The Completeness Problem for Modal Logic*

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Abstract

We introduce the completeness problem for Modal Logic and examine its complexity. For a definition of completeness for formulas, given a formula of a modal logic, the completeness problem asks whether the formula is complete for that logic. We discover that completeness and validity have the same complexity — with certain exceptions for which there are, in general, no complete formulas. To prove upper bounds, we present a two-player game which combines bisimulation games and tableaux, and which determines whether a formula is complete. The game then yields an alternating polynomial-time algorithm.

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1 Introduction

For a modal logic l, we call a modal formula φ *complete* when for every modal formula ψ on the same propositional variables as φ, we can derive from φ in l either the formula ψ or its negation. For different modal logics l, we examine the following problem: given a modal formula φ, is it complete for l? We call this the completeness problem for l and we examine its complexity. Our main results show that the completeness problem has the same complexity as provability, at least for the logics we consider.

Modal Logic is a very well-known family of logics. When one uses it to formally describe a situation, it may be of importance to be able to determine whether the formula/finite theory one uses as a description formalizes exactly one setting (i.e. it is complete), or it leaves room for several instances consistent with this description. Given Modal Logic's wide area of applications and the importance of logical completeness in general, we find it surprising that, to the best of our knowledge, the completeness problem for Modal Logic has not been studied as a computational problem so far. On the other hand, the complexity of satisfiability (and thus validity) for Modal Logic has been studied extensively — for example, see [11, 12, 14].

We examine the completeness problem for several well-known modal logics, namely the extensions of **K** by the axioms Factivity, Consistency, Positive Introspection, and Negative Introspection (also known as T, D, 4, and 5, respectively) — i.e. the ones between **K** and **S5**. We discover that the complexity of provability and completeness tend to be the same: the completeness problem is PSPACE-complete if the logic does not have Negative Introspection and it is coNP-complete otherwise. There are exceptions: for certain logics (**D** and **T**), the completeness problem as we define it is trivial, as these logics have no finite complete theories.

Our motivation partly comes from [3], where Artemov raises the following issue. It is the usual practice in Game Theory (and Epistemic Game Theory) to reason about a game

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based on a model of the game description. On the other hand, it is often the case in an epistemic setting that the game specification is not complete, thus any conclusions reached by examining any single model are precarious. He proposes a syntactic, proof-centered approach, which is more robust and general, and which is based on a syntactic formal description of the game specification. Artemov's approach is more sound, in that it allows one to draw only conclusions that can be safely derived from the game specification, but on the other hand, the model-based approach has been largely successful in Game Theory for a long time. If we can determine that the syntactic specification of a game is complete, then the syntactic and semantic approaches are equivalent and we can describe the game using one model.

For a formula–specification φ (for example, a syntactic description of a game), if we are interested in the formulas we can derive from φ (the conclusions we can draw from the game description), knowing that φ is complete can give a significant computational advantage. If φ is complete and consistent, for a model $\mathcal{M}$ for φ, ψ can be derived from φ exactly when ψ is satisfied in $\mathcal{M}$ (at the same state as φ). Thus, knowing that φ is complete effectively allows us to reduce a derivability problem to a model checking problem, which is easier to solve (see, for example, [11]). This approach may be useful when we need to examine multiple conclusions, especially if the model for φ happens to be fairly small. On the other hand, if φ is discovered to be incomplete, then, as a specification it may need to be refined. We can make similar claims for other areas where Modal Logic is used as a specification language.

Notions similar to complete formulas have been studied before. Characteristic formulas allow one to characterize a state's equivalence class for a certain equivalence relation. In our case, the equivalence relation is bisimulation on states of (finite) Kripke models and the notions of characteristic and complete formulas collapse, by the Hennessy-Milner Theorem [13], in that a formula is complete for one of the logics we consider if and only if it is characteristic for a state in a model for that logic. A construction of characteristic formulas for variants of CCS processes [16] was introduced in [10]. This construction allows one to verify that two CCS processes are equivalent by reducing this problem to model checking. Similar constructions were studied later in [17,21] for instance and in a more general manner in [1,2].

Normal forms for Modal Logic were introduced by Fine [8] and they can be used to prove soundness, completeness, and the finite frame property for several modal logics with respect to their classes of frames. Normal forms are modal formulas that completely describe the behavior of a Kripke model up to a certain distance from a state, with respect to a certain number of propositional variables. Therefore a formula which is complete is also equivalent to a normal form, but not all normal forms are complete, as they may be agnostic with respect to states located further away. We may define that a formula is complete up to depth d for logic l when it is equivalent to a normal form of modal depth (the nesting depth of a formula's modalities) at most d. We discuss these topics more in Section 5.

We focus on a definition of completeness which emphasizes on the formula's ability to either affirm or reject every possible conclusion. We can also consider a version of the problem which asks to determine if a formula is complete up to its modal depth — that is, whether it is equivalent to a normal form. If we are interested in completely describing a setting, the definition we use for completeness is more appropriate. However, it is not hard to imagine situations where this variation of completeness is the notion that fits better, either as an approximation on the epistemic depth agents reason with, or, perhaps, as a description of process behavior for a limited amount of time. We briefly examine this variation in Section 5.

Overview Section 2 provides background on Modal Logic, bisimulation, and relevant complexity results. We draw our first conclusions about the completeness problem in relation to bisimulation and give our first complexity result. In Section 3, we examine different logics and in which cases for each of these logics the completeness problem is non-trivial. In Section 4, we examine the complexity of the completeness problem. We first present a general lower bound. For logics with Negative Introspection we prove coNP-completeness. For the remaining logics — the ones without Negative Introspection for which the problem is not trivial — we present a two-player game which determines whether a formula is complete. The game is then converted into a polynomial-time alternating algorithm which uses an oracle from PSPACE, proving that the completeness problem for these cases is PSPACE-complete. These complexity results are summarized in Table 1. In Section 5, we consider variations of the problem and draw further conclusions.

2 Background

We present needed background on Modal Logic, its complexity, and bisimulation, and we introduce the completeness problem. For an overview of Modal Logic and its complexity, we refer the reader to [4, 6, 11]. We do not provide background on Computational Complexity, but the reader can see [20].

2.1 Modal Logic

We assume a countably infinite set of propositional variables $p_1, p_2, \ldots$. Literals are all p and $\neg p$, where p is a propositional variable. Modal formulas are constructed from literals, the constants $\bot, \top$, the usual operators for conjunction and disjunction $\wedge, \vee$ of propositional logic, and the dual modal operators, $\Box$ and $\Diamond$:

$$\varphi ::= \bot \mid \top \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi.$$

The negation $\neg\varphi$ of a modal formula, implication $\varphi \to \psi$, and $\varphi \leftrightarrow \psi$ are constructed as usual. The language described by the grammar above is called L.

For a finite set of propositional variables P, $L(P) \subseteq L$ is the set of formulas that use only variables from P. For a formula φ, $P(\varphi)$ is the set of propositional variables that appear in φ, so $\varphi \in L(P(\varphi))$. If $\varphi \in L$, then $sub(\varphi)$ is the set of subformulas of φ and $\overline{sub}(\varphi) = sub(\varphi) \cup \{\neg\psi \mid \psi \in sub(\varphi)\}$. For Φ a nonempty finite subset of L, $\bigwedge \Phi$ is a conjunction of all elements of Φ and $\bigwedge \emptyset = \top$; we define $\bigvee \Phi$ similarly. The modal depth $md(\varphi)$ of φ is the largest nesting depth of its modal operators; the size of φ is $|\varphi| = |sub(\varphi)|$.

Normal modal logics use all propositional tautologies and axiom K, Modus Ponens, and the Necessitation Rule:

$$K : \Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi; \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}; \qquad \frac{\varphi}{\Box\varphi}.$$

The logic which has *exactly* these axioms and rules is the smallest normal modal logic, **K**. We can extend **K** with more axioms:

$$D : \Diamond\top; \qquad T : \Box\varphi \to \varphi; \qquad 4 : \Box\varphi \to \Box\Box\varphi; \qquad 5 : \Diamond\varphi \to \Box\Diamond\varphi.$$

We consider modal logics which are formed from a combination of these axioms. Of course, not all combinations make sense: axiom D (also called the Consistency axiom) is a special case of T (the Factivity axiom). Axiom 4 is called Positive Introspection and 5 is called Negative Introspection. Given a logic l and axiom a, $l + a$ is the logic which has as axioms all the axioms of l and a. Logic **D** is $\mathbf{K} + D$, **T** is $\mathbf{K} + T$, $\mathbf{K4} = \mathbf{K} + 4$, $\mathbf{D4} = \mathbf{K} + D + 4 = \mathbf{D} + 4$,

$\mathbf{S4} = \mathbf{K} + T + 4 = \mathbf{T} + 4 = \mathbf{K4} + T$, $\mathbf{KD45} = \mathbf{D4} + 5$, and $\mathbf{S5} = \mathbf{S4} + 5$. From now on, unless we explicitly say otherwise, by a logic or a modal logic, we mean one of the logics we defined above. We use $\vdash_l \varphi$ to mean that φ can be derived from the axioms and rules of l; when l is clear from te context, we may drop the subscript and just write $\vdash$.

