Maximum Likelihood Estimates (MLEs)

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Part of the Quantopian Lecture Series:

- www.quantopian.com/lectures (https://www.quantopian.com/lectures)
- github.com/quantopian/research_public (https://github.com/quantopian/research_public)

In this tutorial notebook we'll do the following things:

1. Compute the MLE for a normal distribution.
2. Compute the MLE for an exponential distribution.
3. Fit a normal distribution to asset returns using MLE.

First we need to import some libraries

In [13]:
```
import math
import matplotlib.pyplot as plt
import numpy as np
import scipy
import scipy.stats
```

Normal Distribution

We'll start by sampling some data from a normal distribution.

In [14]:
```
TRUE_MEAN = 40
TRUE_STD = 10
X = np.random.normal(TRUE_MEAN, TRUE_STD, 1000)
```

Now we'll define functions that given our data, will compute the MLE for the μ and σ parameters of the normal distribution.

Recall that

$$\hat{\sigma} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(x_t - \hat{\mu})^2}$$

In [15]:
```
def normal_mu_MLE(X):
    # Get the number of observations
    T = len(X)
    # Sum the observations
    s = sum(X)
    return 1.0/T * s

def normal_sigma_MLE(X):
    T = len(X)
    # Get the mu MLE
    mu = normal_mu_MLE(X)
    # Sum the square of the differences
    s = sum( np.power((X - mu), 2) )
    # Compute sigma^2
    sigma_squared = 1.0/T * s
    return math.sqrt(sigma_squared)
```

Now let's try our functions out on our sample data and see how they compare to the built-in `np.mean` and `np.std`

In [16]:
```
print "Mean Estimation"
print normal_mu_MLE(X)
print np.mean(X)
print "Standard Deviation Estimation"
print normal_sigma_MLE(X)
print np.std(X)
```

```
Mean Estimation
39.5829392522
39.5829392522
Standard Deviation Estimation
10.2920350139
10.2920350139
```

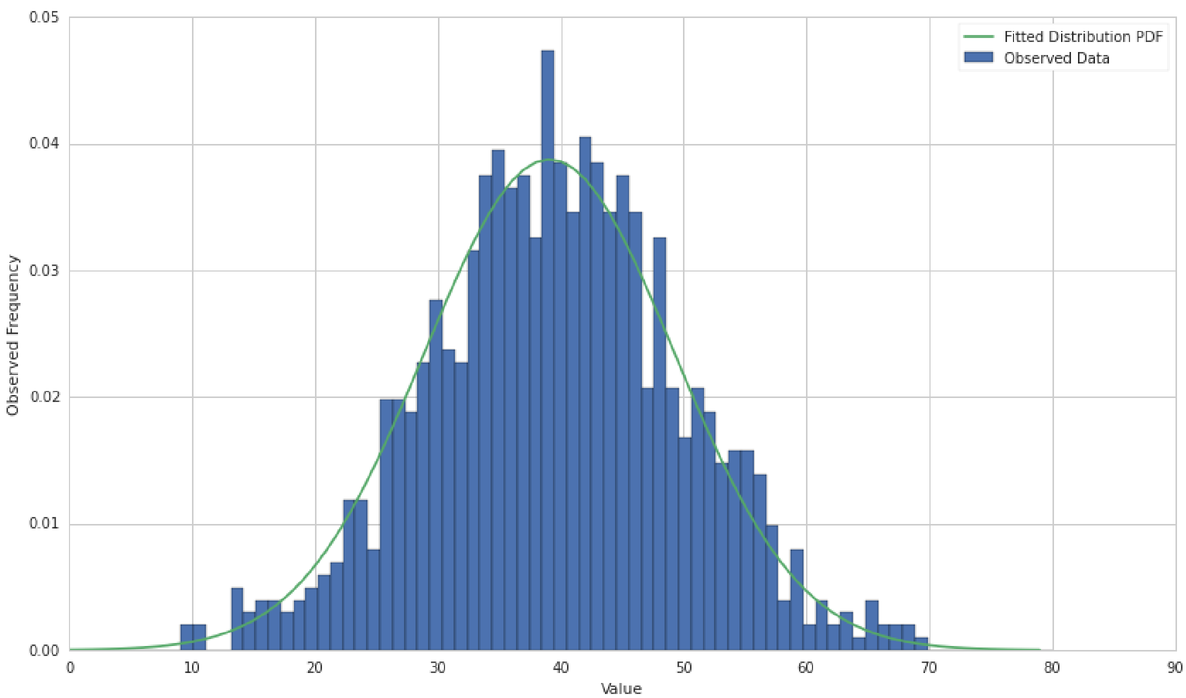
Now let's estimate both parameters at once with scipy's built in `fit()` function.

