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Branch: master research_public / lectures / MIT Sloan ARCH, GARCH, and GMM.ipynb

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Maxwell Margenot MAINT: Added compliance disclaimer a59600a on Apr 6

1 contributor

842 lines (842 sloc) 579 KB

Raw Blame History

Generalized Method of Moments with ARCH and GARCH Models

By Delaney Granizo-Mackenzie and Andrei Kirilenko developed as part of the Masters of Finance curriculum at MIT Sloan.

AutoRegressive Conditionally Heteroskedastic (ARCH) occurs when the volatility of a time series is also autoregressive.

```
In [1]: import cvxopt
        from functools import partial
        import math
        import numpy as np
        import scipy
        from scipy import stats
        import statsmodels as sm
        from statsmodels.stats.stattools import jarque_bera

        import matplotlib.pyplot as plt
```

Simulating a GARCH(1, 1) Case

We'll start by using Monte Carlo sampling to simulate a GARCH(1, 1) process. Our dynamics will be

$$\epsilon \sim \mathcal{N}(0, 1)$$

Our parameters will be $a_0 = 1$, $a_1 = 0.1$, and $b_1 = 0.8$. We will drop the first 10% (burn-in) of our simulated values.

```
In [2]: # Define parameters
        a0 = 1.0
        a1 = 0.1
        b1 = 0.8
        sigma1 = math.sqrt(a0 / (1 - a1 - b1))
```

```
In [3]: def simulate_GARCH(T, a0, a1, b1, sigma1):

            # Initialize our values
            X = np.ndarray(T)
            sigma = np.ndarray(T)
            sigma[0] = sigma1

            for t in range(1, T):
                # Draw the next x_t
                X[t - 1] = sigma[t - 1] * np.random.normal(0, 1)
                # Draw the next sigma_t
                sigma[t] = math.sqrt(a0 + b1 * sigma[t - 1]**2 + a1 * X[t - 1]**2)

            X[T - 1] = sigma[T - 1] * np.random.normal(0, 1)

            return X, sigma
```

Now we'll compare the tails of the GARCH(1, 1) process with normally distributed values. We expect to see fatter tails, as the GARCH(1, 1) process will experience extreme values more often.

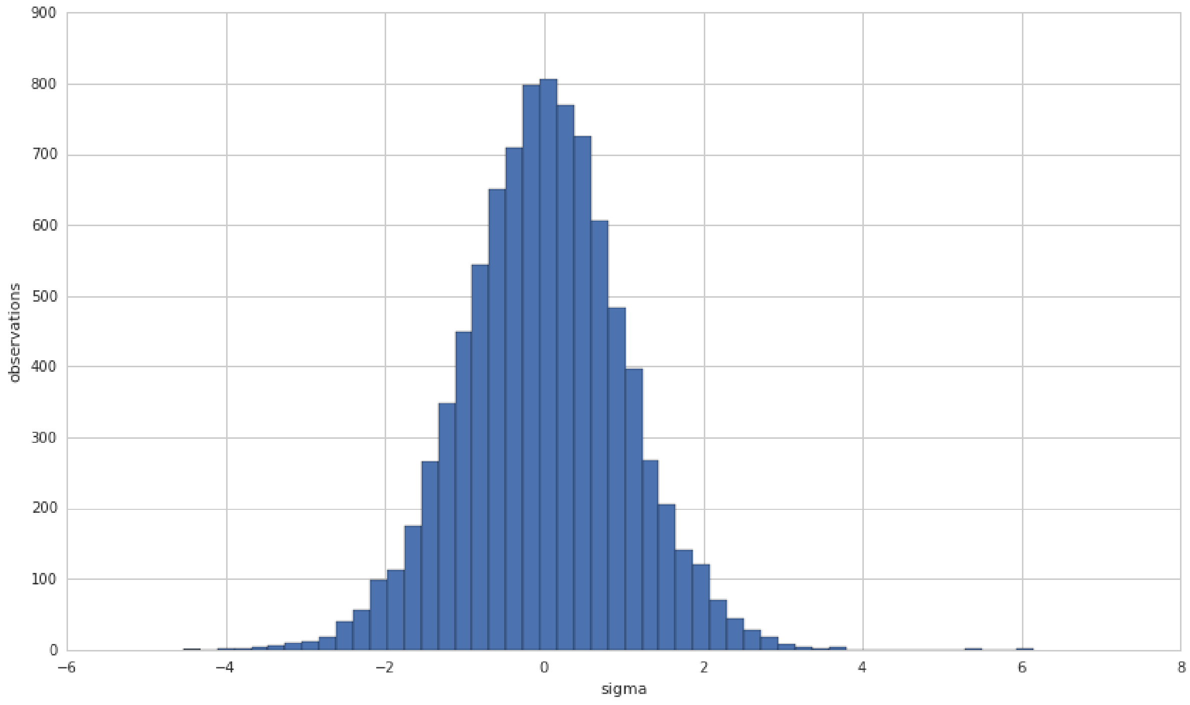
```
In [4]: X, _ = simulate_GARCH(10000, a0, a1, b1, sigma1)
        X = X[1000:] # Drop burn in
        X = X / np.std(X) # Normalize X

        def compare_tails_to_normal(X):
            # Define matrix to store comparisons
            A = np.zeros((2,4))
            for k in range(4):
                A[0, k] = len(X[X > (k + 1)]) / float(len(X)) # Estimate tails of X
                A[1, k] = 1 - stats.norm.cdf(k + 1) # Compare to Gaussian distribution
            return A
```

```
compare_tails_to_normal(X)
```

```
Out[4]: array([[  1.49444444e-01,   2.34444444e-02,   1.88888889e-03,
                  2.22222222e-04],
               [  1.58655254e-01,   2.27501319e-02,   1.34989803e-03,
                  3.16712418e-05]])
```

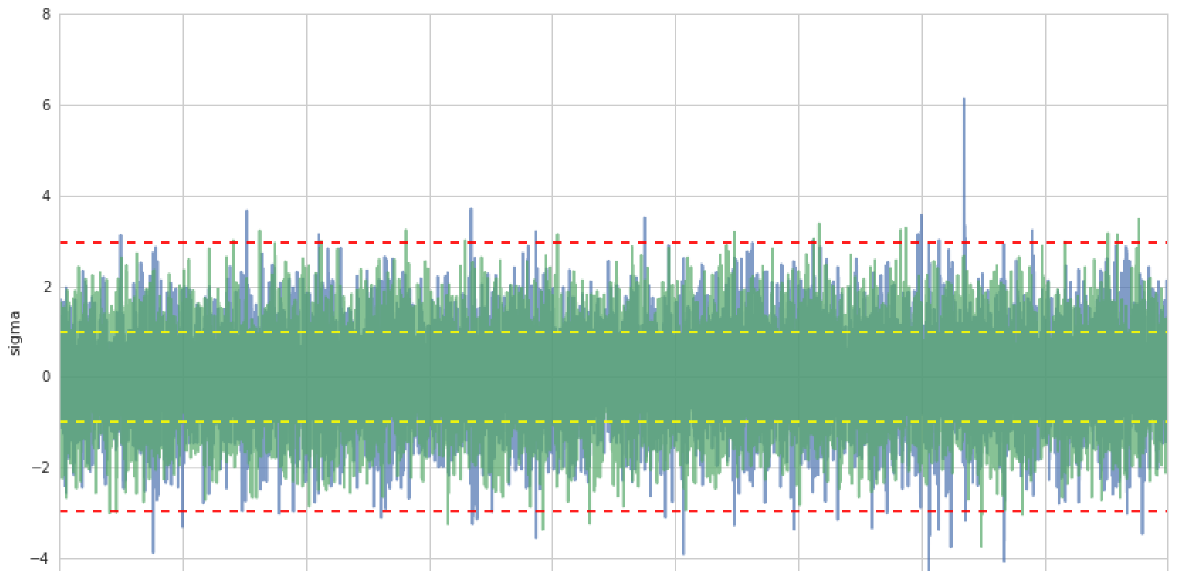
Sure enough, the tails of the GARCH(1, 1) process are fatter. We can also look at this graphically, although it's a little tricky to see.

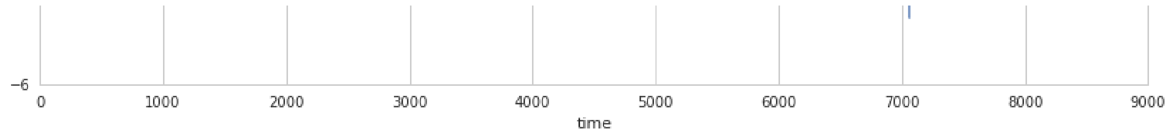
```
In [5]: plt.hist(X, bins=50)
        plt.xlabel('sigma')
        plt.ylabel('observations');
```



```
In [6]: # Sample values from a normal distribution
        X2 = np.random.normal(0, 1, 9000)
        both = np.matrix([X, X2])
```

```
In [7]: # Plot both the GARCH and normal values
        plt.plot(both.T, alpha=.7);
        plt.axhline(X2.std(), color='yellow', linestyle='--')
        plt.axhline(-X2.std(), color='yellow', linestyle='--')
        plt.axhline(3*X2.std(), color='red', linestyle='--')
        plt.axhline(-3*X2.std(), color='red', linestyle='--')
        plt.xlabel('time')
        plt.ylabel('sigma');
```





What we're looking at here is the GARCH process in blue and the normal process in green. The 1 and 3 std bars are drawn on the plot. We can see that the blue GARCH process tends to cross the 3 std bar much more often than the green normal one.

Testing for ARCH Behavior

The first step is to test for ARCH conditions. To do this we run a regression on x_t fitting the following model.

$$x_t^2 = a_0 + a_1 x_{t-1}^2 + \ldots + a_p x_{t-p}^2$$

We use OLS to estimate $\hat{\theta} = (\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_p)$ and the covariance matrix $\hat{\Omega}$. We can then compute the test statistic

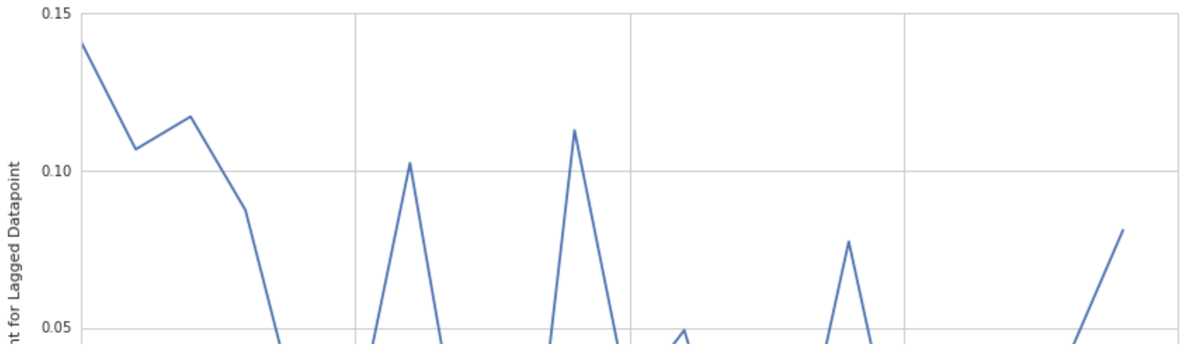
$$F = \hat{\theta}\hat{\Omega}^{-1}\hat{\theta}'$$

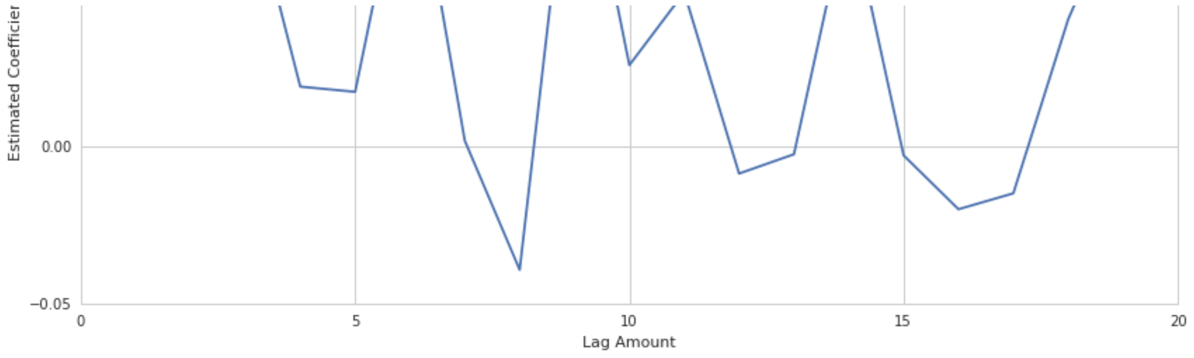
We will reject if F is greater than the 95% confidence bars in the (scr;X) 2(p)\mathcal(X)^2(p) distribution.

To test, we'll set $p = 20$ and see what we get.

In [8]:
```
X, _ = simulate_GARCH(1100, a0, a1, b1, sigma1)
X = X[100:] # Drop burn in

p = 20

# Drop the first 20 so we have a lag of p's
Y2 = (X**2)[p:]
X2 = np.ndarray((980, p))
for i in range(p, 1000):
    X2[i - p, :] = np.asarray((X**2)[i-p:i])[::-1]

model = sm.regression.linear_model.OLS(Y2, X2)
model = model.fit()
theta = np.matrix(model.params)
omega = np.matrix(model.cov_HC0)
F = np.asscalar(theta * np.linalg.inv(omega) * theta.T)

print np.asarray(theta.T).shape

plt.plot(range(20), np.asarray(theta.T))
plt.xlabel('Lag Amount')
plt.ylabel('Estimated Coefficient for Lagged Datapoint')

print 'F = ' + str(F)

chi2dist = scipy.stats.chi2(p)
pvalue = 1-chi2dist.cdf(F)
print 'p-value = ' + str(pvalue)

# Finally let's look at the significance of each a_p as measured by the standard deviations away from 0
print theta/np.diag(omega)
```

```
(20, 1)
F = 380.645595779
p-value = 0.0
[[ 66.55194039  69.87117113  74.67024832  58.88503994  14.11740836
   11.83421598  47.10420145   1.4313501  -36.55771396  69.15283656
   23.23590451  37.72347901  -7.80824905  -1.93944403  55.54508858
   -2.40144257 -19.23312596 -12.97539207  29.57300021  58.19284553]]
```





Fitting GARCH(1, 1) with MLE

Once we've decided that the data might have an underlying GARCH(1, 1) model, we would like to fit GARCH(1, 1) to the data by estimating parameters.

To do this we need the log-likelihood function

$$\mathscr{L}(\theta) = \sum_{t=1}^{T} - \ln\sqrt{2\pi} - \frac{x_t^2}{2\sigma_t^2} - \frac{1}{2}\ln(\sigma_t^2)$$

To evaluate this function we need x_t and σ_t for $1 \leq t \leq T$. We have x_t, but we need to compute σ_t. To do this we need to make a guess for σ_1. Our guess will be $\sigma_1^2 = \hat{E}[x_t^2]$. Once we have our initial guess we compute the rest of the σ's using the equation

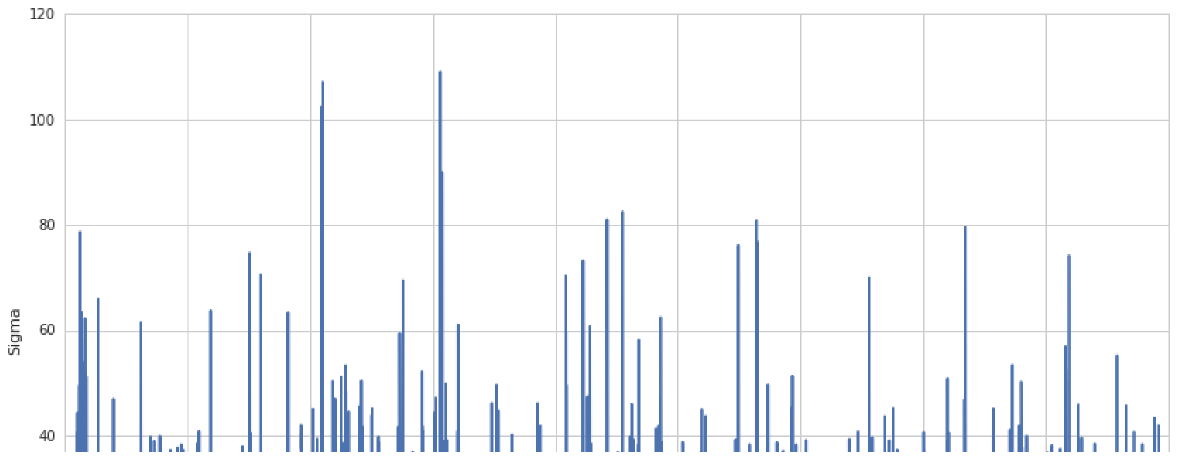
$$\sigma_t^2 = a_0 + a_1 x_{t-1}^2 + b_1 \sigma_{t-1}^2$$

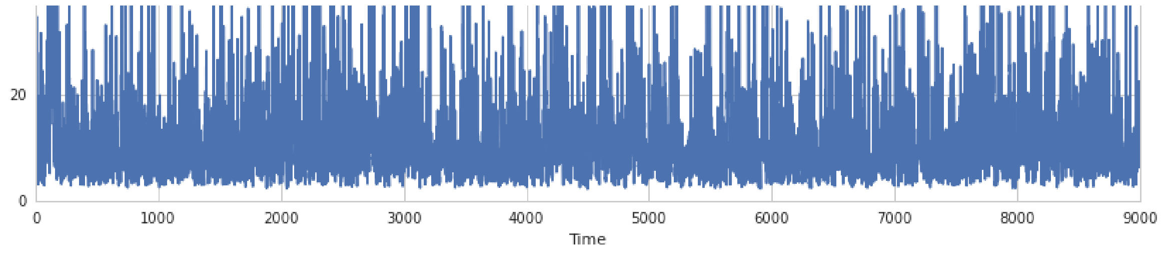
In [9]:
```
X, _ = simulate_GARCH(10000, a0, a1, b1, sigma1)
X = X[1000:] # Drop burn in
```

In [10]:
```
# Here's our function to compute the sigmas given the initial guess
def compute_squared_sigmas(X, initial_sigma, theta):

    a0 = theta[0]
    a1 = theta[1]
    b1 = theta[2]

    T = len(X)
    sigma2 = np.ndarray(T)

    sigma2[0] = initial_sigma ** 2

    for t in range(1, T):
        # Here's where we apply the equation
        sigma2[t] = a0 + a1 * X[t-1]**2 + b1 * sigma2[t-1]

    return sigma2
```

Let's look at the sigmas we just generated.

In [11]:
```
plt.plot(range(len(X)), compute_squared_sigmas(X, np.sqrt(np.mean(X**2)), (1, 0.5, 0.5)))
plt.xlabel('Time')
plt.ylabel('Sigma');
```





Now that we can compute the σ_t's, we'll define the actual log likelihood function. This function will take as input our observations x and θ and return $-\mathscr{L}(\theta)$. It is important to note that we return the negative log likelihood, as this way our numerical optimizer can minimize the function while maximizing the log likelihood.

Note that we are constantly re-computing the σ_t's in this function.

```
In [12]: def negative_log_likelihood(X, theta):

             T = len(X)

             # Estimate initial sigma squared
             initial_sigma = np.sqrt(np.mean(X ** 2))

             # Generate the squared sigma values
             sigma2 = compute_squared_sigmas(X, initial_sigma, theta)

             # Now actually compute
             return -sum(
                 [-np.log(np.sqrt(2.0 * np.pi)) -
                 (X[t] ** 2) / (2.0 * sigma2[t]) -
                 0.5 * np.log(sigma2[t]) for
                  t in range(T)]
             )
```

Now we perform numerical optimization to find our estimate for

$$\hat{\theta} = \operatorname{argmax}_{(a_0,\ a_1,\ b_1)} \mathscr{L}(\theta) = \operatorname{argmin}_{(a_0,\ a_1,\ b_1)} - \mathscr{L}(\theta)$$

We have some constraints on this

$$a_1 \geq 0,\ b_1 \geq 0,\ a_1 + b_1 < 1$$

```
In [13]: # Make our objective function by plugging X into our log likelihood function
         objective = partial(negative_log_likelihood, X)

         # Define the constraints for our minimizer
         def constraint1(theta):
             return np.array([1 - (theta[1] + theta[2])])

         def constraint2(theta):
             return np.array([theta[1]])

         def constraint3(theta):
             return np.array([theta[2]])

         cons = ({'type': 'ineq', 'fun': constraint1},
                 {'type': 'ineq', 'fun': constraint2},
                 {'type': 'ineq', 'fun': constraint3})

         # Actually do the minimization
         result = scipy.optimize.minimize(objective, (1, 0.5, 0.5),
                                 method='SLSQP',
                                 constraints = cons)
         theta_mle = result.x
         print 'theta MLE: ' + str(theta_mle)

         theta MLE: [ 1.28589303  0.0893921   0.77758464]
```

Now we would like a way to check our estimate. We'll look at two things:

1. How fat are the tails of the residuals.
2. How normal are the residuals under the Jarque-Bera normality test.

We'll do both in our `check_theta_estimate` function.

```
In [14]: def check_theta_estimate(X, theta_estimate):
             initial_sigma = np.sqrt(np.mean(X ** 2))
             sigma = np.sqrt(compute_squared_sigmas(X, initial_sigma, theta_estimate))
             epsilon = X / sigma
             print 'Tails table'
             print compare_tails_to_normal(epsilon / np.std(epsilon))
             print ''

             _, pvalue, _, _ = jarque_bera(epsilon)
             print 'Jarque-Bera probability normal: ' + str(pvalue)

         check_theta_estimate(X, theta_mle)
```

```
Tails table
[[  1.52666667e-01   2.23333333e-02   1.77777778e-03   0.00000000e+00]
 [  1.58655254e-01   2.27501319e-02   1.34989803e-03   3.16712418e-05]]

Jarque-Bera probability normal: 0.786796937059
```

GMM for Estimating GARCH(1, 1) Parameters

We've just computed an estimate using MLE, but we can also use Generalized Method of Moments (GMM) to estimate the GARCH(1, 1) parameters.

To do this we need to define our moments. We'll use 4.

1. The residual $\hat{\epsilon}_t = x_t / \hat{\sigma}_t$
2. The variance of the residual $\hat{\epsilon}_t^2$
3. The skew moment $\mu_3 / \hat{\sigma}_t^3 = (\hat{\epsilon}_t - E[\hat{\epsilon}_t])^3 / \hat{\sigma}_t^3$
4. The kurtosis moment $\mu_4 / \hat{\sigma}_t^4 = (\hat{\epsilon}_t - E[\hat{\epsilon}_t])^4 / \hat{\sigma}_t^4$

```
In [15]: # The n-th standardized moment
         # skewness is 3, kurtosis is 4
         def standardized_moment(x, mu, sigma, n):
             return ((x - mu) ** n) / (sigma ** n)
```

GMM now has three steps.

Start with W as the identity matrix.

1. Estimate $\hat{\theta}_1$ by using numerical optimization to minimize

$$\min_{\theta \in \Theta}\left(\frac{1}{T}\sum_{t=1}^{T} g(x_t, \hat{\theta})\right)' W\left(\frac{1}{T}\sum_{t=1}^{T} g(x_t, \hat{\theta})\right)$$

2. Recompute W based on the covariances of the estimated θ. (Focus more on parameters with explanatory power)

$$\hat{W}_{i+1} = \left(\frac{1}{T}\sum_{t=1}^{T} g(x_t, \hat{\theta}_i) g(x_t, \hat{\theta}_i)'\right)^{-1}$$

3. Repeat until $|\hat{\theta}_{i+1} - \hat{\theta}_i| < \epsilon$ or we reach an iteration threshold.

Initialize W and T and define the objective function we need to minimize.

```
In [16]: def gmm_objective(X, W, theta):
             # Compute the residuals for X and theta
             initial_sigma = np.sqrt(np.mean(X ** 2))
             sigma = np.sqrt(compute_squared_sigmas(X, initial_sigma, theta))
             e = X / sigma

             # Compute the mean moments
             m1 = np.mean(e)
             m2 = np.mean(e ** 2) - 1
             m3 = np.mean(standardized_moment(e, np.mean(e), np.std(e), 3))
             m4 = np.mean(standardized_moment(e, np.mean(e), np.std(e), 4) - 3)

             G = np.matrix([m1, m2, m3, m4]).T

             return np.asscalar(G.T * W * G)

         def gmm_variance(X, theta):
             # Compute the residuals for X and theta
             initial_sigma = np.sqrt(np.mean(X ** 2))
             sigma = np.sqrt(compute_squared_sigmas(X, initial_sigma, theta))
             e = X / sigma

             # Compute the squared moments
             m1 = e ** 2
```

```
    m2 = (e ** 2 - 1) ** 2
    m3 = standardized_moment(e, np.mean(e), np.std(e), 3) ** 2
    m4 = (standardized_moment(e, np.mean(e), np.std(e), 4) - 3) ** 2

    # Compute the covariance matrix g * g'
    T = len(X)
    s = np.ndarray((4, 1))
    for t in range(T):
        G = np.matrix([m1[t], m2[t], m3[t], m4[t]]).T
        s = s + G * G.T

    return s / T
```

Now we're ready to the do the iterated minimization step.

In [17]:
```
# Initialize GMM parameters
W = np.identity(4)
gmm_iterations = 10

# First guess
theta_gmm_estimate = theta_mle

# Perform iterated GMM
for i in range(gmm_iterations):
    # Estimate new theta
    objective = partial(gmm_objective, X, W)
    result = scipy.optimize.minimize(objective, theta_gmm_estimate, constraints=cons)
    theta_gmm_estimate = result.x
    print 'Iteration ' + str(i) + ' theta: ' + str(theta_gmm_estimate)

    # Recompute W
    W = np.linalg.inv(gmm_variance(X, theta_gmm_estimate))


check_theta_estimate(X, theta_gmm_estimate)
```

```
Iteration 0 theta: [ 0.92784847  0.07017489  0.83347953]
Iteration 1 theta: [ 0.9278816   0.07039501  0.83372203]
Iteration 2 theta: [ 0.92788126  0.07039224  0.83371928]
Iteration 3 theta: [ 0.92788126  0.07039224  0.83371928]
Iteration 4 theta: [ 0.92788126  0.07039224  0.83371928]
Iteration 5 theta: [ 0.92788126  0.07039224  0.83371928]
Iteration 6 theta: [ 0.92788126  0.07039224  0.83371928]
Iteration 7 theta: [ 0.92788126  0.07039224  0.83371928]
Iteration 8 theta: [ 0.92788126  0.07039224  0.83371928]
Iteration 9 theta: [ 0.92788126  0.07039224  0.83371928]
Tails table
[[  1.53666667e-01   2.24444444e-02   1.77777778e-03   0.00000000e+00]
 [  1.58655254e-01   2.27501319e-02   1.34989803e-03   3.16712418e-05]]

Jarque-Bera probability normal: 0.834347015917
```

Predicting the Future: How to actually use what we've done

Now that we've fitted a model to our observations, we'd like to be able to predict what the future volatility will look like. To do this, we can just simulate more values using our original GARCH dynamics and the estimated parameters.

The first thing we'll do is compute an initial σ_t. We'll compute our squared sigmas and take the last one.

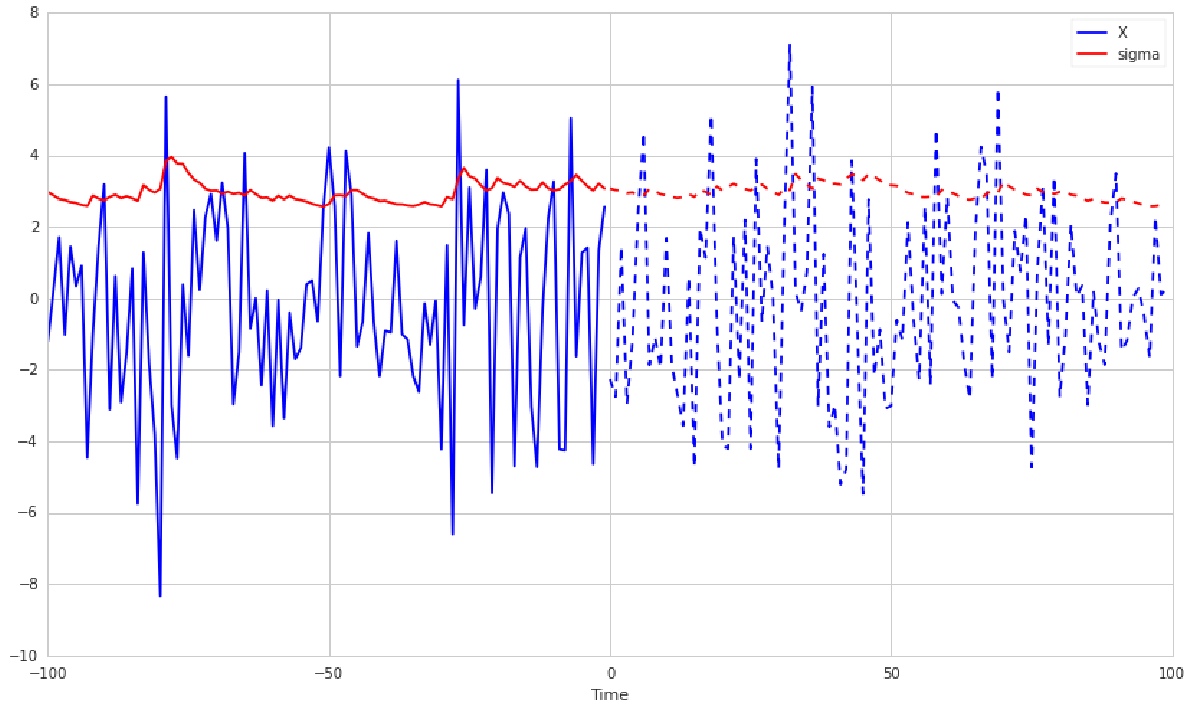
In [18]:
```
sigma_hats = np.sqrt(compute_squared_sigmas(X, np.sqrt(np.mean(X**2)), theta_mle))
initial_sigma = sigma_hats[-1]
initial_sigma
```

Out[18]:
```
3.0753430189459459
```

Now we'll just sample values walking forward.

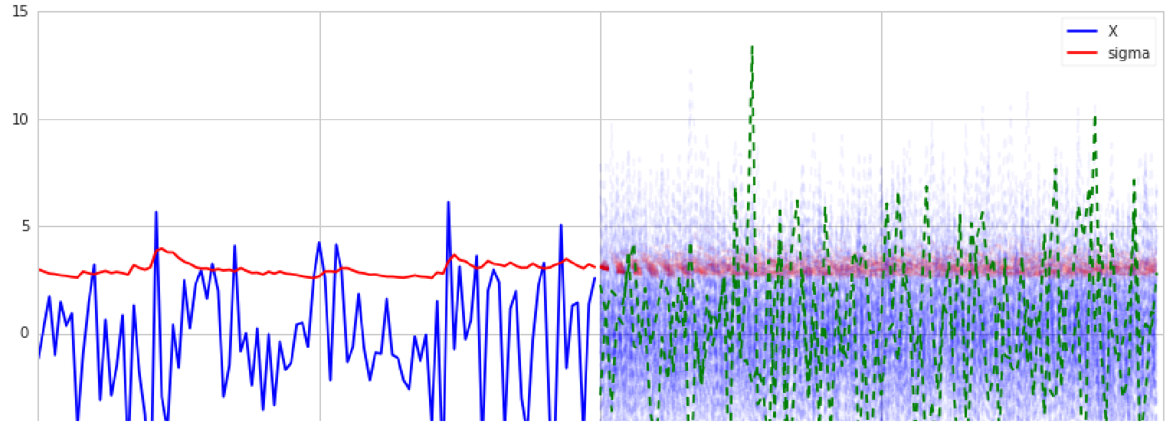
In [19]:
```
a0_estimate = theta_gmm_estimate[0]
a1_estimate = theta_gmm_estimate[1]
b1_estimate = theta_gmm_estimate[2]

X_forecast, sigma_forecast = simulate_GARCH(100, a0_estimate, a1_estimate, b1_estimate, initial_sigma)
```

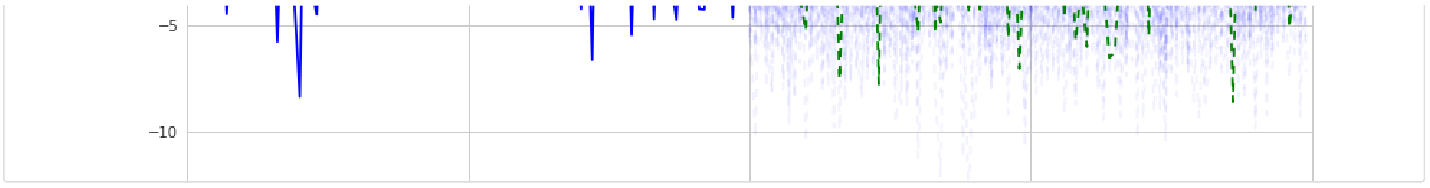
In [20]:
```
plt.plot(range(-100, 0), X[-100:], 'b-')
plt.plot(range(-100, 0), sigma_hats[-100:], 'r-')
plt.plot(range(0, 100), X_forecast, 'b--')
plt.plot(range(0, 100), sigma_forecast, 'r--')
plt.xlabel('Time')
plt.legend(['X', 'sigma']);
```



One should note that because we are moving foward using a random walk, this analysis is supposed to give us a sense of the magnitude of sigma and therefore the risk we could face. It is not supposed to accurately model future values of X. In practice you would probably want to use Monte Carlo sampling to generate thousands of future scenarios, and then look at the potential range of outputs. We'll try that now. Keep in mind that this is a fairly simplistic way of doing this analysis, and that better techniques, such as Bayesian cones, exist.

In [21]:
```
plt.plot(range(-100, 0), X[-100:], 'b-')
plt.plot(range(-100, 0), sigma_hats[-100:], 'r-')
plt.xlabel('Time')
plt.legend(['X', 'sigma'])


max_X = [-np.inf]
min_X = [np.inf]
for i in range(100):
    X_forecast, sigma_forecast = simulate_GARCH(100, a0_estimate, a1_estimate, b1_estimate, initial_sigma)
    if max(X_forecast) > max(max_X):
        max_X = X_forecast
    elif min(X_forecast) < min(max_X):
        min_X = X_forecast
    plt.plot(range(0, 100), X_forecast, 'b--', alpha=0.05)
    plt.plot(range(0, 100), sigma_forecast, 'r--', alpha=0.05)

# Draw the most extreme X values specially
plt.plot(range(0, 100), max_X, 'g--', alpha=1.0)
plt.plot(range(0, 100), min_X, 'g--', alpha=1.0);
```




−5
−10