A Kripke model is a triple $\mathcal{M} = (W, R, V)$, where W is a nonempty set of states (or worlds), $R \subseteq W \times W$ is an accessibility relation and V is a function which assigns to each state in W a set of propositional variables. If P is a set of propositional variables, then for every $a \in W$, $V_P(a) = V(a) \cap P$. To ease notation, when $(s,t) \in R$ we usually write sRt.

Truth in a Kripke model is defined through relation $\models$ in the following way: $\mathcal{M}, a \models p$ iff $p \in V(a)$, while boolean connectives are treated as usual; finally,

$\mathcal{M}, a \models \Diamond\varphi$ iff there is some $b \in W$ such that aRb and $\mathcal{M}, b \models \varphi$; and

$\mathcal{M}, a \models \Box\varphi$ iff for all $b \in W$ such that aRb it is the case that $\mathcal{M}, b \models \varphi$.

If $\mathcal{M}, a \models \varphi$, we say that φ is true/satisfied in a of $\mathcal{M}$. (W, R) is called a *frame*. We call a Kripke model (W, R, V) (resp. frame (W, R)) finite if W is finite.[1] If $\mathcal{M}$ is a model (for logic l) and a is a state of $\mathcal{M}$, then $(\mathcal{M}, a)$ is a pointed model (resp. for l). For a state $x \in W$ of a frame (W, R), $\mathit{Reach}(x) \subseteq W$ is the set of states reachable from x; i.e. it is the smallest set such that $x \in \mathit{Reach}(x)$ and if $y \in \mathit{Reach}(x)$ and yRz, then $z \in \mathit{Reach}(x)$.

Each modal logic l is associated with a class of frames $F(l)$, that includes all frames (W, R) for which R meets certain conditions, depending on the logic's axioms. If l has axiom:

***D*,** then R must be serial (for every state $a \in W$ there must be some $b \in W$ such that aRb);
***T*,** then R must be reflexive (for all $a \in W$, aRa);
4, then R must be transitive (if $aRbRc$, then aRc);
5, then R must be euclidean (if aRb and aRc, then bRc).

A model (W, R, V) is a model for a logic l if and only if $(W, R) \in F(l)$. We call a formula satisfiable for a modal logic l, if it is satisfied in a state of a model for l. We call a formula valid for a modal logic l, if it is satisfied in all states of all models for l.

▶ **Theorem 1** (Completeness, Finite Frame Property)**.** *A formula φ is valid for l if and only if it is provable in l; φ is satisfiable for l if and only if it is satisfied in a finite model for l.*

For the remainder of this paper we only consider finite Kripke models and frames. For a finite model $\mathcal{M} = (W, R, V)$, we define $|\mathcal{M}| = |W| + |R|$.

▶ **Definition 2.** A formula φ is called *complete* for logic l when for every $\psi \in L(P(\varphi))$, $\vdash_l \varphi \to \psi$ or $\vdash_l \varphi \to \neg\psi$. Formula φ is *incomplete* for l if it is not complete for l.

By Theorem 1, φ is complete for l exactly when for every $\psi \in L(P(\varphi))$, either ψ or its negation is true at every pointed model for l that satisfies φ.

2.2 Bisimulation

An important notion in Modal Logic (and other areas) is that of bisimulation. Let P be a (finite) set of propositional variables. For Kripke models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$, a non-empty relation $\mathcal{R} \subseteq W \times W'$ is a *bisimulation* (respectively, bisimulation modulo P) from $\mathcal{M}$ to $\mathcal{M}'$ when the following conditions are satisfied for all $(s, s') \in \mathcal{R}$:

- $V(s) = V'(s')$ (resp. $V_P(s) = V'_P(s')$).

[1] According to our definition, for a finite model $\mathcal{M} = (W, R, V)$ and $a \in W$, $V(a)$ can be infinite. However, we are mainly interested in (W, R, V_P) for finite sets of propositions P, which justifies calling $\mathcal{M}$ finite.

- For all $t \in W$ such that sRt, there exists $t' \in W'$ such that $(t, t') \in \mathcal{R}$ and $s'R't'$.
- For all $t' \in W'$ such that $s'R't'$, there exists $t \in W$ such that $(t, t') \in \mathcal{R}$ and sRt.

We call pointed models $(\mathcal{M}, a), (\mathcal{M}', a')$ are bisimilar (resp. bisimilar modulo P) and write $(\mathcal{M}, a) \sim (\mathcal{M}', a')$ (resp. $(\mathcal{M}, a) \sim_P (\mathcal{M}', a')$) if there is a bisimulation (resp. bisimulation modulo P) $\mathcal{R}$ from $\mathcal{M}$ to $\mathcal{M}'$, such that $a\mathcal{R}a'$. If $(\mathcal{M}, a)$ is a pointed model, and P a set of propositional variables, then $Th_P(\mathcal{M}, a) = \{\varphi \in L(P) \mid \mathcal{M}, a \models \varphi\}$. We say that two pointed models are equivalent and write $(\mathcal{M}, a) \equiv_P (\mathcal{M}', a')$ when $Th_P(\mathcal{M}, a) = Th_P(\mathcal{M}', a')$.

The following simplification of the Hennessy-Milner Theorem [13] gives a very useful characterization of pointed model equivalence; Proposition 4 is its direct consequence.

▶ **Theorem 3** (Hennessy-Milner Theorem). *If $(\mathcal{M}, a)$, $(\mathcal{M}', a')$ are finite pointed models, then*

$$(\mathcal{M}, a) \equiv_P (\mathcal{M}', a') \text{ if and only if } (\mathcal{M}, a) \sim_P (\mathcal{M}', a').$$

▶ Proposition 4. A formula φ is complete for a logic l if and only if for every two pointed models $(\mathcal{M}, a)$ and $(\mathcal{M}', a')$ for l, if $\mathcal{M}, a \models \varphi$ and $\mathcal{M}', a' \models \varphi$, then $(\mathcal{M}, a) \sim_P (\mathcal{M}', a')$.

Paige and Tarjan in [19] give an efficient algorithm for checking whether two pointed models are bisimilar. Theorem 5 is a variation on their result to account for receiving the set P of propositional variables as part of the algorithm's input.

▶ **Theorem 5.** *There is an algorithm which, given two pointed models $(\mathcal{M}, a)$ and $(\mathcal{M}', a')$ and finite set of propositional variables P, determines whether $(\mathcal{M}, a) \sim_P (\mathcal{M}', a')$ in time $O(|P| \cdot (|\mathcal{M}| + |\mathcal{M}'|) \cdot \log(|\mathcal{M}| + |\mathcal{M}'|))$.*

2.3 Problems of Modal Logic, Complexity, and other Constructions

For logic l, the satisfiability problem for l, or l-satisfiability is the problem which asks, given a formula φ, if φ is satisfiable. Similarly, the provability problem for l asks if $\vdash_l \varphi$; and *the completeness problem for l* asks if φ is complete for l.

The classical complexity results for Modal Logic are due to Ladner [14], who established PSPACE-completeness for the satisfiability of **K**, **T**, **D**, **K4**, **D4**, and **S4** and NP-completeness for the satisfiability of **S5**. Halpern and Rêgo later characterized the NP–PSPACE gap by the presence or absence of Negative Introspection [12], resulting in Theorem 6.

▶ **Theorem 6.** *If $l \in \{\boldsymbol{K}, \boldsymbol{T}, \boldsymbol{D}, \boldsymbol{K4}, \boldsymbol{D4}, \boldsymbol{S4}\}$, then l-provability is* PSPACE*-complete and $l + 5$-provability is* coNP*-complete.*

In the course of proving the coNP upper bound for logics with Negative Introspection, Halpern and Rêgo give in [12] a construction which provides a small model for a satisfiable formula. From parts of their construction, we can extract Lemma 7 and Corollary 8.

For a logic $l + 5$, we call a pointed model $(\mathcal{M}, s)$ for $l + 5$ *flat* when
- $\mathcal{M} = (\{s\} \cup W, R, V)$;
- $R = R_1 \cup R_2$, where $R_1 \subseteq \{s\} \times W$ and R_2 is an equivalence relation on W; and
- if $l \in \{\mathbf{T}, \mathbf{S4}\}$, then $s \in W$.

Lemma 7 informs us that flat models are a normal form for models of logics with axiom 5.

▶ **Lemma 7.** *Every pointed $l + 5$-model $(\mathcal{M}, s)$ is bisimilar to a flat pointed $l + 5$-model.*

Proof. Let W' be the set of states of $\mathcal{M}$ reachable from s and R the restriction of the accessibility relation of $\mathcal{M}$ on W'. It is easy to see that the identity relation is a bisimulation from $\mathcal{M}$ to $\mathcal{M}'$, so $(\mathcal{M}, s) \sim (\mathcal{M}', s)$; let $W = \{w \in W' \mid \exists w' R w\}$. Therefore $W' = W \cup \{s\}$

and if $l \in \{\mathbf{T}, \mathbf{S4}\}$, then $s \in W$. Since $\mathcal{M}$ is an $l+5$-model, R is euclidean. Therefore, the restriction of R on W is reflexive. This in turn means R is symmetric in W: if $a, b \in W$ and aRb, since aRa, we also have bRa. Finally, R is transitive in W: if $aRbRc$ and $a, b, c \in W$, then bRa, so aRc. Therefore R is an equivalence relation when restricted on W. ◀

The construction from [14] and [12] continues to filter the states of the flat model, resulting in a small model for a formula φ. Using this construction, Halpern and Rêgo prove Corollary 8 [12]; the NP upper bound for $l+5$-satisfiability of Theorem 6 is a direct consequence.

▶ **Corollary 8.** *Formula φ is $l+5$-satisfiable if and only if it is satisfied in a flat $l+5$-model of $O(|\varphi|)$ states.*

Since we are asking whether a formula is complete, instead of whether it is satisfiable, we want to be able to find two small non-bisimilar models for φ when φ is incomplete. For this, we need a characterization of bisimilarity between flat models.

▶ **Lemma 9.** *Flat pointed models $(\mathcal{M}, a) = (\{a\} \cup W, R, V)$ and $(\mathcal{M}', a') = (\{a'\} \cup W', R', V')$ are bisimilar modulo P if and only if $V_P(a) = V_P(a')$ and:*

- *for every $b \in W$, there is some $b' \in W'$ such that $V_P(b) = V'_P(b')$;*
- *for every $b' \in W'$, there is some $b \in W'$ such that $V_P(b) = V'_P(b')$;*
- *for every $b \in W$, if aRb, then there is a $b' \in W'$ such that $a'Rb'$ and $V_P(b) = V'_P(b')$; and*
- *for every $b' \in W'$ such that $a'Rb'$, there is a $b \in W'$ such that $V_P(b) = V'_P(b')$ and aRb.*

Proof. If these conditions are met, we can define bisimulation $\mathcal{R}$ such that $a\mathcal{R}a'$ and for $b \in W$ and $b' \in W'$, $b\mathcal{R}b'$ iff $V_P(b) = V'_P(b')$; on the other hand, if there is a bisimulation, then it is not hard to see by the definition of bisimulation that these conditions hold — for both claims, notice that the conditions above, given the form of the models, correspond exactly to the conditions from the definition of bisimulation. ◀

This gives us Corollary 10, which is a useful characterization of incomplete formulas.

▶ **Corollary 10.** *Formula φ is incomplete for $l+5$ if and only if it has two non-bisimilar flat pointed models for $l+5$ of at most $O(|\varphi|)$ states.*

Proof. If φ has two non-bisimilar pointed models for $l+5$, then by Theorem 3, it is incomplete. On the other hand, if φ is incomplete, again by Theorem 3 and Lemma 7, φ has two non-bisimilar flat pointed models, $(\mathcal{M}, a) = (\{a\} \cup W, R, V)$ and $(\mathcal{M}', a') = (\{a'\} \cup W', R', V')$. By Lemma 9 and without loss of generality, we can distinguish three cases:

- there is some $p \in V_P(a) \setminus V_P(a')$: in this case let $\psi = p$;
- there is some $b \in W$, such that for all $b' \in W'$, $V_P(b) \neq V'_P(b')$: in this case let $\psi = \Diamond\Diamond \bigwedge V_P(b) \wedge \bigwedge(P \setminus V_P(b))$;
- there is some $b \in W$, such that aRb and for all $b' \in W'$ such that $a'Rb'$, $V_P(b) \neq V'_P(b')$: in this case let $\psi = \Diamond \bigwedge V_P(b) \wedge \bigwedge(P \setminus V_P(b))$.

In all these cases, both $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are satisfiable and of size $O(|\varphi|)$, so by Corollary 8 they are satisfied in non-bisimilar flat pointed models for $l+5$ of at most $O(|\varphi|)$ states. ◀

Our first complexity result is a direct consequence of Corollary 10 and Theorem 5:

▶ **Corollary 11.** *The completeness problem for logic $l+5$ is in* coNP.

In the following, when P is evident from the context, we will often omit any reference to it and instead of bisimulation modulo P, we will call the relation simply bisimulation.

3 The Completeness Problem and Triviality

The first question we need to answer concerning the completeness problem for l is whether there are any satisfiable and complete formulas for l.If the answer is negative, then the problem is trivial. We examine this question with parameters the logic and whether P, the set of propositional variables we use, is empty or not. If for some logic l the problem is nontrivial, then we give a complete formula φ^l_P, which depends on P and uses exactly the propositional variables in P. We see that for $P = \emptyset$, completeness can become trivial for another reason: for some logics, when $P = \emptyset$, all formulas are complete. On the other hand, when $P \neq \emptyset$, $\bigwedge P$ is incomplete for every logic.

Whether $P = \emptyset$ or not, completeness is nontrivial for **K** and **K4**: let $\varphi^{\mathbf{K}}_P = \varphi^{\mathbf{K4}}_P = \bigwedge P \wedge \Box\bot$ for every finite P. Formula $\top$ is incomplete for **K** and **K4**.

▶ **Lemma 12.** *Formula $\bigwedge P \wedge \Box\bot$ is complete and satisfiable for **K** and for **K4**.*

Proof. A model which satisfies $\varphi^{\mathbf{K}}_P$ is $\mathcal{M} = (\{a\}, \emptyset, V)$, where $V(a) = P$. If there is another model $\mathcal{M}', a' \models \varphi^{\mathbf{K}}_P$, then $\mathcal{M}', a' \models \Box\bot$, so there are no accessible worlds from a' in $\mathcal{M}'$; therefore, $\mathcal{R} = \{(a, a')\}$ is a bisimulation. ◀

Notice that if φ is complete for l, then it is complete for every extension of l. Therefore, $\varphi^{\mathbf{K}}_P$ is complete for all other logics. However, we are looking for *satisfiable and complete* formulas for the different logics, so finding one complete formula for **K** is not enough. On the other hand, if l' is an extension of l (by a set of axioms) and a formula φ is complete and satisfiable for l', then we know that φ is satisfiable and complete for all logics between (and including) l and l'. Unfortunately, the following lemma demonstrates that we cannot use this convenient observation to reuse $\varphi^{\mathbf{K}}_P$ — except perhaps for **K5** and **K45**, but these can be handled just as easily together with the remaining logics with Negative Introspection.

When l has axiom T or D, but not 4 or 5, P determines if a satisfiable formula is complete:

▶ **Lemma 13.** *Let l be either **D** or **T**. A satisfiable formula $\varphi \in L$ is complete with respect to l if and only if $P(\varphi) = \emptyset$.*

Proof. When $P = \emptyset$, all models are bisimilar through the total bisimulation; therefore, all formulas φ, where $P(\varphi) = \emptyset$ are trivially complete. We now consider the case for $P \neq \emptyset$; notice that we can assume that $l = \mathbf{D}$, as **D** is contained in **T**. Let the modal depth of φ be d and let $\mathcal{M}, a \models \varphi$, where $\mathcal{M} = (W, R, V)$; let $x \notin W^*$, $a_0 = a$, and

$$\Pi_d = \{a_0 \cdots a_k \in W^* \mid k \leq d \text{ and for all } 0 \leq i < k,\ a_i R a_{i+1}\}.$$

Then, we define $\mathcal{M}'_1 = (W', R', V'_1)$ and $\mathcal{M}'_2 = (W', R', V'_2)$, where

$$\begin{aligned} W' &= \Pi_d \cup \{x\}; \\ R' &= \{(\alpha, \alpha b) \in W'^2 \mid b \in W\} \ \cup\ \{(a_0 a_1 \cdots a_d, x) \in W'^2\} \ \cup\ \{(x, x)\} \\ V'_i(\alpha b) &= V(b),\ \text{for } i = 1, 2,\ 0 \leq |\alpha| < d; \\ V'_1(x) &= \emptyset; \quad \text{and} \quad V'_2(x) \ = \ P. \end{aligned}$$

To prove that $\mathcal{M}'_1, a \models \varphi$ and $\mathcal{M}'_2, a \models \varphi$, we prove that for $\psi \in sub(\varphi)$, for every $i = 1, 2$ and $w = a_0 \cdots a_k \in \Pi_d$, where $k \leq d - md(\psi)$, $\mathcal{M}'_i, w \models \psi$ if and only if $\mathcal{M}, a_k \models \psi$. We use induction on ψ. If ψ is a literal or a constant, the claim is immediate and so are the cases of the $\wedge, \vee$ connectives. If $\psi = \Box\psi'$, then $md(\psi') = md(\psi) - 1$; $\mathcal{M}'_i, w \models \psi$ iff for every $wR'w'$, $\mathcal{M}'_i, w' \models \psi'$ iff for every $a_k R' b$, $\mathcal{M}, b \models \psi'$ (by the Inductive Hypothesis) iff $\mathcal{M}, a_k \models \psi$; the case of $\psi = \Diamond\psi'$ is symmetric.

If $(\mathcal{M}'_1, a) \sim (\mathcal{M}'_2, a)$ through bisimulation $\mathcal{R}$ from $\mathcal{M}'_1$ to $\mathcal{M}'_2$, then by the conditions of bisimulation, there must be a path of length $d+2$ from a to some $b \in W'$ such that $b\mathcal{R}x$. Since from b there is a path to x and the paths from x only end up to x, $x\mathcal{R}x$, which is a contradiction, since $V'_1(x) \neq V'_2(x)$. So, φ is satisfied in two non-bisimilar models for l. ◀

For every finite P, let $\varphi_P^{\mathbf{D4}} = \varphi_P^{\mathbf{S4}} = \bigwedge P \wedge \Box \bigwedge P$. As the following lemma demonstrates, $\varphi_P^{\mathbf{D4}}$ is a complete formula for **D4** and **S4**.

▶ **Lemma 14.** *For every finite P, $\varphi_P^{\boldsymbol{D4}}$ is complete for **D4** and **S4**; all formulas in $L(\emptyset)$ are complete for **D4** and **S4**.*

Proof. Let $\mathcal{M}, a \models \varphi_P^{\mathbf{D4}}$ and $\mathcal{M}', a' \models \varphi_P^{\mathbf{D4}}$; let $\mathcal{R}$ be the relation that connects all states of $\mathcal{M}$ that are reachable from a (including a) to all states of $\mathcal{M}'$ that are reachable from a' (including a'); it is not hard to verify that $\mathcal{R}$ is a bisimulation. Notice that if $P = \emptyset$, $\varphi_P^{\mathbf{D4}}$ is a tautology, thus all formulas are complete. ◀

It is straightforward to see that $\varphi_P^{\mathbf{D4}}$ is satisfiable for every modal logic l: consider a model based on any frame for l, where $\bigwedge P$ holds at every state of the model. Therefore:

▶ **Corollary 15.** *$\varphi^{\boldsymbol{D4}}$ is satisfiable and complete for every extension of **D4**.*[2]

For logic $l = l' + 5$, let $\varphi_P^l = \bigwedge P \wedge \Diamond\Box \bigwedge P$.

▶ **Lemma 16.** *φ_P^l is a satisfiable complete formula for l.*

Proof. By Lemma 7, φ_P^l is complete. It is satisfied in $(\{a\}, \{(a,a)\}, V)$, where $V(a) = P$. ◀

When $P = \emptyset$, we can distinguish two cases. If $l' \in \{\mathbf{D}, \mathbf{D4}, \mathbf{T}, \mathbf{S4}\}$, then $\varphi_\emptyset^l$ is a tautology, therefore all formulas in $L(P)$ are complete for l. If $l' \in \{\mathbf{K}, \mathbf{K4}\}$, by Lemma 7, an l-model would either satisfy φ_P^l or $\Box\bot$, depending on whether the accessibility relation is empty or not. Therefore, if $P = \emptyset$ the completeness problem for **K5** and **K45** is not trivial, but it is easy to solve: a formula with no propositional variables is complete for $l \in \{\mathbf{K5}, \mathbf{K45}\}$ if it is satisfied in at most one of the two non-bisimilar modulo $\emptyset$ models for l.

▶ **Corollary 17.** *If $P = \emptyset$, the completeness problem for **K5** and **K45** is in* P.

A logic l has a nontrivial completeness problem if for $P \neq \emptyset$, there are complete formulas for l. From the logics we examined, only **D** and **T** have trivial completeness problems. Table 1 summarizes the results of this section and of Section 4 regarding the completeness problem.

4 The Complexity of Completeness

Our main result is that for a modal logic l, the completeness problem has the same complexity as provability for l, as long as we allow for propositional variables in a formula and the completeness problem for l is nontrivial (see also Table 1). For the lower bounds, we consider hardness under polynomial-time reductions. As the hardness results are relative to complexity classes that include coNP, these reductions suffice.

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[2] Although for the purposes of this paper we only consider a specific set of modal logics, it is interesting to note that the corollary can be extended to a much larger class of logics.

Modal Logic	$P = \emptyset$	$P \neq \emptyset$
K, **K4**	PSPACE-complete	PSPACE-complete
D, **T**	trivial (all)	trivial (none)
D4, **S4**	trivial (all)	PSPACE-complete
$l + 5$, $l \neq \mathbf{K}, \mathbf{K4}$	trivial (all)	coNP-complete
K5, **K45**	in P	coNP-complete

Table 1 The complexity of the completeness problem for different modal logics. Trivial (all) indicates that all formulas in this case are complete for the logic; trivial (none) indicates that no satisfiable formula in this case is complete for the logic.

4.1 A Lower Bound

We present a lower bound for the complexity of the completeness problem: we show that the completeness problem is at least as hard as provability for a logic, as long as it is nontrivial.

▶ **Theorem 18.** *Let l be a logic that has a nontrivial completeness problem and let C be a complexity class. If l-provability is C-hard, then the completeness problem for l is C-hard.*

Proof. To prove the theorem we present a reduction from l-provability to the completeness problem for l. From a formula φ, the reduction constructs in polynomial time a formula φ_c, such that φ is provable if and only is φ_c is complete. For each logic l with nontrivial completeness and finite set of propositional variables P, in Section 3 we provided a complete formula φ^l_P. This formula is satisfied in a model of at most two states, which can be generated in time $O(|P|)$. Let $(\mathcal{M}_l, a_l)$ be such a pointed model for φ^l_P.

Any pointed model which satisfies φ^l_P is bisimilar to $(\mathcal{M}_l, a_l)$. Given a formula $\varphi \in L(P)$, we can determine in linear time if $\mathcal{M}_l, a_l \models \varphi$. Then, there are two cases:

- $\mathcal{M}_l, a_l \not\models \varphi$, in which case φ is not provable and we set $\varphi_c = \bigwedge P$.
- $\mathcal{M}_l, a_l \models \varphi$, so $\neg\varphi \wedge \varphi^l_P$ is not satisfiable, in which case we set $\varphi_c = \varphi \to \varphi^l_P$. We demonstrate that φ is provable if and only if $\varphi \to \varphi^l_P$ is complete. If φ is provable, then $\varphi \to \varphi^l_P$ is equivalent to φ^l_P, which is complete. On the other hand, if $\varphi \to \varphi^l_P$ is complete and $(\mathcal{M}, a)$ is any pointed model, we show that $\mathcal{M}, a \models \varphi$, implying that if $\varphi \to \varphi^l_P$ is complete, then φ is provable. If $(M, a) \sim_P (M_l, a_l)$, then from our assumptions $\mathcal{M}, a \not\models \neg\varphi$, thus $\mathcal{M}, a \models \varphi$. On the other hand, if $(M, a) \not\sim_P (M_l, a_l)$, since $(M_l, a_l) \models \varphi \to \varphi^l_P$ and $\varphi \to \varphi^l_P$ is complete, $\mathcal{M}, a \not\models \varphi \to \varphi^l_P$, therefore $\mathcal{M}, a \models \varphi$. ◀

Theorem 18 applies to more than the modal logics that we have defined in Section 2. For Propositional Logic, completeness amounts to the problem of determining whether a formula does not have *two* distinct satisfying assignments, therefore it is coNP-complete. Of course, for First-order Logic, completeness is undecidable, as satisfiability is undecidable.

4.2 Upper Bounds

The easiest cases are the logics with axiom 5. Directly from Theorem 18 and Corollary 11:

▶ Proposition 19. The completeness problem for logic $l + 5$ is coNP-complete.

For the logics without axiom 5, by Theorem 6, satisfiability and provability are both PSPACE-complete. So, completeness is PSPACE-hard, if it is nontrivial. It remains to show that it is also in PSPACE. To this end we present a family of games to decide completeness for a modal formula. We call them Model Constructing-Collapsing Games. They are essentially bisimulation games [22] combined with the tableaux by Fitting [9] and Massacci [15] for

Modal Logic. For more on tableaux the reader can see [7]. These games can then be directly turned into an alternating polynomial time algorithm, thus establishing that the completeness problems for these logics is in PSPACE. We have treated that case for logics with axiom 5 and the completeness problem for **D** and **T** is trivial. Therefore, form now on, we fix a logic l, which can either be **K**, or have axiom 4 and be one of **K4**, **D4**, and **S4**.

Model Constructing-Collapsing Game for Modal Logic l on φ

The game is played by two players: the Constructor and the Collapser. Intuitively, the Constructor tries to construct two models for φ and at the same time demonstrate that these models are not bisimilar. On the other hand the Collapser's goal is to demonstrate that the models the Constructor provides are bisimilar (thus collapsing these to one model). We first give a few definitions which are needed to describe the game.

In the game, *states* are sets of formulas from $\overline{sub}(\varphi)$. A game is played on a pair S_L, S_R, called the *sides* of the game and uses an integer counter $k \geq 0$ of the game's points. The game's points essentially determine how many moves are left in the game for the Constructor. Each side S is a pair $(p(S), C(S))$ of a (possibly empty) set $C(S)$ of states, that are called the *children-states* of that side and a distinguished state $p(S)$ called the *parent-state* of that side. Each side is allowed to have up to $|\varphi|$ children-states. S_L is called the left side and S_R is the right side.

▶ **Definition 20.** We call a set s of formulas l-*closed* if the following conditions hold:
- if $\varphi_1 \wedge \varphi_2 \in s$, then $\varphi_1, \varphi_2 \in s$;
- if $\varphi_1 \vee \varphi_2 \in s$, then $\varphi_1 \in s$ or $\varphi_2 \in s$;
- if $\Box\psi \in s$ and l has axiom T, then $\psi \in s$;
- for every $p \in P$, either $p \in s$ or $\neg p \in s$.

We call a side S l-*complete* (or *complete* if l is fixed) if the following conditions hold:
- the parent-state and every child-state of that side are l-closed;
- for every $\Diamond\psi \in p(S)$, $\psi \in \bigcup C(S)$;
- for every $\Box\psi \in p(S)$, $\psi \in \bigcap C(S)$;
- if l has axiom 4, then for every $\Box\psi \in p(S)$, $\Box\psi \in \bigcap C(S)$;
- if l has axiom D, then $C(S) \neq \emptyset$.

For state a, $th(a) = \bigwedge a$. For states a, b, we define $a \leq b$ to mean $\vdash th(b) \to th(a)$. State $a \in C(S)$ is *maximal* for S if it is maximal in $C(S)$ with respect to $\leq$. State $a \subseteq \overline{sub}(\varphi)$ is 4-*maximal* if it is a maximally consistent subset of $\overline{sub}(\varphi)$. For states a, b, we write $a \simeq_P b$ when they have the same set of literals. A side S is *consistent* when every state of S is a consistent set of formulas. A side S' *completes* side S when: S' is l-complete; $p(S) \subseteq p(S')$; for every $a \in C(S)$ there is an $a' \in C(S')$ such that $a \subseteq a'$; and: if $l = \mathbf{K}$, then every $a' \in C(S')$ is maximal; if l has axiom 4, then every $a' \in C(S')$ is 4-maximal.

Each side gives a local view of a model, as long as it is consistent. During the game it is the Constructor's responsibility to maintain that each side is complete — so that all relevant information is present in each side — and consistent — so that the side indeed represents parts of a model. Since the Constructor maintains that the parent-states can represent non-bisimilar states of two models (say, a and b), they need to provide a child, representing a state accessible from a or b that is not bisimilar to any state accessible from b or a, respectively. The Collapser maintains that a and b are bisimilar, they must counter by choosing a child they claim to be bisimilar to the Constructor's pick from the other side.

The game description:

At the beginning of the game each side S is such that $p(S) = \emptyset$ and $C(S) = \{\{\varphi\}\}$. The game starts with k points, for a nonnegative integer k. The players take turns and move according to the rules below:

The Constructor's Move: The Constructor replaces each side S by a consistent side S' that completes S. The Constructor then must use a game point and pick a side A and a child-state a from $C(A)$. Let B be the other side that the Constructor did not pick.

The Collapser's Move: *If l does not have axiom 4:* The Collapser must pick a child-state b from B such that $b \simeq_P a$.
If l has axiom 4: The Collapser must either pick a child-state b from B such that $b \simeq_P a$, or if $\vdash th(p(B)) \to \Diamond th(a)$, then the Collapser can alternatively pick $b = a$. The game continues with the Constructor's move on sides $(a, \emptyset)$ and $(b, \emptyset)$.

The game continues until one of the players has no move to play. Then, the other player wins the game. State $\{\varphi\}$ is a child in each side at the beginning of the game to ensure that the Constructor can provide consistent and complete states that include φ, but have different literals, if there are any. Then, the Collapser will not be able to pick a state, resulting in the Constructor's win — consider $\varphi = (p \vee \neg p) \wedge \Box\bot$ as an example. More formally:

▶ **Definition 21.** A *play of length n* on formula φ is a sequence $\pi = X_0 X_1 \cdots X_n$, such that $n \geq 0$ and for every $1 \leq i \leq n$ the following conditions hold:

- $X_0 = (S_L(0), S_R(0), k(0))$ is the initial configuration, where $S_L(0) = S_R(0) = (\emptyset, \{\{\varphi\}\})$ and $k(0) \geq 0$. We say that the play π starts with $k(0)$ points.
- If i is odd, then $X_i = (S_L(i), S_R(i), s(i), a(i), k(i))$, such that
  $S_L(i)$, $S_R(i)$ are consistent sides that respectively complete $S_L(i-1)$, $S_R(i-1)$,
  $s(i) \in \{L, R\}$ and $a(i)$ is a child-state of side $S_{s(i)}(i)$, and
  $k(i) = k(i-1) - 1 \geq 0$.
- If i is even, then let $t \in \{L, R\} \setminus \{s(i-1)\}$.

$$X_i = (S_L(i), S_R(i), k(i)) = ((a(i-1), \emptyset), (b(i), \emptyset), k(i-1)),$$

  where $a(i-1) \simeq_P b(i)$ and if l has axiom 4 and $\vdash th(p(S_t(i-1))) \to \Diamond th(a(i-1))$, then $b(i) \in C(S_t(i-1)) \cup \{a(i-1)\}$; otherwise, $b(i) \in C(S_t(i-1))$.

The X_i's are the results of each player's move. Play π is *completed* if there is no X such that πX is a play. A completed play is a *winning play* for the Collapser when its length is even; otherwise, it is a winning play for the Constructor. A *strategy* is a partial function f from plays to plays, such that if $f(\pi) = \pi'$, then $\pi' = \pi X$ for some X. The Constructor (respectively the Collapser) conforms to strategy f during play $\pi = X_0 X_1 \cdots X_n$ when for every odd (resp. even) $1 \leq i \leq n$, if $\pi = \pi' X_i \pi''$ and π' is of length $i-1$, then $f(\pi') = \pi' X_i$. A strategy f is a *winning strategy* for a player on (φ, k) if every completed play that starts with k points on φ during which the player conforms to f is a winning play for that player.

▶ **Lemma 22.** *A play on formula φ that starts at k points can be of length at most $2k$.*

Proof. The game starts with k points, so each player can play a move up to k times. ◀

Since the game is finite, it is also determined:

▶ **Corollary 23.** *For every formula φ and $k \geq 0$, one of the players has a winning strategy on (φ, k).*

This section's main theorem is Theorem 24 (proof in the Appendix) and informs us that these games determine the completeness of formula φ. That the completeness problem for logics without axiom 5 is in PSPACE is a direct corollary.

▶ **Theorem 24.** *The Collapser has a winning strategy on* $(\varphi, 4|\varphi|)$ *if and only if* φ *is complete.*

▶ **Corollary 25.** *The completeness problem for* ***K****,* ***K4****,* ***D4****, and* ***S4*** *is* PSPACE*-complete.*

Proof. PSPACE-hardness is a consequence of Theorem 18. The Constructor-Collapser game can be turned into an alternating polynomial-time algorithm with an oracle from PSPACE. The algorithm simply runs the game for φ on the logic, starting from X_0; from a play π of even length it uses an existential choice to guess the next move X of the Constructor and from a play π of odd length it uses a universal choice to guess the next move X of the Collapser. To verify each time that X is a valid move, the algorithm needs to verify the conditions for πX to be a valid play. These conditions are either closure conditions or conditions for the consistency or provability of formulas of polynomial size with respect to $|\varphi|$; therefore, they can be verified either directly or with an oracle from PSPACE. Thus, the completeness problem for these logics is in $\mathsf{AP}^{\mathsf{PSPACE}} = \mathsf{PSPACE}$. ◀

5 Variations and Other Considerations

There are several variations one may consider for the completeness problem. One may define the completeness of a formula in a different way, or consider a different logic, depending on the intended application. One may also wonder whether we could attempt a solution to the completeness problem by using normal forms [8].

5.1 Satisfiable and Complete Formulas

It may be more appropriate, depending on the case, to check whether a formula is *satisfiable and complete*. In this case, if the modal logic does not have axiom 5, we can simply alter the Constructor-Collapser games so that the Constructor wins right away if the formula is not satisfiable. Therefore, the problem remains in PSPACE; for PSPACE-completeness, notice that the reduction for Theorem 18 constructs satisfiable formulas.

For logics with axiom 5 (and plain Propositional Logic), the language of satisfiable and complete formulas is US-complete, where a language U is in US when there is a nondeterministic Turing machine T, so for every instance x for U, $x \in U$ if and only if T has exactly one accepting computation path for x[3] [5]: UniqueSAT is a complete problem for US and a special case of this variation of the completeness problem.

5.2 Completeness with Respect to a Model

A natural variation of the completeness problem would be to consider completeness of a formula over a satisfying model. That is, the problem would ask: given a formula φ and model $(\mathcal{M}, s)$, such that $\mathcal{M}, s \models \varphi$, is the formula complete? For this variation, we are given one of φ's pointed models, so it is a reasonable expectation that the problem became easier. Note that in many cases, this problem may even be more natural than the original one, as

[3] We note that US is different from UP; for UP, if T has an accepting path for x, then it is *guaranteed* that it has a unique accepting path for x.

we are now testing whether the formula completely describes the pointed model (that is, the formula is characteristic for the model).

Unfortunately, this variation has exactly the same complexity as the original completeness problem. We can easily reduce completeness with respect to a model to plain completeness by dropping the model from the input. On the other hand, the reduction from provability to completeness of Section 4 still works in this case, as it can easily be adjusted to additionally provide the satisfying model of the complete formula φ^l_P.

5.3 Completeness and Normal Forms for Modal Logic

In [8], Fine introduced normal forms for Modal Logic. The sets F^d_P are defined recursively on the depth d, which is a nonnegative integer and depend on the set of propositional variables P (we use a variation on the presentation from [18]):

$$F^0_P = \left\{ \bigwedge_{p \in S} p \wedge \bigwedge_{p \notin S} \neg p \mid S \subseteq P \right\}; \quad \text{and}$$

$$F^{d+1}_P = \left\{ \varphi_0 \wedge \bigwedge_{\varphi \in S} \Diamond \varphi \wedge \Box \bigvee_{\varphi \in S} \varphi \mid S \subseteq F^d_P, \ \varphi_0 \in F^0_P \right\}.$$

For example, formula $\varphi^{\mathbf{K}}_P$ from Section 3 is a normal form in F^1_P.

▶ **Theorem 26** (from [8]). *For every modal formula φ of modal depth at most d, if φ is consistent for $\boldsymbol{K}$, then there is some $S \subseteq F^d_P$, so that $\vdash_{\boldsymbol{K}} \varphi \leftrightarrow \bigvee S$.*

Furthermore, as Fine [8] demonstrated, normal forms are mutually exclusive: no two distinct normal forms from F^d_P can be true at the same state of a model. Normal forms are not necessarily complete by our definition (for example, consider $p \wedge \Diamond p \wedge \Box p$ for $P = \{p\}$), but, at least for $\mathbf{K}$, it is not hard to distinguish the complete ones; by induction on d, $\varphi \in F^d_P$ is complete for $\mathbf{K}$ if and only if $md(\varphi) < d$. Therefore, for $\mathbf{K}$, the satisfiable and complete formulas are exactly the ones which are equivalent to such a complete normal form. However, we cannot use this observation to test formulas for completeness by guessing a complete normal form and verifying that it is equivalent to our input formula, as normal forms can be of very large size: $|F^0_P| = 2^{|P|}$; $|F^{d+1}_P| = |P| \cdot 2^{|F^d_P|}$; and if $\psi \in F^d_P$, $|\psi|$ can be up to $|P| + 2|F^{d-1}_P|$. We would be guaranteed a normal form of reasonable (that is, polynomial w.r.to $|\varphi|$) size to compare to φ only if φ uses a small (logarithmic with respect to $|\varphi|$) number of variables and its modal depth is very small compared to $|\varphi|$ (that is, $md(\varphi) = O(\log^*(|\varphi|))$).

5.4 Completeness up to Depth

Fine's normal forms [8] can inspire us to consider a relaxation of the definition of completeness. We call a formula φ *complete up to its depth* for a logic l exactly when for every formula $\psi \in L(P(\varphi))$ of modal depth at most $md(\varphi)$, either $\vdash_l \varphi \to \psi$ or $\vdash_l \varphi \to \neg\psi$.

▶ **Lemma 27.** *Formula φ is complete up to its depth for logic l if and only if it is equivalent in l to a normal form from $F^{md(\varphi)}_P$.*

Proof. From Theorem 26, if φ is complete up to its depth, then it is equivalent to some $\bigvee S$, where $S \subseteq F^{md(\varphi)}_P$, but it can also derive one of the normal forms in S. For the other direction, notice that every normal form in $F^{md(\varphi)}_P$ is either complete or has the same modal depth as φ. By Theorem 26, all normal forms are complete up to their depths. ◀

Therefore, all modal logics have formulas which are complete up to their depth. In fact, for finite set of propositional variables P and $d \geq 0$, we can define $\varphi_P^d = \bigwedge_{i=0}^{d} \Box^i \bigwedge P$, which is equivalent in $\mathbf{T}$ and $\mathbf{D}$ to a normal form (by induction on d). Then, we can use a reduction similar to the one from the proof of Theorem 18 to prove that for every modal logic, completeness up to depth is as hard as provability.

▶ Proposition 28. For any complexity class C and logic l, if l-provability is C-hard, then completeness up to depth is C-hard.

Proof. The proof is similar to the one for Theorem 18. For $l \neq \mathbf{T}, \mathbf{D}$, $\varphi^l_P(d) = \varphi^l_P$ as defined in Section 3; for $l = \mathbf{T}$ or $\mathbf{D}$, let $\varphi^l_P(d) = \varphi^d_P$ as defined above. We also assume an appropriate $M_l, a_l \models \varphi^l_P(d)$. If $M_l, a_l \not\models \varphi$, let $\varphi_c = \bigwedge P \wedge \Box\top$; otherwise, let $\varphi_c = \varphi \to \varphi^l_P(d)$. For the second case, if φ is provable, then φ_c is equivalent to $\varphi^l_P(d)$, which is complete up to its depth. If φ_c is complete up to its depth, then by Lemma 27, it is equivalent to a normal form $\psi \in F^d_P$. So, ψ is equivalent to $\varphi_c = \varphi \to \varphi^l_P(d)$, which is equivalent to $\neg \bigvee S \vee \varphi^l_P$ for some $S \subseteq F^d_P$, by Theorem 26. Since normal forms are mutually exclusive, $\bigvee S$ is equivalent to $\neg \bigvee (F^d_P \setminus S)$, so ψ is equivalent to $\bigvee (F^d_P \setminus S) \vee \varphi^l_P(d)$; therefore, either $S = F^d_P$ and $\psi = \varphi^l_P(d)$, or $F^d_P \setminus S$ is a singleton of a normal form equivalent to $\varphi^l_P(d)$. In the first case, φ is provable, because for any model $\mathcal{M}, a$, by Theorem 26, $\mathcal{M}, a \models \bigvee F^d_P$, so $\mathcal{M}, a \models \varphi$. The second case cannot hold, because $M_l, a_l \not\models \varphi$. ◀

We demonstrate that this variation is in PSPACE when the logic is $\mathbf{K}$; it seems plausible that one can follow similar approaches that use normal forms for the remaining modal logics.

▶ Proposition 29. A formula φ is complete up to its depth for $\mathbf{K}$ if and only if $\varphi \wedge \Box^{md(\varphi)+1}\bot$ is complete for $\mathbf{K}$.

Proof. Let $\psi \in F^d_P$ be a normal form. Then, $\psi \wedge \Box^{d+1}\bot$ is equivalent in $\mathbf{K}$ to $\psi^{+1} \in F^{d+1}_P$, which is ψ after we replace all $\Diamond\psi'$ in ψ by $\Diamond(\psi' \wedge \Box\bot)$, where $\psi' \in F^0_P$. If $\psi_1, \psi_2 \in F^d_P$ are distinct normal forms, then ψ_1^{+1}, ψ_2^{+1} are distinct normal forms in F^r_P for every $r > d$. So, φ is complete up to its depth for $\mathbf{K}$ if and only if $\varphi \wedge \Box^{md(\varphi)+1}\bot$ is complete for $\mathbf{K}$. ◀

5.5 More Logics

There is more to Modal Logic, so perhaps there is also more to discover about the completeness problem. We based the decision procedure for the completeness problem for each logic on a decision procedure for satisfiability. We distinguished two cases:

- If the logic has axiom 5, then to test satisfiability we guess a small model and we use model checking to verify that the model satisfies the formula. This procedure uses the small model property of these logics (Corollary 8). To test for completeness, we guess *two* small models; we verify that they satisfy the formula and that they are non-bisimilar. We could try to use a similar approach for another logic based on a decision procedure for satisfiability based on a small model property (for, perhaps, another meaning for "small"). To do so successfully, a small model property may not suffice. We need to first demonstrate that for this logic, a formula which is satisfiable and incomplete has *two* small non-bisimilar models.
- For the other logics, we can use a tableau to test for satisfiability. We were able to combine the tableaux for these logics with bisimulation games to provide an optimal — when the completeness problem is not trivial — procedure for testing for completeness. For logics where a tableau gives an optimal procedure for testing for satisfiability, this is, perhaps, a promising approach to also test for completeness.

Another direction of interest would be to consider axiom schemes as part of the input — as we have seen, axiom 5 together with $\varphi^{\mathbf{S5}}$ is complete for $\mathbf{T}$, when no modal formula is.

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A Appendix: The Proof of Theorem 24

We prove that the Constructor has a winning strategy on $(\varphi, 4|\varphi|)$ if and only if φ is satisfied in two non-bisimilar models. By Corollary 23 and Theorem 3, the theorem follows.

We first assume that there are two non-bisimilar pointed models (A, a) and (B, b), such that $A, a \models \varphi$ and $B, b \models \varphi$. We prove that the Constructor has a winning strategy on $(\varphi, 4|\varphi|)$. We call these models the underlying models; the states of the underlying models are called model states to distinguish them from states of the game. Let $A = (W^A, R^A, V^A)$ and $B = (W^B, R^B, V^B)$; we can assume that $W^A \cap W^B = \emptyset$. We construct a strategy f.

Strategy f can be constructed recursively, given a partial mapping g from plays of even length to pairs of model-states and a partial mapping r, from plays of even length to pairs $(\mathcal{M}, x)$, where $\mathcal{M}$ is either A or B and x is a state of $\mathcal{M}$. Strategy f is defined only on plays of even length. For every model-state $c \in W^{\mathsf{A}}$, where $\mathcal{M} \in \{A, B\}$, and game-state s, if $\mathcal{M}, c \models th(s)$, then let $s^{\mathcal{M},c}$ be minimal such that $s \subseteq s^{\mathcal{M},c}$, $s^{\mathcal{M},c}$ is l-closed, and $\mathcal{M}, c \models s^{A,c}$. The base case is:

$$f(X_0) = X_0((\emptyset, \{\{\varphi\}^{A,a}\}), (\emptyset, \{\{\varphi\}^{B,b}\}), L, \{\varphi\}^{A,a}, k(0) - 1).$$

For even $i > 1$ and $\pi = X_0 X_1 \cdots X_i$, if the following conditions hold:

1. $k(i) > 0$,
2. $g(\pi) = (a_\pi, b_\pi) \in W^A \times W^B$, so that $A, a_\pi \models th(p(S_L(i)))$ and $B, b_\pi \models th(p(S_R(i)))$, and
3. $r(\pi) = (A, x)$, so that $a_\pi R^A x$, or $r(\pi) = (B, x)$, so that $b_\pi R^B x$,

then we construct $f(\pi) = \pi X$, where $X = (S_L, S_R, s, c, k(i) - 1)$, in the following way. We describe $f(\pi)$ for the case where $r(\pi) = (A, a')$, as the case where $r(\pi) = (B, b')$ is symmetric. If l has axiom D, we also fix some $b' \in W^B$, such that $b_\pi R^B b'$.

- $p(S_L) = (p(S_L(i)))^{A,a_\pi}$ and $p(S_R) = (p(S_R(i)))^{B,b_\pi}$.
- For every $\Diamond\psi \in p(S_L)$, if $A, a' \models \psi$, then set $a^\psi = a'$; otherwise, $A, a_\pi \models \Diamond\psi$, so set $a^\psi \in W^B$ to be such that $a_\pi R^A a^\psi$ and $A, a^\psi \models \psi$. If l does not have axiom 4, then let $c^L_\psi = (c^L_\psi[0])^{A,a_\psi}$, where

$$c^L_\psi[0] = \begin{cases} \{\psi\} \cup \{\Box\psi', \psi' \mid \Box\psi' \in p(S_L)\}, & \text{if } l \text{ has axiom 4;} \\ \{\psi\} \cup \{\psi' \mid \Box\psi' \in p(S_L)\}, & \text{otherwise;} \end{cases}$$

  if l has axiom 4, then let let $c^L_\psi = \{\chi \in \overline{sub}(\varphi) \mid A, a_\psi \models \chi\}$.
- For every $\Diamond\psi \in p(S_R)$, if l has axiom D and $B, b' \models \psi$, then set $b^\psi = b'$; otherwise, set $b^\psi \in W^B$ to be such that $b_\pi R^B b^\psi$ and $B, b^\psi \models \psi$. If l does not have axiom 4, then let $c^R_\psi = (c^R_\psi[0])^{B,b_\psi}$, where

$$c^R_\psi[0] = \begin{cases} \{\psi\} \cup \{\Box\psi', \psi' \mid \Box\psi' \in p(S_R)\}, & \text{if } l \text{ has axiom 4;} \\ \{\psi\} \cup \{\psi' \mid \Box\psi' \in p(S_R)\}, & \text{otherwise;} \end{cases}$$

  if l has axiom 4, then let let $c^R_\psi = \{\chi \in \overline{sub}(\varphi) \mid B, b_\psi \models \chi\}$.
- Let $\Psi_L[0] = \max_\leq \{c^L_\psi \mid \Diamond\psi \in p(S_L)\}$, the set of the maximal c^L_ψ's, and similarly, $\Psi_R[0] = \max_\leq \{c^R_\psi \mid \Diamond\psi \in p(S_R)\}$. Then,

$$\Psi_L = \left\{ c^L_\psi \cup \bigcup_{\substack{\Diamond\psi' \in p(S_L) \\ c^L_{\psi'} \leq c^L_\psi}} c^L_{\psi'} \mid c^L_\psi \in \Psi_L[0] \right\}, \quad \Psi_R = \left\{ c^R_\psi \cup \bigcup_{\substack{\Diamond\psi' \in p(S_R) \\ c^R_{\psi'} \leq c^R_\psi}} c^R_{\psi'} \mid c^R_\psi \in \Psi_R[0] \right\}.$$

- If there is some $c^L_\psi \in \Psi_L$, $A, a' \models \psi$, then $a' = a_\psi$ and we set $c^L_\top = c^L_\psi$. Otherwise, let $c^L_\top = \{\chi \in \overline{sub}(\varphi) \mid A, a' \models \chi\}$.
- If l has axiom D and there is some $c^R_\psi \in \Psi_R$, then we can assume $b' = b_\psi$ and we set $c^R_\top = c^R_\psi$. Otherwise, if l has axiom D and $\Psi_R = \emptyset$, let $c^R_\top = \{\chi \in \overline{sub}(\varphi) \mid B, b' \models \chi\}$.
- Then, $C(S_L) = \Psi_L \cup \{c^L_\top\}$; if l has axiom D, then $C(S_R) = \Psi_R \cup \{c^R_\top\}$, otherwise $C(S_R) = \Psi_R$.
- $s = L$, and $c = c^L_\top$.

Observe that if $i > 0$ is even, then no side of X_i in a valid play has any children states. So, it is not hard to verify that S_L and S_R are consistent and complete $S_L(i)$ and $S_R(i)$, respectively. So, if conditions 1, 2, and 3 hold, then f presents a valid move for the Constructor. We can define $g(f(X_0)X_2) = (a, b)$, for every valid play $f(X_0)X_2$; $g(f(X_0)X_2)$ satisfies condition 2. Furthermore, for even $i > 1$, play π of length i, if conditions 1, 2, and 3 are satisfied for π, then for every valid play $f(\pi)X_{i+2}$, if $r(\pi) = (A, x)$ and $p(S_R(i+2)) = c^R_\psi$ as in the construction above, then $g(f(\pi)X_{i+2}) = (x, b_\psi)$; if $r(\pi) = (B, x)$ and $p(S_L(i+2)) = c^L_\psi$ as in the construction above, then $g(f(\pi)X_{i+2}) = (a_\psi, x)$. Then, in both cases, $g(f(\pi)X_{i+2})$ satisfies condition 2 as well. We fix the above definitions of f and g, relative to an r.

We first prove that there is some r, so that if the Constructor conforms to f, then conditions 2 and 3 always hold for plays of even length. For a valid play π of even positive length, if $(A, a_\pi) \not\sim (B, b_\pi)$, then we know that there is some $x \in A$, so that $a_\pi R^A x$ and for every $y \in W^B$, $A, x \not\sim B, y$, or there is some $y \in B$, so that $b_\pi R^B y$ and for every $x \in W^A$, $A, x \not\sim B, y$; in the first case, we can have $r(\pi) = (A, x)$ and in the second case, $r(\pi) = (B, y)$. We call such an r, *appropriate*. By induction, for an appropriate r, if the Constructor conforms to f, then conditions 2 and 3 always hold for plays of even length.

We see, then, that for an appropriate r, if the Constructor conforms to f, the strategy always provides a move to play, as long as the game does not run out of points. If l does not have axiom 4, then we also see that with every move of the Collapser after the first, the maximum modal depth of each parent is reduced by at least one. Therefore, if there are no more points remaining, l does not have axiom 4, and the Constructor has not won the play yet, we can extend f to a winning strategy, which construct one child at one side and no children at the other side, when all formulas in the parent-states are of modal depth 0.

We now examine the cases where l has axiom 4. We can extend the definition of a play: an infinite sequence $\pi = X_0X_1\cdots$ is a play if for every $n \geq 0$, $X_0X_1\cdots X_n$ is a play. As we have argued above, the Constructor, given an infinite amount of initial points and non-bisimilar pointed models (A, a) and (B, b) of φ, always has a move to play according to f, given an appropriate r. If for every choice of an appropriate r, there is an infinite valid play on which the Constructor conforms to f, then we show that $(A, a) \sim (B, b)$, reaching a contradiction. Let $\mathcal{R} = \sim \cup Z$, where $\sim$ is the bisimilarity relation between the states of A and the states of B, and xZy when for an appropriate r, in an infinite play $X_0X_1\cdots X_iX_{i+1}\cdots$ where $i > 0$ is even, $g(X_0X_1\cdots X_i) = (z, y)$. If $x\mathcal{R}y$, either $(A, x) \sim (B, y)$, so $V^A_P(x) = V^B_P(y)$, or xZy, so $V^A_P(x) = V^B_P(y)$, because the play is infinite. If $x\mathcal{R}y$ and xR^Ax', then if $(A, x) \sim (B, y)$, immediately there is some yR^By' so that $(A, x') \sim (B, y')$; if $g(\pi) = (x, y)$ for some π of even length, then either there is some yR^By', so that $(A, x') \sim (B, y')$, or $x' = r(\pi)$ for an appropriate r — the case for yR^By' is symmetrical. Therefore, there is a strategy f, so that if the Constructor conforms to f during π, then π is finite; furthermore, f is a winning strategy for the Constructor. We call a strategy σ a *finite winning* strategy when it is the case that if the Constructor conforms to σ during π, then π is finite and σ is a winning strategy for the Constructor. The *length* of a finite winning strategy is the length of the longest play during which the Constructor conforms to the strategy.

We now demonstrate that there is a finite winning strategy for the Constructor of length at most $4|\varphi|$. For even $i > 1$, $j = L, R$, and play $\pi = X_0 X_1 \cdots X_i$, let $D_j(\pi) = \{\Diamond\psi \in \overline{sub}(\varphi) \mid \vdash p(S_j(i)) \to \Diamond\psi\}$ and $B_j(\pi) = \{\Box\psi \in p(S_j(i))\}$. We can modify the definition of f above, so that $p(S_L) = (p(S_L(i)))^{A,a_\pi} \cup D_L(\pi)$ and similarly for $p(S_R)$. This modification does not affect any of the arguments above. Notice that now

- the set of children-states that can appear on a consistent side that completes $S_j(i)$ completely depends on $D_j(\pi)$ and $B_j(\pi)$, as all conditions for children depend on these sets; and
- for any formula of the form $\Diamond\chi$, whether $\vdash th(p(S_j(i))) \to \Diamond\chi$ or not completely depends on $D_j(\pi)$ and $B_j(\pi)$ as well, since all formulas in $p(S_j(i))$ are boolean combinations of $D_j(\pi)$ and $B_j(\pi)$ and literals.

Let $\pi = \pi'\pi''$, where $\pi \neq \pi'$ are plays of even positive length, so that for all $j = L, R$, $D_j(\pi) = D_j(\pi')$ and $B_j(\pi) = B_j(\pi')$. From the above, if $\pi\rho$ is a play, then $\pi'\rho$ is a play. Therefore, a finite winning strategy which results in a play π as above does not have the minimum length of finite winning strategies on φ. Notice that $B_j(\pi)$ and $D_j(\pi)$ are monotonic with respect to π and $0 \leq B_j(\pi) \leq k_1$ and $0 \leq D_j(\pi) \leq 2k_2$ for some $k_1 + k_2 \leq |\varphi| - 1$, from which we can conclude that any finite winning strategy of minimum length has length less that $4|\varphi|$.

On the other hand, we prove that if φ is complete, then the Collapser has a winning strategy. For this, we use the following two lemmata:

▶ **Lemma 30.** *If a side S is consistent and complete and $C(S) \neq \emptyset$, then*

- *if l does not have axiom 4 ($l = \boldsymbol{K}$), then the following formula is consistent:*

$$th(p(S)) \wedge \bigwedge_{c \in C(S)} \Diamond th(c) \wedge \Box \bigvee_{c \in C(S)} th(c);$$

- *if l has axiom 4 ($l \in \{\boldsymbol{K4}, \boldsymbol{D4}, \boldsymbol{S4}\}$), then the following formula is consistent:*

$$th(p(S)) \wedge \bigwedge_{c \in C(S)} \Diamond th(c).$$

Proof. For $c \in C(S)$, let $\mathcal{M}_c = (W_c, R_c, V_c)$ and $a_c \in W_c$, such that $\mathcal{M}_c, a_c \models th(c)$; we assume $p(S) \notin W_c$. Then let $\mathcal{M} = (W, R, V)$, where

$$W = \{p(S)\} \cup \bigcup_{c \in C(S)} W_c, \qquad R' = \{(p(S), a_c) \mid c \in C(S)\} \cup \bigcup_{c \in C(S)} R_c,$$

R is the transitive (if l has axiom 4 and not T) or reflexive and transitive (if l has both axioms) closure of R', or just R' (if l has neither axiom), and

$$V(p(S)) = \{p \in P \mid p \in p(S)\}$$

and for $b \in W_c$, $V(b) = V_c(b)$. Now, $\mathcal{M}, a_c \models th(c)$, since it is not hard to see that $(\mathcal{M}, a_c) \sim (\mathcal{M}_c, a_c)$. By straightforward induction on ψ, we can see that for all $\psi \in p(S)$, $\mathcal{M}, p(S) \models \psi$, from which we can conclude that

$$\mathcal{M}, p(S) \models th(p(S)) \wedge \bigwedge_{c \in C} \Diamond th(c) \wedge \Box \bigvee_{c \in C} th(c), \qquad \text{if } l \text{ does not have axiom 4; or}$$

$$\mathcal{M}, p(S) \models th(p(S)) \wedge \bigwedge_{c \in C} \Diamond th(c), \qquad \text{if } l \text{ has axiom 4.}$$

◀

▶ **Lemma 31.** *If a state s is consistent and complete,*

$$C = \{s' \subseteq \overline{sub}(\varphi) \mid \ s' \text{ is 4-maximal and } \vdash_l th(s) \to \Diamond th(s')\},$$

there is a formula ψ and a $d \in C$, so that $th(d)$ is not equivalent to $th(s)$ and $d' = d \cup \{\Box\psi\}$ is consistent, then $th(s) \wedge \Box(\neg th(d) \vee \Box\psi)$ is consistent.

Proof. If l does not have axiom 4, then we can use Lemma 30; otherwise, we can use a similar construction as for Lemma 30. Let $\mathcal{M} = (W, R, V)$, where: $W = \{s\} \cup C$; for $a, b \in W$, aRb iff for every $\Box\chi \in a$, $\chi, \Box\chi \in b$; and $V(a) = P \cap a$. $\mathcal{M}$ is an l-model: R is transitive, since if $\Box\chi \in a$ and aRb, also $\Box\chi \in b$; if l has axiom D, then R is serial, because for $a \in W$, $\{\chi \mid \Box\chi \in a\}$ is consistent; and if l has axiom T, then R is reflexive, since for a to be maximally consistent, if $\Box\chi \in a$, then $\chi \in a$. Then, by induction on the formulas, for every $\chi \in \overline{sub}(\phi)$ and $a \in W$, $\chi \in a$ iff $\mathcal{M}, a \models \chi$: constants, literals and boolean connectives are immediate; if $\Box\chi \in a$, then for every aRb, $\chi \in b$, so $\mathcal{M}, a \models \Box\chi$; if $\Diamond\chi \in a$, then $v = \{\chi \mid \Box\chi \in a\} \cup \{\chi\}$ is consistent and $\vdash_l th(a) \to \Diamond th(v)$, so $\vdash_l th(s) \to \Diamond th(v)$ (by axiom 4), therefore there is some $b \supseteq v$ in C and aRb.

Let $D, x \models th(d')$, where $D = (W_d, R_d, V_d)$ is an l-model; let $\mathcal{M}' = (W \cup W_d, R', V')$, where $V'(a) = V(a)$ if $a \in W$ and $V_d(a)$ otherwise and R' is the transitive closure of

$$\{(a, b) \in R \mid a = d \text{ or } b \neq d\} \cup \{(a, x) \mid a \neq d \text{ and } aRd\} \cup R_d.$$

Then, by induction on the formulas, for every $\chi \in \overline{sub}(\varphi)$ and $a \in W$, $\mathcal{M}', a \models \chi$ iff $\mathcal{M}, a \models \chi$: constants, literals, boolean connectives are immediate; if $\chi = \Box\chi'$, then if not aRd, nothing changed and if aRd, $\chi, \chi' \in d$, so $D, x \models \chi, \chi'$, meaning that if aRy, then $\mathcal{M}', y \models \chi'$; the case for $\chi = \Diamond\chi'$ is more straightforward. Therefore, $\mathcal{M}', s \models th(s) \wedge \Box(\neg th(d) \vee \Box\psi)$. ◀

▶ **Lemma 32.** *For a consistent and complete side S with a child c, such that c is maximal and if l has axiom 4, then all children-states are 4-maximal, if $th(p(S))$ is complete, then so is $th(c)$.*

Proof. If $th(c)$ is not complete, then $\nvdash_l th(c) \to th(p(S))$ and there is some ψ such that $th(c) \nvdash \psi$ and $th(c) \nvdash \neg\psi$. Since c is maximal, for every other $b \in C(S)$, $b \cup \{\neg th(c)\}$ is consistent. Therefore there is a a consistent side S_1 that completes S and has a child-state $c_1 \supseteq \{\psi\} \cup c$ and a consistent side S_2 that completes S, which has a child-state $c_2 \supseteq \{\neg\psi\} \cup c$ and $\neg th(c)$ is an element of every other child-state of S_2. By Lemma 30, if l does not have axiom 4, then $th(p(S)) \wedge \Diamond(th(c) \wedge \psi)$ is consistent and so is $th(p(S)) \wedge \neg\Diamond(th(c) \wedge \psi)$, which is a contradiction.

If l has axiom 4, then for every other child (or maximally consistent subset of $\overline{sub}(\varphi)$) c', $\vdash_l th(c') \to \neg th(c)$; c is 4-maximal and we can assume $\psi = \Box\chi$. By Lemma 31, $th(p(S)) \wedge \Diamond(th(c) \wedge \Diamond\neg\chi)$ is consistent. By Lemma 31, so is $th(p(S)) \wedge \Box(\neg th(c) \vee \psi)$, again a contradiction. ◀

The strategy f of the Collapser is defined on plays $\pi = X_0X_1 \cdots X_i$ of odd length i, as long as $\vdash th(p(S_L(X_i))) \leftrightarrow th(p(S_R(X_i)))$ and $th(p(S_L(X_i)))$ is complete. We assume that $s(i-1) = L$ and the case for $s(i-1) = R$ is similar. Then, $f(\pi) = \pi X$, where $X = ((a(i), \emptyset), (b, \emptyset), k(i-1))$, so we need to determine b. Because of the conditions for f and Lemma 30, $\vdash th(p(S_R(X_i))) \to \Diamond th(a(i))$. If l does not have axiom 4, then by Lemmata 30 and 32, there is a child-state b' of $S_R(X_i)$, such that $\vdash th(b) \leftrightarrow th(a(i))$, so we can set $b = b'$. If l has axiom 4, then by the above, we can set $b = a(i)$. Notice that by Lemma 32, $th(b)$ is complete; so, every consistent $S_L(i+2)$ and $S_R(i+2)$ that complete $(a(i), \emptyset)$ and $(b, \emptyset)$ respectively, must also be complete. Therefore, there is always a move for the Collapser to play, conforming to f. ◀