In [17]:
```
mu, std = scipy.stats.norm.fit(X)
print "mu estimate: " + str(mu)
print "std estimate: " + str(std)
```

```
mu estimate: 39.5829392522
std estimate: 10.2920350139
```

Now let's plot the distribution PDF along with the data to see how well it fits. We can do that by accessing the pdf provided in `scipy.stats.norm.pdf`.

In [18]:
```
pdf = scipy.stats.norm.pdf
# We would like to plot our data along an x-axis ranging from 0-80 with 80 intervals
# (increments of 1)
x = np.linspace(0, 80, 80)
plt.hist(X, bins=x, normed='true')
plt.plot(pdf(x, loc=mu, scale=std))
plt.xlabel('Value')
plt.ylabel('Observed Frequency')
plt.legend(['Fitted Distribution PDF', 'Observed Data', ]);
```



Exponential Distribution

Let's do the same thing, but for the exponential distribution. We'll start by sampling some data.

In [37]:
```
TRUE_LAMBDA = 5
X = np.random.exponential(TRUE_LAMBDA, 1000)
```

numpy defines the exponential distribution as

$$\frac{1}{\lambda}e^{-\frac{x}{\lambda}}$$

So we need to invert the MLE from the lecture notes. There it is

$$\hat{\lambda} = \frac{T}{\sum_{t=1}^{T} x_t}$$

Here it's just the reciprocal, so

$$\hat{\lambda} = \frac{\sum_{t=1}^{T} x_t}{T}$$

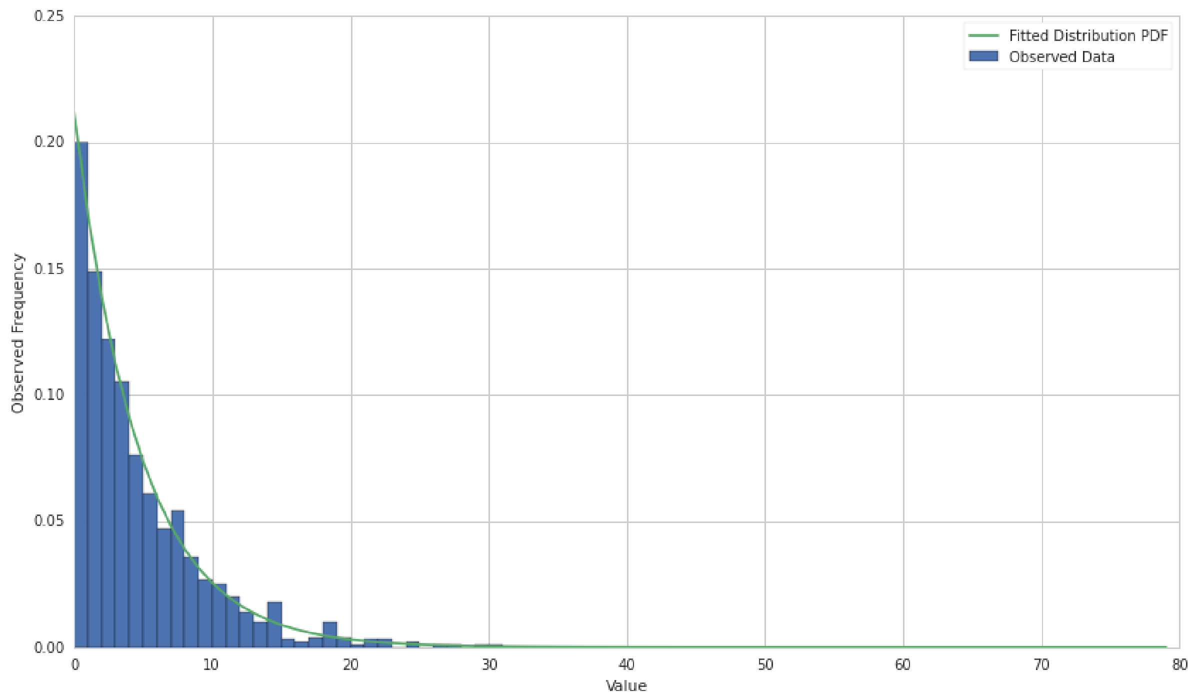
In [38]:
```
def exp_lamda_MLE(X):
    T = len(X)
    s = sum(X)
    return s/T
```

In [39]:
```
print "lambda estimate: " + str(exp_lamda_MLE(X))
```

```
lambda estimate: 4.71513802853
```

In [40]:
```
# The scipy version of the exponential distribution has a location parameter
# that can skew the distribution. We ignore this by fixing the location
# parameter to 0 with floc=0
_, l = scipy.stats.expon.fit(X, floc=0)
```

