{\alpha} u'$ and $\sigma(u') \leftrightarrow p_n \| \sigma(t')$. We proceed to show that this case leads to a contradiction. We distinguish two cases:
  - $\sigma(t') \leftrightarrow \mathbf{0}$. Thus $\sigma(u') \leftrightarrow p_n$ and we can rewrite $u' = \sum_{h \in H} v_h$ for some terms v_h that do not have $+$ as head operator. Moreover, since u not having $\mathbf{0}$ summands nor factors implies that neither u_j no u' have some, the same holds for all the v_h. Since n is lager than the size of u, and thus than that of u', by Lemma 19 $\sigma(u') \leftrightarrow p_n$ implies that there is one index $h \in H$ such that $v_h = y$ for some variable y and $\sigma(y) \leftrightarrow \bar{\alpha}\alpha^{\leq i_1} + \cdots + \bar{\alpha}\alpha^{\leq i_m}$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m \leq n$. However, by the choice of σ, all variables but x are mapped to $\mathbf{0}$, and moreover $\sigma(x) \not\leftrightarrow \bar{\alpha}\alpha^{\leq i_1} + \cdots + \bar{\alpha}\alpha^{\leq i_m}$ thus contradicting $\sigma(u') \leftrightarrow p_n$.
  - $\sigma(t') \not\leftrightarrow \mathbf{0}$. Consequently, $\sigma(t') \leftrightarrow \sum_{h \in H} \mu_h q_h$ for some actions $\mu_h \in \{a, \bar{a}, \tau\}$ and closed terms q_h. We can therefore apply the expansion law for parallel composition obtaining

$$\begin{aligned} \sigma(u') &\leftrightarrow p_n \| \sigma(t') \\ &\leftrightarrow \sum_{i=1}^{n} \bar{\alpha}(\alpha^{\leq i} \| \sigma(t')) + \sum_{h \in H} \mu_h (p_n \| q_h) + \sum_{\substack{i=1,\dots,n \\ h \in H \text{ s.t. } \mu_h = \alpha}} \tau(\alpha^{\leq i} \| q_h). \end{aligned}$$

    We notice that the first term in the expansion has size at least $n+1$ and therefore greater than the size of u and in particular of u'. Moreover $\alpha^{\leq i} \| \sigma(t') \not\leftrightarrow \alpha^{\leq j} \| \sigma(t')$ whenever $i \neq j$. Therefore, by Lemma 18 there is a variable $y \in var(u')$ such that $\sigma(y) \leftrightarrow \bar{\alpha}(\alpha^{\leq i_1} \| \sigma(t')) + \cdots + \bar{\alpha}(\alpha^{\leq i_m} \| \sigma(t')) + r$ for some $m > 1$ and $1 \leq i_1 < \cdots < i_m$ and closed term r. However, $\sigma(y) = \mathbf{0}$ whenever $y \neq x$ and $\sigma(x) \not\leftrightarrow \bar{\alpha}(\alpha^{\leq i_1} \| \sigma(t')) + \cdots + \bar{\alpha}(\alpha^{\leq i_m} \| \sigma(t')) + r$, for any closed term r, thus contradicting $\sigma(u') \leftrightarrow p_n \| \sigma(t')$.

We have therefore obtained that whenever $x \lhd_1^\alpha t$ then also $x \lhd_1^\alpha u$.

Assume now that t has a summand x. We aim to show that u has a summand x as well. Since $x \lhd_1^\alpha x$ gives $x \lhd_1^\alpha t$, by the first part of the Proposition we get $x \lhd_1^\alpha u$ and thus there is an index $j \in J$ such that $x \lhd_1^\alpha u_j$. Notice that since we are considering the case of f that does not distribute with respect to either of its arguments, there must be at least one action $\mu \in \{a, \bar{a}, \tau\}$ such that R^f_μ holds. Assume such an action μ. Again, let n be greater than the size of t and consider the substitution

$$\sigma_1(y) = \begin{cases} \alpha\alpha^{\leq n} & \text{if } y = x \\ \alpha + \mu & \text{otherwise.} \end{cases}$$

Thus $\sigma_1(x) \xrightarrow{\alpha} \alpha^{\leq n}$ and consequently $\sigma_1(t) \xrightarrow{\alpha} \alpha^{\leq n}$. Since $\sigma_1(t) \leftrightarrow \sigma_1(u)$ it must hold that $\sigma_1(u) \xrightarrow{\alpha} q$ for some $q \leftrightarrow \alpha^{\leq n}$. As n is greater than the size of u, one can infer that u can have a summand given by at most $\lfloor \frac{n-2}{2} \rfloor$ nested occurrences of f (which is a binary operator of size at least 3). Since, moreover, all variables but x are mapped into a term of depth 1, we can infer that the only term that can be responsible for the α-move to q is the summand u_j such that $x \lhd_1^\alpha u_j$. To show $u_j = x$ we show that the only other possible case, namely $u_j = f(u', u'')$ with $x \lhd_1^\alpha u'$ leads to a contradiction. Recall that by the proviso of the Proposition u has no $\mathbf{0}$ factors, which implies that $u', u'' \not\leftrightarrow \mathbf{0}$. Since moreover, $x \lhd_{\mathrm{l}}^{\alpha} u'$, by Lemma 17 and Lemma 16 we get $u' \xrightarrow{x_{\mathrm{l}}}_{\alpha} c$ and thus $u_j \xrightarrow{x_{\mathrm{l}}}_{\alpha} c \| u''$ for some configuration $c \leftrightarrow x_{\mathrm{d}} \| u'''$ for some term u''', so that $\sigma_1(u_j) \xrightarrow{\alpha} \sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\leq n}](c) \| \sigma_1(u'') = q$. However, $u'' \not\leftrightarrow \mathbf{0}$ implies that either there is a term v such that $u'' \xrightarrow{\nu} v$, for some action ν, or in u'' at least one variable occurs unguarded. Hence, by the choice of σ_1, as both L_{α}^{f} and R_{μ}^{f} hold, we can infer that $depth(\sigma_1(u'')) \geq 1$ which gives

$$\begin{aligned}
n &= depth(\alpha^{\leq n}) \\
&= depth(q) \\
&= depth(\sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\leq n}](c) \| \sigma_1(u'')) \\
&= depth(\sigma_1[x_{\mathrm{d}} \mapsto \alpha^{\leq n}](c)) + depth(\sigma_1(u'')) \\
&\geq depth(\alpha^{\leq n}) + depth(\sigma_1(u'')) \\
&\geq n+1
\end{aligned}$$

thus contradicting $q \leftrightarrow \alpha^{\leq n}$. □

D.7 Proof of Lemma 20

Proof of Lemma 20. The proof proceeds by structural induction over t and a case analysis over $\mathrm{w} \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$. The only interesting case is the inductive step corresponding to $t = f(t_1, t_2)$ which we expand below for the case of $\mathrm{w} = \mathrm{l}$. The other cases can be obtained by applying a similar reasoning.

Moreover, always for sake of simplicity, assume that there is only one action μ such that $x \lhd_{\mathrm{l}}^{\mu} t$, so that $\mathrm{init}(\sigma(x)) = \{\mu\}$. Once again, the general case can be easily derived from this one. Notice that this implies the existence of a closed term q such that $\sigma(x) \xrightarrow{\mu} q$ and $depth(\sigma(x)) = depth(q) + 1$. We have that $x \lhd_{\mathrm{l}}^{\mu} f(t_1, t_2)$ can be derived only by $x \lhd_{\mathrm{l}}^{\mu} t_1$. Hence, structural induction over t_1 gives $depth(\sigma(t_1)) \geq depth(\sigma(x))$. Moreover, by Lemma 17 we obtain that $t_1 \xrightarrow{x_{\mathrm{l}}}_{\mu} c_1$ for some $c_1 \leftrightarrow x_{\mathrm{d}} \| t'$ for some term t'. Furthermore, $\sigma(x) \xrightarrow{\mu} q$ together with Lemma 15 gives $\sigma(t_1) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto q](c_1)$. Then we can infer that $\sigma(t) \xrightarrow{\mu} \sigma[x_{\mathrm{d}} \mapsto q](c_1) \| \sigma(t_2) \leftrightarrow q \| (\sigma(t') \| \sigma(t_2))$. We have therefore obtained

$$\begin{aligned}
depth(\sigma(t)) &\geq 1 + depth(q \| (\sigma(t') \| \sigma(t_2))) \\
&= 1 + depth(q) + depth(\sigma(t') \| \sigma(t_2)) \\
&\geq 1 + depth(q) \\
&= depth(\sigma(x)).
\end{aligned}$$

□

E Proofs of results in Section 8

E.1 Proof of Lemma 21

Proof of Lemma 21. Since $f(\alpha, p_n)$ is not bisimilar to $\mathbf{0}$, to prove the statement it suffices only to show that $f(\alpha, p_n)$ is irreducible for $n \geq 0$.

If $n = 0$ then $f(\alpha, p_n) = f(\alpha, \mathbf{0})$ is a term of depth 1, and is therefore irreducible as claimed.

Let $n \geq 1$. Assume, towards a contradiction, that $f(\alpha, p_n) \underline{\leftrightarrow} p \| q$ for two closed terms p and q with $p \not\underline{\leftrightarrow} \mathbf{0}$ and $q \not\underline{\leftrightarrow} \mathbf{0}$—that is, $f(\alpha, p_n)$ is *not* irreducible. We have that

$$f(\alpha, p_n) \xrightarrow{\alpha} \mathbf{0} \| p_n \underline{\leftrightarrow} p_n \ .$$

As $f(\alpha, p_n) \underline{\leftrightarrow} p \| q$, there is a transition $p \| q \xrightarrow{\alpha} r$ for some $r \underline{\leftrightarrow} p_n$. Without loss of generality, we may assume that $p \xrightarrow{\alpha} p'$ and $r = p' \| q$. Since we have assumed that $n \geq 1$, by statement 2 and our assumption that $q \not\underline{\leftrightarrow} \mathbf{0}$, we have that $p' \underline{\leftrightarrow} \mathbf{0}$ and $q \underline{\leftrightarrow} p_n$. Again using that $n \geq 1$, it follows that $q \xrightarrow{\bar{\alpha}} q'$ for some q'. This means that $p \| q \xrightarrow{\bar{\alpha}}$, contradicting the assumption that $f(\alpha, p_n) \underline{\leftrightarrow} p \| q$. Thus $f(\alpha, p_n)$ is irreducible, which was to be shown. □

E.2 Proof of Lemma 22

Proof of Lemma 22. Since $f(p,q) \underline{\leftrightarrow} f(\alpha, p_n)$ and $f(\alpha, p_n) \xrightarrow{\alpha} \mathbf{0} \| p_n \underline{\leftrightarrow} p_n$, there is a p' such that $p \xrightarrow{\alpha} p'$ and $p' \| q \underline{\leftrightarrow} p_n$. It follows that $q \underline{\leftrightarrow} p_n$ and $p' \underline{\leftrightarrow} \mathbf{0}$, because p_n is prime (Lemma 13(2)) and $q \not\underline{\leftrightarrow} \mathbf{0}$. We are therefore left to prove that p is bisimilar to α. To this end, note, first of all, that, as $\underline{\leftrightarrow}$ is a congruence over the language CCS_f, we have that

$$f(p, p_n) \underline{\leftrightarrow} f(\alpha, p_n) \ .$$

Assume now that $p \xrightarrow{\mu} p''$ for some action μ and closed term p''. In light of the above equivalence, one of the following two cases may arise:

1. $\mu = \alpha$ and $p'' \| p_n \underline{\leftrightarrow} p_n$ or
2. $\mu = \tau$ and $p'' \| p_n \underline{\leftrightarrow} \alpha^{\leq i}$, for some $i \in \{1, \ldots, n\}$.

In the former case, p'' must have depth 0 and is thus bisimilar to $\mathbf{0}$. The latter case is impossible, because the depth of $p'' \| p_n$ is at least $n + 1$.

We may therefore conclude that every transition of p is of the form $p \xrightarrow{\alpha} p''$, for some $p'' \underline{\leftrightarrow} \mathbf{0}$. Since we have already seen that p affords an α-labelled transition leading to $\mathbf{0}$, modulo bisimulation equivalence, it follows that $p \underline{\leftrightarrow} \alpha$, which was to be shown. □

F Proofs of results in Section 9

F.1 Proof of Lemma 25

Proof of Lemma 25. Since $f(p,q) \underline{\leftrightarrow} f(\alpha,q_n)$ and $f(\alpha,q_n) \xrightarrow{\alpha} \mathbf{0} \| q_n \underline{\leftrightarrow} q_n$, we can distinguish the following two cases depending on whether a matching transition from $f(p,q)$ stems from p or q:

- There is a p' such that $p \xrightarrow{\alpha} p'$ and $p' \| q \underline{\leftrightarrow} q_n$. It follows that $q \underline{\leftrightarrow} q_n$ and $p' \underline{\leftrightarrow} \mathbf{0}$, because q_n is prime (Lemma 13(2)) and $q \not\underline{\leftrightarrow} \mathbf{0}$. We are therefore left to prove that p is bisimilar to α. To this end, note, first of all, that, as $\underline{\leftrightarrow}$ is a congruence over the language CCS$_f$, we have that

$$f(p,q_n) \underline{\leftrightarrow} f(\alpha,q_n) \ .$$

  First of all, notice that the equivalence above implies that $depth(p) = 1$. We proceed to prove that $p \underline{\leftrightarrow} \alpha$. Assume towards a contradiction that $p \not\underline{\leftrightarrow} \alpha$ and thus that $p \xrightarrow{\mu} \mathbf{0}$ for some $\mu \neq \alpha$. We can distinguish two cases, according to whether the predicate L^f_μ holds or not.

  – Assume first that L^f_μ holds. Then we would have $\mathrm{init}(f(p,q_n)) = \{\alpha,\mu\}$ and $\mathrm{init}(f(\alpha,q_n)) = \{\alpha\}$, thus contradicting $f(p,q_n) \underline{\leftrightarrow} f(\alpha,q_n)$.

  – Assume now that L^f_μ does not hold. Then, in light of the above equivalence, from $f(\alpha,q_n) \xrightarrow{\alpha} \alpha \| \bar{\alpha}^{\leq n}$ and the fact that $q_n \not\underline{\leftrightarrow} \bar{\alpha}^{\leq n}$, we can infer that $f(p,q_n) \xrightarrow{\alpha} p \| \bar{\alpha}^{\leq n}$ and $p \| \bar{\alpha}^{\leq n} \underline{\leftrightarrow} \alpha \| \bar{\alpha}^{\leq n}$. Now, if $\mu = \tau$, then $p \| \bar{\alpha}^{\leq n} \xrightarrow{\tau} \mathbf{0} \| \bar{\alpha}^{\leq n} \underline{\leftrightarrow} \bar{\alpha}^{\leq n}$. However, $\alpha \| \bar{\alpha}^{\leq n}$ can perform a τ-move only due to a synchronization between α and one of the $\bar{\alpha}$, thus implying that $\alpha \| \bar{\alpha}^{\leq n} \xrightarrow{\tau} \mathbf{0} \| \bar{\alpha}^i \underline{\leftrightarrow} \bar{\alpha}^i$ for some $i \in \{0,\dots,n-1\}$. Since there is no such index i such that $\bar{\alpha}^{\leq n} \underline{\leftrightarrow} \bar{\alpha}^i$, we obtain a contradiction with $f(p,q_n) \underline{\leftrightarrow} f(\alpha,q_n)$.
    Similarly, if $\mu = \bar{\alpha}$, then $p \| \bar{\alpha}^{\leq n}$ could perform a sequence of $n+1$ transitions all with label $\bar{\alpha}$, whereas $\alpha \| \bar{\alpha}^{\leq n}$ can perform at most n $\bar{\alpha}$-moves in a row. Therefore, also this case is in contradiction with $f(p,q_n) \underline{\leftrightarrow} f(\alpha,q_n)$.

  We may therefore conclude that every transition of p is of the form $p \xrightarrow{\alpha} p''$, for some $p'' \underline{\leftrightarrow} \mathbf{0}$. Since we have already seen that p affords an α-labelled transition leading to $\mathbf{0}$, modulo bisimulation equivalence, it follows that $p \underline{\leftrightarrow} \alpha$, which was to be shown.

- There is a q' such that $q \xrightarrow{\alpha} q'$ and $p \| q' \underline{\leftrightarrow} q_n$. This case can be treated similarly to the previous case and allows us to conclude that $q \underline{\leftrightarrow} \alpha$ and $p \underline{\leftrightarrow} q_n$.

□

G Proofs of results in Section 11

G.1 Proof of Lemma 29

Proof of Lemma 29. The proof is analogous to that of Lemma 22. We remark that the τ-transition by $f(\tau,q_n)$ can be mimicked only by a τ-move by p. To see this, we show that any other case would lead to a contradiction with the proviso of the lemma $f(p,q) \underline{\leftrightarrow} f(\tau,q_n)$. In particular, we distinguish three cases, according to which rule of type (6) is available for f and whether the predicates R^f_τ holds or not.

- Assume $p \xrightarrow{\alpha} p'$ and $q \xrightarrow{\bar{\alpha}} q'$ with $p' \| q' \underline{\leftrightarrow} q_n$. This would contradict $f(\tau,q_n) \underline{\leftrightarrow} f(p,q)$ since $f(p,q) \xrightarrow{\bar{\alpha}} p \| q'$, whereas $f(\tau,q_n) \stackrel{\bar{\alpha}}{\not\to}$.
- Assume $p \xrightarrow{\bar{\alpha}} p'$ and $q \xrightarrow{\alpha} q'$ with $p' \| q' \underline{\leftrightarrow} q_n$. Notice that since q_n is prime, then we have that either $p' \underline{\leftrightarrow} \mathbf{0}$ and $q' \underline{\leftrightarrow} q_n$, or $p' \underline{\leftrightarrow} q_n$ and $q' \underline{\leftrightarrow} \mathbf{0}$. The latter case contradicts $f(p,q) \underline{\leftrightarrow} f(\tau,q_n)$ since the transition $f(p,q) \xrightarrow{\alpha} p \| q' \underline{\leftrightarrow} p \| q_n$ cannot be mimicked by $f(\tau,q_n)$. The former case also contradicts the proviso of the lemma, since we would have $f(p,q) \xrightarrow{\alpha} p \| q' \underline{\leftrightarrow} p \xrightarrow{\bar{\alpha}} p' \underline{\leftrightarrow} q_n$, whereas $f(\tau,q_n) \xrightarrow{\alpha} \tau \| \bar{\alpha}^{\leq i}$, for some $i \in \{1,\dots,n\}$, and there is no r such that $\tau \| \bar{\alpha}^{\leq i} \xrightarrow{\bar{\alpha}} r$ and $r \underline{\leftrightarrow} q_n$, for any $i \in \{1,\dots,n\}$.
- Finally, assume that the predicate R^f_μ holds, and thus that f has a rule of type (8) with label τ. Hence, assume $q \xrightarrow{\tau} q'$, for some q', so that $f(p,q) \xrightarrow{\tau} p \| q' \underline{\leftrightarrow} q_n$. Since q_n is prime and $p \not\underline{\leftrightarrow} \mathbf{0}$, we have that $p \underline{\leftrightarrow} q_n$ and $q' \underline{\leftrightarrow} \mathbf{0}$. So, by congruence closure, we get $f(p,q) \underline{\leftrightarrow} f(q_n,q) \underline{\leftrightarrow} f(\tau,q_n)$. Since $f(\tau,q_n) \xrightarrow{\alpha} \tau \| \bar{\alpha}^{\leq n}$ and only R^f_α holds, we have that $q \xrightarrow{\alpha} q_1$ for some process term q_1 such that $q_n \| q_1 \underline{\leftrightarrow} \tau \| \bar{\alpha}^{\leq n}$, which is a contradiction as $q_n \xrightarrow{\alpha}$ implies $q_n \| q_1 \xrightarrow{\alpha}$, whereas $\tau \| \bar{\alpha}^{\leq n} \stackrel{\alpha}{\not\to}$.

□