In [41]:
```
pdf = scipy.stats.expon.pdf
x = range(0, 80)
plt.hist(X, bins=x, normed='true')
plt.plot(pdf(x, scale=l))
plt.xlabel('Value')
plt.ylabel('Observed Frequency')
plt.legend(['Fitted Distribution PDF', 'Observed Data', ]);
```



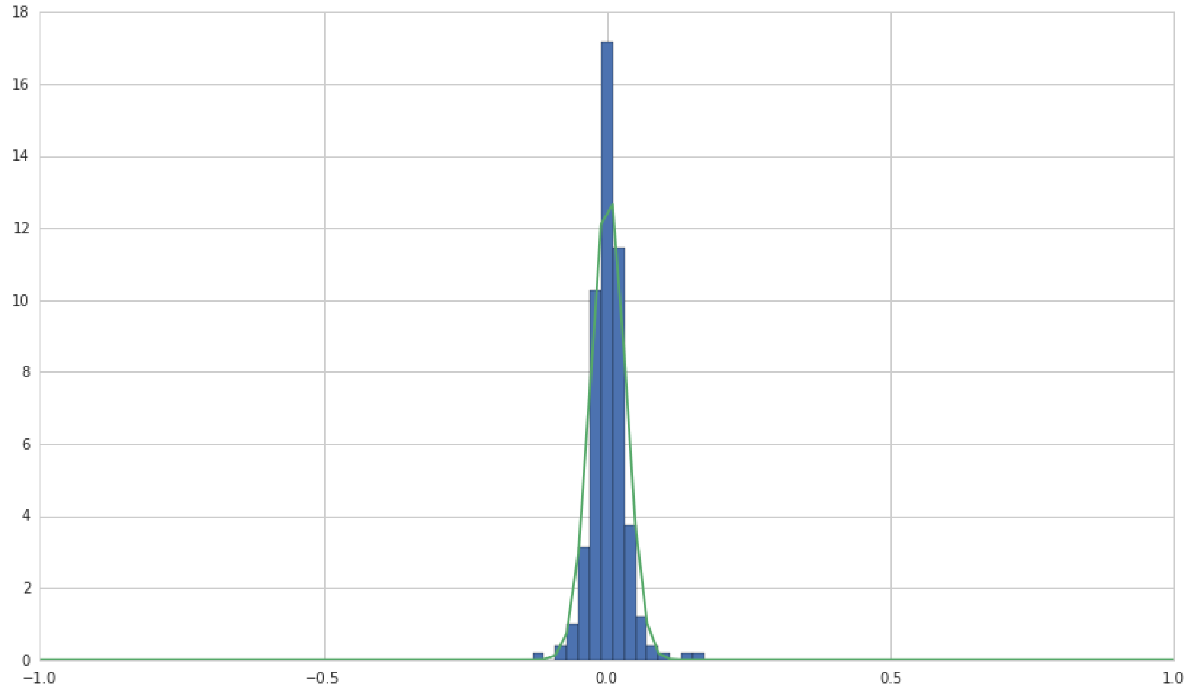
MLE for Asset Returns

Now we'll fetch some real returns and try to fit a normal distribution to them using MLE.

In [42]:
```
prices = get_pricing('TSLA', fields='price', start_date='2014-01-01', end_date='2015-01-01')
# This will give us the number of dollars returned each day
absolute_returns = np.diff(prices)
# This will give us the percentage return over the last day's value
# the [:-1] notation gives us all but the last item in the array
# We do this because there are no returns on the final price in the array.
returns = absolute_returns/prices[:-1]
```

Let's use `scipy`'s fit function to get the μ and σ MLEs.

In [43]:
```
mu, std = scipy.stats.norm.fit(returns)
pdf = scipy.stats.norm.pdf
x = np.linspace(-1,1, num=100)
h = plt.hist(returns, bins=x, normed='true')
l = plt.plot(x, pdf(x, loc=mu, scale=std))
```



Of course, this fit is meaningless unless we've tested that they obey a normal distribution first. We can test this using the Jarque-Bera normality test. The Jarque-Bera test will reject the hypothesis of a normal distribution if the p-value is under a c

In [45]:
```
from statsmodels.stats.stattools import jarque_bera
jarque_bera(returns)
```

Out[45]:
```
(242.53773931956744,
 2.1557534018848578e-53,
 0.7405363946340933,
 7.582279228661398)
```

In [46]:
```
jarque_bera(np.random.normal(0, 1, 100))
```

Out[46]:
```
(0.9856724005093672,
 0.61089132471452023,
 -0.10806345762002881,
 2.5642815902086933)
```

In [ ]: